Fluid accretion onto a spherical black hole: Relativistic description versus the Bondi model

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We describe general relativistically a spherically symmetric stationary fluid accretion onto a black hole. Relativistic effects enhance mass accretion, in comparison to the Bondi model predictions, in the case when back reaction is neglected. That enhancement depends on the adiabatic index and the asymptotic gas temperature and it can magnify accretion by one order in the ultrarelativistic regime. [S0556-2821(99)01120-0]

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I. INTRODUCTION

In this paper we re-examine the spherical gas accretion onto a black hole, paralleling previous studies of fluid accretion of Michel and Shapiro and Teukolsky [1,2]. It is shown that relativistic effects can lead to a bigger mass accretion than that predicted by the corresponding Bondi model [3].

The order of this paper is as follows. Section II presents spherically symmetric Einstein equations expressed in the language of extrinsic curvatures. A suitable choice of a gauge condition leads to a ‘‘comoving coordinates’’ [4] formulation that is particularly suitable for the description of self-gravitating fluids. In Sec. III we show that the original set of integrodifferential equations can be reduced to an integroalgebraic problem, whose solution would constitute a new stationary, general-relativistic solution of self-gravitating polytropic fluids. That model is complete—it includes the back effect exerted by matter onto a metric; therefore it is capable of describing a stationary phase of the interaction of (even) heavy clouds of gas with a relatively light center. Section IV discusses a case when back reaction can be neglected. Under some circumstances, an accretion is described by a set of purely algebraic equations. Section V proves several quantitative and qualitative properties of accreting solutions. It is shown that the Bondi model relation between the asymptotic and sonic speeds of sound appears as a limiting case of relativistic formulas. Section VI compares predictions of the Bondi model and of the relativistic solution without back reaction. Relativistic magnification of the mass accretion becomes noticeable in the case of infall of a hot gas, when the correction factor can be bigger than $2.4(1+1/\Gamma)$, where Γ is the polytropic index.

II. EQUATIONS

We will use a spherically symmetric line element,

$$ds^2=-N^2dt^2+a\,dr^2+R^2(d\theta^2+\sin^2\theta)d\phi^2, \tag{2.1}$$

where N,a, and R depend on t (asymptotic time variable) and a coordinate radius r. We will work in extrinsic curvature variables. Thus we need the mean curvature of centered two-spheres in a Cauchy slice,

$$p=\frac{2\partial_r R}{\sqrt{a}R} \tag{2.2}$$

and the extrinsic curvature

$$\mathrm{tr}\,K=\frac{\partial_t(\sqrt{a}R^2)}{N\sqrt{a}R^2},$$

$$K_r^r=\frac{1}{2Na}\partial_0 a,$$

$$K_\phi^\phi=K_\theta^\theta=\frac{\partial_0 R}{NR}=\frac{1}{2}(\mathrm{tr}\,K-K_r^r). \tag{2.3}$$

Let T_μ^ν be the energy-momentum tensor of matter fields, $\rho=-T_0^0$, and $j_r=NT_r^0$, $R_{\mu\nu}$ the Ricci tensor, and R the Ricci scalar.

The Einstein constraint equations $R_{0\mu}-g_{0\mu}R/2=8\pi T_{0\mu}$ can be integrated to yield, assuming asymptotic flatness,

$$Rp=2\sqrt{1-\frac{2m}{R}+\frac{8\pi}{R}\int_R^\infty \tilde{R}^2\rho\,d\tilde{R}+\tau}, \tag{2.4}$$

$$RK_r^r-R\,\mathrm{tr}\,K=\frac{C_1-8\pi\int_0^R(2\tilde{R}^2j_r/\sqrt{a}p)d\tilde{R}}{R^2}-2\frac{\int_0^R \mathrm{tr}\,K\tilde{R}^2d\tilde{R}}{R^2}, \tag{2.5}$$

where m is the asymptotic mass and

$$\tau=\frac{3}{4R}\int_R^\infty \tilde{R}^2(K_r^r)^2d\tilde{R}-\frac{1}{4R}\int_R^\infty \tilde{R}^2(\mathrm{tr}\,K)^2d\tilde{R}-\frac{1}{2R}\int_R^\infty \mathrm{tr}\,KK_r^r\tilde{R}^2d\tilde{R}. \tag{2.6}$$

By imposing the integral gauge condition

$$\tau=\left(\frac{R(\mathrm{tr}\,K-K_r^r)}{2}\right)^2, \tag{2.7}$$

where τ is given in Eq. (2.6), one can show that in vacuum, Eq. (2.7) is satisfied identically.

Differentiation of Eq. (2.7) with respect r yields, after some algebra,

$$R(\mathrm{tr}\,K-K_r^r)\frac{16\pi j_r R}{p}=0 \tag{2.8}$$

which implies

$$j_r=0=U_i \tag{2.9}$$

in geometries without minimal surfaces and with $\mathrm{tr}\,K\neq K_r^r$. Thus in this gauge coordinates are ''comoving''—each particle of matter carries a fixed value of a radial coordinate ''r.''

The energy-momentum tensor of a self-gravitating fluid reads, in comoving coordinates,

$$T_{\mu\nu}=(\rho+\tilde{p})U_\mu U_\nu+\tilde{p}g_{\mu\nu}. \tag{2.10}$$

Here $U_\mu U^\mu=-1$. Notice that the pressure is $\tilde{p}=T_r^r=T_\theta^\theta$. This space-time foliation is regular even at the vicinity of the boundary of a black hole, in contrast with other approaches [1,2] in which the Schwarzschild geometry is foliated by polar gauge slices.

