

Title	Importance-based semantics of polynomial comparative preference inference
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Publication date	2012
Original Citation	WILSON, N. 2012. Importance-based semantics of polynomial comparative preference inference. In: DE RAEDT, L., BESSIÈRE, C., DUBOIS, D., DOHERTY, P., FRASCONI, P., HEINTZ, F. & LUCAS, P. (eds.) ECAI 2012: 20th European Conference on Artificial Intelligence. Montpellier, France, 27-31 August. Amsterdam: IOS Press, pp. 852-857. doi: 10.3233/978-1-61499-098-7-852
Type of publication	Conference item
Link to publisher's version	http://ebooks-iospress.nl/publication/7081 - 10.3233/978-1-61499-098-7-852
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Download date	2024-04-26 16:18:44
Item downloaded from	https://hdl.handle.net/10468/1386



Importance-based Semantics of Polynomial Comparative Preference Inference

Nic Wilson¹

Abstract. A basic task in preference reasoning is inferring a preference between a pair of outcomes (alternatives) from an input set of preference statements. This preference inference task for comparative preferences has been shown to be computationally very hard for the standard kind of inference. Recently, a new kind of preference inference has been developed, which is polynomial for relatively expressive preference languages, and has the additional property of being much less conservative; this can be a major advantage, since it will tend to make the number of undominated outcomes smaller. It derives from a semantics where models are weak orders that are generated by objects called *cp-trees*, which represent a kind of conditional lexicographic order. We show that there are simple conditions, based on the notion of importance, that determine whether a weak order can be generated by a cp-tree of the given form. This enables a simple characterisation of the less conservative preference inference. We go on to study the importance properties satisfied by a simple kind of cp-tree, leading to another characterisation of the corresponding preference inference.

1 INTRODUCTION

A key task for preference reasoning is inferring a preference $\alpha \geq \beta$ between alternatives (outcomes) α and β , given a set Γ of input preferences. Here we are interested in comparative preferences, as expressed by languages related e.g., to CP-nets; an example comparative preference statement, expressing one aspect of my preferences about the car I am looking to buy, is the following: *If I buy a Toyota, I'd prefer a hatchback to a saloon, irrespective of the colour.* Suppose that all we assume about the decision maker is that their preference relation over outcomes is a weak order (i.e., a total pre-order). Then we can deduce $\alpha \geq \beta$ from Γ if and only if $\alpha \succcurlyeq \beta$ holds for all weak orders satisfying (every element of) Γ . We write this as $\Gamma \models \alpha \geq \beta$. This corresponds to the usual inferred preference relation for CP-nets, TCP-nets and other related formalisms. However, this preference relation has some major disadvantages. Firstly, there are serious computational problems, even for the special case of CP-nets, when it is PSPACE-complete in general [3, 10, 13]. This issue is especially important for constrained optimisation, or when one has a substantial database of possible choices, since one will need to perform a large number of comparisons between outcomes [4, 14, 19].

Secondly, the inference is rather weak. In a recommender system context, such as that described in [14], the system displays a number of outcomes (corresponding to products) to the user. If the preference inference is too weak, we can have a huge number of undominated outcomes. For this purpose it can often be helpful to have some form

of plausible inference for preferences, that is stronger, hence reducing the set of undominated alternatives to a manageable size [14].

The preference inference relation defined in [16] overcomes these disadvantages, since it is polynomial for relatively expressive comparative preference languages, making it very much faster (this is backed up by experimental results [14, 19]), and is a considerably stronger (i.e., larger) relation. It uses the same definition of inference, except that it only considers a subset of weak orders, those generated by a structure called a cp-tree, or, more specifically, a \mathcal{Y} -cp-tree, where \mathcal{Y} is a set of small subsets of V , the set of variables. However, what is not so obvious is what it means to restrict weak orders to \mathcal{Y} -cp-trees: in particular, which weak orders can be generated by some \mathcal{Y} -cp-tree?

The main aim of this paper is to characterise weak orders that are generated by \mathcal{Y} -cp-trees, and hence characterise the associated preference inference relation $\models_{\mathcal{Y}}$. The basis of our characterisation is the notion of (preferential) importance, the significance of which has been pointed out by Brafman, Domshlak and Shimony [7] (see also [5, 9]). Loosely speaking, one set of variables S is more important than another set T , if the preference between outcomes that differ on S does not depend at all on their values on T . For example, in a standard lexicographic ordering based on the sequence X_1, \dots, X_n , variables, each X_i is more important than variables $\{X_{i+1}, \dots, X_n\}$.

Let b be an assignment to some subset B of the variables V . We say that set of variables S has *overall importance given b* if, given b , S is more important than $V - (B \cup S)$. A weak order \succcurlyeq is then said to satisfy *overall importance with respect to \mathcal{Y}* if for any partial assignment b there exists some element of \mathcal{Y} that has overall importance given b . Theorem 1 in Section 4 states that, for any weak order \succcurlyeq , there exists a \mathcal{Y} -cp-tree with ordering \succcurlyeq if and only if \succcurlyeq satisfies overall importance with respect to \mathcal{Y} . This implies that the $\models_{\mathcal{Y}}$ preference inference relation can be expressed in a very simple way (Corollary 1): $\Gamma \models_{\mathcal{Y}} \alpha \geq \beta$ if and only if every weak order satisfying Γ and overall importance w.r.t. \mathcal{Y} also satisfies $\alpha \geq \beta$.

