Sorted-Pareto Dominance and Qualitative Notions of Optimality

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Abstract. Pareto dominance is often used in decision making to compare decisions that have multiple preference values – however it can produce an unmanageably large number of Pareto optimal decisions. When preference value scales can be made commensurate, then the Sorted-Pareto relation produces a smaller, more manageable set of decisions that are still Pareto optimal. Sorted-Pareto relies only on qualitative or ordinal preference information, which can be easier to obtain than quantitative information. This leads to a partial order on the decisions, and in such partially-ordered settings, there can be many different natural notions of optimality. In this paper, we look at these natural notions of optimality, applied to the Sorted-Pareto and min-sum of weights case; the Sorted-Pareto ordering has a semantics in decision making under uncertainty, being consistent with any possible order-preserving function that maps an ordinal scale to a numerical one. We show that these optimality classes and the relationships between them provide a meaningful way to categorise optimal decisions for presenting to a decision maker.

1 Introduction

In a decision-making task, it is often the case that the basis for comparing decisions involves more than one preference value (e.g., evaluations of multiple criteria in multi-criteria decision making, evaluations by more than one agent in multi-agent decision making, or considerations of different states in decision making under uncertainty), and therefore we have a preference vector for each decision. In these cases, Pareto dominance is an often used preference relation, where a decision Pareto dominates another if it is at least as good as the other in every component (comparing the preference vectors component-wise), and a decision is Pareto optimal if it is not Pareto dominated by any other [18, Ch. 2]. For example, for minimising costs, where costs are on an ordered scale $T = (low, med, hi)$, and we have three decisions with preference vectors: $a = (low, hi)$, $b = (med, low)$ and $c = (med, hi)$, we can see that both a and b Pareto dominate c, and also, since a and b do not Pareto dominate each other, they are both

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Pareto optimal. This relation is not very discerning though, and often the set of Pareto optimal decisions is very large. However, if the preference scales used in each component are commensurate (or can be normalised as such), then we can compare decisions by sorting the preference vectors first and then performing the component-wise comparison – which leads to a more discerning relation. For example, if a and b are sorted in non-descending order, i.e., (low, hi) and (low, med) respectively, the second vector now dominates the first, and therefore we now have only one undominated decision w.r.t. this new relation, which we call the Sorted-Pareto relation. This leads to a smaller, more manageable set of Sorted-Pareto optimal solutions, that are still Pareto optimal, as shown in [13].

If the scale T is quantitative, or we have information that gives a quantitative mapping for T, e.g., we have a mapping $f : T \to I\!R^+$, then the decisions could be compared by summing the preference vector values and seeing which decisions have the smallest sum of costs, i.e., the min-sum of weights. However, often the preference information available is only of an ordinal or qualitative nature, as it can be easier to obtain such information, e.g., there may be uncertainty over exact values, or it may be easier to elicit qualitative preference information from a decision maker [12]. Sorted-Pareto relies only on ordinal or qualitative information, and therefore can be used in these qualitative decision making situations. In addition, for any mapping $f : T \to I\!R^+$, where f is order-preserving w.r.t. scale T, we show that Sorted-Pareto is compatible with any such mapping.

In a partially ordered setting, such as in the situation just described, there can be different natural notions of optimality. The framework in [21] describes some of these notions, for qualitative decision making under uncertainty, where there are different possible scenarios in a given problem. This gives us classes of decisions that are not dominated by any other decision, decisions that are possibly optimal or possibly strictly optimal, (i.e., optimal in some scenario), and decisions that are optimal in all scenarios. Sorted-Pareto connects to Weighted Constraints Satisfaction Problems (WCSP) [17, Ch. 9] and Bayesian Networks [15] where we only have ordinal information, and in these frameworks the possibly optimal decisions are those that are min-sum optimal for some compatible WCSP, or are the complete assignments that are most probable in some compatible Bayesian Network. In this paper, we look at the relationship between Sorted-Pareto and min-sum of weights in Section 3, and in Section 4 we then examine these different natural notions of optimality from [21] and apply them to the Sorted-Pareto and min-sum of weights case. In Section 5, we show how to generate these optimality classes for Sorted-Pareto, and in Section 6 we present some experimental results.

2 Preliminaries

We assume a minimising context, where lower preference values are preferred. A preference relation $\preceq$ on a set $\mathcal{A}$ is a binary relation that gives an ordering over $\mathcal{A}$, i.e., given any $\alpha, \beta \in \mathcal{A}$, if $\alpha \preceq \beta$, then α is preferred to β according to $\preceq$. Relation $\preceq$ is a *preorder*, if it is reflexive ($\alpha \preceq \alpha$, for all $\alpha \in \mathcal{A}$) and transitive (i.e., if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$). Relation $\preceq$ is a *total preorder*, if it is complete (i.e., either $\alpha \preceq \beta$, or $\beta \preceq \alpha$, or both, for all $\alpha, \beta \in \mathcal{A}$) and transitive. For a preorder $\preceq$ on $\mathcal{A}$, we have a corresponding strict relation $\prec$, and a corresponding equivalence relation $\equiv$, defined respectively as: $\alpha \prec \beta$ if and only if $\alpha \preceq \beta$ and $\beta \not\preceq \alpha$; and $\alpha \equiv \beta$ if and only if $\alpha \preceq \beta$ and $\beta \preceq \alpha$.

