Title	Affine connections on complex manifolds with algebraic dimension zero
Author(s)	McKay, Benjamin; Dumitrescu, Sorin
Publication date	2016
Original citation	Dumitrescu, S. and McKay, B. (2016) 'Affine connections on complex manifolds with algebraic dimension zero', Moscow Mathematical Journal, 16(4), pp.675-689.
Type of publication	Article (peer-reviewed)
Link to publisher's version	http://www.mathjournals.org/mmj/2016-016-004/2016-016-004-006.html Access to the full text of the published version may require a subscription.
Rights	**© 2016, Moscow Center for Continuous Mathematical Education, Independent University of Moscow and Higher School of Economics, Moscow. Materials in this journal may be reproduced by any means for educational and scientific purposes without fee or permission (provided that the customary acknowledgment of the source is given). This consent does not extend to other kinds of copying for general distribution, for advertising or promotional purposes, or for resale.** http://www.mathjournals.org/mmj/
Item downloaded from	http://hdl.handle.net/10468/4027

AFFINE CONNECTIONS ON COMPLEX MANIFOLDS OF ALGEBRAIC DIMENSION ZERO

SORIN DUMITRESCU AND BENJAMIN MCKAY

Abstract. We prove that any compact complex manifold with finite fundamental group and algebraic dimension zero admits no holomorphic affine connection or holomorphic conformal structure.

Contents

1. Introduction

We conjecture that any compact complex manifold of finite fundamental group with a holomorphic Cartan geometry is isomorphic to the model of its Cartan geometry, a homogeneous bundle of complex tori over a rational homogeneous variety. Roughly, the dynamics of the fundamental group on the universal cover form the essential ingredient in the classification of holomorphic Cartan geometries, so trivial dynamics gives a trivial geometry. As a first step, we try to study this question in the extreme case of manifolds of algebraic dimension zero, where there are few tools available from algebraic geometry.

A holomorphic affine connection is a holomorphic connection on the holomorphic tangent bundle of a complex manifold. A compact Kähler manifold admits a holomorphic affine connection just when it has a finite holomorphic unramified covering by a complex torus [13]. In this case the holomorphic affine connection pulls back to the complex torus to a translation invariant affine connection.

Nevertheless, some interesting compact complex manifolds which are not Kähler admit holomorphic affine connections. Think, for example, of the Hopf manifold associated to a linear contraction of complex Euclidean space. Also the parallelizable manifolds G/Γ associated to a complex Lie group G and a lattice $\Gamma \subset G$ admit holomorphic affine connections. In [11] Ghys constructs exotic deformations of quotients $SL(2,\mathbb{C})/\Gamma$ which are nonparallelizable complex manifolds admitting holomorphic affine connections. Moreover Ghys' quotients do not admit nonconstant meromorphic functions.

Date: September 8, 2015.
2000 *Mathematics Subject Classification.* Primary 53B21; Secondary 53C56, 53A55.
Key words and phrases. holomorphic affine connections, algebraic dimension.

Our conjecture predicts that holomorphic affine connections do not exist on compact complex manifolds of finite fundamental group. We prove this here under the additional hypothesis that our manifold's meromorphic functions are constant.

2. Notation and main result

The *algebraic dimension* of a complex manifold M is the transcendence degree of the field of meromorphic functions of M over the field of complex numbers. A compact complex manifold M has algebraic dimension zero just when every meromorphic function on M is constant.

Theorem 1. *No compact complex manifold with finite fundamental group and algebraic dimension zero admits a holomorphic affine connection.*

The main theorem will be proved in section 4. The principal ingredient of the proof is the following result which might be of independent interest.

Theorem 2. *If a complex abelian Lie algebra acts holomorphically on a complex manifold M with a dense open orbit preserving a holomorphic affine connection, then it also preserves a* flat *torsion-free holomorphic affine connection.*

The same proof yields the obvious analogue of Theorem 2 in the real analytic category, but the smooth category remains a mystery. The more general result, for Lie algebras which might not be abelian, is not true in higher dimension: the canonical action of $SL(2,\mathbb{C})$ on $SL(2,\mathbb{C})/\Gamma$ preserves the standard connection for which the right-invariant vector fields are parallel, but there are no torsion-free flat affine connections on $SL(2,\mathbb{C})/\Gamma$ [7]. Nevertheless, we conjecture that a finite dimensional complex Lie algebra acting holomorphically with a dense open orbit and preserving a holomorphic affine connection always preserves a *locally homogeneous* holomorphic affine connection. In the real analytic category and for surfaces this result was proved in [10].

3. Holomorphic geometry in algebraic dimension zero

Holomorphic affine connections are geometric structures of algebraic type which are rigid in Gromov's sense; see the nice expository survey [5] for the precise definition. Roughly speaking the rigidity comes from the fact that local biholomorphisms fixing a point and preserving a connection linearize in exponential coordinates, so they are completely determined by their differential at the fixed point. More generally, the local biholomorphisms preserving a rigid geometric structure are completely determined by a finite jet [5]. Also Gromov noticed that all known geometric structures are of algebraic kind, in the sense that the natural action of jets of local bihomorphisms on the jets of the geometric structure is algebraic.

So holomorphic affine connection and, in particular, holomorphic parallelizations of the holomorphic tangent bundle are examples of rigid holomorphic geometric structures of algebraic type in Gromov's sense. Other important examples of holomorphic rigid geometric structures of algebraic type are holomorphic Riemannian metrics and holomorphic conformal structures. They will be defined in section 5.

In this section we will use the result obtained by the first author in [9] (see Corollary 2.2 on page 35) asserting that on complex manifolds with algebraic dimension zero, rigid meromorphic geometric structures of algebraic type are locally homogeneous away from a nowhere dense analytic subset (see also [6]). This means that the set of local holomorphic vector fields preserving a rigid meromorphic geometric structure is transitive away from a nowhere dense analytic subset.

