

**UCC Library and UCC researchers have made this item openly available.  
Please [let us know](#) how this has helped you. Thanks!**

<b>Title</b>	Pseudospectra of elements of reduced Banach algebras
<b>Author(s)</b>	Krishnan, Arundhathi; Kulkarni, S. H.
<b>Publication date</b>	2017
<b>Original citation</b>	Krishnan, A. and Kulkarni, S. H. (2017) 'Pseudospectra of elements of reduced Banach algebras', <i>Advances in Operator Theory</i> , 2(4), pp. 475-493. doi: 10.22034/aot.1702-1112
<b>Type of publication</b>	Article (peer-reviewed)
<b>Link to publisher's version</b>	<a href="http://www.aot-math.org/article_48321.html">http://www.aot-math.org/article_48321.html</a> <a href="http://dx.doi.org/10.22034/aot.1702-1112">http://dx.doi.org/10.22034/aot.1702-1112</a> Access to the full text of the published version may require a subscription.
<b>Rights</b>	© 2017 Tusi Math. Research Group (TMRG)
<b>Item downloaded from</b>	<a href="http://hdl.handle.net/10468/7428">http://hdl.handle.net/10468/7428</a>

Downloaded on 2021-01-20T17:45:48Z



**UCC**

University College Cork, Ireland  
Coláiste na hOllscoile Corcaigh

# PSEUDOSPECTRA OF ELEMENTS OF REDUCED BANACH ALGEBRAS

ARUNDHATHI KRISHNAN<sup>1</sup> \* and S. H. KULKARNI<sup>2</sup>

ABSTRACT. Let  $A$  be a Banach algebra with identity 1 and  $p \in A$  be a non-trivial idempotent. Then  $q = 1 - p$  is also an idempotent. The subalgebras  $pAp$  and  $qAq$  are Banach algebras, called reduced Banach algebras, with identities  $p$  and  $q$  respectively. For  $a \in A$  and  $\varepsilon > 0$ , we examine the relationship between the  $\varepsilon$ -pseudospectrum of  $a \in A$ ,  $\Lambda_\varepsilon(A, a)$  and  $\varepsilon$ -pseudospectra of  $pap \in pAp$ ,  $\Lambda_\varepsilon(pAp, pap)$  and of  $qaq \in qAq$ ,  $\Lambda_\varepsilon(qAq, qaq)$ . We also extend this study by considering a finite number of idempotents  $p_1, \dots, p_n$ , as well as an arbitrary family of idempotents satisfying certain conditions.

## 1. INTRODUCTION

In this note, we study the decomposition of the spectrum and pseudospectrum of an element in a Banach algebra into the spectra and pseudospectra of its corresponding elements in reduced Banach algebras.

Suppose  $T$  is a bounded operator on a Hilbert space  $H$  that can be expressed as the direct sum of two closed subspaces  $H_1$  and  $H_2$ . Let  $P_1$  and  $P_2 = I - P_1$  be the bounded linear projections onto  $H_1$  and  $H_2$  respectively. The operator  $T$  commutes with  $P_1$  and  $P_2$  iff  $H_1$  and  $H_2$  are invariant under  $T$ . In this case  $T$  can be expressed as the direct sum  $T_1 \oplus T_2$ , where  $T_1 = P_1T \upharpoonright_{H_1}$  and  $T_2 = P_2T \upharpoonright_{H_2}$ . It is easily seen that  $\sigma(B(H), T) = \sigma(B(H_1), T_1) \cup \sigma(B(H_2), T_2)$ . This can be extended to a finite direct sum of operators. See Problem 98 in [6]. The case of infinite direct summands is a little different. See Theorem 2.3 in [4].

This can also be extended to the decomposition of the spectrum of an element in an arbitrary Banach algebra. If  $p \in A$  is an idempotent in a Banach algebra, then  $pAp = \{pap : a \in A\}$  is a closed subalgebra of  $A$  called a reduced Banach algebra. If  $q = 1 - p$ , then  $q$  is an idempotent too, and thus  $qAq$  is also a closed subalgebra of  $A$ . Then for  $a \in A$  that commutes with  $p$ ,  $\sigma(A, a) = \sigma(pAp, pa) \cup \sigma(qAq, qa)$ . A similar result is true for the case of a finite number of idempotents  $\{p_i\}_{i=1}^n$  satisfying  $\sum_{i=1}^n p_i = 1$  (see Lemmas 3.1 and 3.2).

Further, we would like to examine if such a decomposition holds for the  $\varepsilon$ -pseudospectrum  $\Lambda_\varepsilon(A, a)$  of  $a \in A$ . We show that if  $p_1, \dots, p_n$  are idempotents

---

Copyright 2016 by the Tusi Mathematical Research Group.

