Event horizons and apparent horizons in spherically symmetric geometries

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Spherical configurations that are very massive must be surrounded by apparent horizons. These, in turn, when placed outside a collapsing body, have a fixed area and must propagate outward with a velocity equal to the velocity of radially outgoing photons. That proves, within the framework of the 1+3 formalism and without resorting to the Birkhoff theorem, that apparent horizons coincide with event horizons in an electrovacuum. The existence of the maximal slicing of electrovacuum is proved and an explicit line element is found in the maximal foliation.

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I. INTRODUCTION

The cosmic censorship hypothesis (CCH) [1] is certainly the most challenging open problem of classical general relativity. From the dynamical point of view any successful attempt to prove a weak version of the CCH in a space-time generated by an isolated self-gravitating object must consist of the following points: (i) state smooth initial data (if this is done then space-time can be unambiguously split into time and space directions, initially at least, once the local Cauchy problem is solved); (ii) prove that if a singularity emerges then it must be hidden inside an event horizon, so as not to influence the asymptotically flat open end; and (iii) prove the global Cauchy problem in the asymptotically flat region outside the event horizon (this implies that space-time splitting is possible globally in the region and we are guaranteed the existence of the line element inside it).

From this perspective the proof of the CCH seems to be technically unattainable (at least nowadays) in the general case of a nonspherical collapse, since the global Cauchy problem is almost intractable at present [2]. Even in the spherically symmetric collapse only partial results are known [3].

It is natural to assume the solvability of the global Cauchy problem, that is, to assume the global existence of a space-time metric, in order to test the remaining steps of the above program. In this case Israel has proven the confining property of apparent horizons [4] in spherically symmetric geometries [5]. Israel's result opens a way to prove the CCH for those singularities that must be hidden inside apparent horizons. The characterization of the formation of apparent horizons, in turn, in spherically symmetric space-times has been completely solved in the initial value formulation [6] and [7], thus accomplishing the proof of steps (i) and (ii) for a version of the CCH in spherically symmetric geometries.

The intention of this paper is to provide another proof of Israel's result in the framework of a dynamical 1+3 description of a spherical collapse. I prove a confining property of apparent horizons that are placed in an electrovacuum and complement this with a proof that a relevant solution of the global Cauchy problem exists and that in an electrovacuum apparent horizons coincide with a sphere of the areal radius $m+\sqrt{m^2-q^2}$.

The confining property of apparent horizons in an electrovacuum is not a surprise. It is well known, thanks to the Birkhoff theorem [8], that once the areal radius $\mathbf{R}$ of a charged collapsing body becomes equal to $m+\sqrt{m^2-q^2}$, where m is the asymptotic (Einstein-Freund-Arnowitt-Deser-Misner) mass and q is the total charge, then the body hides within an event horizon that coincides with a sphere of the areal radius $\mathbf{R}$ which in turn is the locus of an apparent horizon. This conclusion appears correct despite the fact that the Birkhoff transformation does not exist in the situation of interest when a geometry contains apparent horizons (see a discussion in the Appendix of [9] in which a generalized version of the Birkhoff theorem is described).

In the first two sections I will assume the existence of a global Cauchy solution, that is, the existence of an asymptically flat space-time with the spherically symmetric metric line element

$$ds^2 = -\alpha^2(r,t)dt^2 + a(r,t)dr^2 + b(r,t)r^2 d\Omega^2. \qquad (1)$$

r is a coordinate radius and $br^2d\Omega^2$ is a standard two-sphere metric element. We assume the maximal gauge condition in which components of the extrinsic curvature K_{ij} of the hypersurface Σ_t (defined as a set of points having a fixed coordinate time t = const) satisfy the equations

$$K_r^r = \frac{\partial_t a}{2a\alpha} = -2K_\phi^\phi = -2K_\theta^\theta = -\frac{\partial_t b}{b\alpha}. \qquad (2)$$

We use the standard convention [8] in which quantities supplied with greek indices refer to the four-geometry while quantities with latin labels refer to the geometry of the hypersurface Σ_t. The Einstein summation convention is applied in some places, with the exception concerning the label r, whose repetition is assumed never to mean

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summation. Below, ∇_i denotes the covariant derivative on Σ_t. The Einstein equations read

$$-\frac{2}{abr^2}\partial_r(r^2\partial_r b)+\frac{(\partial_r b)^2}{2ab^2}+\frac{\partial_r b\partial_r a}{a^2 b}+\frac{2\partial_r a}{a^2 r}$$

$$-\frac{2\partial_r b}{abr}+\frac{2}{r^2}\left(\frac{1}{b}-\frac{1}{a}\right)=K_{ij}K^{ij}+16\pi\rho, \quad (3)$$

$$\nabla_i K^{ir}=-8\pi j^r, \quad (4)$$

$$\nabla_i\partial^i\alpha=K_{ij}K^{ij}\alpha+4\pi\left[-T_0^0+T_i^i\right]\alpha, \quad (5)$$

$$\partial_t K_r^r=-\alpha R_r^{(3)r}-8\pi\alpha\left[-T_r^r+\frac{T_\mu^\mu}{2}\right]+\nabla_r\partial^r\alpha. \quad (6)$$

Above T_μ^ν is the energy-momentum tensor of matter generating the gravitational field, $\rho=-T_0^0$ and $j_i=T_i^0\alpha$. $R_r^{(3)r}$ denotes a radial component of the three-dimensional Ricci tensor which might be expressed (using the Hamiltonian constraint) as

$$R_r^{(3)r}=-8\pi T_0^0+\frac{K_{ij}K^{ij}}{2}+\frac{(\partial_r b)^2}{4ab^2}+\frac{\partial_r b}{abr}-\frac{1}{br^2}+\frac{1}{ar^2}. \quad (7)$$