We consider situations where the following preference information is available for some finite set of decisions $\mathcal{A}$. Let $\mathcal{S} = \{1, \ldots, m\}$ be a finite set, where each $i \in \mathcal{S}$ labels some aspect of the decisions in $\mathcal{A}$ for which a preference can be expressed. Let T be a scale, totally ordered by relation $\leq$. Let $\alpha_i \in T$ represent a preference value for decision $\alpha \in \mathcal{A}$ in aspect i. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be the preference vector of m preference values (to ease notation, we interchangeably use "α" as meaning a decision $\alpha \in \mathcal{A}$, or as meaning the evaluation vector $\alpha = (\alpha_1, \ldots, \alpha_m)$). Let $\alpha^{\uparrow} = (\alpha_{(1)}, \ldots, \alpha_{(m)})$ be the sorted preference vector such that $\alpha_{(1)} \leq \ldots \leq \alpha_{(m)}$, i.e., the values are ordered w.r.t. the scale T. For any two preference vectors α and β: $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$ for all $i \in \{1, \ldots, m\}$; and $\alpha < \beta$ if and only if $\alpha_i \leq \beta_i$ for all $i \in \{1, \ldots, m\}$, and there exists $j \in \{i, \ldots, m\}$ such that $\alpha_j < \beta_j$.

3 Sorted-Pareto and Min-Sum of Weights

In this section, we recall definitions for Sorted-Pareto dominance from [13], and show how this ordering relates to min-sum of weights. For all $\alpha, \beta \in \mathcal{A}$, decision α *Weak Sorted-Pareto dominates* β, written as $\alpha \preccurlyeq_{\text{SP}} \beta$, if and only if $\alpha^{\uparrow} \leq \beta^{\uparrow}$. Decision α *Sorted-Pareto dominates* β, written as $\alpha \prec_{\text{SP}} \beta$, if and only if $\alpha^{\uparrow} < \beta^{\uparrow}$, or in terms of $\preccurlyeq_{\text{SP}}$, if and only if $\alpha \preccurlyeq_{\text{SP}} \beta$ and $\beta \not\preccurlyeq_{\text{SP}} \alpha$. Decision α is *Sorted-Pareto equivalent* to β, written as $\alpha \equiv_{\text{SP}} \beta$, if and only if $\alpha^{\uparrow} = \beta^{\uparrow}$, or in terms of $\preccurlyeq_{\text{SP}}$, if and only if $\alpha \preccurlyeq_{\text{SP}} \beta$ and $\beta \preccurlyeq_{\text{SP}} \alpha$. Let $[\alpha]_{\text{SP}}$ denote the SP-equivalence class of $\alpha \in \mathcal{A}$, where $[\alpha]_{\text{SP}} = \{\beta \in \mathcal{A} : \alpha \equiv_{\text{SP}} \beta\}$ Decision α is *Sorted-Pareto optimal* (or *undominated*) if and only if there is no $\beta \in \mathcal{A}$ such that $\beta \prec_{\text{SP}} \alpha$.

Min-Sum of Weights. We consider situations in which there is additional quantitative preference information available, i.e., we have a function $f : T \to I\!R^+$. In such cases, we can order the set of decisions by using the min-sum of weights, defined as follows.

For some $f : T \to I\!R^+$, for all $\alpha, \beta \in \mathcal{A}$, decision α is *min-sum preferred* to β, written as $\alpha \leq_f \beta$, if and only if $\sum_{i=1}^{m} f(\alpha_i) \leq \sum_{i=1}^{m} f(\beta_i)$. Decision α is *strictly min-sum preferred* to β, written as $\alpha <_f \beta$, if and only if $\sum_{i=1}^{m} f(\alpha_i) < \sum_{i=1}^{m} f(\beta_i)$. Decision α is *min-sum equivalent* to β, written as $\alpha \equiv_f \beta$, if and only if $\sum_{i=1}^{m} f(\alpha_i) = \sum_{i=1}^{m} f(\beta_i)$. The relation $\leq_f$ forms a total preorder on a set of decisions $\mathcal{A}$. Decision α is *min-sum-optimal* for f if and only if for all $\beta \in \mathcal{A}$, $\alpha \leq_f \beta$.

3.1 Relating Sorted-Pareto and Min-Sum of Weights.

Let F be the set of all possible weight functions $f : T \to \mathbb{R}^+$ such that $f \in F$ if and only if f is monotonic w.r.t. T, i.e., $u \leq v \Leftrightarrow f(u) \leq f(v)$ for all $u, v \in T$.

Define the order relation $\leq_F$ on $\mathcal{A}$ as, for all $\alpha, \beta \in \mathcal{A}$, $\alpha \leq_F \beta \Leftrightarrow \alpha \leq_f \beta$, for all f monotonic w.r.t. T. From Theorem 1 in [13], we have that $\leq_F = \preccurlyeq_{\text{SP}}$.

Now, let F' be the set of all possible weight functions such that $f \in F'$ if and only if f is *strictly* monotonic w.r.t. T, i.e., $u < v \Leftrightarrow f(u) < f(v)$ for all $u, v \in T$. Define the order relation $\leq_{F'}$ as, for all $\alpha, \beta \in \mathcal{A}$, $\alpha \leq_{F'} \beta \Leftrightarrow \alpha \leq_f \beta$, for all f strictly monotonic w.r.t. T. Define $<_{\cap_{F'}}$ as the intersection of all $<_f$ such that $f \in F'$, i.e., $<_{\cap_{F'}} = \bigcap_{f \in F'} <_f$ so for all $\alpha, \beta \in \mathcal{A}$, $\alpha <_{\cap_{F'}} \beta$ if and only if for all $f \in F'$, $\alpha <_f \beta$. We have the following results (proofs are in an extended version of the paper [14]).

