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ON THE ELLIPTIC SINH-GORDON EQUATION WITH INTEGRABLE BOUNDARY CONDITIONS

M. KILIAN AND G. SMITH

Dedicated to Ulrich Pinkall on the occasion of his 65th birthday.

ABSTRACT. We adapt Sklyanin's K-matrix formalism to the sinh-Gordon equation, and prove that all free boundary constant mean curvature (CMC) annuli in the unit ball in $\mathbb{R}^3$ are of finite type.

1. CMC immersions and the sinh-Gordon equation

Let $\Omega \subseteq \mathbb{C}$ be an open subset with smooth boundary $\partial\Omega$. For $r > 0$, let $B_r(0)$ denote the open ball of radius r about the origin in $\mathbb{R}^3$. A minimal or CMC immersion $f : \overline{\Omega} \to B_r(0)$ is said to have *free boundary* whenever it meets $S_r(0) := \partial B_r(0)$ orthogonally along $\partial\Omega$. Free boundary minimal and CMC surfaces have attracted the interest of geometric analysts since the work [9, 10] of Fraser–Schoen. The purpose of this paper is to open the way to applying integrable systems techniques to the study of free boundary CMC annuli.

In order to better explain the ideas studied in the sequel, we briefly review the case of CMC tori in $\mathbb{R}^3$. Here, the modern use of integrable systems techniques traces its roots to Wente [20], and further developed by Abresch [1]. These ideas were extended by Pinkall–Sterling [17] by showing that, modulo closing conditions, the study of immersed CMC tori in $\mathbb{R}^3$ is equivalent to the study of real, doubly-periodic solutions of the elliptic sinh-Gordon equation. Independently, a similar technique was developed by Hitchin [14] for classifying all harmonic tori in the 3-sphere. In this paper, we will follow Pinkall–Sterling's approach.

Pinkall–Sterling proceed as follows. Let $f : \mathbb{C} \to \mathbb{R}^3$ be a smooth, doubly-periodic immersion of non-zero constant mean curvature H (where here we define the mean curvature to be equal to the algebraic mean of the two principal curvatures). We suppose that f is conformal, so that the metric it induces over Ω is given by

$$g := e^{2\omega} dz d\overline{z}, \tag{1.1}$$

for some smooth, real-valued function ω, which we call the *conformal factor* of f. Recall from [15] that the *Hopf differential*

$$Q := \phi\, dz dz$$

of f is constant. Thus, upon rescaling the domain and the codomain if necessary, we may suppose that

$$H = \frac{1}{2} \ \text{and} \ \ |\phi| = \frac{1}{4}, \tag{1.2}$$

and the Gauss–Codazzi equations for f are then equivalent to

$$\omega_{z\overline{z}} + \frac{1}{8}\sinh(2\omega) = 0, \tag{1.3}$$

which is the *elliptic sinh–Gordon equation*.

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Conversely, by the fundamental theorem of surface theory, given H and Q satisfying (1.2) and a doubly-periodic function $\omega : \mathbb{C} \to \mathbb{R}$ satisfying (1.3), there exists, up to rigid motions of $\mathbb{R}^3$, a unique CMC-(1/2) immersion $f : \mathbb{C} \to \mathbb{R}$ with Hopf differential Q and conformal factor ω. This immersion is, furthermore, a torus provided that two further closing conditions are satisfied. This gives the desired equivalence, modulo closing conditions, between immersed CMC tori on the one hand, and doubly periodic solutions of the elliptic sinh–Gordon equation, on the other.

2. Finite type solutions

Bobenko [3] observes that the key steps in both Pinkall–Sterling's and Hitchin's work lie in showing that all doubly-periodic solutions of the elliptic sinh–Gordon equation over $\mathbb{C}$ are of finite type. Heuristically, this means that all such solutions are completely determined by polynomial data (c.f. [14]). However, the formal statement of the finite-type property is rather technical, with different authors using different definitions. In this paper, we adopt the perspective of Adler–Kostant–Symes theory (c.f. [4] and [5]). Recall first that the elliptic sinh–Gordon equation translates into the integrability condition of the Lax pair

$$\begin{aligned} \alpha_z(\lambda,\gamma) &= \frac{1}{2}\omega_z\sigma_0 + \frac{i}{4\lambda}e^{\omega}\sigma_+ + \frac{i\gamma}{4}e^{-\omega}\sigma_- \text{ and} \\ \alpha_{\overline{z}}(\lambda,\gamma) &= -\frac{1}{2}\omega_{\overline{z}}\sigma_0 + \frac{i}{4\gamma}e^{-\omega}\sigma_+ + \frac{i\lambda}{4}e^{\omega}\sigma_-, \end{aligned} \tag{2.1}$$

where

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

and $\lambda, \gamma \in \mathbb{C}^*$ are non-zero complex parameters called the *spectral* respectively *torsion* parameters. A *Killing field* of ω is defined to be a map $\Phi : \mathbb{C}^* \times \mathbb{C}^* \to C^\infty(\Omega, \mathrm{sl}_2(\mathbb{C}))$ which solves the system of partial differential equations

$$d\Phi(\lambda,\gamma) = [\Phi(\lambda,\gamma), \alpha(\lambda,\gamma)], \tag{2.3}$$

where

$$\alpha(\lambda,\gamma) := \alpha_z(\lambda,\gamma)\, dz + \alpha_{\overline{z}}(\lambda,\gamma)\, d\overline{z}. \tag{2.4}$$

We say that a Killing field is *polynomial* whenever it takes the form

$$\Phi(\lambda) = \sum_{(m,n)\in A} \Phi_{m,n}\lambda^m\gamma^n, \tag{2.5}$$

for some *finite* subset A of $\mathbb{Z}\times\mathbb{Z}$ where, for all (m,n), $\Phi_{m,n} : \Omega \to \mathrm{sl}_2(\mathbb{C})$ is a smooth function. A solution ω of the elliptic sinh–Gordon equation is of *finite type* if it admits a polynomial Killing field.

