The inverse limit of GIT quotients of Grassmannians by the maximal torus

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I, Vahid Yazdanpanah, certify that this thesis is my own work and I have not obtained a degree in this university or elsewhere on the basis of the work submitted in this thesis.

Vahid Yazdanpanah

Aknowledgement

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Abstract

There are finitely many GIT quotients of $G(3, n)$ by maximal torus and between each two there is a birational map. These GIT quotients and the flips between them form an inverse system. This thesis describes this inverse system first and then, describes the inverse limit of this inverse system as a moduli space.

An open set in this scheme represents the functor of arrangements of lines in planes. We show how to enrich this functor such that it is represented by the above inverse limit.

Chapter 1

Introduction

In the study of moduli spaces in algebraic geometry, geometric invariant theory (GIT) is an important tool. For various group actions, the investigation of the invariants was one of the most active areas in mathematics in the nineteenth century. Finding explicit invariants is not an easy task. This classical period of invariant theory ended with the fundamental result of Hilbert which says that the ring $Sym(V^*)^{SL(V)}$ of invariant polynomials for a linear representation V is finitely generated.

The subject of invariant theory was rediscovered by Mumford in the sixties. His work [6] showed how to use GIT without explicitly knowing the invariants. It was well-known, but essentially ignored that for a variety X and a group G acting on it, the construction of a quotient $X/\!\!/_{\mathcal{L}}G$, depends on the choice of a G-linearized ample line bundle $\mathcal{L}$ on X. A renewal of GIT occurred in the early nineties, when Thaddeus [9] and independently Dolgachev and Hu [3] analysed this dependence and showed that it is well-behaved.

This creates a family of GIT quotients connected through flips. From this we can create an inverse system of birational morphisms. We would like to describe the inverse limit as a fine moduli space. Therefore we are interested in the universal family over it.

Alexeev in [1] introduced the notion of stable toric varieties which are constructed from gluing toric varieties along equivariant Weil divisors. He also constructed a moduli space of $(\mathbb{C}^*)^n$–equivariant morphisms from stable toric varieties into a variety Y. Tevelev, Keel and Hacking in [4] constructed a moduli space of hyperplane arrangements using Alexeev's moduli space in the case when a maximal torus is acting on Y.

In this thesis we introduce a more detailed construction of the moduli space of line arrangements in rational surfaces and a detailed description of the fibers of the universal family based on the family of GIT quotients of $G(3, n)$. Our method of constructing the moduli space differs from Tevelev, Keel and Hacking's as we don't use Alexeev's results. Instead we use a detailed analysis of the birational transformations between GIT quotients.

All schemes and morphisms in this work are over $\mathbb{C}$ and also by a point on a scheme the author means a closed point. This paper is organized as follows. Chapter 2 recalls the definition of Grassmannian varieties and the natural torus action on them. Chapter 3 summarizes the definitions of different quotients of a variety and reviews a few properties of GIT quotients. Chapter 4 introduces a moment map and describes its image in terms of chambers and walls. This is used to describe a flip of two GIT quotients as a moduli problem in Chapter 5. Chapter 6 focuses on building up the universal family over the inverse limit of GIT quotient of Grassmannian variety by maximal torus.

Chapter 2

Grassmannian Varieties

Grassmannian varieties are fundamental objects in mathematics especially in algebraic geometry. In this chapter the Grassmanninan varieties are defined as projective varieties and in Chapter 5 as a moduli space.

2.1 Grassmannian Varieties

We let $G(m, n)$ denote the set of (m)-dimensional linear subspaces of the vector space $\mathbb{C}^n$ which is the same as the set of $(m-1)$–planes in the corresponding projective space $\mathbb{P}^{n-1}$.

We describe the Grassmannian variety first as a subset of projective space. If $W \subset \mathbb{C}^n$ is the m-dimensional linear subspace spanned by vectors $e_1, ..., e_m$, we can associate to W the multivector

$$w = e_1 \wedge ... \wedge e_m \in \bigwedge^m \mathbb{C}^n,$$

where w is determined up to scalars by W. If we chose a different basis, the corresponding vector w would simply be multiplied by the determinant of the matrix of the change of basis. So we have a well-defined map of sets

$$\Psi : G(m, n) \to \mathbb{P}(\bigwedge^m \mathbb{C}^n).$$

In fact, this is an inclusion. For any $[w] = \Psi(W)$ in the image, we can recover the corresponding subspace W as the space of vectors v in $\mathbb{C}^n$ such that $v \wedge w = 0 \in \wedge^{m+1}\mathbb{C}^n$. This inclusion is called the Plücker embedding of $G(m, n)$.

The homogeneous coordinates on $\mathbb{P}(\wedge^m \mathbb{C}^n)$ are called Plücker coordinates. We can represent the hyperplane W by the $m \times n$ matrix Λ_W whose rows are the vectors $e_1, ..., e_m$; the matrix Λ_W, is determined up to multiplication on the left by an invertible $m \times m$ matrix i.e. an element in $GL(m)$. The Plücker coordinates are then just the maximal minors of the matrix Λ_W.

We have described $G(m,n)$ as a subset of $\mathbb{P}(\bigwedge^m \mathbb{C}^n)$; we should now check that it is indeed a subvariety. This amounts to characterising the subset of *totally decomposable* vectors $w \in \wedge^m \mathbb{C}^n$, that is, as a product $w = v_1 \wedge ... \wedge v_m$ of linear factors. We begin with a basic observation: given a multivector $w \in \wedge^m \mathbb{C}^n$ and a vector $v \in \mathbb{C}^n$, the vector v will divide w – that is, w will be expressible as $v \wedge u$ for some $u \in \wedge^{m-1}\mathbb{C}^n$ – if and only if the wedge product $w \wedge v = 0$. Moreover, a multivector w will be totally decomposable if and only if the space of vectors v, dividing it, is m-dimensional. Thus $[w]$ will lie in the Grassmannian if and only if the rank of the map

$$\begin{aligned} \Phi(w) : \mathbb{C}^n &\rightarrow \wedge^{m+1}\mathbb{C}^n \\ v &\mapsto w \wedge v \end{aligned}$$

is $n - m$. Since the rank of $\Phi(w)$ is never strictly less than $n - m$, we can say

$$[w] \in G(m,n) \Leftrightarrow \text{rank}(\Phi(w)) \leq n - m.$$

Now, the map $\wedge^m \mathbb{C}^n \rightarrow \text{Hom}(\mathbb{C}^n, \wedge^{m+1}\mathbb{C}^n)$ sending w to $\Phi(w)$ is linear, that is, the entries of the matrix $\Phi(w) \in \text{Hom}(\mathbb{C}^n, \wedge^{m+1}\mathbb{C}^n)$ are homogeneous coordinates on $\mathbb{P}(\wedge^{m+1}\mathbb{C}^n)$. W can say that $G(m,n) \subset \mathbb{P}(\wedge^m \mathbb{C}^n)$ is the subvariety defined by the vanishing of the $(n - m + 1) \times (n - m + 1)$ minors of this matrix.

This is the simplest way to see that $G(m,n)$ is a subvariety of $\mathbb{P}(\wedge^m \mathbb{C}^n)$, but the polynomials we get in this way are far from the simplest possible. To find the actual generators of the ideal, we need to invoke also the natural identification of $\wedge^m \mathbb{C}^n$ with the exterior power $\wedge^{n-m}(\mathbb{C}^n)^*$ of the dual space $(\mathbb{C}^n)^*$ (this is natural only up to scalars, but this is acceptable for our purpose). In particular, an element $w \in \wedge^m \mathbb{C}^n$ corresponding to $w^* \in \wedge^{n-m}(\mathbb{C}^n)^*$ gives rise in this way to a map

$$\begin{aligned} \Psi(w) : (\mathbb{C}^n)^* &\rightarrow \wedge^{n-m+1}(\mathbb{C}^n)^* \\ v^* &\mapsto v^* \wedge w^*. \end{aligned}$$

By the same argument w will be totally decomposable if and only if the map $\Psi(w)$ has rank at most m. In case w is totally decomposable, the kernel of the map $\Phi(w)$;

equivalently, the images of the transposed maps

$$\Phi(w)^t : \wedge^{m+1}(\mathbb{C}^n)^* \to (\mathbb{C}^n)^*$$

and

$$\Psi(w)^t : \wedge^{n-m+1}\mathbb{C}^n \to \mathbb{C}^n$$

annihilate each other. In sum, then, we see that $[w] \in G(m,n)$ if and only if for every pair $\alpha \in \wedge^{m+1}(\mathbb{C}^n)^*$ and $\beta \in \wedge^{n-m+1}(\mathbb{C}^n)$, the contraction

$$\Xi_{\alpha,\beta}(w) = \langle \Phi(w)^t(\alpha), \Psi(w)^t(\beta) \rangle = 0 \tag{2.1}$$

The $\Xi_{\alpha,\beta}$ are thus quadratics whose common zero locus is the Grassmannian $G(m,n)$. The equations from 2.1 are called the Plücker relations, and they do in fact generate the homogeneous ideal of $G(m,n)$, though we will not prove that here.

2.2 The Natural Torus Action on Grassmannian Varieties

The natural action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^{n-1}$ keeps the coordinate hyperplanes invariant. This action induces an action on $G(m,n)$ as below. For an arbitrary $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ and an arbitrary element

$$[Q] = \left[\begin{pmatrix} r_{1,1} & r_{1,2} & \ldots & r_{1,n} \\ r_{2,1} & r_{2,2} & \ldots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \ldots & r_{m,n} \end{pmatrix}\right] \in G(m,n),$$

we define the action as

$$t.[Q] = \left[\begin{pmatrix} t_1r_{1,1} & t_2r_{1,2} & \ldots & t_nr_{1,n} \\ t_1r_{2,1} & t_2r_{2,2} & \ldots & t_nr_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ t_1r_{m,1} & t_2r_{m,2} & \ldots & t_nr_{m,n} \end{pmatrix}\right].$$

Note that in the rest of the thesis we consider the torus $(\mathbb{C}^*)^{n-1}$ as the quotient of $(\mathbb{C}^*)^n$ by $\mathbb{C}^*$, the diagonal in $(\mathbb{C}^*)^n$.

Chapter 3

Geometric Invariant Theory

For k a positive integer, the torus $(\mathbb{C}^*)^k$ is the group whose action is of interest to us. Denote orbits by $O(x)$ or $(\mathbb{C}^*)^k.x$ for a point x and their closures by $\overline{O(x)}$ or $\overline{(\mathbb{C}^*)^k.x}$, and also denote stabilizers by $Stab(x)$. In the case $k = 1$, for a $\mathbb{C}^*$–action on a variety X and a point $x \in X$, we say that the orbit $\mathbb{C}^*.x$ goes into a fixed locus Y if $\lim_{t\to\infty} t.x = y \in Y$, and it goes out of a fixed locus Y' if $\lim_{t\to 0} t.x = y' \in Y'$. This section aims to give a brief overview of the standard GIT as developed by Mumford. We also represent a few required notations from first chapter of [6].

3.1 Affine Quotients.

When $(\mathbb{C}^*)^n$ acts on an affine variety $X = SpecR$ a fundamental result from Hilbert says that the ring of invariants $R^{(\mathbb{C}^*)^n} \subset R$ is a finitely generated $\mathbb{C}$-algebra. Thus, it is natural to define the quotient to be an affine variety:

$$X/(\mathbb{C}^*)^n := SpecR^{(\mathbb{C}^*)^n}. \tag{3.1}$$

In fact Hilbert stated this in general for an arbitrary reductive group.

3.2 Categorical and Geometric Quotient

The $(\mathbb{C}^*)^n$-invariant projection $\pi : X \to Y$ is a categorical quotient if any $(\mathbb{C}^*)^n$-invariant morphism $f : X \to Z$ factors through Y:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & \searrow^{f} & \downarrow \exists ! \bar{f} \\ & & Z. \end{array}$$

A categorical quotient π is a universal categorical quotient if it is stable under base change: for any $Y' \to Y$, $\pi' : X' = X \times_Y Y' \to Y'$ is a categorical quotient.

A categorical quotient $\pi : X \to Y$ is called a geometric quotient if it satisfies the following properties:

- π is surjective and the image of the morphism

$$\begin{aligned} \psi : (\mathbb{C}^*)^n \times X &\to X \times X \\ (t, x) &\mapsto (t.x, x), \end{aligned}$$

  equals $X \times_Y X$ (equivalently, the geometric fibers of π are precisely the orbits of the points of X).

- for any subset $U \subset Y$, the inverse image $\pi^{-1}(U)$ is open if and only if U is open.

- for any open subset U of Y, the natural homomorphism $\pi^* : \mathcal{O}(U) \to \mathcal{O}(\pi^{-1}(U))$ is an isomorphism onto the subring $\mathcal{O}(\pi^{-1}(U))^{(\mathbb{C}^*)^n}$ of $(\mathbb{C}^*)^n$–invariant functions.

However the categorical quotient $X/(\mathbb{C}^*)^n$ is typically not a geometric quotient. In general, there is no one-to-one correspondence between the points of $X/(\mathbb{C}^*)^n$ and the orbits of $(\mathbb{C}^*)^n$. As a simple example, consider the action of $\mathbb{C}^*$ on $X = \mathbb{A}^1_{\mathbb{C}} \cong \mathbb{C}$ given by the natural multiplication $(t, x) \in \mathbb{C}^* \times \mathbb{C} \to t.x \in \mathbb{C}$. Since $\mathbb{C}[x]^{\mathbb{C}^*} \cong \mathbb{C}$ (the only invariants are the constants), the quotient $X/\mathbb{C}^*$ is a point and the quotient map π is the trivial projection $\mathbb{C} \to \{pt\}$. On the other hand, the action of $\mathbb{C}^*$ on X has two orbits: $\{0\} = \mathbb{C}^*.0$ and $\mathbb{C} \setminus \{0\} = \mathbb{C}^*.1$. The issue here is that the fibers of π are always closed in X. In our example, the orbit 0 is closed, while the closure of the orbit $\mathbb{C} \setminus \{0\}$ is the affine line, which contains the closed orbit $\{0\}$. Thus, the two orbits map to the same point via π, showing that the orbits are not always separated by the quotient map.

3.3 Closed Orbits

In some sense, the failure of the categorical quotient $X/(\mathbb{C}^*)^n$ to be a geometric quotient is always of the type exemplified above. Namely, we recall the following fact about the orbits of torus actions: each $(\mathbb{C}^*)^n$-orbit is smooth, locally closed in X, and its boundary is a union of orbits of strictly lower dimension. This easily gives that each fiber of π contains a unique closed orbit, namely, the orbit of minimal dimension in that fiber. Furthermore, if $(\mathbb{C}^*)^n.x_0$ is a closed orbit in X, then for all $x \in \pi^{-1}(\pi(x_0))$ we have $(\mathbb{C}^*)^n.x_0 \subset \overline{(\mathbb{C}^*)^n.x}$.

3.4 Projective Quotients

Mumford constructed a GIT quotient considering an ample line-bundle together with a $(\mathbb{C}^*)^n$–linearisation (i.e. essentially a lift of the $(\mathbb{C}^*)^n$-action from X to $\mathcal{L}$). This choice gives an embedding

$$i : X \longrightarrow \mathbb{P}^N \text{ for some N } \gg 0$$

such that $(\mathbb{C}^*)^n$ acts linearly on $\mathbb{P}^N$ and the embedding i is $(\mathbb{C}^*)^n$-equivariant. By considering affine cones, one can reduce the case to the affine situation. Concretely the definition of a GIT quotient in the projective case is as follows.

Definition 3.4.1 *Let $(\mathbb{C}^*)^n$ be a torus acting on a projective variety X. For $\mathcal{L}$ an ample $(\mathbb{C}^*)^n$-linearized line bundle, the associated GIT quotient is the projective variety:*

$$X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n := Proj \oplus_{i\geq 0} H^0(X, \mathcal{L}^{\otimes i})^{(\mathbb{C}^*)^n}.$$

3.5 Semistable Points

The linear systems $H^0(X, \mathcal{L}^{\otimes m})^{(\mathbb{C}^*)^n}$ for large enough m defines a rational map $\pi : X \to X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n$ the domain of definition of π is precisely the set of points $x \in X$ such that there exists a $(\mathbb{C}^*)^n$-invariant section $\sigma \in H^0(X, \mathcal{L}^{\otimes m})^{(\mathbb{C}^*)^n}$ (for some m) with $\sigma(x) \neq 0$. We call such points semistable and denote by $X^{ss}(\mathcal{L}) \subset X$ the corresponding open set. The points in $X^{us}(\mathcal{L}) = X \setminus X^{ss}(\mathcal{L})$ are called unstable and are excluded from the GIT analysis.

3.6 Stable Points

In moduli theory, one is particularly interested in geometric quotients. It follows that for points $x \in X^{ss}(\mathcal{L})$ such that the orbit $(\mathbb{C}^*)^n.x$ is closed in X^{ss} and of maximal dimension (i.e. $\dim((\mathbb{C}^*)^n.x) = \dim(\mathbb{C}^*)^n$, or equivalently the stabilizer $(\mathbb{C}^*)^n_x$ is finite) one has $\pi^{-1}(\pi(x)) = (\mathbb{C}^*)^n.x$ where π is the natural quotient map above (see Theorem 1.10 [6]). Such points are called stable points and form $X^s(\mathcal{L}) \subset X^{ss}(\mathcal{L})$ an open $(\mathbb{C}^*)^n$-invariant subset such that the induced quotient $X^s(\mathcal{L})/(\mathbb{C}^*)^n$ is both a geometric and a categorical quotient. If $X^s(\mathcal{L}) \neq \varnothing$, then $X^s(\mathcal{L})/(\mathbb{C}^*)^n$ is an open dense subset of $X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n$.

3.7 Some Facts About Standard GIT

For a torus $(\mathbb{C}^*)^n$ acting on a projective variety X and $\mathcal{L}$ an ample $(\mathbb{C}^*)^n$–linearized line bundle on X:

- $X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n$ is a projective variety.
- Each fiber of the quotient map $\pi : X^{ss} \to X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n$ contains a unique closed orbits in X^{ss}. Furthermore, $\pi(x) = \pi(y)$ iff $\overline{(\mathbb{C}^*)^n.x} \cap \overline{(\mathbb{C}^*)^n.y} \cap X^{ss} \neq \varnothing$.
- $X^s(\mathcal{L}) \to X^s(\mathcal{L})/(\mathbb{C}^*)^n$ is a geometric quotient; it is an open, non-empty and dense subset of $X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n$. In particular, $Stab(x)$ is finite and $\pi^{-1}(\pi(x)) = (\mathbb{C}^*)^n.x$ for $x \in X^s(\mathcal{L})$.
- In cases coming from moduli problems, the most desirable case is

  $$X^s(\mathcal{L}) = X^{ss}(\mathcal{L}).$$

  Namely, one gets that the quotient $X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n = X^s(\mathcal{L})/(\mathbb{C}^*)^n$ is both a projective variety and a geometric quotient. Unfortunately, the natural GIT set-up for many moduli problems gives situations with $X^s \subsetneq X^{ss}$. Even in those situations the fact that $X/\!\!/_{\mathcal{L}}(\mathbb{C}^*)^n$ is projective can be used to one's advantage.

Chapter 4

The Moment Map for a Torus Action

This chapter provides a detailed description of the moment map for the action of a torus on a projective variety. We describe the image of such a moment map as a convex polytope divided into chambers, and separated by walls. We define the one-parameter subgroup associated to a wall and give a concrete description of the walls and chambers in the case of maximal torus acting on the Grassmannian variety. This description is needed for the next chapter which is dedicated to the description of flips as a moduli problem.

4.1 Moment Map

Suppose for $t = (t_1, \ldots, t_n) \in (\mathbb{C}^*)^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$,

$$t.x = (t_1 x_1, \ldots, t_n x_n).$$

This action induces an action of $(\mathbb{C}^*)^n/\Delta$ (where here Δ is the diagonal) on $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$.

We define the moment map $\mu_{(\mathbb{C}^*)^n}$ on $\mathbb{P}^{n-1}$ as below:

$$\begin{aligned} \mu_{(\mathbb{C}^*)^n} : \mathbb{P}^{n-1} &\rightarrow \mathbb{R}^n \\ x &\mapsto \left(\frac{|x_1|^2}{\sum\limits_{i=1}^{n} |x_i|^2}, \ldots, \frac{|x_n|^2}{\sum\limits_{i=1}^{n} |x_i|^2} \right). \end{aligned}$$

Note that the image of this map is the $(n-1)$-dimensional simplex

$$\Delta^{n-1} := \{(r_1, ..., r_n) \in \mathbb{R}^n; r_i \geq 0 \text{ for all } i \in \{1, ..., n\} \text{ and } \sum_{i=1}^{n} r_i = 1\},$$

and the fiber over $\mu_{(\mathbb{C}^*)^n}(x)$ is the orbit of x under the natural action of $(S^1)^n$. Hence we can think of Δ^{n-1} as the topological quotient of $\mathbb{P}^{n-1}$ by $(S^1)^n$.

Also note

$$\mu_{(\mathbb{C}^*)^n}(t.x) = \left(\frac{|\ t_1\ |^2\ |x_1|^2}{\sum_{i=1}^{n} |\ t_i\ |^2\ |x_i|^2}, \ldots, \frac{|\ t_n\ |^2\ |x_n|^2}{\sum_{i=1}^{n} |\ t_i\ |^2\ |x_i|^2} \right),$$

so that the orbit of a generic point $x \in \mathbb{P}^{n-1}$ (on which $(\mathbb{C}^*)^n/\Delta \cong (\mathbb{C}^*)^{n-1}$) is mapped surjectively to the interior of the simplex Δ^{n-1}, while the orbits of the points with stabilizers are mapped to the faces of Δ^{n-1}.

Let $\{v_1, ..., v_n\}$ denote the standard basis in $\mathbb{R}^n$. Consider a $\mathbb{C}$–basis $\{e_1, ..., e_n\} \subset Re(\mathcal{T}_{1,(\mathbb{C}^*)^n})$ for $\mathcal{T}_{1,(\mathbb{C}^*)^n} \cong \mathbb{C}^n$ such that for $i = 1, ..., n$

$$d\mu_{(\mathbb{C}^*)^n}(e_i) = v_i,$$

where here $\mathcal{T}_{1,(\mathbb{C}^*)^n}$ is the tangent space of $\mathbb{C}^*)^n$ at 1.

Lemma 4.1.1 *Consider an algebraic morphism of groups $\lambda : (\mathbb{C}^*)^k \to (\mathbb{C}^*)^n$. Then for a suitable choice of coordinates on $(\mathbb{C}^*)^k$ and $(\mathbb{C}^*)^n$, the morphism λ can be written as*

$$\lambda(t_1, \ldots, t_k) = (\prod_{j=1}^{k} t_j^{a_j^1}, \ldots, \prod_{j=1}^{k} t_j^{a_j^n}),$$

where $A = (a_j^i)_{j\in\{1,...,k\},i\in\{1,...,n\}}$ is a $k \times n$– matrix with integer entries and such that $A \cdot A^t = I$.

This Lemma is proved in page 43 in [8].

Definition 4.1.1 *Let $i : X \hookrightarrow \mathbb{P}^{n-1}$ and $(\mathbb{C}^*)^k$ a subgroup of $(\mathbb{C}^*)^n$. The $(\mathbb{C}^*)^n$–action on $\mathbb{P}^{n-1}$ induces an action of $(\mathbb{C}^*)^k$ on $\mathbb{P}^{n-1}$. We assume that this restricts to an action* *of $(\mathbb{C}^*)^k$ on X. We define the moment map associated to the action of $(\mathbb{C}^*)^k$ on X by*

$$\begin{aligned} \mu_{(\mathbb{C}^*)^k} : X &\to \mathbb{R}^k \\ x &\mapsto \left(\frac{\sum_{i=1}^{n} a_1^i |x_i|^2}{\sum_{i=1}^{n} |x_i|^2}, \ldots, \frac{\sum_{i=1}^{n} a_k^i |x_i|^2}{\sum_{i=1}^{n} |x_i|^2} \right), \end{aligned}$$

that is, the composition

$$\mu_{(\mathbb{C}^*)^k} = A^t \circ \mu_{(\mathbb{C}^*)^n}{}_{|X}. \tag{4.1}$$

Thus the image of μ_λ is $A^t(\Delta^n)$, which is a polytope in $\mathbb{R}^k$ (See Theorem 8.9. [6]). Note that

$$\mu_\lambda(t.x) = \left(\frac{\sum_{i=1}^{n} a_1^i \mid \prod_{j=1}^{k} t_j^{a_j^i} \mid^2 |x_i|^2}{\sum_{i=1}^{n} \mid \prod_{j=1}^{k} t_j^{a_j^i} \mid^2 |x_i|^2}, \ldots, \frac{\sum_{i=1}^{n} a_k^i \mid \prod_{j=1}^{k} t_j^{a_j^i} \mid^2 |x_i|^2}{\sum_{i=1}^{n} \mid \prod_{j=1}^{k} t_j^{a_j^i} \mid^2 |x_i|^2} \right).$$

In particular, for a one-parameter subgroup $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^n$ for any $t \in \mathbb{C}^*$

$$t.x = \lambda(t).x = (t^{a_1} x_1 : \cdots : t^{a_n} x_n),$$

we have

$$\mu_\lambda : X \to \mathbb{R} \tag{4.2}$$

$$x \mapsto \frac{\sum_{i=1}^{n} a_i |x_i|^2}{\sum_{i=1}^{n} |x_i|^2}. \tag{4.3}$$

Example 4.1.1 *Let's assume $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^2$ as follows; $t.x = (x_0 : t_1 x_1 : t_2^{-1} x_2)$ where $t = (t_1, t_2) \in (\mathbb{C}^*)^2$ and $x = (x_0 : x_1 : x_2) \in \mathbb{P}^2$. Then the moment map associated to this action is*

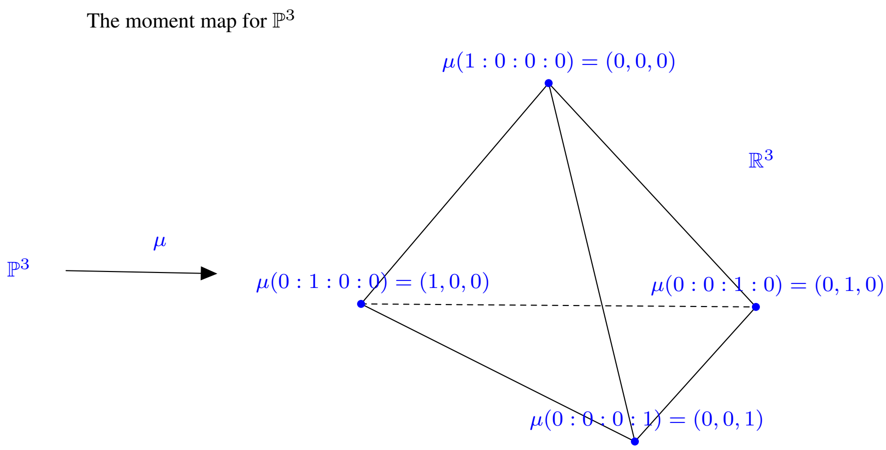
$$\begin{gathered} \mu : X \to \mathbb{R}^2 \\ x \mapsto \left(\tfrac{|x_1|^2}{|x_0|^2+|x_1|^2+|x_2|^2}, \tfrac{-|x_2|^2}{|x_0|^2+|x_1|^2+|x_2|^2}\right). \end{gathered}$$

Example 4.1.2 *Suppose $(\mathbb{C}^*)^3$ acts on $\mathbb{P}^3$ as below:*

$$\begin{aligned} &\forall (t_1, t_2, t_3) \in (\mathbb{C}^*)^3, \quad \forall (x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 \\ &(t_1, t_2, t_3)(x_0 : x_1 : x_2 : x_3) = (x_0 : t_1 x_1 : t_2 x_2 : t_3 x_3). \end{aligned}$$

Then the moment map is as below:

$$\begin{aligned} \mu : \mathbb{P}^3 &\rightarrow \mathbb{R}^3 \\ x &\mapsto \Big(\frac{|x_1|^2}{|x_0|^2+|x_1|^2+|x_2|^2+|x_3|^2}, \frac{|x_2|^2}{|x_0|^2+|x_1|^2+|x_2|^2+|x_3|^2}, \\ &\quad \frac{|x_3|^2}{|x_0|^2+|x_1|^2+|x_2|^2+|x_3|^2}\Big). \end{aligned}$$



Proposition 4.1.2 *Given an embedding $i : X \hookrightarrow \mathbb{P}^{n-1}$ (for $n > 1$) which is $(\mathbb{C}^*)^n$-equivariant, and the action of $\mathbb{C}^*$ on X induced by a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow (\mathbb{C}^*)^n$, we have a naturally induced $\mathbb{C}^*$–linearisation on $i^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) = \mathcal{L}$. For this linearisation we have*

$$X^{ss}_\lambda = \{x \in X : \mu_\lambda^{-1}(0) \cap \overline{O(x)} \neq \emptyset\}.$$

Proof: Suppose $x \in X$ is fixed and for a one-parameter subgroup

$$\begin{aligned} \lambda : \mathbb{C}^* &\hookrightarrow (\mathbb{C}^*)^n \\ t &\mapsto (t^{a_1}, \dots, t^{a_n}), \end{aligned}$$

there exists a point $y = (y_1 : \cdots : y_n) \in \overline{O(x)} \cap \mu_\lambda^{-1}(0)$. Hence $\mu_\lambda(y) = 0$ and we have three cases:

$$\begin{cases} \text{Case I} & y = \lim_{t\to 0} \lambda(t).x = \lim_{t\to 0}(t^{a_1}.x_1 : \ldots : t^{a_n}.x_n) \\ \text{Case II} & y = \lim_{t\to\infty} \lambda(t).x \\ \text{Case III} & y = \lambda(t).x \text{ for some } t. \end{cases}$$

Case I. Since μ_λ is a continuous map

$$\begin{aligned} \mu_\lambda(y) &= \mu_\lambda(\lim_{t\to 0}(t^{a_1}.x_1 : \ldots : t^{a_n}.x_n)) \\ &= \lim_{t\to 0} \mu_\lambda((t^{a_i}.x_i)_i) \\ &= \lim_{t\to 0} \frac{\sum_{i=1}^{n} a_i |t^{2a_i}||x_i|^2}{\sum_{i=1}^{n} |t^{2a_i}||x_i|^2} \\ &= \min\{a_i; x_i \neq 0\} = 0. \end{aligned}$$

Without loss of generality let us assume $a_1 = min\{a_i : x_i \neq 0\}$. Hence $a_1 = 0$ and therefore $a_i \geq 0$ for all $i = 2, 3, \ldots, n$. Now consider a global section in $\mathcal{O}_X(1)$

$$F(X_1, \ldots, X_n) = X_1.$$

$F(x) \neq 0$ and F is invariant under the $\mathbb{C}^*-$action which means that $x \in X_\lambda^{ss}$.

The Case II is treated similarly, but with the minimum replaced by maximum.

In Case III we have $\mu_\lambda(\lambda(t).x) = 0$ hence either all $a_i = 0$ whenever $x_i \neq 0$, or there exist $a_i > 0$ and $a_j < 0$ with x_i and $x_j \neq 0$. In this case we consider the global section $X_j^{|a_i|} X_i^{|a_j|}$ in $H^0(X, \mathcal{L}^{|a_i|+|a_j|})$, which is $\mathbb{C}^*$–equivariant and non-zero at x. Hence in all the cases, $x \in X_\lambda^{ss}$.

For the converse let $x = (x_1, ..., x_n) \in X_\lambda^{ss}$. This means that there exists a $\mathbb{C}^*$–invariant section F in $H^0(X, \mathcal{L}^N)^{\mathbb{C}^*}$ for some integer $N > 0$ such that $F(x) \neq 0$. We can assume that F is a monomial

$$F(X_1, \ldots, X_n) = X_1^{b_1} X_2^{b_2} \ldots X_m^{b_m},$$

with $b_j > 0$ and that $x_l \neq 0$ for all $l \leq m$. On the other hand

$$\mu_\lambda(x) = \sum_{i=1}^{m} a_i |x_i|^2 / \sum_{i=1}^{m} |x_i|^2.$$

If $\mu_\lambda(x) = 0$ then $x \in \overline{O(x)} \cap \mu_\lambda^{-1}(0)$ and therefore

$$\overline{O(x)} \cap \mu_\lambda^{-1}(0) \neq \emptyset.$$

Otherwise it is enough to find a solution, t', for $\mu_\lambda(\lambda(t').x) = 0$. Note that

$$\begin{aligned}
F(\lambda(t).x) &= F(t^{a_1}x_1 : \dots : t^{a_n}x_n) \\
&= (t^{a_1}x_1)^{b_1}(t^{a_2}x_2)^{b_2}\dots(t^{a_m}x_m)^{b_m} \\
&= t^{a_1b_1+a_2b_2+\dots+a_mb_m}x_1^{b_1}x_2^{b_2}\dots x_m^{b_m} \\
&= t^{\sum_{i=1}^{m} a_ib_i}F(x) \\
&\Rightarrow \sum_{i=1}^{m} a_ib_i = 0 \text{ and } b_i > 0 \\
&\Rightarrow \min\{a_j; x_j \neq 0\} < 0 \text{ and } \max\{a_j; x_j \neq 0\} > 0.
\end{aligned}$$

Since $\mu_\lambda(\lambda(t).x)$, as a function of t, depends only on $|t|$ and

$$\lim_{t\to 0}\mu_\lambda(\lambda(t).x) = \frac{\sum_{i=0}^{n} a_i|t^{a_i}|^2|x_i|^2}{\sum_{i=0}^{n}|t^{a_i}|^2|x_i|^2} = \min\{a_j; x_j \neq 0\} < 0,$$

and

$$\lim_{t\to\infty}\mu_\lambda(\lambda(t).x) = \frac{\sum_{i=0}^{n} a_i|t^{a_i}|^2|x_i|^2}{\sum_{i=0}^{n}|t^{a_i}|^2|x_i|^2} = \max\{a_j; x_j \neq 0\} > 0,$$

the equation $\mu_\lambda(\lambda(t).x) = 0$ should have at least one solution, by the Intermediate Value Theorem. Therefore

$$\overline{O(x)} \cap \mu_\lambda^{-1}(0) \neq \emptyset.$$

□

Theorem 4.1.3 *For a one-parameter subgroup λ*

$$\begin{aligned}
\lambda : \mathbb{C}^* &\to (\mathbb{C}^*)^n &(4.4)\\
t &\mapsto (t^{a_1}, \dots, t^{a_n}), &(4.5)
\end{aligned}$$

where $\mathbb{C}^*$ via λ acts nontrivially on X, we have $X_\lambda^{ss} = \{x \in X : \exists i, j$ such that $x_i \neq 0$ and $a_i \leq 0$ and $x_j \neq 0$ and $a_j \geq 0\}$.

Proof: Consider the moment map associated to λ:

$$\begin{aligned} \mu_\lambda \quad &: \quad X \to \mathbb{R} \\ x \quad &\mapsto \quad \frac{\sum_{i=1}^{n} a_i |x_i|^2}{\sum_{i=0}^{n} |x_i|^2}. \end{aligned}$$

Note

$$\lim_{t\to\infty} \mu_\lambda(t.x) = \frac{\sum_{i=1}^{n} a_i |t^{a_i}|^2 |x_i|^2}{\sum_{i=1}^{n} |t^{a_i}|^2 |x_i|^2} = \max\{a_i : x_i \neq 0\}$$

and

$$\lim_{t\to 0} \mu_\lambda(t.x) = \frac{\sum_{i=1}^{n} a_i |t^{a_i}|^2 |x_i|^2}{\sum_{i=1}^{n} |t^{a_i}|^2 |x_i|^2} = \min\{a_i : x_i \neq 0\}.$$

Assume $a_i > 0$, for all $i \in \{1, ..., n\}$ such that $x_i \neq 0$. Then $\lim_{t\to\infty} \mu_\lambda(tx) > 0$ and $\lim_{t\to 0} \mu_\lambda(tx) > 0$. So $0 \notin \mu_\lambda(\overline{O(x)})$ i.e. $\mu_\lambda^{-1}(0) \cap \overline{O(x)} = \emptyset$ and therefore $x \notin X_\lambda^{ss}$. The same argument works for the case when all $a_i < 0$.

Now to prove the converse, suppose there exist $a_i \geq 0$ and $a_j \leq 0$. Therefore since μ_λ is continuous, so there exist $t_0 \in \mathbb{C}^*$ such that either $\mu_\lambda(t_0 x) = 0$ or $\lim_{t\to\infty} \mu_\lambda(tx) = 0$ or $\lim_{t\to 0} \mu_\lambda(tx) = 0$. Hence

$$\begin{aligned} t_0 x \in \mu_\lambda^{-1}(0) \cap \overline{O(x)} \quad &\Rightarrow \quad \mu_\lambda^{-1}(0) \cap \overline{O(x)} \neq \emptyset \\ &\Rightarrow \quad x \in X_\lambda^{ss}. \end{aligned}$$

□

We can use the Hilbert-Mumford numerical criterion from the proof of the Proposition 4.1.2 as below:

Theorem 4.1.4 *Let $X \hookrightarrow \mathbb{P}^n$ and $(\mathbb{C}^*)^k \subset (\mathbb{C}^*)^{n+1}$ a subgroup of the maximal torus actin on $\mathbb{P}^n$. We assume that the action of $(\mathbb{C}^*)^k$ on $\mathbb{P}^n$ is the restriction of its action on X. The rational action of $(\mathbb{C}^*)^{n+1}$ on $\mathbb{A}^{n+1}$ induces an action $(\mathbb{C}^*)^k$ on $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^n}(1)$.*

With respect to this linearization on $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^n}(1)$,

- $X^{ss} = \{x; \mu_\lambda(x) \leq 0$ *for all 1-parameter subgroups* $\lambda\}$.
- $X^s = \{x; \mu_\lambda(x) < 0$ *for all 1-parameter subgroups* $\lambda\}$.

4.2 Description of Walls

Consider the projective variety $X \subset \mathbb{P}^{n-1}$ with a $(\mathbb{C}^*)^{n-1}$ acting on $\mathbb{P}^{n-1}$ such that X be invariant under the action of a subgroup $(\mathbb{C}^*)^k$ of $(\mathbb{C}^*)^{n-1}$. We consider the torus $(\mathbb{C}^*)^{n-1}$ as the quotient of $(\mathbb{C}^*)^n$ by $\mathbb{C}^*$, the diagonal in $(\mathbb{C}^*)^n$.

There exists a natural decomposition $\mu(X) = \bigcup_l F_l$ into a union of convex polyhedra, such that the complement of the union of the top-dimensional polyhedra is the image of the set

$$\{x \in X \text{ such that } \dim(\text{Stab}(x)) > 0\},$$

where $\text{Stab}(x)$ denotes the stabilizer of x.

The top dimensional polyhedra are called chambers, and their boundary consists of walls.