We go on, in Section 5, to construct a different characterisation of $\models_{\mathcal{Y}}$, for the special case when \mathcal{Y} is $\mathcal{Y}(1)$, the set of singleton subsets of V . We call such a \mathcal{Y} -cp-tree a *1-cp-tree*. The idea is to consider properties of the importance relation. One natural property is Right Union: If S is more important than T and U then it's more important than $T \cup U$. Another is transitivity, for instance, if X_1 is more important than X_2 which is more important than X_3 , then X_1 is more important than X_3 . We also consider a completeness property, roughly speaking that, in any given context, either X_1 is more important than X_2 or vice versa. None of these three properties hold universally. We consider forms of these three properties and show (Theorem 2) that \succcurlyeq can be generated by a 1-cp-tree if and only if the \succcurlyeq -importance

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satisfies the three properties. This leads to another characterisation of the $\models_{\mathcal{Y}(1)}$ preference inference relation (Corollary 2).

The remainder of the paper is structured as follows. Section 2 describes the $\models_{\mathcal{Y}}$ preference inference relation; Section 3 defines and gives some general properties of importance. Section 4 gives the characterisation of $\models_{\mathcal{Y}}$ in terms of overall importance (the corollary of Theorem 1); Section 5 gives the further characterisation of the $\models_{\mathcal{Y}(1)}$ relation (the corollary of Theorem 2). Section 6 goes into details about cp-trees, and gives results that lead to a proof of Theorem 1. Section 7 defines ‘‘before-statements’’, which relate to the variable orderings in different paths in a cp-trees, building to a proof of Theorem 2. Section 8 concludes. More complete versions of the proofs are available in [18].

Terminology. Throughout the paper, we focus on a fixed finite set of variables V . For each $X \in V$ let $D(X)$ be the set of possible values of X ; we assume $D(X)$ has at least two elements. For subset of variables $A \subseteq V$ let $\underline{A} = \prod_{X \in A} D(X)$ be the set of possible assignments to set of variables A . The assignment to the empty set of variables is written \top . An **outcome** is an element of \underline{V} , i.e., an assignment to all the variables. If $a \in \underline{A}$ is an assignment to A , and $b \in \underline{B}$, where $A \cap B = \emptyset$, then we may write ab as the assignment to $A \cup B$ that combines a and b . For partial tuples $a \in \underline{A}$ and $u \in \underline{U}$, we may write $a \models u$, or say a extends u , if $A \supseteq U$ and $a(U) = u$, i.e., a projected to U gives u . More generally, we say that a is compatible with u if u and a agree on common variables, i.e., $u(A \cap U) = a(A \cap U)$. Binary relation \succcurlyeq is said to be a **weak order** (also known as a *total pre-order*) if it is transitive and complete, so that (i) $\alpha \succcurlyeq \beta$ and $\beta \succcurlyeq \gamma$ implies $\alpha \succcurlyeq \gamma$, and (ii) for all outcomes α and β , either $\alpha \succcurlyeq \beta$ or $\beta \succcurlyeq \alpha$. We say that α and β are \succcurlyeq -equivalent if both $\alpha \succcurlyeq \beta$ and $\beta \succcurlyeq \alpha$.

2 COMPARATIVE PREFERENCES

2.1 Comparative Preference Statements

A number of languages of comparative preference have been defined in recent years, for example, CP-nets [3], TCP-nets [7], cp-theories [17], feature vector rules [12], CI-nets [5] and more general languages [11, 1]. Of particular interest to us is the language from [16], which can express CP-nets, TCP-nets, cp-theories, feature vector rules and CI-nets, and for which inference is polynomial for the preference inference from [16] (see below). It considers comparative preference statements of the form $p \geq q \parallel T$, where P, Q and T are subsets of V , and $p \in \underline{P}$ is an assignment to P , and $q \in \underline{Q}$. Informally, the statement $p \geq q \parallel T$ represents the following: p is preferred to q if T is held constant.

We use a model-based semantics for preference inference, in which the models are weak orders on the set \underline{V} of outcomes (as in [8, 16]). A weak order \succcurlyeq satisfies $p \geq q \parallel T$ if and only if $\alpha \succcurlyeq \beta$ holds for all outcomes α and β such that α extends p , and β extends q , and α and β agree on T : $\alpha(T) = \beta(T)$. As shown in [16], such statements can be used to represent CP-nets [2, 3], TCP-nets [6, 7], feature vector rules [12] and cp-theories [15, 17]. It can also represent a preference of one outcome, α , over another, β : as a statement $\alpha \geq \beta \parallel \emptyset$, which we abbreviate to just $\alpha \geq \beta$; this can be useful, for instance, for application to recommender systems [14].

2.2 $\models_{\mathcal{Y}}$ Inference for Comparative Preferences

The $\models_{\mathcal{Y}}$ inference from [16] is parameterised by a set \mathcal{Y} of subsets of V , so that different \mathcal{Y} will give rise to different preference inference relations $\models_{\mathcal{Y}}$. The smaller \mathcal{Y} is, the stronger the relation $\models_{\mathcal{Y}}$. If \mathcal{Y} is 2^V then $\models_{\mathcal{Y}}$ is just the standard inference \models , defined early in Section 1. We are interested in cases where \mathcal{Y} only contains small subsets. For example, \mathcal{Y} might be defined to be all singleton subsets of V (i.e., sets with cardinality of one), or, alternatively, all subsets of cardinality at most two etc.

Let \mathcal{Y} be a set of non-empty subsets of the set of variables V . We say that \mathcal{Y} is a valid family (for cp-trees) if it satisfies the following properties: (i) $\bigcup \mathcal{Y} = V$, so that every variable in V appears in some element of \mathcal{Y} ; and (ii) it is closed under the subset relation, i.e., if $Y \in \mathcal{Y}$ and non-empty Y' is a subset of Y then $Y' \in \mathcal{Y}$.