3. Free boundary CMC annuli

We will show that conformal factors of free boundary CMC annuli are finite type solutions of the sinh-Gordon equation (1.3). Consider therefore a periodic, conformal CMC immersion f defined over the ribbon $\Omega := \mathbb{R}\times[-T,T]$ with free boundary in $B_r(0)$. By classical surface theory, the Hopf differential Q of f has constant argument along each of the boundary components. It follows by the Schwarz reflection principle that it extends to a bounded, holomorphic form over the whole of $\mathbb{C}$ which, by Liouville's theorem, is constant. The conformal factor ω satisfies the non-linear boundary condition

$$\omega_y = \frac{\epsilon}{r}e^{\omega}, \tag{3.1}$$

where ϵ is equal to $+1$ along the upper boundary component and -1 along the lower boundary component. More generally, non-linear boundary conditions of the form

$$\omega_y = Ae^{\omega} + Be^{-\omega}, \tag{3.2}$$

where A and B are constant along each boundary component, are known as *integrable boundary conditions*, first considered for the sine-Gordon equation by Ghoshal-Zamolodchikov [12], and subsequently for other integrable equations, e.g [7, 8].

We show

Theorem 1. *If $\omega : \mathbb{R} \times [-T, T] \to \mathbb{R}$ is a singly-periodic solution of the elliptic sinh–Gordon equation with integrable boundary conditions, then ω is of finite type.*

Our proof also yields deeper information about the structure of the polynomial Killing fields of such ω. First, we say that a Killing field Φ satisfies the *Sklyanin condition for fields* whenever, at every point of $\partial\Omega$,

$$K(\lambda, \gamma)\Phi(\lambda, \gamma) = \overline{\Phi(\overline{\lambda}, \overline{\gamma})}^t K(\lambda, \gamma), \tag{3.3}$$

for all $\lambda, \gamma \in \mathbb{C}^*$, where

$$K(\lambda, \gamma) := \begin{pmatrix} 4A\gamma - 4B\lambda & \frac{\lambda}{\gamma} - \frac{\gamma}{\lambda} \\ \frac{\lambda}{\gamma} - \frac{\gamma}{\lambda} & \frac{4A}{\gamma} - \frac{4B}{\lambda} \end{pmatrix} \tag{3.4}$$

is Sklyanin's K-matrix (see [18, 19], see also [16] for an excellent treatment of Sklyanin's ideas). Theorem 1 follows from the following theorem, proven in Section 11.

Theorem 2. *If $\omega : \mathbb{R} \times [-T, T] \to \mathbb{R}$ is a periodic solution of the elliptic sinh–Gordon equation, then ω satisfies the integrable boundary conditions if and only if it admits a polynomial Killing field Φ which satisfies the Sklyanin condition for fields.*

4. Killing fields

As before, let $\Omega := \mathbb{R} \times [-T, T]$ and let $\omega : \Omega \to \mathbb{R}$ be a real solution of the elliptic sinh–Gordon equation. Throughout the current chapter, we will consider λ as a variable and γ as a constant. Let X be a manifold. Let E be a complex vector space. For $k \in \mathbb{Z}$, a series of degree k over X taking values in E is defined to be a (formal) series of the form

$$\Phi(\lambda) := \sum_{m=k}^{\infty} \Phi_m \lambda^m,$$

where, for all m, Φ_m is a smooth function over X taking values in E and Φ_k is non-zero. We say that the trivial series $\Phi = 0$ is a series of degree $+\infty$. Let $\mathcal{L}(X, E)$ denote the space of such series over X taking values in E. In addition, denote

$$\mathcal{L}(X) := \mathcal{L}(X, \mathbb{C}) \text{ and } \mathcal{L} := \mathcal{L}(\{x\}, \mathbb{C}), \tag{4.1}$$

where $\{x\}$ denotes the manifold consisting of a single point. We readily verify

Lemma 4.1.
(1) $\mathcal{L}(X, E)$ is a complex vector space;
(2) $\mathcal{L}(X)$ is a unitary commutative algebra;
(3) $\mathcal{L}$ is an algebraic field; and
(4) $\mathcal{L}(X, E)$ is a module over $\mathcal{L}(X)$ and a vector space over $\mathcal{L}$.

For $k \le l \in \mathbb{N}$, a polynomial of bidegree (k, l) over X is a finite series of the form

$$\Phi(\lambda) := \sum_{m=k}^{l} \Phi_m \lambda^m,$$

where Φ_k and Φ_l are non-zero. Again we call the trivial series $\Phi = 0$ a polynomial of bidegree $(+\infty, +\infty)$. Let $\mathcal{P}(X, E)$ denote the space of finite series over X taking values in E.

The Lax pair (2.1) defines the 1-form (2.4) over Ω which maps vector fields into $\mathcal{P}(\Omega, \mathrm{sl}(2, \mathbb{C}))$. A (formal) Killing field of ω over Ω is a series $\Phi \in \mathcal{L}(\Omega, \mathrm{sl}(2, \mathbb{C}))$ which satisfies

$$d\Phi = [\Phi, \alpha] \tag{4.2}$$

at every point of Ω (c.f. [4]). Upon evaluating (4.2) term by term, we obtain (see [13])

Lemma 4.2. *Let ω be a solution of the sinh–Gordon equation. Let*

$$\Phi := \sum_{m=k}^{\infty} \begin{pmatrix} u_m & e^{\omega} t_m \\ e^{\omega} s_m & -u_m \end{pmatrix} \lambda^m$$

be a series over Ω taking values in $\mathrm{sl}(2, \mathbb{C})$. *Then Φ is a Killing field if and only if, at every point of Ω and for all m,*

$$4u_{m,z} + ie^{2\omega} s_{m+1} - i\gamma t_m = 0, \tag{4.3}$$

$$4u_{m,\bar{z}} + i\gamma^{-1} s_m - ie^{2\omega} t_{m-1} = 0, \tag{4.4}$$

$$4\omega_z t_m + 2t_{m,z} - iu_{m+1} = 0, \tag{4.5}$$

$$2e^{\omega} t_{m,\bar{z}} - i\gamma^{-1} e^{-\omega} u_m = 0, \tag{4.6}$$

$$2e^{\omega} s_{m,z} + i\gamma e^{-\omega} u_m = 0 \text{ and} \tag{4.7}$$

$$4\omega_{\bar{z}} s_m + 2s_{m,\bar{z}} + iu_{m-1} = 0. \tag{4.8}$$

5. The space of Killing fields

Let $\mathcal{K}(\Omega)$ denote the space of Killing fields of ω over Ω. It is non-trivial as Pinkall–Sterling [17] construct an explict, non-trivial Killing field of ω over Ω, which we henceforth refer to as the *Pinkall–Sterling field*[1]. We need their iteration to construct this field. First, define

$$u_0 := 0 \text{ and } \psi_0 := -\frac{1}{2}. \tag{5.1}$$

Next, after having determined $u_1, \cdots, u_m$ and $\psi_1, \cdots, \psi_{m-1}$, define

$$\psi_m := \begin{cases} \gamma u_k^2 + 2\sum_{n=1}^{k-1} \theta_{n,m-n} & \text{if } m = 2k-1, \text{ and} \\ \gamma u_k u_{k+1} + \theta_{k,k} + 2\sum_{n=1}^{k-1} \theta_{n,m-n} & \text{if } m = 2k, \end{cases} \tag{5.2}$$

and

$$u_{m+1} := \frac{1}{\gamma}\big(- 4u_{m,zz} + 4i\omega_z \psi_m \big), \tag{5.3}$$

where, for all p and for all q,

$$\theta_{p,q} := \gamma u_p u_{q+1} + 4u_{p,z} u_{q,z} + \psi_p \psi_q. \tag{5.4}$$