Equations (2) and (6) are dynamical ones while (3) and (4) are the Hamiltonian and momentum constraints [8], respectively. As initial data one may take, for instance, $a(r,0)=b(r,0)$ at a time $t=0$; given the matter distribution one obtains from (3) and (4) the three-geometry of Σ and the extrinsic curvature K_r^r and from (5) the lapse α. Equations (2) and (6) determine the rate of change of the three-geometry and of K_r^r. Of the two functions a,b only one is independent. In the maximal slicing $\partial_t(b^2a)=0$; taking into account the above initial condition one gets

$$b^2(r,t)=\frac{b^3(r,0)}{a(r,t)}. \quad (8)$$

The order of the rest of this paper is the following. In Sec. II it is proved that the apparent horizon is null-like in an electrovacuum, has a constant area and coincides with the event horizon. Section III contains a proof of a version of the global Cauchy problem. Section IV shows an explicit solution of the Einstein equations in the maximal foliation. Its properties coincide with those proven in preceding sections. In the last section I comment on the significance of the results and their (possible) generalization.

II. MAIN CALCULATIONS

Let the boundary of a collapsing body be a sphere of a coordinate radius r_0. Let us define $c=\frac{-q^2+m^2}{4}$. First, let us notice that there exists a solution (Appendix A) that is manifestly static outside a collapsing body [10]:

$$a=b=\left(1+\frac{m}{r}+\frac{c}{r^2}\right)^2, \quad (9)$$

$$\alpha=\frac{r^2-c}{r^2+mr+c}. \quad (10)$$

α vanishes at $r=\sqrt{c}$ like $r-\sqrt{c}$. From this one readily infers that at a surface S placed at a coordinate radius $r=\sqrt{c}$ (that is, at an areal radius $\mathbf{R}=m+\sqrt{m^2-q^2}$) exists an event horizon; no signal can traverse through S in a finite coordinate time t. In the case of vanishing total charge q the corresponding line element coincides with the Schwarzschild line element in isotropic coordinates [8]. The solution (9) and (10) will not be considered in the rest of this paper.

Let us assume that a part of a spherically symmetric space-time generated by a collapsing body can be foliated by maximal slices Σ_t, $0\le t\le t_0$, that are asymptotically flat. Assume also that there exists a smooth continuation of the above band of slices, hypersurfaces $\Sigma_t^{\rm out}$, that are maximal outside a region of compact support and that cover a region with the outermost apparent horizon (if it exists). The coordinate time t is a parameter that labels maximal slices but it coincides with a proper time of an external observer that is localized very far from a collapsing body.

It was proven in the last reference of [6] that, when the amount of matter minus a total radial momentum exceeds the proper radius, then apparent horizons must form. Let us assume that there exists an apparent horizon outside a (neutral or charged) collapsing body of a compact support. (We do not exclude electrovacuum; i.e., there might exist long-ranged potentials outside a body, with an electrostatic Coulomb-like energy density.)

Under these conditions, one proves that the Penrose [11,12] inequality (which actually becomes an equality) holds true that, at the surface of an apparent horizon,

$$m=\sqrt{\frac{S}{16\pi}}+q^2\sqrt{\frac{\pi}{S}}. \quad (11)$$

It will be convenient to prove (11) in an isotropic system of coordinates in which $a=b=\phi^4$. It is easy to prove that this form of a metric can be achieved just by performing a suitable change of a radial coordinate on a fixed Cauchy slice. Morever, the final result [Eq. (11)] is already expressed in a coordinate-independent way.

The proof goes as follows. In an electrovacuum the Hamiltonian constraint reads [13]

$$\hat{\Delta}\phi=-\frac{1}{4}\hat{E}_i\hat{E}^i\phi^{-3}-\frac{K_{ij}K^{ij}\phi^5}{8}. \quad (12)$$

Here the quantities with carets refer to the flat background metric and $\hat{E}^r=\frac{q}{r^2}, E^\theta=E^\phi=0$. From the momentum constraints one gets [6]

$$K_{ij}=\left(n_i n_j-\frac{g_{ij}}{3}\right)\frac{C}{\phi^6 R^3}, \quad (13)$$

where n_j is a unit normal vector in the physical metric without a caret. C depends on time but it is constant on the part of a fixed Cauchy slice that is exterior to the collapsing body.

Equation (12) has a conserved (r independent, $\partial_r E = 0$) quantity

$$E = \frac{r}{8}(2r\partial_r\phi + \phi)^2 - \frac{r\phi^2}{8} - \frac{q^2}{8r\phi^2} - \frac{C^2}{72r^3\phi^6}. \quad (14)$$

Assuming asymptotic flatness one finds that $E = -\frac{m}{4}$, where m is the asymptotic mass. Notice also that $r\phi^2$ is equal to an areal radius $R = r\phi^2 = \sqrt{\frac{S}{4\pi}}$. After some rearrangements one might write (14) as

$$\begin{aligned} m - \left[\sqrt{\frac{S}{16\pi}} + q^2\sqrt{\frac{\pi}{S}}\right] &= -\frac{r}{2}\left[2r\partial_r\phi + \phi - \frac{C}{3r^2\phi^3}\right] \\ &\quad \times \left[2r\partial_r\phi + \phi + \frac{C}{3r^2\phi^3}\right] \\ &= -\frac{\mathbf{R}^3}{8}\theta(S)\theta'(S). \end{aligned} \quad (15)$$

Here $\theta(S)$ is the divergence of outgoing light rays, $\theta(S) = \frac{1}{\alpha}\frac{d}{dt}\ln S$ (the derivative in the direction of outgoing photons), and $\theta'(S) = -\frac{1}{\alpha}\frac{d}{dt}\ln S$ (the derivative in the direction of ingoing photons) [7] is the convergence of ingoing light rays. If S is an apparent horizon, then $\theta(S)$ vanishes, which proves (11).