Define a mass function

$$m(R(r))=2\pi\int_0^r \tilde{R}^3 p\rho dx, \tag{2.11}$$

where $\tilde{R}$ is an areal radius. The mass evolves as follows:

$$\partial_0 m(R(r))=-2\pi[NR^3(\mathrm{tr}\,K-K_r^r)\tilde{p}](r). \tag{2.12}$$

Direct differentiation of $m(R)$ gives

$$\frac{\partial_r m(R)}{\sqrt{a}}=2\pi R^3 p\rho. \tag{2.13}$$

The remaining relevant equations are the two continuity equations

$$N\partial_r\tilde{p}+\partial_r N(\tilde{p}+\rho)=0, \tag{2.14}$$

$$\partial_0\rho=-N\,\mathrm{tr}\,K(\tilde{p}+\rho), \tag{2.15}$$

and the Einstein evolution equation

$$\partial_t(K_r^r-\mathrm{tr}\,K)=\frac{3N}{4}(K_r^r)^2-\frac{Np^2}{4}-\frac{p}{\sqrt{a}}\partial_r N+\frac{N}{R^2}+8\pi N T_r^r+\frac{3}{4}N(\mathrm{tr}\,K)^2-\frac{3N}{2}\mathrm{tr}\,K K_r^r. \tag{2.16}$$

The rate of accretion $\dot{m}$ of mass along orbits of a constant areal radius R is equal to

$$\dot{m}(R)\equiv(\partial_0-\dot{R}\partial_R)m(R(r))=-4\pi NR^2U(\tilde{p}+\rho), \tag{2.17}$$

where

$$U\equiv\partial_0 R/N=\frac{R}{2}(\mathrm{tr}\,K-K_r^r). \tag{2.18}$$

III. STATIONARY DESCRIPTION OF A SELF-GRAVITATING FLUID

All results of this section hold true for systems with collapsing or exploding matter. Assume a compact cloud of a fluid. We will say that an accretion (explosion) is stationary if (i) the mass accretion

$$\dot{m}\equiv(\partial_0-\dot{R}\partial_R)m(R(r))|_{R=\mathrm{const}}$$

on a central body is constant in time, (ii) the radial fluid velocity $U=\partial_0 R/N$ is constant at a fixed value of the areal radius R, $(\partial_0-\dot{R}\partial_R)U=0$, (iii) the energy density at a fixed areal radius does not change in time, and (iv) asymptotically, i.e., close to the outer boundary of the collapsing (exploding) gas, its speed U is much smaller than the speed of sound $a^2=\partial_\rho\tilde{p}$, $U_\infty\approx 0$. (Expanding fluid, in turn, would be subject to a condition $U\approx 0$ at the inner boundary [1].)

At first, we shall prove the following fact:

Theorem 1. Under conditions (i)–(iii), $\dot{m}$ does not depend on R within the fluid filled zone, $\partial_R\dot{m}=0$.

Proof. Equations (2.5) and (2.18) yield

$$\partial_R(UR^2)=R^2\mathrm{tr}\,K. \tag{3.1}$$

From $\dot{m}(R)=-4\pi NUR^2(\rho+\tilde{p})$ we obtain $\partial_R\dot{m}=I+II+III$, where

$$I=N(\rho+\tilde{p})\partial_R(UR^2),$$

$$II=NUR^2\partial_R\tilde{p}+UR^2(\tilde{p}+\rho)\partial_R N,$$

$$III=NUR^2\partial_R\rho. \tag{3.2}$$

Using Eq. (3.1) one writes $I=NR^2\mathrm{tr}\,K(\tilde{p}+\rho)$, while the stationarity condition (ii) allows one to write $III=R^2\partial_0\rho=-NR^2\mathrm{tr}\,K(\tilde{p}+\rho)$ [the second equality follows from Eq. (2.15)]. Thus $I+III=0$; since $II=0$ [due to the momentum conservation Eq. (2.14)], we arrive at $\partial_R\dot{m}(R)=0$.

Assume that the equation of state

$$\tilde{p}=K\rho^\Gamma, \tag{3.3}$$

Γ being a constant, and define the speed of sound as $a^2=\partial_\rho\tilde{p}$. We assume that $1\leqslant\Gamma\leqslant 5/3$, since we are primarily interested in comparing predictions with the Bondi model, but it is quite likely that much of the forthcoming analysis applies to adiabatic indices in the standard in astrophysics range [1, 2].

Let us point out that astrophysicists [2] use a different equation of state, $\tilde{p}=Cn^{\Gamma}$ (where n is the baryon number density); this reads in our notation

$$\tilde{p}=C\times\exp\left(\Gamma\int d\rho\,\frac{1}{\rho+K\rho^{\Gamma}}\right). \tag{3.4}$$

In the Newtonian limit both approaches agree when $\Gamma\neq 5/3$, but they disagree for $\Gamma=5/3$. The momentum conservation equation (2.14) can be integrated,

$$a^2=-\Gamma+\frac{\Gamma+a_\infty^2}{N^{\kappa}}, \tag{3.5}$$

where $\kappa=(\Gamma-1)/\Gamma$ and the integration constant a_∞^2 is equal to the asymptotic speed of sound of a fluid.