A cp-tree σ has a set of variables Y_r associated with each node r . (The full definition is given later, in Section 6.1, and is taken from [16].) Associated with a cp-tree σ is a weak order \succcurlyeq_{σ} on outcomes. We say that σ satisfies set of comparative preference statements Γ if \succcurlyeq_{σ} satisfies (every element of) Γ .

Definition 1 (\mathcal{Y} -cp-tree) Let \mathcal{Y} be a valid family of subsets of V . A \mathcal{Y} -cp-tree is defined to be a cp-tree σ such that for any node r of σ , we have $Y_r \in \mathcal{Y}$. Let $\mathcal{Y}(1)$ be the set of singleton subsets of V , i.e., $\{\{X\} : X \in V\}$. A 1-cp-tree is defined to be a $\mathcal{Y}(1)$ -cp-tree, and so has a single variable associated with each node.

Example 1: Figure 1 gives an example of a cp-tree, with its associated weak order on outcomes. If, for example, \mathcal{Y} is the set of proper subsets of V then σ is a \mathcal{Y} -cp-tree. However, it is not a $\mathcal{Y}(1)$ -cp-tree because the leftmost node r has $Y_r = \{X_2, X_3\}$, and thus is not a singleton.

\mathcal{Y} -entailment $\models_{\mathcal{Y}}$. We assume a set \mathcal{L} , the elements of which are called *comparative preference statements*, and a satisfaction relation between weak orders (over \underline{V}) and \mathcal{L} . The preference inference relation based on \mathcal{Y} -cp-trees is defined as follows. Let $\Gamma \subseteq \mathcal{L}$ be a set of comparative preference statements, and let α and β be outcomes. $\Gamma \models_{\mathcal{Y}} \alpha \geq \beta$ holds if and only if every \mathcal{Y} -cp-tree σ satisfying Γ also satisfies $\alpha \succcurlyeq_{\sigma} \beta$. For a given \mathcal{Y} consisting of sets of bounded cardinality, determining an inference of the form $\Gamma \models_{\mathcal{Y}} \alpha \geq \beta$ can be done using a fairly simple algorithm in polynomial time when Γ consists of statements of the above form $p \geq q \parallel T$ [16]. It is important that the sets in \mathcal{Y} are small, since the computation is exponential in the cardinality of the largest set in \mathcal{Y} .

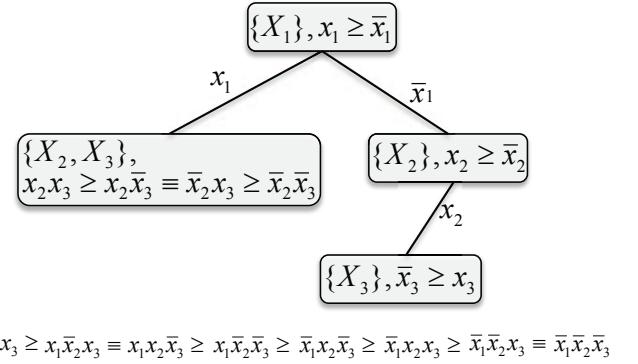


Figure 1. An example cp-tree σ over binary variables $V = \{X_1, X_2, X_3\}$, and its associated weak ordering \succcurlyeq_{σ} on outcomes. For each node r we are including its associated set Y_r and ordering \geq_r .

3 IMPORTANCE

We define and describe some properties of preferential importance (Section 3.1). Our definition differs somewhat from that given in [7], and applies more generally, but the intuition behind both definitions seems similar. A special case of importance is what we call *overall importance* (Section 3.2), where, given a partial tuple, one set of variables is more important than all the remaining variables. This property relates strongly to cp-trees, as shown in Section 4.

3.1 Some Properties of Importance

We consider importance statements on V which are of the form $b : S \sqsupseteq T$, where $b \in \underline{B}$ is an assignment to variables B , and B, S and T are mutually disjoint subsets of V . Such a statement may be read as: *Given b , S is more important than T* . For the case when B is empty, so that $b = \top$, we may abbreviate $\top : S \sqsupseteq T$ to just $S \sqsupseteq T$.

Let U be the other variables, i.e., $U = V - (B \cup S \cup T)$. A weak order \succcurlyeq is said to satisfy $b : S \sqsupseteq T$ if and only if for all $u \in \underline{U}$, for all $s, s' \in \underline{S}$ such that $s \neq s'$, for all $t_1, t_2, t_3, t_4 \in \underline{T}$,

$$bus_1 \succcurlyeq bus't_2 \iff bust_3 \succcurlyeq bus't_4.$$

Another way of saying this is that if two outcomes α and β both extend b , agree on $V - (S \cup T)$ and differ on S then the preference between α and β (i.e., if $\alpha \succcurlyeq \beta$ or $\beta \succcurlyeq \alpha$) does not depend on the values of α and β on T . That is: if $\alpha(V - T) = \alpha'(V - T)$ and $\beta(V - T) = \beta'(V - T)$ then $\alpha \succcurlyeq \beta \iff \alpha' \succcurlyeq \beta'$. It is a strong notion of importance: the variables S dominate the variables T , making T irrelevant, except if the pair of outcomes agree on S .