Equation (11) actually holds on all maximal slices or on their time developments that are maximal in an asymptotically flat region (up to the apparent horizons) as far as the apparent horizon remains outside the collapsing body. That means that the areal radius $\mathbf{R} = r\sqrt{b}$ [we are coming back to the original metric notation (1)] of the apparent horizon must be conserved in time, since the mass m is conserved in time in asymptotically flat systems. That is, the full time derivative of $\mathbf{R}$ must vanish, which leads to the equality

$$V\sqrt{b}\frac{2b + r\partial_r b}{2b} + r\frac{\partial_t b}{2\sqrt{b}} = 0. \quad (16)$$

Here $V = dr/dt$ is the coordinate velocity expansion of the apparent horizon. Now, the condition for the apparent horizon reads

$$2b + r\partial_r b = \frac{brK_{rr}}{\sqrt{a}}; \quad (17)$$

inserting this into (16) and using (2) we obtain

$$\frac{\sqrt{b}K_{rr}r}{\sqrt{a}}\left[V - \frac{|\alpha|}{\sqrt{a}}\right] = 0. \quad (18)$$

We conclude that the apparent horizon expands with a radial velocity

$$V = \frac{|\alpha|}{\sqrt{a}}. \quad (19)$$

But, from Eq. (1) we know that $\frac{|\alpha|}{\sqrt{a}}$ is equal to the velocity of radially outgoing photons. Therefore, no material object can escape from within the apparent horizon; it just coincides with the event horizon. We can say more. Actually, Eq. (15) might be interpreted in the following way: if an areal radius of a sphere S satisfies (11), then S is an apparent horizon. By continuity, all Cauchy slices must contain surfaces satisfying (11), and the surfaces must remain in vacuum, since they move with the velocity of light. That means that, as long as maximal slices exist (or a time development of an initially maximal slice that is partially maximal later on, that is, it remains maximal in an open end containing the apparent horizon) and under conditions stated previously, the apparent horizon must exist forever and it coincides with an event horizon. The existence of the suitable version of the Cauchy problem is shown in Sec. III. In this way we have proven in the framework of the 1+3 formalism, without resorting to the Birkhoff theorem, that there exists an event horizon that intersects each Cauchy slice along a sphere of the areal radius $\mathbf{R} = m + \sqrt{m^2 - q^2}$.

Let us remark that one can prove, in a similar way, that when a locus of points with $\theta' = 0$ (i.e., the past apparent horizon) is placed in an electrovacuum, then it moves to the center with the velocity of light. This set of points is impenetrable from outside and its area is constant; hence, it constitutes a boundary of a white hole [14]. The solution (9) and (10) is simultaneously a white hole and a black hole since $\theta = \theta' = 0$ at $r = \sqrt{c}$. The velocity of the boundary $r = \sqrt{c}$ is equal to zero.

A more plausible way of expressing the above facts is as follows. The areal speed of radially ingoing or outgoing photons is given by formulas

$$V_{\text{in}} = \frac{d}{dt_{\text{in}}}R = -\alpha R\theta', \qquad V_{\text{out}} = \frac{d}{dt_{\text{out}}}R = \alpha R\theta,$$

respectively. At the boundary of a realistic black hole we have $\theta = 0, \theta' > 0$; therefore it can be penetrated from outside ($V_{\text{in}} < 0$) but nothing can leave its interior ($V_{\text{out}} = 0$). Exactly the opposite is true at the boundary of the white hole if $\theta > 0$; now $V_{\text{in}} = 0$ but $V_{\text{out}} > 0$. If θ is negative then $V_{\text{out}} < 0$, but still a photon gets out of the white hole. This is because the mean curvature p is now negative at the horizon and, therefore, the area of outer spheres next to the boundary of the white hole is smaller than of the boundary itself. There must exist, however, a black hole outside such a white hole that traps eventually everything that is inside. In the case when $\theta = \theta' = 0$ permanently at a surface S [as in the solution (9) and (10)] both V_{out} and V_{in} must vanish; the inner and outer parts of the white and/or black hole are permanently separated. Let us notice that the same happens when both scalar optical invariants θ and θ' are nonzero, but the lapse function is zero.

Remark. The above statements are true provided that $V_{\text{in}}, V_{\text{out}}$ vanish like $O(R - R_{\text{AH}})$ close to the horizons, which is not hard to prove.

There exists yet another possibility to prove that, if the evolution of collapsing matter is smooth outside a region of compact support (the latter can contain singularities), then if an apparent horizon exists at a time t it must exist forever. This becomes obvious if we notice that the development of the divergence θ in the direction of outgoing photons is given by the Raychaudhury equation [9], which in an electrovacuum reads, in 1+3 splitting [15],

$$\frac{d}{dt}\theta_{\mathrm{out}} = \frac{\theta}{\sqrt{a}}(\partial_r \alpha + \alpha\sqrt{a}K_r^r) - \frac{\alpha\theta^2}{2}. \tag{20}$$

From (20) one can deduce that a surface with vanishing divergence θ moves outward with the velocity of light. The reasoning is as follows. Assume that a photon is situated at a surface of vanishing θ. (20) implies that the rate of change of θ along a trajectory of the photon is exactly zero; the photon must forever remain on the surface with $\theta = 0$. The extension of this reasoning to cover cases with matter crossing an outermost apparent horizon is immediate; all that we need is the weak energy condition. If matter falls through the apparent horizon then it moves faster than light and the event horizon must exist out of it [15].

Equation (11) provides a well known necessary and (simultaneously) sufficient condition for the formation of event horizons formulated in terms of asymptotic quantities m, q and an area S. One can obtain also criteria in terms of quasilocal quantities, by combining results of this paper with some of the theorems of [6]. One can, for instance, fórmulate the following statement, proven in the last paper of [6].

