# EXACT, FREE-SURFACE EQUATORIAL FLOWS WITH GENERAL STRATIFICATION IN SPHERICAL COORDINATES

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Abstract. This paper is concerned with the construction of a new exact solution to the geophysical fluid dynamics governing equations for inviscid and incompressible fluid in the equatorial region. This solution represents a steady purely–azimuthal flow with a free-surface. The novel aspect of the solution we derive is that the flow it prescribes accommodates a general fluid stratification: the density may vary both with depth, and with latitude. The solution is presented in the terms of spherical coordinates, hence at no stage do we invoke approximations by way of simplifying the geometry in the governing equations. Following the construction of our explicit solution, we employ functional analytic considerations to prove that the pressure at the free–surface defines implicitly the shape of the free–surface distortion in a unique way, exhibiting also the expected monotonicity properties. Finally, using a short-wavelength stability analysis we prove that certain flows defined by our exact solution are stable for a specific choice of the density distribution.

## 1. Introduction

This paper is concerned with the construction of a new exact solution to the geophysical fluid dynamics (GFD) governing equations for inviscid, incompressible and stratified fluid in the equatorial region, which represents a steady purely azimuthal flow. The solution is presented in the terms of spherical coordinates, hence at no stage do we invoke approximations through simplifying the geometry in the governing equations. In this regime, the GFD equations of motion incorporate Coriolis and centripetal forces in the Euler equation, and accordingly are strongly nonlinear and intractable [7,12,17,35]. The remarkable aspect of the solution we derive is that the flow it prescribes accommodates a general fluid stratification: the density may vary both with depth, and with latitude.

Stratification plays a fundamental role in GFD and— particularly in the equatorial region— large scale oceanic processes exhibit and experience pronounced density variations, most commonly due to fluctuations in the fluid temperature or salinity, cf. [4, 6, 11, 14, 29, 34]. Achieving a detailed theoretical understanding of stratified flows is therefore of the utmost practical importance. However, counter to this, accommodating variable density in the fluid vastly complicates the mathematical analysis of an already intractable problem. This assertion is illustrated by observing

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that even in the simpler setting of two-dimensional gravity water waves (where the Coriolis and centripetal effects of the earth's rotation are neglected) stratified fluids remained impervious to rigorous mathematical analysis approaches until recently—a selection of recent advances in this field is given by [3, 9, 10, 13, 21, 22, 32, 36, 37].

Exact solutions in fluid dynamics are extremely rare and, while they are important and useful in and of themselves, in an oceanographical context they are commonly regarded as robust and reliable starting points from which to generate more physically realistic and observable flows by way of asymptotic, or multiple scale, methods. The validity of this approach hinges on our ability to glean detailed information regarding fine properties of the structure of such solutions. In this paper we construct an exact solution which accommodates general stratification, and which furthermore admits a velocity profile beneath the surface which is arbitrarily prescribable with depth. The resulting fluid flow possesses a free-surface whose form is intricately linked to the pressure distribution at the surface. Accordingly, the resulting solution is very rich structurally from both the physical perspective, and the viewpoint of mathematical analysis.

This work builds on a number of important recent developments. In [7, 8] it was first shown that exact solutions to the full GFD governing equations can be constructed in terms of spherical coordinates which represent purely-azimuthal, depth-varying flows, and that these solutions can be chosen to model both the equatorial undercurrent (EUC), and the Antarctic Circumpolar Current (ACC), respectively. These solutions model purely homogeneous fluids with no stratification permitted. Subsequently, exact equatorial flow solutions were constructed by the authors [19, 20] which do permit stratification, but of a relatively simple form; namely, the fluid density varies linearly with depth and is independent of the latitude. The solution we construct below is remarkable mathematically, and exciting physically, in its ability to accommodate a general fluid stratification that varies both with depth and latitude. Furthermore, the form of the exact solution we construct possesses quite a bit of flexibility and freedom in its prescription; in particular, it offers a model which also captures the salient features of the EUC. Unsurprisingly, this level of generality has the drawback of generating quite a number of technical complications in the mathematical analysis that we must handle with some care.

The layout of this paper is as follows. Following the presentation of the governing equations in Section 2, the velocity and pressure fields for our exact solution are constructed in Section 3.1. As a by-product of the derivation process we obtain a Bernoulli-type relation at the free-surface, which is key to all further considerations. This somewhat convoluted relation provides an implicit prescription of the relationship between the imposed pressure, and the resulting surface distortion, at the free-surface. In Section 3.2 this Bernoulli relation is recast into a functional operator formulation, which is then subjected to a careful implementation of the implicit function theorem in order to establish that the relationship between the imposed pressure at the free-surface and the resulting distortion of the surface's shape is well-defined. In Section 3.3 we prove that the imposed pressure, and related surface–distortion, exhibit the desired monotonicity properties. We conclude the analysis in Section 3.4

by subjecting our exact solution to a short-wavelength stability analysis. Verifying the stability (or otherwise) of a given fluid motion is a question of the utmost physical importance, which is typically incredibly difficult to establish mathematically. Although the underlying mathematical analysis is intractable in general for our solution, we prove in Theorem 3.3 that, with a specified choice of density distribution, the exact solution is linearly stable to short-wavelength perturbations.

## 2. Governing equations

A noteworthy, and important, aspect of the work we present here is that the exact solution we derive satisfies the full governing equations expressed in a spherical coordinate system which is fixed at a point of the Earth's surface. At no stage do we invoke standard simplification techniques– such as reverting to cylindrical, or tangent plane, coordinates– and as a result the structure of our solution captures in full detail the curvature of the Earth's geometry. The one compromise we enforce in order to achieve this goal is to assume an azimuthal invariance, and accordingly any resulting flows will assume a jet-like structure. Guided by Maslowe's observation [33] that the Reynolds number is, in general, extremely large for oceanic flows of the type we are interested in, we consider incompressible and inviscid flows. Therefore the appropriate GFD governing equations comprise an Euler equation complemented with free–surface and bottom boundary conditions.