The same result was also proved to be true for holomorphic Cartan connections with algebraic model in [8]. In particular, the result applies to holomorphic affine connections on manifolds with algebraic dimension zero. Here we give a more precise result.

A *toroidal structure*, or *toroidal action*, on a complex manifold M of complex dimension $n := \dim_{\mathbb{C}} M$ is a holomorphic group action of the *toroidal group* $(\mathbb{C}^*)^n$ on M with a dense open orbit.

Example 1. We thank Misha Verbitsky for suggesting to us the following simple example of toroidal simply connected compact manifolds of algebraic dimension zero. This example is constructed by deformation of the standard complex structure on a simply connected Calabi-Eckmann manifold and it is a very particular case of the family of moment-angle manifolds constructed in [21, 22] as a generalization of the LMVB manifolds. Recall that the LMVB manifolds, constructed and studied by Lopez de Medrano, Meersseman, Verjovsky and Bosio in [3, 4, 19] (see also [18, 23, 14]) are toroidal and many of them admit complex affine structures. In [22] the authors prove that generic moment-angles manifolds are of algebraic dimension zero.

Consider the quotient of $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})$ by the $\mathbb{C}$-action given by the one-parameter group $\begin{pmatrix} e^t & 0 \\ 0 & e^{\alpha t} \end{pmatrix}$, with $\alpha \in \mathbb{C} \setminus \mathbb{R}$. This action is holomorphic, proper and free so the quotient is a complex manifold.

The embedding of S^3 as the unit sphere in $\mathbb{C}^2$ shows that the quotient M is diffeomorphic to $S^3 \times S^3$. This complex structure on $S^3 \times S^3$ fibers over $P^1(\mathbb{C})) \times P^1(\mathbb{C})$. Now deform the previous $\mathbb{C}$-action in an action of a one-parameter semi-simple subgroup in $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$. Then the corresponding complex structure on M is of algebraic dimension zero, as proved in [22]. Moreover, since the $\mathbb{C}$-action is semi-simple, its centralizer in $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ is isomorphic to $(\mathbb{C}^*)^4$. This induces a holomorphic $(\mathbb{C}^*)^3$-action with an open dense orbit on the quotient M: a toroidal structure.

The *characteristic subvariety* $S \subset M$ of a toroidal structure is the complement of the dense open orbit. If we write out vector fields $\{ v_a \}$ spanning $\mathfrak{g}$, then S is precisely the set where

$$0 = v_1 \wedge v_2 \wedge \cdots \wedge v_n.$$

In particular, $S \subset M$ is a closed complex hypersurface representing the anticanonical bundle of M.

Consider the *standard torus action*

$$t \in G, z \in \mathbb{C}^n \mapsto tz := (t_1 z_1, t_2 z_2, \dots, t_n z_n).$$

The standard torus action acts by affine transformations, and therefore on the bundle of frames of $\mathbb{C}^n$, which is identified equivariantly with the affine group, the standard torus action is free and proper. The standard torus action is therefore a rigid geometric structure in Gromov's sense (see [5], p. 70, 5.12B). The coordinate hyperplanes through the origin constitute the characteristic subvariety of the standard torus action. In particular, the characteristic subvariety of the standard torus action is an immersed complex manifold.

Denote by $G_0 := \left(S^1\right)^n \subset G$ the compact real form, by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}_0$ that of G_0. For a complex manifold M with a toroidal structure, an *adapted coordinate system* near a point $m_0 \in M$ is a holomorphic coordinate system

$$z \colon U_M \subset M \to U_{\mathbb{C}^n} \subset \mathbb{C}^n$$

defined in an open neighborhood of m_0, and a complex Lie group automorphism $\alpha \colon G \to G$ so that

$$\alpha(t) z(m) = z(tm)$$

for t in an open subset of $U_G \subset G$ containing the G_0-stabilizer of $z_0 := z(m_0)$.

Proposition 1. *Suppose that M is a complex manifold with a toroidal structure. Then every point of M lies in the domain of an adapted coordinate system. Every toroidal structure is a holomorphic rigid geometric structure of algebraic type in Gromov's sense.*

Proof. Pick a point $s_0 \in S$ in the characteristic subvariety of M. Because G is abelian, the stabilizer of any point of M is the same throughout each G-orbit. Another way to say this: the fixed point locus

$$M^{s_0} := M^{G^{s_0}} \subset M$$

of any stabilizer G^{s_0} contains the G-orbit $Gs_0 \subset M^{s_0}$.

By Bochner's theorem [2] p. 375, the action of the G_0-stabilizer $G_0^{s_0}$is linearizable near s_0, giving us holomorphic coordinates, called *Bochner coordinates*, in which $G_0^{s_0}$ acts as a subtorus of the unitary group. Since the Cartan subgroup is the maximal torus, we can assume that $G_0^{s_0}$ acts in Bochner coordinates by diagonal unitary matrices. It follows that the associated vector fields of $\mathfrak{g}_0$ are similarly represented in those coordinates as diagonal matrices with imaginary entries. Complexifying these vector fields, all of the vector fields of $\mathfrak{g}$ are linear and represented by diagonal matrices. In Bochner coordinates, M^{s_0} is a linear subspace in Bochner coordinates, so is smooth near s_0 with tangent space

$$T_{s_0}\left(M^{s_0}\right) = T_{s_0}\left(M^{G^{s_0}}\right) = \left(T_{s_0}M\right)^{G^{s_0}}.$$

Let $n = \dim_{\mathbb{C}} M = p + q$ where $q = \dim_{\mathbb{C}} Gs_0$ is the orbit dimension. We can reorder our Bochner coordinates to get

$$\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^q = \mathbb{C}^p \oplus T_{s_0}Gs_0,$$

so that the elements of G^{s_0} are diagonal matrices, with 1's in the last q entries, and some complex numbers in the first p entries.