Example 1 continued: The weak ordering on outcomes in Figure 1 satisfies $\bar{x}_1 : \{X_2\} \sqsupseteq \{X_3\}$. This is because, for different assignments s and s' to X_2 , and arbitrary assignments t_1 and t_2 to X_3 , $\bar{x}_1 s t_1 \succcurlyeq \bar{x}_1 s' t_2$ if and only if $s = x_2$ and $s' = \bar{x}_2$, and so the choices of assignments to X_3 (t_1 and t_2) are irrelevant.

The definitions immediately imply the following property, showing that importance is monotonic with respect to changes in the tuple and the sets.

Proposition 1 Let B, S and T be mutually disjoint subsets of V , and let $S' \subseteq S$ and $T' \subseteq T$, and let $B' \supseteq B$ be a superset of B that is disjoint from $S' \cup T'$. Also, let b be an assignment to B , and let b' be an assignment to B' extending b , i.e., such that $b'(B) = b$. Then for any weak order \succcurlyeq on \underline{V} ,

$$\text{if } \succcurlyeq \text{ satisfies } b : S \sqsupseteq T \text{ then } \succcurlyeq \text{ satisfies } b' : S' \sqsupseteq T'.$$

Let $C \subseteq V$ be a set of variables, and let $c \in \underline{C}$ be an assignment to C . For convenience, we also will use the notation $[c] : S \sqsupseteq T$ as an abbreviation for the statement $c(C - (S \cup T)) : S \sqsupseteq T$. (Recall that $c(C - (S \cup T))$ is c with any assignments to variables in $S \cup T$ deleted.) The following result shows that the importance statements satisfied by a weak order are determined by those of the form $[\alpha] : S \sqsupseteq T$, for outcomes α .

Proposition 2 Let \succcurlyeq be a weak order on the set of outcomes \underline{V} , and let B, S and T be mutually disjoint subsets of V . Let $b \in \underline{B}$ be an assignment to B . Then \succcurlyeq satisfies $b : S \sqsupseteq T$ if and only if for all outcomes α extending b , weak order \succcurlyeq satisfies $[\alpha] : S \sqsupseteq T$.

However, there are apparently natural properties of Importance that do not always hold. For instance, if X_1 is more important X_2 and X_3 then one might expect that it is more important than $\{X_2, X_3\}$ (a “Right Union” property). Also, if X_1 is more important than X_2

which is more important than X_3 then one might expect that X_1 would be more important than X_3 . The following two examples show that neither property always holds.

Example 2 (Failure of Right Union Property): Let $V = \{X_1, X_2, X_3\}$, with each variable having boolean domain $\{0, 1\}$. Let us define $weight(\alpha) = 4\alpha(X_1) + 3\alpha(X_2) + 2\alpha(X_3)$. This defines a weak order \succcurlyeq given by $\alpha \succcurlyeq \beta$ if and only $weight(\alpha) \geq weight(\beta)$. For example, if $\alpha = (0, 1, 1)$ and $\beta = (1, 0, 0)$ then we have $weight(\alpha) = 3 + 2 = 5$, and $weight(\beta) = 4$, and so $\alpha \succcurlyeq \beta$. It can be seen that \succcurlyeq satisfies $\{X_1\} \sqsupseteq \{X_2\}$, and $\{X_1\} \sqsupseteq \{X_3\}$ (and also $\{X_2\} \sqsupseteq \{X_3\}$), but \succcurlyeq does not satisfy $\{X_1\} \sqsupseteq \{X_2, X_3\}$. This is because we have, for example, $(1, 1, 1) \succcurlyeq (0, 1, 1)$, but not $(1, 0, 0) \succcurlyeq (0, 1, 1)$. Thus we do not have $b : S \sqsupseteq T$ and $b : S \sqsupseteq U$ implies $b : S \sqsupseteq T \cup U$.

Example 3 (Failure of Transitivity Property): Consider the weak (indeed, total) order \succcurlyeq defined by the transitive and reflexive closure of: $111 \succcurlyeq 110 \succcurlyeq 100 \succcurlyeq 000 \succcurlyeq 101 \succcurlyeq 001 \succcurlyeq 011 \succcurlyeq 010$. \succcurlyeq satisfies $\{X_1\} \sqsupseteq \{X_2\}$ and $\{X_2\} \sqsupseteq \{X_3\}$, but not $\{X_1\} \sqsupseteq \{X_3\}$. (In fact, \succcurlyeq even satisfies $(X_2 = 0) : \{X_3\} \sqsupseteq \{X_1\}$.)

3.2 Overall Importance

A special type of importance statement $b : S \sqsupseteq T$ (with $b \in \underline{B}$) is when $B = V - (S \cup T)$, so that B, S and T partition the set of variables V .

Definition 2 Let $b \in \underline{B}$, where $B \subseteq V$, and let S be a non-empty subset of $V - B$. Let \succcurlyeq be a weak order on \underline{V} . We say that, for \succcurlyeq , S has *overall importance given b* if \succcurlyeq satisfies $b : S \sqsupseteq V - (B \cup S)$.

For instance, if S has overall importance given \top , then to determine which of outcomes α and β are preferred (w.r.t. \succcurlyeq), only the variables in S are relevant if α and β differ on S .