Theorem 1. $\preccurlyeq_{\text{SP}} = \leq_F = \leq_{F'}$

Corollary 1. $\prec_{\text{SP}} = <_{\cap_{F'}}$

4 Qualitative Notions of Optimality

In this section, we look at the different notions of optimality from the qualitative decision making framework in [21], which we use to describe the relationship between Sorted-Pareto and min-sum of weights. A Multiple Ordering Decision Structure (MODS) is a tuple $\mathcal{G} = \langle \mathcal{A}, \mathcal{P}, \{\preccurlyeq_p : p \in \mathcal{P}\} \rangle$, where $\mathcal{A}$ is a set of decisions, $\mathcal{P}$ is a set of possible scenarios, and for each $p \in \mathcal{P}$, relation $\preccurlyeq_p$ is a total preorder on $\mathcal{A}$, with corresponding strict and equivalence relations $\prec_p$ and $\equiv_p$ respectively.

For any instance of this framework, we have the following relations that always hold in general. Decision α *necessarily dominates* β, written $\alpha \preccurlyeq_N \beta$, if and only if $\alpha \preccurlyeq_p \beta$, for all $p \in \mathcal{P}$. Relation $\preccurlyeq_{\text{N}}$ is the intersection of $\preccurlyeq_p$ over all $p \in \mathcal{P}$. Relation $\preccurlyeq_N$ has corresponding strict and equivalence relations $\prec_N$ and $\equiv_N$ respectively. Let $[\alpha]_{\text{N}}$ denote the N-equivalence class of $\alpha \in \mathcal{A}$, where $[\alpha]_{\text{N}} = \{\beta \in \mathcal{A} : \alpha \equiv_N \beta\}$. Decision α *necessarily strictly dominates* β, written $\alpha \prec_{\text{NS}} \beta$, if and only if $\alpha \prec_p \beta$ for all $p \in \mathcal{P}$. Relation $\prec_{\text{NS}}$ is the intersection of $\prec_p$ over all $p \in \mathcal{P}$.

Optimality Classes. We now look at different notions of optimality for the general case. Decision α is *necessarily optimal* if and only if $\alpha \preccurlyeq_{\text{N}} \beta$ for all $\beta \in \mathcal{A}$. The set of these decisions is denoted by $\text{NO}(\mathcal{G})$. Decision α is *necessarily strictly optimal* if and only if $\alpha \prec_{\text{NS}} \beta$ for all $\beta \in \mathcal{A} \setminus [\alpha]_{\text{N}}$. The set of these decisions is denoted by $\text{NSO}(\mathcal{G})$. Decision α is *possibly optimal* if and only if there exists $p \in \mathcal{P}$ such that $\alpha \preccurlyeq_p \beta$ for all $\beta \in \mathcal{A}$. The set of these decisions is denoted by $\text{PO}(\mathcal{G})$. Decision α is *possibly strictly optimal* if and only if there exists $p \in \mathcal{P}$ such that $\alpha \prec_p \beta$ for all $\beta \in \mathcal{A} \setminus [\alpha]_{\text{N}}$. The set of these decisions is denoted by $\text{PSO}(\mathcal{G})$. A decision α is in $\text{CD}(\mathcal{G})$, if and only if for all $\beta \in \mathcal{A}$, there exists $p \in \mathcal{P}$ such that $\alpha \preccurlyeq_p \beta$. $\text{CD}(\mathcal{G})$ are the decisions that are undominated w.r.t. $\prec_{\text{NS}}$. A decision α is in $\text{CSD}(\mathcal{G})$ if and only if for all $\beta \in \mathcal{A} \setminus [\alpha]_{\text{N}}$, there exists $p \in \mathcal{P}$ such that $\alpha \prec_p \beta$. $\text{CSD}(\mathcal{G})$ are the decisions that are undominated w.r.t. $\prec_{\text{N}}$. We also have the following optimality classes, which are intersections between existing classes. NOPSO($\mathcal{G}$) is the intersection of NO($\mathcal{G}$) and PSO($\mathcal{G}$). PO′($\mathcal{G}$) is the intersection of PO($\mathcal{G}$) and CSD($\mathcal{G}$).

Figure 1 shows precisely the subclass relationships between these optimality classes that always hold in the general case, as given by Theorem 1 in [21]. [21] also gives an example of strict subclass relationships between each of the optimality classes.

$$\begin{tikzcd}
 & & & \mathrm{NO}(\mathcal{G}) \arrow[dr] & & \mathrm{PO}(\mathcal{G}) \arrow[dr] & & \\
\emptyset \arrow[r] & \mathrm{NSO}(\mathcal{G}) \arrow[r] & \mathrm{NOPSO}(\mathcal{G}) \arrow[ur] \arrow[dr] & & \mathrm{PO}'(\mathcal{G}) \arrow[ur] \arrow[dr] & & \mathrm{CD}(\mathcal{G}) \arrow[r] & \mathcal{A} \\
 & & & \mathrm{PSO}(\mathcal{G}) \arrow[ur] & & \mathrm{CSD}(\mathcal{G}) \arrow[ur] & &
\end{tikzcd}$$

Fig. 1. Subclass relationships ($\subseteq$) between classes that always hold in general.