Every orbit is $Gs_0 = G/G^{s_0}$, so $q = n - \dim_{\mathbb{C}} G^{s_0}$, i.e. $\dim_{\mathbb{C}} G^{s_0} = p$. So the complex numbers in the first p entries of elements of G^{s_0} can be arbitrary, i.e. G^{s_0} is the group of diagonal $n \times n$ matrices whose last q entries are 1's. But therefore M^{s_0} and Gs_0 are both identified with the $\lambda = 1$-eigenspace of G^{s_0}, $0 \oplus \mathbb{C}^q \subset \mathbb{C}^n$. In particular, $Gs_0 \subset M^{s_0}$ is an open subset.

Write out Bochner coordinates x^i, y^μ so that $i = 1, 2, \dots, p$ and $\mu = p+1, \dots, p+q = n$. The Lie algebra $\mathfrak{g}^{s_0}$ of G^{s_0} is the span of

$$x^1\partial_{x^1}, x^2\partial_{x^2}, \dots, x^p\partial_{x^p}.$$

Take a basis of $\mathfrak{g}$

$$x^1\partial_{x^1}, x^2\partial_{x^2}, \dots, x^p\partial_{x^p}, v_\mu.$$

The vectors $v_\mu(0)$ form a basis of $T_{s_0}Gs_0$, the tangent space to the G-orbit, so there are q linear independent commuting vector fields v_μ, linearly independent at s_0. Pick local coordinates so that

$$v_\mu(s_0) = \partial_{y^\mu}.$$

Expand out

$$0 = \left[x^i\partial_{x_i}, v_\mu\right]$$

to see that

$$v_\mu = \sum_i a^i_\mu(y)x^i\partial_{x_i} + \sum_\nu b^\nu_\mu(y)\partial_{y^\nu}.$$

The projections of these into the y-coordinates give linearly independent commuting vector fields

$$\sum_\nu b^\nu_\mu(y)\partial_{y^\nu}.$$

Change the y-variables to arrange that these become

$$\partial_{y^\mu}.$$

So the original vector fields are now

$$v_\mu = \partial_{y^\mu} + \sum_i a^i_\mu(y) x^i \partial_{x^i}.$$

The condition that these commute is precisely

$$\frac{\partial a^i_\nu}{\partial y^\mu} = \frac{\partial a^i_\mu}{\partial y^\nu},$$

i.e. locally there are functions $a^i(y)$ so that

$$a^i_\mu = \frac{\partial a^i}{\partial y^\mu}.$$

Define meromorphic 1-forms ω^a on M by

$$\omega^a(\xi) := \frac{v_1 \wedge v_2 \wedge \cdots \wedge v_{a-1} \wedge \xi \wedge v_{a+1} \wedge \cdots \wedge v_n}{v_1 \wedge v_2 \wedge \cdots \wedge v_n}.$$

Calculate that

$$\omega^i = \frac{dx^i}{x^i} - da^i,$$
$$\omega^\mu = dy^\mu.$$

Therefore we make new coordinates by replacing x^i by $x^i e^{-a^i}$, and find that in our new coordinates, the action of $G_0^{s_0}$ is unchanged, while the action of $\mathfrak{g}$ is given by the vector fields

$$x^1 \partial_{x^1}, x^2 \partial_{x^2}, \ldots, x^p \partial_{x^p}, \partial_{y^1}, \partial_{y^2}, \ldots, \partial_{y^q}.$$

Following the same steps, applied to the standard torus action, rather than the action on M, we find the same Lie algebra action and the same stabilizer action, near any point $z_0 \in \mathbb{C}^n$ with p zero entries and q nonzero entries. Therefore the Lie algebra actions, and stabilizer actions, are locally isomorphic. Since the vector fields of the standard torus action constitute a holomorphic rigid geometric structure of algebraic type in Gromov's sense, the local isomorphism of the Lie algebra actions already ensures that the toroidal structure on M is also a holomorphic rigid geometric structure of algebraic type. □

A by-product of the proof of Proposition 1 is the fact that any toroidal action is localy linearizable. This is not true for general holomorphic $\mathbb{C}^n$-actions with an open dense orbit. Think at the standard $\mathbb{C}^n$-action by translations on $P^n(\mathbb{C})$ in the neighborhood of points situated on the divisor at the infinity. Nevertheless Theorem 2 can be seen as a global linearization result for $\mathbb{C}^n$-actions preserving a holomorphic affine connection and admitting an open dense orbit.

Dealing with non abelian locally free holomorphic actions admitting an open dense orbit, Guillot classified in [12] holomorphic equivariant compactifications of quotients of $SL(2,\mathbb{C})/\Gamma$, where Γ is a discrete subgroup in $SL(2,\mathbb{C})$. Unlike toroidal actions, those $SL(2,\mathbb{C})$-actions are nonrigid holomorphic geometric structures of algebraic type (they are rigid only on the open dense orbit U of the $SL(2,\mathbb{C})$-action). Indeed, here the local symmetries of the $SL(2,\mathbb{C})$-action on U pull-back in $SL(2,\mathbb{C})$ as left-invariant vector fields: they generate right translations in $SL(2,\mathbb{C})$ and generically they do not extend to all of the compactification space. The extendability of local symmetries do not stand for those holomorphic geometric structures: they must be nonrigid by a result of Gromov ([5], p. 73, 5.15).