We assume in this paragraph that for \succcurlyeq , non-empty S has overall importance given b . \succcurlyeq then induces an ordering \succcurlyeq_b^S on the set \underline{S} of assignments to S . Define the reflexive relation \succcurlyeq_b^S as follows, where s and s' are arbitrary different elements of \underline{S} . $s \succcurlyeq_b^S s'$ if and only if for some (or any) $w_1, w_2 \in V - (B \cup S)$, $bsw_1 \succcurlyeq_b bs'w_2$. (This makes sense by overall importance of S given b .) Consider any outcomes α and β that extend b and differ on S . Then to see if $\alpha \succcurlyeq \beta$ we just need consider variables S . We have: $\alpha \succcurlyeq \beta \iff \alpha(S) \succcurlyeq_b^S \beta(S)$. An extreme case is given in the following definition:

In Example 1 (Figure 1), $\{X_1\}$ has overall importance given \top . Also, $\{X_2\}$ has overall importance given \bar{x}_1 . Relation $\succcurlyeq_{\bar{x}_1}^{\{X_2\}}$ is given by $x_2 \succcurlyeq_{\bar{x}_1}^{\{X_2\}} \bar{x}_2$. In Example 2, $\{X_2\}$ has overall importance given either x_1 or \bar{x}_1 . If instead we were to define $weight(\alpha) = 6\alpha(X_1) + 3\alpha(X_2) + 2\alpha(X_3)$, then $\{X_1\}$ would have overall importance given \top .

Definition 3 Let \succcurlyeq be a weak order on \underline{V} , let $b \in \underline{B}$ be an assignment to set of variables $B \subseteq V$. Let $U = V - B$. We say that U are all \succcurlyeq -equivalent given b if for all $u, u' \in \underline{U}$, $bu \succcurlyeq bu'$.

Note that if $V - B$ are all \succcurlyeq -equivalent given b then any non-empty subset S of $V - B$ has overall importance given b .

The following result gives some properties, relating to the local ordering \succcurlyeq_b^S , that we will use in proving Theorem 1.

Lemma 1 Let $b \in \underline{B}$ be an assignment to set of variables $B \subseteq V$, and let S be a non-empty subset of $V - B$. Let \succcurlyeq be a weak order on \underline{V} . Suppose that, for \succcurlyeq , S has overall importance given b . Then the following hold:

- (i) \succsim_b^S is a weak order on \underline{S} .
- (ii) Suppose α and β are outcomes extending b such that $\alpha(S) \neq \beta(S)$. Then $\alpha \succsim \beta \iff \alpha(S) \succsim_b^S \beta(S)$.
- (iii) Let $s \in \underline{S}$, and suppose that there exists some different \succsim_b^S -equivalent element $s' \in \underline{S}$, i.e., such that $s \succsim_b^S s'$ and $s' \succsim_b^S s$. Then $V - (B \cup S)$ are all \succsim -equivalent given bs .
- (iv) If \succsim_b^S is the full relation, i.e., it is such that for all $s, s' \in \underline{S}$, $s \succsim_b^S s'$, then $V - B$ are all \succsim -equivalent given b .

4 A CHARACTERISATION OF $\models_{\mathcal{Y}}$

Let \mathcal{Y} be a valid family of subsets of V . Theorem 1 below shows exactly which weak orders can be generated by a \mathcal{Y} -cp-tree.

We say that \succsim satisfies *overall importance* with respect to \mathcal{Y} if for all proper subsets A of V , and all assignments $a \in \underline{A}$, there exists some $Y \in \mathcal{Y}$ which, for \succsim , has overall importance given a .

Theorem 1 (representation of cp-tree orders) *Let \mathcal{Y} be a valid family of subsets of V , and let \succsim be a weak order on \underline{V} . There exists a \mathcal{Y} -cp-tree σ with $\succsim_\sigma = \succsim$ if and only if \succsim satisfies overall importance w.r.t. \mathcal{Y} .*

Theorem 1 immediately implies a characterisation of \mathcal{Y} -entailment for preference inference:

Corollary 1 *Let Γ be a set of comparative preference statements, and let α and β be outcomes. Then $\Gamma \models_{\mathcal{Y}} \alpha \geq \beta$ if and only if every weak order satisfying Γ and overall importance w.r.t. \mathcal{Y} also satisfies $\alpha \geq \beta$.*

This result gives a simpler way of defining the polynomial plausible inference $\models_{\mathcal{Y}}$: it's the inference one obtains by restricting the set of models to weak orders satisfying overall importance w.r.t. \mathcal{Y} . For example, if \mathcal{Y} is just $\mathcal{Y}(1)$, the set of singleton subsets of V , then we are assuming that the user's unknown preference ordering is such that, given any partial tuple a , there exists a variable X which has overall importance.

5 1-cp-TREE INFERENCE VIA PROPERTIES OF IMPORTANCE

In this section we take another approach to the characterisation of cp-tree orderings, specifically for 1-cp-trees (where there is a single variable associated with each node); this then characterises the $\models_{\mathcal{Y}(1)}$ preference entailment. It was pointed out in Section 3.1 that Importance, in general, fails to satisfy some apparently natural properties. The approach we take is, in the semantics, to restrict models to being weak orders whose importance relation satisfies certain nice properties (particular right union, transitivity and completeness properties)—see Section 5.1. We show, in Section 5.2, that the obtained set of weak orders is precisely the set of weak orders generated by 1-cp-trees, thus giving another characterisation of the $\models_{\mathcal{Y}(1)}$ preference inference.

5.1 Additional Conditions On Importance

We say that \succsim -Importance satisfies **Strong Right Union** if for any subsets S, T_1 and T_2 of V with S disjoint from $T_1 \cup T_2$, and for any $C \subseteq V$ and partial assignment $c \in \underline{C}$,

if \succsim satisfies $[c] : S \supseteq T_1$ and $[c] : S \supseteq T_2$
then \succsim satisfies $[c] : S \supseteq T_1 \cup T_2$.