4.1 Sorted-Pareto MODS

Recall from Section 3, where we define F be the set of all possible weight functions $f : T \to I\!R^+$ such that f is monotonic w.r.t. T. We also have that for all $\alpha, \beta, \in \mathcal{A}$, $\alpha \preccurlyeq_{\text{SP}} \beta \Leftrightarrow \alpha \leq_f \beta$ for all $f \in F$, i.e., α Weak Sorted-Pareto dominates β if and only if α is min-sum-preferred to β for all $f \in F$. This gives us the Sorted Pareto MODS $\mathcal{S} = \langle \mathcal{A}, F, \{\leq_f : f \in F\}\rangle$, where the set of scenarios is the set F of possible order-preserving weight functions $f : T \to I\!R^+$, and the set of possible orderings is that given by the min-sum of weights orderings for all possible weight functions, i.e., the set $\{\leq_f : f \in F\}$.

For the Sorted-Pareto MODS $\mathcal{S}$, we have the following relations. Decision α necessarily dominates β if and only if $\alpha \leq_f \beta$ for all $f \in F$. Since $\alpha \leq_f \beta$, for all $f \in F \Leftrightarrow \alpha \preccurlyeq_{\text{SP}} \beta$, this gives us the result in Proposition 1.

Proposition 1. *For* MODS $\mathcal{S}$, $\preccurlyeq_{\text{N}} = \preccurlyeq_{\text{SP}}$

Since we have that $\alpha \prec_{\text{SP}} \beta$ if and only if $\alpha \preccurlyeq_{\text{SP}} \beta$ and $\beta \not\preccurlyeq_{\text{SP}} \alpha$, then we also have that $\prec_{\text{N}} = \prec_{\text{SP}}$. Decision α necessarily strictly dominates β if and only if $\alpha <_f \beta$ for all $f \in F$. Since from Corollary 1, $<_{\cap_{F'}} = \prec_{\text{SP}}$, and $<_{\cap_{F'}}$ is defined as the intersection of all $<_f$ such that $f \in F'$ (and $F' \subseteq F$), then we have the result in Proposition 2.

Proposition 2. *For* MODS $\mathcal{S}$, $\prec_{\text{NS}} = \prec_{\text{N}} = \prec_{\text{SP}}$

We have an equivalence relation for each $f \in F$, i.e., $\alpha \equiv_f \beta$ if and only if $\sum_{i=1}^{m} f(\alpha_i) = \sum_{i=1}^{m} f(\beta_i)$. We also have an equivalence relation $\equiv_F$, which is the intersection of $\equiv_f$ over all $f \in F$, i.e., $\equiv_F$ is equal to $\bigcap_{f\in F} \equiv_f$, so $\alpha \equiv_F \beta$ if and only if they are equivalent over all possible choice of f. Let $[\alpha]_F$ denote the F-equivalence class of $\alpha \in \mathcal{A}$, where $[\alpha]_F = \{\beta \in \mathcal{A} : \alpha \equiv_F \beta\}$. Since we have from Theorem 1 in [13] that $\leq_F$ is equal to $\preccurlyeq_{\text{SP}}$, i.e., $\preccurlyeq_{\text{SP}} = \bigcap_{f\in F} \leq_f$, then we have that $\equiv_F$ is equal to $\equiv_{\text{SP}}$, i.e., $\equiv_{\text{SP}}$ is the intersection of $\equiv_f$ over all $f \in F$, which gives us the result in Proposition 3.

Proposition 3. *For* MODS $\mathcal{S}$, $\equiv_N = \equiv_{\text{SP}}$

Sorted-Pareto Optimality classes. We now look at the notions of optimality that are applicable for the Sorted-Pareto MODS $\mathcal{S}$. Decision α is in NO($\mathcal{S}$) if and only if for all $\beta \in \mathcal{A}$, for all $f \in F$, $\alpha \leq_f \beta$, i.e., if and only if $\alpha \preccurlyeq_{\text{SP}} \beta$ for all $\beta \in \mathcal{A}$. Decision α is in NSO($\mathcal{S}$) if and only if for all $\beta \in \mathcal{A} \setminus [\alpha]_F$, for all $f \in F$, $\alpha <_f \beta$, i.e., if and only if $\alpha \prec_{\text{SP}} \beta$ for all $\beta \in \mathcal{A} \setminus [\alpha]_{\text{SP}}$.

These definitions and Proposition 2 give us the result in Proposition 4.

Proposition 4. *For* MODS $\mathcal{S}$*,* NSO($\mathcal{S}$) = NOPSO = NO($\mathcal{S}$)

Decision α is in CD($\mathcal{S}$) if and only if for all $\beta \in \mathcal{A}$, there exists $f \in F$ such that $\alpha \leq_f \beta$. Decision α is in CSD($\mathcal{S}$) if and only if for all $\beta \in \mathcal{A} \setminus [\alpha]_F$, there exists $f \in F$ such that $\alpha <_f \beta$.