We work in a system of right–handed coordinates $(r, \theta, \varphi)$ where $r$ denotes the distance to the centre of the sphere, $\theta \in [0, \pi]$ is the polar angle (the convention being that $\pi/2 - \theta$ is the angle of latitude) $\varphi \in [0, 2\pi]$ is the azimuthal angle (the angle of longitude). In this coordinate system the North and South poles are located at $\theta = 0, \pi$, respectively, while the Equator sits on $\theta = \pi/2$. The unit vectors in this system are $\mathbf{e}_r = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\mathbf{e}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$, $\mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, with $\mathbf{e}_\varphi$ pointing from West to East and $\mathbf{e}_\theta$ from North to South. Denoting by $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\varphi$ the velocity field we have that the governing equations in the $(r, \theta, \varphi)$ coordinate system are the Euler's equations,

$$\begin{aligned} u_t + uu_r + \tfrac{v}{r}u_\theta + \tfrac{w}{r\sin\theta}u_\varphi - \tfrac{1}{r}(v^2 + w^2) &= -\tfrac{1}{\rho}p_r + F_r \\ v_t + uv_r + \tfrac{v}{r}v_\theta + \tfrac{w}{r\sin\theta}v_\varphi + \tfrac{1}{r}(uv - w^2\cos\theta) &= -\tfrac{1}{\rho}\tfrac{1}{r}p_\theta + F_\theta \\ w_t + uw_r + \tfrac{v}{r}w_\theta + \tfrac{w}{r\sin\theta}w_\varphi + \tfrac{1}{r}(uw + vw\cot\theta) &= -\tfrac{1}{\rho}\tfrac{1}{r\sin\theta}p_\varphi + F_\varphi, \end{aligned} \tag{2.1a}$$

where $p(r, \theta, \varphi)$ is denotes the pressure in the fluid and $(F_r, F_\theta, F_\varphi)$ is the body-force vector, and the equation of mass conservation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial(\rho w)}{\partial\varphi} = 0. \tag{2.1b}$$

The solution we construct in Section 3 below will incorporate a variable density distribution of the form $\rho = \rho(r, \theta)$. To capture the effect of the Earth's rotation we associate $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ with a fixed point on the sphere which is rotating about its polar axis. Consequently, we introduce the Coriolis term $2\mathbf{\Omega} \times \mathbf{u}$ on the left-hand side of (2.1a), where

$$\mathbf{\Omega} = \Omega(\mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta),$$

(with $\Omega \approx 7.29 \times 10^{-5}$ rad $s^{-1}$ being the constant rotational speed of Earth) and the centripetal acceleration $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$, where $\mathbf{r} = r\mathbf{e}_r$. The latter two quantities combine, with respect to the $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ basis, to give

$$2\Omega\big[-w(\sin\theta)\mathbf{e}_r - w(\cos\theta)\mathbf{e}_\theta + (u\sin\theta + v\cos\theta)\mathbf{e}_\varphi\big] \\ - r\Omega^2\big[(\sin^2\theta)\mathbf{e}_r + (\sin\theta\cos\theta)\mathbf{e}_\theta\big].$$

We assume that the external body-force is due to gravity alone, hence the body-force vector is given by $-g\mathbf{e}_r$. The equations of motion are completed by the boundary conditions as follows. On the free surface $r = R + h(\theta, \varphi)$ we have the dynamic boundary condition

$$p = P(\theta, \varphi), \tag{2.1c}$$

and the kinematic boundary condition

$$u = \frac{v}{r}\frac{\partial h}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial h}{\partial \varphi}. \tag{2.1d}$$

At the bottom of the ocean, described by $r = d(\theta, \varphi)$, the kinematic boundary condition is given by

$$u = \frac{v}{r}\frac{\partial d}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial d}{\partial \varphi}, \tag{2.1e}$$

which ensures that the sea-bed is an impermeable, and impenetrable, boundary.

## 3. Exact solutions

This section has three primary aims. Firstly, in Section 3.1 we construct an exact solution of the equations of motion and the boundary conditions presented in (2.1). This solution prescribes the velocity field and the pressure distribution for the flow and, due on the one hand to the complex fluid stratification, and on the other to the usage of spherical coordinates, the elaborate expression for the pressure function we obtain in (3.14) makes it impossible to obtain an explicit formula for the function $h$ representing the free surface. Nevertheless, an important by-product of the derivation process is a complex Bernoulli-type relation at the free-surface which provides an implicit prescription of the relationship between the imposed pressure, and the resulting surface distortion, at the free-surface. In Section 3.2 we subject this Bernoulli relation (3.15) to functional analytic considerations in order to establish that the relationship between the imposed pressure at the free-surface and the resulting distortion of the surface's shape is well-defined (yielding an existence and uniqueness result). Finally, in Section 3.3 we prove that the free-surface distortion, and the pressure distribution, exhibit the desired monotonicity properties.