Proposition 2. *Suppose that M is a compact, connected and simply connected complex manifold of complex dimension n and of algebraic dimension zero. Then M admits a holomorphic rigid geometric structure of algebraic type if and only if M admits a toroidal structure. The toroidal structure is then unique. The toroidal group is a cover of the identity component G of the biholomorphism group of M. The toroidal action preserves* all *meromorphic geometric structures of algebraic type on M.*

Every bimeromorphism of M is a biholomorphism of the open toroidal orbit. The bimeromorphism group of M is a semidirect product $G \rtimes \Gamma$ where Γ is the discrete subgroup of the bimeromorphism group fixing some point $m_0 \in M$ of the dense open toroidal orbit. Each element of Γ is determined by its action on $T_{m_0}M$, giving an injective morphism of Lie groups $\Gamma \to \mathrm{GL}(T_{m_0}M) \cong \mathrm{GL}(n, \mathbb{C})$. The complement in M of the open toroidal orbit is a complex hypersurface containing at least n analytic components.

Proof. By the main theorem in [9] (see also [8]) any holomorphic rigid geometric structure of algebraic type g on M is locally homogeneous on an open dense set (away from a nowhere dense analytic subset S), meaning that the Lie algebra of local holomorphic vector fields on M preserving g is transitive on an open dense set in M. Moreover, since M is simply connected, by a result due to Nomizu [20] and generalized by Amores [1] and then by Gromov [5] (p. 73, 5.15) these local vector fields preserving g extend to all of M to form a finite dimensional complex Lie algebra $\mathfrak{g}$ of (globally defined) holomorphic vector fields v_i acting with a dense open orbit in M and preserving g.

Now put together g and the v_i to form another rigid holomorphic geometric structure of algebraic type $g' = (g, v_i)$ (see [5] for details about the fact that the juxtaposition of a rigid geometric structure with another geometric structure is still a rigid geometric structure in Gromov's sense). Considering g' instead of g and repeating the same proof as before, the complex Lie algebra $\mathfrak{g}'$ of those holomorphic vector fields preserving g' acts with a dense open orbit in M. But preserving g' means preserving both g and the v_i. Hence $\mathfrak{g}'$ lies in the center of $\mathfrak{g}$. In particular $\mathfrak{g}'$ is a complex abelian Lie algebra acting with a dense open orbit in M. At each point m_0 in the open $\mathfrak{g}'$-orbit, the values $v(m)$ of the vector fields $v \in \mathfrak{g}'$ span the tangent space to the $\mathfrak{g}'$-orbit, i.e. span the tangent space. Any linear relation between the values $v(m)$ of the vector fields $v \in \mathfrak{g}'$ is $\mathfrak{g}'$-invariant, so holds throughout the open orbit, and so holds everywhere. Therefore all vector fields in $\mathfrak{g}'$ are linearly independent at every point of the open orbit of $\mathfrak{g}'$.

Pick a basis $v_1, v_2, \dots, v_n \in \mathfrak{g}'$. Notice that any holomorphic vector field commuting with all v_i, for $i \in \{1, \dots, n\}$, is a constant coefficient linear combination of those v_i: this is true on the open $\mathfrak{g}'$-orbit and, consequently, on all of M. It follows that the centralizer of the Lie algebra $\mathfrak{g}'$ in the Lie algebra of all holomorphic vector fields on M is exactly $\mathfrak{g}'$. This implies that $\mathfrak{g}' = \mathfrak{g}$ and hence $\mathfrak{g}$ is a complex abelian Lie algebra of dimension n acting with a dense open orbit.

The proof is the same if we replace g' by the extra rigid meromorphic geometric structure of algebraic type g'' on M which is the juxtaposition of g', with any meromorphic geometric structure of algebraic type globally defined on M. This implies that any meromorphic geometric structure on M is $\mathfrak{g}$-invariant.

Therefore all meromorphic vector fields on M belong to $\mathfrak{g}$. If a connected Lie group acts by bimeromorphisms, then its Lie algebra acts by meromorphic vector fields, so as a Lie subalgebra of $\mathfrak{g}$.

Define meromorphic 1-forms ω^a on M by

$$\omega^a(\xi) := \frac{v_1 \wedge v_2 \wedge \cdots \wedge v_{a-1} \wedge \xi \wedge v_{a+1} \wedge \cdots \wedge v_n}{v_1 \wedge v_2 \wedge \cdots \wedge v_n}.$$

Take a bimeromorphic map $f\colon M \to M$. Then $f^*\omega^a = h^a_b \omega^b$, for some meromorphic functions h^a_b forming an invertible matrix. But meromorphic functions on M are constant, so f acts on $\mathfrak{g}$ by a linear transformation. In particular, f extends to be defined on the dense open $\mathfrak{g}$-orbit, which is therefore invariant under bimeromorphism. If f acts trivially on $\mathfrak{g}$, then we can pick any point m_0 in the dense open $\mathfrak{g}$-orbit and we can find some element $e^v \in e^{\mathfrak{g}}$ in the biholomorphism group of M which takes m_0 to $f(m_0)$, so we can write $f = e^v g$, so that $g(m_0) = m_0$ for some bimeromorphism g of M. But then g is uniquely determined by its action on $\mathfrak{g}$, i.e. on the tangent space $T_{m_0}M$, since it commutes with exponentiation of the vector fields, so we have a unique decomposition of the bimeromorphism group of M into a semidirect product $G \rtimes \Gamma$ where $\Gamma \subset \mathrm{GL}(\mathfrak{g}) = \mathrm{GL}(T_{m_0}M) = \mathrm{GL}(n, \mathbb{C})$ and $G = e^{\mathfrak{g}}$. Notice that compactness of M is only required here in order to ensure completeness of vector fields in $\mathfrak{g}$.

Every meromorphic differential form on M is closed because it is $\mathfrak{g}$-invariant. In particular, all meromorphic 1-form on M are closed, while only 0 is exact. The Albanese dimension of M is zero. The indefinite integral of any meromorphic 1-form over the simply connected manifold M is a meromorphic function on some covering space of the complement of the simple poles of the 1-form. Therefore every meromorphic 1-form has a simple pole on some component of S.