Since in the general case CD($\mathcal{G}$) are the decisions that are undominated w.r.t. $\prec_{\text{NS}}$ and CSD($\mathcal{G}$) are the decisions that are undominated w.r.t. $\prec_{\text{N}}$, and also from Proposition 1 we have $\prec_{\text{NS}} = \prec_{\text{N}}$, then this gives us the result in Proposition 5.

Proposition 5. *For* MODS $\mathcal{S}$*,* CSD($\mathcal{S}$) = CD($\mathcal{S}$)

Decision α is in PO($\mathcal{S}$) if and only if there exists $f \in F$ such that for all $\beta \in \mathcal{A}$, $\alpha \leq_f \beta$. Decision α is in PSO($\mathcal{S}$) if and only if there exists $f \in F$ such that for all $\beta \in \mathcal{A} \setminus [\alpha]$, $\alpha \leq_f \beta$. Let PO$'$($\mathcal{S}$) = PO($\mathcal{S}$) $\cap$ CSD($\mathcal{S}$) and NOPSO($\mathcal{S}$) = NO($\mathcal{S}$) $\cap$ PSO($\mathcal{S}$).

Since in the general case PO($\mathcal{G}$) $\subseteq$ CD($\mathcal{G}$), and since we have from Proposition 5 that CSD($\mathcal{S}$) = CD($\mathcal{S}$), this gives us the result in Proposition 6.

Proposition 6. *For* MODS $\mathcal{S}$*,* PO($\mathcal{S}$) $\subseteq$ CSD($\mathcal{S}$).

Given these results, we now look at the subclass relationship between the optimality classes for the Sorted-Pareto instance of the MODS framework. Propositions 1-6 and definitions give us that (NSO($\mathcal{S}$) = NOPSO($\mathcal{S}$) = NO($\mathcal{S}$)) $\subseteq$ PSO($\mathcal{S}$) $\subseteq$ (PO($\mathcal{S}$) = PO$'$($\mathcal{S}$)) $\subseteq$ (CSD($\mathcal{S}$) = CD($\mathcal{S}$)), as shown in Figure 2.

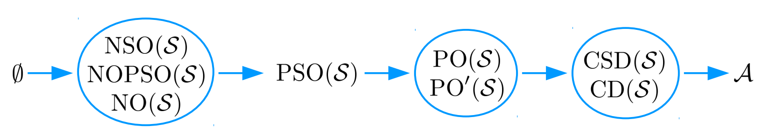

Fig. 2. Subclass relationships ($\subseteq$) between classes for MODS $\mathcal{S}$.

Now we consider the case where there exists a decision that is necessarily optimal, i.e., when NO($\mathcal{S}$) $\neq \emptyset$. Proposition 5 in [21] gives us that if NO($\mathcal{S}$) $\neq \emptyset$, then we have NO($\mathcal{S}$) = CSD($\mathcal{S}$), and therefore we have a single SP-equivalence class, where the decisions are all equivalent. This gives us the result in Proposition 7.

Proposition 7. *For* MODS $\mathcal{S}$*, if* NO($\mathcal{S}$) $\neq \emptyset$*, then* NSO($\mathcal{S}$) = NO($\mathcal{S}$) = PSO($\mathcal{S}$) = PO($\mathcal{S}$) = CSD($\mathcal{S}$) = CD($\mathcal{S}$) $\subseteq \mathcal{A}$

5 Computing Optimality classes for MODS $\mathcal{S}$

In this section, we look at methods for generating the different optimality classes for Sorted-Pareto MODS $\mathcal{S}$. Here we assume that there is some procedure to generate CSD($\mathcal{S}$), i.e., that calculates the preference vectors for each decision and compares them using Sorted-Pareto dominance to generate the set of decisions that are non-dominated. For example, the branch and bound search algorithms detailed in [13] do exactly this; however other search procedures can be used. From CSD($\mathcal{S}$), NO($\mathcal{S}$) can be calculated by comparing all the solutions in CSD($\mathcal{S}$) with one another to see if any Sorted-Pareto dominate all others. In our experimental results in Section 6, we use the procedure outlined in [13] to calculate CSD($\mathcal{S}$), where the algorithm has been modified to maintain the set of currently non-dominated preference vectors, each preference vector mapping to the corresponding equivalence class of decisions. This leads to a substantial improvement in computation times as it results in a reduction in the number of dominance checks performed by the algorithm.

Calculating PO($\mathcal{S}$) and PSO($\mathcal{S}$). We want to determine if some decision α in CSD($\mathcal{S}$) is possibly optimal, i.e., there exists some weight function $f \in F$ such that $\alpha \leq_f \beta$ for all $\beta \in$ CSD($\mathcal{S}$). We can formulate this problem as a linear program P, as follows. Only certain elements on the scale T appear in any of the preference vectors for the decisions in CSD($\mathcal{S}$); let T' denote this set, i.e., $T' = \{i \in \beta : \beta \in \text{CSD}(\mathcal{S})\}$. For each of these elements $i \in T'$ we have a linear program variable w_i, representing an unknown weight. Since the scale T is totally ordered, then on these weights we have constraints of the form $w_i < w_j$, where $i < j$. For all $\beta \in$ CSD($\mathcal{S}$), we have a linear expression $\omega(\beta)$ as a sum in terms of the unknown weight variables, i.e., $\omega(\beta) = \sum_{i \in \beta} w_i$. For α to be possibly optimal, we require, for each $\beta \in$ CSD($\mathcal{S}$), that $\omega(\alpha) \leq \omega(\beta)$. Therefore we have a set P of linear inequalities, which consists of, (i) $w_i < w_j$, for all $i, j \in T'$, where $i < j$, and (ii) $\omega(\alpha) \leq \omega(\beta)$, for all $\beta \in$ CSD($\mathcal{S}$). If P has a feasible solution, then there exists some weights that make $\alpha \leq \beta$ for all $\beta \in$ CSD($\mathcal{S}$), i.e., α is possibly optimal.