[1] We remark that the formalism of [17] is slightly different from our own, but can be transformed into our own by replacing their variable z with the variable $\zeta := iz/2$ so that $\partial_\zeta = -2i\partial_z$ and $\partial_{\bar{\zeta}} = 2i\partial_{\bar{z}}$.

The sequences t_m and s_m are now determined by

$$t_m := \frac{1}{\gamma}\big(-2iu_{m,z} - \psi_m\big) \text{ and } s_m := e^{-2\omega}\big(2iu_{m-1,z} - \psi_{m-1}\big). \tag{5.5}$$

The Pinkall–Sterling field is then the series

$$\Phi := \sum_{m=0}^{\infty} \begin{pmatrix} u_m & e^{\omega} t_m \\ e^{\omega} s_m & -u_m \end{pmatrix} \lambda^m. \tag{5.6}$$

We now show that $\mathcal{K}(\Omega)$ is 1-dimensional over $\mathcal{L}$. To this end, we show

Lemma 5.1. *Let*

$$\Phi := \sum_{m=0}^{\infty} \begin{pmatrix} u_m & e^{\omega} t_m \\ e^{\omega} s_m & -u_m \end{pmatrix} \lambda^m$$

be a Killing field of ω over Ω. For all m,

(1) if $u_m = 0$, then t_m and s_m are constant;
(2) if $u_m = t_m = 0$, then $s_{m+1} = u_{m+1} = 0$; and
(3) if $u_m = s_m = 0$, then $u_{m-1} = t_{m-1} = 0$.

Proof. Suppose that $u_m = 0$. By (4.6), we have $t_{m,\overline{z}} = 0$. Next, by (4.3),

$$e^{\omega} s_{m+1} = \gamma e^{-\omega} t_m,$$

and by (4.7),

$$\begin{aligned} \gamma e^{-\omega} u_{m+1} &= 2ie^{\omega} s_{m+1,z} = 2i(e^{\omega} s_{m+1})_z - 2i\omega_z e^{\omega} s_{m+1} \\ &= 2i\gamma(e^{-\omega} t_m)_z - 2i\gamma\omega_z e^{-\omega} t_m = 2i\gamma e^{-\omega} t_{m,z} - 4i\gamma\omega_z e^{-\omega} t_m, \end{aligned}.$$

Hence by (4.5), we obtain $t_{m,z} = 0$. Thus t_m is constant. In the same manner s_m is constant, and (1) follows. If $u_m = t_m = 0$, then it follows by (4.3) and (4.5) that $s_{m+1} = u_{m+1} = 0$, and (2) follows. If $s_m = u_m = 0$, then it follows by (4.4) and (4.8) that $u_{m-1} = t_{m-1} = 0$, and (3) follows. This completes the proof. □

Lemma 5.2. *$\mathcal{K}(\Omega)$ is the 1-dimensional vector space over $\mathcal{L}$ generated by the Pinkall–Sterling field.*

Remark 5.3. *In particular in the present framework a solution of the elliptic sinh–Gordon equation is of finite type if and only if its space of Killing fields is generated by a finite series.*

Proof. Let Φ be an arbitrary Killing field of ω over Ω and let Ψ be the Pinkall–Sterling field. We construct recursively a series f such that

$$\Phi = f\Psi.$$

Let k be the degree of Φ. As Ψ has degree 0, the series f must also be of degree k. Suppose we have already determined the coefficients $f_k, f_{k+1}, ..., f_{k+l-1}$ in such a manner that the series

$$\tilde{\Phi} := \Phi - f_{(l)}\Psi$$

is of degree $k + l$, where

$$f_{(l)} := \sum_{m=k}^{k+l-1} f_m \lambda^m.$$

For all m, denote

$$\tilde{\Phi}_m := \begin{pmatrix} u_m & e^{\omega} \tau_m \\ e^{\omega} \sigma_m & -u_m \end{pmatrix}.$$

As $\tilde{\Phi}$ is also a solution of the Killing field equation, it follows by Lemma 5.1 that $s_{k+l} = u_{k+l} = 0$ and $\tau_{k+l} = c$ is constant. The result now follows by setting $f_{k+l} := 2c$. □

6. The determinant

We now characterise the Pinkall–Sterling field amongst all Killing fields of ω over Ω. Observe first that, since elements of $\mathcal{K}(\Omega)$ are 2×2 matrices with coefficients in $\mathcal{L}(\Omega)$, they have well-defined determinants which are also elements of $\mathcal{L}(\Omega)$.

Lemma 6.1. *For every Killing field Φ of ω over Ω, $\mathrm{Det}(\Phi)$ is constant over Ω, that is,*

$$\mathrm{Det}(\Phi) \in \mathcal{L}.$$

Proof. Indeed, $d\mathrm{Det}(\Phi) = \mathrm{Tr}(\mathrm{Adj}(\Phi)d\Phi) = \mathrm{Tr}(\mathrm{Adj}(\Phi)[\Phi,\alpha]) = 0$, as desired. □

Lemma 6.2. *For all γ, the Pinkall–Sterling field is, up to a choice of sign, the unique Killing field Φ of ω over Ω such that*

$$\mathrm{Det}(\Phi) = -\frac{\lambda}{4\gamma}. \tag{6.1}$$

Proof. Let Φ be the Pinkall–Sterling field. We first show uniqueness. Let Ψ be another Killing field of ω over Ω which satisfies (6.1). Since $\mathcal{K}(\Omega)$ is generated by Φ, there exists $f \in \mathcal{L}$ such that $\Psi = f\Phi$. In particular,

$$-\frac{\lambda}{4\gamma} = \mathrm{Det}(\Psi) = \mathrm{Det}(\alpha\Phi) = -\frac{\lambda}{4\gamma}f^2.$$

Since $\mathcal{L}$ is an algebraic field, it follows that $f = \pm 1$, and uniqueness follows.