We say that \succsim -**Importance on V is transitive** if the following implication holds for every outcome $\alpha \in \underline{V}$, and for different variables $X_1, X_2, X_3 \in V$:

If \succsim satisfies $[\alpha] : \{X_1\} \supseteq \{X_2\}$ and $[\alpha] : \{X_2\} \supseteq \{X_3\}$
then it satisfies $[\alpha] : \{X_1\} \supseteq \{X_3\}$.

We say that \succsim -**Importance on V is complete** if for all $\alpha \in \underline{V}$, and all different $X_1, X_2 \in V$,

\succsim satisfies either $[\alpha] : \{X_1\} \supseteq \{X_2\}$ or $[\alpha] : \{X_2\} \supseteq \{X_1\}$.

The following result shows that the above three conditions on \succsim -Importance are sufficient to imply that, given any partial assignment, there always exists a variable with overall importance.

Lemma 2 *Let \succsim be a weak order on the set of outcomes \underline{V} such that \succsim -Importance satisfies Strong Right Union and \succsim -Importance on V is transitive and complete. Then, \succsim satisfies overall importance w.r.t. the set $\mathcal{Y}(1)$ of singleton subsets of V .*

5.2 Representation Results for 1-cp-trees

The following result shows that a weak order equals a 1-cp-tree ordering if and only if \succsim -Importance satisfies the three conditions from Section 5.1.

Theorem 2 *Let \succsim be a weak order on the set of outcomes \underline{V} . Then there exists a 1-cp-tree σ with $\succsim_\sigma = \succsim$ if and only if \succsim -Importance satisfies Strong Right Union and \succsim -Importance on V is transitive and complete.*

Similarly to Theorem 1, this gives a characterisation of the preference inference $\models_{\mathcal{Y}}$, in this case for when $\mathcal{Y} = \mathcal{Y}(1)$, the set of singleton subsets of V .

Corollary 2 *Let Γ be a set of comparative preference statements, and let α and β be outcomes. Then $\Gamma \models_{\mathcal{Y}(1)} \alpha \geq \beta$ if and only if $\alpha \succsim \beta$ for every weak order \succsim satisfying Γ and such that \succsim -Importance satisfies Strong Right Union and \succsim -Importance on V is transitive and complete.*

6 cp-TREES AND OVERALL IMPORTANCE

This section first defines cp-trees and their associated weak orders (Section 6.1), and then, in Section 6.2 gives properties that will enable us to prove Theorem 1 in Section 6.3.

6.1 cp-trees and their Weak Orders

A cp-tree [16] (over the set of variables V) is a rooted directed tree, which we picture being drawn with the root at the top, and children below parents (see Figure 1). Associated with each node r in the tree is a set of variables Y_r , which is instantiated with a different assignment in each of the node's children (if it has any), and also a weak order \succsim_r of the values of Y_r .

More formally, define a **cp-node** r (usually abbreviated to just “node”) to be a tuple $\langle A_r, a_r, Y_r, \succsim_r \rangle$, where $A_r \subseteq V$ is a set of variables, $a_r \in \underline{A_r}$ is an assignment to those variables, and $Y_r \subseteq V - A_r$ is a non-empty set of other variables; \succsim_r is a weak order on the set Y_r of values of Y_r which is not equal to the trivial full relation on Y , i.e., there exists some $y, y' \in \underline{Y}$ with $y \not\succsim_r y'$.

For example, in the cp-tree σ in Figure 1, the leftmost node r has $A_r = \{X_1\}$, which is the set of variables assigned above the node, with assignment $a_r = x_1$; also, $Y_r = \{X_2, X_3\}$, and the node weak order \succsim_r is given by $x_2x_3 \succsim_r x_2\bar{x}_3 \equiv_r \bar{x}_2x_3 \succsim_r \bar{x}_2\bar{x}_3$.

A **cp-tree** is defined to be a directed tree, where edges are directed away from a root node, root, so that all nodes apart from the root node have a unique parent node. The *ancestors* of a node r are the nodes on the path from root to the parent node of r . Each node is identified with a unique cp-node a_r . Let $r \rightarrow r'$ be an edge in the cp-tree from a node r to one of its children r' . Associated with this edge is an assignment y to variables Y_r . This is different from the assignment y' associated with any other edges from node r . $A_{r'} = A_r \cup Y_r$, and $a_{r'}$ is a_r extended with the assignment $Y_r = y$. We also have $A_{\text{root}} = \emptyset$. Therefore A_r is the union of sets $Y_{r''}$ over all ancestors r'' of r ; and a_r consists of all assignments made on the path from the root to r . The root node has $a_{\text{root}} = \top$, the assignment to the empty set of variables.

It is also assumed that \geq_r satisfies the following condition, for each node r in the cp-tree (to ensure that the associated ordering on outcomes is transitive): if there exists a child of node r associated with instantiation $Y_r = y$, then y is not \geq_r -equivalent to any other value of Y , so that $y \geq_r y' \geq_r y$ only if $y' = y$.

The weak order \geq_σ associated with a cp-tree σ

For outcome α , define the *path to α* to consist of all nodes r such that α extends a_r . To generate this, for each node r , starting from the root, we choose the child associated with the instantiation $Y_r = \alpha(Y_r)$ (there is at most one such child); the path finishes when there exists no such child. Node r is said to *decide outcomes α and β* if it is the deepest node (i.e., furthest from the root) which is both on the path to α and on the path to β . Hence α and β both extend the tuple a_r (but they may differ on variable Y_r). We compare α and β by using \geq_r , where r is the unique node that decides α and β .