Let $\Delta := H_1(M - S, \mathbb{Z}) = H_1(G, \mathbb{Z})$. Then $\Delta \subset \mathfrak{g}$ and $G = \mathfrak{g}/\Delta$. Pair $\gamma \in \Delta, \omega \in \mathfrak{g}^* \mapsto \int_\gamma \omega \in \mathbb{C}$. If this vanishes for some ω, for every γ, then ω integrates around each component of S to a meromorphic function. But then $\omega = 0$. Therefore the pairing is nondegenerate. Define an injection $\gamma \in \Delta \mapsto v_\gamma \in \mathfrak{g}$ by $\Delta \subset \mathfrak{g}$. Then Δ spans $\mathfrak{g}$ over $\mathbb{C}$. Therefore Δ contains a complex basis of $\mathfrak{g}$, so the action of $\mathfrak{g}$ drops to an action of $\left(\mathbb{C}^*\right)^n$. If some meromorphic 1-form integrates to zero around each analytic component of S, then its indefinite integral is meromorphic. So picking one $\gamma \in \Delta$ around each analytic component of S, the associated $v_\gamma \in \mathfrak{g}$ already span $\mathfrak{g}$. Therefore the number of analytic components of S is at least n. ☐

4. Abelian groups preserving holomorphic connections

Suppose that M is a complex manifold and A is an abelian Lie algebra of holomorphic vector fields, acting with a dense open orbit (or equivalently, acting with an open orbit on every component of M) and preserving a holomorphic connection ∇ on TM. In this section we will prove theorem 2 on page 2: that A also preserves a flat holomorphic affine connection. Every holomorphic affine connection induces a unique torsion-free affine connection with the same geodesics (see for instance [13]). We assume that ∇ is torsion-free and that M is connected without loss of generality.

The complement of the open orbit is a complex hypersurface $S \subset M$, possibly singular, where the vector fields $v \in A$ are not linearly independent, representing the anticanonical line bundle. If this hypersurface is empty, then in terms of any basis $v_a \in A$,

$$\bar{\nabla} v_a \left(X^b v_b\right) := \left(\mathcal{L}_{v_a} X^b\right) v_b.$$

is an invariant flat torsion-free connection. So we assume that S is not empty.

Lemma 1. *We make A into a commutative, perhaps nonassociative, algebra by $vw := \nabla_v w$.*

Proof. Take any basis $\{ v_a \} \subset A$. Then on $M - S$, these $\{ v_a \}$ form a parallelism, so we can write

$$\nabla_{v_a} v_b = \Gamma^c_{ab} v_c$$

for unique functions Γ^c_{ab}. By A-invariance, Γ^c_{ab} are constants. By continuity, the relation $\nabla_{v_a} v_b = \Gamma^c_{ab} v_c$ continues to hold everywhere on M. The torsion of ∇ is

$$\nabla_v w - \nabla_w v - [v, w],$$

so by torsion-freedom $vw = wv$. □

Lemma 2. *Pick a point $s_0 \in S$ and let*

$$I_{s_0} := \{ v \in A \mid v(s_0) = 0 \} \subset A.$$

Then $I_{s_0} \subset A$ is an ideal.

Proof. If $v \in I_{s_0}$ and $w \in A$ then clearly $\nabla_v w$ vanishes wherever v vanishes. In other words, $I_{s_0} \subset A$ is an ideal. □

Lemma 3. *Take an ideal $I \subset A$. For each point $m \in M$, let*

$$I(m) := \{ v(m) \mid v \in I \} \subset T_m M.$$

There is a unique nowhere singular holomorphic foliation F_I on M so that the tangent spaces of the leaves are

$$T_m F_I = I(m)$$

for every $m \in M - S$. The normal bundle of F_I is flat along each leaf of F_I, with trivial leafwise holonomy.

Proof. Every tangent vector to $M - S$ is $v(m)$ for a unique $v \in A$. For any $w \in I$, the vector field $\nabla_v w$ lies in I, so $\nabla_{v(m)} \colon I(m) \to I(m)$. In other words, through every point $m \in M - S$, the subspace $I(m)$ is invariant under parallel transport in every direction. Therefore the orbit of the vector fields in I through any such point is totally geodesic. Parallel transport extends the subspaces $I(m)$ to be defined at every point $m \in M$, continuously and therefore holomorphically, *but* $I(m) \neq \{ v(m) \mid v \in I \}$ for any $m \in S$. These subspaces $I(m)$ are the tangent spaces to the I-orbits throughout $M - S$, and therefore form a bracket closed subbundle of the tangent bundle on $M - S$. This ensures bracket closure on all of M of the subbundle spanned by these $I(m)$ by continuity. Let $F = F_I$ be the associated holomorphic totally geodesic foliation of M whose tangent spaces are the spaces $I(m)$. Let $\pi \colon TM \to TM/TF$ be the quotient to the normal bundle of the foliation. If $w \in A$ and $v \in I$ then we let $n := \pi \circ w$ be the associated quotient section of the normal bundle, and compute

$$\begin{aligned} \nabla_v n &= \nabla_v \pi \circ w, \\ &= \pi \circ \nabla_v w, \end{aligned}$$

because π is parallel, as the foliation is totally geodesic,

$$= 0$$

because $\nabla_v w \in I$ is tangent to the foliation F. Therefore the quotient sections $n = \pi \circ w$ of the normal bundle of the foliation F are parallel along the leaves of F. So the quotient bundle TM/TF is flat along the leaves, with trivial leafwise holonomy, wherever $A/I \mapsto A(m)/I(m)$ is injective, and in particular on $M - S$. But this is a dense open set, so TM/TF is flat along every leaf, with trivial leafwise holonomy. □