In order to check this using a standard linear program solver, we need to convert to an equivalent problem which only has non-strict inequalities. Therefore, we create a linear program P' as follows, where $c > 0$ is some arbitrary strictly positive real number, for example, let us choose $c = 1$. Then, for any constraint in P of the form $w_i < w_j$, we have a constraint in P' with the form $w_j - w_i \geq c$, and for any constraint in P of the form $\omega(\alpha) \leq \omega(\beta)$, we have a constraint of the form $\omega(\beta) - \omega(\alpha) \geq 0$. We then solve the linear program P', and this has a solution if and only if P has a solution, and α is possibly optimal.

We can also determine if some solution is possibly strictly optimal, i.e., there exists f such that for all $\beta \in \text{CSD}(\mathcal{S}) \setminus [\alpha]$, $\alpha <_f \beta$. We have a set Q of linear inequalities for this problem, which consists of, (i) $w_i < w_j$, for all $i, j \in T'$, where $i < j$ and, (ii) $\omega(\alpha) < \omega(\beta)$, for all $\beta \in$ CSD($\mathcal{S}$). We again create a modified linear program Q' as follows: for any constraint in Q of the form $w_i < w_j$, we have a constraint in Q' with the form $w_j - w_i \geq c$, and for any constraint in Q of the form $\omega(\alpha) < \omega(\beta)$, we have a constraint in Q' of the form $\omega(\beta) - \omega(\alpha) \geq c$. We then solve the linear program Q', and this has a solution if and only if Q has a solution, and α is possibly strictly optimal.

Proposition 8. *The set of linear inequalities P has a solution if and only if linear program P' has a solution. The set of linear inequalities Q has a solution if and only if linear program Q' has a solution.*

6 Experimental Results

In this section, we calculate the optimality classes CSD($\mathcal{S}$), PO($\mathcal{S}$), PSO($\mathcal{S}$), and NO($\mathcal{S}$) for some randomly generated and benchmark instances (details of the generation process are in the extended version of the paper [14]). As detailed in Section 5, we use the branch and bound algorithm from [13] to generate CSD($\mathcal{S}$) and NO($\mathcal{S}$), and we solve linear programs to generate PO($\mathcal{S}$) and PSO($\mathcal{S}$). The instances used are Weighted Constraint Satisfaction problems (WCSP) [17, Ch.9], where, for a set of problem variables, each variable can be assigned a value from its domain, and a complete assignment to all of the variables is a solution to the problem (which corresponds to a decision). There is also a set of weighted constraints which associate weights to these assignments, and these correspond to the preference levels of the solutions.

Table 1. Average size of optimality sets over 50 random instances, n denotes problem size, sc denotes size of preference vector (increasing), $|T|$ denotes size of scale.

		CSD($\mathcal{S}$)			PO($\mathcal{S}$)		
n	sc	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$
20	48	9.12 (2.90)	9.48 (6.78)	21.68 (19.28)	7.60 (2.62)	8.22 (5.88)	17.44 (15.70)
24	69	9.94 (3.70)	11.76 (10.18)	55.18 (51.30)	8.90 (3.46)	9.72 (8.36)	36.88 (34.50)
28	95	10.12 (4.52)	15.96 (14.04)	67.54 (63.88)	8.32 (3.86)	11.54 (10.32)	40.30 (38.44)
32	124	9.56 (5.00)	21.44 (19.28)	114.58 (110.96)	6.76 (3.86)	13.44 (12.42)	58.38 (56.82)
36	158	10.60 (5.68)	30.42 (27.04)	145.36 (143.02)	8.22 (4.64)	18.24 (16.58)	68.16 (67.06)
40	195	10.20 (5.38)	24.12 (23.00)	135.14 (134.44)	8.50 (4.62)	15.94 (15.38)	63.96 (63.84)

		PSO($\mathcal{S}$)			NO($\mathcal{S}$)		
n	sc	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$
20	48	6.52 (2.30)	8.02 (5.78)	17.42 (15.68)	0.20 (0.10)	0.00 (0.00)	0.00 (0.00)
24	69	6.58 (2.92)	9.44 (8.18)	36.80 (34.46)	0.02 (0.02)	0.00 (0.00)	0.00 (0.00)
28	95	6.14 (3.24)	11.36 (10.14)	40.24 (38.40)	0.02 (0.02)	0.00 (0.00)	0.00 (0.00)
32	124	5.60 (3.36)	13.12 (12.16)	58.14 (56.68)	0.02 (0.02)	0.00 (0.00)	0.00 (0.00)
36	158	5.44 (3.72)	17.88 (16.26)	67.96 (66.92)	0.02 (0.02)	0.00 (0.00)	0.00 (0.00)
40	195	6.18 (3.78)	15.32 (14.92)	63.86 (63.76)	0.00 (0.00)	0.00 (0.00)	0.00 (0.00)

Random Instances. For these random instances: n denotes problem size, i.e., the number of variables; sc denotes the size of the preference vector for each solution, i.e., the number of weighted constraints; and $|T|$ denotes the size of the ordinal scale used. Each set of problems was generated with 3 different scales, with $|T| = 3, 5$ and 7.