Lemma 4.2.1 *Consider the action of $(\mathbb{C}^*)^k$ on $\mathbb{P}^{n-1}$. Let $x \in \mathbb{P}^{n-1}$ be a point which doesn't have infinite stabilizer. Then the image of the orbit $(\mathbb{C}^*)^k \cdot x$ through the moment map $\mu_{(\mathbb{C}^*)^k}$ is an open set in $\mathbb{R}^k$.*

Proof: Consider a point $x \in \mathbb{P}^{n-1}$ and the map $i_x : (\mathbb{C}^*)^k \to \mathbb{P}^{n-1}$ given by the action of $(\mathbb{C}^*)^k$ on x. Note that the image of this map is in $(\mathbb{C}^*)^{n-1}$ because x has no infinite stabilizers. The restriction of the map $\mu_{(\mathbb{C}^*)^n} : \mathbb{P}^{n-1} \to \mathbb{R}^n$ to $(\mathbb{C}^*)^{n-1}$ has fibers isomorphic to $(S^1)^{n-1}$, and their preimages through i_x are isomorphic to $(S^1)^k$. Hence we have a commutative diagram:

$$\begin{array}{ccccccc}
(\mathbb{C}^*)^k & \xrightarrow{\quad i_x \quad} & (\mathbb{C}^*)^{n-1} & \xrightarrow{\mu_{(\mathbb{C}^*)^n}} & \mathbb{R}^n & \xrightarrow{A^t} & \mathbb{R}^k \\
\downarrow & & \downarrow & \nearrow_{j} & & & \\
(\mathbb{C}^*)^k/(S^1)^k & \xrightarrow{\overline{i_x}} & (\mathbb{C}^*)^{n-1}/(S^1)^{n-1} & & & &
\end{array},$$

where the quotients are smooth manifolds and the vertical maps are smooth maps. We can think of j as identifying $(\mathbb{C}^*)^{n-1}/(S^1)^{n-1}$ with $\text{int}(\triangle^{n-1})$. Hence $\phi = j \circ \overline{i_x}$ is an embedding with $d\phi = A$. (Recall that $A \cdot A^t = I_k$ by Lemma 4.1.1 hence $\text{rank } A = \text{rank } A^t = k$.) Hence $d(A^t \circ \phi) = A^t \cdot A = I_k$ and so $A \circ \phi$ is a local diffeomorphism. $\square$

Lemma 4.2.2 *The fixed point loci for any $\mathbb{C}^*$–action on a smooth variety are smooth.*

Proof: Let's denote a component of the $\mathbb{C}^*$–fixed locus in a variety X by Y. Every

$t \in \mathbb{C}^*$ induces the map

$$\begin{array}{rcl} t. : X & \longrightarrow & X \\ x & \mapsto & t.x, \end{array}$$

which for every $y \in Y$ induces $dt : \mathcal{T}_{y,X} \to \mathcal{T}_{y,X}$. Thus the restriction of $\mathcal{T}_X$ to Y decomposes into sheaves of eigenspaces $\mathcal{T}_{X_{|Y}} = \oplus_{i=0}^{\ell} E_i$, where $E_0 = T_Y$. Since $\mathcal{T}_X$ is a vector bundle, its restriction to Y is also a vector bundle of constant rank.

From Theorem A.0.11, $\dim_{\mathbb{C}(y)} E_{iy} \otimes_{\mathcal{O}_X} \mathbb{C}(y)$ is *upper semi-continuous*. Since the rank of $\mathcal{T}_{y,X}$, is constant, therefore the rank of each summand E_{iy} must be constant. □

Theorem 4.2.3 *Consider an action of the torus $(\mathbb{C}^*)^k$ on a smooth variety $X \subset \mathbb{P}^{n-1}$ such that the stabilizer of the generic point of X is trivial. If Y_i is an irreducible component of maximal dimension of $Y = \{x \in X : \dim(\mathit{Stab}(x)) > 0\}$, then there exists a unique subgroup $\mathbb{C}^*$ of $(\mathbb{C}^*)^k$ such that $t.y = y$ for any $t \in \mathbb{C}^*$ and $y \in Y_i$.*

Proof: We will denote $T = (\mathbb{C}^*)^k$. Consider the following diagram

$$\begin{array}{ccc} f^{-1}(\triangle) & \longrightarrow & \triangle = \{(x,y); x = y\} \\ \downarrow & & \downarrow \\ T \times X & \xrightarrow{f} & X \times X, \end{array}$$

where

$$\begin{array}{ccc} (t,x); tx = x & \longmapsto & (tx,x); tx = x \\ \downarrow & & \downarrow \\ (t,x) & \longmapsto & (t.x,x), \end{array}$$

where $Y = \{x \in X;\ \dim f^{-1}((x,x)) > 0\}$ i.e. Y represents the points with infinite stabilizers. Let $Y = \bigcup_j Y_j$ where Y_i are the irreducible components of Y. There exists a $\mathbb{C}^* \subset (\mathbb{C}^*)^k$ that fixes all the points of Y_i because all the $\mathbb{C}^*$ that fix points of Y should vary continuously when we move continuously on Y but all the $\mathbb{C}^* \to (\mathbb{C}^*)^{n-1}$ are discretely distributed. So there should be one $\mathbb{C}^*$ which fixes all of these points. Let $y \in Y$ be a generic point (could be smooth). We want to prove that y is fixed just by one $\mathbb{C}^*$. Suppose that there exist a bigger torus T_1 such that $y.T_1 = y$, then $\mathcal{T}_{y,Y}$ is fixed by T_1. We can find a $\mathbb{C}^*$ in T_1 such that it fixes some other directions (other than $\mathcal{T}_{y,Y}$) in $\mathcal{T}_{y,X}$. So T_1 fixes something of dimension more than $\dim(\mathcal{T}_{y,Y}) = \dim(Y)$.

To finish the proof we only need to show that $X^{\mathbb{C}^*}$ is smooth. Let $x \in X^{\mathbb{C}^*}$. There exist a hyperplane H that doesn't contain x. So we can reduce our argument to affine case, $X = \mathit{Spec}(R)$. Let's assume $X^{\mathbb{C}^*} = \mathit{Spec}(R/J)$ where J is the smallest ideal is

R such that R/J has trivial $\mathbb{C}^*$–action on it. It means that $Spec(R/J)$ is the maximal scheme which is fixed by $\mathbb{C}^*$. The weights of the $\mathbb{C}^*$–action creates a grading on R. Lets assume $m_x = \langle x_1, ..., x_n \rangle \lhd R$ and $J = \langle x_i, ..., x_n \rangle$ is generated by all the generators of m_x which are not fixed by $\mathbb{C}^*$–action. Any homogeneous element in R that is not fixed by $\mathbb{C}^*$–action has at least one component which at least one of the generators of J is dividing it and it has to be in J. Hence $\dim(R/J) = i - 1$ and therefore $X^{\mathbb{C}^*}$ is smooth. □

Definition 4.2.1 *We call $\mu(Y_i)$ in the previous theorem a wall.*

Let C be a chamber. As a result of the Theorem 4.1.4 we have the following correspondence.

Proposition 4.2.4 *For any point x with finite stabilizer $\mu(Tx)$ is a polytope. A point x is stable if and only if $\mu(Tx) \supset C$.*

Proof: From Theorem 4.1.4, for any 1-parameter subgroup λ we have $\mu_\lambda(Tx)$ contains the projection of C on $T_{\mathbb{C}^*_\lambda}$.

Theorem 4.2.5 *Let m and n be natural numbers with $m < n$ and the $(\mathbb{C}^*)^{n-1}$ acting on the Grassmanian $G(m, n)$ induced from natural action on $\mathbb{P}^{n-1}$. For an arbitrary wall $Y_i \subset G(m, n)$ (as in the previous definition) there exists a unique subgroup*

$$\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^{n-1},$$

where λ can be taken to act as below:

$$t \mapsto (t^{a_1}, \ldots, t^{a_n}),$$

and a partition $I \cup J = \{1, \ldots, n\}$ with

$$a_i = \begin{cases} 1 & i \in I \\ -1 & i \in J \end{cases}$$

and I and J are nonempty, such that $\mathbb{C}^*$ fixes the points of Y_i.

Proof: Note that the diagonal of $(\mathbb{C}^*)^n$ acts trivially on $G(m, n)$. The quotient of $(\mathbb{C}^*)^n$ by the diagonal, which is isomorphic to $(\mathbb{C}^*)^{n-1}$, satisfies the conditions in the previous theorem. In that context we simply can work with $(\mathbb{C}^*)^n$ considering $\{x \in G(m, n); \dim(stab(x)) > 1\}$ instead of $\{x \in G(m, n); \dim(stab(x)) > 0\}$. Now from the previous theorem there exists Y, an irreducible component of maximal dimension of $\{x \in G(m, n); \dim(stab(x)) > 1\}$ and there exists a unique $\mathbb{C}^*$ which fixes Y

point-wise and therefore $\mu(Y) = W$. Let's assume this one parameter subgroup is as below:

$$\begin{array}{rcl} \lambda : \mathbb{C}^* & \to & (\mathbb{C}^*)^n \\ t & \mapsto & (t^{a_1}, \dots, t^{a_n}). \end{array}$$

The action of this $\mathbb{C}^*$ on $G(m, n)$ is induced from the action of $\mathbb{C}^*$ on $\mathbb{P}^{n-1}$. More precisely, if $[Q] \in G(m, n)$ where Q is an $m \times n$ matrix, we just multiply the i–th column by t^{a_i}. Without loss of generality we can reduce the argument on an open set $\mathbb{A}^{m(n-m)} \subset G(m, n)$. Therefore let

$$[Q] = \left[\begin{pmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,n-m} & 1 & 0 & \dots & 0 \\ r_{2,1} & r_{2,2} & \dots & r_{2,n-m} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \dots & r_{m,n-m} & 0 & 0 & \dots & 1 \end{pmatrix}_{m \times n} \right] = [(R|I_{m \times m})],$$

where R is an $m \times (n - m)$–matrix and $I_{m \times m}$ is the identity matrix. Then $t.[Q] = [(t.R|I)]$ where t acts on R by multiplying the i–th column by t^{a_i} and the j–th row by $t^{-a_{n-m+j}}$.

Note that in other open sets the action looks different. The condition $t.[Q] = [Q]$ is equivalent to

$$\begin{cases} a_1 = a_{n-m+1}, & \text{if } r_{1,1} \neq 0; \\ a_2 = a_{n-m+2}, & \text{if } r_{2,2} \neq 0; \\ a_1 = a_{n-m+2}, & \text{if } r_{2,1} \neq 0; \\ & \vdots \end{cases}$$

Therefore

$$r_{i,j} \neq 0 \Rightarrow a_j = a_{n-m+i}.$$

On the other hand, if $a_j = a_{n-m+i}$ then the projection on the $r_{i,j}$-th coordinate

$$Y \bigcap \mathbb{A}^{m(n-m)} \to \mathbb{A}^1,$$

is surjective.

Let $\Gamma = \{a_1, a_2, ..., a_n\}$ represent the set of weights.

First claim. There exist $a, b \in \mathbb{Z}$ such that $\Gamma = \{a, b\}$.

Proof of claim. Let's assume that $\{a, b, c\} \subset \Gamma$. Then there exists a one-parameter subgroup

$$\begin{array}{rcl} \lambda' : \mathbb{C}^* & \to & (\mathbb{C}^*)^n \\ t & \mapsto & (t^{b_1}, \ldots, t^{b_n}), \end{array}$$

with

$$b_i = \begin{cases} a & \text{if } a_i = a; \\ b & \text{otherwise,} \end{cases}$$

which also fixes Y, which is a contradiction with the uniqueness of the one dimensional sub-torus which fixes a wall.

Second claim. We can choose the one-parameter subgroup λ such that $a = -b$.

Proof of claim. This is true because λ and

$$\begin{array}{rcl} \lambda_\ell : \mathbb{C}^* & \hookrightarrow & (\mathbb{C}^*)^n \\ t & \mapsto & (t^{a_1+\ell}, \ldots, t^{a_n+\ell}), \end{array}$$

define the same action on the projective space (which contains the wall). Hence after choosing $l = -(a + b)/2 \in \mathbb{Z}$ we obtain $a' = (a + l) = -(b + l) = -b'$. Note that if $a + b$ is odd, one need to replace t by t^2 to make this sum even and then proceed as before.

Third claim. $\{a, b\} = \{1, -1\}$.

Proof of claim. Otherwise λ is not an embedding but a multiple cover of the one-parameter subgroup for which $\{a, b\} = \{1, -1\}$.

□

From the proof of Theorem 4.2.5 we have the following statement.

Theorem 4.2.6 *With the assumptions of the previous Theorem, for a wall W there exist disjoint sets I, J and also disjoint sets I', J' such that $I \cup J = \{1, 2, ..., n\}$ and $I' \cup J' = \{1, 2, ..., m\}$ and*

$$Y = \{[Q] \in G(m, n) : \quad Q = (a_{ij})_{1 \le i \le m, 1 \le j \le m}, a_{ij} = 0 \text{ if } (i \in I' \wedge j \in I) \vee (i \in J' \wedge j \in J)\},$$

is fixed by the $\mathbb{C}^*$ corresponding to W.

Proof: Let $[Q] \in G(m,n)$ where Q is an $m \times n$ matrix. Without loss of generality we can reduce the argument on an open set $\mathbb{A}^{m(n-m)} \subset G(m,n)$. Therefore let

$$[Q] = \left[\begin{pmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,n-m} & 1 & 0 & \dots & 0 \\ r_{2,1} & r_{2,2} & \dots & r_{2,n-m} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & r_{m,2} & \dots & r_{m,n-m} & 0 & 0 & \dots & 1 \end{pmatrix}_{m \times n} \right] = [(R|I_{m\times m})],$$

where R is an $m \times (n-m)$–matrix and $I_{m\times m}$ is the identity matrix. To have [Q] fixed by $\mathbb{C}^*$–action if $r_{ij} \neq 0$ then $a_j = a_{n-m+i}$. Hence from the previous Theorem we are done. □

From this proof we have the following corollary.

Corollary 4.2.7 *With the assumptions and notations of the previous Theorem,* $F = (\mu^{-1}(W))^{\mathbb{C}^*}$.

Note $\mathbb{C}^*$ acting on $G(m,n)$, induces an action with the same weights as its action on $G(m,n)$, on the universal family and $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$.

Corollary 4.2.8 *Under the same assumptions as the last theorem, the set*

$$\{x \in G(m,n) : \lim_{t\to\infty} \lambda(t).x \in F\},$$

is as below:

$$Y^+ := \{[Q] \in G(m,n) : Q = (a_{ij})_{1\le i\le m, 1\le j\le m}, a_{ij} = 0 \textit{ if } i \in I' \wedge j \in I\},$$

and the set

$$\{x \in G(m,n) : \lim_{t\to 0} \lambda(t).x \in F\},$$

is as follow

$$Y^- := \{[Q] \in G(m,n) : Q = (a_{ij})_{1\le i\le m, 1\le j\le n}, a_{ij} = 0 \textit{ if } i \in J' \wedge j \in J\}.$$

Example 4.2.1 *In* $G(3,n)$ *a wall corresponds to the following one-parameter subgroup*

$$\begin{aligned} \lambda : \mathbb{C}^* &\hookrightarrow (\mathbb{C}^*)^n \\ t &\mapsto (t,t,t,t^{-1},t^{-1}\dots,t^{-1},t,t,t^{-1}), \end{aligned}$$

where

$$\begin{cases} a_1 = a_2 = a_3 = a_{n-2} = a_{n-1} = 1 \in \mathbb{Z}, \\ a_4 = a_5 = \cdots = a_{n-3} = a_n = -1 \in \mathbb{Z}, \end{cases}$$

This wall corresponds to the set of all the points $[Q] \in \mathbb{A}^{3(n-3)}$ *which are fixed by the* $\mathbb{C}^*$*–action above, namely*

$$Q = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ r_{2,1} & r_{2,2} & r_{2,3} & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r_{3,4} & r_{3,5} & \dots & r_{3,n-3} & 0 & 0 & 1 \end{pmatrix}.$$

Note that in general such a point in $G(3,n)$ *is like this*

$$Q = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & 0 & 0 & \dots & 0 & r_{1,n-2} & r_{1,n-1} & 0 \\ r_{2,1} & r_{2,2} & r_{2,3} & 0 & 0 & \dots & 0 & r_{2,n-2} & r_{2,n-1} & 0 \\ 0 & 0 & 0 & r_{3,4} & r_{3,5} & \dots & r_{3,n-3} & 0 & 0 & r_{3,n} \end{pmatrix}.$$

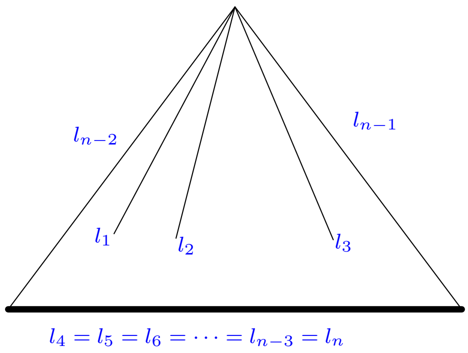
The points whose orbits are going into this fixed loci when $t \to \infty$ *are like below:*

$$Q = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & r_{1,4} & r_{1,5} & \dots & r_{1,n-3} & r_{1,n-2} & r_{1,n-1} & r_{1,n} \\ r_{2,1} & r_{2,2} & r_{2,3} & r_{2,4} & r_{2,5} & \dots & r_{2,n-3} & r_{2,n-2} & r_{2,n-1} & r_{2,n} \\ 0 & 0 & 0 & r_{3,4} & r_{3,5} & \dots & r_{3,n-3} & 0 & 0 & r_{3,n} \end{pmatrix},$$

and the points whose orbits are coming into this fixed loci when $t \to 0$ *are like below:*

$$Q = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} & 0 & 0 & \dots & 0 & r_{1,n-2} & r_{1,n-1} & 0 \\ r_{2,1} & r_{2,2} & r_{2,3} & 0 & 0 & \dots & 0 & r_{2,n-2} & r_{2,n-1} & 0 \\ r_{3,1} & r_{3.2} & r_{3,3} & r_{3,4} & r_{3,5} & \dots & r_{3,n-3} & r_{3,n-2} & r_{3,n-1} & r_{3,n} \end{pmatrix}.$$

In chapter 5 *we will see how this wall corresponds to points in* $G(3,n)$*, each parameterizing plane together with a set of* k *lines passing through a point and* $n-k$ *lines which coincide. We note that here* k *is the number of columns which have 0 in their third row.*



Remark 4.2.9 *From the following diagram*

$$\begin{array}{ccc} \mathbb{P}^{(\wedge^3 V-1)} & \longrightarrow & \mathbb{R}^{(\wedge^3 V-1)} \\ & \searrow^{\mu} & \downarrow \\ & & \mathbb{R}^{n-1}, \end{array}$$

the image of an orbit $O(x)$ is a polytope with vertices a subset of the vertices of the image of Grassmannian variety under the moment map.

Each wall corresponds to a partition $I \cup J = \{1, 2, ..., n\}$ where I corresponds to the columns with positive weights and J corresponds to the columns with negative weights. We denote the wall correspondent to this partition by W_I and the moment map associated to the $\mathbb{C}^*$ by μ_I.

Proposition 4.2.10 *Let I_1 and I_2 two subsets of $\{1, 2, ..., n\}$. For two walls W_{I_1} and W_{I_2} (correspondent to I_1 and I_2) consider the corresponding moment maps μ_{I_1} and μ_{I_2}. If $I_1 \subset I_2$ then $\mu_{I_1}(W_{I_1}) \geq \mu_{I_1}(W_{I_2})$ and also if $J_1 \supset J_2$ then $\mu_{I_1}(W_{I_1}) \geq \mu_{I_1}(W_{I_2})$.*

Proof: Assume $I_1 \subset I_2$ and also assume W_{I_1} corresponds to the lines $l_1, ...l_k$ in the fiber that are passing through a point and $u_1, ..., u_{n-k}$ lines in the fiber coinciding. Let (l_1, l_i, l_k) be the matrix having the coefficient of the lines l_1, l_i and l_k in the 1th, ith and kth columns and 0 for the other columns. Note that the image of an orbit under moment map is the convex hull(polytope) with vertices (v_i, v_s, v_t) where v_i, v_s and v_t are the lines. For example (l_1, l_i, l_k) corresponds to a vertex of $\mu_{I_1}(Y_I)$. We have $\mu_{I_1}(l_1, l_i, l_k) > 0$ also $\mu_{I_1}(l_1, u_i, l_k) > 0$, $\mu_{I_1}(l_1, l_i, l_k) > 0$, $\mu_{I_1}(u_1, u_i, l_k) < 0$, etc which gives us the result we wanted to prove. □

Chapter 5

Moduli Problems

To describe the GIT quotient of a Grassmannian variety by the maximal torus (the largest torus action on the variety) as a moduli space, one needs to describe its universal family. The first section provides a proof that a GIT quotient of a projective bundle is a projective bundle. The next two sections focus on describing a GIT quotient of the Grassmannian variety by the maximal torus as a moduli problem and in the final section, on describing a flip as a moduli problem.

5.1 GIT Quotient of a Projective Bundle

We want to show that under suitable conditions, the GIT quotient of a projective bundle over a scheme, by an equivariant torus action, is a projective bundle.

Lemma 5.1.1 *Let's assume $X \subset \mathbb{P}^n$ is a scheme and $\phi : \mathbb{P}(\mathcal{E}) \to X$ is a projective bundle, where $\mathcal{E}$ is a locally free coherent sheaf on X and a $(\mathbb{C}^*)^k$ acts on X with all finite stabilizers being trivial. We consider a linearization on $\mathbb{E}$ which induces an action of $(\mathbb{C}^*)^k$ on $\mathbb{P}(\mathcal{E})$. Let's assume that for a line bundle $\mathcal{L}$ on X and a linearisation on $\mathcal{L}$ we have $X^s(\mathcal{L}) = X^{ss}(\mathcal{L})$. Then there exists an ample line bundle $\mathcal{L}'$ on $\mathbb{P}(\mathcal{E})$ and a linearisation of $\mathcal{L}'$ such that $\mathbb{P}(\mathcal{E})^s(\mathcal{L}') = \phi^{-1}(X^s(\mathcal{L}))$.*

Proof: We want to show that if $x \in X^s(\mathcal{L})$ for a line bundle $\mathcal{L}$, then there exists a line bundle $\mathcal{L}'$ on $\mathbb{P}(\mathcal{E})$ such that $\phi^{-1}(x) \subset \mathbb{P}(\mathcal{E})^s(\mathcal{L}')$.

Consider a 1-parameter subgroup $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^n$. Let $\mathcal{L}' = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \phi^*(\mathcal{L}^{\otimes N})$ for a large enough positive integer N. For $y \in \phi^{-1}(x)$, let z be a point in the fiber $\mathcal{L}'_{|y}$. This fiber is isomorphic to $\mathbb{C}$. Therefore we can assume that we have $\lambda(t).z = t^{b+Na}z$ where a is the weight of the λ group on $\phi^*(\mathcal{L})$ and b is the weight of λ on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Now from Hilbert-Mumford's Criterion since $x \in X^s(\mathcal{L})$ we have $a < 0$. So for large enough positive integer N we have $Na < -b$ which means that $b + Na < 0$. We can chose N large enough so that the previous relation is true for all 1-parameter subgroups λ, hence $y \in \mathbb{P}(\mathcal{E})^s(\mathcal{L}')$ and therefore $\phi^{-1}(x) \subset \mathbb{P}(\mathcal{E})^s(\mathcal{L}')$. □

Theorem 5.1.2 . *Let's assume $X \subset \mathbb{P}^n$ is a scheme and $\phi : \mathbb{P}(\mathcal{E}) \to X$ is a projective bundle where $\mathcal{E}$ is a locally free coherent sheaf on X and $(\mathbb{C}^*)^k$ acts on X with no non-trivial finite stabilizers $\mathbb{P}(\mathcal{E})$ has an equivariant Cartier divisor $D \subset \mathbb{P}(\mathcal{E})$ and we fix the T–action induced on $\mathbb{P}(\mathcal{E})$ such that $\mathcal{L}(D) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then for any GIT quotient $X/\!/_{\mathcal{L}}T$ satisfying the conditions of the lemma, there is a GIT quotient $\mathbb{P}(\mathcal{E})/\!/T$ such that $\overline{\phi}' : \mathbb{P}(\mathcal{E})/\!/T \to X/\!/T$ is a projective bundle.*

Proof: First we prove that the orbits of the action on $\mathbb{P}(\mathcal{E})$ are transverse to the fibers of ϕ. From the previous lemma the following map

$$\overline{\phi} : \phi^{-1}(X^s(\mathcal{L}))/T \to X^s(\mathcal{L})/T$$
$$[y] \longmapsto [\phi(y)],$$

satisfies $\phi^{-1}(T.x)/T \cong \mathbb{P}^{rank(\mathcal{E})-1}$. Given $x \in X^s(\mathcal{L})$ such that its stabilizer stab$(x) = \{1\}$ we can deduce that for every $y \in \phi^{-1}(x)$ we have stab$(y) = \{1\}$: if for y_1 and y_2 in $\phi^{-1}(x)$ there exist a nontrivial $t_0 \in T$ such that $t_0.y_1 = y_2$, then $t_0.x = x$ which is a contradiction. Therefore $\phi^{-1}(x) \cap T.y = \{y\}$.

Now we want to show that $\mathcal{T}_{y,T.y} \cap \mathcal{T}_{y,\phi^{-1}(x)} = \{0\}$. The torus action gives us the following diagram

$$T \to^{.y} \mathbb{P}(\mathcal{E})^s(\mathcal{L}') \to^{\phi} X^s(\mathcal{L}),$$

where the composition of $.y$ and ϕ is multiplication by x. Both x and y have no non-trivial stabilizers. Hence both maps $.x$ and $.y$ are injective and therefore the restriction of ϕ to $T.y$ is an embedding. The last diagram induces the following diagram of the tangent spaces:

$$\mathcal{T}_{y,T.y} \cong \mathcal{T}_{1,T} \to \mathcal{T}_{y,\mathbb{P}(\mathcal{E})} \to^{d\phi} \mathcal{T}_{x,X},$$

where in $\mathcal{T}_{1,T}$ the 1 is the identity of torus T. Since ϕ over $\phi^{-1}(x) \subset \mathbb{P}(\mathcal{E})$ is constant, x, hence $d\phi_{\mathcal{T}_{y,\phi^{-1}(x)}} = 0$. Therefore if $v \in \mathcal{T}_{y,\phi^{-1}(x)} \cap \mathcal{T}_{y,T.y}$ then $d\phi(v) = 0$ and therefore $v = 0$ (since $d\phi$ is injective).

Now in the following diagram we want to show that $d\iota = 0$ where the vertical map i is an inclusion and u and ι are quotient maps which are smooth since the torus is acting freely (no non-trivial stabilizers) on $\mathbb{P}(\mathcal{E})^s(\mathcal{L}')$:

$$\begin{array}{ccc} \phi^{-1}(x) & \xrightarrow{\iota} & \mathbb{P}(\mathcal{E})^s(\mathcal{L}')/\!\!/_{\mathcal{L}'}T \\ \downarrow i & \nearrow u & \\ \mathbb{P}(\mathcal{E})^s(\mathcal{L}') & & \end{array} .$$

Thus the differential map has maximum rank which means that the kernel of the differential map is equal to the tangent space to the fiber. The previous diagram induces the following diagram at the level of tangent spaces:

$$\begin{array}{ccc} \mathcal{T}_{y,\phi^{-1}(x)} & \xrightarrow{d\iota} & \mathcal{T}_{[y],\mathbb{P}(\mathcal{E})^s(\mathcal{L}')/T} \\ \downarrow j & \nearrow du & \\ \mathcal{T}_{y,\mathbb{P}(\mathcal{E})^s(\mathcal{L}')} & & \end{array} ,$$

where the vertical map, j, is still an inclusion as $\mathcal{T}_{y,\phi^{-1}(x)}$ ia a sub space of $\mathcal{T}_{y,\mathbb{P}(\mathcal{E})^s(\mathcal{L}')}$. Now $d\iota$ being injective is equivalent to $\text{Image}(j) \cap \ker(du) = \{0\}$ and this follows from the previous part as we proved that $\mathcal{T}_{y,\phi^{-1}(x)} \cap \mathcal{T}_{y,T.y} = \{0\}$. Hence $d\iota$ is injective and therefore ι is an embedding.

Next we want to show that $\overline{\phi}_*(\mathcal{L}(D/\!\!/T)) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ where $\mathcal{L}(D/T)$ is the line bundle associated to the divisor D/T. As the quotient map is smooth and D is a certain divisor so is D/T. First we need to show that $\overline{\phi}^{-1}([x]) \cong \mathbb{P}^{\text{rank}\mathcal{E}-1}$ for every $x \in X^s$. In the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E})^s(\mathcal{L}') & \xrightarrow{\phi} & X^s(\mathcal{L}) \\ \downarrow u' & & \downarrow u \\ \mathbb{P}(\mathcal{E})^s(\mathcal{L}')/T & \xrightarrow{\overline{\phi}} & X^s(\mathcal{L})/T. \end{array}$$

the vertical maps are the quotient maps and $\overline{\phi}$ is the map induced by ϕ after taking quotient by T. Since u, u' and ϕ are smooth, $\overline{\phi}$ is smooth too which means that the differentiation of $\overline{\phi}$ is surjective and also $\overline{\phi}$ is flat. Now

$$\begin{aligned} \overline{\phi}^{-1}([x]) &= u'\phi^{-1}(u^{-1}[x]) \\ u'\phi^{-1}(T.x) &= u'(T.\phi^{-1}(x)) \\ &= u'(\phi^{-1}(x)), \end{aligned}$$

and since $u'|_{\phi^{-1}(x)} = i$ which is an embedding then

$$u'(\phi^{-1}(x)) \cong \overline{\phi}^{-1}([x]) \cong \mathbb{P}^{(\mathrm{rank}\mathcal{E}-1)}.$$

We need a vector bundle $\mathcal{E}'$ over $X^s(\mathcal{L})/T$ such that we have $\mathbb{P}(\mathcal{E}') = \mathbb{P}(\mathcal{E})^s/T$. For a point $[x] \in X^s(\mathcal{L})/T$ we showed that $\overline{\phi}^{-1}([x]) \cong \mathbb{P}^{(\mathrm{rank}\mathcal{E}-1)}$. Therefore from Theorem A.0.15, since

$$H^1(\mathbb{P}^{(\mathrm{rank}\mathcal{E}-1)}, \mathcal{L}(D/T)) = H^1(\mathbb{P}^{(\mathrm{rank}\mathcal{E}-1)}, \mathcal{O}_{\mathbb{P}^{(\mathrm{rank}\mathcal{E}-1)}}(1)) = 0,$$

we have

$$\overline{\phi}_*\mathcal{L}(D/T)_{[x]} = H^0(\overline{\phi}^{-1}([x]), \mathcal{O}_{\mathbb{P}^{(\mathrm{rank}\mathcal{E}-1)}}(1)) \cong \mathbb{C}^{(\mathrm{rank}\mathcal{E})},$$

and

$$\overline{\phi}_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong \mathcal{E}',$$

is a vector bundle on $X^s(\mathcal{L})/T$ of rank equal to rank$(\mathcal{E})$. □

Now we want to show that points in the Grassmaninan varieties with a torus action on them have trivial stabilizers.

Lemma 5.1.3 . *Let the torus $T = (\mathbb{C}^*)^{n-1}$ act on the Grassmannian variety $G(m,n)$ and $x \in G(m,n)$ an arbitrary point such that the isotropy group Stab(x) is finite (x has finite stabilizers). Then Stab$(x) = \{1\}$ (Stabilizers of x are trivial).*

Proof: . Let's assume Stab$(x) = \{t_1, ..., t_s\} \subset T$. Set $i = 1$ for example. Since x has finitely many stabilizers, then there exists a one-parameter subgroup

$$\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^{n-1}$$

but we work with the lift of this action to $(\mathbb{C}^*)^n$–action on $G(m,n)$, hence

$$t \mapsto (t^{a_1}, ..., t^{a_n}),$$

such that $t_1 = \lambda(t_0)$ for some $t_0 \in \mathbb{C}^*$ and $\lim_{t\to\infty} \lambda(t).x = y \in Y$, where Y is a fixed point locus of $\mathbb{C}^*$. By Theorem 4.1 and the Remark before it in [2] , we can identify x with a point $x = (x_1, ..., x_k)$ in $\mathcal{N}^-_{Y/G(m,n)_y}$ where $\mathcal{N}^-_{Y/G(m,n)}$ is the sub-bundle of the normal bundle $\mathcal{N}_{Y/G(m,n)}$ which corresponds to λ–orbits "coming into" Y. But $\mathcal{N}^-_{Y/G(m,n)}$ decomposes into positive weights $b_1, ..., b_k$ such that

$$t_1.(x_1, ..., x_k) = (t_0^{b_1}x_1, ..., t_0^{b_k}x_k).$$

Hence from Theorem 4.2.5, $t_0 = 1$ and $t_1 = (1, ..., 1)$. With the same argument for $t_2, ..., t_s$ we have Stab(x) is trivial. □

The description of Flips for the projective bundle could be done for all weights in general but in our case weights are only $+1$ or -1.

5.2 The Moduli Problem

A moduli space $\mathcal{M}$ for (equivalence classes of) geometric objects of a given type consists of:

1. A set $\mathcal{M}$ whose points are in bijective correspondence with the objects we wish to parameterize.
2. The notion of "good" functions to $\mathcal{M}$ described in terms of families of objects.

Definition 5.2.1 ***A Moduli Problem.*** *A moduli functor is a contravariant functor from the category of schemes to the category of sets that associates to a scheme B the equivalence classes of families of geometric objects with certain properties parameterized by B.*

Definition 5.2.2 ***A Representable Moduli Problem and its Fine Moduli Space.***

Given a moduli functor

$$\mathcal{G} : \mathfrak{Sch} \to \mathfrak{Set}$$

we say that $(\mathcal{M},\mathcal{U})$, where $\mathcal{U} \in \mathcal{G}(\mathcal{M})$ (the universal family), finely represents the moduli functor $\mathcal{G}$ if for any scheme B and any $v \in \mathcal{G}(B)$ there exists a unique

$$\phi : B \to \mathcal{M} \text{ such that } v = \mathcal{G}(\phi)(\mathcal{U}).$$

Example 5.2.1 ***(The Grassmannian Functor).*** *Let $0 < m < n$ be two positive integers. Let*

$$G(m, n) : \mathfrak{Sch} \quad \to \quad \mathfrak{Set},$$

be the contravariant functor that associates to a scheme B the set of sub-vector bundles of $B \times_{Spec(\mathbb{C})} \mathbb{C}^n$ of rank m, or, equivalently, the set of projective sub-bundles of $B \times_{Spec(\mathbb{C})} \mathbb{P}^{n-1}$ with fiber $\mathbb{P}^{m-1}$. We will work with this second interpretation throughout this thesis.

To a morphism of schemes $f : B' \to B$, the functor associates their pull-back.

This functor is represented by the Grassmannian variety $G(m,n)$*. Indeed, the universal family over* $G(m,n)$ *is the flag variety* $F = F(1,m,n)$ *defined as follows*

$$F(1,m,n) \quad = \quad \{(x,L) \in \mathbb{P}^{n-1} \times G(m,n) : x \text{ is a point in } L\}.$$

The flag variety comes with two natural morphisms, which are the restrictions to F *of the two projections from* $\mathbb{P}^{n-1} \times G(m,n)$ *to its factors:*

$$\begin{array}{ccccc} (x,L) & \longrightarrow & & & x \\ \Big\downarrow & & F(1,m,n) & \xrightarrow{p_1} & \mathbb{P}^{n-1} \\ & & {\scriptstyle p_2}\Big\downarrow & & \\ L & & G(m,n) & & . \end{array}$$

Let $\mathbb{P}^{n-1} = \mathrm{Proj}(\mathbb{C}[y_1,...,y_n])$*. To represent* $(x,L) \in F = F(1,m,n)$ *in coordinates, we may choose* $(\bar{v},\bar{L}) \in (\mathbb{A}^m \setminus \{0\}) \times M(m,n)$ *where* $\bar{L}$ *is a* $m \times n$ *matrix of rank* m*. For* $i = 1,2,...,m$ *let* $R_i \in \mathbb{A}^n$ *be the* $i-$*th row of the matrix* $\bar{L}$*. Let* $\bar{v} = (v_1, v_2, ..., v_m)$*. Then we can define* $L = \mathbb{P}(V)$*, as the projectivisation of the vector sub-space* $V =< R_1, R_2, ..., R_m > \subset \mathbb{C}^n$ *generated by* $R_1, R_2, ..., R_m$*, and* $x = \mathbb{P}(<\bar{x}>)$ *where* $\bar{x} = \sum_{i=1}^m v_i R_i \in V$*.*

Alternatively, we can think of $\bar{L}$ *as a* $1-1$ *linear transformation*

$$\bar{L} : \mathbb{C}^m \longrightarrow \mathbb{C}^n.$$

Then V *is the image of* $\bar{L}$ *and* $\bar{v}.\bar{L} = \bar{x}$*.*

Note that $(\bar{v},\bar{L})$ *define* (x,L) *uniquely up to a change of basis* $g \in GL(m)$ *for* V *and multiplication by a scalar for* $\bar{v}$*. Hence*

$$F(1,m,n) \cong \{(\bar{v},\bar{L}) \in (\mathbb{C}^m \setminus \{0\}) \times M(m,n) : \bar{L} \text{ is of rank } m\}/(\mathbb{C}^* \times GL(m)),$$

where the $\mathbb{C}^* \times GL(m)-$*action is as follow; for any* $(\bar{v},\bar{L})$ *in* $(\mathbb{C}^m \setminus \{0\}) \times M(m,n)$ *and for any* (t,g) *in* $\mathbb{C}^* \times GL(m)$

$$(t,g).(\bar{v},\bar{L}) = (t.\bar{v}g^{-1}, g\bar{L}),$$

where $t \cdot \bar{v}g^{-1}$ *is the multiplication of the vector* $\bar{v}g^{-1}$ *by the scalar* t *and* $\bar{v}g^{-1}$ *is the multiplication of matrices. Indeed,* $\mathbb{C}^* \times GL(m)$ *acts freely on* $\{(\bar{v},\bar{L}) \in (\mathbb{C}^m \setminus \{0\}) \times M(m,n) : \bar{L} \text{ is of rank } m\}$ *hence the quotient is a smooth variety.*

Definition 5.2.3 ***The Moduli Problem of Planes with n given lines in general position.*** *Consider the functor*

$$\begin{aligned} \mathcal{G} : \mathfrak{Sch} &\rightarrow \mathfrak{Set} \\ B &\mapsto [(\pi : P \rightarrow B, D_1, D_2, ..., D_n)], \end{aligned}$$

associating to each scheme B an isomorphism class of tuples

$$(\pi : P \rightarrow B, D_1, D_2, ..., D_n),$$

consisting of a flat morphism $\pi : P \rightarrow B$ whose fibers P_b are projective planes $\mathbb{P}^2$, and for each $i \in \{1, ..., n\}$, a divisor D_i of P whose intersection $\ell_{i,b}$ with each fiber P_b is a line $\ell_{i,b} \subset P_b$ such that

- *$\ell_{i,b} \neq \ell_{j,b}$ for $i \neq j$.*
- *No three lines $\ell_{i,b}, \ell_{j,b}, \ell_{k,b}$ intersect at the same point in P_b.*

Two tuples $(\pi : P \rightarrow B, D_1, D_2, ..., D_n)$ and $(\pi' : P' \rightarrow B', D'_1, D'_2, ..., D'_n)$ are isomorphic to each other if there exists a map $\phi : B' \rightarrow B$ and an isomorphism $\tilde{\phi} : P' \rightarrow \phi^*(P)$

$$\begin{array}{ccc} P' \cong \phi^*(P) & & P \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\phi} & B, \end{array}$$

such that $\tilde{\phi}$ restricts to an isomorphism $D'_i \cong \phi^*(D_i)$ for each $i \in \{1, ..., n\}$.