In the quest to derive purely–azimuthal equatorial flow solutions, we seek solutions that represent a steady flow which does not vary in the azimuthal direction; hence, all components of the solution will be independent of $\varphi$. The resulting velocity field is characterised by $u = v = 0$ and $w = w(r, \theta)$, and moreover $p = p(r, \theta)$, $h = h(\theta)$, $d = d(\theta)$. Consequently, we observe that the two kinematic boundary conditions (2.1d) and (2.1e) and the equation of mass conservation (2.1b) are satisfied automatically, whereas the Euler equations (2.1a) reduce to the following form:

$$\begin{cases} -\frac{w^2}{r} - 2\Omega w \sin\theta - r\Omega^2 \sin^2\theta &= -\frac{1}{\rho} p_r - g, \\ -\frac{w^2}{r}\cot\theta - 2\Omega w \cos\theta - r\Omega^2 \sin\theta\cos\theta &= -\frac{1}{\rho r} p_\theta, \\ 0 &= -\frac{1}{\rho}\frac{1}{r\sin\theta} p_\varphi. \end{cases} \tag{3.1}$$

The solution we now derive represents a considerable advancement on previous results in the GFD setting [7,8,20,31] in the sense that we incorporate a general density stratification in the flow of the form $\rho = \rho(r, \theta)$. Note that this is the most general density formulation we can aim for in our setting, since it follows from the equation of mass conservation (2.1b) that a density depending on $\varphi$ cannot be accommodated by such azimuthal flows.

3.1. **Explicit solutions for the velocity field and the pressure.** Firstly, we observe from the third equation in (3.1) that $p = p(r, \theta)$. Note that the system (3.1) can now be simplified as

$$\rho \frac{(w + \Omega r \sin\theta)^2}{r} = p_r + g\rho, \qquad \rho r \cot\theta \frac{(w + \Omega r \sin\theta)^2}{r} = p_\theta.$$

Let us denote, for simplicity,

$$Z = Z(r, \theta) := \frac{(w + \Omega r \sin\theta)^2}{r}$$

and obtain from the above system

$$\rho Z = p_r + g\rho, \qquad (\rho r \cot\theta) Z = p_\theta.$$

The elimination of the pressure $p$ from the latter system leads to

$$(\rho Z)_\theta - (\rho r \cot\theta Z)_r = g\rho_\theta$$

which can be rewritten as

$$-r\cos\theta\,(\rho r Z)_r + \sin\theta\,(\rho r Z)_\theta = (r\sin\theta)\,[g\rho_\theta(r, \theta)]\,.$$

We further set $U := \rho r Z$ and appeal to the method of characteristics in order to solve the equation

$$-(r\cos\theta)U_r + (\sin\theta)U_\theta = (r\sin\theta)\,[g\rho_\theta(r, \theta)]\,. \tag{3.2}$$

Accordingly, we seek curves $s \to (r(s), \theta(s))$ satisfying

$$\dot{r}(s) = -r(s)\cos\theta(s), \qquad \dot{\theta}(s) = \sin\theta(s). \tag{3.3}$$

Note that any $(r(s), \theta(s))$ that satisfy the previous system also obeys the equation

$$\frac{d}{ds}\Big(r(s)\sin\theta(s)\Big) = 0 \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.4}$$

The choice (3.3) transforms equation (3.2) into

$$\frac{d}{ds}\left(U(r(s),\theta(s))\right) = (r(s)\sin\theta(s))\left[g\rho_\theta(r(s),\theta(s))\right], \tag{3.5}$$

whose resolution depends upon finding suitable solutions to the characteristic equations (3.3). The integration of (3.3) yields the general solution

$$\tilde{r}(s) = c_1 e^s + c_2 e^{-s}, \quad c_1, c_2 \in \mathbb{R},$$
$$\tilde{\theta}(s) = \arccos\left(\frac{1 - ce^{2s}}{1 + ce^{2s}}\right), \quad c \geq 0,$$

where $c, c_1, c_2$ are constants of integration. To find $U$ as a function of $r$ and $\theta$ we proceed as follows: for given $(r, \theta)$, we search for an $s_0$ such that

$$\tilde{\theta}(s_0) = \theta, \quad \tilde{r}(s_0) = r. \tag{3.6}$$

We choose now the constant $c$ such that $\tilde{\theta}(0) = \frac{\pi}{2}$. This gives $c = 1$, that is

$$\tilde{\theta}(s) = \arccos\left(\frac{1 - e^{2s}}{1 + e^{2s}}\right),$$

From the property (3.4) we observe that

$$\tilde{r}(0)\sin\left(\tilde{\theta}(0)\right) = \tilde{r}(s_0)\sin\left(\tilde{\theta}(s_0)\right) = r\sin\theta,$$

which, after using $\tilde{\theta}(0) = \frac{\pi}{2}$, gives $\tilde{r}(0) = r\sin\theta$. It is easy to see that

$$s_0 = \frac{1}{2}\ln\frac{1 - \cos\theta}{1 + \cos\theta} =: f(\theta)$$

is the unique element satisfying the first equation in (3.6). To determine $c_1$ and $c_2$ we solve

$$\tilde{r}(f(\theta)) = r, \tag{3.7}$$
$$\tilde{r}(0) = r\sin\theta, \tag{3.8}$$

which is equivalent to

$$c_1 + c_2 + (c_2 - c_1)\cos\theta = r\sin\theta, \qquad c_1 + c_2 = r\sin\theta,$$

whose unique solution (for $\theta \neq \pi/2$) is $c_1 = c_2 = \frac{r\sin\theta}{2}$. Let us denote with $(\overline{r}(s), \overline{\theta}(s))$ the special characteristic solution to (3.3) that satisfies (3.6). That is,

$$\overline{r}(s) = \frac{r\sin\theta}{2}(e^s + e^{-s}), \qquad \overline{\theta}(s) = \arccos\left(\frac{1 - e^{2s}}{1 + e^{2s}}\right). \tag{3.9}$$