Table 1 shows the average size of the optimality classes (and the average number of equivalence classes in parentheses) for 50 random instances for problem size $n = 20, 24, \ldots, 40$. The size of the preference vector sc (i.e., the number of weighted constraints) is varied as a parameter of the problem size. It can be observed that $\mathrm{PO}(\mathcal{S})$ is usually smaller than $\mathrm{CSD}(\mathcal{S})$, with $\mathrm{PSO}(\mathcal{S})$ smaller again, and in nearly all cases $\mathrm{NO}(\mathcal{S})$ is empty.

Table 2 shows the average size of the optimality classes (and the average number of equivalence classes in parentheses) for 50 random instances for problem size $n = 20, 24, \ldots, 40$. The size of the preference vector sc is fixed at 10 for all instances. In these problems, the size of the $\mathrm{CSD}(\mathcal{S})$ sets are much larger than in Table 1, and often the same size as the $\mathrm{PO}(\mathcal{S})$ set. However the equivalence classes are much smaller, indicating that for these problems there are a large number of equivalent optimal solutions in each optimality class. Often $\mathrm{NO}(\mathcal{S})$ is non-empty, indicating a single equivalence class of necessarily optimal solutions, and in these cases we have $\mathrm{CSD}(\mathcal{S}) = \mathrm{PO}(\mathcal{S}) = \mathrm{PSO}(\mathcal{S}) = \mathrm{NO}(\mathcal{S})$.

Table 2. Average size of optimality sets over 50 random instances, n denotes problem size, sc denotes size of preference vector (fixed), $|T|$ denotes size of scale.

		CSD($\mathcal{S}$)			PO($\mathcal{S}$)		
n	sc	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$
20	10	190.16 (1.44)	121.38 (2.12)	116.84 (3.56)	190.16 (1.44)	118.50 (2.06)	115.12 (3.42)
24	10	330.88 (1.66)	191.08 (2.36)	242.48 (3.98)	330.88 (1.66)	190.84 (2.32)	227.26 (3.78)
28	10	379.14 (1.52)	196.32 (1.96)	201.68 (3.14)	373.86 (1.50)	186.06 (1.86)	191.22 (3.00)
32	10	642.56 (1.62)	393.72 (2.32)	354.36 (3.72)	642.56 (1.62)	393.72 (2.32)	344.16 (3.54)
36	10	925.92 (1.48)	709.56 (2.08)	663.32 (3.20)	925.92 (1.48)	697.56 (2.02)	652.30 (3.14)
40	10	1177.10 (1.54)	904.24 (2.18)	779.72 (3.06)	1177.10 (1.54)	904.24 (2.18)	779.72 (3.06)

		PSO($\mathcal{S}$)			NO($\mathcal{S}$)		
n	sc	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$	$\|T\| = 3$	$\|T\| = 5$	$\|T\| = 7$
20	10	167.54 (1.38)	118.50 (2.06)	114.02 (3.40)	86.94 (0.62)	36.40 (0.38)	15.36 (0.14)
24	10	275.30 (1.58)	190.84 (2.32)	227.26 (3.78)	96.10 (0.44)	15.50 (0.22)	3.88 (0.12)
28	10	373.06 (1.48)	186.06 (1.86)	191.22 (3.00)	164.94 (0.52)	80.56 (0.38)	18.14 (0.22)
32	10	638.78 (1.60)	392.76 (2.30)	344.04 (3.52)	183.22 (0.42)	11.42 (0.18)	5.92 (0.08)
36	10	925.92 (1.48)	697.56 (2.02)	652.30 (3.14)	150.98 (0.52)	78.00 (0.32)	18.32 (0.12)
40	10	1166.54 (1.52)	904.24 (2.18)	779.72 (3.06)	383.24 (0.50)	265.18 (0.32)	96.92 (0.14)

Benchmark Instances. Table 3 shows the size of the optimality classes (and the number of equivalence classes in parentheses), when applied to some modified WCSP instances from the Celar Radio-Link Frequency Assignment problem benchmark, where again problem size is denoted by n, sc denotes the size of the preference vector, and T denotes the size of the scale. These instances have been modified by adding random binary hard constraints to the problem, to limit the expected number of solutions to around $10,000$. $\mathrm{PO}(\mathcal{S})$ is usually smaller than $\mathrm{CSD}(\mathcal{S})$, but $\mathrm{PSO}(\mathcal{S})$ is only very seldom smaller than $\mathrm{PO}(\mathcal{S})$. In all of these instances, $\mathrm{NO}(\mathcal{S})$ is empty.

Table 3. Size of optimality sets for modified CELAR benchmark instances, n denotes problem size, sc denotes size of preference vector, $|T|$ denotes size of scale.