Equation (3.5) asymptotically ($m/R\ll 1$) yields the Bernoulli equation, hence it can be regarded as the general-relativistic version of the latter.

From the relation between pressure and energy density, one obtains, using Eq. (3.5)

$$\rho=\rho_\infty(a/a_\infty)^{2/(\Gamma-1)}=\rho_\infty\left[-\frac{\Gamma}{a_\infty^2}+\frac{\Gamma/a_\infty^2+1}{N^{\kappa}}\right]^{1/(\Gamma-1)}, \tag{3.6}$$

where the constant ρ_∞ is equal to the asymptotic mass density of a collapsing fluid. From the evolution equation one obtains, using the stationarity assumption,

$$\dot{R}\partial_R U=\frac{1}{4}(pR)^2\partial_R N-m(R)\frac{N}{R^2}-4\pi RN\tilde{p}. \tag{3.7}$$

Equations (2.4) and (2.5) give

$$U\partial_R U=\frac{pR}{4}\partial_R(pR)-\frac{m(R)}{R^2}+4\pi\rho R. \tag{3.8}$$

Comparison of Eq. (3.8) with Eq. (3.7) yields an ordinary differential equation

$$\partial_R\ln\left(\frac{N}{pR}\right)=\frac{16\pi}{p^2R}(\rho+\tilde{p}). \tag{3.9}$$

Integration of this, with the asymptotic condition at spatial infinity $N=pR/2=1$, leads to the following relation between the lapse N and the mean curvature pR:

$$N=\frac{pR}{2}\beta(R), \tag{3.10}$$

where

$$\beta(r)=\exp\left(\int_r^\infty 16\pi(-\tilde{p}-\rho)\frac{1}{p^2 s}ds\right). \tag{3.11}$$

The substitution of $\operatorname{tr}K$ [as calculated from the continuity Equation (2.15)] into Eq. (3.1) gives, employing the stationarity condition,

$$\partial_R\ln(|U|R^2)=-\frac{\partial_R\rho}{\tilde{p}+\rho}. \tag{3.12}$$

Notice that

$$-\frac{\partial_R\rho}{\tilde{p}+\rho}=-\frac{\partial_R(\rho+\tilde{p})}{\tilde{p}+\rho}+\frac{\partial_R\tilde{p}}{\tilde{p}+\rho}.$$

The last term can be presented in another form (due to relations between a^2,ρ and $\tilde{p}$),

$$\frac{\partial_R\tilde{p}}{\tilde{p}+\rho}=\frac{\Gamma}{\Gamma-1}\partial_R\ln(a^2/\Gamma+1).$$

That leads to the following solution of Eq. (3.12):

$$U=C\frac{(a^2/\Gamma+1)^{1/(\Gamma-1)}}{R^2a^{2/(\Gamma-1)}}. \tag{3.13}$$

The whole set of equations describing the collapsing stationary fluid is given by Eq. (3.13) and the previously written Eqs. (3.5), (3.10), and (3.11). Calculation of $\partial_R\ln(a^2+\Gamma)$, with a^2 given by Eq. (3.5) and N being specified above, yields

$$\partial_R\ln(a^2+\Gamma)=\frac{-4\kappa}{p^2R^3}\left(\frac{m(R)}{R}+4\pi R^2\tilde{p}-2U^2+\frac{1}{2R^3}\partial_R(U^2R^4)\right). \tag{3.14}$$

One easily obtains from Eq. (3.13)

$$\frac{1}{R^4}\partial_R(U^2R^4)=-\frac{2U^2}{\kappa a^2}\partial_R\ln(a^2+\Gamma). \tag{3.15}$$

The insertion of Eq. (3.15) into Eq. (3.14) gives

$$\partial_R(U^2R^4)\left(1-\frac{4U^2}{a^2p^2R^2}\right)=\frac{16R}{a^2p^2}\left(\frac{m(R)}{2R}+2\pi R^2\tilde{p}-U^2\right). \tag{3.16}$$

We define sonic points as such where the equality $U=(pR/2)a$ holds true. Let R_* be a radius of a sonic point. Equation (3.16) yields the relation

$$a_*^2\left(1-\frac{3m_*}{2R_*}+c_*\right)=U_*^2=\frac{m_*}{2R_*}+c_*, \tag{3.17}$$

where $c_*=2\pi R_*^2\tilde{p}_*$, $a_*^2=a^2(R_*)$, $m_*=m(R_*)$, and $U_*^2=U^2(R_*)$.

The constant C in formula (3.13) can be expressed in terms of a_*, U_*, and R_*, that is as a function of c_*, $m(R_*)$, and R_*. The infall velocity U reads

$$U=U_* \frac{R_*^2}{R^2}\left(\frac{1+\Gamma/a^2}{1+\Gamma/a_*^2}\right)^{1/(\Gamma-1)}. \tag{3.18}$$

Above U_* means a negative square root in the case of falloff towards a gravity center and a positive square root in the case of exploding gas.