Inserting first $(\overline{r}(s), \overline{\theta}(s))$ in the formula (3.5), then integrating with respect to $s$ from 0 to $f(\theta)$, and taking into account (3.7)-(3.8), we obtain

$$U(r,\theta) - U(r\sin\theta, \pi/2) = gr\sin\theta \int_0^{f(\theta)} \left[\rho_\theta(\overline{r}(s), \overline{\theta}(s))\right] ds,$$

where we have used the fact that $\frac{d}{ds}\left(\overline{r}(s)\sin\overline{\theta}(s)\right) = 0$ for all $s$, which implies that $\overline{r}(s)\sin\left(\overline{\theta}(s)\right) = \overline{r}(0)\sin\left(\overline{\theta}(0)\right) = r\sin\theta$ for all $s$. From the definitions of the functions $U$ and $Z$ we finally obtain that the azimuthal velocity $w$ is given by the formula

$$w(r,\theta) = -\Omega r\sin\theta + \sqrt{\frac{F(r\sin\theta) + gr\sin\theta\int_0^{f(\theta)}\left[\rho_\theta(\overline{r}(s), \overline{\theta}(s))\right] ds}{\rho(r,\theta)}}, \tag{3.10}$$

where $t \to F(t)$ denotes some arbitrary smooth function. To determine the pressure we see first that

$$p_r + g\rho = \frac{U}{r} = \frac{F(r\sin\theta) + gr\sin\theta\int_0^{f(\theta)}\left[\rho_\theta(\overline{r}(s), \overline{\theta}(s))\right] ds}{r} \tag{3.11}$$

and

$$p_\theta = U(r,\theta)\cot\theta = \cot\theta\left[F(r\sin\theta) + r\sin\theta\int_0^{f(\theta)}\left[g\rho_\theta(\overline{r}(s), \overline{\theta}(s))\right] ds\right] \tag{3.12}$$

Integrating with respect to $r$ in (3.11) we obtain

$$p(r,\theta) = C(\theta) - g\int_a^r \rho(\tilde{r},\theta)d\tilde{r} + \int_{a\sin\theta}^{r\sin\theta}\left[\frac{F(y)}{y} + \mathcal{F}(y,\theta)\right] dy, \tag{3.13}$$

where $a$ is an arbitrary constant, $\theta \to C(\theta)$ is a function (to be determined) and

$$\mathcal{F}(y,\theta) := \int_0^{f(\theta)}\left[g\rho_\theta\left(y\cdot\frac{e^s + e^{-s}}{2}, \overline{\theta}(s)\right)\right] ds.$$

Differentiating with respect to $\theta$ in (3.13) yields

$$\begin{aligned} p_\theta =& C'(\theta) - g\int_a^r \rho_\theta(\tilde{r},\theta)d\tilde{r} + \cot\theta[F(r\sin\theta) - F(a\sin\theta)] \\ &+ \mathcal{F}(r\sin\theta)r\cos\theta - \mathcal{F}(a\sin\theta)a\cos\theta + \int_{a\sin\theta}^{r\sin\theta}\mathcal{F}_\theta(y,\theta)dy. \end{aligned}$$

Note that

$$\mathcal{F}_\theta(y,\theta) = \left[g\rho_\theta\left(y\cdot\frac{e^s + e^{-s}}{2}, \overline{\theta}(s)\right)\right]\Big|_{s=f(\theta)} f'(\theta) = \frac{1}{\sin\theta}\left[g\rho_\theta\left(\frac{y}{\sin\theta}, \theta\right)\right],$$

and consequently

$$\int_{a\sin\theta}^{r\sin\theta}\mathcal{F}_\theta(y,\theta)dy = \int_a^r [g\rho_\theta(\tilde{r},\theta)]d\tilde{r},$$

which implies that

$$p_\theta = C'(\theta) + \cot\theta[F(r\sin\theta) - F(a\sin\theta)] + \mathcal{F}(r\sin\theta)r\cos\theta - \mathcal{F}(a\sin\theta)a\cos\theta.$$

Comparing now the latter relation with (3.12) we obtain that

$$C'(\theta) = F(a\sin\theta)\cot\theta + \mathcal{F}(a\sin\theta)a\cos\theta.$$

In summary, we have derived the following formula for the pressure distribution in the fluid:

$$\begin{aligned} p(r,\theta) =& b - g\int_a^r \rho(\tilde{r},\theta)d\tilde{r} + \int_{a\sin\theta}^{r\sin\theta}\left[\frac{F(y)}{y} + \mathcal{F}(y,\theta)\right]dy, \\ &+ \int_{\pi/2}^{\theta}[F(a\sin\tilde{\theta})\cot\tilde{\theta} + \mathcal{F}(a\sin\tilde{\theta})a\cos\tilde{\theta}]d\tilde{\theta} \end{aligned} \tag{3.14}$$

where $a$ and $b$ are real constants, $F$ is an arbitrary smooth function, and $\mathcal{F}$ is as given previously.

*Remark* 3.1. Having derived the mathematical formulation of the velocity field, we remark that the flow prescribed by (3.10) is applicable for modelling any number of depth-varying geophysical flows in the equatorial region: it follows from (3.10) that the azimuthal flow velocity is determined by prescribing it at the equator by setting $w(r,\pi/2) = W(r)$, where $W(r)$ is a given depth-varying velocity profile. In particular, it is possible to capture the salient features of the equatorial undercurrent (EUC) by way of our solution (3.10). The EUC is a celebrated and remarkable current which runs the entire extent of the Pacific equator. Owing to the prevailing trade-winds, its surface flow is predominantly westward yet a flow reversal occurs beneath the surface leading to an eastward-flowing jet, which is itself confined to depths of $100-200$m, cf. [6, 7, 26, 27].