Theorem 1. Let Σ be a maximal spherically symmetric Cauchy hypersurface. Assume the weak energy condition $\rho \geq \mid j \mid$. If at a centered sphere S of a proper radius L the following inequality holds true,

$$M - P > L \tag{21}$$

(here $M = \int_{V(S)} dV \rho$ is a total mass and $P = \int_{V(S)} dV j_r$ is a total radial momentum inside S), then S must be trapped.

From the previous statement it follows that, if a trapped S is a boundary of a collapsing body, then S coincides with an event horizon that surrounds the body.

Thus, if the energy content inside a ball of a fixed radius becomes large, then it hides under an event horizon. That proves a version of the cosmic censorship hypothesis [1] (CCH) in which *singularities* are supplied with a qualifier *massive*; *massive singularities are hidden under an event horizon*—this is a version of CCH that looks plausible.

The formulation in terms of quasilocal quantities is of interest, since it can be pursued further to cover cases in which the standard approach fails. As pointed out above, in spherically symmetric geometries (asymptotically flat and in some cosmological models) event horizons must exist if apparent horizons are present [15]. The quasilocal conditions that imply the formation of apparent horizons in spherically symmetric geometries are already known [6,7]. Using them, one can obtain a number of conditions for the formation of event horizons inside collapsing matter (in asymptotically flat geometries) and in cosmological models.

III. THE CAUCHY PROBLEM

In the above considerations I have assumed the existence of a global maximal Cauchy surface which possesses a maximal extension at least in the part of a space-time that is exterior to the apparent horizon and which includes the latter. Let me point out that in standard proofs of the Birkhoff theorem one usually assumes the existence of that part of space-time that is exterior to the collapsing body, that is, merely equivalent with my conditions. Nevertheless there exists a possibility to get rid of the assumption. Below I sketch a line of reasoning that should lead to a proof of a version of the global Cauchy problem.

To pursue this further we will need the spherically symmetric Einstein equations in an electrovacuum. In electrovacuum some of the matter-related terms (i.e., j_r, T^μ_μ) of Eqs. (4) and (6) vanish. Notice that $K_{ij}K^{ij}$ can be written (due to spherical symmetry) as $\frac{3}{2}(K_r^r)^2$. Below appears the mean curvature p of a sphere as embedded in a hypersurface Σ_t:

$$p = \frac{(r\partial_r b + 2b)}{\sqrt{a}br}. \tag{22}$$

The energy density ρ contains only a contribution from the electrostatic field, $\rho = \frac{q^2}{8\pi r^4 b^2} = \frac{q^2}{8\pi R^4}$.

The Einstein equations read

$$-\frac{2}{abr^2}\partial_r(r^2\partial_r b) + \frac{(\partial_r b)^2}{2ab^2} + \frac{\partial_r b \partial_r a}{a^2 b} + \frac{2\partial_r a}{a^2 r} - \frac{2\partial_r b}{abr} + \frac{2}{r^2}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{3}{2}(K_r^r)^2 + 16\pi\rho, \tag{23}$$

$$\partial_r K_r^r = -\frac{3\sqrt{a}}{2} p K_r^r, \tag{24}$$

$$\frac{1}{a^{1/2}br^2}\partial_r(a^{-1/2}br^2\partial_r\alpha) = \left[\frac{3}{2}(K_r^r)^2 + 8\pi\rho\right]\alpha, \tag{25}$$

$$\partial_t a = 2\alpha a K_r^r, \tag{26}$$

$$\partial_t K_r^r = \frac{3\alpha}{4}(K_r^r)^2 - \frac{\alpha p^2}{4} - \frac{p\partial_r\alpha}{\sqrt{a}} + \frac{\alpha}{br^2} - 8\pi\alpha\rho. \tag{27}$$

A. The initial data

Assume that Σ_0 is a global Cauchy hypersurface. As pointed out at the end of Sec. I, initial data are determined by the initial distribution of matter and momentum; without loss of generality we can assume that initially $a(r) = b(r)$. If Σ_0 contains an apparent horizon, then a singularity will develop in the future; therefore, the best we can hope to prove is the existence of a solution outside the apparent horizon. We wish to consider the Cauchy problem outside the apparent horizon; this seems to be reasonable, since the apparent horizon moves outward with the speed of light and nothing that happens inside it can casually influence its exterior.

Let $\Sigma_t^{\rm out}$ be a Cauchy maximal hypersurface that evolves from Σ_0 in the region outside the cylinder enclosed by the apparent horizon. $\Sigma_0^{\rm out}$ concides obviously with a corresponding part of Σ_0; hence, the Cauchy data K_r^r and a are fixed. [The function b might be determined from (8) and is not an independent dynamical quantity.] The Hamiltonian constraint constitutes an elliptic equation. From asymptotic flatness we have to set $a(\infty) = b(\infty) = 1$, but this condition is not sufficient to ensure the uniqueness of solutions of Eq. (23) on $\Sigma_t^{\rm out}$. However, the asymptotic mass m must be constant on all slices and it is determined by the geometry of Σ_0. Therefore we must demand that on all Cauchy slices $\Sigma_t^{\rm out}$ the following asymptotic condition is met:

$$\lim_{r\to\infty} r^2\partial_r a = \lim_{r\to\infty} r^2\partial_r b = -2m. \tag{28}$$

In what follows we assume that the convergence $\theta' = K_r^r + p$ of ingoing light rays is strictly positive on Σ_0; this implies that $\theta' = K_r^r + p$ is strictly positive on all future slices $\Sigma_t^{\rm out}$ (Appendix B). If there is an apparent horizon, then $\theta(r) = K_r^r - p$ vanishes at a centered sphere in $\Sigma_t^{\rm out}$; the preceding assumption $\theta' > 0$ implies that $K_r^r > 0$ at the apparent horizon and out of it, since K_r^r does not change sign in an electrovacuum [see formula (13)]. Now we can conclude that at the outermost apparent horizon and out of it we must have $p > 0$, i.e., there is no minimal surface in $\Sigma_t^{\rm out}$. Let $r = r_t$ be a position of the outermost apparent horizon. We may invoke the calculation performed in the previous section, which led to Eq. (15). (The calculation is based on the assumption that $a = b = \phi^4$ but it is only a technical condition and there is no loss of generality.) At the apparent horizon (15) yields

$$m - \left[\sqrt{\frac{S}{16\pi}} + q^2\sqrt{\frac{\pi}{S}}\right] = 0. \tag{29}$$