Date: Received: xxxxxx; Accepted: zzzzzz.

\*Corresponding author.

2010 *Mathematics Subject Classification*. Primary 47A10; Secondary 46H05, 47A12.

*Key words and phrases*. Banach algebra, Direct Sum, Reduced Banach algebra, Idempotent, Pseudospectrum, Spectrum.

in  $A$  such that  $\sum_{i=1}^n p_i = 1$ , and  $a \in A$  commutes with each  $p_i$ , then

$$\Lambda_{\frac{\varepsilon}{n}}(A, a) \subseteq \bigcup_{i=1}^n \Lambda_{\varepsilon}(p_i A p_i, p_i a) \subseteq \Lambda_{\max_i \|p_i\| \varepsilon}(A, a).$$

See Theorem 3.3.

We then study specific cases in which

$$\Lambda_{\varepsilon}(A, a) = \bigcup_{i=1}^n \Lambda_{\varepsilon}(p_i A p_i, p_i a). \quad (1.1)$$

This happens, for instance, if  $a$  is a  $G_1$  element or if each  $p_i$  is a self adjoint projection in a  $C^*$ -algebra. See Theorems 3.6 and 3.7. The decomposition (1.1) occurs precisely when  $\|(\lambda - a)^{-1}\| = \max_i \|p_i(\lambda - a)^{-1}\|$ . We hence consider those Banach algebras and idempotents  $\{p_i\}$  that satisfy  $\|a\| = \max_i \|p_i a\| \forall a \in A$ . We use a result from [11] to show that for the Banach algebra of bounded linear operators on a Banach space  $X$ , and  $P_i \in B(X)$  idempotent operators such that  $P_i P_j = 0$ ,  $i \neq j$  and  $\sum_{i=1}^n P_i = I$ , this property is satisfied precisely when the norm on  $X$  satisfies the following:

$$\|P_i x\| = \|P_i y\| \forall i \Rightarrow \|x\| = \|y\|.$$

Suppose  $A$  is a Banach algebra,  $p_1, \dots, p_n \in A$  are idempotents such that  $\sum_{i=1}^n p_i = 1$  and  $p_i p_j = 0$ ,  $i \neq j$  and  $\|a\| = \max_i \|p_i a\|$ ,  $\forall a \in A$ . Then we show that each  $p_i$  is Hermitian and (1.1) is satisfied. See Theorems 3.17 and 3.18. However, each  $p_i$  being Hermitian is not sufficient for (1.1) to occur as seen in Example 3.15.

In Section 4, we examine the infinite case. We consider a family of idempotent operators on a Hilbert space or a Banach space  $X$ , that converges strongly to the identity operator and consider the decomposition of the spectrum and pseudospectrum of  $T \in B(X)$ . The spectrum of  $T \in B(X)$  is in general larger than the union of the spectra of  $P_n T$ , the difference in the two sets arising from those complex numbers  $\lambda$  for which  $\sup_n \|(\lambda P_n - P_n T)^{-1}\| = \infty$ . Similarly, the pseudospectrum of  $T$  may properly contain the union of the pseudospectra of its reduced components. The difference in the two sets arises from those complex numbers  $\lambda$  for which  $\|(\lambda - T)^{-1}\| = \frac{1}{\varepsilon}$ . In special cases, the pseudospectrum of  $T$  is equal to the closure of the union of the pseudospectra of its reduced components. See Theorem 4.4 and Remark 4.5.

The primary objective of this note is to show that the pseudospectra of certain elements of Banach algebras can be decomposed into the pseudospectra of elements of certain reduced subalgebras. This could make it easier to compute the pseudospectra of certain operators. Examples 4.10 and 4.11 are simple illustrations of this.

## 2. DEFINITIONS

Let  $A$  be a complex Banach algebra with unit 1. For  $\lambda \in \mathbb{C}$ ,  $\lambda \cdot 1$  is identified with  $\lambda$ . Let  $\text{Inv}(A) = \{x \in A : x \text{ is invertible in } A\}$  and  $\text{Sing}(A) = \{x \in A : x \text{ is not invertible in } A\}$ .

**Definition 2.1.** The *spectrum* of an element  $a \in A$  is defined as:

$$\sigma(A, a) := \{\lambda \in \mathbb{C} : \lambda - a \in \text{Sing}(A)\}.$$

**Definition 2.2.** The *spectral radius* of an element  $a \in A$  is defined as:

$$r(A, a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

**Definition 2.3.** Let  $\varepsilon > 0$ . The  $\varepsilon$ -*pseudospectrum*  $\Lambda_\varepsilon(a)$  of  $a \in A$  is defined by

$$\Lambda_\varepsilon(A, a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geq \varepsilon^{-1}\}$$

with the convention that  $\|(\lambda - a)^{-1}\| = \infty$  if  $\lambda - a$  is not invertible.