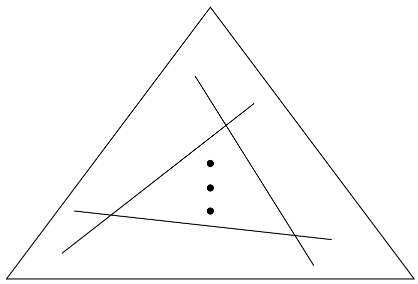
We will prove that this functor is representable. First we need some preliminary data. Previously we have defined the flag variety $F = F(1, m, n)$ over $G(m, n)$ which is the universal family over $G(m, n)$. F provides a $\mathbb{P}^{m-1}$–fibration over $G(m, n)$. Let $D_i = p_2^*(\{y_i = 0\})$ for $i = 1, 2, ..., n$. Then for $[\Lambda] \in G(m, n)$ such that $\Lambda = \begin{pmatrix} r_{1,1} & & r_{1,n} \\ \vdots & \ddots & \vdots \\ r_{m,1} & & r_{m,n} \end{pmatrix} \in M(m, n)$, let $\ell_{i,\Lambda} = F_\Lambda \cap D_i$ where $F_\Lambda \cong \mathbb{P}^{m-1}$ is the fiber of F over $[\Lambda]$. Now let $[x_1 : x_2 : ... : x_m]$ be homogeneous coordinates on F_Λ. Then the restriction of p_2 to F_Λ is given by

$$F_\Lambda \to \mathbb{P}^{n-1}$$
$$[x_1 : x_2 : ... : x_m] \mapsto [\sum_{j=1}^{m} r_{j,i} x_j]_{i=1,...,n},$$

and $\sum_{j=1}^{m} r_{j,i} x_j = p_2^*(y_i)$ represents the equation of $\ell_{i,\Lambda}$. Note that this equation is unique only up to multiplication by a constant $t_i \in \mathbb{C}^*$.

In particular in the case $m = 3$, we get $p_1^{-1}([\Lambda]) = F_\Lambda$ is a plane, and if Λ has no zero column, we constructed n lines $\ell_{1,\Lambda}, ..., \ell_{n,\Lambda}$ where

$$\ell_{i,\Lambda} : r_{1,i}x_1 + r_{2,i}x_2 + r_{3,i}x_3 = 0.$$



(n) lines in $\mathbb{P}^2$

For $1 \leq i < j < k \leq n$, the lines $\ell_{i,\Lambda}, \ell_{j,\Lambda}$ coincide if and only if

$$\text{rank } p_{ij}([\Lambda]) := \begin{pmatrix} r_{1,i} & r_{1,j} \\ r_{2,i} & r_{2,j} \\ r_{3,i} & r_{3,j} \end{pmatrix} \leq 1,$$

and $\ell_{i,\Lambda}, \ell_{j\Lambda}$ and $\ell_{k\Lambda}$ meet at a point (or two coincide) if and only if

$$\begin{vmatrix} r_{1,i} & r_{1,j} & r_{1k} \\ r_{2,i} & r_{1,j} & r_{2,k} \\ r_{3,i} & r_{1,j} & r_{3,k} \end{vmatrix} = 0.$$

Theorem 5.2.1 *There exists an open set $U' \subset G(3, n)$ such that the geometric quotient $U'/(\mathbb{C}^*)^n$ represents the functor defined above.*

Proof: Let U' be the open subset in $G(3,n)$ given by the conditions $p_{ijk}([\Lambda]) \neq 0$ for all $[\Lambda] \in U'$ and all sets of indices $1 \leq i < j < k \leq n$. By the discussion above, the points $[\Lambda] \in U'$ represent planes $F_\Lambda \hookrightarrow \mathbb{P}^{n-1}$ such that the hyperplane sections $\ell_{i,\Lambda}$ satisfy the conditions in Definition 5.2.3.

Recall that for each $i \in \{1,...,n\}$, the equations of $\ell_{i,\Lambda}$ in F_Λ were defined uniquely only up to multiplication by a constant $t_i \in \mathbb{C}^*$. Indeed, the natural action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^{n-1}$ keeps the coordinate hyperplanes invariant. This action induces an action on $G(3,n)$ as described in Section 2.2 and hence on $F = F(1,3,n) \hookrightarrow \mathbb{P}^{n-1} \times G(3,n)$.

Hence each $t \in (\mathbb{C}^*)^n$ gives an isomorphism of $\mathbb{P}^{n-1}$ which by restriction induces an isomorphism

$$(F_\Lambda, \ell_{1,\Lambda}, ..., \ell_{n,\Lambda}) \to (F_{t\cdot\Lambda}, \ell_{1,t\cdot\Lambda}, ..., \ell_{n,t\cdot\Lambda}).$$

Note that U' is invariant under the action of $(\mathbb{C}^*)^n$ because

$$p_{ijk}(t\cdot[\Lambda]) = t_i t_j t_k p_{ijk}([\Lambda]) \neq 0,$$

for all $[\Lambda] \in U'$, where $t = (t_1, ..., t_n)$.

We will prove that $U'/(\mathbb{C}^*)^n$ represents our given functor.

First we note that by Theorem 4.2.5, U' is the set of points in $G(3,n)$ whose $(\mathbb{C}^*)^n$ orbits, when mapped via moment map, contain in the boundary all the external walls (walls which separate chambers). Therefore the points in U' are the stable points for every linearisation of the action of $(\mathbb{C}^*)^n$ on $G(3,n)$. Hence $U'/(\mathbb{C}^*)^n$ is a geometric quotient.

We claim that the family $(p_1^{-1}(U')/(\mathbb{C}^*)^n \to U'/(\mathbb{C}^*)^n, \ell_1, ..., \ell_n)$ is the universal family over $U'/(\mathbb{C}^*)^n$, for the given moduli problem. Equivalently, we wish to show that for any tuple

$$(p : \mathcal{F} \to B, D_1, D_2, ..., D_n)$$

satisfying the conditions in Definition 5.2.3; there is a unique map $\psi : B \to U'/(\mathbb{C}^*)^n$ such that

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & p_1^{-1}(U')/(\mathbb{C}^*)^n \\ \downarrow{\scriptstyle p} & & \downarrow \\ B & \xrightarrow{\psi} & U'/(\mathbb{C}^*)^n, \end{array}$$

is a fiber product i.e.

$$\mathcal{F} \cong \psi^*(p_1^{-1}(U')/(\mathbb{C}^*)^n).$$

$\mathcal{F}$ is a $\mathbb{P}^2$–family over B, whose hyperplane divisors satisfy the required properties. A specific torus bundle $\tilde{T} \to B$ will be constructed below so that it works. We first wish to construct a map $\iota : \tilde{T} \times_B \mathcal{F} \to \mathbb{P}^{n-1} = \text{Proj}(\mathbb{C}[y_1, ..., y_n])$ where y_i is a global section in $\mathcal{L}(D_i)$ for $i = 1, ..., n$ and $\tilde{T}$ is a torus bundle over B.

Each D_i is the zero locus of a global section in $\mathcal{L}(D_i)$. We only want one line bundle but we have n of them: $\mathcal{L}(D_1), \mathcal{L}(D_2), ..., \mathcal{L}(D_n)$ which might not all be the same. Note that on each fiber $\mathcal{F}_b \cong \mathbb{P}^2$ we have $L(D_i)_{|\mathcal{F}_b} \cong \mathcal{O}_{\mathcal{F}_b}(1)$ hence $\mathcal{L}(D_1) \otimes \mathcal{L}(D_i)^{-1}$ is trivial on the fibers $\mathcal{F}_b$ of p for each $i = 2, 3, ..., n$.

Let $\mathcal{L}_i = \mathcal{L}(D_1) \otimes \mathcal{L}(D_i)^{-1}$ for $i = 2, 3, ..., n$. We claim that $p_*(\mathcal{L}_i)$ are line bundles over B.

For $b \in B$ the map $\phi^1(b) : R^1p_*(\mathcal{L}_i) \otimes \mathbb{C}(b) \to H^1(\mathcal{F}_b, \mathcal{L}_{ib})$ ia a surjective map because

$H^1(\mathcal{F}_b, \mathcal{L}_{ib}) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$ for $i = 2, 3, ..., n$. From the Cohomology and Base Change Theorem A.0.15 the map $\phi^1(b)$ is an isomorphism for $i = 2, 3, ..., n$ which means that $R^1p_*(\mathcal{L}_i) \otimes \mathbb{C}(b) \cong 0$. Hence from part (b) of the same theorem $\phi^0(b)$ is also surjective where $\phi^0(b) : R^0p_*(\mathcal{L}_i) \otimes \mathbb{C}(b) \to H^0(\mathcal{F}_b, \mathcal{L}_{ib}) = \Gamma(\mathcal{F}_b, \mathcal{L}_{ib})$ and again from the same theorem part (a), $\phi^0(b)$ is an isomorphism. So

$$p_*(\mathcal{L}_i) \otimes \mathbb{C}(b) \cong \Gamma(\mathcal{F}_b, \mathcal{L}_{ib}) = \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}),$$

for $i = 2, 3, ..., n$. Since $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})$ is of dimension one, we can deduce that over each point $b \in B$ the fibers $p_*(\mathcal{L}_i) \otimes \mathbb{C}(b) = p_*(\mathcal{L}_i)_b$ have dimension equal to one.

We claim $p^*p_*(\mathcal{L}_i) \cong \mathcal{L}_i$ for $i = 2, 3, ..., n$. Indeed, as p^* is left adjoint to p_*, we have a morphism of line bundle $p^*p_*(\mathcal{L}_i) \to \mathcal{L}_i$, which is an isomorphism on the fibers of p, hence an isomorphism (by Nakayama's Lemma).

Now we would like to have a nowhere zero section in $p_*(\mathcal{L}_i)$ for $i = 2, 3, ..., n$. This does not necessarily happen on B but it is true for the pull-back of $p_*(\mathcal{L}_i)$ on

$$f : \tilde{T} := (p_*(\mathcal{L}_2) \setminus \text{ (zero section) }) \times_B \cdots \times_B (p_*(\mathcal{L}_n) \setminus \text{(zero section)}) \to B.$$

Hence $f^*(p_*(\mathcal{L}_i))$ has a nowhere zero section on $\tilde{T}$ for $i = 2, 3, ..., n$. In other word $f^*(p_*(\mathcal{L}_i)) \cong \mathcal{O}_{\tilde{T}}$, for $i = 2, 3, ..., n$.

$q_1^* f^*(p_*(\mathcal{L}_i))$ is trivial too where $q_1 : \tilde{T} \times_B \mathcal{F}$ is projection on the first component. On the other hand from the following fiber product diagram

$$\begin{array}{ccc} \tilde{T}\times_B \mathcal{F} & \xrightarrow{q_1} & \tilde{T} \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle f} \\ \mathcal{F} & \xrightarrow{p} & B, \end{array}$$

we have $q_1^* f^* p_* \mathcal{L}_i = q_2^* p^* p_* \mathcal{L}_i$, and from $p^* p_*(\mathcal{L}_i) \cong \mathcal{L}_i$ we have $q_2^*(\mathcal{L}_i)$ is trivial and therefore

$$q_2^*(\mathcal{L}(D_i)) \cong q_2^*(\mathcal{L}(D_1)),$$

for $i = 2, 3, ..., n$, and it has n global sections $s_1, ..., s_n$ induced by $D_1, ..., D_n$.

It means that we have the following map

$$\iota : \tilde{T} \times_B \mathcal{F} \to \mathrm{Proj}(\mathbb{C}[s_1, ..., s_n]),$$

which maps each fiber of q_1 into a linear projective subspace $\mathbb{P}^2 \subset \mathbb{P}^{n-1}$. As D_i are quotient divisors for the $(\mathbb{C}^*)n-1$ action on $\tilde{T} \times_B \mathcal{F}$, the map i is $(\mathbb{C}^*)n-1$-equivariant. Now the following diagram

$$\begin{array}{ccc} \tilde{T}\times_B \mathcal{F} & \xrightarrow{\iota} & \mathbb{P}^{n-1} \\ \downarrow{\scriptstyle q_2} & & \\ \tilde{T} & & , \end{array}$$

induces an equivariant map $\psi : \tilde{T} \to G(3, n)$. (Note that the $(\mathbb{C}^*)^{n-1}$–action is induced from the action on $\mathbb{P}^{n-1}$ and similarly the action on $\tilde{T}$). Hence from the universal property $\tilde{T} \times_B \mathcal{F} \cong \psi^* F$ and we have the following fiber product diagram

$$\begin{array}{ccc} \tilde{T}\times_B \mathcal{F} & \longrightarrow & F \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle \pi} \\ \tilde{T} & \xrightarrow{\psi} & G(3,n), \end{array}$$

and after taking a quotient by $(\mathbb{C}^*)^n$ we have the following fiber product

$$\begin{array}{ccc} \mathcal{F} \cong (\tilde{T}\times_B \mathcal{F})/(\mathbb{C}^*)^{n-1} & \longrightarrow & F/\!\!/T = \pi^{-1}(U')/(\mathbb{C}^*)^{n-1} \\ \downarrow & & \downarrow \\ B \cong \tilde{T}/(\mathbb{C}^*)^{n-1} & \longrightarrow & G(3,n)/\!\!/(\mathbb{C}^*)^n = U'/(\mathbb{C}^*)^{n-1}, \end{array}$$

as all the quotient maps are $(\mathbb{C}^*)n-1$-fiberation. □

This moduli space $U'/(\mathbb{C}^*)^n$ is not compact. We would like to find a compactification which is canonical, i.e. such that if $(\ell_1, \ldots, \ell_n)$ is an arrangement of lines in plane parameterized by a point in this compactification, then for any permutation $\sigma \in S_n$, the tuple $(\ell_{\sigma(1)}, \ell_{\sigma(2)}, \ldots, \ell_{\sigma(n)})$ is another arrangement of lines in a plane also parameterized by a point in this compactification.

5.3 The Moduli Problem Represented by GIT quotients of the Grassmannian Variety $G(3,n)$ by the maximal torus

Dolgachev and Hu [3] and Thaddeus [9] proved that there are finitely many chambers and walls in the image of a moment map. For the moment map associated to the action of $(\mathbb{C}^*)^n$ on $G(m,n)$, let's denote the set of chambers by $\{C_k\}_k$ and the set of walls by $\{W_j\}_j$, where $j \in \{1, 2, ..., q\}$. The image of each orbit is a convex hull. The vertices of this convex hull correspond to the lines in general positions. From Corollary 4.2.6, for each wall W_l there exists a partition $\{1, ..., n\} = I_l \cup J_l$ and a partition $\{1, ..., m\} = I'_l \cup J'_l$ such that

$$(\mu^{-1}(W_l))^{\mathbb{C}^*} = \{[Q] \in G(m,n) : \quad Q = (r_{ij})_{m\times n}, \text{ with } r_{ij} = 0 \text{ if } (i \in I'_l \wedge j \in I_l) \vee (i \in J'_l \wedge j \in J_l)\},$$

where $\mathbb{C}^*$ is the unique subtorus associated to W_l and μ is the moment map. From Corollary 4.2.8 the set $Y_l^+ := \{x \in G(m,n) : \lim_{t\to\infty} \lambda(t).x \in (\mu^{-1}(W_l))^{\mathbb{C}^*}\}$ is as below:

$$Y_l^+ = \{[Q] \in G(m,n) : Q = (r_{ij})_{1\le i\le m, 1\le j\le n}, r_{ij} = 0 \text{ if } i \in I'_l \wedge j \in I_l\},$$

and the set $\{x \in G(m,n) : \lim_{t\to 0} \lambda(t).x \in (\mu^{-1}(W_l))^{\mathbb{C}^*}\}$ is as follow

$$Y_l^- = \{[Q] \in G(m,n) : Q = (r_{ij})_{1\le i\le m, 1\le j\le n}, r_{ij} = 0 \text{ if } i \in J'_l \wedge j \in J_l\},$$

and each chamber having W_l as a wall is contained in only one of $\mu(Y_l^+)$ and $\mu(Y_l^-)$. For a chamber C_k consider

$$\mathcal{A}_k^+ = \{I_l : \mu(Y_l^+) \supset C_k\},$$

and

$$\mathcal{A}_k^- = \{J_{l'} : \mu(Y_{l'}^-) \supset C_k\}.$$

Similarly we can define $\mathcal{A}^+_{\mu(O(x))}$ and $\mathcal{A}^-_{\mu(O(x))}$ for $\mu(O(x))$ for a point x.

Remark 5.3.1 *Consider an arbitrary orbit $O(x)$ and a chamber C_k. From Proposition 4.2.10, we have $C_k \subset \mu(O(x))$ iff for every $I \in \mathcal{A}^+_{\mu(O(x))}$ there exist $I' \in \mathcal{A}_k^+$ such that $I \subset I'$ and for every $J \in \mathcal{A}^-_{\mu(O(x))}$ there exist $J' \in \mathcal{A}_k^-$ such that $J \subset J'$.*

Previously we have defined the flag variety $F = F(1,m,n)$ over $G(m,n)$ which is the universal family over $G(m,n)$.

F provides a $\mathbb{P}^{m-1}$–fibration over $G(m,n)$. Let $D_i = p_2^*(\{y_i = 0\})$ for $i = 1,2,...,n$. Then for $[\Lambda] \in G(m,n)$ such that $\Lambda = \begin{pmatrix} r_{1,1} & & r_{1,n} \\ \vdots & \ddots & \vdots \\ r_{m,1} & & r_{m,n} \end{pmatrix} \in M(m,n)$, let $\ell_{i,\Lambda} = F_\Lambda \cap D_i$ where $F_\Lambda \cong \mathbb{P}^{m-1}$ is the fiber of F over $[\Lambda]$. Now let $[x_1 : x_2 : ... : x_m]$ be homogeneous coordinates on F_Λ. Then the restriction of p_2 to F_Λ is given by

$$F_\Lambda \to \mathbb{P}^n$$
$$[x_1 : x_2 : ... : x_m] \mapsto [\sum_{j=1}^{m} r_{j,i}x_j]_{i=1,...,n},$$

and $\sum_{j=1}^m r_{j,i}x_j = p_2^*(y_i)$ represents the equation of $\ell_{i,\Lambda}$. Note that this equation is unique only up to multiplication by a constant $t_i \in \mathbb{C}^*$.

Note that this thesis works with $G(3,n)$. Hence $p_1^{-1}([\Lambda]) = F_\Lambda$ is a plane, and we constructed n lines $\ell_{1,\Lambda}, ..., \ell_{n,\Lambda}$ where

$$\ell_{i,\Lambda} : r_{1,i}x_1 + r_{2,i}x_2 + r_{3,i}x_3 = 0.$$

We will exclude the cases where $r_{1,i} = r_{2,i} = r_{3,i} = 0$ as such Λ are maped in the boundary of $\mu(G(m,n)$ and are not relevant to our analysis. Let's consider a wall W_l of a fixed chamber. Without loss of generality consider $I'_l = \{1,2\}$ and $J'_l = \{3\}$. For $r, s \in I_l$ and all $i \in I'_l$ we have $r_{ir} = r_{is} = 0$ i.e. $\ell_{r,\Lambda} : x_3 = 0$ and $\ell_{s,\Lambda} : x_3 = 0$. For $t \in J_l$ we have $\ell_{t,\Lambda} : r_{1t}x_1 + r_{2t}x_2 = 0$ as $r_{3t} = 0$.

In general, whenever we have a partition $I \cup J = \{1,2,3\}$, we will assume $|I| = 2$ and $|J| = 1$.

Definition 5.3.1 ***The Moduli Problem for a Chamber.*** *For a fixed chamber C_k con-*

sider the functor

$$\begin{array}{rcl} \mathcal{C}_k : \mathfrak{Sch} & \to & \mathfrak{Set} \\ B & \mapsto & [(\pi : P \to B, D_1, D_2, ..., D_n)], \end{array}$$

associating to each scheme B an isomorphism class of tuples $(\pi : P \to B, D_1, ..., D_n)$ consisting of a flat morphism $\pi : P \to B$ whose fibers P_b are projective planes $\mathbb{P}^2$, and for each $i \in \{1, ..., n\}$, a divisor D_i of P whose intersection $\ell_{i,b}$ with each fiber P_b is a line $\ell_{i,b} \subset P_b$ such that one of the following cases holds:

- *If $\ell_{i_1,b} = \ell_{i_2,b} = ... = \ell_{i_c,b}$, then $\{i_1, i_2, ..., i_c\} \subseteq I_h$ for some $I_h \in \mathcal{A}_k^+$.*
- *If all the lines $\ell_{j_1,b}, \ell_{j_2,b}, ..., \ell_{j_c,b}$ have a point in common, then $\{j_1, j_2, ..., j_c\} \subseteq J_l$ for some $J_l \in \mathcal{A}_k^-$.*

Two tuples $(\pi : P \to B; D_1, D_2, ..., D_n)$ and $(\pi' : P' \to B'; D'_1, D'_2, ..., D'_n)$ are isomorphic to each other if there exists a map $\phi : B' \to B$ and an isomorphism $\tilde{\phi} : P' \to \phi^*(P)$

$$\begin{array}{ccc} P' \cong \phi^*(P) & & P \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\phi} & B \end{array}$$

such that $\tilde{\phi}$ restricts to an isomorphism $D'_i \cong \phi^*(D_i)$ for each $i \in \{1, ..., n\}$.

Theorem 5.3.2 *For every chamber C_k, there exists an open set $U''_k \subset G(3, n)$ such that the geometric quotient $U''_k/(\mathbb{C}^*)^n$ represents the functor $\mathcal{C}_k$ defined above.*

Proof: Let

$$U''_k = (\mathbb{C}^*)^n \mu^{-1}(C_k^\circ),$$

where C_k° is the interior of the chamber C_k. Thus U''_k is the set of points $b \in G(3, n)$ with the property that the intersections of the divisors $D_i = p_2^*(\{y_i = 0\})$ with the fiber F_b, for $i = 1, 2, ..., n$ are the lines, $\ell_{i,b}$, which satisfy the two conditions listed in Definition 5.3.1. The rest of the proof is similar to the proof of Theorem 5.2.1. □

Now it is time to describe a flip of the universal bundle (i.e. the flag $F := F(1, 3, n)$) over the Grassmannian variety $G(3, n)$. The flip corresponds to passing through a wall from a chamber C_k to a chamber C_j. It corresponds to a birational map as below:

$$U''_k/(\mathbb{C}^*)^n \dashrightarrow U''_j/(\mathbb{C}^*)^n.$$

Lemma 5.3.3 *If $\mathbb{C}^*$ acts with two different weights on $\mathbb{P}^n$ it has two fixed loci of complementary dimensions.*

Proof: For an arbitrary $\lambda \in \mathbb{C}^*$ and $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n$

$$\lambda.x = (\lambda^\alpha.x_0 : \cdots : \lambda^\alpha.x_k : \lambda^\beta.x_{k+1} : \cdots : \lambda^\beta.x_n),$$

where $\alpha < \beta$ then the fixed loci is $\{x \in \mathbb{P}^n; x = (x_0 : \cdots : x_k : 0 : \cdots : 0)\} \cup \{x \in \mathbb{P}^n; x = (0 : \cdots : 0 : x_{k+1} : \cdots : x_n)\}$. □

For the moment map μ associated to the action of $(\mathbb{C}^*)^n$ on $G(m,n)$ and a wall W, by Theorem 4.2.3 there exists a unique one-parameter subgroup

$$\begin{aligned} \lambda : \mathbb{C}^* &\to (\mathbb{C}^*)^n \\ t &\mapsto (t^{a_1}, \ldots, t^{a_n}), \end{aligned}$$

such that W is the image through μ of the locus fixed by $\mathbb{C}^*$. Recall that the flag variety over the Grassmannian is $F = \mathbb{P}(\mathcal{G})$ where $\mathcal{G}$ is the universal sub-bundle, which is a rank 3 vector bundle and $\mathbb{C}^*$ acts on each fiber of

$$\mathcal{G}|_{(\mu^{-1}(W))^{\mathbb{C}^*}} := \mathcal{G}_W,$$

and so we can write it as decomposition of eigenspaces

$$\mathcal{G}_W = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where $\mathcal{V}_1$ is a rank one and $\mathcal{V}_2$ is a rank two vector bundle. Note that $\mathbb{P}(\mathcal{V}_1)$ and $\mathbb{P}(\mathcal{V}_2)$ are fixed by the $\mathbb{C}^*$.

Recall that for each wall W there exists a partition $\{1,...,n\} = I \cup J$ and a partition $\{1,...,m\} = I' \cup J'$ and a unique sub-torus $\mathbb{C}^*$ such that

$$(\mu^{-1}(W))^{\mathbb{C}^*} = \{[Q] \in G(m,n) : \ Q = (a_{ij})_{m\times n}, \text{ with } a_{ij} = 0 \text{ if } (i \in I' \wedge j \in I) \vee (i \in J' \wedge j \in J)\}.$$

From Lemma 5.3.3 the fixed locus of the fiber F_x over each point $x \in \mu^{-1}(W)^{\mathbb{C}^*}$ is isomorphic to the union of a point, $\mathbb{P}(\mathcal{V}_1)_x$, and a line $\mathbb{P}(\mathcal{V}_2)_x$, in the projective plane F_x .

Consider a fixed linearisation on $G(3, n)$. Recall that in Lemma 5.1.1 we proved that the pre-image of stable points via π is a set F^s, the set of stable points in F for a suitably chosen linearisation on $F = F(1, 3, n)$. Hence for a chamber C_i and projection on first component p_1, $\pi^{-1}((\mathbb{C}^*)^n \mu^{-1}(C_i^\circ))$ is the set of stable points for an induced linearisation on F. Let's denote by U_i the GIT quotient of F by $(\mathbb{C}^*)^n$ corresponding to that linearisation and by M_i, the GIT quotient of the Grassmannian variety. In the image of moment maps to go from one chamber to a next one (which shares a boundary with it) is the same as having a birational morphism between their corresponding GIT quotients which is called a flip. Dolgachev and Hu in [3] for general case and Taddeus [9] for an specific case showed that a flip corresponds to a blow-up followed by a blow down. Mustata in [7] has described it in more details.

Lemma 5.3.4 *Assume two chambers C_i and C_j share a wall W_l. Unlike C_i and C_j, in the image of moment map for F and $(\mathbb{C}^*)^n$, the chambers corresponding to U_i and U_j do not share a common wall but there is a third chamber between them, whose corresponding GIT quotient we denote by U_{ij}^0.*

Proof: Denote $T = (\mathbb{C}^*)^n$, $X = G(3, n)$, $X_i^{ss} = T.\mu^{-1}(C_i)$, $M_i = X_i^{ss} /\!\!/ T$, $F_i^{ss} = \pi^{-1}(T.\mu^{-1}(C_i))$, $U_i = F_i^{ss} /\!\!/ T$, with $p_i : U_i \to M_i$. Let $\mu' : F \to \mathbb{R}^n$ denote the moment map for F with the T–action and $\widetilde{C_i} = \bigcap_{y' \in F_i^{ss}} \mu'(T.y')$ is the chamber corresponding to U_i. Apply similar notations for j instead of i.

Let $Y = \mu^{-1}(W_l)^{\mathbb{C}^*}$. We may assume $X_i^{ss} \setminus X_j^{ss} \subset Y^- = \{y; \lim_{t\to\infty} t.y \in Y\}$ and $X_j^{ss} \setminus X_i^{ss} \subset Y^+ = \{y; \lim_{t\to 0} t.y \in Y\}$. Indeed if $\mu(Tx \supset C_i$ but $C_j \not\subset \mu(Tx)$ then W_l is in the boundary of $\mu(Tx)$, hence $\lim_{t\to\infty} t.x \in Y$. We know that $\pi^{-1}(Y)$ contains two $\mathbb{C}^*$–fixed loci $\mathbb{P}(\mathcal{V}_1)$ and $\mathbb{P}(\mathcal{V}_2)$. Define $\widetilde{C_{ij}} = \mu'(\pi^{-1}(Y))$, and $W_1 = \mu'(\mathbb{P}(\mathcal{V}_1))$ and $W_2 = \mu'(\mathbb{P}(\mathcal{V}_2))$.

Consider the sub-torus $\mathbb{C}^*$ which fixes Y and denote by $\mu'_{\mathbb{C}^*}$ the moment map for the induced action of $\mathbb{C}^*$ on the flag variety F. From Equation (4.1) we have $\mu'_{\mathbb{C}^*} = (d\lambda_{t=1})^t \circ \mu'$ where $\lambda' : \mathbb{R}^n \to \mathbb{R}$ is projection map induced by the embedding $\mathbb{C}^* \to (\mathbb{C}^*)^n$. Let $\lambda' = (d\lambda_{t=1})^t$.

If Z is a locus fixed by $\mathbb{C}^*$ we denote

$$Z^- = \{z; \lim_{t\to\infty} t.z \in Z\}$$

and

$$Z^+ = \{z; \lim_{t\to 0} t.z \in Z\}.$$

Claims:

(a) $\lambda'(\widetilde{C_{ij}}) = (\mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1)), \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_2)))$.

(b) $\lambda'(\widetilde{C_i}) = (a, \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1)))$ for some $a \in \mathbb{R}$, $a < \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1))$.

(c) $\lambda'(\widetilde{C_j}) = (\mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1)), b)$ for some $b \in \mathbb{R}$, $b > \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1))$.

(d) $\lambda'(\widetilde{C_{ij}})$ is either a chamber in $\mu'(F)$ or a union of chambers each having walls contained in W_1 or W_2 i.e. $\widetilde{C_{ij}} \subset \mu'(T.y')$ for some y' s.t. $W_1 \subset \overline{\mu'(T.y')}, W_2 \subset \overline{\mu'(T.y')}$.

Proof of (a): Every point $y' \in \pi^{-1}(Y)$ satisfies $\lim_{t\to 0} t.y' \in \mathbb{P}(\mathcal{V}_1)$ and $\lim_{t\to\infty} t.y' \in \mathbb{P}(\mathcal{V}_2)$, for the given $\mathbb{C}^*$–action on F. Claim (a) follows from the fact that $\mu : \mathbb{C}^* y' \to \mathbb{R}$ factors through an embedding of $\mathbb{C}^* y'/S' \hookrightarrow \mathbb{R}$.

Proof of (b): We first show $\exists y' \in F_i^{ss}$ s.t. $\lim_{t\to\infty} t.y' \in \mathbb{P}(\mathcal{V}_1)$. Indeed as $X_i^{ss} \backslash X_j^{ss} \subset Y^-$ and $\pi^{-1}(Y)^{\mathbb{C}^*} = \mathbb{P}(\mathcal{V}_1) \cup \mathbb{P}(\mathcal{V}_2)$, we have

$$F_i^{ss} \setminus F_j^{ss} \subset \mathbb{P}(\mathcal{V}_1)^- \cup \mathbb{P}(\mathcal{V}_2)^-. \tag{5.1}$$

Moreover $F_i^{ss} \cap \mathbb{P}(\mathcal{V}_2)^- = \mathbb{P}(\mathcal{V}_2)^- \setminus \pi^{-1}(Y)$.

We knew that $\mathbb{P}(\mathcal{V}_2)^- \setminus \mathbb{P}(\mathcal{V}_2)/\!\!/\mathbb{C}^*$ is a projective bundle, compact and $\pi^{-1}(Y) \setminus \mathbb{P}(\mathcal{V}_2)/\!\!/\mathbb{C}^*$ is a $\mathbb{P}^1$–bundle. Hence $(\mathbb{P}(\mathcal{V}_2)^- \setminus \pi^{-1}(Y))/\!\!/\mathbb{C}^*$ is not compact, and in consequence $F_i^{ss} \cap \mathbb{P}(\mathcal{V}_2)^- = \mathbb{P}(\mathcal{V}_2)^- \setminus \pi^{-1}(Y) \neq \mathbb{P}(\mathcal{V}_2)^-$.

As the $\lim_{t\to\infty} t.y \in \mathbb{P}(\mathcal{V}_1) \cup \mathbb{P}(\mathcal{V}_2)$ for any $y \in \pi^{-1}(Y)$ we have $\exists y' \in F_i^{ss}$ such that $\lim_{t\to\infty} t.y' \in \mathbb{P}(\mathcal{V}_1)$. Moreover we saw in Equation (5.1) that for all $y' \in F_i^{ss} \setminus F_j^{ss}$, we have $\mu'_{\mathbb{C}^*}(T.y') = (a_{y'}, \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1)))$ or $\mu'_{\mathbb{C}^*}(T.y') = (a_{y'}, \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_2)))$. As $\mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1))) < \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_2)))$, it follows that

$$\lambda'(\tilde{C}_i) = \bigcap_{y' \in F_i^{ss}} \mu'_{\mathbb{C}^*}(T.y') = (a, \mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_1))).$$

Note that as the map $\mu_{\mathbb{C}^*} : \mathbb{C}^* x \to \mathbb{R}$ factors through an embedding of $\mathbb{C}^* x/S^1$ in $\mathbb{R}$ and $\lim_{t\to 0} \mu(tx) < \lim_{t\to\infty} \mu(tx)$ we have $\mu(tx) < \lim_{t\emptyset\infty} tx$.

Proof of (c): It is similar to the proof of (b).

Proof of (d): Let $y' \in F$ and $\pi(y') = y \in X$. If $W_1^\circ \cap \mu'(T.y') \neq \emptyset$ and $y' \notin F_i^{ss}$ then either $T.y' \subset \pi^{-1}(Y)$ or $C_j \subset \mu(T.y)$. Equivalently, $y' \in \pi^{-1}(Y)$ or $T.y \subset X_j^{ss}$. The last inclusion implies $T.y' \subset F_j^{ss}$. In both cases, using $(a), (b), (c)$ and the fact that W_2 is the wall sent to $\mu'_{\mathbb{C}^*}(\mathbb{P}(\mathcal{V}_2))$, we have $W_2 \cap \widetilde{\mu'(T.y')} \neq \emptyset$.

□

Correspondingly there is a birational map between U_i and U_j which we will shortly show it made of two flips.

Consider an affine open set $U_{uvw} = (\{x_{uvw} \neq 0\})$ which corresponds to all the elements in $G(3,n)$ whose minor of columns (u,v,w) is non-zero. From Corollary 4.2.7 we have

$$\begin{aligned}(\pi^{-1}(\mu^{-1}(W)^{\mathbb{C}^*} \bigcap U_{uvw}))^{\mathbb{C}^*} = &\mathbb{P}_{(\mu^{-1}(W))^{\mathbb{C}^*} \bigcap U_{uvw}}(\mathcal{V}_1) \bigsqcup \mathbb{P}_{(\mu^{-1}(W))^{\mathbb{C}^*} \bigcap U_{uvw}}(\mathcal{V}_2)\\ =&((\mu^{-1}(W))^{\mathbb{C}^*} \bigcap U_{uvw}) \times (\mathbb{P}^0 \bigsqcup \mathbb{P}^1) \subset ((\mu^{-1}(W))^{\mathbb{C}^*} \bigcap U_{uvw}) \times \mathbb{P}^2,\end{aligned} \tag{5.2}$$

where $\pi : F \to G(3,n)$ is the natural projection. Hence from the description of flags in Example 5.2.1, locally on U_{uvw}, we have the following

$$\begin{aligned}\mathbb{P}(\mathcal{V}_1)^+ \cap \pi^{-1}(U_{uvw}) :=&\{y = ([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2;\\ &[Q'] := \lim_{t\to 0} \lambda(t).[Q] \in (\mu^{-1}(W))^{\mathbb{C}^*}, \lim_{t\to 0} \lambda(t).y \in \mathbb{P}(\mathcal{V}_1)_{[Q']}\}\\ =&\{([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2 :\\ &v_3 \neq 0, Q = (a_{ij})_{1\le i\le 3, 1\le j\le n-3}, a_{ij} = 0 \text{ if } i \in J' \wedge j \in J\},\end{aligned} \tag{5.3}$$

which are the $\mathbb{C}^*$–orbits coming out of the first fixed locus $\mathbb{P}(\mathcal{V}_1)$, and similarly

$$\begin{aligned}\mathbb{P}(\mathcal{V}_1)^- \cap \pi^{-1}(U_{uvw}) :=&\{y = ([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2;\\ &[Q'] := \lim_{t\to \infty} \lambda(t).[Q] \in (\mu^{-1}(W))^{\mathbb{C}^*}, \lim_{t\to \infty} \lambda(t).y \in \mathbb{P}(\mathcal{V}_1)_{[Q']}\}\\ =&\{([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2 : v_1 = v_2 = 0,\\ &Q = (a_{ij})_{1\le i\le 3, 1\le j\le n-3}, a_{ij} = 0 \text{ if } i \in I' \wedge j \in I\},\end{aligned} \tag{5.4}$$

which are the $\mathbb{C}^*$–orbits going into the first fixed loci, $\mathbb{P}(\mathcal{V}_1)$. For $\mathbb{P}(\mathcal{V}_2)$ we have the following

$$\begin{aligned}\mathbb{P}(\mathcal{V}_2)^+ \cap \pi^{-1}(U_{uvw}) :=& \{y = ([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2;\\ &[Q'] := \lim_{t\to 0} \lambda(t).[Q] \in (\mu^{-1}(W))^{\mathbb{C}^*}, \lim_{t\to 0} \lambda(t).y \in \mathbb{P}(\mathcal{V}_2)_{[Q']}\}\\ =&\{([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2 :\\ &v_3 = 0, Q = (a_{ij})_{1\le i\le 3, 1\le j\le n-3}, a_{ij} = 0 \text{ if } i \in I' \wedge j \in I\},\end{aligned} \tag{5.5}$$

which are the $\mathbb{C}^*$–orbits coming out of the first fixed loci, $\mathbb{P}(\mathcal{V}_1)$, and similarly

$$\begin{aligned}\mathbb{P}(\mathcal{V}_2)^- \cap \pi^{-1}(U_{uvw}) :=& \{y = ([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2;\\ &[Q'] := \lim_{t\to \infty} \lambda(t).[Q] \in (\mu^{-1}(W))^{\mathbb{C}^*}, \lim_{t\to \infty} \lambda(t).y \in \mathbb{P}(\mathcal{V}_2)_{[Q']}\}\\ =&\{([Q], (v_1 : v_2 : v_3)) \in \mathbb{A}^{3(n-3)} \times \mathbb{P}^2 : v_1 \neq 0 \vee v_2 \neq 0,\\ &Q = (a_{ij})_{1\le i\le 3, 1\le j\le n-3}, a_{ij} = 0 \text{ if } i \in I' \wedge j \in I\}.\end{aligned} \tag{5.6}$$

The following diagram illustrates the flips between the GIT quotients of $G(m,n)$ and between the GIT quotients of F:

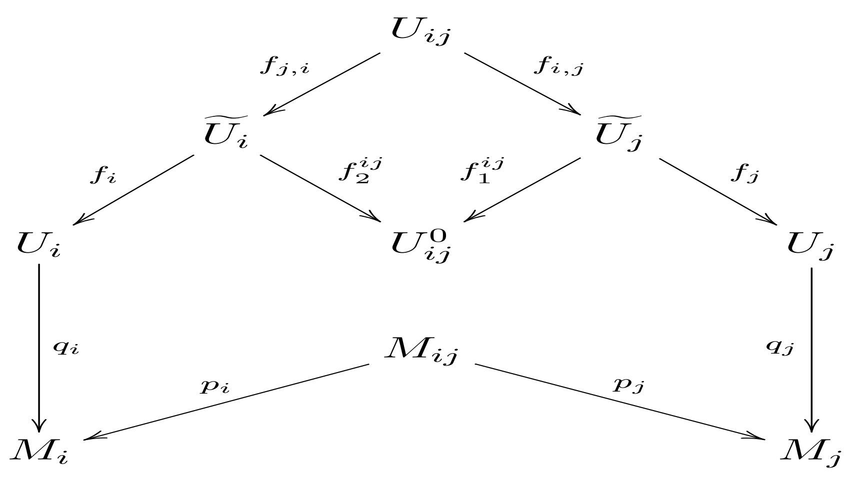


Figure 5.1: Flip of the universal families over two GIT quotients

where
$M_i = T.\mu^{-1}(C_i^{\circ})/T$,

$M_j = T.\mu^{-1}(C_j^\circ)/T$,
$M_{ij} = Bl_{Y^-/T}M_i = Bl_{Y^+/T}M_j$,
$U_i = \pi^{-1}(T.\mu^{-1}(C_i^\circ))/T$,
$U_j = \pi^{-1}(T.\mu^{-1}(C_j^\circ))/T$,
$\widetilde{U_i} = Bl_{\mathbb{P}(\mathcal{V}_1)^-/T}U_i$,
$\widetilde{U_j} = Bl_{\mathbb{P}(\mathcal{V}_2)^+/T}U_j$, and

$$U_{ij} = Bl_{\widetilde{\mathbb{P}(\mathcal{V}_1)^+/T}}\widetilde{U_j} = Bl_{\widetilde{\mathbb{P}(\mathcal{V}_2)^-/T}}\widetilde{U_i} = \widetilde{U_i} \times_{U_{ij}^0} \widetilde{U_j}. \tag{5.7}$$

Detailed description for these type of flips can be found in first section (and also Example 1.16) of Thaddeus [9].