In an electrovacuum there are two solutions of Eq. (29):

$$r_t\phi^2(r_t)|_2^1 = m^{+}_{-}\sqrt{m^2 - q^2}; \tag{30}$$

taking into account the above conditions we have to choose

$$\phi(r_t) = \sqrt{\frac{1}{r_t}(m + \sqrt{m^2 - q^2})}, \tag{31}$$

since otherwise there could exist a minimal surface at some $r > r_t$. The Hamiltonian constraint is now supplied with the standard Dirichlet boundary conditions $\phi(r_t), \phi(\infty) = 1$ on $\Sigma_t^{\rm out}$ and it is easy to prove that there is a unique solution.

Thus, under the above conditions, fixing the asymptotic mass m uniquely determines the conformal factor ϕ (and, consequently, a, b, if a relation between the two functions is determined on an initial slice) at the surface of the apparent horizon. But m is determined by the initial geometry Σ_0; hence we have no freedom left in specifying the solutions of the Hamiltonian constraint in the exterior region.

The corresponding boundary problem for the lapse equation (25) contains, however, an arbitrariness. The condition

$$\alpha(\infty) = 1 \tag{32}$$

does not specify uniquely a solution of (25); we still can impose

$$\frac{\partial_r\alpha}{\sqrt{a}} = f(t) \tag{33}$$

at the surface of an outermost apparent horizon. I assume that the function $f(t)$ is smooth and strictly positive. We have defined the exterior Cauchy problem (i.e., with data on $\Sigma_t^{\rm out}$) as a restriction of the global Cauchy problem with data on Σ_t. There is an obvious loss of information during such a restriction, since we do not control the collapse of matter fields that are enclosed inside the apparent horizon. The freedom in choosing $f(t)$ corresponds to our unawareness about the full state of the collapsing system. This arbitrariness cannot be noticed by an external observer that has access to local data only; once the existence of exterior geometry is proven, one can always cast (using the Birkhoff transformation) the exterior of the event horizon in standard Reissner-Nordström coordinates.

In summary, the initial value problem of the electrovacuum Einstein equations outside the apparent horizon can be determined by prescribing the asymptotic mass m, a relation $a(r, t = 0) = b(r, t = 0)$, an initial datum K_r^r at a time $t = 0$, and a condition (33) for the lapse function.

B. The exterior Cauchy solution exists globally

Theorem 2. Let $H_s(\Sigma_t^{\rm out})$ be a Sobolev space of functions defined on $\Sigma_t^{\rm out}$. Let $\Sigma_{t=0}$ be a global maximal Cauchy surface with initial data $a(t = 0) = b(t = 0)$ such that $\partial_r a(r, t = 0)\epsilon H_2(\Sigma_{t=0})$, $K_r^r(t = 0)\epsilon H_2(\Sigma_{t=0})$ generated by a given initial distribution of matter. Assume that the convergence $\theta'(S)$ of the ingoing light rays is strictly positive for any centered sphere S on $\Sigma_0^{\rm out}$ and that there exists an apparent horizon that is placed in a vacuum or in an electrovacuum. Let the lapse function satisfy the boundary conditions (32), (33) with $f(t) \geq 0$ at the outermost apparent horizon. Then there exists a unique solution of the global Cauchy problem in the region exterior to the apparent horizon, including the apparent horizon itself.

Sketch of the proof. There exist theorems [17] from which one infers the existence of a solution $[K_r^r\epsilon H_s(\Sigma_t), \partial_r a\epsilon H_s(\Sigma_t)$ or $K_r^r, \partial_r a\epsilon H_s^{\rm loc}(\Sigma_t)]$ for $s \geq 2$ and sufficiently short intervals of time. The lower bound on the index s is due to the Schaüder ring property [18] which is satisfied in three spatial dimensions if $s \geq 2$.

Assuming the local existence, I will estimate (step 1) the pointwise growth of $a(t)$,$K^r_r(t), \partial_r K^r_r, p, \partial_r p, \partial_r \alpha$. Using those estimates and the method of energy estimates [19] one can show (step 2) that the Sobolev norms in question $[K^r_r \epsilon H_s(\Sigma^{\rm out}_t), \partial_r a \epsilon H_s(\Sigma^{\rm out}_t)]$ do not blow up in a finite time t, thus accomplishing the final goal.

Step 1. The L_∞ estimates. In an electrovacuum the right-hand side of (25) is non-negative. Invoking the maximum principle and using the boundary condition that the lapse satisfies at the apparent horizon we conclude that

$$\frac{\partial_r \alpha}{\sqrt{a}} \geq f(t) \geq 0 \tag{34}$$

everywhere in $\Sigma^{\rm out}_t$. As pointed out in Sec. III A, if $\theta' > 0$ then

$$p \geq K^r_r > 0 \tag{35}$$

everywhere in $\Sigma^{\rm out}_t$ [but at spatial infinity both functions vanish, $K^r_r = O(1/r^3)$ and $p = 0(1/r)$]. Notice also that

$$\begin{aligned} a(r,t) &= a(r,0) + 2\int_0^t a\alpha K^r_r ds \\ &\geq a(r,0) = b(r,0) \\ &\geq b(r,0) - 2\int_0^t b\alpha K^r_r ds \\ &= b(r,t). \end{aligned} \tag{36}$$

Using the inequalities (34) and (35) one gets from (24) and (27) that

$$\begin{aligned} &\left(\partial_t + \frac{\alpha}{\sqrt{a}}\partial_r\right)(RK^r_r) \\ &= -pR_{\rm AH} f(t) + \frac{\alpha}{R_{\rm AH}}[1 - (K^r_r R_{\rm AH})^2] - \frac{\alpha q^2}{R^3} \\ &\leq \frac{\alpha}{R_{\rm AH}}[1 - (K^r_r R_{\rm AH})^2] \end{aligned} \tag{37}$$

at the apparent horizon. Above, $R_{\rm AH}$ is the areal radius of the apparent horizon, $R_{\rm AH} = m + \sqrt{m^2 - q^2}$.