We now show that Φ satisfies (6.1). To this end, denote

$$U := \sum_{k=0}^{\infty} u_k\lambda^k,\ S := \sum_{k=0}^{\infty} s_k\lambda^k,\ T := \sum_{k=0}^{\infty} t_k\lambda^k \text{ and } \Psi := \sum_{k=0}^{\infty} \psi_k\lambda^k,$$

where ψ_m is the sequence constructed in (5.2). By (5.5)

$$T = \frac{1}{\gamma}\big(-2iU_z - \Psi\big) \text{ and } S = \lambda e^{-2\omega}\big(2iU_z - \Psi\big),$$

so that

$$\begin{aligned} Det(\Phi) &= -U^2 - e^{2\omega}ST = -U^2 + \frac{\lambda}{\gamma}(2iU_z - \Psi)(2iU_z + \Psi) \\ &= -U^2 - 4\frac{\lambda}{\gamma}U_z^2 - \frac{\lambda}{\gamma}\Psi^2. \end{aligned}$$

Since the coefficients of the expression

$$-U^2 - 4\,\frac{\lambda}{\gamma}\,U_z^2 - \frac{\lambda}{\gamma}\,\Psi^2 = -\frac{\lambda}{4\gamma}$$

are precisely the recurrence relations (5.2) and (5.4), the result follows. □

7. The Sklyanin matrix

We now recall how Sklyanin translates the integrable boundary conditions into boundary conditions for the Lax pair (c.f. [18] and [19]). The real component of the Lax pair is

$$\alpha_x = -\frac{i}{2}\omega_y\sigma_0 + \left(\frac{i}{4\lambda}e^{\omega} + \frac{i}{4\gamma}e^{-\omega}\right)\sigma_+ + \left(\frac{i\gamma}{4}e^{-\omega} + \frac{i\lambda}{4}e^{\omega}\right)\sigma_-. \tag{7.1}$$

We say that α_x satisfies the *Sklyanin condition* for Lax pairs at a point of $\partial\Omega$ whenever

$$K(\lambda,\,\gamma)\,\alpha_x(\lambda,\,\gamma) = \alpha_x\big(\lambda^{-1},\,\gamma^{-1}\big)\,K(\lambda,\,\gamma) \tag{7.2}$$

at this point, for all $\lambda, \gamma \in \mathbb{C}^*$, where $K(\lambda,\gamma)$ is *Sklyanin's K-matrix*, given by (3.4).

Lemma 7.1. *The function ω satisfies the boundary condition*

$$\omega_y = Ae^{\omega} + Be^{-\omega}$$

at a point of $\partial\Omega$ if and only if the real part α_x of its Lax pair satisfies the Sklyanin condition for Lax pairs at this point.

Proof. Make the ansatz $K := a\,\mathrm{Id} + b\,\sigma_0 + c\,(\sigma_+ + \sigma_-)$ where the coefficients a, b and c only depend on γ and λ. The relation (7.2) holds if and only if

$$c\omega_y + \left(\frac{(a+b)}{4\lambda} - \frac{(a-b)\lambda}{4}\right)e^{\omega} + \left(\frac{(a+b)}{4\gamma} - \frac{(a-b)\gamma}{4}\right)e^{-\omega} = 0.$$

Setting

$$a+b := 4A\gamma - 4B\lambda,, \ a-b := \frac{4A}{\gamma} - \frac{4B}{\lambda} \text{ and } c := \frac{\lambda}{\gamma} - \frac{\gamma}{\lambda},$$

we see that (7.2) is satisfied at a point of $\partial\Omega$ if and only if

$$\omega_y = Ae^{\omega} + Be^{-\omega}$$

at this point, which is precisely our integrable boundary condition. Finally, substituting a, b and c into the ansatz for K yields (3.4) as desired. □

8. The Sklyanin condition

We conclude this chapter by showing that if ω satisfies the integrable boundary conditions on $\partial\Omega$, then its Pinkall–Sterling field satisfies the Sklyanin condition for fields on $\partial\Omega$. To this end, let $\partial\Omega_0$ be one of the two connected components of $\partial\Omega$. We define a *Killing field* of ω on $\partial\Omega_0$ to be a series Φ in $\mathcal{L}(\partial\Omega_0, \mathrm{sl}(2,\mathbb{C}))$ which satisfies the Lax equation

$$\Phi_x = [\Phi, \alpha_x]. \tag{8.1}$$

Let $\mathcal{K}(\partial\Omega_0)$ denote the space of Killing fields of ω over $\partial\Omega_0$. Trivially, every Killing field of ω over Ω restricts to a Killing field of ω over $\partial\Omega_0$. In the one-dimensional case, we have the following weaker version of Lemma 5.1.

Lemma 8.1. *Let*

$$\Phi := \sum_{m=0}^{\infty} \begin{pmatrix} u_m & e^{\omega}t_m \\ e^{\omega}s_m & -u_m \end{pmatrix} \lambda^m$$

be a Killing field of ω over $\partial\Omega_0$. If

$$t_{k-2} = s_{k-1} = u_{k-1} = t_{k-1} = 0,$$

then $s_k = u_k = 0$, and t_k is constant.

Proof. By (7.1) and (8.1), for all m,

$$u_{m,x} = \frac{i\gamma}{4}t_m + \frac{i}{4}e^{2\omega}t_{m-1} - \frac{i}{4}e^{2\omega}s_{m+1} - \frac{i}{4\gamma}s_m, \tag{8.2}$$

$$e^{\omega}s_{m,x} = -2e^{\omega}\omega_{\overline{z}}s_m - \frac{i\gamma}{2}e^{-\omega}u_m - \frac{i}{2}e^{\omega}u_{m-1}, \text{ and} \tag{8.3}$$

$$e^{\omega}t_{m,x} = -2e^{\omega}\omega_z t_m + \frac{i}{2}e^{\omega}u_{m+1} + \frac{i}{2\gamma}e^{-\omega}u_m. \tag{8.4}$$

By (8.2) and (8.4), We obtain $s_k = u_k = 0$. These three relations together yield

$$\begin{aligned} e^{2\omega} s_{k+1} &= \gamma t_k, \\ u_{k+1} &= \frac{1}{\gamma}\big(2ie^{2\omega}s_{k+1,x} + 4ie^{2\omega}\omega_{\bar z}s_{k+1}\big), \text{ and} \\ e^{\omega}t_{k,x} &= -2e^{\omega}\omega_z t_k + \frac{i}{2}e^{\omega}u_{k+1}. \end{aligned}$$