Indeed,

$$\begin{aligned}\mathbb{P}(\mathcal{V}_1)^+ \cap \mathbb{P}(\mathcal{V}_2)^- \cap \pi^{-1}(U_{uvw}) = \{&([(a_{ij})_{1\leq i\leq m, 1\leq j\leq(n-m)}], (v_1 : v_2 : v_3)) \in \\ &\mathbb{A}^{m(n-m)} \times \mathbb{P}^2; (v_1 \neq 0 \vee v_2 \neq 0) \wedge (v_3 \neq 0) \wedge \\ &(a_{ij} = 0 \text{ if } (i \in J' \wedge j \in J) \vee (i \in I' \wedge j \in I))\} \\ &= \pi^{-1}(Y \cap U_{uvw}) \setminus (\mathbb{P}(\mathcal{V}_1) \cup \mathbb{P}(\mathcal{V}_2)).\end{aligned} \tag{5.8}$$

On the other hand

$$\begin{aligned}\dim(\mathbb{P}(\mathcal{V}_1)^+) &= \dim(\mathbb{A}^{3|I|+|I'||J|} \times \mathbb{P}^2) = 3|I| + 2|J| + 2, \\ \dim(\mathbb{P}(\mathcal{V}_2)^-) &= \dim(\mathbb{A}^{3|J|+|J'||I|} \times \mathbb{P}^2 \setminus \{(0:0:1)\}) = 3|J| + |I| + 2,\end{aligned} \tag{5.9}$$

and also

$$\begin{aligned}\dim(\mathbb{P}(\mathcal{V}_1)^+ \cap \mathbb{P}(\mathcal{V}_2)^-) =& \dim(\mathbb{A}^{|J'||J|+|J||I'|} \times (\mathbb{P}^2 \setminus (\mathbb{P}^1 \cup \{(0:0:1)\})) \\ =& 2|J| + |I| + 2.\end{aligned} \tag{5.10}$$

Hence

$$\begin{aligned}\dim(\mathbb{A}^{3(n-3)} \times \mathbb{P}^2) = \dim(\mathbb{P}(\mathcal{V}_1)^+) + \dim(\mathbb{P}(\mathcal{V}_2)^-) - \\ \dim(\mathbb{P}(\mathcal{V}_1)^+ \cap \mathbb{P}(\mathcal{V}_2)^-) = 3(n-3) + 2.\end{aligned} \tag{5.11}$$

Moreover, from the local description of $\mathbb{P}(\mathcal{V}_1)^+$ and $\mathbb{P}(\mathcal{V}_2)^-$ above, we see that these intersect transversely at $\pi^{-1}(Y) \setminus (\mathbb{P}(\mathcal{V}_1) \cup \mathbb{P}(\mathcal{V}_2))$. Transverseness is preserved when taking GIT quotients by T, which explains Equation (5.7).

Next, we construct a map $q_{ij} : U_{ij} \to M_{ij}$ such that $p_j \circ q_{ij} = \widetilde{q_i \circ f_i} \circ f_{ji}$ and $\widetilde{p_i \circ q_{ij}} = q_j \circ f_j \circ f_{ij}$. Note that $U_{ij} = Bl_{\widetilde{\mathbb{P}(\mathcal{V}_2)^-}/T} Bl_{\mathbb{P}(\mathcal{V}_1)^-/T} U_i$ and $\mathbb{P}(\mathcal{V}_2)^-/T = \pi^{-1}(Y^-)/T$ while $\mathbb{P}(\mathcal{V}_1)^-/T \subset \pi^{-1}(Y^-)/T \subset U_i$ are regular embedding. It is because some of the orbits which go into $\mathbb{P}(\mathcal{V}_2)$, pass through $\mathbb{P}(\mathcal{V}_1)$ first. Hence we may apply the following lemma:

Lemma 5.3.5 *Consider X, Y and Z smooth manifolds, with $Z \subset Y \subset X$ regular embeddings. Then*

$$Bl_{\tilde{Y}} Bl_Z X \cong Bl_{\pi^{-1}(Z)} Bl_Y X,$$

where $\pi : Bl_Y X \to X$ is the blow-up map and $\tilde{Y} = Bl_Z Y$ is the strict transform of Y in $Bl_Z X$. Moreover, the exceptional divisor in $Bl_{\pi^{-1}(Z)} Bl_Y X$ is a $\mathbb{P}^k$–bundle over $\pi^{-1}(Z)$ where $k = codim_Y Z$.

Proof: Working locally, assume $X = Spec(A)$, $Y = V(I_Y)$ where $I_Y = (t_1, \ldots, t_d)$, $Z = V(I_Z)$ where $I_Z = (t_1, \ldots, t_d, t_{d+1}, \ldots, t_n)$ and $\mathbb{P}^{n-1} = \mathrm{Proj}(\mathbb{C}[T_1, \ldots, T_n])$. Then

$$\begin{aligned} Bl_Z X &= \tilde{X} \\ &= V(t_i T_j - t_j T_i : i, j \in \{1, ..., n\}, i < j) \\ &\subset X \times \mathbb{P}^{n-1} = \mathrm{Proj}(A[T_1, \ldots, T_n]), \end{aligned} \tag{5.12}$$

and

$$\tilde{Y} = V(T_1, ..., T_d, \bar{t_i} T_j - \bar{t_j} T_i : d < i < j) \subset Y \times \mathbb{P}^{n-1},$$

for $\bar{t_i} = f(t_i)$ where $f : A \to B = A/I_Y$. Now let $U_k \subset \tilde{X}$ a standard affine cover; $(T_k \neq 0)$ and $T_j/T_k = x_j$;

$$\begin{aligned} U_k &= Spec(A[x_1, \ldots, \hat{x_k}, \ldots, x_n]/(t_i - t_k x_i : i \neq k)) \\ &= Spec(A[\frac{t_1}{t_k}, ..., \widehat{\frac{t_k}{t_k}}, ..., \frac{t_n}{t_k}]). \end{aligned}$$

We consider the case $k \leq d$, as the other case is quite trivial. In this case, in U_k we have

$$I_{\tilde{Y}} = (\frac{t_1}{t_k}, ..., \widehat{\frac{t_k}{t_k}}, ..., \frac{t_d}{t_k}),$$

if $k \leq d$. Then an affine cover for

$$Bl_{\tilde{Y}} \tilde{X},$$

is as below:

$$\mathrm{Spec}(A[\frac{t_1}{t_k}, ..., \widehat{\frac{t_k}{t_k}}, ..., \frac{t_n}{t_k}][\frac{t_1}{t_s}, ..., \widehat{\frac{t_s}{t_s}}, ..., \widehat{\frac{t_k}{t_s}}, ..., \frac{t_d}{t_s}]),$$

for some $s \in \{1, ..., d\}$. To describe $Bl_{\pi^{-1}(Z)} Bl_Y X$ we first describe $Bl_Y X$ locally as

$$Spec(A[\frac{t_1}{t_s}, ..., \widehat{\frac{t_s}{t_s}}, ..., \frac{t_d}{t_s}]).$$

If $k \leq d$, then $I_{\pi^{-1}(Z)}$ is generated by $t_S x_i$ where $x_i = \frac{t_i}{t_s}$ for $i \in \{1, ..., d\}$ and t_l for $l \in \{d+1, ..., n\}$. When $k \leq d$, an affine cover for $Bl_{\pi^{-1}(Z)} Bl_Y X$ is

$$\mathrm{Spec}(A[\frac{t_1}{t_s}, ..., \widehat{\frac{t_s}{t_s}}, ..., \frac{t_n}{t_s}][\frac{t_1}{t_k}, ..., \widehat{\frac{t_k}{t_k}}, ..., \widehat{\frac{t_s}{t_k}}, ..., \frac{t_n}{t_k}]),$$

which is the same as the affine cover of $Bl_{\tilde{Y}} \tilde{X}$ descried above which proves the theorem.

Finally, $k = codim_{Bl_Y X}(\pi^{-1}(Z)) - 1 = codim_X Z - codim_X Y + 1 - 1 = codim_Y Z$.

□

The last Lemma gives us the following Remark.

Corollary 5.3.6 *The following diagram is commutative i.e.*

$$U_{ij} = Bl_{\widetilde{\mathbb{P}(\mathcal{V}_2)^-}/T} Bl_{\mathbb{P}(\mathcal{V}_1)^-/T}(U_i) \cong Bl_{\rho^{-1}(\mathbb{P}(\mathcal{V}_1)^-/T)} p_i^* U_i,$$

where on the left hand side the second blow-up is along the whole fibers over the preimage of the wall which creates the flip.

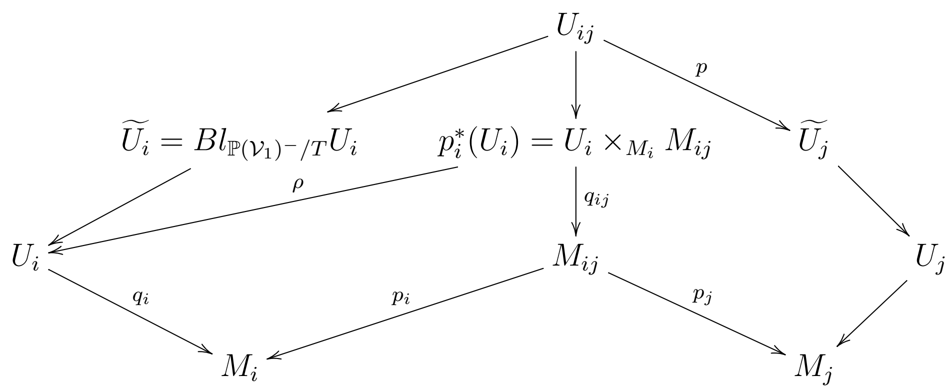


Figure 5.2: Flip of universal family over two GIT quotients

Indeed, we apply the previous Lemma to $\mathbb{P}(\mathcal{V}_2)^-/T \subset \pi^{-1}(Y^-)/T \subset U_i$, keeping in mind that $\widetilde{\pi^{-1}(Y^-)/T} = \widetilde{\mathbb{P}(\mathcal{V}_2)^-/T}$ in $\widetilde{U_i}$ and that $p_i^*(U_i) = Bl_{\pi^{-1}(Y^-)/T} U_i$. Note that $\widetilde{\pi^{-1}(Y^-)}/T = \widetilde{\mathbb{P}(\mathcal{V}_2)^-)}^{T}$ as they coincide outside the exceptional divisor and their intersection with the exceptional divisor is $\pi^{-1}(Y)/(T \times_{\mathbb{P}(\mathcal{V}_1)} \mathbb{P}(\mathcal{V}_1)^-)$

Recall that $M_{ij} = Bl_{Y^-/T} M_i$. For a point y in the exceptional divisor $\widetilde{Y^-/T}$, the fiber $q_{ij}^{-1}(y)$ of the map $U_{ij} \to M_{ij}$ consists of 2 components;

- $\widetilde{q_i^{-1}(y)} = Bl_{point}\mathbb{P}^2$, the strict transform of $q_i^{-1}(y) = \mathbb{P}^2$ under the blow-up p_i.
- a fiber $\mathbb{P}^2$ of the exceptional divisor E of p_i.

Indeed, $\rho^{-1}(\mathbb{P}(\mathcal{V}_1)^-/T) \cap q_i^{-1}(y) = \{point\}$ and $codim_{\pi^{-1}(Y^-)}(\mathbb{P}(\mathcal{V}_1)^-) = 2$. The two components $\mathbb{P}^2$ and $Bl_{point}\mathbb{P}^2$ intersect along a line $\mathbb{P}^1$, the exceptional divisor of $Bl_{point}\mathbb{P}^2$.

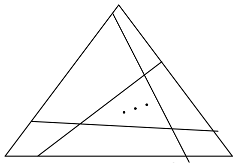
In the final step we want use these facts and show that all the fibers of q_{ij} have the same Hilbert polynomial. In other words U_{ij} is flat over M_{ij}.

Let $\mathcal{L} = \mathcal{O}(\sum_{i=1}^n D_i)$.

On a general fiber of the global sections of $\mathcal{L}^{\otimes m}_{|\mathbb{P}^2}$ on $\mathbb{P}^2 = Proj(\mathbb{C}[y_1, y_2, y_3])$ ($\mathbb{P}^2$ is a fiber of the universal family) are generated by monomials $y_1^{d_1} y_2^{d_2} y_3^{d_3}$ where $d_1 + d_2 + d_3 = mn$. There are $\binom{mn+2}{2}$ different global sections which means that for $\mathbb{P}^2$ and $m \gg 0$

$$\dim(H^0(\mathbb{P}^2, \mathcal{L}^{\otimes m}_{|\mathbb{P}^2})) = \dim \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(mn)) = \binom{mn+2}{2}.$$

The restriction of $\sum_i D_i$ to a fiber $\mathbb{P}^2$ of $F \to G(3, n)$:



n lines in $\mathbb{P}^2$

For $\mathbb{P}^1$, the global sections of $\mathcal{L}^{\otimes m}_{|\mathbb{P}^1}$ on $\mathbb{P}^1 = Proj(\mathbb{C}[y_1, y_2])$ are generated by monomials $y_1^{d_1} y_2^{d_2}$ where $d_1 + d_2 = mn$. There are $\binom{mn+1}{1}$ different combinations. Hence

$$dim(H^0(\mathbb{P}^1, \mathcal{L}^{\otimes m}_{|\mathbb{P}^1})) = \dim \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(mn)) = \binom{mn+1}{1}.$$

Consider $\mathcal{L}_i = \mathcal{O}(\sum_{l=1}^{n} D_l)$, the line bundle on U_i involved in the presentation of M_i as a moduli space. Let $g_{ij} : U_{ij} \to U_i$ be the composition of blow-ups described above, and $\mathcal{L}_{ij} := g_{ij}^* \mathcal{L}_i$. We will calculate the Hilbert polynomial of the fiber $q_{ij}^{-1}(y)$ of the family $q_{ij} : U_{ij} \to M_{ij}$ with respect to the ample line bundle $\mathcal{L}_{ij}|_{q_{ij}^{-1}(y)}$. Consider y in the exceptional locus of M_{ij} and $p_i(y)$ its image in M_i. Recall that $q_{ij}^{-1}(y) = Bl_{point}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$. With the notations introduced earlier, in the fiber $g_i^{-1}(p_i(y)) = \mathbb{P}^2$, the divisor $\sum_{l=1}^{n} D_l$ contains t lines passing through $(0:0:1)$, where $t = |J|$. Recall that $(0:0:1)$ is the point which was blown up. Pull-back through $g_i \circ g_{ij}$ these lines yield t divisors $F_1, ..., F_t$ in $Bl_{point}\mathbb{P}^2$, and t other lines in $\mathbb{P}^2$. Let $F := \sum_{l=1}^{t} F_l$ and $L = \sum_{l=1}^{n} \widetilde{D_l} - F$ on $Bl_{point}\mathbb{P}^2 \subset g_{ij}^{-1}(y)$. Thus $\mathcal{L}_{ij}$ is $\mathcal{O}(F+L)$ on $Bl_{point}\mathbb{P}^2$ and $\mathcal{O}_{\mathbb{P}^2}(t)$ on $\mathbb{P}^2$, and $\mathcal{O}_{\mathbb{P}^1}(t)$ on $Bl_{point}\mathbb{P}^2 \cap \mathbb{P}^2 = \mathbb{P}^1$. Hence

$$\begin{aligned} h^0(g_{ij}^{-1}(y), \mathcal{L}_{ij}(m)) = h^0(Bl_{point}\mathbb{P}^2, \mathcal{O}_{Bl_{point}\mathbb{P}^2}(mF + mL)) + \\ h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(mt)) - h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(mt)). \end{aligned} \tag{5.13}$$

Note that the pullback maps

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(mt)) \to h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(mt)),$$

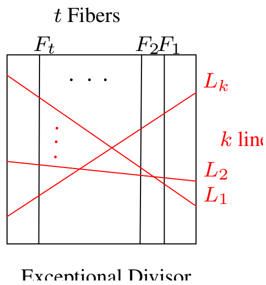
and

$$h^0(Bl_{point}\mathbb{P}^2, \mathcal{O}_{Bl_{point}\mathbb{P}^2}(mF + mL)) \to h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(mt)),$$

are surjective.

We calculate each summand separately. Note that $Bl_{point}\mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ is a projective bundle over $\mathbb{P}^1$ with $f : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$ and $\mathcal{O}_{Bl_{point}\mathbb{P}^2}(F) = f^*(\mathcal{O}_{\mathbb{P}^1}(t))$ while $\mathcal{O}_{Bl_{point}\mathbb{P}^2}(L) = \iota^*\mathcal{O}_{\mathbb{P}^2}(k) = \mathcal{O}_{Bl_{point}\mathbb{P}^2}(k)$ for the blow-up map $\iota : Bl_{point}\mathbb{P}^2 \to \mathbb{P}^2$.

Now consider k lines and t fibers in $Bl_{point}\mathbb{P}^2$ which are the pull-back of lines in $\mathbb{P}^2$ via the blow-up map.



For $m \gg 0$

$$0 \to I_{\cup F_i} \hookrightarrow \mathcal{O}_{Bl_{point}\mathbb{P}^2} \to \mathcal{O}_{mF} = \mathcal{O}_{Bl_{point}\mathbb{P}^2}/I_{\cup F_i} \to 0,$$

where $I_{\cup F_i} = \mathcal{O}_{Bl_{point}\mathbb{P}^2}(-mF)$. Multiplying this short exact sequence by $\mathcal{O}(mF + mL)$ we have

$$0 \to \mathcal{O}(mL) \to \mathcal{O}(mF + mL) \to \mathcal{O}_{\cup F_i} \otimes \mathcal{O}(mF + mL) \to 0. \tag{5.14}$$

Since $F_1, ..., F_t$ are fibers of the projection $f : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$, we have $F^2 = 0$ and hence $\mathcal{O}_{\cup F_i} \otimes \mathcal{O}(mF) = \bigoplus^{mt} \mathcal{O}_{\mathbb{P}^1}$.

As well,

$$\begin{aligned} f_*\mathcal{O}_{Bl_{point}\mathbb{P}^2}(mk) &= Symm^{mk}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \\ &= \mathcal{O}_{\mathbb{P}^1}(mk) \oplus \mathcal{O}_{\mathbb{P}^1}(mk-1) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1}, \end{aligned}$$

and $R^i f_*\mathcal{O}_{Bl_{point}\mathbb{P}^2}(mk) = 0$ for $i > 0$. Hence

$$H^i(Bl_{point}\mathbb{P}^2, \mathcal{O}_{Bl_{point}\mathbb{P}^2}(mk)) = H^i(\mathbb{P}^1, f_*\mathcal{O}_{Bl_{point}\mathbb{P}^2}(mk)) = 0,$$

for $i > 0$ and thus the short exact sequence 5.14 yields a short exact sequence for the spaces of global sections

$$\begin{aligned} 0 \to H^0(Bl_{point}\mathbb{P}^2, \mathcal{O}_{Bl_{point}\mathbb{P}^2}(mk)) \to \ & H^0(Bl_{point}\mathbb{P}^2, \mathcal{O}(mF+mL)) \to \\ & H^0(\mathbb{P}^1, \bigoplus^{mt} \mathcal{O}_{\mathbb{P}^1}(mk)) \to 0. \end{aligned}$$

Hence

$$\begin{aligned} \dim(H^0(Bl_{point}\mathbb{P}^2, \mathcal{O}(mF+mL))) &= \dim(H^0(\mathbb{P}^1, \bigoplus^{mt} \mathcal{O}_{\mathbb{P}^1}(mk))) + \\ & \quad dim(H^0(Bl_{point}\mathbb{P}^2, \mathcal{O}_{Bl_{point}\mathbb{P}^2}(mk))) \\ &= \binom{mk+2}{2} + mt\binom{mk+1}{1} \\ &= (mk+2)(mk+1)/2 + mt(mk+1). \end{aligned}$$

Hence the Hilbert Polynomial of $q_{ij}^{-1}(y) = \mathbb{P}^2 \cup Bl_{point}\mathbb{P}^2$ with respect to $\mathcal{L}_{ij}$

$$P_X(m) = \binom{mt+2}{2} + \binom{mk+2}{2} + mt\binom{mk+1}{1} - \binom{mt+1}{1} \quad (5.15)$$

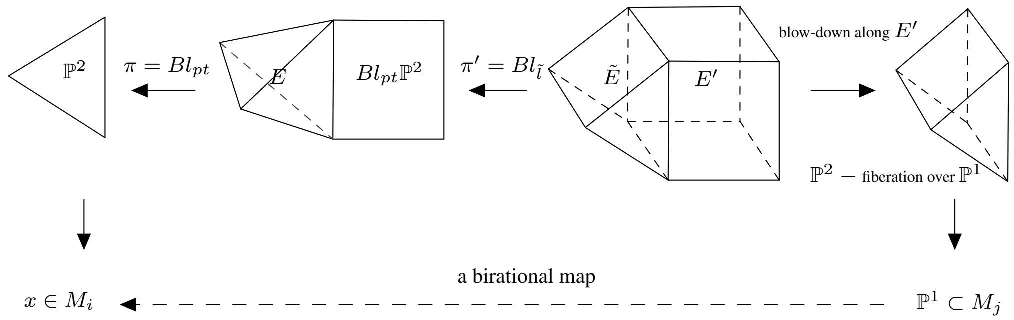
$$= m^2(\frac{t^2+k^2+2kt}{2}) + m(\frac{3t+3k+2t-2t}{2}) + 1 \quad (5.16)$$

$$= m^2(\frac{(t+k)^2}{2}) + m(\frac{3(k+t)}{2}) + 1, \quad (5.17)$$

which is the same as the Hilbert Polynomial of $\mathbb{P}^2$ with the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(t+k)$ as below:

$$\begin{aligned} P_{\mathbb{P}^2}(m) &= \binom{m(t+k)+2}{2} \\ &= m^2(\frac{(t+k)^2}{2}) + m(\frac{3(k+t)}{2}) + 1. \end{aligned}$$

As an example the following diagram shows what happens to the fibers over a flip:



and the divisors in the fibers change as below:

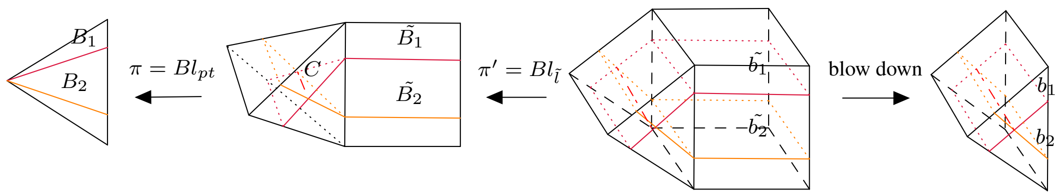


Figure 5.3: Flip of universal family over two GIT quotients

To connect these figures to Fig 6.3, π is the restriction of the blow down map $\widetilde{U_i} \to U_i$, and π' is restriction of $U_{ij} \to \widetilde{U_i}$.

5.4 The Moduli Problem for a Flip

Definition 5.4.1 ***The Moduli Problem for a Flip.*** *Consider two fixed chambers C_k and C_j which share a wall W_l. Recall the sets $I_l, J_l \subset \{1, 2, ..., n\}$ associated to W_l in Corollary 4.2.6 and beginning of Section 5.3. We define the functor*

$$\begin{aligned} \mathcal{U}_{jk} : \mathfrak{Sch} &\to \mathfrak{Set} \\ B &\mapsto [(\pi : P \to B, D_1, ..., D_n)], \end{aligned}$$

associating to each scheme B an isomorphism class of tuples

$$(\pi : P \to B, D_1, ..., D_n),$$

consisting of a flat morphism $\pi : P \to B$ whose fiber P_b are either $\mathbb{P}^2$ or $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{point}\mathbb{P}^2$ (where the gluing is done along the exceptional divisors in the blow-ups, and along some special lines in $\mathbb{P}^2$s) and such that the restrictions $\ell_{1,b}, \ell_{2,b}, ..., \ell_{n,b}$ of the divisors $D_1, D_2, ..., D_n$ to the fiber P_b satisfy:

if $P_b = \mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{point}\mathbb{P}^2$, then

- *for each $i \in I_l$, the $L_i := \ell_{i,b} \subset Bl_{point}\mathbb{P}^2$ is a section of the projection $\phi : Bl_{point}\mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$, different from the exceptional divisor (or in other words pull-back of a line in $\mathbb{P}^2$),*
- *while for each $j \in J_l$, the restriction $\ell_{j,b|Bl_{point}\mathbb{P}^2} =: F_j$ is a fiber of ϕ, and $\ell_{j,b|\mathbb{P}^2}$ is a line intersecting F_j at a point on the exceptional divisor E of $Bl_{point}\mathbb{P}^2$,*

if $P_b = \mathbb{P}^2$, then , $\ell_{i,b}$ is a line.

For all fibers P_b

- *if $\ell_{i_1,b} = \ell_{i_2,b} = ... = \ell_{i_a,b}$ then $\{i_1, ..., i_a\} \subseteq I_h$ for some $I_h \in \mathcal{A}_k^+ \cup \mathcal{A}_j^+$, and $I_h \neq \{i_1, ..., i_a\}$,*
- *if $\ell_{j_1,b}, \ell_{j_2,b}, ..., \ell_{j_c,b}$ have a point in common then $\{j_1, j_2, ..., j_c\} \subseteq J_h$ for some $J_h \in \mathcal{A}_k^- \cup \mathcal{A}_j^-$, and $J_h \neq \{j_1, j_2, ..., j_c\}$,*

where $\mathcal{A}_k^+, \mathcal{A}_k^-, \mathcal{A}_j^+$ and $\mathcal{A}_j^-$ are as defined in Section 5.3. Assume $|I_l| = s$ and $|J_l| = t$, where $s + t = n$.

Two tuples $(\pi : P \to B, D_1, D_2, ..., D_n)$ and $(\pi' : P' \to B', D'_1, D'_2, ..., D'_n)$ are isomorphic to each other if there exists a map $\phi : B' \to B$ and an isomorphism $\tilde{\phi} : P' \to \phi^*(P)$

$$\begin{array}{ccc} P' \cong \phi^*(P) & & P \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\phi} & B, \end{array}$$

such that $\tilde{\phi}$ restricts to an isomorphism $D'_i \cong \phi^*(D_i)$ for each $i \in \{1, ..., n\}$.

Note that the last conditions are equivalent to the fact that

- if $P_b \cong \mathbb{P}^2$ the data $(P_b, \ell_1, ..., \ell_n)$ corresponds to an orbit Tx in $G(3, n)$ with $\mu(Tx) \supset C_k \cup C_j$,

- if $P_b \cong \mathbb{P}^2 \cup Bl_{point}\mathbb{P}^2$ the first $\mathbb{P}^2$ corresponds to $Tx \subset G(3,n)$ with $\mu(Tx) \supset C_i$ and $C_i \cap C_j$ in the boundary of $\mu(Tx)$. Also the contraction of $Bl_{point}\mathbb{P}^2$ corresponds to an orbit Tx with $\mu(Tx) \supset C_j$ and $C_i \cap C_j$ is in the boundary of $\mu(Tx)$.

Theorem 5.4.1 *For every two chambers C_i and C_j which share a wall W_l, the scheme M_{ij} defined earlier represents the functor defined above.*

Proof: We will prove that the family $(U_{ij} \to M_{ij}, D_1, ..., D_n)$ is the universal family over M_{ij} for the given moduli problem. First we note that any one of the special fibers of $U_{ij} \to M_{ij}$ is of the form $S_b = Bl_{point}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$. The restriction of the divisors $D_1, ..., D_n$ to the fiber S_b determine:

- s sections $L_1, ..., L_s$ of $Bl_{point}\mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$.
- t fibers $F_1, ..., F_t$ in $Bl_{point}\mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^1$.
- t lines in $\mathbb{P}^2$, intersecting the fibers above in t points $p_1, ..., p_t$.

The line bundle $\mathcal{O}_{Bl_{point}\mathbb{P}^2}(F_1)$ and $\mathcal{O}_{Bl_{point}\mathbb{P}^2}(L_1)$, with sections chosen to correspond to the above divisors determine an embedding $Bl_{point}\mathbb{P}^2 \hookrightarrow \mathbb{P}^{t-1} \times \mathbb{P}^{s-1}$, while the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ with suitably chosen sections gives a map $\mathbb{P}^2 \to \mathbb{P}^{t-1}$. Note that the composition $Bl_{point}\mathbb{P}^2 \hookrightarrow \mathbb{P}^{t-1} \times \mathbb{P}^{s-1} \to^{\pi_2} \mathbb{P}^{s-1}$ contracts the exceptional divisor E of $Bl_{point}\mathbb{P}^2$ to a point q as $[L_1] = [E] + [F]$ and $[E]^2 = -1$. Thus the embeddings $Bl_{point}\mathbb{P}^2 \hookrightarrow \mathbb{P}^{t-1} \times \mathbb{P}^{s-1}$ and $\mathbb{P}^2 \to \mathbb{P}^{t-1} \times \{q\}$ glue to an embedding of $S = Bl_{point}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2 \hookrightarrow \mathbb{P}^{t-1} \times \mathbb{P}^{s-1}$, which, after the Segre embedding, has Hilbert polynomial $P(m)$ as calculated in Equations (5.17).

Based on the discussion above, our strategy of proof will consist in identifying M_{ij} with a GIT quotient of the Hilbert Scheme $\mathcal{H} = Hilb(\mathbb{P}^{t-1} \times \mathbb{P}^{s-1}, P(m))$. We will proceed in the following steps:

(i) For any family $(P \to B, D_1, ..., D_n)$ satisfying the conditions of the moduli problem, we construct two torus bundle T_1 and T_2 over B, of ranks $t-1$ and $s-1$ respectively, such that for $T = T_1 \times_B T_2$, there is an embedding $T \times_B P \to \mathbb{P}^{t-1} \times \mathbb{P}^{s-1} \times T$. This is done just like in the proof of Theorem 5.2.1 except that we work separately with $\{D_i\}_{i \in I_l}$ and $\{D_j\}_{j \in J_l}$.

(ii) Hence there is a map $T \to \mathcal{H} = Hilb(\mathbb{P}^{t-1} \times \mathbb{P}^{s-1}, P(m))$, such that $T \times_B P$ is the pull-back of the universal family on $\mathcal{H}$. In the case when $P \to B$ is exactly the family $U_{ij} \to M_{ij}$, we get a torus bundle T_{ij} on M_{ij} and an injective map $\phi_{ij} : T_{ij} \to \mathcal{H}$, which we claim to be an open embedding. We check this by comparing the dimensions of the tangent spaces $T_{ij,b}$ and $\mathcal{H}_{\phi_{ij}(b)} =$

$H^0(S_b, \mathcal{N}_{S_b/\mathbb{P}^{t-1}\times\mathbb{P}^{s-1}})$, while also checking that $\phi_{ij}(b)$ is a smooth point of $\mathcal{H}$, as $H^1(S_b, \mathcal{N}_{S_b/\mathbb{P}^{t-1}\times\mathbb{P}^{s-1}}) = 0$. We do these calculations in the steps (numbered from 1 to 5) following this proof.

(iii) For any other family $P \to B$, the corresponding map $T \to \mathcal{H}$ has image embedded in $\phi_{ij}(T_{ij})$, since this open set must parameterize all schemes S_b satisfying the conditions of the moduli problem. Hence taking quotient by $(\mathbb{C}^*)^{n-2}$ gives

$$\begin{array}{ccc} P & \longrightarrow & U_{ij} \\ \downarrow & & \downarrow \\ B & \longrightarrow & M_{ij}, \end{array}$$

as described.

Let $S = S_1 \cup_{\mathbb{P}^1} S_2$ with $S_1 = Bl_{point}\mathbb{P}^2$ and $S_2 = \mathbb{P}^2$, let $\mathbb{P}^{t-1}\times\mathbb{P}^{s-1} = Z$. To calculate $H^i(S, \mathcal{N}_{S/\mathbb{P}^{t-1}\times\mathbb{P}^{s-1}})$ we need a few lemmas as follows.

Lemma 5.4.2 *For all* $i \geq 0$

$$H^i(S_1, \mathcal{T}_W|_{S_1}) \cong H^i(S_1, \mathcal{T}_{S_1}).$$

Proof: Embed $S \subset W$ where $W := Bl_{\mathbb{P}^1\times\{a\}}(\mathbb{P}^2 \times \mathbb{P}^1) \to^f \mathbb{P}^1$ and $S = f^{-1}(a)$. Indeed, $\mathcal{N}_{\mathbb{P}^1\times\{a\}/\mathbb{P}^2\times\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}$. Hence the exceptional divisor

$$\mathbb{P}(\mathcal{N}_{\mathbb{P}^1\times\{a\}/\mathbb{P}^2\times\mathbb{P}^1}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}) \cong Bl_{point}\mathbb{P}^2 = S_1,$$

while $\widetilde{\mathbb{P}^2 \times \{a\}} \cong \mathbb{P}^2 = S_2$. S is a fiber in the flat family f. Hence

$$\mathcal{N}_{S/W} \cong \mathcal{O}_S. \tag{5.18}$$

Since S is a normal crossing divisor in W, we have

$$(\mathcal{N}_{S/W})_{|S_i} \cong \mathcal{N}_{S_i/W} \otimes \mathcal{O}_{S_i}(D), \tag{5.19}$$

where $D = S_1 \cap S_2 \cong \mathbb{P}^1$. Indeed, $\mathcal{N}_{S/W} \cong \mathcal{O}_S(S) = \mathcal{O}_S(S_1 + S_2)$ and restriction to S_i gives the Formula 5.19 above. From Formulas 5.18 and 5.19 we have

$$\mathcal{N}_{S_1/W} \cong \mathcal{O}_{Bl_{point}\mathbb{P}^2}(-D) = g^*\mathcal{O}_{\mathbb{P}^2}(-1) \otimes \phi^*\mathcal{O}_{\mathbb{P}^1}(-1),$$

and $\mathcal{N}_{S_2/W} \cong \mathcal{O}_{\mathbb{P}^2}(-D) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$, where $g : S_1 \to \mathbb{P}^2$. Hence

$$H^i(S_j, \mathcal{N}_{S_j/W}) = 0,$$

for all i and j.

Thus the short exact sequence

$$0 \to \mathcal{T}_{S_1} \to \mathcal{T}_W|_{S_1} \to \mathcal{N}_{S_1/W} \to 0,$$

leads to

$$H^i(S_1, \mathcal{T}_W|_{S_1}) \cong H^i(S_1, \mathcal{T}_{S_1}),$$

for all $i \geq 0$.

Lemma 5.4.3 *$h^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}) = 8$ and $h^i(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}) = 0$ otherwise.*

Proof: From Euler's sequence for $\mathbb{P}^2$:

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{T}_{\mathbb{P}^2} \to 0,$$

we have:
$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 1$,
$h^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$ for $i > 0$,
$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 3$,
$h^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 0$ for $i > 0$,
$h^0(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}) = 3h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = (3)(3) - 1 = 8$ and
$h^i(\mathbb{P}^2, \mathcal{T}_{\mathbb{P}^2}) = 0$ otherwise.