Instance	n	sc	$\|T\|$	CSD($\mathcal{S}$)	PO($\mathcal{S}$)	PSO($\mathcal{S}$)	NO($\mathcal{S}$)
CELAR6-SUB0*	16	207	5	17 (16)	12 (11)	12 (11)	0 (0)
CELAR6-SUB1*	14	300	5	24 (20)	13 (11)	11 (9)	0 (0)
CELAR6-SUB2*	16	353	5	20 (12)	19 (11)	19 (11)	0 (0)
CELAR6-SUB3*	18	421	5	4 (3)	4 (3)	4 (3)	0 (0)
CELAR6-SUB4*	22	477	5	6 (6)	6 (6)	6 (6)	0 (0)
CELAR7-SUB0*	16	188	5	10 (10)	8 (8)	8 (8)	0 (0)
CELAR7-SUB1*	14	300	5	15 (11)	14 (10)	14 (10)	0 (0)
CELAR7-SUB2*	16	353	5	10 (8)	7 (5)	7 (5)	0 (0)
CELAR7-SUB3*	18	421	5	19 (15)	13 (9)	13 (9)	0 (0)
CELAR7-SUB4*	22	477	5	8 (8)	5 (5)	5 (5)	0 (0)

Discussion. One possible approach to choosing which decisions to present to a decision maker is to calculate CSD($\mathcal{S}$) first, and from this set, NO($\mathcal{S}$) can be easily derived. If NO($\mathcal{S}$) is not empty, then there are one or more equivalent decisions which are preferred to all other decisions for any choice of f, and these are prime candidates for presenting to a decisions maker. However, if NO($\mathcal{S}$) is empty, then PO($\mathcal{S}$) or PSO($\mathcal{S}$) can be computed and presented, these sets are often much smaller than CSD($\mathcal{S}$). PO($\mathcal{S}$) is the set of decisions that are min-sum optimal for some possible f, and thus are good candidates to present to a decision maker. If the PO($\mathcal{S}$) set is large, and there is a small number of equivalence classes, then a representative solution for each equivalence class could be chosen to present to a decision maker, since this would give a decision maker a choice between non-equivalent solutions that are possibly min-sum-optimal.

7 Related Work

As well as our own work [13,21], on which this work builds, Larichev and Moshkovich [11] use Sorted-Pareto in the context of normalising different criteria scales, and Kaci and Prade [10] use it in preference handling using possibilistic logic. Both Perny and Spanjaard [16] and Bossong and Schweigert [4] look at preference based search for generating sets of optimal solutions for shortest path problems, which is related to Sorted-Pareto as previously outlined in [13]. The Sorted Pareto relation extends the Pareto dominance relation [18], and computing the Sorted-Pareto optimal set is viable when preference level scales are commensurate, since calculating the Pareto optimal set can be prohibitive. Some works that approximate the Pareto optimal set in constraints problems include Torrens and Faltings [20], however this requires quantitative information as it performs a sum of weights on the preference vector, and Gavanelli [8] uses a branch and bound algorithm similar to what is used in [13]. Sorted-Pareto is reminiscent of Lorenz dominance [19], and is extended by preference relations that perform a lexicographic comparison on reordered vectors of preference levels, such as Lexicographic Min-Max [7] in multicriteria optimisation, and Leximin [6]. These lexicographic orderings place excessive emphasis on the worse preference values, since they ignore better values when comparing two decisions, whereas Sorted-Pareto compares over all values. Bouveret and Lemaître [5] looks at depth first branch and bound algorithms for the computation of Leximin optimal solutions. The notions of optimality in the MODS framework are partly inspired by Gelain et al. [9], who investigate optimality for interval-valued constraints, however we assume only qualitative or ordinal information. Also, the MODS framework relates to decision making under complete uncertainty or ignorance (such as in Arrow [1]), since there is no quantitative information assumed on the importance or likelihood of scenarios.

8 Conclusion

In this paper, we looked at Sorted-Pareto dominance, a preference relation that assumes only qualitative information, and based on the correspondence between Sorted-Pareto and decision making under uncertainty, we argue that there are other natural notions of optimal decision. Specifically, we look at decisions that are undominated, i.e, CSD($\mathcal{S}$), the solutions that are optimal and strictly optimal in one (or more) scenarios, i.e., PO($\mathcal{S}$) and PSO($\mathcal{S}$) respectively, and the solutions that are optimal in all scenarios, i.e., NO($\mathcal{S}$). We explore the relations between these notions of optimality and show how to compute them for the Sorted-Pareto ordering and the min-sum of weights case. The experimental results show, that in some cases, NO($\mathcal{S}$) is non-empty, and these are the decisions that would be of most interest to a decision maker. However, in other cases, no such decisions exist, and then PO($\mathcal{S}$) and PSO($\mathcal{S}$) are of interest to a decision maker since these are the decisions that are optimal or strictly optimal in some scenario. The Sorted-Pareto ordering connects with Weighted Constraint Satisfaction problems (WCSP) [17, Ch. 9] (or similarly, with Generalised Additive Independence decompositions [2]), where a problem has only weights on an ordinal scale T; each such problem has a set of compatible proper weighted constraints problems, based on mapping the ordinal scale $T \to I\!R^+$. Sorted-Pareto is also connected to Bayesian Networks [15], where in a given network we only have ordinal probabilistic information and therefore we have an associated set of compatible Bayesian Networks. In a Weighted CSP with ordinal weights, the decisions that are possibly optimal are those that are min-sum optimal in some compatible weighted constraints problem, and in a Bayesian Network with ordinal probabilities, the possibly optimal decisions are those assignments that are most probable in some compatible Bayesian Network. In the context of decision making under uncertainty, we argue that these decisions would certainly be of interest to a decision maker.

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