Definition 4 Let σ be a cp-tree. The associated relation \geq_σ on outcomes is defined as follows: for outcomes $\alpha, \beta \in \underline{V}$, we define $\alpha \geq_\sigma \beta$ to hold if and only if $\alpha(Y_r) \geq_r \beta(Y_r)$, where r is the node that decides α and β .

This ordering is similar to a lexicographic ordering in that two outcomes are compared on the first variable on which they differ.

For comparative preference statement φ and set of comparative statements Γ , we say that σ *satisfies* φ (respectively, Γ) if and only if \geq_σ satisfies φ (respectively, Γ).

The empty cp-tree: For technical reasons we allow the empty set (of nodes) to be a cp-tree. Its associated ordering is thus the full relation on outcomes, with $\alpha \geq \beta$ for all outcomes α and β .

The following basic property is used in the proof of Theorem 1.

Lemma 3 Let σ be a cp-tree, let $\alpha, \beta \in \underline{V}$ be outcomes, and let r be the node of σ that decides α and β . If $\alpha(Y_r) = \beta(Y_r)$ then $\alpha \equiv_\sigma \beta$, i.e., $\alpha \geq_\sigma \beta$ and $\beta \geq_\sigma \alpha$. Also, there exists no node r' of σ with $a_{r'}$ extending $a_r \alpha(Y_r)$.

6.2 cp-trees and Importance

Proposition 3 below implies half of Theorem 1, i.e., that \geq_σ satisfies overall importance w.r.t. \mathcal{Y} , for any \mathcal{Y} -cp-tree σ . The idea behind the proof is as follows. If $\underline{V} - B$ are all \geq_σ -equivalent given b then set Y to be $\{X\}$ for some $X \in \underline{V} - B$. Else, we iteratively pick edges with assignments compatible with b until we reach a node r with $Y_r \not\subseteq B$; we then set $Y = Y_r - B$. In either case, $Y \in \mathcal{Y}$ and Y has overall importance given b .

Proposition 3 Consider a \mathcal{Y} -cp-tree σ (for valid family \mathcal{Y}). Then, for all proper subsets B of V , and all assignments $b \in \underline{B}$ there exists some $Y \in \mathcal{Y}$ which, for \geq_σ , has overall importance given b .

6.3 Proof of Theorem 1

\Rightarrow : This is shown by Proposition 3.

\Leftarrow : Suppose that for all proper subsets A of V , and all assignments $a \in \underline{A}$, there exists some $Y \in \mathcal{Y}$ which, for \geq , has overall importance given a .

Consider any proper subset A of V , and any $a \in \underline{A}$ such that it is not the case that $\underline{V} - A$ are all \geq -equivalent given a . By our assumptions, there exists $Y \in \mathcal{Y}$ that has overall importance given a . (If there is more than one such Y , we choose Y in some canonical way, e.g., based on a total ordering of subsets.) Define $Y_a = Y$. We also define \geq_a on \underline{Y} to be the relation \geq_a^Y defined in Section 3.2. This is a weak order by Lemma 1(i). Also, \geq is not the trivial full relation, by Lemma 1(iv) and the fact that it is not the case that $\underline{V} - A$ are all \geq -equivalent given a . Let $r(a)$ be the tuple $\langle A, a, Y_a, \geq_a \rangle$. We have shown that $r(a)$ is a valid cp-node.

We will construct a \mathcal{Y} -cp-tree σ . If \geq is equal to $\underline{V} \times \underline{V}$ we let σ be the empty cp-tree. Otherwise, we define σ iteratively as follows. Let the root node be $r(\top)$. We continue iteratively: for each node $\langle A, a, Y, \geq \rangle$ we've defined, and each $y \in \underline{Y}$ such that it is not the case that $\underline{V} - (A \cup Y)$ are all \geq -equivalent given ay , we generate a child node $r(ay)$ (as defined above).

To show that this does indeed generate a cp-tree, we still need to show that if there exists a child of node r associated with instantiation $Y_r = y$, then y is not \geq_r -equivalent to any other value of Y . This follows from Lemma 1(iii), since it is not the case that $\underline{V} - (A \cup Y)$ are all \geq -equivalent given ay .

We shall show that \geq_σ equals \geq . Consider any $\alpha, \beta \in \underline{V}$, and let r be the node of σ that decides α and β . By construction of the node r , set Y_r has, for \geq , overall importance given a_r .

First suppose that $\alpha(Y_r) \neq \beta(Y_r)$. We have $\alpha \geq_\sigma \beta$ if and only if $\alpha(Y_r) \geq_r \beta(Y_r)$, which, by the definition of the nodes, is if and only if $\alpha(Y_r) \geq_{a_r}^Y \beta(Y_r)$. Using Lemma 1(ii), this is iff $\alpha \geq \beta$.

Now consider the case where $\alpha(Y_r) = \beta(Y_r)$. We will show that α and β are equivalent with respect to both \geq_σ and \geq . Lemma 3 implies that $\alpha \equiv_\sigma \beta$. Lemma 3 also implies that σ has no node r' with $a_{r'}$ extending $a_r \alpha(Y_r)$. The construction of σ then implies that $\underline{V} - (A_r \cup Y_r)$ are all \geq -equivalent given $a_r \alpha(Y_r)$. Since α and β agree on $A_r \cup Y_r$ we have that α and β are \geq -equivalent.

In either case we have $\alpha \geq_\sigma \beta \iff \alpha \geq \beta$, showing that $\geq_\sigma = \geq$, completing the proof. \square

7 BEFORE-STATEMENTS AND IMPORTANCE

For cp-trees, importance is related to the ordering of variables in the different branches. We formalise this with the notion of *before-statement*. We show (Proposition 6) that a 1-cp-tree σ satisfies a before-statement if and only if \geq_σ satisfies the corresponding importance statement. We use this to show that a cp-tree ordering satisfies the properties required for Theorem 2.