(37) implies that at the apparent horizon $K^r_r R_{\rm AH}$ is less that the greater of the two numbers $(1, (K^r_r R_{\rm AH})(t=0))$. (In fact the bound is 1, since one can show that if the apparent horizon lies on a smooth Cauchy hypersurface then the product $(K^r_r R_{\rm AH})(t=0)$ cannot exceed 1 [15].)

Equation (24) can be solved on each slice $\Sigma^{\rm out}_t$ to give

$$K^r_r(r,t) = \frac{C(t)}{R^3}. \tag{38}$$

At the apparent horizon $p = K^r_r$ and $\theta' = p - K^r_r$ is positive during the collapse (Appendix B) if it is positive on $\Sigma^{\rm out}_0$; this implies that K^r_r is positive at the horizon and from (38) also $C(t)$ is positive.

The extrinsic curvature K^r_r decreases on a fixed slice $\Sigma^{\rm out}_t$ and achieves its largest value at the apparent horizon.

Define

$$C_0 = R^2_{\rm AH} \sup(1, (K^r_r R_{\rm AH})(t=0)) = \sup(R^2_{\rm AH}, C(0)). \tag{39}$$

(37) and (38) lead to the final estimations

$$C(t) \leq C_0, \quad K^r_r(r,t) \leq \frac{C_0}{R^3}. \tag{40}$$

(2) and (40) imply an estimation

$$\partial_t a(r,t) = 2a(r,t)\alpha(r,t)K^r_r(r,t) \leq a(r,t)2\frac{C_0}{R^3_{\rm AH}}, \tag{41}$$

which in turn gives the following estimation of the metric coefficient $a(r,t)$:

$$a(r,t) \leq a(r,0)e^{\frac{2C_0 t}{R^3_{\rm AH}}}. \tag{42}$$

A straightforward calculation gives

$$\begin{aligned} \partial_t p(r,t) &= -\frac{\partial_r \alpha(r,t)}{\sqrt{a}}K^r_r(r,t) + \frac{\alpha K^r_r}{2}p \\ &\leq \frac{\alpha K^r_r}{2}p \leq \frac{C_0}{2R^3_{\rm AH}}p; \end{aligned} \tag{43}$$

above, I employed the inequalities $\partial_r \alpha \geq 0$, $\alpha \leq 1$ and (40).

One can show also that

$$\begin{aligned} &\partial_r(p(r,t)R) \\ &= -\sqrt{a}R\left(8\pi\rho + \frac{3}{4}[K^r_r(r,t)]^2 + \frac{1}{4}p^2 - \frac{1}{R^2}\right). \end{aligned} \tag{44}$$

If the product pR exceeds 2, then the right-hand side of (44) becomes negative, which means that pR has to decrease. Therefore $pR \leq C_2 = \sup(2, C_1)$, where C_1 is the value of pR at the apparent horizon. At the apparent horizon $pR = K^r_r R$ which means that $C_1 = \frac{C_0}{R^2_{\rm AH}}$.

Finally we get

$$p(r,t) \leq \frac{C_2}{R(r,t)} = \sup\left[\frac{2}{R(r,t)}, \frac{C_0}{R(r,t)R^2_{\rm AH}}\right]. \tag{45}$$

The energy density is positive and bounded from above by $\frac{q^2}{8\pi R^4_{\rm AH}}$; using the preceding information one easily infers that $\mid \frac{\partial_r p}{\sqrt{a}} \mid$ must be uniformly bounded on all slices $\Sigma^{\rm out}_t$ by $\frac{q^2}{R^3_{\rm AH}} + \frac{3C_0^2}{4R^5_{\rm AH}} + \frac{|4-C_2^2|}{4R_{\rm AH}}$.

A combination of (40), (41), and (45) together with the momentum constraint (24) allows one to obtain a bound for $\partial_r K^r_r(r,t)$:

$$\left|\frac{\partial_r K^r_r(r,t)}{\sqrt{a(r,t)}}\right| \leq \frac{3C_0 C_2}{2R^4}. \tag{46}$$

The energy density is positive and bounded from above by $\frac{q^2}{8\pi R^4_{\rm AH}}$; using the preceding information one easily infers that $\mid \frac{\partial_r p}{\sqrt{a}} \mid$ must be uniformly bounded on all slices

$\Sigma_t^{\rm out}$ by $\frac{q^2}{R_{\rm AH}}+\frac{3C_0^2}{4R_{\rm AH}^5}$.

In order to get estimations of $\partial_r\alpha(r,t), \partial_r^2\alpha(r,t)$ one should analyze the lapse equation (25) using the above information about the evolution of the extrinsic curvature. A lengthy and not particularly illuminating calculation gives a bound

$$\left(\frac{\partial_r\alpha}{\sqrt{a}}\right)(r,t)\le f(t)+\frac{4\pi q^2}{RC(t)}+\frac{3\pi C(t)}{R^2R_{\rm AH}}. \quad (47)$$

The right-hand side of (47) is finite since $C(t)$ is strictly greater than 0 in any finite time t (Appendix B) and satisfies (40).

