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## Boolean Rings are Definitely Commutative!

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ABSTRACT. A ring  $\{R, +, \cdot\}$  is called Boolean if  $r^2 = r$  for all  $r \in R$ . We present four proofs that a Boolean ring is commutative.

A ring  $\{R, +, \cdot\}$  is called Boolean if  $r^2 = r$  for all  $r \in R$ . In this bicentenary year of Boole's birth we present four proofs that a Boolean ring is commutative. Our first proof is the standard one found in many textbooks.

*Proof 1.* For all  $r \in R$  we have  $r = r^2 = (-r)^2 = -r$ , so  $r + r = 0$ . Next, for all  $x$  and  $y$  in  $R$ ,  $x + y = (x + y)^2 = x^2 + xy + yx + y^2$ , so by cancellation in the group  $\{R, +\}$ , we have  $xy + yx = 0 = xy + xy$ , by the above. Again by cancellation we have  $xy = yx$ , as required.  $\square$

*Proof 2.* As in Proof 1,  $xy + yx = 0$ , for all  $x$  and  $y$  in  $R$ . Since for all  $r \in R$ ,  $0.r = 0 = r.0$  we have  $(xy + yx)x = x(xy + yx)$  or  $xyx + y.x^2 = x^2.y + xyx$ . Cancelling  $xyx$  and remembering that  $x^2 = x$ , we get  $xy = yx$ , as required.  $\square$

*Proof 3.* Since for all  $r$ ,  $r^2 = r$  it follows that if  $r^2 = 0$  then  $r = 0$ . Now for all  $x$  and  $y$  in  $R$  we have  $(xy - xyx)^2 = xyxy + xyxxyx - xyxyx - xyxxy = xyxy + xyxxyx - xyxyx - xyxxy = 0$ . So  $xy - xyx = 0$  and  $xy = xyx$ . Then  $(yx - xyx)^2 = yxyx + xyxxyx - yxxyx - xyxyx = yxyx + xyxxyx - yxxyx - xyxyx = 0$ . So  $yx - xyx = 0$  and  $yx = xyx$ . Thus  $xy = yx$  as required.  $\square$

*Proof 4.* For  $a, b \in R$  if  $ab = 0$ , then  $ba = (ba)^2 = b(ab)a = 0$ . Now,  $0 = xy - xy = xy - x^2y = x(y - xy)$ , so  $0 = (y - xy)x = yx - xyx$ . Also,  $0 = yx - yx = yx - yx^2 = (y - yx)x$ , so  $0 = x(y - yx) = xy - xyx$ . Thus  $xy = yx$  for all  $x$  and  $y$  in  $R$ .  $\square$

We note it is immediate in all four proofs that  $xy = yx = xyx = yxy$ , for all  $x$  and  $y$ .

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