The rate of accretion of mass in Eq. (2.17) can be conveniently expressed by characteristics of the sonic point R_*,

$$\dot{m}=-4\pi R_*^2\rho_\infty\left(1-\frac{3m_*}{2R_*}+c_*\right)^{1/2}U_* \times\left(\frac{a(R_*)}{a_\infty}\right)^{2/(\Gamma-1)}\left(1+\frac{a_*^2}{\Gamma}\right)\beta(R). \tag{3.19}$$

For the sake of completeness we write down the space-time line element with the areal radius chosen as the radial coordinate,

$$ds^2=dt^2\left(-N^2+\frac{4N^2U^2}{(pR)^2}\right)-4\beta\frac{U}{pR}dtdR + \frac{4}{(pR)^2}dR^2+R^2d\Omega^2. \tag{3.20}$$

IV. RELATIVISTIC ACCRETION: NEGLECTING BACK REACTION

The quasistationary accretion shall apply to the description of black holes interacting with a fluid. The description of the accretion onto other compact bodies (say, neutron stars) is more complex, since there can appear shocks that are excluded in our picture. The above model can be valid only if shocks are absent, for instance, when the inner boundary of a collapsing shell of gas is disconnected from the surface of a compact body.

All hitherto proven results are exact and—under the preceding reservation—they refer to a fully nonlinear stationary system consisting of a central mass and a cloud of gas that would dynamically influence a geometry through a back reaction. If the gas is heavy, compared with the central mass, then $\beta(R)$ is nonconstant; metric functions do depend on the infalling matter. That means that back reaction should be taken into account in the description of such a system.

If the contribution of a fluid to the total asymptotic mass of a system is negligible, i.e.,

$$m_f\equiv 4\pi\int_{R>2m} dr r^2\rho \ll m \tag{4.1}$$

and $\tilde{p}\ll\rho$, then $\beta\approx 1$ and

$$N\approx\frac{pR}{2}=\approx\sqrt{1-\frac{2m}{R}+U^2}. \tag{4.2}$$

That would suggest that in this case the standard Schwarzschildean geometry constitutes a valid approximation. There is, however, one subtle point. The reasoning of the former section shows that in order to neglect the effect of back reaction the condition

$$c_*=2\pi R^2\tilde{p}\ll\frac{2m_*}{R} \tag{4.3}$$

must hold at a sonic point. That can be interpreted as the demand that not only N be close to $pR/2$ but also $\partial_R N$ shall be approximated by $\partial_R(pR)/2$.

We will say that back reaction is negligible if both Eq. (4.1) and Eq. (4.3) hold true. In such a case $m\approx m_*$ and one obtains

$$a_*^2\left(1-\frac{3m}{2R_*}\right)=U_*^2=\frac{m}{2R_*} \tag{4.4}$$

at the sonic point. The remaining two equations describing accretion are

$$U^2=\frac{R_*^3 m}{2R^4}\left(\frac{1}{1+\Gamma/a_*^2}\right)^{2/(\Gamma-1)}\left(1+\frac{\Gamma}{a^2}\right)^{2/(\Gamma-1)} \tag{4.5}$$

and

$$a^2=-\Gamma+\frac{\Gamma+a_\infty^2}{N^\kappa}. \tag{4.6}$$

This is a purely algebraic system of equations, describing the fluid accretion in a fixed space-time (Schwarzschild) geometry.

V. RELATIVISTIC ACCRETION WITHOUT BACK REACTION

Numerical analysis demonstrates—as pointed out by Michel [1]—the existence of two branches of solutions of the relativistic fluid equations. An analytic proof is given below.

In the first part we prove the existence of a sonic point in a black hole spacetime endowed with a Schwarzschild geometry. That black–hole–fluid system is shown to possess a sonic point, which leads, through a construction outlayed in the second step, to the existence of two accreting solutions.

A. Sonic points

Define

$$L=a^2+\Gamma,$$

$$P=\frac{a_\infty^2+\Gamma}{[1-2m/R+U^2]^{(\Gamma-1)/(2\Gamma)}} \tag{5.1}$$

where U^2 is given by Eq. (4.5) with parameters a_* and U_*^2 specified by Eq. (4.4).

The equation $L(R_*)=P(R_*)$ for a sonic point can be written as $1+y(3\Gamma-1)=3(a_\infty^2+\Gamma)y^{(\Gamma+1)/(2\Gamma)}$, where $y=1-3m/(2R_*)$. One has to demand that $y>0$ (i.e., $R_*>3m/2$), since at $y=0$ (or $R_*=3m/2$) the coordinate system breaks down. Notice a numerical mistake in [1] which led Michel to the wrong claim that sonic points must exist outside a sphere of a radius $6m$. In fact they may exist even inside a black hole, although—as we point out below—that would contradict established views on properties of matter.

The left hand side of the equation in question is bigger than its right hand side at $y=0$ while at $y=1$ the opposite holds true. Since both sides are continuous in x, their graphs must intersect somewhere. Since $1+y(3\Gamma-1)$ increases at a lower rate than $3(a_\infty^2+\Gamma)y^{(\Gamma+1)/(2\Gamma)}$ for $\Gamma\leqslant 5/3$, there exists a unique sonic point characterized by y_*. The case with $y_*<1/2$ (i.e., when $R_*<2m$) is physically noninteresting. In that case the speed of sound would be bigger than the velocity of light and the dominant energy condition [6] would be broken, even outside of a black hole. One easily infers that y_* is a monotonously decreasing function of the asymptotic sound density a_∞^2. Therefore there exists a critical value of a_∞^2 which separates solutions that are subluminal from unphysical solutions that become superluminal.