Lemma 5.4.4 *$h^0(S_1, \mathcal{T}_{S_1}) = 6$ and $h^i(S_1, \phi^*\mathcal{T}_{S_1}) = 0$ for $i > 0$.*

Proof: We have

$$S_1 = Bl_{point}\mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \to^{\phi} \mathbb{P}^1,$$

$h^0(S_1, \mathcal{O}_{S_1}) = 1$ and

$$h^i(S_1, \mathcal{O}_{S_1}) = h^i(\mathbb{P}^1, \phi_*\mathcal{O}_{S_1}) = 0,$$

for $i > 0$, (as $h^i(\phi^{-1}(s), \mathcal{O}_{S_1|\phi^{-1}(s)}) = h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for $i > 0$ hence $R^i\phi_*\mathcal{O}_{S_1} = 0$ for $i > 0$). Similarly, $h^i(S_1, \mathcal{O}_{S_1}(-1)) = 0$ for $i \geq 0$. The relative Euler's sequence for $S_1 = Bl_{point}\mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to^{\phi} \mathbb{P}^1$ is as follow;

$$0 \to \mathcal{O}_{S_1} \to \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D) \to \mathcal{T}_{S_1/\mathbb{P}^1} \to 0. \tag{5.20}$$

Hence

$$h^0(S_1, \mathcal{T}_{S_1/\mathbb{P}^1}) = -h^0(S_1, \mathcal{O}_{S_1}) + h^0(S_1, \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D)), \tag{5.21}$$

and

$$h^i(S_1, \mathcal{T}_{S_1/\mathbb{P}^1}) = h^i(S_1, \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D)),$$

for $i > 0$. To calculate $h^i(S_1, \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D))$, we use the short exact sequence

$$0 \to \mathcal{O}_{S_1}(-D) \to \mathcal{O}_{S_1} \to \mathcal{O}_D \to 0,$$

tensored with $\phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D)$, which yields:

$$\begin{aligned} 0 \to \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D) \to \\ (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_D(D) \quad \to 0. \end{aligned}$$

As $\mathcal{O}_D(D) = \mathcal{O}_{\mathbb{P}^1}(-1)$ for the exceptional divisor D, we have

$$(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_D(D) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1} = 0.$$

On the other hand,

$$R^i\phi_*\phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes R^i\phi_*\mathcal{O}_{S_1}, \tag{5.22}$$

for $i > 0$. So

$$\begin{aligned} h^i(S_1, \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))) &= h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})) \\ &= \begin{cases} 3 \text{ if } i = 0, \\ 0 \text{ otherwise.} \end{cases} \end{aligned}$$

In conclusion,

$$\begin{aligned} h^i(S_1, \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D))) &= h^i(S_1, \phi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})) + \\ &\quad h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \\ &= 3 + 1 = 4, \end{aligned}$$

and

$$\begin{aligned} h^i(S_1, \phi^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{S_1}(D)) &= h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}) \\ &= 0 \text{ for } i > 0. \end{aligned}$$

Hence in the Equation (5.21),

$$h^0(S_1, \mathcal{T}_{S_1/\mathbb{P}^1}) = 4 - 1 = 3,$$

and

$$h^i(S_1, \mathcal{T}_{S_1/\mathbb{P}^1}) = 0.$$

Finally, from the relative tangent sequence;

$$0 \to \mathcal{T}_{S_1/\mathbb{P}^1} \to \mathcal{T}_{S_1} \to \phi^*\mathcal{T}_{\mathbb{P}^1} \to 0, \tag{5.23}$$

we have

$$h^0(S_1, \mathcal{T}_{S_1}) = h^0(S_1, \mathcal{T}_{S_1/\mathbb{P}^1}) + h^0(S_1, \phi^*\mathcal{T}_{\mathbb{P}^1}), \tag{5.24}$$

and for $i > 0$ we have $h^i(S_1, \mathcal{T}_{S_1}) = h^i(S_1, \phi^*\mathcal{T}_{\mathbb{P}^1})$. But

$$0 \to \phi^*\mathcal{O}_{\mathbb{P}^1} \to \bigoplus_{i=1}^{2} \phi^*\mathcal{O}_{\mathbb{P}^1}(1) \to \phi^*\mathcal{T}_{\mathbb{P}^1} \to 0, \tag{5.25}$$

and $R^i\phi_*\phi^*\mathcal{F} = \mathcal{F} \otimes R^i\phi_*\mathcal{O}_{S_1} = 0$ for all bundles $\mathcal{F}$ and $i > 0$. Therefore
$h^0(S_1, \phi^*\mathcal{O}_{\mathbb{P}^1}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1$,
$h^0(S_1, \phi^*\mathcal{O}_{\mathbb{P}^1}(1)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2$ and
$h^i(S_1, \phi^*\mathcal{O}_{\mathbb{P}^1}) = h^i(S_1, \phi^*\mathcal{O}_{\mathbb{P}^1}(1)) = 0$ for $i > 0$. Hence

$$h^0(S_1, \phi^*\mathcal{T}_{\mathbb{P}^1}) = 2h^0(S_1, \phi^*\mathcal{O}_{\mathbb{P}^1}(1)) - h^0(S_1, \mathcal{O}_{\mathbb{P}^1}) = (2)(2) - 1 = 3,$$

and $h^0(S_1, \phi^*\mathcal{T}_{\mathbb{P}^1}) = 0$ otherwise. Therefore from Equation (5.24) $h^0(S_1, \mathcal{T}_{S_1}) = 6$ and $h^i(S_1, \mathcal{T}_{S_1}) = 0$ for $i > 0$.

Lemma 5.4.5

$$h^0(D, \mathcal{T}_W|_D) = h^0(D, \mathcal{T}_D) + h^0(D, \mathcal{N}_{D/W}) = 5,$$

and $h^i(D, \mathcal{T}_W|_D) = 0$ *for* $i > 0$.

Proof: As well, for $\mathbb{P}^1 = D = S_1 \cap S_2$ we have

$$0 \to \mathcal{T}_D \to \mathcal{T}_W|_D \to \mathcal{N}_{D/W} \to 0, \tag{5.26}$$

and

$$0 \to \mathcal{N}_{D/S_2} \to \mathcal{N}_{D/W} \to (\mathcal{N}_{S_2/W})_{|D} \to 0, \tag{5.27}$$

where

$$\mathcal{N}_{D/S_2} = \mathcal{O}_{\mathbb{P}^1}(1),$$

and

$$(\mathcal{N}_{S_2/W})_{|D} = \mathcal{O}_D(-1) \cong \mathcal{O}_{\mathbb{P}^1}(-1).$$

Since

$$h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0,$$

for $i > 0$, we get $h^i(D, \mathcal{N}_{D/W}) = 0$ for $i > 0$ and

$$h^0(D, \mathcal{N}_{D/W}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 2.$$

Also, $\mathcal{T}_D = \mathcal{T}_{\mathbb{P}^1} = \bigoplus_{i=1}^{2} \mathcal{O}_{\mathbb{P}^1}(1)/\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, so $h^0(D, \mathcal{T}_D) = 3$ and

$$h^i(D, \mathcal{T}_D) = 0$$

for $i > 0$. Hence Short exact Sequence 5.26 yields:

$$h^0(D, \mathcal{T}_W|_D) = h^0(D, \mathcal{T}_D) + h^0(D, \mathcal{N}_{D/W}) = 5,$$

and $h^i(D, \mathcal{T}_W|_D) = 0$ for $i > 0$.

Continuation of the proof:

Putting calculations in Lemma 5.4.3, Lemma 5.4.4 and Lemma 5.4.5 together and using the exact sequence

$$0 \to \mathcal{T}_W|_S \to i_{1*}\mathcal{T}_W|_{S_1} \oplus i_{2*}\mathcal{T}_W|_{S_2} \to \mathcal{T}_W|_D \to 0, \tag{5.28}$$

(where $i_1 : S_1 \to S$ and $i_2 : S_2 \to S$) we get:

$$0 \to H^0(S, \mathcal{T}_W|_S) \to \bigoplus_{i=1}^{2} H^0(S_i, \mathcal{T}_W|_{S_i}) \to H^0(D, \mathcal{T}_W|_D) \to^{\delta}$$
$$H^1(S, \mathcal{T}_W|_S) \to 0,$$

where $\bigoplus_{i=1}^{2} H^0(S_i, \mathcal{T}_W|_{S_i}) \cong \mathbb{C}^6 \times \mathbb{C}^8$ and $H^0(D, \mathcal{T}_W|_D) \cong \mathbb{C}^5$. Hence

$$h^0(S, \mathcal{T}_W|_S) - h^1(S, \mathcal{T}_W|_S) = 9.$$

Now the short exact sequence

$$0 \to \mathcal{T}_S \to \mathcal{T}_W|_S \to \mathcal{N}_{S/W} \to 0, \tag{5.29}$$

together with the equalities $\mathcal{N}_{S/W} = \mathcal{O}_S$, $h^0(S, \mathcal{O}_S) = 1$ and $h^1(S, \mathcal{O}_S) = 0$, imply

$$h^0(S, \mathcal{T}_S) - h^1(S, \mathcal{T}_S) = h^0(S, \mathcal{T}_W|_S) - h^1(S, \mathcal{T}_W|_S) - 1 = 8.$$

Let $P = \mathbb{P}^{t-1} \times \mathbb{P}^{s-1}$ where $t + s = n$ and

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & \mathbb{P}^{s-1} \\ \downarrow{\scriptstyle \pi_1} & & \\ \mathbb{P}^{t-1} & & \end{array}$$

are the projections. We have a short exact sequence of fiber bundles

$$0 \to \pi_1^*\mathcal{O}_{\mathbb{P}^{t-1}} \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^{s-1}} \to \pi_1^*(\oplus^t \mathcal{O}_{\mathbb{P}^{t-1}}(1)) \oplus \pi_2^*(\oplus^s \mathcal{O}_{\mathbb{P}^{s-1}}(1)) \to \mathcal{T}_P \to 0. \tag{5.30}$$

Tensoring this with $\mathcal{O}_{S_1}$ and $\mathcal{O}_{S_2}$, respectively, we get

$$0 \to \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_1} \to (\oplus^t \mathcal{O}_{S_1}(F_1)) \oplus (\oplus^s \mathcal{O}_{S_1}(L_1)) \to \mathcal{T}_P|_{S_1} \to 0,$$

and

$$0 \to \mathcal{O}_{S_2} \oplus \mathcal{O}_{S_2} \to (\oplus^t \mathcal{O}_{S_2}(1)) \oplus (\oplus^s \mathcal{O}_{S_2}) \to \mathcal{T}_P|_{S_2} \to 0.$$

Using $\mathcal{O}_{S_1}(F_1) = \phi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and $\mathcal{O}_{S_1}(L_1) = g^*(\mathcal{O}_{\mathbb{P}^2}(1))$ we get $h^0(S_1, \mathcal{T}_P|_{S_1}) = 2t + 3s - 2$,
$h^i(S_1, \mathcal{T}_P|_{S_1}) = 0$ for $i > 0$,
$h^0(S_2, \mathcal{T}_P|_{S_2}) = 3t + s - 2$,
$h^i(S_2, \mathcal{T}_P|_{S_2}) = 0$ for $i > 0$.
From tensoring the Short Exact Sequence 5.30 with $\mathcal{O}_D$ for $D = S_1 \cap S_2 \cong \mathbb{P}^1$ we have;

$$0 \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to (\oplus^t \mathcal{O}_{\mathbb{P}^1}(1)) \oplus (\oplus^s \mathcal{O}_{\mathbb{P}^1}) \to \mathcal{T}_P|_D \to 0.$$

So $h^0(D, \mathcal{T}_P|_D) = 2t + s - 2$ and $h^i(D, \mathcal{T}_P|_D) = 0$ for $i > 0$.

Moreover, from the short exact sequences on global sections induced by the above presentation, we see that the short exact sequence

$$0 \to \mathcal{O}_S \to i_{1*}(\mathcal{O}_{S_1}) \oplus i_{2*}(\mathcal{O}_{S_2}) \to \mathcal{O}_D \to 0,$$

tensored by $\mathcal{T}_P$, yields an exact sequence of global sections:

$$0 \to H^0(S, \mathcal{T}_P|_S) \to H^0(S, \mathcal{T}_P|_{S_1}) \oplus H^0(S, \mathcal{T}_P|_{S_2}) \to H^0(D, \mathcal{T}_P|_D) \to 0.$$

Hence

$$\begin{aligned} h^0(S, \mathcal{T}_P|_S) &= h^0(S_1, \mathcal{T}_P|_{S_1}) + h^0(S_2, \mathcal{T}_P|_{S_2}) - h^0(D, \mathcal{T}_P|_D) \\ &= 2t + 3s - 2 + 3t + s - 2 - 2t - s + 2 \\ &= 3n - 2, \end{aligned}$$

and $h^i(S, \mathcal{T}_P|_S) = 0$ for $i > 0$.

Finally, the short exact sequence for $i : S \to Z$ is as folow;

$$0 \to \mathcal{T}_S \to i^*(\mathcal{T}_Z) \to \mathcal{N}_{S/Z} \to 0.$$

Hence

$$\begin{aligned} 0 \to \quad & H^0(S, \mathcal{T}_S) \to H^0(S, i^*(\mathcal{T}_Z)) \to H^0(S, \mathcal{N}_{S/Z}) \to H^1(S, \mathcal{T}_S) \to \\ & \quad H^1(S, i^*(\mathcal{T}_Z)) = 0 \to H^1(S, \mathcal{N}_{S/Z}) \to H^2(S, \mathcal{T}_S) = 0, \end{aligned}$$

Hence $h^1(S, \mathcal{N}_{S/Z}) = 0$ and

$$\begin{aligned} h^0(S, \mathcal{N}_{S/Z}) &= h^0(S, i^*(\mathcal{T}_Z)) - h^0(S, \mathcal{T}_S) + h^1(S, \mathcal{T}_S) \\ &= -8 + 3n - 2 \\ &= 3n - 10, \end{aligned}$$

while the torus bundle T_{ij} over M_{ij} has dimension $(n-2)+3(n-3)-(n-1) = 3n-10$.

□

Chapter 6

Compactification of Arrangements of Lines in Planes as Inverse Limit

In the previous chapters we have constructed various GIT quotients of $G(3, n)$ by $(\mathbb{C}^*)^n$ and their flips, and we presented these as moduli problem of arrangements of lines in planes or other rational surfaces. The universal families for these moduli spaces exhibited lines which were allowed to coincide or intersect in special ways. We would like to construct a compact family of planes with the property that all lines are in general position. We do not want that in a plane three or more lines pass through a point and also we do not want in a plane two or more lines to coincide. Hence we have to replace planes which exhibit at least one of these problems with other surfaces which do not have such a problem. Thus this replacement is a compactification of the space of arrangements of lines in planes in general position constructed in Definition 5.2.3.

The first section describes the image of $G(3, n)/(\mathbb{C}^*)^{n-3}$ under moment map. This description is necessary for understanding special loci in the inverse limit of GIT quotients of $G(3, n)$ by maximal torus. The second section defines a functor which satisfies the desired properties and the third section states the main result of this thesis for the Grassmannian variety $G(3, n)$ namely, we construct a scheme which represents the functor.

6.1 The Moment Map of $(\mathbb{P}^2)^{n-3}$

We start by considering the family of GIT quotients of $G(3, n)$ which are also described as $(\mathbb{P}^2)^{n-3}/(\mathbb{C}^*)^2$.

Consider the natural torus action of $T = (\mathbb{C}^*)^n$ on $G(3,n)$ as described in Section 2.2 and the sub-torus $T' = (\mathbb{C}^*)^{n-3}$ which omits the triple t_i, t_j and t_k for fixed $1 \le i < j < k \le n$. A natural unique GIT quotient of $G(3,n)$ by T' (The linearisation coming from the standard linearisation on $G(3,n)$ induced by the action of torus on $\mathbb{P}^{n-1}$) is

$$\begin{aligned} G(3,n)/\!\!/T' &= U_{i,j,k}/T' \\ &\cong \underbrace{\mathbb{P}^2 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^2}_{n-3}, \end{aligned}$$

where $U_{i,j,k} = \{[\Lambda] \in G(3,n)\ ;\ p_{ijk}([\Lambda]) \neq 0; \Lambda \text{ has no zero column }\}$.

The quotient of F by T' is a universal family over $G(3,n)/\!\!/T'$

$$\begin{array}{c} F/\!\!/T' \\ \Big\downarrow{\scriptstyle p_1} \\ G(3,n)/\!\!/T'. \end{array}$$

The induced $(\mathbb{C}^*)^2$–action on $(\mathbb{P}^2)^{n-3}$ is given by

$$(u', v').((r_{1,l} : r_{2,l} : r_{3,l}))_l = ((u'^{-1} r_{1,l} : v'^{-1} r_{2,l} : r_{3,l}))_l.$$

for an arbitrarily chosen $(u', v') \in (\mathbb{C}^*)^2$. So $G(3,n)/\!\!/T'$ can be understood as the moduli space parameterizing $n-3$ lines in $\mathbb{P}^2$. The choice of the triple t_i, t_j and t_k count for the 3 lines. Note that for this moduli space 2 sets of $n-3$ lines determine the same point in the moduli space if and only if the lines are the same (it is not enough condition to have an automorphism of $\mathbb{P}^2$ sending one set of lines to the other set).

Next we will construct the moment map for the $(\mathbb{C}^*)^2$–action on $(\mathbb{P}^2)^{n-3}$. Consider the Segre embedding

$$(\mathbb{P}^2)^{n-3} \longrightarrow \mathbb{P}^N \tag{6.1}$$

$$((r_{1l} : r_{2,l} : r_{3,l}))_l \mapsto ((\prod_{l=1}^{n-3} r_{i_l l}); i_l = 1, 2, 3), \tag{6.2}$$

where $N = (3)^{n-3} - 1$. We can construct the moment map as in Equation (4.1). Recall that $\mu = A^t \circ \mu_{(\mathbb{C}^*)^N}|_{(\mathbb{P}^2)^{n-3}}$ where A is the matrix given by weights of the $(\mathbb{C}^*)^2$–action on $\mathbb{P}^N$.

Lemma 6.1.1 *The image of μ after tilting the Y coordinate is as bellow and each chamber is a triangle where sides are $X = i, Y = j$ and $Z = k$.*

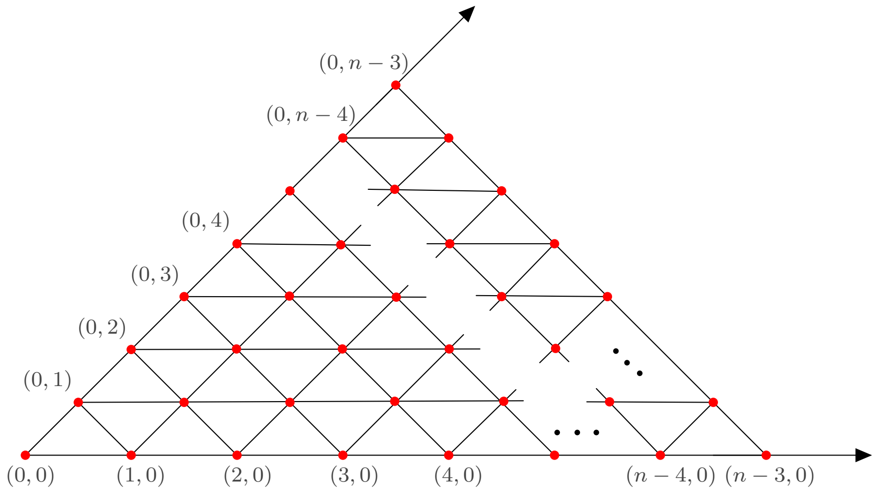


Figure 6.1: Image of Moment Map

For each point $P(i, j)$ in the net above let $k = n - 3 - i - j$. We can think of (i, j, k) as the trilinear coordinates of P.

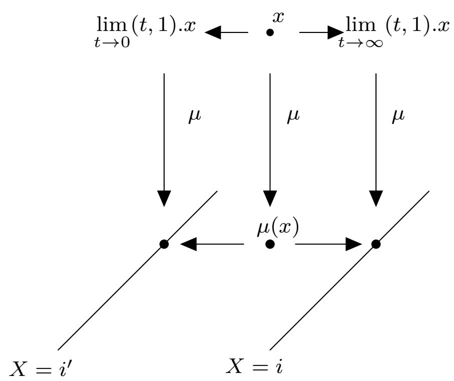
For every point $P(i, j, k)$ with i, j and k positive integers so that $i + j + k = n - 3$, we have $P = \mu(x)$ where $x = ((r_{1l} : r_{2l} : r_{3l}))_{l\in\{1,2,3,\ldots,n-3\}}$ has i coordinates triples of the form $(1 : 0 : 0)$, j coordinates triples of the form $(0 : 1 : 0)$ and k coordinates triples of the form $(0 : 0 : 1)$. Note that there are $\frac{(n-3)!}{i!j!k!}$ such points. Hence the nodes of the net are the images of all the

$$\sum_{i+j+k=n-3} \frac{(n-3)!}{i!j!k!} = 3^{n-3},$$

$(\mathbb{C}^*)^2$–fixed points. Note that $(\mathbb{C}^*)^2$ has three important sub-tori; $\mathbb{C}^* \times \{1\}, \{1\} \times \mathbb{C}^*$ and $\triangle = \{(u, u) : u \in \mathbb{C}^*\}$. The walls in this net are the images through μ of the fixed point loci with respect to these sub-tori. Note that the three sub-tori naturally correspond to three directions in the net. The net segment given by the equation $X = i$ is the image through μ of the loci given by $(r_{1l} : r_{2l} : r_{3l}) = (1 : 0 : 0)$ for i values of l and $(r_{1l} : r_{2l} : r_{3l}) = (0 : r_{2l} : r_{3l})$ for the remaining values of l. Note that these are fixed loci for $\mathbb{C}^* \times \{1\}$. Similarly for the other trilinear coordinates j and k.

Now we describe the generic point for each orbit mapped by μ to the region between

$X = i$ and $X = i'$. For other fixed loci the argument would be similar.



Here $\lim_{t\to 0}(t,1).x$ is a point in $(\mathbb{P}^2)^{n-3}$ given by $(n-3)$ triples $(r_{1l} : r_{2l} : r_{3l})$, among which $(1 : 0 : 0)$ appears i' times and $r_{1l} = 0$ for the remaining triples. As well $\lim_{t\to\infty} t.x$ is a point in $(\mathbb{P}^2)^{n-3}$ given by $(n-3)$ triples $(r_{1l} : r_{2l} : r_{3l})$, among which $r_{1l} = 0$ exactly $n-3-i$ times and all the other triples are $(1 : 0 : 0)$.

So we have

$$\begin{aligned}\{x =&((r_{1,l} : r_{2,l} : r_{3,l}))_l \in (\mathbb{P}^2)^{n-3} : \mu(\lim_{t\to 0}(t,1).x) \in (X = i')\} = \\ &\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l : \text{ there exist a partition of } \{1,2,...,n-3\} = I \cup J \\ &\quad \text{such that } r_{2,l} = r_{3,l} = 0 \text{ for every } l \in I \text{ and } r_{2,l} \neq 0 \text{ and } r_{3,l} \neq 0 \text{ for every } l \in J \\ &\quad \text{where } |I| = i'\}.\end{aligned} \tag{6.3}$$

Also

$$\begin{aligned}\{x =&((r_{1,l} : r_{2,l} : r_{3,l}))_l \in (\mathbb{P}^2)^{n-3} : \mu(\lim_{t\to\infty}(t,1).x) \in (X = i)\} = \\ &\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l : \text{ there exist a partition of } \{1,2,...,n-3\} = I \cup J \\ &\quad \text{such that } r_{1,l} \neq 0 \text{ for every } l \in I \text{ and } r_{1,l} = 0 \text{ for every } l \in J \text{ where} \\ &\quad |I| = i \text{ and } |J| = n-3-i\}.\end{aligned} \tag{6.4}$$

In conclusion points x such that $\mu(\mathbb{C}^* x)$ is the entire interval (i', i), will have exactly i' triples $(1 : 0 : 0)$ and exactly $n-3-i$ triples with $r_{1,l} = 0$.

Using a similar argument we can describe the generic points for each $(\mathbb{C}^*)^1$–orbit mapped between two fixed loci $Z = k$ and $Z = k'$ and between two fixed loci $Y = j$ and $Y = j'$. Indeed, for $(Y = j)$ we have

$$\begin{aligned}
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l \in (\mathbb{P}^2)^{n-3} : \mu(\lim_{t\to 0}(1,t).x) \in (Y = j)\} = \\
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l : \text{ there exist a partition of } \{1,2,...,n-3\} = I \cup J \\
&\qquad \text{such that } r_{1,l} = r_{3,l} = 0 \text{ for every } l \in I \text{ and } r_{1,l} \neq 0, r_{3,l} \neq 0 \text{ for every } l \in J \\
&\qquad \text{where } |I| = j \text{ and } |J| = n-3-j\}.
\end{aligned} \tag{6.5}$$

Also

$$\begin{aligned}
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l \in (\mathbb{P}^2)^{n-3} : \mu(\lim_{t\to \infty}(1,t).x) \in (Y = j)\} = \\
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l : \text{ there exist a partition of } \{1,2,...,n-3\} = I \cup J \\
&\qquad \text{such that } r_{2,l} \neq 0 \text{ for every } l \in I \text{ and } r_{2,l} = 0 \text{ for every } l \in J \\
&\qquad \text{where } |I| = j \text{ and } |J| = n-3-j\}.
\end{aligned} \tag{6.6}$$

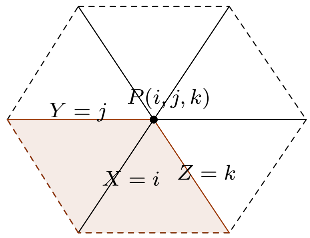
For $(Z = k)$ we have the following:

$$\begin{aligned}
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l \in (\mathbb{P}^2)^{n-3} : \mu(\lim_{t\to \infty}(t,t).x) \in (Z = k)\} = \\
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l : \text{ there exist a partition of } \{1,2,...,n-3\} = I \cup J \\
&\qquad \text{such that } r_{1,l} = r_{2,l} = 0 \text{ for every } l \in I \text{ and } r_{1,l} \neq 0, r_{2,l} \neq 0 \text{ for every } l \in J \\
&\qquad \text{where } |I| = k \text{ and } |J| = n-3-k\}.
\end{aligned} \tag{6.7}$$

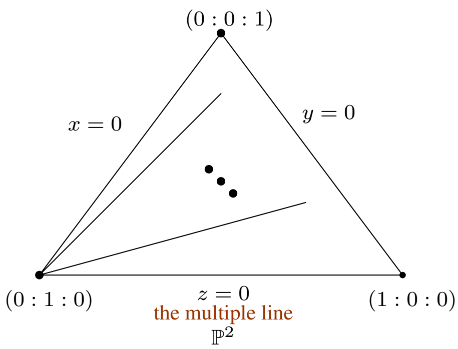
Also

$$\begin{aligned}
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l \in (\mathbb{P}^2)^{n-3} : \mu(\lim_{t\to 0}(t,t).x) \in (Z = k)\} = \\
&\{x = ((r_{1,l} : r_{2,l} : r_{3,l}))_l : \text{ there exist a partition of } \{1,2,...,n-3\} = I \cup J \\
&\qquad \text{such that } r_{3,l} \neq 0 \text{ for every } l \in I \text{ and } r_{3,l} = 0 \text{ for every } l \in J \\
&\qquad \text{where } |I| = k \text{ and } |J| = n-3-k\}.
\end{aligned} \tag{6.8}$$

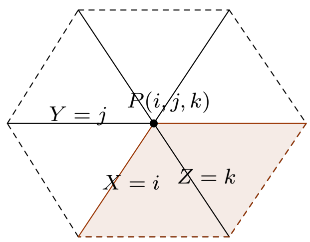
Now we want to describe points mapped between two different nonparallel walls. Consider the points in the shaded region here:



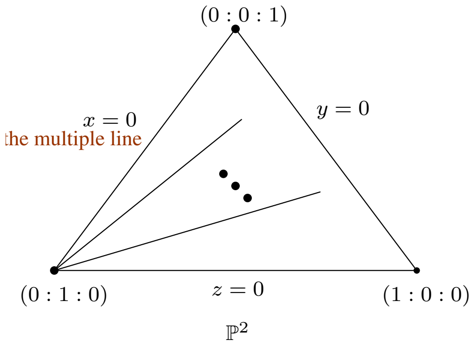
From Equations (6.6) and (6.7), the fiber of the universal family over each point x such that $\mu(\mathbb{C}^* x)$ is the shaded region , contains $n-3-j$ lines with equation $r_{1,l}x+r_{3,l}z = 0$ for $l \in J$ and a k–tuple line $z = 0$ (Since at the limit the orbits meet $Y = j$ and $Z = k$)



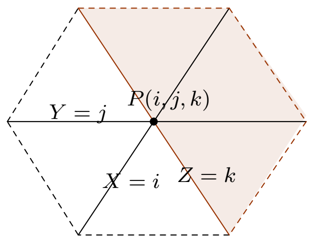
Rotating the shaded region in the hexagon above clockwise by 120 degrees and 240 degrees, respectively, around P, yields similar configurations for the fiber of the universal family (but rotated counter-clockwise by 120 degrees and 240 degrees respectively). The points mapped to the following region are described as follow.



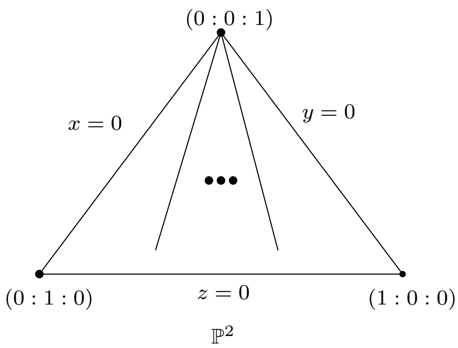
From Equations (6.6) and (6.3) the fiber of the universal family over each point in the pre-image of these points contains $n - 3 - j$ lines with equation $r_{1,l}x + r_{3,l}z = 0$ for $l \in J$ and i–tuple line $x = 0$.



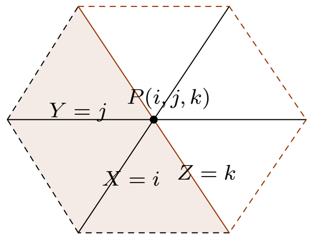
and similarly if we rotate by 120 degrees or 240 degrees around the center. The next case we study is as follow:



From Equation (6.8) the fiber of the universal family over each point in the pre-image of these points contains $n-3-k$ lines with equation $r_{1,l}x + r_{2,l}y = 0$ for $l \in J$.



while



corresponds to a plane with k copies of the line $(z = 0)$. Now for the following case,

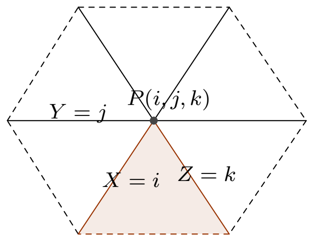
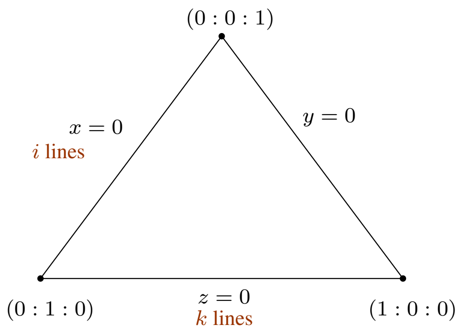


Figure 6.2

From Equations (6.7) and (6.3) the fiber of the universal family over each point in the pre-image of these points contains two groups of special lines. The first group is a set of i lines with equation $x = 0$ (an i–tuple line) and the second is a set of k lines $z = 0$ (a k–tuple line).



Similarly the generic points whose $(\mathbb{C}^*)^2$–orbit maps between two fixed loci $Y = j$ and $Z = k$ have an $(n-3)$–tuple of coordinates where in $r_{2,l} = 0$ for $n - 3 - j$ times and $r_{3,l} = 0$ $n - 3 - k$ times.

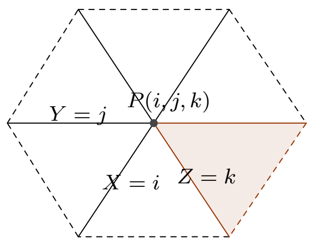
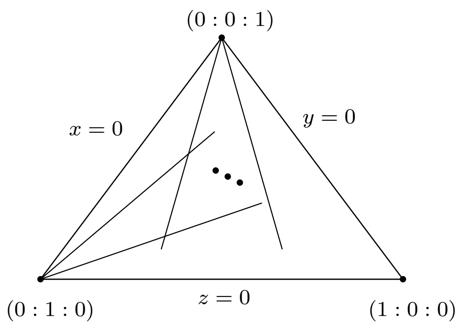


Figure 6.3

In the fiber of universal family over these type of points there are $n-3-j$ lines through $(0:1:0)$ and $n-3-k$ lines through $(0:0:1)$. We obtain similar structures when we rotate the above by 120 degrees or 240 degrees.



In fact this tells us that there are two types of triangular chambers in Figure 6.1. The first group is made of all the triangles with a vertex pointing upwards (same orientation as $\mu((\mathbb{P}^2)^{n-3})$. For a generic point among those whose orbits are mapped by μ inside such a chamber, the fiber of the universal family $F/\!\!/T'$ over x is a projective plane with line arrangements including $(x=0), (y=0)$ and $(z=0)$ with some multiplicity. The second group is made of all the triangles with a vertex pointing downwards. There are no points in X whose orbits are mapped by μ inside such a chamber. Rather, the smallest image of orbits $\mu(O(x))$ containing such a triangle are actually rhombuses. For a generic point of such orbit, the fiber of the universal family $F/\!\!/T'$ is a projective plane with two different sets of lines, each set passing through one of the points $(1:0:0), (0:1:0)$ or $(0:0:1)$.

6.2 Moduli Problem of Compactification of the Moduli Space of Arrangements of Lines in Plane

This sections defines a moduli problem which generalizes the moduli problem of Arrangements of Lines in Plane. We will prove that this functor is represented by the inverse limit of the GIT quotients of $G(3,n)$ by $(\mathbb{C}^*)^{n-1}$. Denote the moment map for this action by $\mu_{(\mathbb{C}^*)^{n-1}} : G(3,n) \to \mathbb{R}^{n-1}$. The notations from the introduction of Section 5.3 will be employed. Recall from Remark 4.2.9 that the image of an orbit via moment map is a convex hull made of union of polytops, with the same vertices as the vertices of the image of Grrassmannian variety via moment map. We call such polytopes Δ–polytopes.

Definition 6.2.1 ***Compactification of the moduli space of Arrangements of lines in plane***

Consider the functor

$$\begin{aligned} \mathcal{C} : \mathfrak{Sch} &\to \mathfrak{Set} \\ B &\mapsto [(\pi : P \to B, D_1, D_2, ..., D_n)], \end{aligned}$$

associating to each scheme B all isomorphism classes of tuples $(\pi, D_1, \ldots, D_n)$ consisting of a flat morphism $\pi : P \to B$ and a set of flat families over B of reduced curves, D_i embedded in P for each $b \in B$ (close point), the fiber P_b is written as $P_b = \cup_{j \in A} S_j$, where each S_j is either $Bl_{r(points)}\mathbb{P}^2$ (where r could be 0 and also these points are distinct) or $\mathbb{P}^1 \times \mathbb{P}^1$, and there exists a corresponding partition $\mu_{(\mathbb{C}^*)^{n-1}}(G(3,n)) = \cup_{j \in A}\Delta_j$ into convex polytopes

$$\Delta_j = (\cap_{l \in L}\mu_{(\mathbb{C}^*)^{n-1}}(Y_l^+)) \cap (\cap_{l' \in L'}\mu_{(\mathbb{C}^*)^{n-1}}(Y_{l'}^-)),$$

and for each $j \in A$, a rational map $\phi_j : P_b \to \mathbb{P}^2$ such that:

(a) For all $j, j' \in A$,

$$S_j \cap S_{j'} = \begin{cases} \mathbb{P}^1 & \Leftrightarrow \Delta_j \cap \Delta_{j'} \text{ is a codimension 1 face}, \\ \textit{point} & \Leftrightarrow \Delta_j \cap \Delta_{j'} \text{ is a codimension 2 face}, \\ \emptyset & \text{otherwise}, \end{cases}$$

and if $S_j \cap S_{j'} \neq \emptyset$ then $S_j \cap S_{j'} \not\subseteq D_i$, and if $S_j \cap S_{j'} \cap S_{j''} \neq \emptyset$ then $S_j \cap S_{j'} \cap S_{j''} \not\subseteq D_i$ for all i.

(b) For all $j \in A$ and $i \in \{1, 2, ..., n\}$ we have $\phi_j(D_i \cap P_b) =$ line in $\mathbb{P}^2$.

(c) With the notations from the introduction of Section 5.3, the sets

$$\{i \in \{1, 2, ..., n\}; \phi_j(D_i) \text{ are all the same line } \},$$

with at least 2 elements, are exactly the sets I_l with $l \in L$. The sets

$$\{i \in \{1, 2, ..., n\}; \phi_j(D_i) \text{ all intersect at the same point}\},$$

with at least 3 elements are exactly the sets $J_{l'}$ with $l' \in L'$.

(d) Consider the set $Z = (\cup_{l \in L, i \in I_l} \phi_j(D_i)) \cup (\cup_{l' \in L'} \cap_{i \in J_{l'}} \phi_j(D_i))$. Then ϕ_j restricts to an isomorphism $P_b \setminus \phi_j^{-1}(Z) \cong \mathbb{P}^2 \setminus Z$.

(e) (1) If $S_j = Bl_{rpoints}\mathbb{P}^2$, with $r \geq 0$, then $\phi_{j_{|S_j}} : S_j \to \mathbb{P}^2$ describes S_j as the blow-up of $\mathbb{P}^2$ along the points where 3 or more lines of the form $\phi_j(D_i)$ intersect.

(2) If $S_j = \mathbb{P}^1 \times \mathbb{P}^1$, then $\phi_{j_{|S_j}} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is the birational map given by projection from a point in $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ and when $S_j \cap S_{j'} \neq \emptyset$ and $S_j \cap D_i \neq \emptyset$, these are only fibers of one of the two projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, when they are nonempty.

(f) For every wall W_l corresponding to $S_i \cap S_j$, there exist a partition $I_l \cup I_j = \{1, 2, ..., n\}$ such that for every $k \in J_l$ we have $\phi_i(D_k) = \phi_i(S_i \cap S_j)$ and for every $k \in I_l$ we have $\phi_i(S_I \cap S_j) \in \phi_i(D_k)$.

Two tuples $(\pi : P \to B, D_1, D_2, ..., D_n)$ and $(\pi' : P' \to B', D'_1, D'_2, ..., D'_n)$ are isomorphic to each other if there exists a map $\phi : B' \to B$ and an isomorphism $\tilde{\phi} : P' \to \phi^*(P)$

$$\begin{array}{ccc} P' \cong \phi^*(P) & & P \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\phi} & B, \end{array}$$

such that $\tilde{\phi}$ restricts to an isomorphism $D'_i \cong \phi^*(D_i)$ for each $i \in \{1, ..., n\}$.

Note that (e) implies that no 3 curves among D_is intersect at a point.

Remark 6.2.1 *For each surface that satisfies the conditions of Definition 6.2.1, there* *is a natural point in the inverse limit of GIT quotients of Grassmannian variety correspondent to it. Each chamber C_k is inside a polytope Δ_i. The chamber corresponds to a GIT quotient M_i. Every Δ_i corresponds to a surface S_i which projects to a $\mathbb{P}^2$ via ϕ_i such that $\phi_i(D_j)$ are lines in $\mathbb{P}^2$. From condition (f) and Proposition 4.2.10 every Δ_i corresponds to a point in the M_k.*

Two disjoint chambers which their images under moment map corresponds to two adjacent polytops (Δ_i, Δ_j) corresponds to a point in the flip. Also two disjoint chambers which their images under moment map corresponds to the same polytope, their corresponding chambers are in the same polytope.