If the algebra  $A$  is fixed or is clear from the context, then we use simplified notations  $\sigma(a)$ ,  $r(a)$  and  $\Lambda_\varepsilon(a)$  in place of  $\sigma(A, a)$ ,  $r(A, a)$  and  $\Lambda_\varepsilon(a)$  respectively. The basic reference for pseudospectrum, especially for matrices, is the book [17]. The  $\varepsilon$ -pseudospectrum of an element of an arbitrary Banach algebra has been studied in [9].

**Definition 2.4.** Let  $A$  be a Banach algebra and  $a \in A$ . The *numerical range* (see Definition 1.10.1 in [2]) of  $a$  is defined by

$$V(a) := \{f(a) : f \in A', f(1) = 1, \|f\| = 1\},$$

where  $A'$  is the dual space of  $A$ .

**Definition 2.5.** Let  $A$  be a Banach algebra and  $a \in A$ . Then  $a$  is said to be *Hermitian* if  $V(a) \subseteq \mathbb{R}$ .

**Definition 2.6.** Let  $A$  be a Banach algebra and  $p \in A$  be an idempotent. Then  $pAp = \{pap : a \in A\}$  is a closed subalgebra of  $A$  called a reduced Banach algebra.

## 3. REDUCED BANACH ALGEBRAS

Let  $A$  be a Banach algebra and  $p \in A$  be an idempotent element, that is,  $p = p^2$ . We shall always assume that  $p$  is non-trivial, that is,  $p \neq 0$  and  $p \neq 1$ . Let  $q = 1 - p$ . Now, suppose  $a \in A$  is such that  $ap = pa$ . Then observe that  $qa = aq$ . We observe that  $\|p\|, \|q\| \geq 1$ . We examine the relationships between the spectrum and pseudospectrum of  $a \in A$  and the spectra and pseudospectra of  $pa = pap \in pAp$  and  $qa = qa \in qAq$ .

We show that  $\sigma(A, a) = \sigma(pAp, pa) \cup \sigma(qAq, qa)$ . This is an elementary result and a version of it in the case of Hilbert space operators can be found in Problem 98 of [6]. We provide the proof here for the sake of completeness.

**Lemma 3.1.** *Let  $A$  be a unital Banach algebra,  $p \in A$  be an idempotent, and  $q = 1 - p$ . Let  $a \in A$  be such that  $ap = pa$ . Then*

$$\sigma(A, a) = \sigma(pAp, pa) \cup \sigma(qAq, qa).$$

*Thus,  $r(a) = \max\{r(pAp, pa), r(qAq, qa)\}$ .*

*Proof.* Without loss of generality, we assume that the element considered in the spectrum or union of spectra is  $\lambda = 0$ .

If  $a$  is invertible in  $A$ , then  $pa$  is invertible in  $pAp$  with inverse  $pa^{-1}$  and  $qa$  is invertible in  $qAq$  with inverse  $qa^{-1}$ . Hence

$$\sigma(pAp, pa) \cup \sigma(qAq, qa) \subseteq \sigma(A, a).$$

Conversely, suppose  $pa$  is invertible in  $pAp$  with inverse  $pbp$  and  $qa$  is invertible in  $qAq$  with inverse  $qcq$ . Then

$$pa(pbp) = pabp = p = pbap = (pbp)ap, \quad (3.1)$$

and

$$qacq = q = qcaq. \quad (3.2)$$

Adding (3.1) and (3.2) gives

$$pabp + qacq = p + q = pbap + qcaq.$$

Simplifying, we get

$$a(pbp + qcq) = 1 = (pbp + qcq)a.$$

Hence  $\sigma(A, a) \subseteq \sigma(pAp, pa) \cup \sigma(qAq, qa)$ . We also observe that the inverses of  $\lambda p - pa$  and  $\lambda q - qa$ , when they exist, in  $pAp$  and  $qAq$  respectively are  $(\lambda - a)^{-1}p$  and  $(\lambda - a)^{-1}q$  respectively.  $\square$

Next, suppose  $p_1, \dots, p_n \in A$  such that  $p_i^2 = p_i \forall i$  and  $\sum_{i=1}^n p_i = 1$ . Suppose  $ap_i = p_i a \forall i$ . Then just as in the case of two idempotents, it can be easily shown that  $\sigma(A, a) = \bigcup_{i=1}^n \sigma(p_i A p_i, p_i a)$ . The same is mentioned in the case of Hilbert space operators in Problem 98 of [6].