An interesting feature of the Bondi model is the simple relation $a_*^2/a_\infty^2=2/(5-3\Gamma)$ for $\Gamma<5/3$ [5]. Below we will show that this relation appears in the nonrelativistic limit of a relativistic formula.

Theorem 2. Let a_∞^2 and a_*^2 be the asymptotic and sonic speeds of sound, respectively. Define $\alpha=(\Gamma-1)/(2\Gamma)$:

$$A=\alpha[(9\Gamma-5)\ln 4-1.5]$$

and

$$B=\frac{3}{2}\alpha^2\Gamma(9\Gamma-7).$$

If sonic points are exterior to a black hole then

$$\frac{1}{(5-3\Gamma)/2+Ba_*^2/(1+3a_*^2)}\geqslant\frac{a_*^2}{a_\infty^2}\geqslant\frac{1}{(5-3\Gamma)/2+Aa_*^2/(1+3a_*^2)}. \tag{5.2}$$

Proof. Define $x\equiv m/2R_*$ [or alternatively, $x=a_*^2/(1+3a_*^2)$], and

$$F\equiv x(1-3x)^\alpha+\Gamma(1-3x)^{1+\alpha}-\Gamma(1-3x)-\frac{5-3\Gamma}{2}x-Bx^2,$$

$$\Psi=x(1-3x)^\alpha+\Gamma(1-3x)^{1+\alpha}-\Gamma(1-3x)-\frac{5-3\Gamma}{2}x-Ax^2. \tag{5.3}$$

The sonic point equation can be written as

$$\frac{a_\infty^2}{a_*^2}-\frac{5-3\Gamma}{2}=Bx+\frac{F}{x}=Ax+\frac{\Psi}{x}. \tag{5.4}$$

It suffices to show that $F\geqslant 0$ and $\Psi\leqslant 0$. We shall deal with the first inequality. The second derivative of F with respect x reads

$$F''=3\frac{\Gamma-1}{2\Gamma}(1-3x)^{\alpha-2}G(x), \tag{5.5}$$

where

$$G(x)=\frac{9\Gamma-7}{2}(1-3x)-3x\frac{\Gamma+1}{2\Gamma}-\frac{1}{2}(\Gamma-1)(9\Gamma-7)(1-3x)^{2-\alpha}. \tag{5.6}$$

One shows that $G'\leqslant(3\Gamma-5)/\Gamma\leqslant 0$; thus $G(x)$ is decreasing for $0\leqslant x\leqslant 1/4$ and $1\leqslant\Gamma\leqslant 5/3$. Therefore if $F''(x_0)=0$ then $F''(x)<0$ for any $x>x_0$. That means, taking into account that $F''(0)>0$ and $F'(0)=0$, that if F' vanishes at a point x_1, then it must be negative in the interval $(x_1,1/4)$. In conclusion, either F is increasing (and then it achieves its minimum at $x=0$) or it has a single extremum (a maximum) in (0, 1/4). Notice now that $F(0)=0$. Thence in order to show that $F(x)\geqslant 0$ it is enough to show that $F(x)$ is nonnegative at $x=1/4$, when

$$F(1/4)=(\Gamma+1)\frac{1}{4^{1+\alpha}}+\frac{\Gamma}{8}-\frac{5}{8}-\frac{3}{128\Gamma}(\Gamma-1)^2(9\Gamma-7). \tag{5.7}$$

A numerical calculation shows that $F(1/4)\geqslant 0$ and the equality is achieved only at $\Gamma=1$.

In a similar vein, one proves the other inequality $\Psi\leqslant 0$. At $x=0$ one has $\Psi=0$. On the other hand,

$$\Psi'=\frac{9\Gamma-5}{2}[1-(1-3x)^\alpha]-3x\alpha(1-3x)^{\alpha-1}-2\alpha x\left((9\Gamma-5)\ln 4-\frac{3}{2}\right). \tag{5.8}$$

Employing the estimate

$$1-(1-3x)^\alpha\leqslant 4\alpha x\ln 4, \tag{5.9}$$

which is valid for $0\leqslant x\leqslant 1/4$ and $0.2\geqslant\alpha\geqslant 0$, one arrives at $\Psi'\leqslant 0$. Thus the function Ψ is non-negative, as desired. That ends the proof.

Let us remark that Eq. (5.2) can be written as, resolving the biquadratic inequalities,

$$\frac{\delta_0-\sqrt{\delta_0^2+4a_\infty^2\delta_2}}{2\delta_2}\geqslant a_*^2\geqslant\frac{\delta_0-\sqrt{\delta_0^2+4a_\infty^2\delta_1}}{2\delta_1}, \tag{5.10}$$

where

$$\delta_0=3a_\infty^2-\frac{5-3\Gamma}{2},$$

$$\delta_1=\frac{3}{2}(5-3\Gamma)+A,$$

$$\delta_2=\frac{3}{2}(5-3\Gamma)+B. \tag{5.11}$$

Asymptotically, i.e., for $x\to 0$, one obtains the Bondi equality $a_*^2/a_\infty^2=2/(5-3\Gamma)$ for $\Gamma<5/3$. If $\Gamma=5/3$ then the above gives asymptotically $1.12a_\infty \geqslant a_*^2 \geqslant 0.8a_\infty$, in a good agreement with the exact formula $a_*^2=\sqrt{5/6}a_\infty$. If a sonic point is located at a horizon of a black hole (that is, $a_*=1$) then Eq. (5.2) (or the above inequalities) yields $0.79\geqslant a_\infty \geqslant 0.5$ for $\Gamma=5/3$. Notice also a rough bound $a_*^2>1.6a_\infty^2$ which is valid for any Γ and a_∞^2; outside of a black hole $a_*^2\leqslant 1$, therefore one infers that the asymptotic speed of sound is less than 1.