6.3 Main Theorem

In this section first we build up the inverse limit of universal families over the inverse limit of the GIT quotients of $G(3, n)$ with the sub-torus $(\mathbb{C}^*)^{n-3}$. We recall the definition of an inverse limit. Let $\{S_i\}_{i\in I}$ be a set of schemes and $\phi_{(i_1,i_2)} : S_{i_1} \to S_{i_2}$, a set of morphisms for any $(i_1, i_2) \in J$. Let f be the morphism

$$f : \prod_{(i_1,i_2)\in J} S_{i_1} \to \prod_{(i_1,i_2)\in J} S_{i_2},$$

where it is defined component wise by $\phi_{(i_1,1_2)}$. Let Δ be the diagonal in $\prod_{(i_1,i_2)\in J} S_{i_2}$. The inverse limit defines as

$$\varprojlim S_i = f^{-1}(\Delta).$$

We will prove that the functor of compactification of arrangements of lines in planes, that we have defined in the previous section, can be represented with the inverse limit of GIT quotients of Grassmannians.

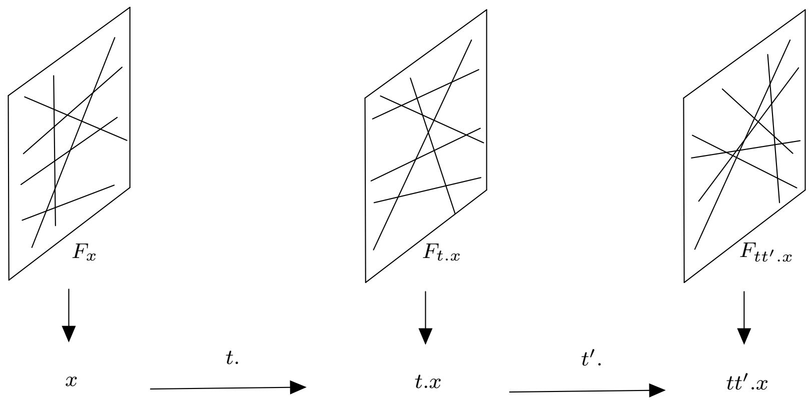
Theorem 6.3.1 *The functor of compactification of arrangements of lines in planes can be represented by the inverse limit of GIT quotients of Grassmannian varieties.*

Now we start building up the inverse limit. Recall the following diagram

$$\begin{array}{ccc} F & \longrightarrow & \mathbb{P}^{n-1} = \text{Proj}(\mathbb{C}[x_1, \dots, x_n]) \\ \downarrow & & \\ G(3, n), & & \end{array}$$

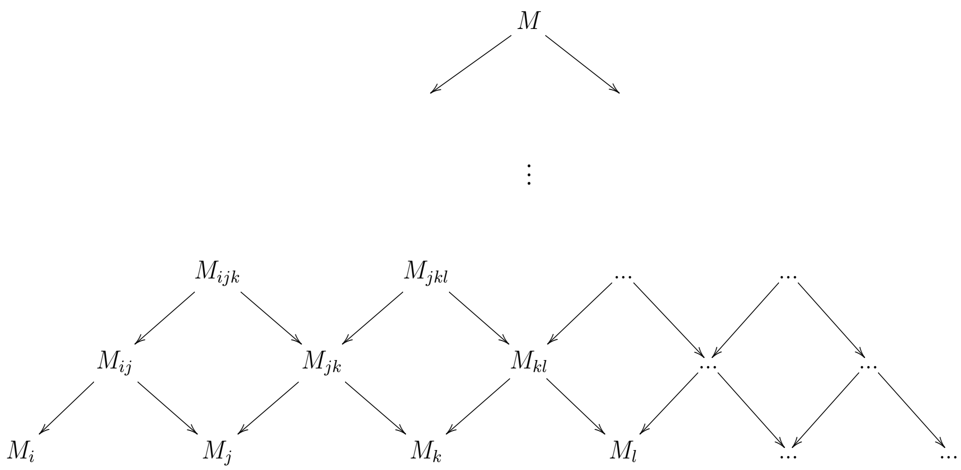
where $F = F(1, 3, n)$ is the Flag variety. Note that each GIT quotient parameterizes sets of n lines in plane. For instance for a point $x \in G(3, n)$ and two arbitrary elements

t and t' in the torus acting on $G(3,n)$ we have a class of planes with lines in them as below:



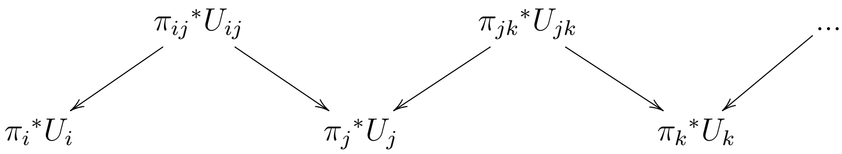
Before going through the proof we need to explain how we construct this inverse limit. Variation of GIT gives us birational morphisms between various GIT quotients. In general we construct the inverse limit using fiber product of GIT quotients via such morphisms.

We start with building up the universal family over inverse limit of GIT quotients of Grassmannian variety. We start with the following inverse system as shown in the following diagram. The elements in the first (lowest) row are the GIT quotients of the Grassmannian variety. Recall that there is finite number of them. The elements in the second row are the elements on the top of the flips which are either the blow-up of an element of the first row or they are isomorphic to the elements on the first row which is a blow-up along a divisor. Then the elements in the next rows are just the fiber product of the elements below them. So we have the following diagram



where $M = \varprojlim M_i$, $M_{ijk} = M_{ij} \times_{M_j} M_{jk}$, $M_{jkl} = M_{jk} \times_{M_k} M_{kl}$ and etc. Note that this diagram is not a tree and there might be loops in there if we make an abstraction of the orientation of the arrow. For example we might have a blow up map from M_{jk} to M_i.

Now we wish to construct a universal family over this limit. A first candidate would seem to be the inverse limit of pull backs of universal families, constructed as follows: let U_i be the universal family over M_i, (enjoying the properties of Definition 5.3.1) and U_{ij} the universal family of M_{ij} (enjoying the properties of Definition 5.4.1). We pull-back the families U_i, U_{ij} and the maps between them, to the inverse limit $M = \varprojlim M_i$. Thus we obtain a system of maps



(Where $\pi_i : M \to M_i$ and $\pi_{ij} : M \to M_{ij}$ are the natural maps). We complete this to an inverse system by taking the fiber products of the existing maps. The inverse limit $\tilde{U}$ of this inverse system has a natural map to M, and the generic fiber is a surface. Unfortunately, some of the fibers of $\tilde{U} \to M$ might be higher dimensional. When the inverse system is generated by three GIT quotients M_i, M_j and M_k, consider the flips M_{ij} and M_{jk} with the universal families U_{ij} and U_{jk} mapping to the universal family U_j over M_j. We take the fiber products like in the diagram:

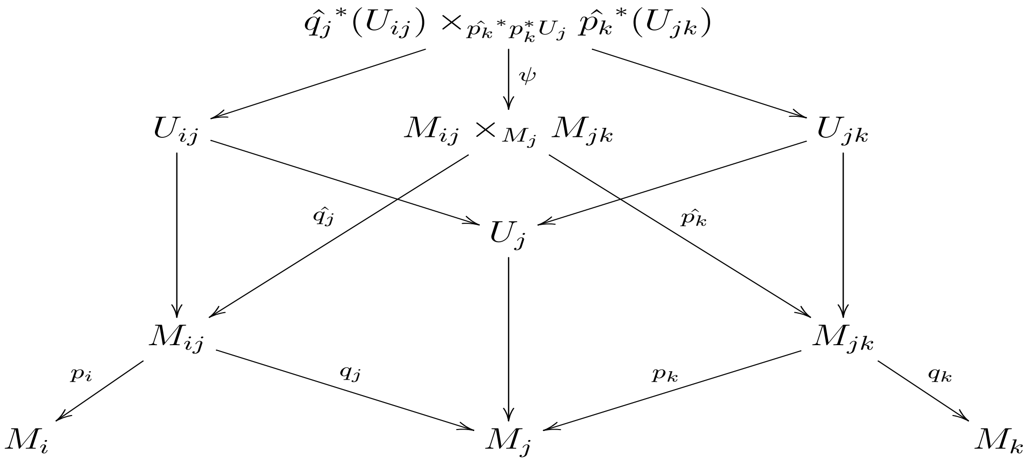


Figure 6.4

In the case of taking fiber product of fibers we may need to refine our construction so that we eventually obtain a family of surfaces $U \to M$. In order to construct U, we will first need to gain a detailed understanding of the special loci in M and the surfaces parameterized by them.

Recall the image of the moment map for the $(\mathbb{C}^*)^2$–action on $G(3,n)/\!\!/T' \cong (\mathbb{P}^2)^{n-3}$:

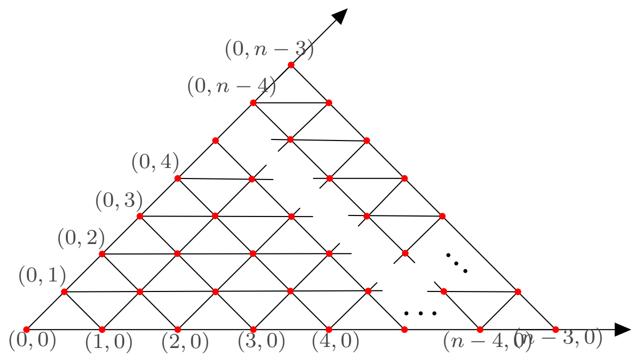
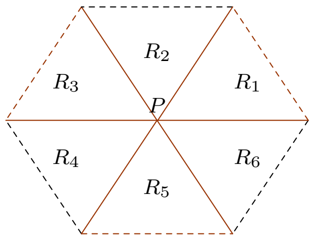


Figure 6.5

For each chamber C_i in $\mu_{(\mathbb{C}^*)^2}((\mathbb{P}^2)^{n-3})$, let M_i be the corresponding GIT quotient $(\mathbb{C}^*)^2\mu^{-1}(C_i^\circ)/\!\!/(\mathbb{C}^*)^2$, and consider the projection $\pi_i : M \to M_i$ from M, the inverse limit of GIT quotients, to M_i. Note that as $(\mathbb{P}^2)^{n-3}$ is the quotient of $G(3,n)$ by $(\mathbb{C}^*)^{n-3}$ and M_i is the quotient of $(\mathbb{P}^2)^{n-3}$ by $(\mathbb{C}^*)^2$, M_i is also the GIT quotient of $G(3,n)$ by $(\mathbb{C}^*)^{n-1}$. Indeed for each $(\mathbb{C}^*)^{n-1}$ linearization of $\mathcal{O}(1)$ on $G(3,n)$ we pick $(\mathbb{C}^*)^{n-3} \subset (\mathbb{C}^*)^{n-1}$ such that $G(3,n)^{ss}_{(\mathbb{C}^*)^{n-1}}(\mathcal{O}(1)) \subset G(3,n)^{ss}_{(\mathbb{C}^*)^{n-3}}(\mathcal{O}(1))$. The $(\mathbb{C}^*)^{n-1}$ linearizatin of $\mathcal{O}(1)$ induces a $(\mathbb{C}^*)^2$ linearization of $\mathcal{O}(1)$ on $(\mathbb{P}^2)^{n-3}$. From the point of view of moment map, the image of the moment map for $\mu_{(\mathbb{C}^*)^2} : (\mathbb{P}^2)^{n-3} \to \mathbb{R}^2$ is the section of the image of the moment map $\mu_{(\mathbb{C}^*)^{n-1}} : (\mathbb{P}^2)^{n-3} \to \mathbb{R}^{n-1}$.

Consider now a node P in $\mathrm{Image}(\mu_{(\mathbb{C}^*)^2})$, that is, the image of a fixed locus of $(\mathbb{C}^*)^2$. We saw that there are 6 regions in $\mathrm{Image}(\mu_{(\mathbb{C}^*)^2})$ around the node which are not on the axis:

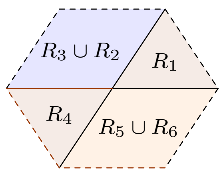


and within each chamber R_i, a chamber C_i corresponding to a GIT quotient M_i ($i \in \{1, ..., 6\}$).

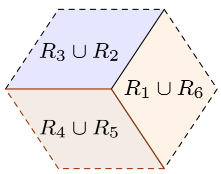
Definition 6.3.1 *To each partition $\mathcal{P} = (\cup_{i\in I} R_i)_I$ we associate a stratum (locally closed subset) $Y_{\mathcal{P}}$ of M as follows:*

- *For each $i \in I \subset \{1, 2, ..., 6\}$, we define $Y^i_I \subset M_i$ to be the locus of points $[x] \in M_i$ such that $\mu((\mathbb{C}^*)^2.x) \subset \cup_{i\in I} R_i$ and $\overline{\mu((\mathbb{C}^*)^2.x)}$ contains the walls bounding $\cup_{i\in I} R_i$ (in the neighborhood of P).*
- *We define $Y_{\mathcal{P}}$ to be the locus of points $[x] \in M$ such that $\pi_i([x]) \in Y^i_I$ for each $i \in \{1, 2, ..., 6\}$ and all I in the partition $\mathcal{P}$.*

The region around the point P can be partitioned into unions of regions bounded by walls, e.g.



or

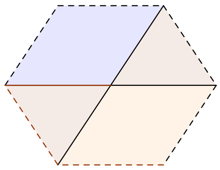


etc.

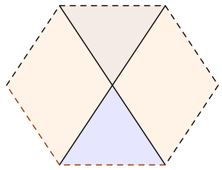
Our purpose is to give a natural description of the fibers of $U \to M$ over generic points in the locus $Y_{\mathcal{P}}$ for various partitions $\mathcal{P}$.

Now we start studying different partitions $\mathcal{P}$ around the image of a locus fixed by

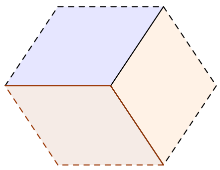
$(\mathbb{C}^*)^2$. Up to rotations by 120° the following are all the possible partitions which are 6 triangles (case h), 1 parallelogram and 4 triangles (case d and g) , 2 parallelograms and 2 triangles (case a, b and f), 2 trapezoids (case e) or 3 parallelograms (case c):



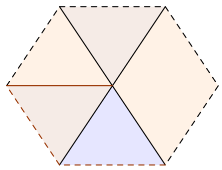
(a)



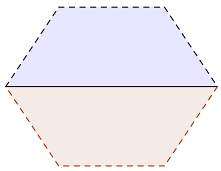
(b)



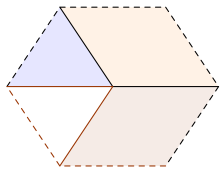
(c)



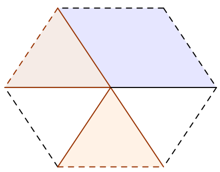
(d)



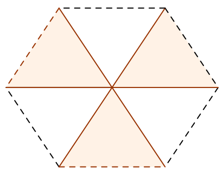
(e)



(f)

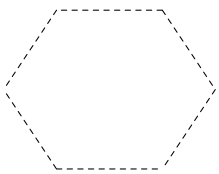


(g)



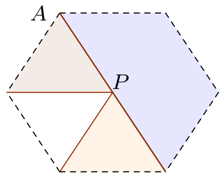
(h)

Case (h) consists of six triangles and case (i), corresponding to $\{1, 2, ..., 6\}$, consists of the entire region around P.



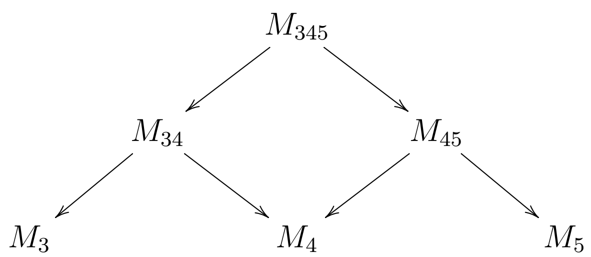
(i)

Note for example that the following case can't happen.



There is no point in $M = \varprojlim M_i$ corresponding to such a combination of orbits. Indeed, following [7], for any $[x]$ in the sub-locus of M represented below, the closures of the orbits represented by $\pi_5([x])$ and $\pi_6([x])$ share a common 1-dimensional orbit fixed by $\mathbb{C}^* \times \{1\} \subset (\mathbb{C}^*)^2$ (see beginning of section 1). Let y be a point on this 1-dimensional orbit such that is a limit point for the orbit represented by $\pi_5[x]$, then via the action of $\Delta = \{(u,u); u \in \mathbb{C}^*\}$ we have $\mu(\lim_{u\to\infty} u.y) = P$, while writing y as a limit point for the orbit represented by $\pi_6[x]$ given $\mu(\lim_{u\to\infty} u.y) = A$ which is a contradiction as $A \neq P$.

To construct the inverse limit of GIT quotients in each case we start with three consecutive chambers. Denote $M_i = (\mathbb{C}^*)^2\mu^{-1}(R_i)/\!/(\mathbb{C}^*)^2$ for $i = 1, 2, ..., 6$. As it is shown in the following diagram we consider the inverse system associated to the 3 regions R_3, R_4 and R_5:



where $I = \{3, 4, 5\}$ and $M_{345} = \varprojlim_{i\in I} M_i = M_{34} \times_{M_4} M_{45}$.

Recall that in Definition 5.4.1 we defined the moduli problem for two GIT quotients and the flip between them. This definition applies to spaces M_{34} and M_{45} above. Now it is time to define the moduli problem for three consecutive chambers where the middle one is of second group (as defined at the end of Section 6.1 Figure 6.3).

In fact, there are two ways to define the moduli problem as follows.

Definition 6.3.2 ***The First Moduli Problem for The Partial Inverse Limit of Three GIT Quotients*** *Consider three fixed chambers C_k, C_h and C_j such that $\{k, h, j\} = \{3, 4, 5\}$ or $\{1, 2, 3\}$ or $\{5, 6, 1\}$ and the first two chambers share a wall W_l and the last two chambers share the wall $W_{l'}$. Consider $I_l \cup J_l = I_{l'} \cup J_{l'} = \{1, 2, ..., n\}$*

partitions associated to the walls W_l and $W_{l'}$. With the conventions used in Section 5.3, note that $I_l \cap I_{l'} = \emptyset$ (as we have 2 different lines given by $x = 0$ and $y = 0$ and the line $\{\ell_i\}_{i \in I_l}$ coincide with $x = 0$ and the lines $\{\ell_i\}_{i \in I_{l'}}$ coincide with $y = 0$), hence $I_l \subset J_{l'}$ and $I_{l'} \subset J_l$. Consider the functor

$$\begin{aligned} \mathcal{U}_{khj} : \mathfrak{Sch} &\rightarrow \mathfrak{Set} \\ B &\mapsto [(\pi : P \rightarrow B, D_1, ..., D_n)], \end{aligned}$$

associating to each scheme B all isomorphism classes of tuples

$$(\pi : P \rightarrow B, D_1, ..., D_n),$$

consisting of a flat morphism $\pi : P \rightarrow B$ whose fibers P_b are all connected and are either $\mathbb{P}^2$ or $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{point}\mathbb{P}^2$ or $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{2(points)}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ ($\mathbb{P}^1$ in $Bl_{point}\mathbb{P}^2$ is the exceptional divisor and two $\mathbb{P}^1$ in $Bl_{2point}\mathbb{P}^2$ are the different exceptional divisors) and such that the restrictions $\ell_{1,b}, \ell_{2,b}, ..., \ell_{n,b}$ of the reduced curves $D_1, D_2, ..., D_n$ to the fiber P_b satisfy:

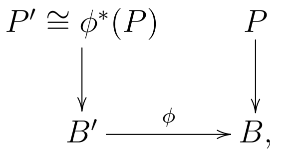
- *if $P_b = \mathbb{P}^2$, then all $\ell_{i,b}$ are lines.*
- *if $P_b = \mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{point}\mathbb{P}^2$, then either for all $i \in I_l$, or for all $i \in I_{l'}$, we have $L_i := \ell_{i,b}$ sections of the projection on the exceptional divisor $Bl_{point}\mathbb{P}^2 \rightarrow \mathbb{P}^1$, and all other $\ell_{i,b}$ are the connected unions of a fiber of $\phi : Bl_{point}\mathbb{P}^2 \rightarrow \mathbb{P}^1$, with a line in $\mathbb{P}^2$.*
- *if $P_b = \mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{2points}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$, then we can denote the two planes by $\mathbb{P}^2_l$ and $\mathbb{P}^2_{l'}$, and the projections to the exceptional divisors $\phi_l : Bl_{2points}\mathbb{P}^2 \rightarrow P^1_l$ and $\phi_{l'} : Bl_{2points}\mathbb{P}^2 \rightarrow P^1_{l'}$ such that $P^1_l \subset \mathbb{P}^2_l$ and $P^1_{l'} \subset \mathbb{P}^2_{l'}$ are the exceptional divisors in $Bl_{2points}\mathbb{P}^2$. Then*
  - *for each $i \in I_l$, the restrictions $\ell_{i,b}$ are connected unions of a fiber of $\phi_{l'}$ with a line in $\mathbb{P}^2_{l'}$, and similarly if we swap l with l'.*
  - *for all $i \in J_l \cap J_{l'}$, $\ell_{i,b}$ is a connected union between a line in $\mathbb{P}^2_l$, a line in $\mathbb{P}^2_{l'}$ and the strict transform of the line PQ in $Bl_{P,Q}\mathbb{P}^2$.*

In general for all fibers P_b

- *if $\ell_{i_1,b} = \ell_{i_2,b} = ... = \ell_{i_a,b}$ then $\{i_1, ..., i_a\} \subseteq I$ for some $I \in \mathcal{A}^+_k \cup \mathcal{A}^+_h \cup \mathcal{A}^+_j$, and $I_l \neq \{i_1, ..., i_a\} \neq I_{l'}$,*
- *if $\ell_{j_1,b}, \ell_{j_2,b}, ..., \ell_{j_c,b}$ have a point in common then $\{j_1, j_2, ..., j_c\} \subseteq J$ for some $J \in \mathcal{A}^-_k \cup \mathcal{A}^-_h \cup \mathcal{A}^-_j$, and $J_l \neq \{j_1, j_2, ..., j_c\} \neq J_{l'}$,*

where $\mathcal{A}_k^+, \mathcal{A}_k^-, \mathcal{A}_h^+, \mathcal{A}_h^-, \mathcal{A}_j^+$ and $\mathcal{A}_j^-$ are as defined in Section 5.3.

Two tuples $(\pi : P \to B, D_1, D_2, ..., D_n)$ and $(\pi' : P' \to B', D'_1, D'_2, ..., D'_n)$ are isomorphic to each other if there exists a map $\phi : B' \to B$ and an isomorphism $\tilde{\phi} : P' \to \phi^*(P)$



such that $\tilde{\phi}$ restricts to an isomorphism $D'_i \cong \phi^*(D_i)$ for each $i \in \{1, ..., n\}$.

Remark 6.3.2 *Note that the strict transform for PQ in $\pi : Bl_{P,Q}\mathbb{P}^2 \to \mathbb{P}^2$ is the unique global section of $\mathcal{O}(\pi^*H - E_1 - E_2)$, while the fibers of $Bl_{P,Q}\mathbb{P}^2 \to E_i$ for $i = 1, 2$ can be thought of as the global sections of $\mathcal{O}(\pi^*H - E_i)$. (here $H = \mathcal{O}_{\mathbb{P}^2}(1)$)*

Theorem 6.3.3 *Let's assume the chambers C_k, C_h and C_j are as in the previous definition. Then the scheme $M_{khj} = M_{kh} \times_{M_h} M_{hj}$ represents the functor defined above, where M_k, M_h and M_j are the GIT quotients corresponding to C_k, C_h and C_j, respectively. Also $U_{khj} = p_{kh}^* U_k \times_{p_h^* U_h} p_{hj}^* U_j$ represents the universal family over M_{khj} where U_k, U_h and U_j are the universal families over M_k, M_h and M_j respectively.*

Proof: Consider the following diagram:

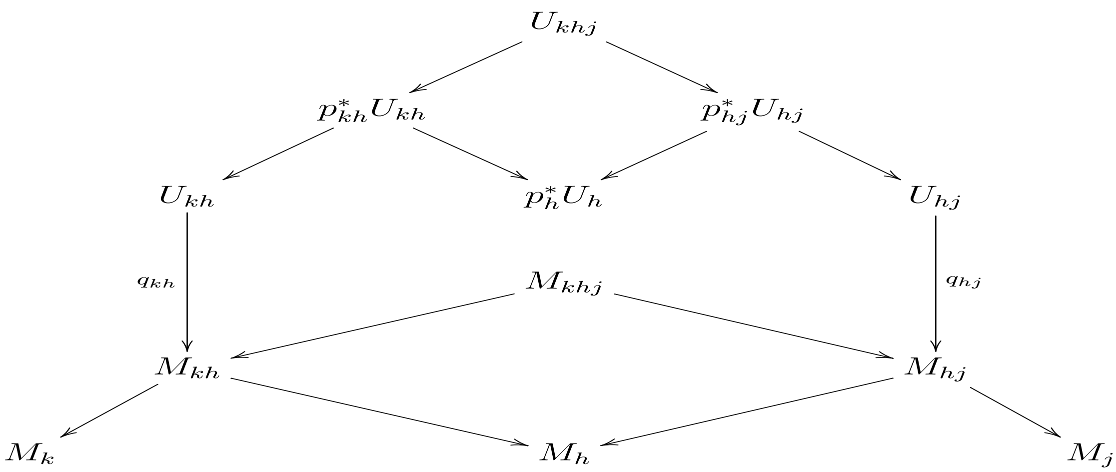


Figure 6.6: Inverse limit of the universal families over the inverse limit of three GIT quotients

where for $T = (\mathbb{C}^*)^2$,

$M_k = T.\mu^{-1}(C_k^\circ)/T$,
$M_h = T.\mu^{-1}(C_h^\circ)/T$,
$M_j = T.\mu^{-1}(C_j^\circ)/T$,
M_{kh} and M_{hj} are the flips, $M_{khj} = M_{kh} \times_{M_h} M_{hj}$, U_{kh} and U_{hj} are universal families over M_{kh} and M_{hj} respectively, where $p_h : M_{khj} \to M_h$, $p_{kh} : M_{khj} \to M_{kh}$ and $p_{hj} : M_{khj} \to M_{hj}$ are the natural contractions, and U_k, U_h and U_j are the universal families over M_k, M_h and M_j respectively.

We claim that the family $(p : U_{khj} \to M_{khj}, \ell_1, ..., \ell_n)$ is the universal family over M_{khj}, for the given moduli problem. Note that U_{khj} is obtained by gluing open sets from $p_{kh}^* U_{kh}$ and $p_{hj}^* U_{hj}$. So it is flat over M_{khj}. Next we want to show that for any tuple

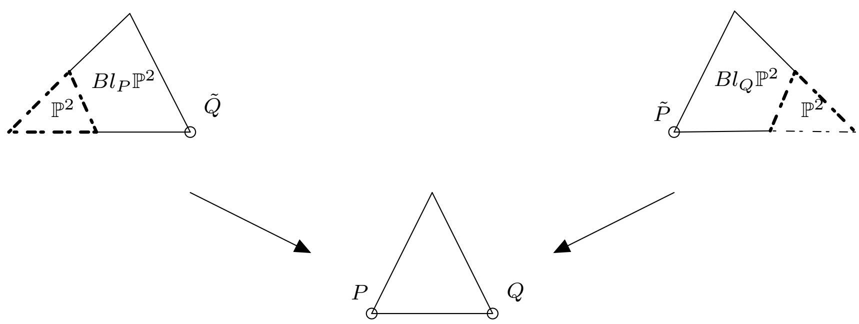
$$(f : \mathcal{F} \to B, D_1, D_2, ..., D_n),$$

satisfying the conditions in Definition 6.3.2; there is a unique map $\psi : B \to M_{khj}$ such that

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & U_{khj} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ B & \xrightarrow{\ \psi\ } & M_{khj}, \end{array}$$

is a fiber product i.e. $\mathcal{F} \cong \psi^*(p_1^{-1}(U')/(\mathbb{C}^*)^n)$ and $D_i \cong \psi^*(\ell_i)$.

We first check that $(p : U_{khj} \to M_{khj}, \ell_1, ..., \ell_n)$ satisfies the conditions in Definition 6.2. To construct the blow-up of the projective plane in two points P and Q, two blow-ups $Bl_P\mathbb{P}^2$ and $Bl_Q\mathbb{P}^2$ can be glued together via $Bl_P\mathbb{P}^2 \setminus (E_1 \cup \widetilde{\{Q\}}) \cong \mathbb{P}^2 \setminus \{P, Q\} \cong Bl_Q\mathbb{P}^2 \setminus (E_2 \cup \widetilde{\{P\}})$ where E_1 and E_2 are the exceptional divisors. As a consequence, $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{P,Q}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ can be regarded as the fiber product of $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{P,Q}\mathbb{P}^2$ and $Bl_{P,Q}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ over $\mathbb{P}^2$, via the maps contracting the two planes to the points P and Q respectively, like in the diagram.



Thus the fibers of $q_{khj} : U_{khj} \to M_{khj}$ satisfy the condition of Definition 6.3.2. In fact if $[x] \in M_{khj}$ is such that $q_{khj}^{-1}[x]$ is $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{P,Q}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$, then the Cartesian diagram at the top of Figure 6.6 restricts over $[x]$ to:

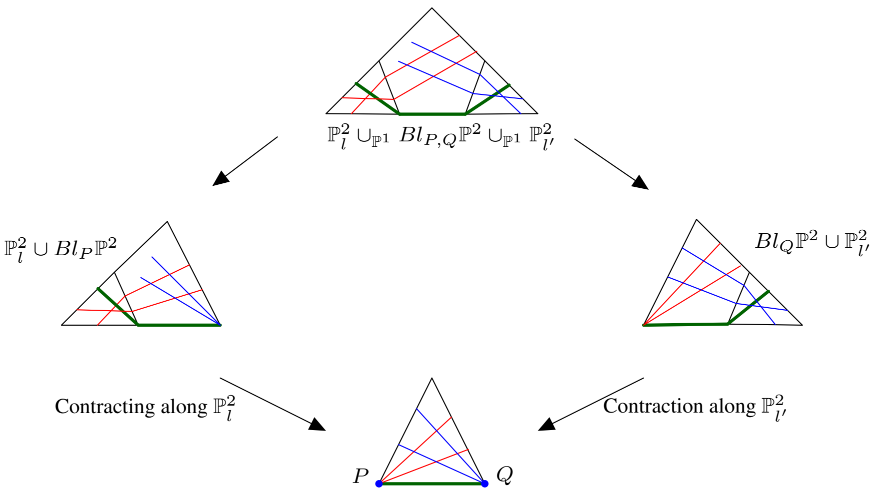


Figure 6.7

where the lines in red represent ℓ_i with $i \in I_{l'}$, those in blue represent ℓ_i with $i \in I_l$, and the green ones ℓ_i with $i \in J_l \cap J_{l'}$.

A flat family $f_{kh} : \mathcal{F}_{kh} \to B$ which satisfy the condition of Definition 5.4.1, is constructed from $f : \mathcal{F} \to B$ by contracting the fibers which are isomorphic to $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{2points}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ along $\mathbb{P}^2_l$. This contraction can be defined as $\mathcal{F} = Proj \oplus_n \mathcal{O}_{\mathcal{F}}(nD_i) \to Proj \oplus_n f_*\mathcal{O}_{\mathcal{F}}(nD_i)$ for some $i \in I_l$ (such that $D_i \cap \mathbb{P}^2_l = \emptyset$). Hence from Theorem 5.4.1, there exists a unique map ψ_{kh} such that

$$\begin{array}{ccc} \mathcal{F}_{kh} & \longrightarrow & U_{kh} \\ {\scriptstyle f_{kh}}\downarrow & & \downarrow \\ B & \xrightarrow{\psi_{kh}} & M_{kh}, \end{array}$$

is a fiber product. Also from definition of M_{khi}, the map ψ_{kh} factors through M_{khj} which provides the following diagram.

$$\begin{array}{ccccc}
\mathcal{F}_{kh} & & \longrightarrow & & U_{kh} \\
 & \searrow & & \nearrow & \\
\downarrow f_{kh} & & p^*_{kh}U_{kh} & & \downarrow \\
 & & \downarrow & & \\
B & & \xrightarrow{\psi_{kh}} & & M_{kh} \\
 & \searrow \psi'_{kh} & & \nearrow & \\
 & & M_{khj} & &
\end{array}$$

Note that the

$$\begin{array}{ccc}
{p_{kh}}^*U_{kh} & \longrightarrow & U_{kh} \\
\downarrow & & \downarrow \\
M_{khj} & \longrightarrow & M_{kh},
\end{array}$$

is a fiber product, hence

$$\begin{array}{ccc}
\mathcal{F}_{kh} & \longrightarrow & p^*_{kh}U_{kh} \\
\downarrow & & \downarrow \\
B & \xrightarrow{\psi'_{kh}} & M_{khj},
\end{array}$$

is a fiber product.

Similarly we create $f_{hj} : \mathcal{F}_{hj} \to B$ of a flat family of surfaces and therefore there exist a unique map ψ'_{hj} such that

$$\begin{array}{ccc}
\mathcal{F}_{hj} & \longrightarrow & p^*_{hj}U_{hj} \\
\downarrow & & \downarrow \\
B & \xrightarrow{\psi'_{hj}} & M_{khj},
\end{array}$$

is a fiber product. We also create the family $f_h : \mathcal{F}_h \to B$ by contracting the other $\mathbb{P}^2$ in the special fibers isomorphic to $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{2points}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$. Thus $\mathcal{F} = \mathcal{F}_{kh} \times_{\mathcal{F}_h} \mathcal{F}_{hj}$.

From the existence of ψ'_{hj}, ψ'_{kh} and ψ'_h, there exist a unique $\psi_{khj} : B \to M_{khj}$ such that

$$\begin{array}{ccc}
\mathcal{F} = \mathcal{F}_{kh} \times_{\mathcal{F}_h} \mathcal{F}_{hj} & \longrightarrow & U_{khj} \\
\downarrow & & \downarrow \\
B & \xrightarrow{\psi_{khj}} & M_{khj} = M_{kh} \times_{M_h} M_{hj},
\end{array}$$

is a fiber product where $\psi_{khj} = (\psi'_{kh}, \psi'_{hj})$ for $M_{khj} = M_{kh} \times_{M_h} M_{hj}$. □

Definition 6.3.3 ***The Second Moduli Problem for The Partial Inverse Limit of Three GIT Quotients*** *Consider three fixed chambers C_k, C_h and C_j as in Definition 6.3.2, such that the first two chambers share a wall W_l and the last two chambers share the wall $W_{l'}$. Consider $I_l \cup J_l = I_{l'} \cup J_{l'} = \{1, 2, ..., n\}$ partitions associated to the walls*

W_l and $W_{l'}$. Consider the functor

$$\begin{aligned} \mathcal{U}'_{khj} : \mathfrak{Sch} &\rightarrow \mathfrak{Set} \\ B &\mapsto [(\pi : P \rightarrow B, D_1, ..., D_n)], \end{aligned}$$

associating to each scheme B the set of all isomorphism classes of tuples

$$(\pi : P \rightarrow B, D_1, ..., D_n),$$

consisting of a flat morphism $\pi : P \rightarrow B$ whose fibers P_b are all connected and are either $\mathbb{P}^2$ or $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{point}\mathbb{P}^2$ or $\mathbb{P}^2 \cup_{\mathbb{P}^1} (\mathbb{P}^1 \times \mathbb{P}^1) \cup_{\mathbb{P}^1} \mathbb{P}^2$ and such that the restrictions $\ell_{1,b}, \ell_{2,b}, ..., \ell_{n,b}$ of the reduced curves $D_1, D_2, ..., D_n$ to the fiber P_b satisfy:

- *if $P_b = \mathbb{P}^2$, then all $\ell_{i,b}$ are lines.*
- *if $P_b = \mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{point}\mathbb{P}^2$, then either for all $i \in I_l$, or for all $i \in I_{l'}$, we have $L_i := \ell_{i,b}$ sections of the projection on the exceptional divisor $Bl_{point}\mathbb{P}^2 \rightarrow \mathbb{P}^1$, and all other $\ell_{i,b}$ are the connected unions of a fiber of $\phi : Bl_{point}\mathbb{P}^2 \rightarrow \mathbb{P}^1$, with a line in $\mathbb{P}^2$ intersecting the fiber at a point on the exceptional divisor of $Bl_{point}\mathbb{P}^2$.*
- *if $P_b = \mathbb{P}^2 \cup_{\mathbb{P}^1_l} (\mathbb{P}^1 \times \mathbb{P}^1) \cup_{\mathbb{P}^1_{l'}} \mathbb{P}^2$, then we can denote the two planes by $\mathbb{P}^2_l$ and $\mathbb{P}^2_{l'}$, and the projections $\phi_l : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1_l$ and $\phi_{l'} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow {\mathbb{P}^1}_{l'}$ such that ${\mathbb{P}^1}_l \subset \mathbb{P}^2_l$ and ${\mathbb{P}^1}_{l'} \subset \mathbb{P}^2_{l'}$. Then*
  - *for all $i \in J_l \cap J_{l'}$, then $\ell_{i,b} \subset \mathbb{P}^1 \times \mathbb{P}^1$ are unions of two lines, one in each $\mathbb{P}^2$, and intersecting at the point of intersection ${\mathbb{P}^1}_l \cap {\mathbb{P}^1}_{l'} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \cap \mathbb{P}^2 \cap \mathbb{P}^2$,*
  - *for all $i \in I_l$, then $\ell_{i,b}$s are connected unions of a fiber of $\phi_{l'}$ and a line in $\mathbb{P}^2_{l'}$, and similarly if we swap l with l',*

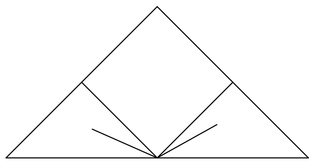
In general for all fibers P_b

- *if $\ell_{i_1,b} = \ell_{i_2,b} = ... = \ell_{i_a,b}$ then $\{i_1, ..., i_a\} \subseteq I$ for some $I \in \mathcal{A}^+_k \cup \mathcal{A}^+_h \cup \mathcal{A}^+_j$, and $I_l \neq \{i_1, ..., i_a\} \neq I_{l'}$,*
- *if $\ell_{j_1,b}, \ell_{j_2,b}, ..., \ell_{j_c,b}$ have a point in common then $\{j_1, j_2, ..., j_c\} \subseteq J$ for some $J \in \mathcal{A}^-_k \cup \mathcal{A}^-_j \cup \mathcal{A}^-_j$, and $J_l \neq \{j_1, j_2, ..., j_c\} \neq J_{l'}$,*

where $\mathcal{A}^+_k, \mathcal{A}^-_k, \mathcal{A}^+_h, \mathcal{A}^-_h, \mathcal{A}^+_j$ and $\mathcal{A}^-_j$ are as defined in Section 5.3.

The isomorphism conditions are as in Definition 6.3.2.