3.2. **Functional analytical considerations for the Bernoulli relation.** As a consequence of the rich physical structure we have incorporated into our fluid model– specifically, the general fluid stratification and the usage of spherical coordinates– the solutions formulated in (3.10) and (3.14) are relatively convoluted and involved. A satisfactory treatment of the full GFD governing equations (2.1) involves not only deriving a formulation for the velocity field and of the pressure function but also elucidating, insofar as is possible, the nature of the resulting free surface. As the flow prescribed by (3.10) and (3.14) is too complicated to achieve an explicit description of the free-surface, to attain this goal we must appeal to functional analytical considerations. The aim of this section is to invoke the implicit function theorem to establish, for a given pressure distribution on the free surface, the existence and uniqueness of the implicitly defined function $h(\theta)$ representing the distortion of the free surface. We begin by addressing the surface boundary condition (2.1c). Setting $p = P(\theta)$ on $r = R + h(\theta)$, gives

$$\begin{aligned} P(\theta) =& b - g\int_a^{R+h(\theta)} \rho(\tilde{r},\theta)d\tilde{r} + \int_{a\sin\theta}^{[R+h(\theta)]\sin\theta}\left[\frac{F(y)}{y} + \mathcal{F}(y,\theta)\right]dy, \\ &+ \int_{\pi/2}^{\theta}[F(a\sin\tilde{\theta})\cot\tilde{\theta} + \mathcal{F}(a\sin\tilde{\theta})a\cos\tilde{\theta}]d\tilde{\theta}. \end{aligned} \tag{3.15}$$

Formula (3.15) is a Bernoulli-type condition, relating the imposed pressure at the surface of the ocean to the resulting deformation of that surface. A rigorous analysis of (3.15) can be performed by recasting it as a functional operator, but first we non-dimensionalise (3.15) in order to compare the physical quantities involved in it in a meaningful way. To this end we set $h \equiv 0$ in (3.15), which corresponds to an undisturbed surface that follows the curvature of the Earth. The pressure required to maintain this shape is given by

$$\begin{aligned} P_0(\theta) =& P(h \equiv 0) = b - g \int_a^R \rho(\tilde{r}, \theta) d\tilde{r} + \int_{a \sin\theta}^{R \sin\theta} \left[ \frac{F(y)}{y} + \mathcal{F}(y, \theta) \right] dy \\ &+ \int_{\pi/2}^{\theta} [F(a \sin\tilde{\theta}) \cot\tilde{\theta} + \mathcal{F}(a \sin\tilde{\theta}) a \cos\tilde{\theta}] d\tilde{\theta}. \end{aligned}$$

Next we observe that if the pressure at the Equator is set to be the constant atmospheric pressure, $P_{atm}$, then it follows from the latter formula that

$$P_{atm} = b - g \int_a^R \rho\left(\tilde{r}, \frac{\pi}{2}\right) d\tilde{r} + \int_a^R \left[ \frac{F(y)}{y} + \mathcal{F}\left(y, \frac{\pi}{2}\right) \right] dy. \tag{3.16}$$

We now non-dimensionalise by dividing (3.15) by $P_{atm}$, resulting in

$$\begin{aligned} &- \mathfrak{P} + \frac{b}{P_a} - \frac{g}{P_a} \int_a^{[1+\mathfrak{h}(\theta)]R} \rho(\tilde{r}, \theta) d\tilde{r} + \frac{1}{P_a} \int_{a \sin\theta}^{[1+\mathfrak{h}(\theta)]R \sin\theta} \left[ \frac{F(y)}{y} + \mathcal{F}(y, \theta) \right] dy \\ &+ \frac{1}{P_a} \int_{\pi/2}^{\theta} [F(a \sin\tilde{\theta}) \cot\tilde{\theta} + \mathcal{F}(a \sin\tilde{\theta}) a \cos\tilde{\theta}] d\tilde{\theta} = 0, \end{aligned} \tag{3.17}$$

where $\mathfrak{h}(\theta) := \frac{h(\theta)}{R}$ and $\mathfrak{P}(\theta) := \frac{P(\theta)}{P_a}$. Equation (3.17) can now be reformulated as a functional operator equation by denoting its left-hand side as $\mathfrak{F}(\mathfrak{h}, \mathfrak{P})$, resulting in the relation

$$\mathfrak{F}(\mathfrak{h}, \mathfrak{P}) = 0, \tag{3.18}$$

where $\mathfrak{F}$ operates from

$$\mathfrak{F} : B \times C\left([\pi/2, \pi/2 + \varepsilon]\right) \to C\left([\pi/2, \pi/2 + \varepsilon]\right).$$

Here, $B$ denotes the open ball of radius $10^{-2}$ from the Banach space

$$C\left(\left[\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right]\right),$$

consisting of continuous functions $f : [\pi/2, \pi/2 + \varepsilon] \to \mathbb{R}$, equipped with the supremum norm

$$||f|| = \sup_{t \in [\pi/2, \pi/2 + \varepsilon]} \{|f(t)|\},$$

and the choice $\epsilon = 0.016$ is appropriate for flows in the equatorial region, corresponding to a strip of 100 km width about the Equator, cf. [7]. In order to apply the implicit function theorem we need to find an elementary solution to (3.18). This is given by the pressure required to keep an undisturbed free surface, that is, we set $h \equiv 0$ in (3.17) and obtain