It follows that

$$u_{k+1} = 2ie^{2\omega}(e^{-2\omega}t_k)_{,x} + 4i\omega_{\bar z}t_k = -4i\omega_x t_k + 2it_{k,x} + 4i\omega_{\bar z}t_k,$$

so that $e^{\omega}t_{k,x} = -2e^{\omega}\omega_z t_k + 2e^{\omega}\omega_x t_k - e^{\omega}t_{k,x} - 2e^{\omega}\omega_{\bar z}t_k$. Thus $t_{k,x} = 0$, completing the proof. □

Via the same argument as in Sections 5 and 6, this yields

Lemma 8.2.
(1) $\mathcal{K}(\partial\Omega_0)$ is the one-dimensional vector space over $\mathcal{L}$ generated by the Pinkall–Sterling field.
(2) The restriction of the Pinkall–Sterling field to $\partial\Omega_0$ is, up to a choice of sign, the unique Killing field Φ of ω over $\partial\Omega_0$ which satisfies

$$\mathrm{Det}(\Phi) = -\frac{\lambda}{4\gamma}.$$

Given a Killing field Φ of ω over $\partial\Omega_0$, denote

$$\tilde\Phi(\lambda,\gamma) := K(\lambda,\gamma)^{-1}\overline{\Phi(\overline{\lambda},\overline{\gamma})}^{\,t}K(\lambda,\gamma). \tag{8.5}$$

Observe that $\tilde\Phi$ is also a Laurent series over $\partial\Omega_0$.

Lemma 8.3. *If the real part α_x of the Lax pair of ω satisfies the Sklyanin condition for Lax pairs along $\partial\Omega_0$, then $\tilde\Phi$ satisfies the Killing field equation* (8.1) *along $\partial\Omega_0$.*

Proof. Observe that

$$\begin{aligned} \overline{\alpha_x(\overline{\lambda}^{-1},\overline{\gamma}^{-1})}^{\,t} &= -\alpha_x(\lambda,\gamma), \\ \overline{K(\overline{\lambda}^{-1},\overline{\gamma}^{-1})} &= D(\lambda,\gamma)K(\lambda,\gamma)^{-1} \text{ and} \\ K(\lambda^{-1},\gamma^{-1}) &= D(\lambda,\gamma)K(\lambda,\gamma)^{-1}, \end{aligned}$$

where

$$D(\lambda,\gamma) := \mathrm{Det}(K(\lambda,\gamma)).$$

Moreover,

$$D(\lambda,\gamma) = D(\lambda^{-1},\gamma^{-1}) = \overline{D(\overline{\lambda}^{-1},\overline{\gamma}^{-1})}.$$

Upon applying the Killing field equation (8.1), we therefore obtain

$$\begin{aligned} \tilde\Phi(\lambda,\gamma)_x &= K(\lambda,\gamma)^{-1}\overline{\Phi(\overline{\lambda},\overline{\gamma})_x}^{\,t}K(\lambda,\gamma) \\ &= K(\lambda,\gamma)^{-1}\overline{\big[\Phi(\overline{\lambda},\overline{\gamma}),\alpha_x(\overline{\lambda},\overline{\gamma})\big]}^{\,t}K(\lambda,\gamma) \\ &= K(\lambda,\gamma)^{-1}\big[\overline{\alpha_x(\overline{\lambda},\overline{\gamma})}^{\,t},\overline{\Phi(\overline{\lambda},\overline{\gamma})}^{\,t}\big]K(\lambda,\gamma) \\ &= -K(\lambda,\gamma)^{-1}\big[\alpha_x(\lambda^{-1},\gamma^{-1}),\overline{\Phi(\overline{\lambda},\overline{\gamma})}^{\,t}\big]K(\lambda,\gamma) \\ &= K(\lambda,\gamma)^{-1}\big[\overline{\Phi(\overline{\lambda},\overline{\gamma})}^{\,t},\alpha_x(\lambda^{-1},\gamma^{-1})\big]K(\lambda,\gamma) \\ &= \big[K(\lambda,\gamma)^{-1}\overline{\Phi(\overline{\lambda},\overline{\gamma})}^{\,t}K(\lambda,\gamma),K(\lambda,\gamma)^{-1}\alpha_x(\lambda^{-1},\gamma^{-1})K(\lambda,\gamma)\big] \\ &= [\tilde\Phi(\lambda,\gamma),\alpha_x(\lambda,\gamma)], \end{aligned}$$

and the result follows. □

Lemma 8.4. *If the real part α_x of the Lax pair of ω satisfies the Sklyanin condition for Lax pairs along $\partial\Omega_0$, then the Pinkall–Sterling field Φ of ω satisfies the Sklyanin condition for fields along $\partial\Omega_0$, that is, for all $\lambda,\gamma\in S^1$,*

$$\Phi(\lambda,\gamma)=\tilde{\Phi}(\lambda,\gamma)=K(\lambda,\gamma)^{-1}\overline{\Phi(\overline{\lambda},\overline{\gamma})}^{t}K(\lambda,\gamma) \tag{8.6}$$

along $\partial\Omega_0$.

Proof. By Lemma 8.3, $\tilde{\Phi}$ is a Killing field of ω over $\partial\Omega_0$. However, for all λ and for all γ,

$$\mathrm{Det}(\tilde{\Phi}(\lambda,\gamma))=\mathrm{Det}\big(K(\lambda,\gamma)^{-1}\overline{\Phi(\overline{\lambda},\overline{\gamma})}^{t}K(\lambda,\gamma)\big)=\overline{\mathrm{Det}\big(\Phi(\overline{\lambda},\overline{\gamma})\big)}=-\frac{\lambda}{4\gamma}.$$

It follows by Lemma 8.2 that $\Phi=\pm\tilde{\Phi}$. Explicitly calculating the first non-zero term of each of these two series yields the claim. □