In summary, the following estimates hold true:

$$||X||_{L_\infty(\Sigma_t^{\rm out})}\le F(t)||X||_{L_\infty(\Sigma_0^{\rm out})}, \quad (48)$$

where X is any of the functions $(a, K_r^r, \frac{\partial_r K_r^r}{\sqrt{a}}, p,)$ and $F(t)$ is a positive function that remains bounded for arbitrarily large but finite values of its argument. The lapse α does not exceed 1, $\frac{\partial_r\alpha}{\sqrt{a}}$ must satisfy (47), and $|\frac{\partial_r p}{\sqrt{a}}|$, $|\frac{\partial_r K_r^r}{\sqrt{a}}|, K_r^r, p$ are all bounded by some constants.

Step 2. The integral estimates. In order to prove the existence of a global solution one has to show that the Sobolev norm $||K_r^r||_{H_2(\Sigma_t^{\rm out})}+||\partial_r a||_{H_2(\Sigma_t^{\rm out})}$ remains bounded for any finite time t. This could be done explicitly, by pursuing the above calculation in order to get pointwise estimates for all quantities in question and then proving the required integrability. I will choose a way that is probably less economic in this particular case but offers a chance for generalization.

Let $\mathbf{X}$ be a vector having six components $\mathbf{X}_i=\partial_r^i a, \mathbf{X}_{i+3}=\partial_r^{i-1}K_r^r$, $i=1,2,3$. Define

$$H(t)=(||K_r^r||^2_{H_2(\Sigma_t^{\rm out})}+||\partial_r a||^2_{H_2(\Sigma_t^{\rm out})}). \quad (49)$$

A simple but laborious calculation shows that

$$\frac{d}{dt}H=\sum_{i,j=1}^{6}\int_{\Sigma_t^{\rm out}}dV\mathbf{X}_i\mathbf{X}_jf_{ij} +\sum_{i=1}^{6}f_iX_i-4\pi\alpha R^2\sum_{i=1}^{6}X_i^2\Big|_{\rm AH}, \quad (50)$$

where f_i, f_{ij} are certain polynomials of finite order that depend on $\sqrt{a}, b, K_r^r, \partial_r K_r^r, p, \partial_r p, \partial_r\alpha, \alpha, \rho$. The functions f_i are square integrable. The crucial point is that the right-hand side of (50) is bilinear in $\mathbf{X}_i$. Direct differentiation of $||K_r^r||^2_{H_2(\Sigma_t^{\rm out})}+||\partial_r a||^2_{H_2(\Sigma_t^{\rm out})}$ with respect to t and the use of evolution equations (26) and (27) gives also some trilinear terms but manipulating with Eqs. (23)–(25) and (8) finally yields the above equation.

The right-hand side of (50) is bounded from above by

$$\sup\{||f_{ij}||_{L_\infty(\Sigma_t^{\rm out})}\}\sum_{i=1}^{6}\int_{\Sigma_t^{\rm out}}dV\mathbf{X}_i\mathbf{X}_i +\sum_{i=1}^{6}\left[\int_{\Sigma_t^{\rm out}}dVf_i^2\right]^{1/2}\sum_{i=1}^{6}\left[\int_{\Sigma_t^{\rm out}}dV\mathbf{X}_i\mathbf{X}_i\right]^{1/2}. \quad (51)$$

Taking into account the estimates of the preceding subsection, one obtains the inequality

$$\partial_t H\le C_2(t)H+C_3(t), \quad (52)$$

where C_2, C_3 are bounded functions of coordinate time t with coefficients depending only on the initial data. (52) readily implies

$$H(t)\le e^{\int_0^t dsC_2(s)}\int_0^t dsC_3(s)e^{-\int_0^t dsC_2(s)} +H(t=0)e^{\int_0^t dsC_2(s)}, \quad (53)$$

thus proving the existence of a solution for any finite time t. That ends the proof of Theorem 2.

Remarks. The Cauchy problem should be investigated in *weighted Sobolev spaces* [20] instead of Sobolev spaces. The hypersurfaces $\Sigma_t^{\rm out}$ are noncompact and one should incorporate suitable falloff conditions at spatial infinity; weighted Sobolev spaces include them automatically, in contrast with the standard Sobolev spaces. It is easy, however, to adapt the above proof to work with $H_{s,\delta}$ instead of H_s and I will not discuss this point.

IV. EXPLICIT SOLUTION

In an electrovacuum the Einstein equations (24), (25), (27), and (44) [the last is identical with the Hamiltonian constraint (23)] can be solved explicitly. The metric (in terms of the areal radius R instead of the cordinate radius r) is given by

$$ds^2=-\frac{\gamma^2}{4}[(pR)^2-(K_r^rR)^2]dt^2+2\gamma\frac{K_r^r}{p}dtdR +\frac{4}{(pR)^2}dR^2+R^2d\Omega^2, \quad (54)$$

where

$$pR=2\sqrt{1-\frac{2m}{R}+\frac{q^2}{R^2}+\frac{[C(t)]^2}{4R^4}},$$
$$\gamma=1+(4\partial_tC)\int_R^\infty\frac{d\tilde R}{\tilde R^2(p\tilde R)^3}. \quad (55)$$

The function $\frac{C(t)}{R_{\rm AH}^2}$ cannot exceed the greater of the two numbers $(1, C_0)$ (see Sec. III); one can show that it cannot be greater than 1 [15]. It cannot reach 0 in a finite time, unless we impose that by a suitable choice of the boundary condition for the lapse function (see Appendix B). The metric (54) actually slices an inner region adjacent to the apparent horizon; the depth of the penetration region depends on the value of $C(t)$. $C(t)$ must satisfy a nonlinear ordinary differential equation (37). In the case of $q = 0$ the above solution coincides with a maximal foliation of the Schwarzschild solution [22]. The spatial part of (54) has been obtained by Duncan [23].