$$\begin{aligned}\mathfrak{P}_0(\theta) =& \frac{b}{P_a} - \frac{g}{P_a}\int_a^R \rho(\tilde{r},\theta)d\tilde{r} + \frac{1}{P_a}\int_{a\sin\theta}^{R\sin\theta}\left[\frac{F(y)}{y} + \mathcal{F}(y,\theta)\right]dy \\ &+ \frac{1}{P_a}\int_{\pi/2}^{\theta}[F(a\sin\tilde{\theta})\cot\tilde{\theta} + \mathcal{F}(a\sin\tilde{\theta})a\cos\tilde{\theta}]d\tilde{\theta},\end{aligned}$$

which satisfies

$$\mathfrak{F}(0,\mathfrak{P}_0) = 0.$$

We compute now the derivative

$$D_{\mathfrak{h}}\mathfrak{F}(0,\mathfrak{P}_0)(\mathfrak{h}) = \lim_{s\to 0}\frac{\mathfrak{F}(s\mathfrak{h},\mathfrak{P}_0) - \mathfrak{F}(0,\mathfrak{P}_0)}{s},$$

obtaining that

$$\begin{aligned}D_{\mathfrak{h}}\mathfrak{F}(0,\mathfrak{P}_0)\mathfrak{h} &= -\frac{gR}{P_a}\rho(R)\mathfrak{h} + \frac{1}{P_a}[F(R\sin\theta) + R\sin\theta\mathcal{F}(R\sin\theta,\theta)]\mathfrak{h} \\ &= \frac{\rho(R)}{P_a}\left[-gR + (w(R,\theta) + \Omega R\sin\theta)^2\right]\mathfrak{h}\end{aligned}$$

where the last equality follows by the formula (3.10) for the azimuthal velocity $w$. Taking into account the sizes of $g, R$ and $w$ we see that there is a constant $\mathfrak{a} < 0$ such that $-gR + (w(R,\theta) + \Omega R\sin\theta)^2 \le \mathfrak{a} < 0$. This fact implies that the operator $D_{\mathfrak{h}}\mathcal{F}(0,\mathfrak{P}_0) : C([\pi/2,\pi/2+\varepsilon]) \to C([\pi/2,\pi/2+\varepsilon])$ is a linear homeomorphism. Invoking now the implicit function theorem [2], we conclude that for any sufficiently small perturbation $\mathfrak{P}$ of $\mathfrak{P}_0$ there exists a unique $\mathfrak{h} \in C([\pi/2,\pi/2+\varepsilon])$ such that (3.17) holds true.

3.3. **Monotonicity properties.** We have proven above that the Bernoulli relation (3.15) uniquely prescribes a relationship between variations in the imposed surface pressure and the resulting distortion of the ocean's free-surface. The aim of this section is to show that this relationship exhibits the physically expected monotonicity properties, namely we will establish that

$$\mathfrak{P}'(\theta) < 0 \quad \text{if} \quad \mathfrak{h}'(\theta) \ge 0 \quad \text{for some} \quad \theta \in (\pi/2,\pi/2+\varepsilon), \tag{3.19}$$

and

$$\mathfrak{h}'(\theta) < 0 \quad \text{if} \quad \mathfrak{P}'(\theta) \ge 0 \quad \text{for some} \quad \theta \in (\pi/2,\pi/2+\varepsilon). \tag{3.20}$$

We first observe that utilising an iterative bootstrapping procedure, cf. [2], smoothness properties of $\mathfrak{P}$ can be transfered to $\mathfrak{h}$. Therefore, we can differentiate with respect to $\theta$ in (3.17) to obtain

$$
\begin{aligned}
\mathfrak{P}'(\theta) = &-\frac{g}{P_a}\int_a^{[1+\mathfrak{h}(\theta)]R}\rho_\theta(\tilde{r},\theta)d\tilde{r} - \frac{g}{P_a}\rho((1+\mathfrak{h})R,\theta)\mathfrak{h}'(\theta)R \\
&+ \frac{\cot\theta}{P_a}F\left((1+\mathfrak{h})R\sin\theta\right) + \frac{\mathfrak{h}'}{P_a}\cdot\frac{F\left((1+\mathfrak{h})R\sin\theta\right)}{1+\mathfrak{h}} - \frac{\cot\theta}{P_a}F(a\sin\theta) \\
&+ \frac{1}{P_a}\cdot\mathcal{F}((1+\mathfrak{h})R\sin\theta)\Big((1+\mathfrak{h})R\cos\theta + \mathfrak{h}'R\sin\theta\Big) - \frac{1}{P_a}\cdot\mathcal{F}(a\sin\theta)a\cos\theta \\
&+ \frac{1}{P_a}\int_a^{(1+\mathfrak{h})R}[g\rho_\theta(\tilde{r},\theta)]d\tilde{r} + \frac{1}{P_a}\cdot\Big(F(a\sin\theta)\cot\theta + \mathcal{F}(a\sin\theta)a\cos\theta\Big) \\
=&\frac{\mathfrak{h}'}{P_a}\left[-gR\rho((1+\mathfrak{h})R,\theta) + R(\sin\theta)\mathcal{F}((1+\mathfrak{h})R\sin\theta) + \frac{F((1+\mathfrak{h})R\sin\theta)}{1+\mathfrak{h}}\right] \\
&+ \frac{\cot\theta}{P_a}\left[F((1+\mathfrak{h})R\sin\theta) + (1+\mathfrak{h})R(\sin\theta)\mathcal{F}((1+\mathfrak{h})R\sin\theta)\right].
\end{aligned} \tag{3.21}
$$

Making use of the formula for the azimuthal velocity (3.10) we obtain

$$
\begin{aligned}
\frac{P_a}{\rho((1+\mathfrak{h})R,\theta)}\mathfrak{P}'(\theta) = &\left[-gR + \frac{\Big(w((1+\mathfrak{h})R,\theta) + \Omega(1+\mathfrak{h})R\sin\theta\Big)^2}{1+\mathfrak{h}}\right]\mathfrak{h}'(\theta) \\
&+ \cot\theta\Big(w((1+\mathfrak{h})R,\theta) + \Omega(1+\mathfrak{h})R\sin\theta\Big)^2.
\end{aligned} \tag{3.22}
$$

We note that for realistic velocities $w$, the quantity

$$
-gR + \frac{\Big(w((1+\mathfrak{h})R,\theta) + \Omega(1+\mathfrak{h})R\sin\theta\Big)^2}{1+\mathfrak{h}}
$$

is strictly negative. Combining this fact with relation (3.22) results in the monotonicity properties (3.19) and (3.20) stated above.