9. Robin boundary conditions

As before, let $\Omega:=\mathbb{R}\times[-T,T]$ and let $\omega:\Omega\to\mathbb{R}$ be a real solution of the elliptic sinh–Gordon equation. In this chapter, we transform the Sklyanin condition for fields into a sequence of equations that allow us to recover the finite type property. Thus, let Φ be the Pinkall–Sterling field of ω. In order to better capture the symmetries of the problem, it now becomes convenient to treat this field as a (formal) series in λ and γ. We thus denote

$$\Phi(\lambda,\gamma):=\sum_{m,n}\begin{pmatrix}u_{m,n} & e^{\omega}t_{m,n}\\ e^{\omega}s_{m,n} & -u_{m,n}\end{pmatrix}\lambda^m\gamma^n. \tag{9.1}$$

Observe that, for all $\lambda,\gamma\in S^1$,

$$\alpha(\lambda,\gamma)=e^{-\frac{\theta}{2}\sigma_0}\,\alpha\big(\lambda\gamma^{-1},1\big)\,e^{\frac{\theta}{2}\sigma_0}, \tag{9.2}$$

where $\theta\in\mathbb{R}$ satisfies

$$e^{2i\theta}=\gamma.$$

It follows upon applying this gauge transformation that

$$\Phi(\lambda,\gamma)=\sum_{m=0}^{\infty}\begin{pmatrix}u_m & \gamma^{-1}e^{\omega}t_m\\ \gamma e^{\omega}s_m & -u_m\end{pmatrix}\lambda^m\gamma^{-m}, \tag{9.3}$$

where

$$\sum_{m=0}^{\infty}\begin{pmatrix}u_m & e^{\omega}t_m\\ e^{\omega}s_m & -u_m\end{pmatrix}\lambda^m:=\Phi(\lambda,1)$$

is the Pinkall–Sterling field of ω with torsion $\gamma=1$. In this framework, Lemma 4.2 becomes

Lemma 9.1. *The sequences $u_{m,n}$, $t_{m,n}$ and $s_{m,n}$ satisfy, for all m and for all n,*

$$4u_{m,n,z}+ie^{2\omega}s_{m+1,n}-it_{m,n-1}=0, \tag{9.4}$$

$$4u_{m,n,\overline{z}}+is_{m,n+1}-ie^{2\omega}t_{m-1,n}=0, \tag{9.5}$$

$$4\omega_z t_{m,n}+2t_{m,n,z}-iu_{m+1,n}=0, \tag{9.6}$$

$$2e^{\omega}t_{m,n,\overline{z}}-ie^{-\omega}u_{m,n+1}=0, \tag{9.7}$$

$$2e^{\omega}s_{m,n,z}+ie^{-\omega}u_{m,n-1}=0 \ \textit{and} \tag{9.8}$$

$$4\omega_{\overline{z}}s_{m,n}+2s_{m,n,\overline{z}}+iu_{m-1,n}=0. \tag{9.9}$$

As far as the Sklyanin condition for fields is concerned, upon equating every coefficient of $(K\Phi-\tilde{\Phi}K)$ with zero, we obtain

Lemma 9.2. *Along $\partial\Omega$, the sequences $u_{m,n}$, $t_{m,n}$ and $s_{m,n}$ satisfy, for all m and for all n,*

$$\operatorname{Im}\big(e^{\omega}s_{m-1,n+1} - e^{\omega}s_{m+1,n-1} + 4Au_{m,n-1} - 4Bu_{m-1,n}\big) = 0, \tag{9.10}$$

$$\operatorname{Im}\big(e^{\omega}t_{m-1,n+1} - e^{\omega}t_{m+1,n-1} - 4Au_{m,n+1} + 4Bu_{m+1,n}\big) = 0, \tag{9.11}$$

$$\begin{aligned}\operatorname{Re}\big(2Ae^{\omega}t_{m,n-1} - 2Be^{\omega}t_{m-1,n} - 2Ae^{\omega}s_{m,n+1} + 2Be^{\omega}s_{m+1,n}\big)&\\ = \operatorname{Re}\big(u_{m-1,n+1} - u_{m+1,n-1}\big) \text{ and}&\end{aligned} \tag{9.12}$$

$$\operatorname{Im}\big(At_{m,n-1} - Bt_{m-1,n} + As_{m,n+1} - Bs_{m+1,n}\big) = 0. \tag{9.13}$$

Lemma 9.3. *The pair of systems of equations* (9.10) *and* (9.11) *is equivalent to the following pair of systems of equations:*

$$\operatorname{Im}\big(e^{\omega}t_{m-1,n} - 4Au_{m,n} + e^{\omega}s_{m+1,n}\big) = 0 \text{ and} \tag{9.14}$$

$$\operatorname{Im}\big(e^{\omega}t_{m,n-1} - 4Bu_{m,n} + e^{\omega}s_{m,n+1}\big) = 0. \tag{9.15}$$

Proof. Indeed, suppose that the equations (9.10) and (9.11) are satisfied for all m and for all n. Then, upon applying recursively (9.10), we obtain the *finite* sum

$$\operatorname{Im}\big(e^{\omega}s_{m,n}\big) = \operatorname{Im}\big(4Au_{m-1,n} - 4Bu_{m-2,n+1} + 4Au_{m-3,n+2} - \cdots\big).$$

In the same manner, (9.11) yields

$$\operatorname{Im}\big(e^{\omega}t_{m,n}\big) = \operatorname{Im}\big(4Bu_{m,n+1} - 4Au_{m-1,n+2} + 4Bu_{m-2,n+3} - \cdots\big).$$

If we now denote these equations respectively by $\alpha(m,n)$ and $\beta(m,n)$, then $\alpha(m+1,n)$ and $\beta(m-1,n)$ together yield (9.14) whilst $\alpha(m,n+1)$ and $\beta(m,n-1)$ together yield (9.15). Since the converse is trivial, this completes the proof. □

The integrability of boundary conditions is itself remarkable, since many classical boundary conditions do not have this property, see e.g. [2].