V. FINAL COMMENTS

The present investigation is based on the 1+3 splitting of space-time that is smooth (initially and possibly also globally, modulo a region of compact support). The world time t can be used globally to parametrize casually related occurrences. The proof that in an electrovacuum event horizons coincide with apparent horizons is done with only minimal reference to specific properties of spherically symmetric geometries. Obviously, the existence of the maximal slicing requires a proof [16], but the presence or absence of spherical symmetry is probably of no great significance for the validity of maximal slicings. The place where the assumption of spherical symmetry plays an important role is the proof of the identity (15), but it is quite likely that (15) survives (in the form of the Penrose-Gibbons inequality) also in nonspherical geometries. Alternatively, one can use the Raychaudhuri equation (20) in order to prove the local confining property of apparent horizons [21], which should be of help in proving the existence of the global Cauchy solution.

The global Cauchy problem poses a serious obstacle in making significant progress in proving the cosmic censorship hypothesis. I hope that the method of Sec. III can be extended (with some modifications, mostly concerning less stringent smoothness conditions) onto general self-gravitating spherically symmetric systems.

The last section of the paper presents an explicit form of the Reissner-Nordström four-metric in the maximal slicing. It possesses all the properties that have been proven in preceding sections.

The application of the above ideas to a more general class of spherically symmetric geometries of collapsing systems will be reported elsewhere [15].

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APPENDIX A

The static Einstein equations reduce to three equations (23), (25), and (27), of which only two are independent. Equation (23) gives the spatial metric, which in a gauge $a(r) = b(r) = \phi^4$ coincides with the solution (9). Inserting this solution into (27) gives (notice that $K_r^r = 0$ and $c = \frac{m^2-q^2}{4}$)

$$\partial_r \ln\alpha = \frac{\sqrt{a}}{pbr^2} - \frac{p\sqrt{a}}{4} - \frac{q^2\sqrt{a}}{b^2r^4p}$$

(here $p = \frac{2(2\partial_r\phi r+\phi)}{\phi^3 r}$)

$$= \frac{mr^2 + 4cr + mc}{(r^2-c)(r^2+mr+c)}.$$

The last equation is solved by (10).

APPENDIX B

Lemma. Under conditions stated in Theorem 2, if $\theta' = p + K_r^r$ is positive on Σ_0 then it must be positive in all future slices $\Sigma_t^{\rm out}$.

Remark. Below I will present a version that can be easily adapted to collapsing systems containing matter. There exists also a very simple proof in the specific case of an electrovacuum which will be shown at the end of the Appendix.

Proof. Assume the contrary, i.e., that there exists a Cauchy (external) hypersurface $\Sigma_t^{\rm out}$ such that somewhere on it θ' crosses through zero.

The evolution of θ' is given by the equation

$$\left(\partial_t - \frac{\alpha}{\sqrt{a}}\partial_r\right)\theta' = (-\partial_r\alpha/\sqrt{a} + \alpha K_r^r)\theta' + \alpha\theta'^2/2.$$

From this equation one infers that the surface with vanishing convergence θ' moves inward with the velocity of light when immersed in vacuum (Sec. II). Therefore it must exist in all preceding Cauchy slices and in particular in the initial hypersurface $\Sigma_0^{\rm out}$. This gives a contradiction which proves our claim.

Corollary. Assume that Σ_0 contains an apparent horizon. Under the conditions of the preceding lemma, $C(t)$ (and, consequently, K_r^r) must be strictly positive for any finite time t.

Proof. From the results of Sec. II, the apparent horizon propagates to the future. If $C(t)$ was equal to zero on a slice $\Sigma_t^{\rm out}$ for some time t, then both θ and θ' would vanish at an apparent horizon, which would imply (due to the above lemma) the existence of a white hole in Σ_0, contrary to the assumption that $\theta' > 0$ in Σ_0.

Another proof of Lemma. Equation (37) can be written as

$$\frac{d}{d\tau}x = -\frac{xf(t)}{\alpha} + \frac{1-x^2}{R_{\rm AH}} - \frac{q^2}{R_{\rm AH}^3}, \tag{B1}$$

where $x = K_r^r R_{\rm AH}, d\tau = \alpha dt$, and $f(t)$ is given by (33). x must be positive on the initial Cauchy hypersurface. Notice that $x = \frac{C(t)}{R_{\rm AH}^2}$; the above equation determines the evolution of $C(t)$.

Notice that $q^2 \le m^2 \le R_{\rm AH}^2$; therefore $\frac{1}{R_{\rm AH}} - -\frac{q^2}{R_{\rm AH}^3} \ge 0$ and the right-hand side of (56) is estimated from below by $\frac{-xf(t)}{\alpha} - \frac{x^2}{R_{\rm AH}} \ge -xf(t) - \frac{x^2}{R_{\rm AH}}$. This in turn is bounded from below by $-xf(t) - \frac{x}{R_{\rm AH}}$, because $x \leq 1$. It is easy to see that $x_1 \leq x \leq x_2$, where $x(0) = x_1(0) = x_2(0) \leq 1$ and x_1, x_2 satisfy one differential equations

$$\frac{d}{dt}x_1 = -x_1\left(f(t) + \frac{1}{R_{\rm AH}}\right),$$

$$\frac{d}{d\tau}x_2 = \frac{1 - x_2^2}{R_{\rm AH}}.$$

Both equations can be explicitly solved to give the estimation

$$\frac{-e^{\frac{-\tau}{R_{\rm AH}}} + Ce^{\frac{\tau}{R_{\rm AH}}}}{e^{\frac{-\tau}{R_{\rm AH}}} + Ce^{\frac{\tau}{R_{\rm AH}}}} \geq x(t) \geq x(t = 0)e^{\int_0^t ds\left(-f(s) - \frac{1}{R_{\rm AH}}\right)}.$$

Here $\tau = \int_0^t \alpha dt$ and $C = \frac{1+x(0)}{1-x(0)}$.

Therefore we conclude that $C(t)$ can never reach 0 in a finite time if f is a bounded function. This proof can be extended into general self-gravitating systems (at least in those cases when the energy density is finite).

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