Let us point out also that Eq. (5.4) implies that the asymptotic sonic points can exist only for models with adiabatic indices $\Gamma<5/3$.

B. Existence proof

We show that at least two solutions $(a(R),U(R))$ bifurcate from R_*. We define a_α as a solution of the equation

$$\left(\frac{1+\Gamma/a_{\alpha^2}}{1+\Gamma/a_*^2}\right)^{2/(\Gamma-1)}=(R/R_*)^{7/2}. \tag{5.12}$$

From that and Eq. (4.5) it follows that $U^2=U_*^2\sqrt{R_*/R}$ and

$$a_\alpha^2=\frac{\Gamma(R_*/R)^\beta}{\delta-(R_*/R)^\beta}, \tag{5.13}$$

where $\beta=\frac{7}{4}(\Gamma-1)$ and $\delta=1+\Gamma/a_*^2$.

A straightforward calculation gives

$$\frac{d}{dR}\ln L(a_\alpha)=-\frac{\beta(R_*/R)^\beta}{R[\delta-(R_*/R)^\beta]} \tag{5.14}$$

and

$$\frac{d}{dR}\ln P(a_\alpha)$$
$$=-\frac{2(\Gamma-1)R_*U_*^2(1-\sqrt{R/R_*}/8)}{\Gamma R^2[1-3m/2R_*+U_*^2(3-4R_*/R+\sqrt{R_*/R})]}. \tag{5.15}$$

L and P are equal at $R=R_*$ and they are decreasing in the vicinity of $R=R_*$. Moreover, $\partial_R L=\partial_R P$ at the sonic point R_*. A careful investigation shows, however, that second derivatives are both locally positive and

$$\left.\frac{d^2}{dR^2}\ln P(a_\alpha)\right|_{R=R_*}=\left.\frac{d^2}{dR^2}\ln L(a_\alpha)\right|_{R=R_*}\frac{29/14+7a_*^2/2}{\beta(1+a_*^2/\Gamma)+1}. \tag{5.16}$$

One observes that

$$\left.\frac{d^2}{dR^2}\ln P(a_\alpha)\right|_{R=R_*}\geqslant\left.\frac{d^2}{dR^2}\ln L(a_\alpha)\right|_{R=R_*}$$

if $\Gamma<79/49$. This reasoning can be valid for adiabatic indices $\Gamma\leqslant 5/3$ assuming that the exponent 7/2 in Eq. (5.12) is replaced by $x\epsilon(-\infty,4.5-\sqrt{1.5})$ [7]. Therefore $\partial_R L<\partial_R P$, for $R>R_*$ and $\partial_R L>\partial_R P$ for $R<R_*$. Thus locally $P\geqslant L$.

On the other hand, notice that $L(a^2=0)>P(a^2=0)$ and $L(a^2=\infty)>P(a^2=\infty)$, for all values of R.

L and P are differentiable functions of their arguments. Combining the above facts one infers that, due to the continuity of L and P, there must exist at least two solutions in a neighborhood of R_*. Those solutions coincide at $R=R_*$, due to the above construction. The set of those points constitutes at least two branches. Since $\partial_{a^2}(L-P)=1-4U^2/(p^2R^2a^2)\neq 0$ at any point of a solution branch with $R\neq R_*$, the implicit function argument would be used to extend the interval of the existence onto a whole bounded domain. Those solutions are differentiable for $R\neq R_*$.

One of the solutions is supersonic below R_* and subsonic above R_* and it can be interpreted as describing collapse of matter onto a black hole. The other solution is subsonic for $R<R_*$ and supersonic above; it can correspond to an exploding gas.

C. Qualitative results

In what follows we shall deal with a solution that is subsonic asymptotically, i.e., describes accretion of a fluid.

Theorem 3. An asymptotically subsonic solution of the system (4.2)–(4.6) satisfies the following conditions, (i) If $R\neq R_*$ then $\partial_R(U^2R^4)>0$ and the speed of sound decreases, $\partial_R a^2\leqslant 0$, with the equality only at spatial infinity, (ii) $U^2>m/2R$ for $R<R_*$ and $U^2<m/2R$ for $R>R_*$. (iii) Inside the supersonic region $a^2(pR)^2/4<U^2\leqslant 2m/R$. (iv) Mass density ρ monotonously decreases and ρ is bounded in the supersonic region, $R<R_*$,

$$\rho\leqslant\rho_\infty\left(1+(\Gamma-1)\frac{4m}{Ra_\infty^2}\right)^{1/(\Gamma-1)}. \tag{5.17}$$

The proof is postponed to the Appendix.