**Lemma 3.2.** *Let  $A$  be a unital Banach algebra and  $p_1, \dots, p_n \in A$  such that  $p_i^2 = p_i \forall i$  and  $\sum_{i=1}^n p_i = 1$ . Suppose  $ap_i = p_i a \forall i$ . Then*

$$\sigma(A, a) = \bigcup_{i=1}^n \sigma(p_i A p_i, p_i a),$$

and

$$r(A, a) = \max_i r(p_i A p_i, p_i a).$$

We now consider the decomposition of the pseudospectrum into the pseudospectra of the corresponding elements of the reduced Banach algebras.

**Theorem 3.3.** *Let  $A$  be a unital Banach algebra and  $p_1, \dots, p_n \in A$  such that  $p_i^2 = p_i \forall i$  and  $\sum_{i=1}^n p_i = 1$ . Suppose  $ap_i = p_i a \forall i$ . Let  $\varepsilon > 0$  and  $K = \max_i \|p_i\|$ . Then*

$$\Lambda_{\frac{\varepsilon}{n}}(A, a) \subseteq \bigcup_{i=1}^n \Lambda_{\varepsilon}(p_i A p_i, p_i a) \subseteq \Lambda_{K\varepsilon}(A, a).$$

*Proof.* Let  $\varepsilon > 0$  and  $\lambda \in \bigcup_{i=1}^n \Lambda_\varepsilon(p_i A p_i, p_i a)$ . If  $\lambda \in \sigma(p_i A p_i, p_i a)$  for some  $i$ , then  $\lambda \in \sigma(A, a) \subseteq \Lambda_\varepsilon(A, a) \subseteq \Lambda_{K\varepsilon}(A, a)$ . On the other hand, if  $\lambda p_i - p_i a$  is invertible for all  $i$ , then  $\lambda - a$  is invertible. We have  $\lambda \in \Lambda_\varepsilon(p_i A p_i, p_i a)$  for some  $i$ . Hence  $\|p_i(\lambda - a)^{-1}\| = \|(\lambda p_i - p_i a)^{-1}\| \geq \frac{1}{\varepsilon}$ . Hence  $\|(\lambda - a)^{-1}\| \geq \frac{1}{\|p_i\| \varepsilon} \geq \frac{1}{K\varepsilon}$ . Thus

$$\bigcup_{i=1}^n \Lambda_\varepsilon(p_i A p_i, p_i a) \subseteq \Lambda_{K\varepsilon}(a).$$

Next, suppose  $\lambda \notin \Lambda_\varepsilon(p_i A p_i, p_i a) \forall i$ . Then each  $\lambda p_i - p_i a$  is invertible with inverse  $p_i b_i p_i$ , say, and  $\|p_i b_i p_i\| < \frac{1}{\varepsilon}$ . Then  $\lambda - a$  is invertible with inverse  $\sum_{i=1}^n p_i b_i p_i$ , and further,

$$\begin{aligned} \|(\lambda - a)^{-1}\| &= \left\| \sum_{i=1}^n p_i b_i p_i \right\| \\ &< n \frac{1}{\varepsilon}. \end{aligned}$$

Hence, we get

$$\Lambda_{\frac{\varepsilon}{n}}(A, a) \subseteq \bigcup_{i=1}^n \Lambda_\varepsilon(p_i A p_i, p_i a) \subseteq \Lambda_{K\varepsilon}(A, a).$$

□

Now, going back to the case  $n = 2$ , suppose  $\|p\| = \|q\| = 1$ . For example, this happens if  $p$  is a non-trivial Hermitian idempotent.

**Corollary 3.4.** *Let  $A$  be a unital Banach algebra,  $p \in A$  be an idempotent and  $q = 1 - p$ . Let  $a \in A$  be such that  $ap = pa$ . Suppose  $\|p\| = \|q\| = 1$ . Then*

$$\Lambda_{\frac{\varepsilon}{2}}(A, a) \subseteq \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) \subseteq \Lambda_\varepsilon(A, a). \quad (3.3)$$

We examine the question of when equality occurs in the second inclusion of (3.3). It is easily observed that equality occurs precisely when

$$\|(\lambda - a)^{-1}\| = \max \{ \|p(\lambda - a)^{-1}\|, \|q(\lambda - a)^{-1}\| \}.$$

**Definition 3.5.** An element  $a \in A$  is said to be a  $G_1$ -element if it satisfies the following equality:

$$\|(z - a)^{-1}\| = \frac{1}{d(z, \sigma(a))} = r((z - a)^{-1}) \quad \forall z \in \mathbb{C} \setminus \sigma(a).$$

See [12].