Lemma 9.4. *For all m and for all n, the imaginary part of the function $u_{m,n}$ satisfies the following Robin boundary condition:*

$$\operatorname{Im}(u_{m,n})_y = Ae^{\omega}\operatorname{Im}(u_{m,n}) - Be^{-\omega}\operatorname{Im}(u_{m,n}). \tag{9.16}$$

Proof. Indeed, by (9.4) and (9.5),

$$\begin{aligned}e^{\omega}s_{m+1,n} - e^{-\omega}t_{m,n-1} &= 4ie^{-\omega}u_{m,n,z} \text{ and}\\ e^{\omega}t_{m-1,n} - e^{-\omega}s_{m,n+1} &= -4ie^{-\omega}u_{m,n,\overline{z}}.\end{aligned}$$

The sum of (9.14) and (9.15) then yields

$$\begin{aligned}\operatorname{Im}\big(4Au_{m,n} - 4Be^{-2\omega}u_{m,n}\big) &= \operatorname{Im}\big(e^{\omega}t_{m-1,n} + e^{\omega}s_{m+1,n} - e^{-\omega}t_{m,n-1} - e^{-\omega}s_{m,n+1}\big)\\ &= \operatorname{Im}\big(4ie^{-\omega}u_{m,n,z} - 4ie^{-\omega}u_{m,n,\overline{z}}\big)\\ &= 4e^{-\omega}\operatorname{Re}\big((u_{m,n} - \overline{u}_{m,n})_z\big)\\ &= 4e^{-\omega}\operatorname{Re}\big(2i\operatorname{Im}(u_{m,n})_z\big)\\ &= 4e^{-\omega}\operatorname{Im}(u_{m,n})_y,\end{aligned}$$

and the result follows. □

10. The solution is of finite type

We now show that the solution is of finite type. Let Φ be as in the previous section. Recall that, upon applying the gauge transformation (9.2) if necessary, we may henceforth suppose that $\gamma = 1$. We first recall a few elementary lemmas.

Lemma 10.1. *Let*

$$\Psi := \sum_{m=0}^{\infty} \begin{pmatrix} u_m & e^{\omega}t_m \\ e^{\omega}s_m & -u_m \end{pmatrix} \lambda^m$$

be a Killing field. If $u_k = 0$ then there exists $f \in \mathcal{L}$ of degree k such that

$$\Psi - f\Phi = \sum_{m=0}^{k-1} \begin{pmatrix} u_m & e^{\omega}t_m \\ e^{\omega}s_m & -u_m \end{pmatrix} \lambda^m + \begin{pmatrix} 0 & 0 \\ e^{\omega}s_k & 0 \end{pmatrix} \lambda^k.$$

In particular, $(\Psi - f\Phi)$ is a polynomial Killing field.

Proof. We construct f by recurrence. Suppose that the coefficients $f_k, \cdots, f_{k+l-1}$ have already been determined such that if

$$\tilde{\Psi} := \Psi - f_{(l)}\Phi := \sum_{m=0}^{\infty} \begin{pmatrix} \tilde{u}_m & e^{\omega}\tilde{t}_m \\ e^{\omega}\tilde{s}_m & -\tilde{u}_m \end{pmatrix} \lambda^m,$$

where

$$f_{(l)} := \sum_{m=k}^{k+l-1} f_m \lambda^m,$$

then,

$$\begin{aligned} &\tilde{u}_m = 0 \ \forall k \leq m \leq k+l, \\ &\tilde{t}_m = 0 \ \forall k \leq m \leq k+l-1 \text{ and} \\ &\tilde{s}_m = 0 \ \forall k+1 \leq m \leq k+l. \end{aligned}$$

Since $\tilde{\Psi}$ is also a Killing field, by Lemma 5.1,

$$\tilde{t}_{k+l} = c$$

is constant. By Lemma 5.1 again, the result follows upon setting $f_{k+l} := -2c$. □

Lemma 10.2. *The solution ω is of finite type if and only if there exists a finite-dimensional vector space $E \subseteq C^{\infty}(\Omega, \mathbb{C})$ such that, for all m and for all n,*

$$u_{m,n} \in E.$$

Proof. This condition is trivially necessary. We now show that it is sufficient. Suppose again that $\gamma = 1$. In particular, by (9.3), for all m, $u_m = u_{m,m} \in E$. If $d := \mathrm{Dim}(E)$, then there exists $f \in \mathcal{P}$ of bidegree $(0, d-1)$ such that if

$$f\Phi := \sum_{m=0}^{\infty} \begin{pmatrix} \tilde{u}_m & e^{\omega}\tilde{t}_m \\ e^{\omega}\tilde{s}_m & -\tilde{u}_m \end{pmatrix} \lambda^m,$$

then $\tilde{u}_d = 0$, and the result follows by Lemma 10.1. □

Lemma 10.3. *For all m, n the function $u_{m,n}$ satisfies the linearised sinh–Gordon equation*

$$\Delta u_{m,n} + \cosh(2\omega)u_{m,n} = 0. \tag{10.1}$$

Proof. Indeed, differentiating (9.4) yields $4u_{m,n,z\overline{z}} + i(e^{2\omega}s_{m+1,n})_{\overline{z}} - it_{m,n-1,\overline{z}} = 0$. Applying (9.7) and (9.9) then yields

$$4u_{m,n,z\overline{z}} + \frac{1}{2}e^{2\omega}u_{m,n} + \frac{1}{2}e^{-2\omega}u_{m,n} = 0,$$

as desired. □

Lemma 10.4. *If $\varphi : \Omega \to \mathbb{C}$ is a periodic, holomorphic function which satisfies*

$$\mathrm{Im}(\varphi)|_{\partial\Omega} = 0,$$

then φ is constant.

Proof. Indeed, by Cauchy's reflection principle, φ extends to a bounded holomorphic function over $\mathbb{C}$ and the result now follows by Liouville's theorem. □

Theorem 3. *If $\omega : \Omega \to \mathbb{R}$ is a periodic solution of the sinh–Gordon equation with integrable boundary conditions, then ω is of finite type.*

Proof. Let Φ be the Pinkall–Sterling field of ω and let $(u_{m,n})$, $(t_{m,n})$ and $(s_{m,n})$ be as in (9.1). By Lemmas 7.1 and 8.4, Φ satisfies the Sklyanin condition for fields along $\partial\Omega$. It follows by Lemma 9.4 that, for all m and for all n,

$$\mathrm{Im}(u_{m,n})_{,y} = Ae^{\omega}\mathrm{Im}(u_{m,n}) - Be^{-\omega}\mathrm{Im}(u_{m,n}),$$

along $\partial\Omega$. Since $\mathrm{Im}(u_{m,n})$ also satisfies the linearised sinh–Gordon equation, it follows by the classical theory of elliptic operators over compact manifolds with boundary (see [11]) that there exists a finite-dimensional subspace $E_1 \subseteq C^\infty(\Omega, \mathbb{R})$ such that, for all m and for all n,

$$\mathrm{Im}(u_{m,n}) \in E_1.$$

By (9.6) and (9.9), for all m and for all n,

$$\left(e^{2\omega}\overline{s}_{m+1,n} - e^{2\omega}t_{m-1,n}\right)_z = -\frac{i}{2}e^{2\omega}\left(u_{m,n} - \overline{u}_{m,n}\right) = e^{2\omega}\mathrm{Im}(u_{m,n}) \in e^{2\omega}E_1,$$

and, by (9.14), along $\partial\Omega$,

$$\mathrm{Im}\left(e^{2\omega}\overline{s}_{m+1,n} - e^{2\omega}t_{m-1,n}\right) = -4Ae^{\omega}\mathrm{Im}(u_{m,n}) \in e^{\omega}E_1|_{\partial\Omega}.$$