The estimates of (ii) and (iii) in theorem 3 require an explanation. It proved to be convenient to define a sonic point by requiring that $a^2(pR)^2/4=U^2$ instead of the condition (used in the Bondi model) $a^2=U^2$. Therefore the speed of sound can be bigger than infall velocity in regions close to horizons if the factor $pR/2$ is significantly smaller than 1. In the traditional terminology such a solution would be called subsonic. Numerical data in the next Section show that the value $|U|/a$ at a horizon depends strongly on the location of a sonic point, on the ratio $R_*/(2m)$, which in turn depends on the asymptotic speed of sound a_∞. $|U|/a$ decreases with the increase of the asymptotic speed of sound.

VI. BONDI MODEL AND THE RELATIVISTIC SOLUTION

The insertion of Eq. (4.4) and (4.6) into Eq. (3.19) (with $c_*=0$) yields the mass accretion rate

TABLE I. Numerical data ($R_n = 1.001\times 2m$).

$R_*/(2m)$	$a^2(R_*)/a_\infty^2$	$a(R_n)$	$U(R_n)$	$\rho(R_n)/\rho_\infty$	$\rho(R_*)/\rho_\infty$	$\rho(2\times R_*)/\rho_\infty$
500 000	2	0.168	0.999	1.41×10^9	8	3.14
50	1.99	0.173	0.92	1595	7.89	3.9
5	1.91	0.34	0.78	56.4	6.95	3.46
1.1	1.62	0.86	0.53	4.82	4.24	2.1

$$\dot{m} = \pi m^2 \frac{\rho_\infty}{a_\infty^3}\left(\frac{a_*^2}{a_\infty^2}\right)^{(5-3\Gamma)/2(\Gamma-1)}\left(1+\frac{a_*^2}{\Gamma}\right)(1+3a_*^2). \tag{6.1}$$

One can write

$$\dot{m} = \Omega \dot{m}_B, \tag{6.2}$$

where

$$\dot{m}_B = \pi m^2 \frac{\rho_\infty}{a_\infty^3}\left(\frac{2}{5-3\Gamma}\right)^{(5-3\Gamma)/2(\Gamma-1)}, \tag{6.3}$$

is the mass accretion rate predicted by the Bondi model and

$$\Omega = \left(\frac{(5-3\Gamma)a_*^2}{2a_\infty^2}\right)^{(5-3\Gamma)/2(\Gamma-1)}(1+3a_*^2)\left(1+\frac{a_*^2}{\Gamma}\right). \tag{6.4}$$

Ω can be interpreted as the relativistic correction factor.

Application of theorem 2 leads to useful estimates for Ω.

Theorem 4. Assume $1 \leqslant \Gamma \leqslant 5/3$. The relativistic correction factor satisfies

$$(1+3a_*^2)\left(1+\frac{a_*^2}{\Gamma}\right) \geqslant \Omega \geqslant (1+3a_*^2)\left(1+\frac{a_*^2}{\Gamma}\right)e^{-C}, \tag{6.5}$$

where

$$C = \frac{a_*^2}{1+3a_*^2}\left[\left(4.5-\frac{2.5}{\Gamma}\right)\ln 4 - \frac{0.75}{\Gamma}\right] = \frac{a_*^2}{1+3a_*^2}\left(6.24-\frac{4.22}{\Gamma}\right). \tag{6.6}$$

Proof. The definition of Ω and theorem 2 yield immediately the forthcoming inequalities:

$$(1+3a_*^2)\frac{(1+a_*^2/\Gamma)}{(1+[2B/(5-3\Gamma)][3a_*^2/(1+3a_*^2)])^{(5-3\Gamma)/2(\Gamma-1)}} \geqslant \Omega \geqslant \frac{(1+3a_*^2)(1+a_*^2/\Gamma)}{(1+[2(\Gamma-1)/(5-3\Gamma)]C)^{(5-3\Gamma)/2(\Gamma-1)}}. \tag{6.7}$$

Bounding from above the left hand side of Eq. (6.7) by $(1+3a_*^2)(1+a_*^2/\Gamma)$ yields the first bound of theorem 4. The proof of the other inequality is based on the obvious estimate

$$\left(1+\frac{2(\Gamma-1)}{5-3\Gamma}C\right)^{(5-3\Gamma)/2(\Gamma-1)} \leqslant e^C. \tag{6.8}$$

Taking into account theorem 2 and its implications stated in Eq. (5.10) and below Eq. (5.11), one can write the above in terms of asymptotic data,

$$\left(1+3\frac{\delta_0-\sqrt{\delta_0^2+4a_\infty^2\delta_2}}{2\delta_2}\right)\left(1+\frac{\delta_0-\sqrt{\delta_0^2+4a_\infty^2\delta_2}}{2\delta_2}\frac{1}{\Gamma}\right) \geqslant \Omega \geqslant (1+4.8a_\infty^2)\left(1+\frac{1.6a_\infty^2}{\Gamma}\right)e^{-C}. \tag{6.9}$$

The relativistic correction factor Ω is close to 1 when $a_*^2 \ll 1$, i.e., when the asymptotic gas temperature is low. Ω is bounded from below by 0.99. Equation (6.5) yields, in the ultrarelativistic regime $a_*^2 \approx 1$,

$$4\left(1+\frac{1}{\Gamma}\right) \geqslant \Omega \geqslant 2.4\left(1+\frac{1}{\Gamma}\right). \tag{6.10}$$

Ultrarelativistic effects enhance accretion, with the strongest effect for the isothermal gas with $\Gamma = 1$. The enhancement is smaller for $\Gamma \approx 5/3$, as seen from the preceding estimate.