Now, suppose  $a \in A$  commutes with  $p$ , and in addition,  $a$  is a  $G_1$  element. Then  $\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) = \Lambda_\varepsilon(A, a)$

**Theorem 3.6.** *Let  $A$  be a unital Banach algebra,  $p \in A$  be an idempotent and  $q = 1 - p$ . Suppose  $\|p\| = \|q\| = 1$ . Let  $a \in A$  be such that  $ap = pa$ , and suppose  $a$  is a  $G_1$ -element. Then*

$$\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) = \Lambda_\varepsilon(A, a).$$

*Proof.* The proof follows since

$$\begin{aligned}
\|(\lambda - a)^{-1}\| &= r((\lambda - a)^{-1}) \\
&= \max \{r(pAp, p(\lambda - a)^{-1}), r(qAq, q(\lambda - a)^{-1})\} \\
&\leq \max \{\|p(\lambda - a)^{-1}\|, \|q(\lambda - a)^{-1}\|\} \\
&\leq \|(\lambda - a)^{-1}\|.
\end{aligned}$$

□

We next consider the case that  $A$  is a  $C^*$ -algebra and  $p \in A$  is a self-adjoint projection. We observe that the notion of a Hermitian idempotent coincides with that of a self-adjoint projection in  $C^*$ -algebras.

**Theorem 3.7.** *Let  $A$  be a unital  $C^*$ -algebra and  $p \in A$  be a self-adjoint projection. Let  $q = 1 - p$ . Let  $a \in A$  be such that  $ap = pa$ . Then*

$$\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa) = \Lambda_\varepsilon(A, a).$$

*Proof.* Without loss of generality, assume that  $\lambda = 0$ . Then

$$\begin{aligned}
\|a^{-1}\|^2 &= \|(a^{-1})^*a^{-1}\| \\
&= r(A, (a^{-1})^*a^{-1}) \\
&= \max \{r(pAp, p(a^{-1})^*a^{-1}p), r(qAq, q(a^{-1})^*a^{-1}q)\} \\
&= \max \{r(pAp, (a^{-1}p)^*(a^{-1}p)), r(qAq, (a^{-1}q)^*(a^{-1}q))\} \\
&\leq \max \{\|(a^{-1}p)^*(a^{-1}p)\|, \|(a^{-1}q)^*(a^{-1}q)\|\} \\
&= \max \{\|pa^{-1}\|^2, \|qa^{-1}\|^2\}
\end{aligned}$$

□

*Remark 3.8.* Theorem 3.7 is stated for the finite dimensional case in (iv) of Theorem 2.4 of [17].

It is easy to see from Lemma 3.2 that Theorems 3.6 and 3.7 can be generalised to the case of finitely many idempotents of norm one satisfying  $\sum_{i=1}^n p_i = 1$ .

**Example 3.9.** We give an example in which  $\|p\| = \|q\| = 1$ , but equality does not occur in the second inclusion in (3.3). Let  $A = (\mathbb{C}^{2 \times 2}, \|\cdot\|_1)$  and  $p = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . Let  $q = 1 - p$ . Then  $p^2 = p$  and  $\|p\| = 1 = \|q\|$ . Now, if  $a \in A$  commutes with  $p$ , it is necessarily of the form  $\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$ , and  $pap = \frac{1}{2}(\alpha + \beta) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ .

If  $pap$  is invertible in  $pAp$ , then its inverse is equal to  $\frac{1}{2(\alpha+\beta)} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . Similarly, if  $qaq$  is invertible in  $qAq$ , then its inverse is equal to  $\frac{1}{2(\alpha-\beta)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ .

The following can be computed:  $\sigma(a) = \{\alpha + \beta, \alpha - \beta\}$ ,  $\|(\lambda - a)^{-1}\| = \frac{|\beta| + |\lambda - \alpha|}{|(\lambda - \alpha)^2 - \beta^2|}$ ,  $\|p(\lambda - \alpha)^{-1}\| = \frac{1}{|\lambda - \alpha - \beta|}$ , and  $\|q(\lambda - a)^{-1}\| = \frac{1}{|\lambda - \alpha + \beta|}$ .

Choosing  $\alpha = 0, \beta = 1$ , we get  $\|(\lambda - a)^{-1}\| = \frac{1+|\lambda|}{|\lambda^2-1|}$ ,  $\|p(\lambda - a)^{-1}\| = \frac{1}{|\lambda-1|}$ , and  $\|q(\lambda - a)^{-1}\| = \frac{1}{|\lambda+1|}$ .

Now, let  $\varepsilon = 1$ . Then it can be verified that  $\lambda = i \in \Lambda_\varepsilon(A, a)$  but  $i \notin \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$ . Hence in this example,  $\Lambda_\varepsilon(A, a)$  is strictly larger than  $\Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$ .