Lemma 4. *For any choice of basis $\{ v_a \} \subset A$, define meromorphic 1-forms ω^a on M by*

$$\omega^a(\xi) := \frac{v_1 \wedge v_2 \wedge \dots \wedge v_{a-1} \wedge \xi \wedge v_{a+1} \wedge \dots \wedge v_n}{v_1 \wedge v_2 \wedge \dots \wedge v_n}.$$

These are the dual 1-forms to our basis of A, i.e. $\omega^a(v_b) = \delta^a_b$. In particular, take a point $s_0 \in S$ and pick our basis $\{ v_a \} \subset A$ by first picking some basis $\{ v_i \} \subset I_{s_0}$ and *then picking a maximal set of vector fields $\{ v_\mu \} \subset A$ taking on linearly independent values in $T_{s_0}M$. Then ω^μ are holomorphic and linearly independent 1-forms near s_0.*

Proof. Clearly ω^i is singular at s_0 because $\omega^i(v_j) = 1$ everywhere, including at s_0, while $v_j(s_0) = 0$. On the other hand, it is less clear whether ω^μ are holomorphic near s_0. Let $I = I_{s_0}$ and $F = F_I$. We can equivalently define the ω^μ near s_0 by the linear holomorphic equations:

(1) $\omega^\mu = 0$ on TF and
(2) $\omega^\nu(n_\nu) = \delta^\mu_\nu$, where $n_\nu := \pi \circ v_\nu$ is the associated parallel basis of the normal bundle of the foliation F.

In particular, since these holomorphic linear equations specify the ω^μ, these ω^μ are holomorphic everywhere near s_0. □

Lemma 5. *For every nontrivial $v \in A$, the zero locus of v is a union of complex analytic irreducible components of S.*

Proof. Pick some $v \in A$ and some point $s_0 \in S$ so that $v(s_0) \neq 0$. We can pick a basis $\{ v_i, v_\mu \} \subset A$ as in the previous lemma, so that v is one of the v_μ. By the Hartogs extension theorem, the associated ω^μ as defined in the previous lemma are holomorphic on M except on certain hypersurfaces. These hypersurfaces lie inside S, because away from S all of the ω^a are holomorphic. So these hypersurfaces form a union of certain complex analytic irreducible components of S not passing through s_0. By the previous lemma, these hypersurfaces coincide with the zero locus of $v = v_\mu$. Therefore the zero locus of v is a union of complex analytic irreducible components of S not passing through s_0. In particular, in the neighborhood of a vanishing point of v which is a smooth point of S, the zero locus of v is exactly S. □

Lemma 6. *For any smooth point $s_0 \in S$, the ideal $I_{s_0} \subset A$ is principal. In other words there is a vector field $v \in A$ so that v vanishes at every point on the complex analytic irreducible component of S through s_0, v doesn't vanish at the generic point of M, and any element of A vanishing at s_0 is a constant multiple of v.*

Proof. In geodesic local coordinates of any connection, any Killing field of the connection is linearized; this a very classical result in the field, see for instance [10, p. 6 lemma 7]. Therefore every $v \in I_{s_0}$ vanishing at s_0 has nonzero linearization at s_0 or is $v = 0$. The commuting of all of the $v \in A$ ensures that these linearizations commute. Write out a basis of these vector fields $v \in I_{s_0}$, say as

$$v_i = C_{ib}^{a} x^b \frac{\partial}{\partial x^a}.$$

The matrices C_i commute and are linearly independent. The points of S near the origin in these coordinates are precisely the points where each v_i vanishes, i.e. the points x where $C_i x = 0$, for any one value of i. Since S is a complex hypersurface, the kernels of all of the C_i must be the same complex hypersurface. Since the kernel of each C_i is a hypersurface, and each C_i is a square matrix, each C_i has 1-dimensional image, so this image is an eigenspace of C_i. Each C_i preserves the eigenspaces of the others, so they all share the same eigenvectors. Hence all of the C_i are scalar multiples of one another. If there is more than one of these C_i matrices, then they are not linearly independent. □

Lemma 7. *Let $I \subset A$ be the ideal generated by all vector fields $v \in A$ which vanish at some point of M. The leaves of the foliation F_I are totally geodesic and the holomorphic affine connection is flat along these leaves. In particular, if $I = A$, then the holomorphic affine connection is already flat everywhere.*

Proof. Take a maximal collection $\{ v_i \} \subset A$ of nonzero vector fields, so that each vanishes on a different locus from any of the others. Since the ideal generated by each element v_i is principal,

$$v_i w = \alpha_i(w) v_i,$$

for $w \in A$, for a unique $\alpha_i \in A^*$. Clearly if $v_i, v_j \in I$ have different vanishing loci, $v_i v_j = \alpha_i(v_j)\, v_i = \alpha_j(v_i)\, v_j$, so $\alpha_i(v_j) = 0$ if $i \neq j$. So we can write $v_i v_j = \delta_{ij} \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$. We calculate the curvature R along any leaf of any F_I at a point of $M - S$:

$$\begin{aligned} R(v_i, v_j)\, v_k &= \nabla_{v_i} \nabla_{v_j} v_k - \nabla_{v_j} \nabla_{v_i} v_k - \nabla_{[v_i, v_j]} v_k, \\ &= \lambda_i \lambda_j \delta_{ij} \left(\delta_{jk} - \delta_{ik} \right) v_k, \\ &= 0. \end{aligned}$$

□

Lemma 8. *Again let $I \subset A$ be the ideal generated by all vector fields $v \in A$ which vanish at some point of M. Take a basis $\{ v_i \} \subset I$. Choose additional elements $\{ v_\mu \} \subset A$ so that $\{ v_i, v_\mu \} \subset A$ is a basis. Choose a leaf L of F_I and a local basis of parallel sections w_i of $TL|_L$ near a point of L. Note that w_i, v_μ span $TM|_L$ near that point. Extend these w_i nearby on M by invariance under the v_μ. Then the connection ∇' defined by*

$$\begin{aligned} 0 &= \nabla'_{v_\mu} v_\nu, \\ 0 &= \nabla'_{w_i} w_j, \\ 0 &= \nabla'_{v_\mu} w_j, \\ 0 &= \nabla'_{w_i} v_\nu \end{aligned}$$

extends uniquely to a flat torsion-free holomorphic A-invariant connection on M.