3.4. **Short-wavelength perturbation stability analysis.** The meaningfulness, and applicability, of the azimuthal flow solutions (3.10), (3.14) that we constructed and analysed in the previous sections are considerably enhanced if we can establish some stability properties that they fulfil. This is the task that we undertake here by employing the short-wavelength perturbation method for general three-dimensional flows developed by Bayly [1], Friedlander and Vishik [15] and Lifschitz and Hameiri [30]. This method investigates the time growth of the amplitude of perturbations to basic flows having a velocity field which satisfies the Euler equations (2.1a) and the equation of mass conservation (2.1b). The stability of the basic flow with respect to short-wavelength perturbations refers to the uniform boundedness in time of the amplitude of the perturbation. The short-wavelength stability method has turned out to be highly-applicable to the analysis of a number of some recently derived exact solutions in the GFD setting, cf. [5, 16, 18, 20, 24, 25]; see also the survey [23]. Whether a given fluid motion is stable, or unstable, is a question of the utmost physical importance which is typically incredibly difficult to establish mathematically. We will show in this section that this approach can be implemented in the highly intricate setting of a stratified fluid with a density that can vary with respect to both $r$ and $\theta$. Although the underlying mathematical analysis is intractable in general for our solution, we prove in Theorem 3.3 that, with a specified choice of density distribution, the exact solution is linearly stable to short-wavelength perturbations.

To begin, we consider perturbations $\mathbf{P}$ of the pressure $p$ and $\mathbf{U} = U\mathbf{e}_r + V\mathbf{e}_\theta + W\mathbf{e}_\varphi$ along the streamlines of the azimuthal flow with velocity vector $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\varphi$ with $w$ given by the formula (3.10) for a specific choice of the density function $(r,\theta) \to \rho(r,\theta)$. More precisely, we consider $\rho(r,\theta) = (b - ar)\sin\theta$, where $a, b$ are positive constants such that $b - ar > 0$ for all $r$. Moreover, we make the choice $F(r\sin\theta) = \mathfrak{c} r\sin\theta$ for some constant $\mathfrak{c}$. Therefore, the azimuthal component of the velocity is

$$w = -\Omega r\sin\theta + \sqrt{\frac{r}{b - ar}\left(\mathfrak{c} + g\int_0^{f(\theta)} \big(b - a\overline{r}(s)\big)\cos\overline{\theta}(s)\, ds\right)}, \tag{3.23}$$

where $\overline{r}(s), \overline{\theta}(s)$ are as given in (3.9). The specific form of the perturbations $\mathbf{U}$ and $\mathbf{P}$ are given by the WKB ansatz

$$\mathbf{U}(t,r,\theta,\varphi) \;=\; \mathbf{A}(t,r,\theta,\varphi)e^{\frac{i}{\epsilon}f(t,r,\theta,\varphi)} \;+\; \mathcal{O}(\epsilon) \tag{3.24a}$$

and

$$P(t,r,\theta,\varphi) \;=\; \epsilon B(t,r,\theta,\varphi)e^{\frac{i}{\epsilon}f(t,r,\theta,\varphi)} \;+\; \mathcal{O}(\epsilon^2), \tag{3.24b}$$

where $\mathbf{A} = A_1\mathbf{e}_r + A_2\mathbf{e}_\theta + A_3\mathbf{e}_\varphi$, $f$ is a scalar function and $\epsilon$ plays the role of a small parameter. The initial condition for the perturbation is

$$\mathbf{U}_0 := \mathbf{U}(0,r,\theta,\varphi) = \mathbf{A}(0,r,\theta,\varphi)e^{\frac{i}{\epsilon}f(0,r,\theta,\varphi)} =: \mathbf{A}_0(r,\theta,\varphi)e^{\frac{i}{\epsilon}f_0(r,\theta,\varphi)}.$$

We say that the basic flow is stable if the amplitude $\mathbf{A}$ remains uniformly bounded in time.

*Remark* 3.2. That the remainder terms in (3.24) are as asserted (that is, uniformly bounded with respect to the parameter $\epsilon$ in the sense of the $L^2$-norm) is a fact that was proved in [20] in the context of a general density distribution.