*Remark 3.10.* Let  $X$  be a Banach space and  $P \in B(X)$  be an idempotent operator. Let  $Q = I - P$ . Then  $X = \text{Range } P \oplus \text{Range } Q$ . If  $T \in B(X)$  is an operator such that  $PT = TP$ , then both  $\text{Range } P$  and  $\text{Range } Q$  are invariant under  $T$ . Hence  $T = T_1 \oplus T_2$ , where  $T_1 = TP \upharpoonright_{\text{Range } P}$  and  $T_2 = TQ \upharpoonright_{\text{Range } Q}$ . Similarly,  $(\lambda - T)^{-1} = (\lambda - T)^{-1}P \upharpoonright_{\text{Range } P} \oplus (\lambda - T)^{-1}Q \upharpoonright_{\text{Range } Q}$ . We use the following theorem due to Lancaster and Farahat (See Theorem 2 of [11]):

Let  $X = X_1 \oplus \dots \oplus X_n$  be a direct sum of  $n$  Banach spaces. Let  $T = T_1 \oplus \dots \oplus T_n$  be a direct sum of operators acting on  $X$ . Then

$$\|T\| = \max_i \|T_i\|$$

iff the norm on  $X$  is absolute, i.e.

$$\|x_i\| = \|y_i\| \quad \forall i \implies \|x\| = \|y\|,$$

where  $x = (x_i)$  and  $y = (y_i)$ .

Hence, in order to check whether  $\|(\lambda - T)^{-1}\| = \max \{\|(\lambda - T)^{-1}P\|, \|(\lambda - T)^{-1}Q\|\}$ , it suffices to check whether  $\|x\|$ , the norm of  $x \in X$  is a function of  $\|Px\|$  and  $\|Qx\|$ .

We see some examples of idempotent operators which satisfy the above norm condition.

**Example 3.11.** Let  $A = (\mathbb{C}^{n \times n}, \|\cdot\|_r)$ ,  $1 \leq r \leq \infty$ ,  $r \neq 2$ . Here,  $\|(x_n)\|_r = (\sum_{i=1}^n |x_n|^r)^{\frac{1}{r}}$  for  $r \neq \infty$  and  $\|(x_n)\|_\infty = \max_n |x_n|$ . The Hermitian idempotent elements of  $A$  can be shown to be diagonal matrices with diagonal entries equal to 0 or 1. Without any change in the norm, we may assume that  $k$  1-s appear first as diagonal entries, followed by  $(n - k)$  0-s. Let  $P$  be such a Hermitian idempotent matrix. If  $T \in A$  commutes with  $P$ , it must be a block diagonal matrix.

It can be checked that  $\|\cdot\|_r$  is an absolute norm on the decomposition of  $\mathbb{C}^n$  into the ranges of  $P$  and  $Q$  in this case. Thus the induced operator norm has the maximum property. This yields  $\|(\lambda - T)^{-1}\| = \max \{\|P(\lambda - T)^{-1}\|, \|Q(\lambda - T)^{-1}\|\}$ ,  $\lambda \notin \sigma(A, T)$ . Hence  $\Lambda_\varepsilon(A, T) = \Lambda_\varepsilon(PAP, PT) \cup \Lambda_\varepsilon(QAQ, QT)$ .

**Example 3.12.** Let  $\Omega$  be a compact Hausdorff space and  $A = C(\Omega)$ , the space of continuous functions on  $X$ . Then an operator  $P \in B(A)$  is Hermitian iff it is of the form:

$$Pf = hf, \quad f \in C(\Omega), \quad h \text{ a real valued function.}$$

(See Theorem 6.29.3 of [3].) Further, if we want  $h$  to be an idempotent continuous function, it must necessarily be a characteristic function of a connected component of  $\Omega$ , say  $\Delta$ . Hence  $\text{Range } P = \{\chi_\Delta f : f \in C(\Omega)\}$  and  $\text{Range } Q =$



Range  $(I - P) = \{\chi_{\Delta^c} f : f \in C(\Omega)\}$ . Clearly, for  $f \in C(\Omega)$ ,  $\|f\|_\infty = \sup_{\omega \in \Omega} |f(\omega)| = \max\{\sup_{\omega \in \Delta} |f(\omega)|, \sup_{\omega \in \Delta^c} |f(\omega)|\} = \max\{\|\chi_{\Delta} f\|_\infty, \|\chi_{\Delta^c} f\|_\infty\} = \max\{\|Pf\|, \|Qf\|\}$ .

Thus if  $\|Pf\| = \|Pg\|$  and  $\|Qf\| = \|Qg\|$ , then  $\|f\| = \|g\|$ . Hence the induced operator norm has the required maximum property. Note that if such a  $P$  exists,  $\Omega$  must be disconnected.

We observe next that if the norm  $\|x\|$  on the Banach space  $X$  is a function of  $\|Px\|$  and  $\|Qx\|$ , then  $P$  (and thus  $Q$ ) must necessarily be Hermitian.