Proof. Since the leaves of F_I are totally geodesic and flat and the vectors v_μ preserve ∇, then (w_i, v_μ) is a local frame of commuting vector fields. By construction this local frame is parallel with respect to ∇'. Hence, in adapted local holomorphic coordinates, ∇' is the standard torsion free flat affine connection. If we change the choice of w_i, we do so only by constant coefficient linear combinations, so ∇' is unchanged.

Notice that ∇ and ∇' agree on the leaves of F_I. The transverse vector fields v_μ are parallel with respect to ∇'; this is not necessarily the case with respect to ∇.

Since the foliation F_I is holomorphic and nonsingular, ∇' is holomorphic on all of M. Moreover, ∇' is A-invariant by construction (since everything in the definition of ∇' is). □

Proposition 2 and Theorem 2 imply that M admits a complex affine structure. If M is simply connected, a developing map of the complex affine structure is a local biholomorphism from M to the complex affine space of the same dimension. But the compactness of M implies that any such holomorphic map must be constant: a contradiction. This finishes the proof of Theorem 1.

5. Holomorphic conformal structures

Definition 1. A holomorphic Riemannian metric on a complex manifold M is a holomorphic section q of the bundle $S^2(T^*M)$ of complex quadratic forms on M such that in any point m in M the quadratic form $q(m)$ is nondegenerate.

As in the real Riemannian and pseudo-Riemannian settings, any holomorphic Riemannian metric q on M determines a unique torsion free holomorphic affine connection with respect to which q is a parallel tensor. Starting with this Levi-Civita connection one computes the curvature tensor of q. Recall that q is called flat if its curvature tensor vanishes everywhere. In this case q is locally isomorphic to $dz_1^2 + dz_2^2 + \ldots + dz_n^2$ on $\mathbb{C}^n$ and M is locally modelled on $\mathbb{C}^n$ with transition maps in $G = O(n, \mathbb{C}) \ltimes \mathbb{C}^n$.

A holomorphic Riemannian metric on M defines an isomorphism between TM and T^*M. Moreover, up to a double cover on M, the canonical bundle and the anticanonical bundle of M are holomorphically trivial (see for instance [9]). Consequently, Proposition 2 implies that compact complex simply connected manifolds with algebraic dimension zero do not admit holomorphic Riemannian metrics. In complex dimension three this result was proved in [9] (see Corollary 4.1 on page 44) without any hypothesis on the algebraic dimension.

A more flexible geometric structure is a holomorphic conformal structure.

Definition 2. A holomorphic conformal structure on a complex manifold M is a holomorphic section ω of the bundle $S^2(T^*M) \otimes L$, where L is a holomorphic line bundle over M, such that at any point m in M the section $\omega(m)$ is nondegenerate.

Roughly speaking this means that M admits an open cover such that on each open set in the cover, M admits a holomorphic Riemannian metric, and on the overlaps of two open sets the two given holomorphic Riemannian metrics agree up to a nonzero multiplicative constant.

Here the flat example is the quadric $z_0^2 + z_1^2 + \ldots + z_{n+1}^2 = 0$ in $P^{n+1}(\mathbb{C})$ with the conformal structure induced by the quadratic form $dz_0^2 + dz_1^2 + \ldots + dz_{n+1}^2$ on the quadric. The automorphism group of the quadric with its canonical conformal structure is $\mathbb{P}O(n+2, \mathbb{C})$.

A classical result due to Gauss asserts that all conformal structures on surfaces are locally isomorphic to the two-dimensional quadric.

Any manifold M of complex dimension $n \geq 3$ bearing a flat holomorphic conformal structure (meaning that the Weyl tensor of curvature vanishes on all of M) is locally modelled on the quadric.

Recall that Kobayashi and Ochiai classified in [17] the complex compact surfaces locally modelled on the quadric. More recently, Jahnke and Radloff classified projective compact complex threefolds bearing holomorphic conformal structures [15] and also projective compact complex manifolds locally modelled on the quadric [16].

We prove here the following result:

Theorem 3. *No compact complex manifold with finite fundamental group and algebraic dimension zero admits a holomorphic conformal structure.*

Proof. For surfaces, the result is a direct consequence of the classification given in [17].

We suppose now that our manifold M is of complex dimension at least 3. Then the holomorphic conformal structure is a rigid geometric structure [5].

Up to a finite cover, we can assume that M is simply connected and, as a consequence of Proposition 2, M is toroidal. On the open dense orbit U of the toroidal group the holomorphic tangent bundle of M is trivial. In particular, its canonical bundle is trivial and hence the holomorphic conformal structure admits a global representative on U which is a holomorphic Riemannian metric q.

Denote by $v_1, v_2, \ldots v_n$ the fundamental vector fields of the toroidal action. Since both the conformal structure and the holomorphic section $v_1 \wedge v_2 \wedge \ldots \wedge v_n$ of the canonical bundle are invariant by the toroidal action, it follows that the holomorphic Riemannian metric q is invariant by the toroidal action. But a holomorphic Riemannian metric invariant by a transitive action of an abelian group is flat. In particular, the conformal structure is flat on U and hence on all of M.