Remark 6.3.4 *Some of D_is might not be Cartier divisors in P as for example in the* *following case*



$\mathbb{P}^2 \cup_{\mathbb{P}^1} (\mathbb{P}^1 \times \mathbb{P}^1) \cup_{\mathbb{P}^1} \mathbb{P}^2$

To understand the transition from $Bl_{P,Q}\mathbb{P}^2$ in Definition 6.3.2 to $\mathbb{P}^1 \times \mathbb{P}^1$ in Definition 6.3.3 we will start with the following lemma.

Lemma 6.3.5 *Given the blow-up $\pi : Bl_{\{P,Q\}}\mathbb{P}^2 \to \mathbb{P}^2$, the strict transform $l := \widetilde{PQ}$ of the line PQ has self-intersection -1.*

Proof: Consider the blow-up morphism $\pi : Bl_{\{P,Q\}}\mathbb{P}^2 \to \mathbb{P}^2$ and denote $l' := PQ$ and $l = \widetilde{PQ}$, and $E = E_1 + E_2$ =the sum of the exceptional divisors. Then

$$1 = (\pi^*(l'))^2 = l^2 + E^2 + 2l.E = l^2 - 2 + 4 = l^2 + 2,$$

hence $l^2 = -1$. □

Theorem 6.3.6 *Both moduli problems introduced in Definition 6.3.2 and Definition 6.3.3, are represented by the same smooth variety.*

Proof: First the smoothness is a consequence of the fact that the strata of the blow-ups $M_{34} \to M_4$ and $M_{45} \to M_4$ intersect transversally. This can be easily seen in the preimage of M_4 in $G(3, n)$.

Now for any scheme B, suppose $(f : \mathcal{F} \to B, D_1, ..., D_n)$ is a family which satisfies conditions of Definition 6.3.2. With the notations from Definition 6.3.2, let $\mathcal{L} := \mathcal{O}_{\mathcal{F}}(\sum_{i\in I_l \cup I_{l'}} D_i)$ and consider the morphism

$$g : Proj(\oplus_n \mathcal{L}^{\otimes n}) \to \mathrm{Proj}(\oplus_n f_* \mathcal{L}^{\otimes n}).$$

Note that $\mathcal{F} = \mathrm{Proj}(\oplus_n \mathcal{L}^{\otimes n})$ and let $\mathcal{F}' := Image(g)$. For those $b \in B$ such that $\mathcal{F}_b = \mathbb{P}^2$ or $\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{point}\mathbb{P}^2$, we can easily check that $\mathcal{L}_{|_{\mathcal{F}_b}}$ is very ample, and therefore $\mathcal{F}_b \cong \mathcal{F}'_b$. Indeed let $r = |I_l| + |I_{l'}|$. Then $\mathcal{L}_{|_{\mathbb{P}^2}} \cong \mathcal{O}_{\mathbb{P}^2}(|I_l|)$, $\mathcal{L}_{|_{Bl_P\mathbb{P}^2}} \cong \mathcal{O}_{Bl_P\mathbb{P}^2}(r.\pi^*H - |I_l|E_1)$ or $\mathcal{O}_{Bl_Q\mathbb{P}^2}(r.\pi^*H - |I_{l'}|E_2)$, and $\sum_{i\in I_l \cup I_{l'}} D_i$ separates points and tangent directions on different components of $\mathcal{F}_b$. As

$$H^0(\mathbb{P}^2, \mathcal{L}_{|_{\mathbb{P}^2}}) \to H^0(\mathbb{P}^2 \cap Bl_P\mathbb{P}^2, \mathcal{L}_{|_{\mathbb{P}^2 \cap Bl_P\mathbb{P}^2}}) \to 0,$$

and

$$H^0(Bl_P\mathbb{P}^2, \mathcal{L}_{|_{Bl_P\mathbb{P}^2}}) \to H^0(\mathbb{P}^2 \cap Bl_P\mathbb{P}^2, \mathcal{L}_{|_{\mathbb{P}^2 \cap Bl_P\mathbb{P}^2}}) \to 0,$$

we deduce that $\mathcal{L}$ is very ample.

For those $b \in B$ such that $\mathcal{F}_b = \mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_{P,Q}\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$, we can prove that $g_b : \mathcal{F}_b \to \mathcal{F}'_b$ is the contraction of $Bl_{P,Q}\mathbb{P}^2$ along $\widetilde{PQ}$, (the strict transform of PQ defined in the previous Lemma), and thus $g_b(Bl_{P,Q}\mathbb{P}^2) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Indeed, in this case

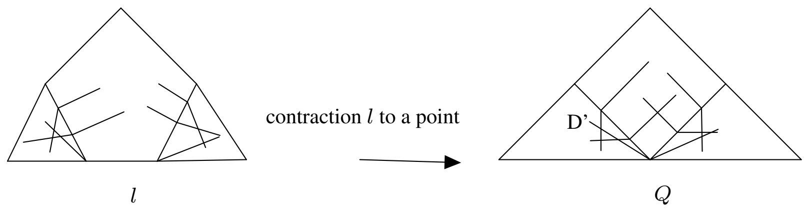
$$\mathcal{L}_{|_{Bl_{P,Q}\mathbb{P}^2}} \cong \mathcal{O}_{Bl_{P,Q}\mathbb{P}^2}(r.\pi^*H - |I_l|E_2 - |I_{l'}|E_1),$$

while $\widetilde{PQ} = \pi^*H - E_2 - E_1$, hence

$$\begin{aligned}\deg(\mathcal{L}_{|_{\widetilde{PQ}}}) &= (r.\pi^*H - |I_l|E_2 - |I_{l'}|E_1)(\pi^*H - E_2 - E_1) \\ &= r - |I_l| - |I_{l'}| = 0,\end{aligned}$$

(as $\pi^*H.E_i = 0$, $E_1.E_2 = 0$, $E_i^2 = -1$). so $\widetilde{PQ}$ is indeed contracted by g_b. On the other hand, points and vectors outside of $\widetilde{PQ}$ are separated by sections of $\mathcal{L}_{|_{Bl_{P,Q}\mathbb{P}^2}}$ as follows from $|I_l| > 0$ and $|I_{l'}| > 0$. We have thus proven that $\mathcal{F}' \to B$ satisfies Definition 6.3.3.

To prove the inverse suppose $(\mathcal{F}' \to B, D'_1, ..., D'_n)$ is a family which satisfies conditions of Definition 6.3.3. Let $D' = \sum_{i \in J_l \cap J_{l'}} D_i$. Hence for every fiber $\mathcal{F}'_b = \mathbb{P}^2 \cup_{\mathbb{P}^1} (\mathbb{P}^1 \times \mathbb{P}^1) \cup_{\mathbb{P}^1} \mathbb{P}^2$, the divisor D' intersects $\mathbb{P}^1 \times \mathbb{P}^1$ in only one point, Q, as we can see in the following figure.



Hence D' is not a Cartier divisor as $D' \cap (\mathbb{P}^1 \times \mathbb{P}^1) = \{Q\}$ is of codimension 2. After blowing up $\mathcal{F}'$ along D', the strict transforms of all D'_is are Cartier divisors and the blow-up $\mathcal{F}$ is a family of flat surfaces which satisfy the conditions in Definition 6.3.2.

□

Remark 6.3.7 *With the notations from Definition 6.3.1, we note that M_{345} is partitioned into 4 strata Y_P^{345}, for (1) $\mathcal{P} = \{\{3\}, \{4, 5\}\}$, (2) $\mathcal{P} = \{\{3, 4\}, \{5\}\}$, (3) $\mathcal{P} = \{\{3\}, \{4\}, \{5\}\}$ and (4) $\mathcal{P} = \{\{3, 4, 5\}\}$. From discussion in the first section of this chapter, the fiber of the restriction $\mathcal{U}_{345|_{Y^{345}_{\mathcal{P}}}}$ in these cases is $Bl_P\mathbb{P}^2 \cup_{\mathbb{P}^1} \mathbb{P}^2$ (cases 1 and 2), $\mathbb{P}^2 \cup_{\mathbb{P}^1} (\mathbb{P}^1 \times \mathbb{P}^1) \cup_{\mathbb{P}^1} \mathbb{P}^2$ (case 3) and $\mathbb{P}^2$ (case 4). Each stratum Y_P^{345} is an inverse limit of strata in M_i and M_{ij} with $i, j \in \{3, 4, 5\}$, as described in Definition 6.3.1. Similarly for M_{123} and M_{561}.*

After constructing the partial inverse limits of GIT quotients corresponding to a point P in $\mu(\prod \mathbb{P}^2)$, it is time to build up the whole inverse limit M_{123456} of the inverse system below:

$$\begin{array}{ccccc} M_{165} & & M_{345} & & M_{123} \\ \downarrow & \searrow\!\!\!\!\swarrow & & \searrow\!\!\!\!\swarrow & \downarrow \\ M_5 & & M_1 & & M_3, \end{array}$$

In other words, M_{123456} is the fiber product in this Cartesian diagram:

$$\begin{array}{ccc} M_{123456} & \longrightarrow & M_{123} \times M_{345} \times M_{561} \\ \downarrow & & \downarrow \\ M_1 \times M_3 \times M_5 & \longrightarrow & (M_1 \times M_3) \times (M_3 \times M_5) \times (M_5 \times M_1), \end{array}$$

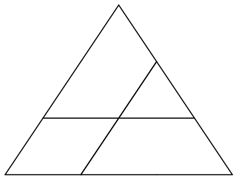
where the lower horizontal map is the diagonal map and $M_{123} \times M_{345} \times M_{561}$ is the inverse limit of the inverse system above. We construct the universal family, $\mathcal{U}_{123456}$, as inverse limit of the inductive system below:

$$\begin{array}{ccccc} p_{165}^*\mathcal{U}_{165} & & p_{345}^*\mathcal{U}_{345} & & p_{123}^*\mathcal{U}_{123} \\ \downarrow & \searrow\!\!\!\!\swarrow & & \searrow\!\!\!\!\swarrow & \downarrow \\ p^*\mathcal{U}_5 & & p^*\mathcal{U}_1 & & p^*\mathcal{U}_3, \end{array}$$

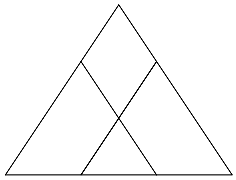
where $\mathcal{U}_I$ is the universal family over M_I for $I \subseteq \{1, 2, ..., 6\}$ and $p_I : M_{123456} \to M_I$ the natural map and p^* is the pull-back over M_{123456}.

Theorem 6.3.8 *Consider the partition of M_{123456} into strata as in Definition 6.3.1. Then for each partition described in the cases following Definition 6.3.1, the restriction of the morphism $\mathcal{U}_{123456} \to M_{123456}$ to the stratum given by that partition has fibers of the following types:*

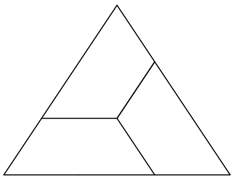
- *in case (a) and (b) the fiber is made of two $Bl_{point}\mathbb{P}^2$s, a $\mathbb{P}^1 \times \mathbb{P}^1$ and a $\mathbb{P}^2$ glued as follows:*



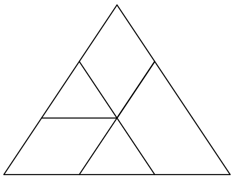
,



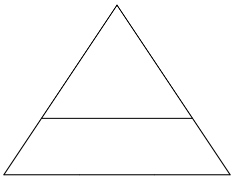
- *in case (c) it is made of three* $Bl_{point}\mathbb{P}^2$*:*



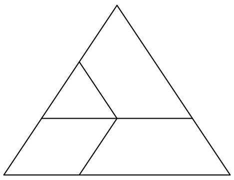
- *in case (d) it is made of a* $Bl_{point}\mathbb{P}^2$*, two* $\mathbb{P}^1 \times \mathbb{P}^1$*s and two* $\mathbb{P}^2$*s:*



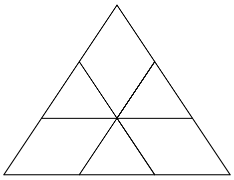
- *in case (e) it is made of a* $Bl_{point}\mathbb{P}^2$ *and a* $\mathbb{P}^2$*:*



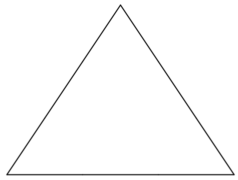
- *in case (f) it is made of two* $Bl_{point}\mathbb{P}^2$*s, a* $\mathbb{P}^1 \times \mathbb{P}^1$ *and a* $\mathbb{P}^2$*:*



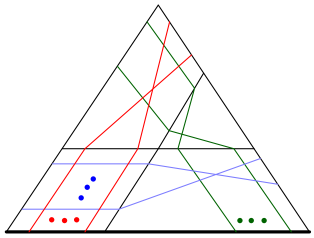
- *in case (g) it is made of three* $\mathbb{P}^1 \times \mathbb{P}^1$*s and three* $\mathbb{P}^2$*s:*



- *in case (h) it is a* $\mathbb{P}^2$*:*

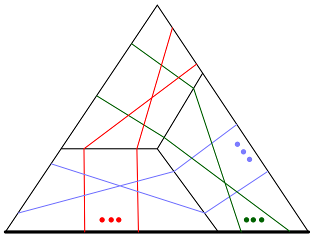


Here each $\mathbb{P}^2$ is represented by a small triangle, each $\mathbb{P}^1 \times \mathbb{P}^1$ by a rhombus and each $Bl_P\mathbb{P}^2$ by a trapezium. Moreover, $\mathcal{U}_{123456} \to M_{123456}$ is a flat morphism and the curves D_is are represented as follow in the fiber in colors (each color represents a group of curves which are intersecting the same surfaces in the fiber):

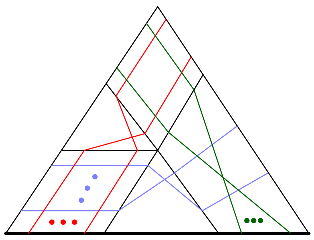


(a)

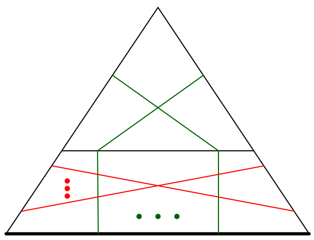
Case (b) is similar to case (a).



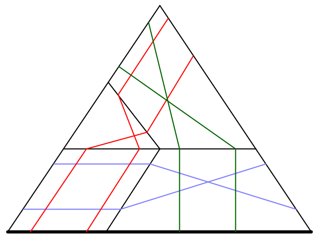
(c)



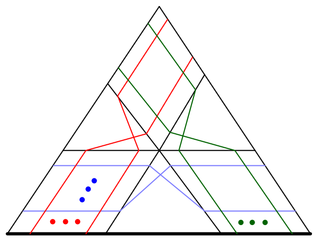
(d)



(e)

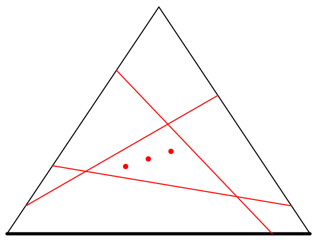


(f)



(g)

and the last case consist of a $\mathbb{P}^2$ and n lines in general position as below:

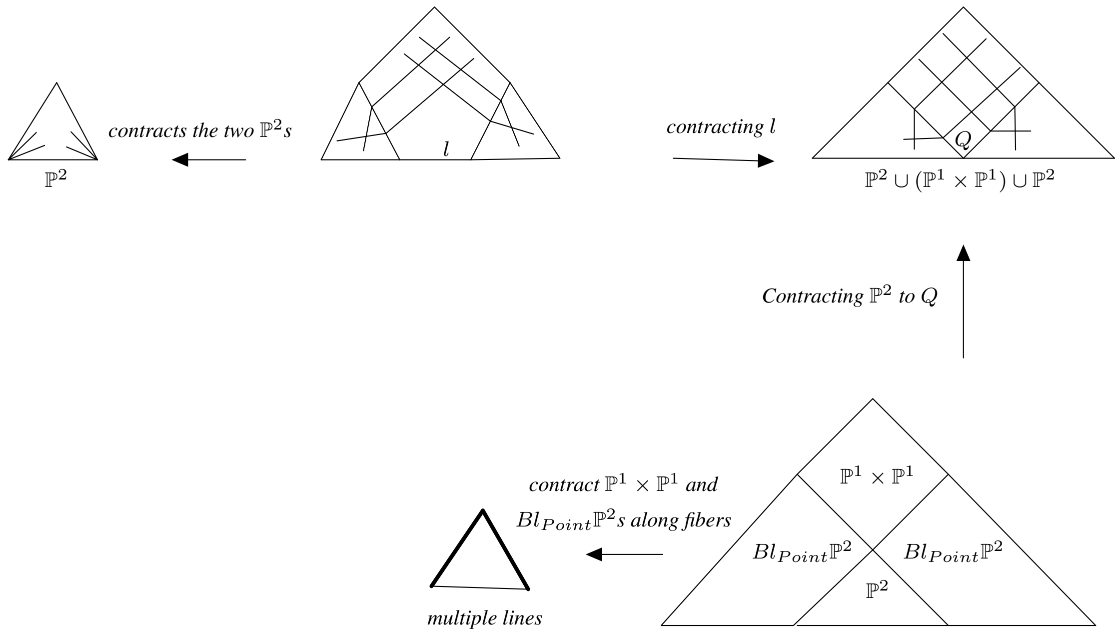


Note that the fibers of the universal family over M_{123456} are obtained by gluing the fibers of the universal families over M_{123}, M_{345} and M_{561} along the components that are pulled back from M_1, M_3 and M_6. However in families over a scheme, the gluing can also be expressed as a fiber product.

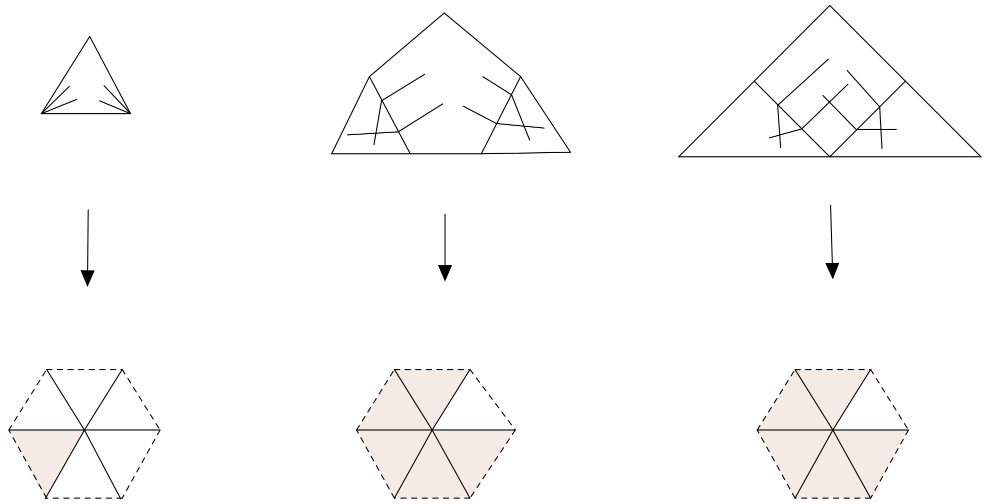
Example 6.3.1 *Consider case (a) for the partition $\mathcal{P} = \{\{1\},\{2,3\},\{4\},\{5,6\}\}$. Then the fibers over $Y_\mathcal{P} \subset M_{123456}$ of each of the universal families in*

$$\begin{array}{ccc} (p_4^*\mathcal{U}_4)_{|Y_\mathcal{P}} & \longleftarrow & (p_{345}^*\mathcal{U}_{345})_{|Y_\mathcal{P}} \\ & & \uparrow \\ (p_1^*\mathcal{U}_1)_{|Y_\mathcal{P}} & \longleftarrow & \mathcal{U}_{123456|Y_\mathcal{P}}, \end{array}$$

are as follows:



The correspondence between fibers of the universal family and the representation of strata via the moment map is a follows:



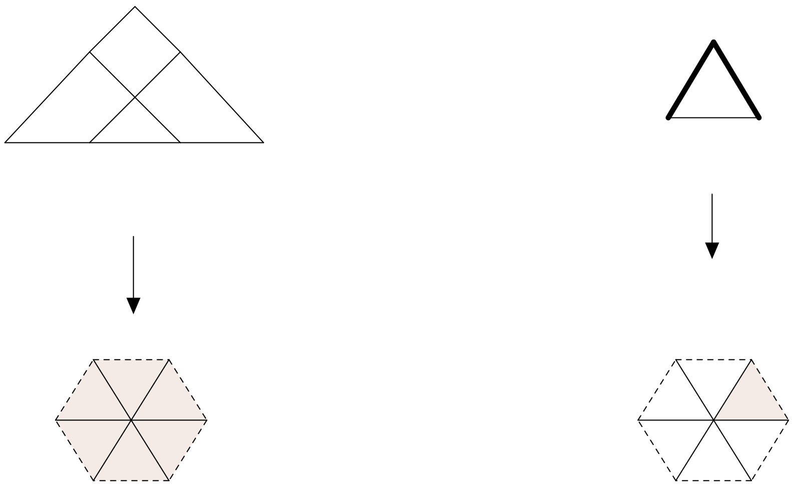


Figure 6.8: Fibers of the universal family versus strata

In each case the shaded area in hexagons show the image of orbits via moment map.

Proof: of Theorem 6.3.8 M_{12345} has indeed a partition into strata as described in Definition 6.3.2. Moreover, each stratum $Y_{\mathcal{P}}$ in M_{12345} is an inverse limit of a system of strata in M_1, M_3, M_5 and $M_{123}, M_{345}, M_{561}$. Thus for $\mathcal{P} = \{\{1\},\{2,3\},\{4\},\{5,6\}\}$ (case (a) in Definition 6.3.2), we have $Y_{\mathcal{P}} =$ the inverse limit of

$$\begin{array}{ccccc} Y^{165}_{\{1\},\{56\}} & & Y^{345}_{\{3\},\{4\},\{5\}} & & Y^{123}_{\{1\},\{23\}} \\ \downarrow & \swarrow \quad \searrow & & \swarrow \quad \searrow & \downarrow \\ Y^{5}_{\{56\}} & & Y^{1}_{\{1\}} & & Y^{3}_{\{23\}}, \end{array}$$

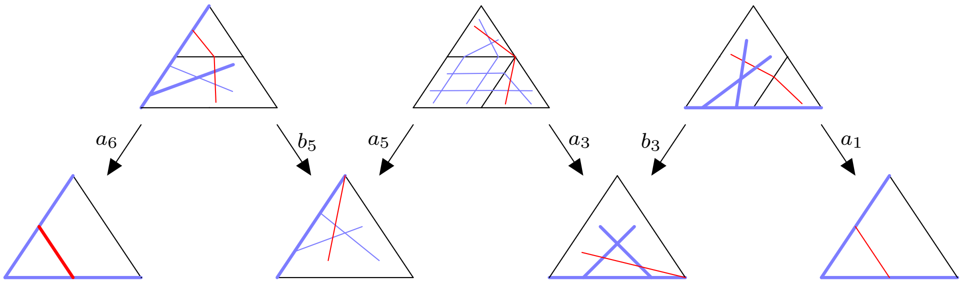
(with the notations from Definition 6.3.2 and Remark 6.3.7). Also,

$$p^*_{345}(\mathcal{U}_{345|_{Y_{\{3\},\{4\},\{5\}}{}^{345}}}) = (p^*_{345}\mathcal{U}_{345})_{|_{Y_{\mathcal{P}}}},$$

and similarly for the other terms in the inductive system of universal families. It follows that the fiber of $\mathcal{U}_{123456|_{Y_{\mathcal{P}}}}$ is the inverse limit of fibers of $\mathcal{U}_1$, $\mathcal{U}_3$, $\mathcal{U}_5$ and $\mathcal{U}_{123}$, $\mathcal{U}_{345}, \mathcal{U}_{561}$ on the strata in the above diagram. From the argument before the start of the proof, at the level of fibers we have the following inductive family:

$$\begin{array}{ccccccc} & \mathbb{P}^2 \cup Bl_{pt}\mathbb{P}^2 & & \mathbb{P}^2 \cup \mathbb{P}^2 \cup (\mathbb{P}^1 \times \mathbb{P}^1) & & \mathbb{P}^2 \cup Bl_{pt}\mathbb{P}^2 & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \\ \mathbb{P}^2 & & \mathbb{P}^2 & & \mathbb{P}^2 & & \mathbb{P}^2, \end{array}$$

where the first and the last $\mathbb{P}^2$s in the second row are the same.



Here the maps b_i contract a $\mathbb{P}^2$ to a point, and the maps a_i contract $Bl_{point}\mathbb{P}^2$, or $(\mathbb{P}^1 \times \mathbb{P}^1) \cup_{\mathbb{P}^1} \mathbb{P}^2$ to $\mathbb{P}^1$ by natural projection maps.

Claim: Inverse limit of this family is the surface $S = (\mathbb{P}^1 \times \mathbb{P}^1) \cup Bl_P\mathbb{P}^2 \cup \mathbb{P}^2 \cup Bl_Q\mathbb{P}^2$ as below:

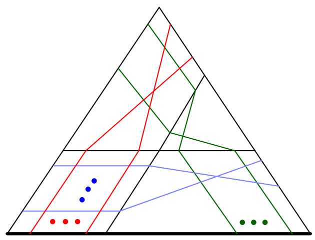
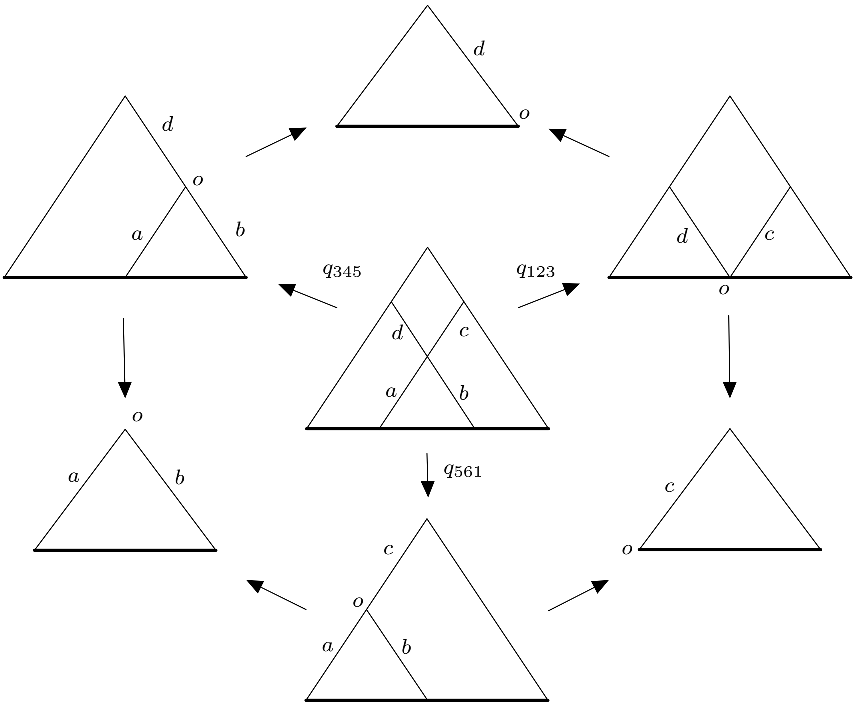


Figure 6.9

Proof of the claim: Note that the following diagram is commutative;



where q_{123} contracts the $\mathbb{P}^2$–component, q_{345} and q_{561} contract $\mathbb{P}^1 \times \mathbb{P}^1$ to a $\mathbb{P}^1$ (along the fibers) and $Bl_{point}\mathbb{P}^2$ along the fibers of $Bl_{point}\mathbb{P}^2 \to \mathbb{P}^1$. Moreover, the diagram above is formed by 3 Cartesian squares, and so S is the inverse limit of the system above. Note that the moduli space M_{123456} is a special example of the moduli space introduced in [7], Proposition 2.7 and Proposition 2.8. The proof for the flatness of the universal family can be done similar to [7]. However it also can be done as follows, directly without any use of toric varieties in an explicit way.

We will now check (by picking points and their images) that the surface S with $\mathcal{L} = \mathcal{O}(\sum_{i=1}^n D_i)$ has the same Hilbert polynomial as $\mathbb{P}^2$ with $\mathcal{O}_{\mathbb{P}^2}(n)$. We partition the set of divisors D_i into 3 subsets: s divisors of $Bl_P\mathbb{P}^2 \cup (\mathbb{P}^1 \times \mathbb{P}^1)$, k divisors of $Bl_Q\mathbb{P}^2 \cup (\mathbb{P}^1 \times \mathbb{P}^1)$, and t divisors of $Bl_P\mathbb{P}^2 \cup \mathbb{P}^2 \cup Bl_Q\mathbb{P}^2$ (as in Figure 6.9). Thus on $\mathbb{P}^1 \times \mathbb{P}^1$, with the two projections $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, we get s divisors which come from sections in $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1))$, and k divisors which come from $\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$. We will denote the first ones by $F_1, ..., F_s$ and the last ones by $L_1, ..., L_k$. Also let $F = \sum_{i=1}^s F_i$ and $L = \sum_{i=1}^k L_i$.

For $m >> 0$

$$0 \to I_{mL} \hookrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \to \mathcal{O}_{mL} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}/I_{mL} \to 0,$$

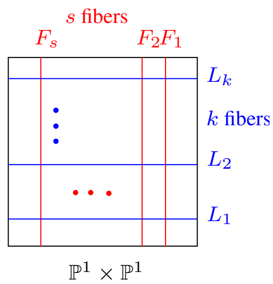
where $I_{mL} = \mathcal{O}(-mL)$. The tensor product of this short exact sequence with $\mathcal{O}(mL)$ gives us

$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} \hookrightarrow \mathcal{O}(mL) \to \mathcal{O}_{mL}(mL) \to 0,$$

where $\mathcal{O}(mL)$ is a locally free coherent sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$. The tensor product of this short exact sequence with $\mathcal{O}(mF)$ gives us

$$0 \to \mathcal{O}(mF) \hookrightarrow \mathcal{O}(mL + mF) \to \mathcal{O}(mF + mL) \otimes \mathcal{O}_{mL} \to 0.$$

Each L_i for $i \in \{1, 2, \ldots, k\}$, meets each F_j for $j \in \{1, 2, \ldots, s\}$ in 1 point.



Moreover $L_i.L_j = 0$ for every i and j. So $\mathcal{O}_{mL}(mL) \cong \mathcal{O}_{mL}$ and

$$\begin{aligned}\dim(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(mF) \otimes \mathcal{O}_{mL})) &= \dim(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \oplus_{i=1}^{mk} \mathcal{O}(mF)|L_i)) \\ &= mk\binom{ms+1}{1} \\ &= mk(ms+1).\end{aligned}$$

Note that $L_i \cong \mathbb{P}^1$ and $\mathcal{O}(mF)$ is a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ and its sections are pull-backs from $\mathbb{P}^1$. Hence

$$dim(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(mF))) = ms + 1.$$

Note that on $\mathbb{P}^1 \times \mathbb{P}^1$ the fibers for one ruling are sections for the other.

$$\begin{aligned} P_{\mathbb{P}^1\times\mathbb{P}^1}(m) &= \dim(H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(mL+mF))) \\ &= (ms+1)+mk(ms+1) \\ &= (mk+1)(ms+1). \end{aligned}$$

The global sections of $\mathcal{L}^{\otimes m}$ on S are constructed by gluing global sections of $\mathcal{L}^{\otimes m}$ on the components of S. Thus the following sequence is exact for $m \gg 0$:

$$\begin{aligned} 0 \to & H^0(S,\mathcal{L}^{\otimes m}) \to H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{L}^{\otimes m}_{|_{\mathbb{P}^1\times\mathbb{P}^1}}) \oplus H^0(\mathbb{P}^2, \mathcal{L}^{\otimes m}_{|_{\mathbb{P}^2}}) \oplus H^0(Bl_P\mathbb{P}^2, \mathcal{L}^{\otimes m}_{|_{Bl_P\mathbb{P}^2}}) \\ & \oplus H^0(Bl_Q\mathbb{P}^2, \mathcal{L}^{\otimes m}_{|_{Bl_Q\mathbb{P}^2}}) \to^F H^0({\mathbb{P}^1}_a, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_a}}) \oplus H^0({\mathbb{P}^1}_b, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_b}}) \\ & \oplus H^0({\mathbb{P}^1}_c, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_c}}) \oplus H^0({\mathbb{P}^1}_d, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_d}}) \to^G H^0(\{o\}, \mathcal{L}^{\otimes m}_{|_{\{o\}}}) \to 0, \end{aligned}$$

where $F(s_1,s_2,s_3,s_4) = (s_3-s_2, s_4-s_2, s_3-s_1, s_4-s_1)$ and $G(t_a,t_b,t_c,t_d) = (t_a-t_b-t_c+t_d)$. Hence for $m \gg 0$:

$$\begin{aligned} P_S(m) = & h^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{L}^{\otimes m}_{|_{\mathbb{P}^1\times\mathbb{P}^1}}) + h^0(\mathbb{P}^2, \mathcal{L}^{\otimes m}_{|_{\mathbb{P}^2}}) + h^0(Bl_P\mathbb{P}^2, \mathcal{L}^{\otimes m}_{|_{Bl_P\mathbb{P}^2}}) + \\ & h^0(Bl_Q\mathbb{P}^2, \mathcal{L}^{\otimes m}_{|_{Bl_Q\mathbb{P}^2}}) - h^0({\mathbb{P}^1}_a, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_a}}) - h^0({\mathbb{P}^1}_b, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_b}}) - h^0({\mathbb{P}^1}_c, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_c}}) - \\ & h^0({\mathbb{P}^1}_d, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_d}}) + h^0(\{o\}, \mathcal{L}^{\otimes m}_{|_{\{o\}}}) \\ = & (mk+1)(ms+1) + \frac{(ms+2)(ms+1)}{2} + mt(ms+1) + \\ & \frac{(mk+2)(mk+1)}{2} + mt(mk+1) + \frac{(mt+2)(mt+1)}{2} - ((ms+1) + \\ & (mt+1)+(mt+1)+(mk+1)) + 1 \\ = & \frac{(s+t+k)^2}{2}m^2 + \frac{3(s+t+k)}{2}m + 1, \end{aligned}$$

which is the same as the Hilbert polynomial of $\mathbb{P}^2$ with $s+t+k$ lines in it:

$$\begin{aligned} P_{\mathbb{P}^2}(m) &= \dim(H^0(\mathbb{P}^2, \mathcal{O}(\Sigma_{i=1}^{s+k+t} D_i)^{\otimes m})) \\ &= \binom{m(s+t+k)+2}{2} \\ &= \frac{(s+t+k)^2}{2} m^2 + \frac{3(s+t+k)m}{2} + 1. \end{aligned}$$

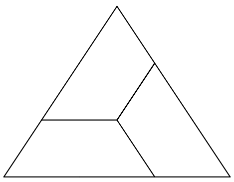
Similarly for the other partial inverse limits we can calculate the Hilbert polynomial and see the same fact about them too. Note that the proof of Proposition 3.3 in [7] implies that M_{123456} is irreducible, hence for flatness of $\mathcal{U}_{123456} \to M_{123456}$ it is enough to check that the Hilbert Polynomial is constant on the the fibers ([5] Chapter 3 Section 9).

Case (c) corresponds to the following family on the fibers:

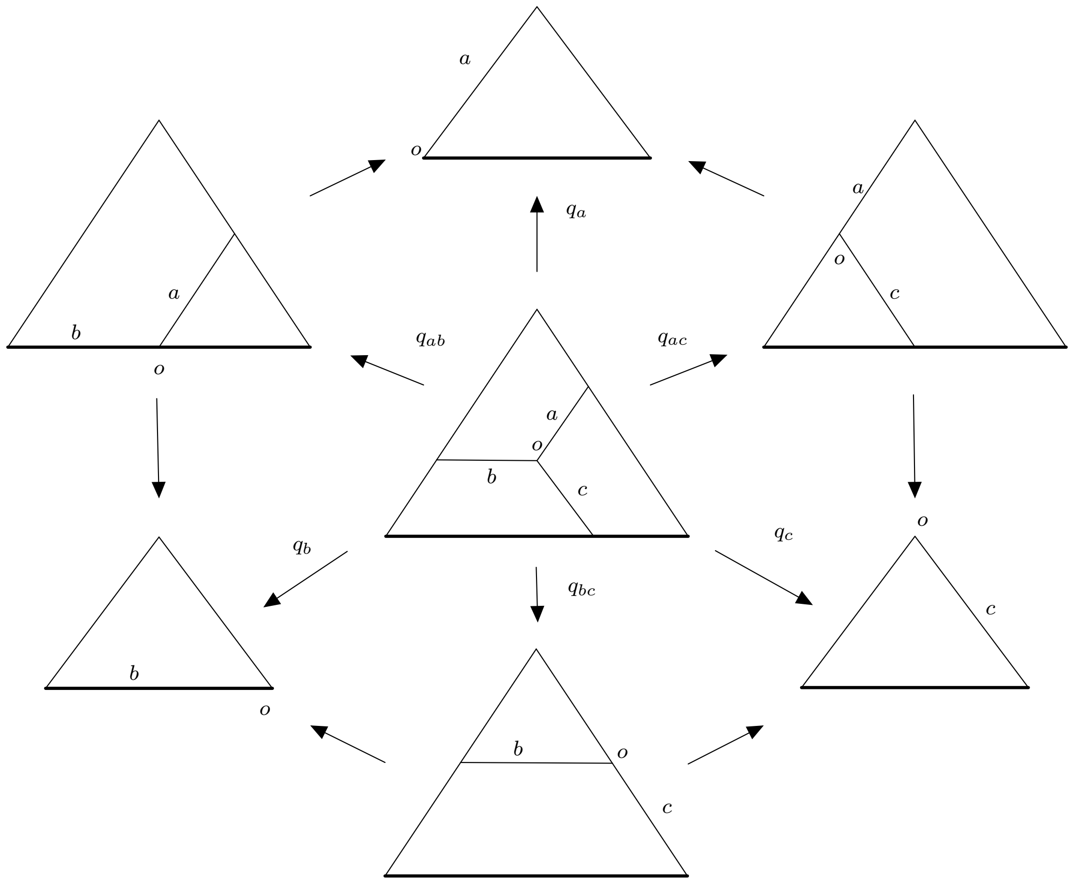
$$\begin{array}{ccccccc} & \mathbb{P}^2 \cup Bl_{pt}\mathbb{P}^2 & & \mathbb{P}^2 \cup Bl_{pt}\mathbb{P}^2 & & \mathbb{P}^2 \cup Bl_{pt}\mathbb{P}^2 & \\ \swarrow & & \searrow \swarrow & & \searrow \swarrow & & \searrow \\ \mathbb{P}^2 & & \mathbb{P}^2 & & \mathbb{P}^2 & & \mathbb{P}^2, \end{array}$$

where the first and the last $\mathbb{P}^2$ in the lower row are the same and the maps to the right are contraction along exceptional divisor in $Bl_{point}\mathbb{P}^2$ and the maps to the left are contraction of $\mathbb{P}^2$ to a point.