It follows by Lemma 10.4 that there exists a finite-dimensional subspace $E_2 \subseteq C^\infty(\Omega, \mathbb{C})$ such that, for all m and for all n,

$$e^{2\omega}\overline{s}_{m+1,n} - e^{2\omega}t_{m-1,n} \in e^{\omega}E_2.$$

In a similar manner, we show by (9.7) and (9.8), that there exists a finite-dimensional subspace $E_3 \subseteq C^\infty(\Omega, \mathbb{C})$ such that, for all m and for all n,

$$s_{m,n+1} - \overline{t}_{m,n-1} \in e^{-\omega}E_3.$$

Finally, by (9.12), along $\partial\Omega$, for all m and for all n,

$$\begin{aligned}\mathrm{Re}\left(u_{m+1,n-1} - u_{m-1,n+1}\right) &= 2Be^{-\omega}\mathrm{Re}\left(e^{2\omega}\overline{s}_{m+1,n} - e^{2\omega}t_{m-1,n}\right)\\ &\quad - 2Ae^{\omega}\mathrm{Re}\left(s_{m,n+1} - \overline{t}_{m,n-1}\right)\\ &\in \mathrm{Re}(E_2 + E_3).\end{aligned}$$

It follows by induction that, along $\partial\Omega$, for all m and for all n,

$$\mathrm{Re}(u_{m,n}) \in \mathrm{Re}(E_2 + E_3).$$

Finally, since $\mathrm{Re}(u_{m,n})$ also satisfies the linearised sinh–Gordon equation, it follows again by the classical theory of elliptic operators over compact manifolds with boundary that there exists a fourth finite-dimensional subspace $E_4 \subseteq C^\infty(\Omega, \mathbb{R})$ such that, for all m and for all n,

$$\mathrm{Re}(u_{m,n}) \in E_4.$$

The result now follows by Lemma 10.2. □

11. Polynomial Killing fields

Finally, we construct polynomial Killing fields of ω over Ω that also satisfy the Sklyanin condition for fields. As in Section 10 we restrict ourselves again to the case where $\gamma = 1$.

Lemma 11.1. *Let*

$$\Psi := \sum_{m=0}^{\infty} \begin{pmatrix} u_m & e^{\omega} t_m \\ e^{\omega} s_m & -u_m \end{pmatrix} \lambda^m$$

be a Killing field of ω over Ω which satisfies the Sklyanin condition for fields. If $u_k = 0$ and if $t_k \notin \mathbb{R}$ then there exists a polynomial Killing field of ω over Ω of bidegree $(0,4)$ which also satisfies the Sklyanin condition for fields.

Proof. Let Φ be the Pinkall–Sterling field. By Lemma 10.1, there exist $f, g \in \mathcal{L}$ with real coefficients such that

$$\Psi - (f + ig)\Phi = P,$$

where P is a polynomial Killing field of ω over Ω. We therefore denote

$$\Psi_1 := g\Phi.$$

Since Φ satisfies the Sklyanin condition for fields, and since g has real coefficients, Ψ_1 also satisfies the Sklyanin condition for fields, that is, for all $\lambda \in \mathbb{C}^*$,

$$K(\lambda,1)\Psi_1(\lambda,1) - \overline{\Psi_1(\overline{\lambda},1)}^t K(\lambda,1) = 0$$

along $\partial\Omega$. Next, since Φ and Ψ both satisfy the Sklyanin condition for fields, and since f has real coefficients, we also have, for all $\lambda \in S^1$,

$$iK(\lambda,1)\Psi_1(\lambda,1) + i\overline{\Psi_1(\overline{\lambda},1)}^t K(\lambda,1) = Q(\lambda,1)$$

along $\partial\Omega$, where

$$Q(\lambda,1) := K(\lambda,1)P(\lambda,1) - \overline{P(\overline{\lambda},1)}^t K(\lambda,1).$$

It follows that

$$\begin{aligned} & K(\lambda,1)\Psi_1(\lambda,1) = -\frac{i}{2}Q(\lambda,1) \\ \Leftrightarrow \quad & D(\lambda,1)\Psi_1(\lambda,1) = -\frac{i}{2}K(\lambda^{-1},1)Q(\lambda,1), \end{aligned}$$

where

$$D(\lambda,1) := \mathrm{Det}(K(\lambda,1)).$$

We thus denote

$$\Psi'(\lambda,1) := \lambda^{2-k}D(\lambda,1)\Psi_1(\lambda,1).$$

Since $D(\lambda,1)$ is a finite series with real coefficients, Ψ' is also a Killing field of ω over Ω which satisfies the Sklyanin condition for fields. Finally, we verify that $Q(\lambda,1)$ is a polynomial of bidegree $(k-1,k+1)$, and the result follows. □

Proof of Theorem 2. In order to prove the existence of a polynomial Killing field of ω over Ω which satisfies the Sklyanin condition for fields, it suffices to repeat the construction of Lemmas 10.1 and 10.2 using only series with real coefficients. The only possible obstruction to this construction is precisely the case studied in Lemma 11.1, where there nonetheless exists a polynomial Killing field of ω over Ω which satisfies the Sklyanin condition for fields. This proves existence.

Conversely, suppose that there exists a polynomial Killing field of ω over Ω

$$\Psi := \sum_{m=0}^{k} \begin{pmatrix} u_m & e^{\omega} t_m \\ e^{\omega} s_m & -u_m \end{pmatrix} \lambda^m$$

which satisfies the Sklyanin condition for fields. Then, by Lemma 5.1, $s_0 = u_0 = 0$. By (9.11), we may suppose that $t_0 = 1/2$ and, by (9.12),

$$\operatorname{Re}\big(u_1 + 2Ae^{\omega}t_0 + 2Be^{\omega}s_1\big) = 0.$$

By (5.1), (5.3) and (5.5), we get $u_1 = -2i\omega_z$, $t_0 = \frac{1}{2}$ and $s_1 = \frac{1}{2}e^{-2\omega}$, yielding the claim. □

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