If follows that M is locally modelled on the quadric. Since M is simply connected, the developing map of the flat conformal structure is a local biholomorphism from M into the quadric. This is impossible since the quadric is an algebraic manifold and M has algebraic dimension zero. □

6. Conclusion

We conjecture that any compact complex manifold with finite fundamental group bearing a holomorphic Cartan geometry is biholomorphic to the model. In particular, this implies that compact complex manifolds bearing holomorphic affine connections have infinite fundamental group. The result is known for Kähler manifolds [13]. Above we prove this fact for manifolds with algebraic dimension zero.

The conjecture is also open and interesting for the particular case of holomorphic projective connections. Our approach here could still work for manifolds with algebraic dimension zero endowed with holomorphic projective connections, if we knew how to prove an equivalent of Theorem 2 on page 2 for projective connections. It seems likely that any holomorphic projective connection invariant under an abelian Lie algebra action with a dense open orbit and preserving no holomorphic affine connection is flat, implying an analogue of Theorem 2 for projective connections.

References

1. A. M. Amores, *Vector fields of a finite type G-structure*, J. Differential Geom. **14** (1979), no. 1, 1–6 (1980). MR 577874 (81k:53033) 6
2. S. Bochner, *Compact groups of differentiable transformations*, Ann. of Math. (2) **46** (1945), 372–381. MR MR0013161 (7,114g) 4
3. Frédéric Bosio, *Variétés complexes compactes: une généralisation de la construction de Meersseman et López de Medrano-Verjovsky*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 5, 1259–1297. MR 1860666 (2002i:32015) 3
4. Frédéric Bosio and Laurent Meersseman, *Real quadrics in* $\mathbf{C}^n$*, complex manifolds and convex polytopes*, Acta Math. **197** (2006), no. 1, 53–127. MR 2285318 (2007j:32037) 3
5. G. D'Ambra and M. Gromov, *Lectures on transformation groups: geometry and dynamics*, Surveys in differential geometry (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991, pp. 19–111. MR 1144526 (93d:58117) 2, 3, 5, 6, 11
6. Sorin Dumitrescu, *Structures géométriques holomorphes sur les variétés complexes compactes*, Ann. Sci. École Norm. Sup. (4) **34** (2001), no. 4, 557–571. MR 1852010 (2003e:32034) 2
7. ———, *Une caractérisation des variétés complexes compactes parallélisables admettant des structures affines*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 19-20, 1183–1187. MR 2567000 (2010j:32032) 2
8. ———, *Killing fields of holomorphic Cartan geometries*, Monatsh. Math. **161** (2010), no. 2, 145–154. MR 2680003 2, 6
9. ———, *Meromorphic almost rigid geometric structures*, Geometry, Rigidity, and Group Actions (Chicago, Il.) (B. Farb and D. Fisher, eds.), Chicago Lectures in Mathematical Sciences, vol. April, University of Chicago Press, 2011, p. 552. 2, 6, 11
10. Sorin Dumitrescu and Adolfo Guillot, *Quasihomogeneous analytic affine connections on surfaces*, J. Topol. Anal. **5** (2013), no. 4, 491–532. 2, 9
11. Étienne Ghys, *Déformations des structures complexes sur les espaces homogènes de* $SL(2,\mathbb{C})$, J. Reine Angew. Math. **468** (1995), 113–138. MR 1361788 (96m:32017) 1
12. Adolfo Guillot, *Sur les équations d'Halphen et les actions de* $\mathrm{SL}_2(\mathbf{C})$, Publ. Math. Inst. Hautes Études Sci. (2007), no. 105, 221–294. MR 2354208 (2009b:32031) 5
13. Masahisa Inoue, Shoshichi Kobayashi, and Takushiro Ochiai, *Holomorphic affine connections on compact complex surfaces*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 247–264. MR 586449 (82g:32033) 1, 7, 12
14. Hiroaki Ishida, *Complex manifolds with maximal torus actions*, ArXiv e-prints (2013). 3
15. Priska Jahnke and Ivo Radloff, *Projective threefolds with holomorphic conformal structure*, Internat. J. Math. **16** (2005), no. 6, 595–607. MR 2153485 (2006c:14062) 11
16. Priska Jahnke and Ivo Radloff, *Projective manifolds modeled after hyperquadrics*, (2015). 11

17. Shoshichi Kobayashi and Takushiro Ochiai, *Holomorphic structures modeled after hyperquadrics*, Tôhoku Math. J. (2) **34** (1982), no. 4, 587–629. MR MR685426 (84b:32039) 11
18. F. Lescure and L. Meersseman, *Compactifications équivariantes non kählériennes d'un groupe algébrique multiplicatif*, Ann. Inst. Fourier (Grenoble) **52** (2002), no. 1, 255–273. MR 1881579 (2002j:32015) 3
19. Laurent Meersseman, *A new geometric construction of compact complex manifolds in any dimension*, Math. Ann. **317** (2000), no. 1, 79–115. MR 1760670 (2001i:32029) 3
20. Katsumi Nomizu, *On local and global existence of Killing vector fields*, Ann. of Math. (2) **72** (1960), 105–120. MR 0119172 (22 #9938) 6
21. Taras Panov and Yuri Ustinovsky, *Complex-analytic structures on moment-angle manifolds*, Mosc. Math. J. **12** (2012), no. 1, 149–172, 216. MR 2952429 3
22. Taras Panov, Yuri Ustinovsky, and Misha Verbitsky, *Complex geometry of moment-angle manifolds*, (2013). 3
23. Jörg Winkelmann, *On manifolds with trivial logarithmic tangent bundle: the non-Kähler case*, Transform. Groups **13** (2008), no. 1, 195–209. MR 2421322 (2009f:32038) 3

Université Côte d'Azur, Université Nice Sophia Antipolis, CNRS, Laboratoire J.A. Dieudonné, Parc Valrose, 06108 NICE Cedex 02, France
E-mail address: dumitres@unice.fr

University College Cork, Cork, Ireland
E-mail address: b.mckay@ucc.ie