Let $S = Bl_P\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_Q\mathbb{P}^2 \cup_{\mathbb{P}^1} Bl_R\mathbb{P}^2$ as below:



The following diagram is commutative;



where here each surface is denoted according to which of lines a, b, c it contains. Here the clockwise maps are projections $Bl_{point}\mathbb{P}^2 \to \mathbb{P}^1$ along with the identity on $\mathbb{P}^2$; the counter clockwise maps are contractions of the $\mathbb{P}^2$ and q_{ac} is made of the identity on $Bl^{ac}_{point}\mathbb{P}^2$, a contraction of b to o on $Bl^{bc}_{pt}\mathbb{P}^2$ i.e. $Bl^{bc}_{pt}\mathbb{P}^2 \to \mathbb{P}^2$ and a contraction $Bl^{ab}_{pt}\mathbb{P}^2 \to \mathbb{P}^1$ on $Bl^{ab}_{pt}\mathbb{P}^2$. Similarly for q_{ab} and q_{bc}.

This gives a morphism $q : S \to \overleftarrow{S}$ into the inverse limit $\overleftarrow{S}$ of the inductive family above. To show that this is an isomorphism, let's consider the morphisms

$$\begin{array}{ccc} S & \xrightarrow{\phi} & \overleftarrow{S}_c \\ & \searrow^{q_c} & \downarrow q \\ & & \mathbb{P}^2_c. \end{array}$$

We will prove that

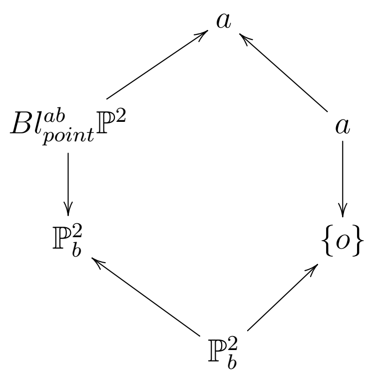
$$\phi|_{q_c^{-1}(\mathbb{P}^2_c \setminus c)} \ : \ q_c^{-1}(\mathbb{P}^2_c \setminus c) \to q^{-1}(\mathbb{P}^2_c \setminus c) \tag{6.9}$$

$$\phi|_{q_c^{-1}(c \setminus \{o\})} \ : \ q_c^{-1}(c \setminus \{o\}) \to q^{-1}(c \setminus \{o\}) \tag{6.10}$$

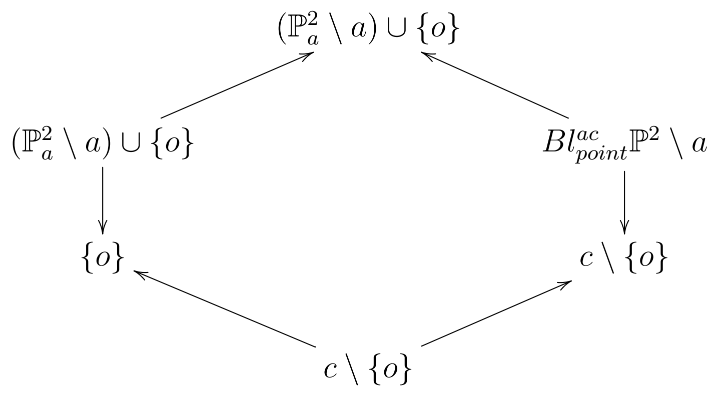
$$\phi|_{q_c^{-1}(\{o\})} \ : \ q_c^{-1}(\{o\}) \to q^{-1}(\{o\}), \tag{6.11}$$

are isomorphism. Note that q_c is made of $Bl^{bc}_{point}\mathbb{P}^2 \to \mathbb{P}^2$, the contraction $Bl^{ac}_{point}\mathbb{P}^2 \to c$, and the contraction $Bl^{ab}_{point}\mathbb{P}^2 \to \{o\}$. Thus $q_c^{-1}(\{o\}) = Bl^{ab}_{point}\mathbb{P}^2$, $q_c^{-1}(c \setminus \{o\}) = Bl^{ac}_{point}\mathbb{P}^2 \setminus a$, $q_c^{-1}(\mathbb{P}^2_c \setminus c) = Bl^{bc}_{point}\mathbb{P}^2 \setminus b \setminus c$.

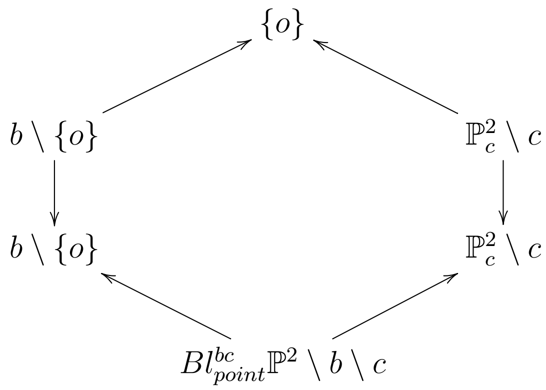
On the other hand, the inductive system corresponding to $q^{-1}(\{o\})$ is



whose limit is indeed $Bl^{ab}_{point}\mathbb{P}^2 \cong q_c^{-1}(\{0\})$. The inductive system for $q_c^{-1}(c \setminus \{o\})$:



which its limit is $q_c^{-1}(c \setminus \{o\}) = Bl^{ac}_{point}\mathbb{P}^2 \setminus a$ and the inductive system for $\mathbb{P}^2_c \setminus c$:



whose limit is $q_c^{-1}(\mathbb{P}^2_c \setminus c) = Bl^{bc}_{point}\mathbb{P}^2 \setminus b \setminus c$.

Now similar to the previous case we want to show that the surface S, the fiber in this case, has the same Hilbert polynomial as $\mathbb{P}^2$.

Thus the following sequence is exact for $m \gg 0$:

$$\begin{aligned} 0 \to & H^0(S, \mathcal{L}^{\otimes m}) \to H^0(Bl_P\mathbb{P}^2, \mathcal{L}^{\otimes m}) \oplus H^0(Bl_Q\mathbb{P}^2, \mathcal{L}^{\otimes m}) \oplus H^0(Bl_R\mathbb{P}^2, \mathcal{L}^{\otimes m}) \\ & \to^F H^0(\mathbb{P}^1{}_a, \mathcal{L}^{\otimes m}_{|_{\mathbb{P}^1{}_a}}) \oplus H^0(\mathbb{P}^1{}_b, \mathcal{L}^{\otimes m}_{|_{\mathbb{P}^1{}_b}}) \oplus H^0(\mathbb{P}^1{}_c, \mathcal{L}^{\otimes m}_{|_{\mathbb{P}^1{}_c}}) \to^G H^0(\{o\}, \mathcal{L}^{\otimes m}_{|_{\{o\}}}) \to 0, \end{aligned}$$

where $F(s_1, s_2, s_3) = (s_1 - s_2, s_2 - s_3, s_3 - s_1)$ and $G(t_1, t_2, t_3) = t_1 + t_2 + t_3$. Hence for $m \gg 0$:

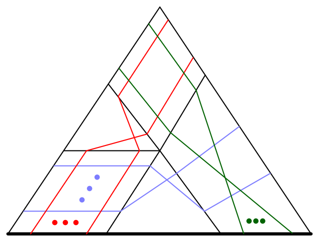
$$\begin{aligned} P_S(m) = & \binom{mk+2}{2} + mt\binom{mk+1}{1} + \binom{mt+2}{2} + ms\binom{mt+1}{1} + \binom{ms+2}{2} \\ & + mk(ms+1) - (ms+1) - (mt+1) - (mk+1) + 1 \\ = & \frac{(mk+2)(mk+1)}{2} + mt(mk+1) + \frac{(mt+2)(mt+1)}{2} + ms(mt+1) \\ & + \frac{(ms+2)(ms+1)}{2} + mk(ms+1) + mk(ms+1) - (ms+1) \\ & - (mt+1) - (mk+1) + 1 \\ = & \frac{(s+t+k)^2}{2}m^2 + \frac{3(s+t+k)}{2}m + 1, \end{aligned}$$

which is the same as the Hilbert polynomial for $\mathbb{P}^2$.

Case (d) consists of two $\mathbb{P}^2$, two $\mathbb{P}^1 \times \mathbb{P}^1$ and a $Bl_{point}\mathbb{P}^2$ which gives us the following inverse system:

$$\mathbb{P}^2 \xleftarrow{} \mathbb{P}^2\cup(\mathbb{P}^1\times\mathbb{P}^1)\cup\mathbb{P}^2 \xrightarrow{} \mathbb{P}^2 \xleftarrow{} \mathbb{P}^2\cup(\mathbb{P}^1\times\mathbb{P}^1)\cup Bl_{point}\mathbb{P}^2 \xrightarrow{} \mathbb{P}^2 \xleftarrow{} \mathbb{P}^2\cup Bl_{point}\mathbb{P}^2 \xrightarrow{} \mathbb{P}^2,$$

which is similar to the previous cases. Its inverse limit is as below:



Thus the following sequence is exact for $m \gg 0$:

$$\begin{aligned}0 \to & H^0(S,\mathcal{L}^{\otimes m}) \to H^0(Bl_{point}\mathbb{P}^2,\mathcal{L}^{\otimes m}) \oplus H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{L}^{\otimes m}) \oplus H^0(\mathbb{P}^2,\mathcal{L}^{\otimes m}) \\ & \oplus H^0(\mathbb{P}^2,\mathcal{L}^{\otimes m}) \oplus H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{L}^{\otimes m}) \to^F H^0({\mathbb{P}^1}_a,\mathcal{L}^{\otimes m}_{|{\mathbb{P}^1}_a}) \oplus H^0({\mathbb{P}^1}_b,\mathcal{L}^{\otimes m}_{|{\mathbb{P}^1}_b}) \\ & \oplus H^0({\mathbb{P}^1}_c,\mathcal{L}^{\otimes m}_{|{\mathbb{P}^1}_c}) \oplus H^0({\mathbb{P}^1}_d,\mathcal{L}^{\otimes m}_{|{\mathbb{P}^1}_d}) \oplus H^0({\mathbb{P}^1}_e,\mathcal{L}^{\otimes m}_{|{\mathbb{P}^1}_e}) \to^G H^0(\{0\},\mathcal{L}^{\otimes m}_{|\{o\}}) \to 0,\end{aligned}$$

where here $F(s_1,s_2,s_3,s_4,s_5) = (s_1-s_2, s_2-s_3, s_3-s_4, s_4-s_5, s_5-s_1)$ and $G(t_1,t_2,t_3,t_4,t_5) = t_1+t_2+t_3+t_4+t_5$. Hence for $m \gg 0$:

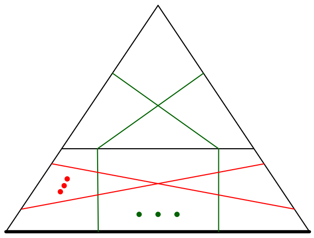
$$\begin{aligned}P_S(m) =& (ms+1)(mk+1) + (mt+1)(ms+1) + \binom{ms+2}{2} + \binom{mt+2}{2} \\ & + \binom{mk+2}{2} + mt\binom{mk+1}{1} + 1 - (mt+1) - (mt+1) \\ & - (ms+1) - (ms+1) - (mk+1) - (mk+1) \\ =& \frac{(mk+2)(mk+1)}{2} + mt(mk+1) + \frac{(mt+2)(mt+1)}{2} + ms(mt+1) \\ =& \frac{(s+t+k)^2}{2}m^2 + \frac{3(s+t+k)}{2}m + 1,\end{aligned}$$

which is the same as the Hilbert polynomial for $\mathbb{P}^2$.

Case (e) consists of a $\mathbb{P}^2$ and a $Bl_{point}\mathbb{P}^2$ which gives us the following inverse system:

$$\mathbb{P}^2 \xleftarrow{} \mathbb{P}^2\cup Bl_{point}\mathbb{P}^2 \xrightarrow{} \mathbb{P}^2.$$

This inverse system has the following inverse limit:



The proof is similar to case (c). The calculation of the Hilbert polynomial is as follow. The following sequence is exact for $m \gg 0$:

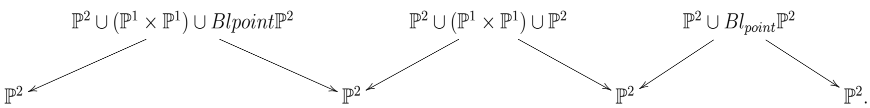
$$
\begin{aligned}
0 \to & H^0(S, \mathcal{L}^{\otimes m}) \to H^0(Bl_{point}\mathbb{P}^2, \mathcal{L}^{\otimes m}) \oplus H^0(\mathbb{P}^2, \mathcal{L}^{\otimes m}) \\
& \to H^0({\mathbb{P}^1}_a, \mathcal{L}^{\otimes m}_{|_{{\mathbb{P}^1}_a}}) \to H^0(\{0\}, \mathcal{L}^{\otimes m}_{|_{\{o\}}}) \to 0.
\end{aligned}
$$

Hence for $m \gg 0$:

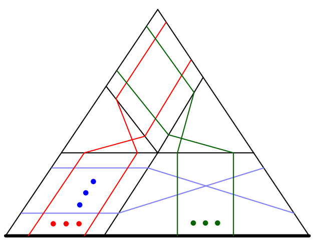
$$
\begin{aligned}
P_S(m) = & \binom{mt+2}{2} + ms\binom{mt+1}{1} + \binom{ms+2}{2} - \binom{ms+1}{1} + 1 \\
= & \frac{(mt+2)(mk+1)}{2} + ms(mt+1) + \frac{(ms+2)(ms+1)}{2} - (ms+1) + 1 \\
= & \frac{(s+t+k)^2}{2} m^2 + \frac{3(s+t+k)}{2} m + 1,
\end{aligned}
$$

which is the same as the Hilbert polynomial for $\mathbb{P}^2$.

Case (f) consists of two $\mathbb{P}^2$s and two $\mathbb{P}^1 \times \mathbb{P}^1$ and a $Bl_{point}\mathbb{P}^2$ which gives us the following inverse system:

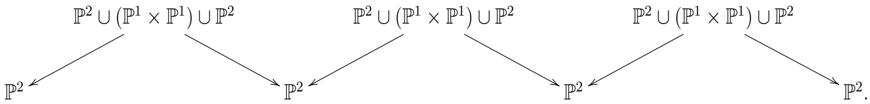


This inverse system has the following inverse limit:

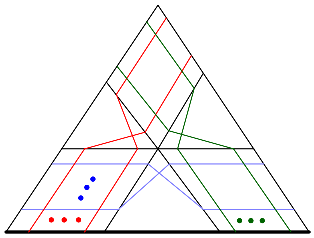


It can be proved similar to case (c).

Case (g) consists of three $\mathbb{P}^2$ and three $\mathbb{P}^1 \times \mathbb{P}^1$ whose inverse system is as below:



Again similarly to the case (c),it can be shown that this inverse system has the following inverse limit:



□

Proof: of the Theorem 6.3.1 Let M denote the inverse limit of all the GIT quotients of $G(3,n)$ by $(\mathbb{C}^*)^{n-1}$ and their flips. We will first construct a universal family $\mathcal{U}$ over M. Recall that the image of the moment map $\mu_{(\mathbb{C}^*)^{n-1}}(G(3,n))$ is divided into chambers separated by walls, and for each chamber C_i there is a moduli space M_i described in Definition 5.3.1. Consider a wall W_l and two codimension 1 faces F_1 and F_2 of W_l. Recall also that for each $a,b,c \in \{1,2,...,n\}$ where $a \neq b \neq c \neq a$ we have an open set $U_{abc} \subset G(3,n)$ such that $U_{abc}/(\mathbb{C}^*)^{n-3} \cong (\mathbb{P}^2)^{n-3}$ and we have the following commutative diagram of moment maps:

$$\begin{array}{ccc} U_{abc} & \xrightarrow{\mu_{(\mathbb{C}^*)^{n-1}}} & \mathbb{R}^{n-1} \\ \downarrow & & \downarrow pr \\ (\mathbb{P}^2)^{n-3} & \xrightarrow{\mu} & \mathbb{R}^2, \end{array}$$

where pr is a linear projection sending the wall W_l to a wall in $\mu((\mathbb{P}^2)^{n-3})$ and each of the F_1 and F_2 to a vertex in the net described in Figure 6.1. This projection sends the decomposition into chambers of $\mu_{(\mathbb{C}^*)^{n-1}}(U_{abc})$ into that of $Im\mu$. The projection pr corresponds to a choice of an inverse of $(\mathbb{C}^*)^{n-1} \to (\mathbb{C}^*)^2$.

(i) If W_l is on the boundary of just one chamber C, then for each F_i we consider the moduli space M_{F_i} corresponding to the inverse limit of GIT quotients associated to chambers containing F_i. Let $\mathcal{U}_{F_i}$ be the corresponding universal family.

(ii) If W_l the wall between two chambers then each M_{F_i} and their universal families $\mathcal{U}_{F_i}$, are as described in Theorem 6.3.8. Let $p_{F_i} : M \to M_{F_i}$ the natural morphism between inverse limits (of the total inductive family of GIT quotients, and the subfamily giving M_{F_i}).

Let C be the union of the chambers adjacent to W_l, F_1 and F_2, and let M_C denote the corresponding moduli space, defined as in Section 5.3 for case (i) above and Section 5.4 for case (ii). Let $p_C : M \to M_C$ be the natural map. Within each universal family $\mathcal{U}_{F_1}, \mathcal{U}_{F_2}$ and $\mathcal{U}_C$, we consider the subspaces $\mathcal{U}'_{F_1}, \mathcal{U}'_{F_2}$ and $\mathcal{U}'_C$, respectively, obtained by removing all intersections of more than 2 divisors from among $(D_i)_{i\in\{1,\ldots,n\}}$, as well as any codimension 1 components of $D_i \cap D_j$ from any fiber of the universal family, for all $i, j \in \{1, ..., n\}$, $i \neq j$. Then $p_C^*\mathcal{U}'_C \subset p_{F_i}^*\mathcal{U}'_{F_i}$ for each $i \in \{1, 2\}$. Indeed, this follows directly from comparing Sections 5.3 ,5.4 with Theorem 6.3.8 and Definition 6.3.2. We can then form the universal family $\mathcal{U}$ by gluing $p_{F_i}^*\mathcal{U}_{F_i}$ along $p_C^*\mathcal{U}'_C$, pairwise for all codimension 2 faces F_i in $\mu_{(\mathbb{C}^*)^{n-1}}(G(3, n))$. Successively as above from Theorem 6.3.8 there exists a rational contraction $p_{F_i}^*U_{F_i} \dashrightarrow p_C^*U_C$ and $p_C^*U'_C$ is the open set where this is in fact an isomorphism.

To prove the representability of the functor in Definition 6.2.1 it is enough to show that any flat family of surfaces $f : \mathcal{F} \to B$ with divisors $(D_i)_{i\in\{1,\ldots,n\}}$ which satisfy the conditions of Definition 6.2.1, there exists a unique morphism $\psi : B \to M$ such that $\psi^*\mathcal{U} \cong \mathcal{F}$.

For any 3 adjacent chambers C_h, C_k and C_m as in Definition 6.3.2, separated by the walls W_l and $W_{l'}$, respectively, we will construct a morphism $\psi_{hkj} : B \to M_{hkj}$, such that the compositions with $M_{hkj} \to M_h$ do not depend on k and j (and similarly for the other indices).

For each chamber $C_h \subset \mu_{(\mathbb{C}^*)^{n-1}}(G(3, n))$, and each $b \in B$, there exists a component $S_{j(h),b}$ in $\mathcal{F}_b$ and a corresponding polytope $\Delta_{j(h),b} \supseteq C_h$. Similarly for k and j.

For each $b \in B$, consider the sets of divisors D_i described in part (c) of Definition 6.2.1. By eliminating some of these divisors, we can obtain $A \subset \{1, 2, ..., n\}$ with the properties:

1. $\phi_{j(h),b}(D_{i_1}) = \phi_{j(h),b}(D_{i_2})$ for some $i_1, i_2 \in A \Leftrightarrow i_1, i_2 \in J$ for some $J \in \mathcal{A}_k^+ \cup \mathcal{A}_h^+ \cup \mathcal{A}_m^+$,
2. $\phi_{j(h),b}(D_{i_1}) \cap \phi_{j(h),b}(D_{i_2}) \cap \phi_{j(h),b}(D_{i_3}) \neq \emptyset$ for some $i_1, i_2, i_3 \in A \Leftrightarrow i_1, i_2, i_3 \in I$ for some $I \in \mathcal{A}_k^- \cup \mathcal{A}_h^- \cup \mathcal{A}_m^-$.

Let $V_{A,b} \subset B$ be the subscheme whose points b' satisfy two properties above with b replaced by b' (i.e. $V_{A,b} = \{b' : b'$satisfies 1 and 2 $\}$).

By the following Lemma 6.3.10, there is a contraction

$$F_A : \mathcal{F} \to \mathcal{F}_A = \mathrm{Proj}(\oplus_m \pi_* \mathcal{L}_A^{\otimes m}),$$

such that $\mathcal{F}_{A|_{V_{A,b}}}$ satisfies the conditions of Definition 6.3.3 where here

$$\mathcal{L}_A := \omega(\Sigma_{j \in A} D_j).$$

Indeed after the contraction, using Lemma 6.3.9, $\mathcal{F}_{A|_{V_{A,b}}}$ satisfies Definition 6.2.1 for the set $\{D_i\}_{i \in A}$. In this context Definition 6.2.1 and Definition 6.3.3 become equivalent. This gives a map $V_{A,b} \to M_{hkm}$ such that $\mathcal{F}_{A|_{V_{A,b}}}$ is the pull-back of U_{hkm}. Moreover, by varying A and b we can cover the entire B.

To check that all $\mathcal{F}_{A|_{V_{A,b}}}$ fit together, it is enough to note that $V_{A,b}$ only depends on A and not on b and for $A_1 \subset A_2$ we have $V_{A_1} \supset V_{A_2}$ for arbitrary b. Moreover $\mathcal{F}_{V_{A_2}}|_{V_{A_1}} = \mathcal{F}_{V_{A_2}}$ as the contraction of $\mathcal{F}_{V_{A_2}}$ on V_{A_1}

is given by $\mathcal{L}_{A_1}$ which is trivial ($\mathcal{L}_{1|\mathcal{F}_{V_{A_2}}|V_{A_1}}$ is relatively ample).

As all constructions are canonical this gives a map $B \to M_{hkm}$ and a contraction $\mathcal{F} \to \mathcal{F}_{hkm}$ such that $\mathcal{F}_{hkm}$ is the pull-back of $\mathcal{U}_{hkm}$.

In conclusion, all the maps $B \to M_{hkm} \to M_h$ (where $M_{hkm} \to M_h$ is only projection on M_h) give a map $\psi : B \to M$ into the inverse limit M, and $\mathcal{F}$ is the pull-back of $\mathcal{U}$ by ψ, as it can be reconstructed from $\mathcal{F}_{hkm}$ in the same way in which $\mathcal{U}$ has been constructed from $\mathcal{U}_{khm}$ in the proof of Theorem 6.3.8 i.e. as an inverse limit of the pull-back of universal families $\mathcal{F}_{hkm}$ and $\mathcal{F}_h$. □

Note that the morphism $M_{hkm} \to M_h$ can also be contracted by using the contraction of $\mathcal{F}_{hkm}$ by an $\mathcal{L}_A$ for a smaller ret A. So the morphism $B \to M_{hkm} \to M_h$ only depends on the choice of A and not on h or k.

Lemma 6.3.9 *Let $(\mathcal{F} \to B, D_1, ..., D_n)$ be a flat family of surfaces satisfying the properties from Definition 6.2.1. Let ω_P denote the relative dualising sheaf of $\mathcal{F}$ sheaf of $\mathcal{F}$ over B. Then $\omega_{\mathcal{F}}(\sum_{i=1}^n D_i)$ is relatively ample.*

Proof: From [4] for each $b \in B$ there exists ω_P such that,

$$\omega_P(\sum_{i=1}^n D_i)_{|_{P_b}} = \mathcal{O}_b(K_{\mathcal{F}_b} + \sum_{i=1}^n D_i),$$

where $K_{\mathcal{F}_b}$ is the canonical divisor on the fiber. To prove that $\omega_P(\sum_{i=1}^n D_i)$ is ample, it will be enough to check it on the components as this will imply that for $N \gg 0$

$$H^1(S_k \cap S_i, \omega^N(\sum_{i=1}^{n} D_i)_{|S_k} \otimes \mathcal{O}_{S_k}(S_k \cap S_i)) = 0,$$

so

$$H^0(S_k \cap S_i, \omega^N(\sum_{i=1}^{n} D_i)_{|S_k}) \to H^0(S_k \cap S_i, \omega^N(\sum_{i=1}^{n} D_i)_{|S_k \cap S_i}) \to 0.$$

We will apply the Nakai-Moishezon criteria of ampleness, namely we check $L := K_{P_b} + \sum_{i=1}^{n} D_i$ is nef and has self-intersection positive. Let $P_b = \cup_j S_j$. Then

$$K_{\mathcal{F}_b|_{S_j}} = K_{S_j} + \sum_k (S_j \cap S_k), \tag{6.12}$$

where the sum is taken after all k s.t. $S_j \cap S_k \cong \mathbb{P}^1$.

Case 1. If $\phi_{j|_{S_j}} : S_j = Bl_{r(points)}\mathbb{P}^2 \to \mathbb{P}^2$ then

$$K_{S_j} = \phi_j^* K_{\mathbb{P}^2} + \sum_a E_a = -3\phi_j^* H + \sum_a E_a, \tag{6.13}$$

where H =the hyperplane divisor in $\mathbb{P}^2$ and $E_a \subset S_j$ are the exceptional divisors of $\phi_{j|_{S_j}}$. Note that by condition (a) in Definition 6.2.1, E_a are among the intersections $S_j \cap S_k$. We'll denote by k' the indices for which $S_j \cap S_{k'} = \mathbb{P}^1$ is not an exceptional divisor in S_j. Then we can rewrite Equations (6.12) and (6.13) as :

$$L_{|S_j} = K_{\mathcal{F}_b|_{S_j}} + \sum_{i=1}^{n} D_{i|_{S_j}} = -3\phi_j^* H + 2\sum_a E_a + \sum_{k'} (S_j \cap S_{k'}) + \sum_{i=1}^{n} D_i \cap S_j. \tag{6.14}$$

Step 1. By condition (b), (d) and (e) in Definition 6.2.1, both $S_j \cap S_{k'}$ and $S_j \cap D_i$ are strict transforms of lines in $\mathbb{P}^2$ (because it is not exceptional divisor), through the map $\phi_j : S_j \to \mathbb{P}^2$. Define

$$\Gamma := \{S_j \cap S_{k'} \neq \text{ exceptional divisor }\} \cup \{D_i; D_i \cap S_j \neq \emptyset\}.$$

Let $m = |\Gamma|$ and $d_a =$ the number of lines of the form $\phi_j(D)$ with $D \in \Gamma$, containing the blow-up point $\phi_j(E_a)$. Then using $\widetilde{H} = \phi_j^* H - \sum_{a \in H} E_a$ for any line H in $\mathbb{P}^2$. We can rewrite 6.14 as:

$$L_{|S_j} = (m-3)\phi_j^* H + \sum_a (2 - d_a) E_a. \tag{6.15}$$

Hence for each a, $L.E_a = d_a - 2 > 0$. (due to (e) in Definition 6.2.1)

Step 2. We can also use the equation $\phi_j^* H_l = \widetilde{H_l} + \sum_{a \in H_l} E_a$ for all line H_l but those four fixes lines of the form $\phi_j(D)$ with $D \in \Gamma$ to write Equation (6.14) as

$$L_{|S_j} = \phi_j^* H + C, \tag{6.16}$$

where C is an effective divisor, $SuppC \subset (\cup_a E_a) \cup (\cup_{k'}(S_j \cap S_{k'})) \cup (\cup_{i=1}^n D_i)$. (Indeed, it is sufficient to choose the 4 lines H_l such that no 3 of them intersect at the same pointbecause $\phi_j(D_i)$ are the date associated to an orbit in $G(3,n)$, Tx with $\mu(Tx) = P_b$. We could pick 4 such lines, other wise $\dim Tx < (\mathbb{C}^*)^{n-1}$.

Equation 6.16 implies $C.C' > 0$ for any irreducible curve C' in S_j which is not among the exceptional divisors. Indeed, $\phi_j^* H.C' > 0$ and we will prove that $\phi_j(C').\phi_j(C) > 0$, and hence $C.C' \geq 0$:

(a) If $C' \not\subset SuppC$, then it is enough to note that $\phi_j(C)$ contains at least 1 line (by (b) and (d) in Definition 6.2.1), and $\phi_j(C')$ is still an irreducible curve.

(b) If $C' = S_j \cap S_{k'}$ or D_i for some i, then we can choose one of the lines H_l above to be $\phi_j(C')$, and then $C' \not\subset SuppC$ for C thus constructed.

Step 3. We will prove $(L_{|S_j})^2 > 0$. Indeed, by Equation (6.15)

$$(L_{|S_j})^2 = (m-3)^2 - \sum_a (d_a - 2)^2 > 0.$$

It can be proved by induction on m. Let $\{H_1, ..., H_{m+1}\}$ be a set of lines in plane and for the moment we will assume that line H_{m+1} (without loss of generality) contains the intersection of at least 2 other pairs of lines $H_1, ..., H_m$. Let $V = \{H_i \cap H_k;$ for any i and k in $\{1, ..., m\}\}$ and $B = \{x \in V : x \in H_{m+1}\}$. We want to check that

$$(m+1-3)^2 - \sum_{a \in V \setminus B} (d_a - 2)^2 - \sum_{a \in B} (d_a + 1 - 2)^2 > 0.$$

By induction

$$(m-3)^2 - \sum_{a \in V} (d_a - 2)^2 > 0.$$

So it is enough to prove that

$$2(m-3) - \sum_{a \in B} (2(d_a - 2) + 1) > 0.$$

Note that $H_{m+1} \notin \{H_1, ..., H_m\}$. None of $H_1, ...H_m$ contains more than 1 points of B.

So $\sum d_a \leq m$. Together with $|B| \geq 2$ this ends the proof. If no H_i for $(i = 1, ..., m+1)$ contains 2 or more points at which two other H_j meet, then

$$(\sum_{a \in v} d_a - 3)^2 > \sum_{a \in V} (d_a - 2)^2,$$

for $d_a \geq 3$ and $|V| \geq 2$, which can be checked.

Case 2. $S_j = \mathbb{P}^1 \times \mathbb{P}^1$, with the two projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, then $K_{S_j} = -2(\pi_1^* H_1 + \pi_2^* H_2)$, where $H_i =$ the class of a point in $\mathbb{P}^1$. Thus

$$L_{|S_j} = -2\pi_1^* H_1 - 2\pi_2^* H_2 + \sum_{i=1}^{n} D_i,$$

and using (e), Definition 6.2.1, we obtain that $L_{|S_j}$ is ample if and only if for each $i \in \{1, 2\}$ at least 3 of the divisors $D_1, D_2, ...D_n$ are fibers for $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, otherwise $\dim D_j < n - 1$ which is a contradiction. □

Lemma 6.3.10 *Let $(\mathcal{F} \to B, D_1, ..., D_n)$ be a flat family of surfaces with divisors as in Definition 6.2.1. Let $I \subset \{1, ..., n\}$ where $|I| \geq 4$ and consider*

$$\mathcal{L}_I := \omega_{\mathcal{F}}(\sum_{j \in I} D_j).$$

Then there is a morphism

$$F_I : \mathcal{F} = \text{Proj}(\oplus_m \mathcal{L}_I^{\otimes m}) \to \text{Proj}(\oplus_m \pi_* \mathcal{L}_I^{\otimes m}),$$

such that $(\mathcal{F}_I = ImF_I \to B, (F_I(D_j))_{j \in I})$, satisfies Definition 6.2.1 with n replaced by $|I|$.

Proof: Working inductively, it is enough to consider the case $n - |I| = 1$. We adapt the steps in the proof of Lemma 6.3.9 for $\omega_{\mathcal{F}}(\sum_{i \in I} D_i) = \mathcal{O}(L_I)$:

Case 1. $S_j = Bl_{r(points)}\mathbb{P}^2$. Then

$$L_{I|S_j} = -3\phi_j^* H + 2\sum_a E_a + \sum_{k'} (S_j \cap S_{k'}) + \sum_{i \in I} D_i \cap S_j.$$

Γ_I is defined similarly to Γ in the proof of Lemma 6.3.9, but with I instead of $\{1, ..., n\}$.
Step 1. We get $L_{I|S_j}.E_a = d_a^I - 2 > 0$ as there are always 2 lines in Γ_I containing $\phi_j(E_a)$. The intersection number is 0 if and only if $d_a^I = 2$.

Step 2. Works as in Lemma 6.3.9 with the exception of the case when

(a) $S_j = Bl_a\mathbb{P}^2$ and all $\phi_j(D_i)_{i\in I}$ pass through the same point. In this case, we can only find 3 lines H_l as in the proof of Lemma 6.3.9, so $L^I = C$ with $SuppC \subseteq E_a \cup (\cup_{k'}(S_j \cap S_{k'})) \cup (\cup_{i\in I} D_i)$. Due to condition (d) in Definition 6.2.1 $L^I.D_i \geq 0$ and $L^I.(S_j \cap S_{k'}) \geq 0$.

(b) $S_j = \mathbb{P}^2$, $|\Gamma^I| = 3$ (i.e. $D_i \cap S_j \neq 0 \Leftrightarrow i \in \{1, ..., n\} \setminus I$).

Case 2. If $S_j = \mathbb{P}^1 \times \mathbb{P}^1$, then $F_{I|_{S_j}}$ is an isomorphism with the exception when $(D_i)_{i\in I}$ contains only 2 lines in the fibers of one of the projection $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. In conclusion, $F_I : \mathcal{F}_b \to \mathcal{F}_b^I$ is an isomorphism every where except at the P_b^I is an isomorphism every where except at the loci C where $L^I_{|C} \cong \mathcal{O}_C$. Moreover, since for $S_j = Bl_{r(points)}\mathbb{P}^2$, the only types of curves contracted by $F_{I|_{S_j}}$ are exceptional divisors the map ϕ_j induces a map ϕ'_j such that

$$\begin{array}{ccc} \mathcal{F}_b & \xrightarrow{\phi_j} & \mathbb{P}^2 \\ {\scriptstyle F_{I|\mathcal{F}_b}}\downarrow & \nearrow{\scriptstyle \phi'_j} & \\ \mathcal{F}_b^I, & & \end{array}$$

is commutative, and ϕ'_j satisfies conditions (b), (d) and (e) in Definition 6.2.1.

Corresponding to $I \subseteq \{1, 2, ..., n\}$, $|I| = n-1$, we have an inclusion $i : G(3, n-1) \hookrightarrow G(3, n)$ compatible with the actions of the maximal tori such that $\mu_{(\mathbb{C}^*)^{n-1}} \circ i$ gives the map $\mu_{(\mathbb{C}^*)^{n-2}} : G(3, n-1) \to \mathbb{R}^{n-2} \subseteq \mathbb{R}^{n-1}$. Thus every fiber $\mathcal{F}_b^I$ corresponds to a partition of $\mu_{(\mathbb{C}^*)^{n-2}}(G(3, n-1))$ into polytopes, obtained by cutting the partition of $\mu_{(\mathbb{C}^*)^{n-1}}(G(3, n))$ induced by $\mathcal{F}_b$ with the hyperplane $\mathbb{R}^{n-2}$. All the incidence conditions in Definition 6.2.1 are preserved, when $\{1, 2, ..., n\}$ is replaces by I. $\square$

Appendix A

A Few Facts

The following definitions and theorems are from Algebraic Geometry by Hartshorne [5].

Definition A.0.4 *Let Y be a topological space. A function $\phi : Y \to \mathbb{Z}$ is upper semicontinuous if for each $y \in Y$, there is an open neighborhood U of y, such that for all $y' \in U$, $\phi(y') \leq \phi(y)$. Intuitively it means that ϕ may get bigger at special points.*

Now we have the following results.

Theorem A.0.11 ***(Hartshorne, Exercise II.5.8)*** *Let X be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf on X. We will consider the function*

$$\phi(x) = \dim_{\mathcal{C}(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{C}(x),$$

where $\mathcal{C}(x) = \mathcal{O}_x/m_x$ is the residue field at the point x. Using Nakayama's lemma it can be shown that the function ϕ is upper semi-continuous.

Theorem A.0.12 . *Let A be a ring and X be a scheme over A. If $\mathcal{L}$ is an invertible sheaf on X, and if $s_1, ..., s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, then there exists a unique A–morphism $\phi : X \to \mathbb{P}^{n-1}_A$ such that $\mathcal{L} \cong \phi^*(\mathcal{O}(1))$ and $s_i = \phi^*(x_i)$ under this isomorphism.*

Proposition A.0.13 *Let X be a noetherian scheme, $\mathcal{E}$ a locally free coherent sheaf on X and $\mathbb{P}(\mathcal{E})$ the associated projective space bundle. Let $g : Y \to X$ be any morphism. Then to give a morphism of Y to $\mathbb{P}(\mathcal{E})$ over X, it is equivalent to give an invertible sheaf $\mathcal{L}$ on Y and a surjective map of sheaves on Y, $\psi : g^*\mathcal{E} \to \mathcal{L}$.*

Proposition A.0.14 *Let X be a projective scheme, and $\phi : X \to \mathbb{P}^n$ be a morphism corresponding to a line bundle $\mathcal{L}$ and $s_0, s_1, ..., s_n \in \Gamma(X, \mathcal{L})$, global sections which* *generate $\mathcal{L}$. Let $V \subseteq \Gamma(X, \mathcal{L})$ be the subspace spanned by the s_i. Then ϕ is a closed immersion if and only if*

1. *elements of V separate points, i.e., for any two distinct closed points $P, Q \in X$, there is an $s \in V$ such that $s \in m_p\mathcal{L}_P$ but $s \notin m_Q\mathcal{L}_Q$, or vice versa, and*
2. *elements of V separate tangent vectors, i.e., for each closed point $P \in X$, the set $\{s \in V : s_P \in m_P\mathcal{L}_P\}$ spans the $\mathbb{C}$–vector space $m_P\mathcal{L}_P/m_P^2\mathcal{L}_P$.*

The next theorem is called Cohomology and Base Change Theorem from the same book.

Theorem A.0.15 . *Let $\pi : \mathcal{P} \to X$ be a projective morphism of noetherian schemes and let F be a coherent sheaf on $\mathcal{P}$, flat over X. Let x be a point of X then:*

(a) *If the natural map*

$$\phi^i(x) : R^i\pi_*(F) \otimes \mathcal{C}(x) \to H^i(\mathcal{P}_x, F_x),$$

is surjective then it is an isomorphism, and the same is true for all x' in a suitable neighborhood of x.

(b) *Assuming that $\phi^i(x)$ is surjective then the following conditions are equivalent;*

(i) *$\phi^{i-1}(x)$ is also surjective.*

(ii) *$R^i\pi_*(F)$ is locally free in a neighborhood of x.*

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