A *before-statement* (on variables V) is defined syntactically to be a statement of the form $b : S \rightsquigarrow T$, where b is an assignment to set of variables B , and sets B, S and T are mutually disjoint subsets of V . The interpretation is that every element of S appears before any element of T on any path compatible with b in the cp-tree. Formally, cp-tree σ satisfies $b : S \rightsquigarrow T$ if and only if for any node r with a_r compatible with b , $Y_r \cap T \neq \emptyset \Rightarrow A_r \supseteq S$.

In Example 1 (Figure 1), the cp-tree satisfies before-statements $\top : \{X_1\} \rightsquigarrow \{X_2, X_3\}$ and $\bar{x}_1 : \{X_2\} \rightsquigarrow \{X_3\}$. The latter holds because on any path compatible with \bar{x}_1 , X_2 is instantiated before X_3 .

Let $c \in \underline{C}$ be an assignment to some arbitrary subset C of V . Analogously with importance statements, we write $[c] : S \rightarrow T$ as an abbreviation for $c(C - (S \cup T)) : S \rightarrow T$.

The following two propositions give properties of the before-statements satisfied by a cp-tree (1-cp-trees for Proposition 5).

Proposition 4 *Let σ be a cp-tree. If σ satisfies the before-statements $[c] : S \rightarrow T_1$ and $[c] : S \rightarrow T_2$ then σ satisfies $[c] : S \rightarrow T_1 \cup T_2$.*

Proposition 5 *Let σ be a 1-cp-tree and let $\alpha \in \underline{V}$ be any outcome. Then σ satisfies the following properties.*

- (i) *For different $X_1, X_2, X_3 \in V$, if σ satisfies $[\alpha] : \{X_1\} \rightarrow \{X_2\}$ and $[\alpha] : \{X_2\} \rightarrow \{X_3\}$ then σ satisfies $[\alpha] : \{X_1\} \rightarrow \{X_3\}$.*
- (ii) *For all different $X_1, X_2 \in V$, σ satisfies either $[\alpha] : X_1 \rightarrow X_2$ or $[\alpha] : X_2 \rightarrow X_1$.*

A before-statement for cp-trees is at least as strong as its corresponding importance statement, and for 1-cp-trees they are equivalent:

Proposition 6 *Let σ be a cp-tree. If σ satisfies before-statement $b : S \rightarrow T$ then \succ_σ satisfies $b : S \sqsupseteq T$. If σ is a 1-cp-tree, then the converse also holds: σ satisfies $b : S \rightarrow T$ if and only if \succ_σ satisfies $b : S \sqsupseteq T$.*

Putting together Proposition 6 and Propositions 4 and 5 we obtain the following, which proves half of Theorem 2.

Proposition 7 *Let σ be a 1-cp-tree, with \succ_σ its associated weak ordering on outcomes. Then \succ_σ -Importance satisfies Strong Right Union and \succ_σ -Importance on V is transitive and complete.*

Proof of Theorem 2: Suppose that there exists a 1-cp-tree σ with $\succ_\sigma = \succ$. Proposition 7 shows that \succ -Importance satisfies Strong Right Union and \succ -Importance on V is transitive and complete.

Conversely, suppose \succ -Importance satisfies Strong Right Union and \succ -Importance on V is transitive and complete. By Lemma 2, \succ satisfies overall importance w.r.t. the set $\mathcal{Y}(1)$ of singleton subsets of V . By Theorem 1, there exists a 1-cp-tree σ with $\succ_\sigma = \succ$. \square

8 SUMMARY AND DISCUSSION

As mentioned in Section 4, the corollary of Theorem 1 implies that we can define \mathcal{Y} -entailment $\models_{\mathcal{Y}}$ in a simpler way: by including in the set of models only the weak orders that satisfy overall importance w.r.t. \mathcal{Y} . The importance-based semantics and the graphical cp-tree semantics complement each other. An apparent weakness of the cp-tree semantics is that the formal definition can seem a little complicated, and the extra condition on the local ordering, ensuring transitivity, sounds perhaps somewhat arbitrary. The importance-based semantics shows that it isn't really arbitrary, and gives an in some ways simpler definition of the preference inference relation.

The corollary of Theorem 2 gives a further characterisation of the $\models_{\mathcal{Y}(1)}$ preference inference relation, showing that if we limit the set of models to only include weak orders whose importance relation satisfies some nice (but strong) properties, then we obtain the $\models_{\mathcal{Y}(1)}$ relation. Such properties could be useful for explaining, to the user, why the system is inferring a preference of one outcome to another. More generally, if the user is unhappy with a $\models_{\mathcal{Y}}$ inference that α is preferred to β , then the set \mathcal{Y} might be automatically increased to remove this inferred preference.

Section 3 described some general properties of the importance relation; it would be interesting to study the general properties of importance further. Proposition 6, identifying connections between before-statements for cp-trees and importance statements, could be a valuable tool for this, since it implies that we only need consider properties that hold of before-statements for 1-cp-trees (which are easier to check). Another potential research direction would be to extend Theorem 2 for the case of other families \mathcal{Y} , in particular, for \mathcal{Y} consisting of all sets of cardinality at most 2.

ACKNOWLEDGEMENTS

This material is based upon works supported by the Science Foundation Ireland under Grant No. 08/PI/I1912.

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