Asking that the perturbed system $\mathbf{U} + \mathbf{u}$, $P + p$ satisfies the conservation of mass and conservation of momentum equations (2.1b) and (2.1a) respectively, and using the WKB ansatz (3.24), we obtain that the amplitude $\mathbf{A}$ satisfies (with respect to the basis $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi$) the vectorial equation

$$\mathbf{A}_t \;+\; (\mathbf{u}\cdot\nabla)\mathbf{A} \;+\; (\mathbf{A}\cdot\nabla)\mathbf{u} \;+\mathfrak{M}(\Omega,\theta)\mathbf{A} = \;-\,i\frac{B}{\rho}\nabla f, \tag{3.25}$$

with

$$\mathfrak{M}(\Omega,\theta) \;:=\; \begin{pmatrix} 0 & 0 & -2\Omega\sin\theta \\ 0 & 0 & -2\Omega\cos\theta \\ 2\Omega\sin\theta & 2\Omega\cos\theta & 0 \end{pmatrix},$$

while the phase $f$ verifies the scalar equation

$$f_t + \frac{w(r,\theta)}{r\sin\theta} f_\varphi = 0. \tag{3.26}$$

Clearly, the general solution of (3.26) is

$$f = \mathcal{G}\left(\varphi \;-\; \int_0^t \frac{w(r(s),\theta(s))}{r(s)\sin\theta(s)}\,ds\right),$$

for some function $\mathcal{G}$.
A simple computation reveals now that the streamlines $(t\to r(t), t\to\theta(t), t\to\varphi(t))$ of the azimuthal flow with $u = v = 0$ and $w$ as in (3.23), are given by

$$r(t)\equiv r_0,\quad \theta(t)\equiv\theta_0,\quad \varphi(t) = \int_0^t \frac{w(r(s),\theta(s))}{r(s)\sin\theta(s)}\,ds \;+\; \varphi_0, \tag{3.27}$$

where $r_0, \theta_0$ and $\varphi_0$ denote initial data. The latter conclusion entails that $\nabla f = 0$ along the streamlines (3.27).

Denoting with $\frac{DA_i}{Dt}$ ($i\in\{1,2,3\}$) the material derivatives of $A_i$ along the streamlines (3.27) we obtain (after some straightforward computations) that equation (3.25) is equivalent to the system

$$\begin{pmatrix} \frac{DA_2}{Dt} \\ \\ \frac{DA_3}{Dt} \end{pmatrix} = \left(\begin{array}{c|c} 0 & 2\left(\frac{w(r_0,\theta_0)}{r_0} + \Omega\sin\theta_0\right)\cot\theta_0 \\ \hline \alpha_1\tan\theta_0 + \beta_1 & 0 \end{array}\right)\begin{pmatrix} A_2 \\ \\ A_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \tilde{c} \end{pmatrix},$$

where

$$\alpha_1 = -\left(w_r(r_0,\theta_0) + \frac{w(r_0,\theta_0)}{r_0} + 2\Omega\sin\theta_0\right),$$

$$\beta_1 = -\left(\frac{w_\theta(r_0,\theta_0)}{r_0} + \frac{w(r_0,\theta_0)\cot\theta_0}{r_0} + 2\Omega\cos\theta_0\right),$$

and $\tilde{c}$ is a constant arising from the initial data. We state now the main result of this section concerning the stability of the basic flow given in (3.23).

**Theorem 3.3.** *The flow with* $u = v = 0$ *and* $w$ *given by* (3.23) *is linearly stable under short-wavelength perturbations.*

*Proof.* Each eigenvalue $\lambda$ of $\mathcal{A}$ is a solution of the equation

$$\lambda^2 - (\alpha_1 + \beta_1 \cot\theta_0)\left(2\frac{w(r_0,\theta_0)}{r_0} + 2\Omega\sin\theta_0\right) = 0. \tag{3.28}$$

It is clear from formula (3.23) that $\frac{w(r_0,\theta_0)}{r_0} + \Omega\sin\theta_0 > 0$. We are going to show that $-(\alpha_1 + \beta_1\cot\theta_0) > 0$, an inequality which, combined with the previous one, leads to the conclusion that the eigenvalues of $\mathcal{A}$ are purely imaginary. The latter fact implies that the amplitude of the perturbed velocity field remains bounded as time progresses, that is, the given azimuthal flow is stable under short-wavelength perturbations. To perform the computations we establish first some notation. We set

$$I(r,\theta) = \int_0^{f(\theta)} \big(b - a\overline{r}(s)\big)\cos\overline{\theta}(s)\,ds \quad \text{and} \quad H(r,\theta) = \sqrt{\frac{r}{b-ar}\,(\mathfrak{c} + gI(r,\theta))}.$$

We then have that

$$-\alpha_1 - \beta_1\cot\theta_0 = H_r(r_0,\theta_0) + \frac{H_\theta(r_0,\theta_0)\cot\theta_0}{r_0} + \frac{H(r_0,\theta_0)(\cot^2\theta_0 + 1)}{r_0} \tag{3.29}$$

A computation reveals that

$$I(r,\theta) = b\log(\sin\theta) + ar(1-\sin\theta),$$

and thus

$$I_\theta(r,\theta) = (b - ar\sin\theta)\cot\theta \quad \text{and} \quad I_r(r,\theta) = a(1-\sin\theta) > 0. \tag{3.30}$$

Hence,

$$H_\theta = g\sqrt{\frac{r}{b-ar}}\cdot\frac{I_\theta(r,\theta)}{2\sqrt{\mathfrak{c} + gI(r,\theta)}}, \tag{3.31}$$

and

$$H_r = g\sqrt{\frac{r}{b-ar}}\cdot\frac{I_r(r,\theta)}{2\sqrt{\mathfrak{c}+gI(r,\theta)}} + \frac{b}{2(b-ar)^2}\cdot\sqrt{\frac{b-ar}{r}}\sqrt{\mathfrak{c}+gI(r,\theta)}.$$

Since $b - ar\sin\theta > 0$ we have (via (3.30) and (3.31)) that $H_\theta(r_0,\theta_0)\cot\theta_0 > 0$. Clearly, $H_r > 0$ and from the above calculations in (3.29) we infer that $-\alpha_1 - \beta_1\cot\theta_0 > 0$. This shows that the eigenvalues of $\mathcal{A}$ are purely imaginary, as asserted. Thus, the azimuthal flow given in the statement of the Theorem is linearly stable under short-wavelength perturbations. □

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