**Theorem 3.13.** *Let  $X$  be a Banach space and  $P \in B(X)$  be an idempotent operator. Let  $Q = I - P$ . Suppose  $\exists f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that*

$$\|x\| = f(\|Px\|, \|Qx\|) \quad \forall x \in X.$$

*Then  $P$  is Hermitian. Further, if  $T \in B(X)$  such that  $PT = TP$ , then for  $\varepsilon > 0$ ,*

$$\Lambda_\varepsilon(A, T) = \Lambda_\varepsilon(PAP, PT) \cup \Lambda_\varepsilon(QAQ, QT).$$

*Proof.* By Corollary 1.10.13 of [2], it suffices to show that  $\|e^{itP}\| = 1 \quad \forall t \in \mathbb{R}$ . Now,  $\|e^{itP}x\| = \|(I + P(e^{it} - 1))x\|$  since  $P$  is an idempotent. By the hypothesis we get

$$\begin{aligned} \|e^{itP}x\| &= f(\|P(I + P(e^{it} - 1))x\|, \|Q(I + P(e^{it} - 1))x\|) \\ &= f(\|(Pe^{it})x\|, \|Qx\|) \\ &= f(\|Px\|, \|Qx\|) \\ &= \|x\| \quad \forall t \in \mathbb{R}. \end{aligned}$$

Hence, in fact, we get that  $e^{itP}$  is an isometry for every real  $t$ , and thus  $P$  is Hermitian. Hence,  $Q$  is also Hermitian and  $\|P\| = \|Q\| = 1$ . It then follows by the theorem of Lancaster and Farahat ([11]) that

$$\|(\lambda - T)^{-1}\| = \max \{ \|(\lambda - T)^{-1}P\|, \|(\lambda - T)^{-1}Q\| \}.$$

Hence  $\Lambda_\varepsilon(A, T) = \Lambda_\varepsilon(PAP, PT) \cup \Lambda_\varepsilon(QAQ, QT)$ . Examples of idempotent operators that satisfy this condition are M- projections and L-projections (See [7]).  $\square$

**Theorem 3.14.** *Let  $A$  be a Banach algebra and  $p \in A$  be an idempotent element. Let  $q = 1 - p$ . Suppose  $\exists f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that*

$$\|a\| = f(\|pa\|, \|qa\|) \quad \forall a \in A.$$

*Then  $p$  is Hermitian and*

$$f(\lambda, \mu) = \max \{ \lambda, \mu \} \quad \lambda, \mu \in \mathbb{R}^+.$$

*Further, if  $a \in A$  commutes with  $p$ , then for  $\varepsilon > 0$ ,  $\Lambda_\varepsilon(A, a) = \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$ .*

*Proof.* We first show that  $p$  must be a Hermitian element. By Corollary 1.10.13 of [2], It suffices to show that  $\|e^{itp}\| = 1 \quad \forall t \in \mathbb{R}$ . Since  $p$  is idempotent and by

the hypothesis, we have  $\forall t \in \mathbb{R}$ ,

$$\begin{aligned} \|e^{itp}\| &= \|1 + p(e^{it} - 1)\| \\ &= f(\|p(1 + p(e^{it} - 1))\|, \|q(1 + p(e^{it} - 1))\|) \\ &= f(\|p\|, \|q\|) \\ &= \|1\| \\ &= 1. \end{aligned}$$

Hence  $p$  is Hermitian, and thus  $\|p\| = 1 = \|q\|$ .

Now, let  $\lambda, \mu > 0$ . Let  $x = \lambda p + \mu q$ . Then  $px = \lambda p$  and  $qx = \mu q$ . We have

$$\begin{aligned} f(\lambda, \mu) &= f(\|px\|, \|qx\|) \\ &= \|x\| \\ &= \|(\lambda - \mu)p + \mu\| \\ &= r((\lambda - \mu)p + \mu) \\ &= \max\{\lambda, \mu\}. \end{aligned}$$

The equality of spectral radius and norm of  $(\lambda - \mu)p + \mu$  follows because  $p$  is Hermitian (See Proposition 2 of [16]). It then follows that  $\|(\lambda - a)^{-1}\| = \max\{\|p(\lambda - a)^{-1}\|, \|q(\lambda - a)^{-1}\|\}$ . By this observation and by Lemma 3.1, it follows that if  $ap = pa$ , then  $\Lambda_\varepsilon(A, a) = \Lambda_\varepsilon(pAp, pa) \cup \Lambda_\varepsilon(qAq, qa)$ .  $\square$

However, in general it is not true that if  $P$  is a Hermitian operator on a Banach space, then  $\|Px\| = \|Py\|$  and  $\|Qx\| = \|Qy\| \implies \|x\| = \|y\|$ . We give an example of a Hermitian idempotent operator on a  $C^*$  algebra, for which the hypothesis of 3.13 does not hold.