The Bondi model fails only in describing the hot gas mode. The correction factor Ω tends quickly to 1 when sonic points are far away from the Schwarzschild sphere, for instance, if $\Gamma = 4/3$, then $\Omega < 7$ at $R_*/m = 2$ but $\Omega < 1.1$ at $R_*/m = 25$.

We analyze numerically a relativistic gas, with the adiabatic index $\Gamma = 4/3$, falling onto a black hole. Results complement analytic estimates and they are comprised in Table I.

Some features of accreting solutions depend in a crucial way on the location of sonic points. When sonic points are close to a horizon, the speed of sound is close to one while the infall velocity at a horizon is smaller and it barely exceeds 1/2. When sonic points are far away from a horizon, $R_* \gg 2m$, the infall velocity nears to the speed of free fall ($U \approx 1$ close to a horizon) while the speed of sound is then much smaller than U. An interesting fact is that the energy density changes quite moderately—by a factor of the order of unity—if sonic points are close to the Schwarzschild sphere. In contrast to that, solutions with $R_* \gg 2m$ are characterized by a rapid growth—up to ten orders—of the energy density near the horizon. The energy density changes by a factor not greater than eight in the region exterior to a sonic point with $R \epsilon (R_*, \infty)$; that type of moderate decay is common for all solutions, irrespective of the value of $2m/R_*$. This actually follows from theorem 2, which bounds a_*^2/a_∞^2 and—consequently—also $\rho(R_*)/\rho_\infty$. Solutions with sonic points close to a horizon have a approaching 1 and they describe a high temperature (circa 10^{10} K) gas, with $a_\infty \approx 0.5$.

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APPENDIX

Proof of Theorem 3.

(1) Equation (3.16) yields, ignoring the back reaction term, the crucial relation

$$\partial_R(U^2R^4)\left(1-\frac{4U^2}{a^2p^2R^2}\right)=\frac{16R}{p^2}\left(\frac{m}{2R}-U^2\right). \quad \text{(A1)}$$

(2) The first observation, which states that signs of $\partial_R a^2$ and of $\partial_R(U^2R)$ are opposite and that they vanish simultaneously at finite values of R, can be drawn from Eq. (3.15).

(3) Let R_* be a position of the sonic point; thus $U^2(R_*)=m/2R_*$. Assume that in the vicinity of R_* the expression $\partial_R(U^2R^4)$ is strictly negative. Then Eq. (A1) yields $U^2R<m/2$ for $R<R_*$ and $U^2R>m/2$ for $R>R_*$. Therefore, U^2R is increasing in the region of interest and that is incompatible with the assumption that $\partial_R(U^2R^4)$ is strictly negative. Therefore, it must be weakly positive at least around the outermost sonic point. This in turn implies that in a neighborhood of a sonic point $2U^2R<m$ for $R>R_*$ and $2U^2R>m$ for $R<R_*$.

(4) The expression $\partial_R(U^2R^4)$ cannot have zeroes. Assume it vanishes at some $R_1>R_*$; Eq. (A1) gives $U^2(R_1)=m/2R_1$ and we would have $\partial_R(U^2R)\geqslant 0$ at R_1. But that is incompatible with the assumption that $\partial_R(U^2R^4)=0$ at R_1.

Let us now consider a region $R<R_*$. If R_1 is a zero point of $\partial_R(U^2R^4)$ but the latter does not change sign at R_1, then $2U^2R$ decreases for $R<R_1$ towards the value m and increases for $R>R_1$ [due to estimates proven in the final part of (3)]. Hence $\partial_R(U^2R)=0$ at R_1. But that contradicts $\partial_R(U^2R^4)=0$ at R_1. Similarly, if $\partial_R(U^2R^4)$ changes sign in the vicinity of R_1, then we are led to the contradiction.

Thus $\partial_R(U^2R^4)>0$ in the domain of existence of the solution. That implies, in conjunction with Eq. (A1), that in the supersonic zone $U^2>m/(2R)$ and that $U^2<m/(2R)$ in the subsonic zone $(R>R_*)$. This accomplishes the proof of (ii).

(5) We rewrite Eq. (3.14), with back reaction terms being dropped out,

$$\partial_R \ln(a^2+\Gamma)=\frac{-4\kappa}{p^2R^3}\left(\frac{m}{R}-\frac{U^2}{2R}+\frac{1}{2}\partial_R(U^2R)\right). \quad \text{(A2)}$$

Let R be the largest point $R<R_*$ such that $U^2R=2m$; then $\partial_R(U^2R)|_R<0$ and from Eq. (A2), it follows $\partial_R a^2|_*>0$, in contradiction with hitherto proven monotonic falloff of a^2. This shows the bounds of (iii)—that $U^2R<2m$. Equation (A1) and (ii) imply, in the supersonic region, $a^2p^2R^2/4<1$. Extremal values of the speed of sound (achieved at a horizon of a black hole) cannot exceed $4/(pR)^2$, while the speed of infalling particles does not exceed 1, the speed of light.

(6) The decrease of the speed of sound together with Eq. (3.6) leads to the conclusion that the mass density also decreases, $\partial_R a^2\leqslant 0$ and $\partial_R\rho\leqslant 0$. The numerical estimates of (iv) are obtained from inserting inequalities proven in (5) for the expression (3.6).

This ends the proof of theorem 3.

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