**Example 3.15.** Let  $X = \mathbb{C}^{2 \times 2}$ , the  $C^*$  algebra endowed with the norm induced by the Euclidean norm on  $\mathbb{C}^2$  and  $p \in X$  be the Hermitian idempotent matrix  $p = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . Then  $q = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ . Let  $L_p \in B(X)$  be the left multiplication operator by  $p$  on  $A$ . Then  $L_p$  is a Hermitian idempotent operator on  $X$ . Let  $L_q$  be the left multiplication by  $q$  on  $X$ . Let  $x = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$  and  $y = \frac{1}{2} \begin{bmatrix} 5 & 5 \\ 3 & 1 \end{bmatrix}$ . Then one can check that  $\|L_p x\| = \|L_p y\| = \frac{5}{\sqrt{2}}$  and  $\|L_q x\| = \|L_q y\| = \sqrt{\frac{5}{2}}$ , but  $\|x\| = \sqrt{\frac{(3+\sqrt{5})5}{2}} \neq \sqrt{\frac{15+\sqrt{200}}{2}} = \|y\|$ .

We now examine the question of equality of pseudospectrum of  $a \in A$  and the union of the pseudospectra of  $pa \in pAp$  and  $qa \in qAq$  in Banach algebras with a special property.

**Theorem 3.16.** *Let  $A$  be a Banach algebra which satisfies the following:*

$$\Lambda_\varepsilon(A, a) = \bigcup_{\|b\| \leq \varepsilon} \sigma(A, a + b) \quad \forall a \in A.$$

This is equivalent to:

$$\forall a \in \text{Inv}(A), \exists b \in \text{Sing}(A) \text{ such that } \|a - b\| = \frac{1}{\|a^{-1}\|}.$$

(See Corollary 2.7 of [9].)

Let  $p \in A$  be a central idempotent element such that  $\|p\| = \|q\| = 1$ , where  $q = 1 - p$ . Then  $pAp = pA$  and

$$\Lambda_\varepsilon(A, a) = \Lambda_\varepsilon(pA, pa) \cup \Lambda_\varepsilon(qA, qa) \quad \forall a \in A.$$

*Proof.* Let  $a \in A$ . Then

$$\begin{aligned} \Lambda_\varepsilon(A, a) &= \bigcup_{\|b\| \leq \varepsilon} \sigma(A, a + b) \\ &= \bigcup_{\|b\| \leq \varepsilon} \sigma(pA, p(a + b)) \cup \bigcup_{\|b\| \leq \varepsilon} \sigma(qA, q(a + b)) \\ &\subseteq \bigcup_{\|pb\| \leq \varepsilon} \sigma(pA, p(a + b)) \cup \bigcup_{\|qb\| \leq \varepsilon} \sigma(qA, q(a + b)) \\ &\subseteq \Lambda_\varepsilon(pA, pa) \cup \Lambda_\varepsilon(qA, qa). \end{aligned}$$

The reverse inclusion is already true, hence the theorem follows. It is easy to see from Lemma 3.2 that Theorem 3.16 can be generalised to the case in which there exists finitely many central idempotents of norm one satisfying  $\sum_{i=1}^n p_i = 1$ .  $\square$

We now extend some of the above results for a finite family of idempotents in a Banach algebra.

**Theorem 3.17.** *Let  $X$  be a Banach space and  $\{P_i\}_{i=1}^n$  be a finite family of idempotents in  $A = B(X)$  such that  $P_i P_j = 0$  if  $i \neq j$ . Suppose  $\exists f : (\mathbb{R}^+)^n \rightarrow \mathbb{R}$  such that*

$$\|x\| = f(\|P_1 x\|, \dots, \|P_n x\|) \quad \forall x \in X.$$

*Then each  $P_i$  is Hermitian. Further, if  $T \in B(X)$  such that  $P_i T = T P_i \forall i$ , then for  $\varepsilon > 0$ ,*

$$\Lambda_\varepsilon(A, T) = \bigcup_{i=1}^n \Lambda_\varepsilon(P_i A P_i, P_i T).$$

*Proof.* For  $i = 1$ , say and  $t \in \mathbb{R}$ ,

$$\begin{aligned} \|e^{itP_1} x\| &= \|(1 + P_1(e^{it} - 1))x\| \\ &= f(\|(P_1 e^{it})x\|, \dots, \|(P_n x)\|) \\ &= f(\|P_1 x\|, \dots, \|P_n x\|) \\ &= \|x\|. \end{aligned}$$

Similarly each  $e^{itP_i}$  is an isometry and hence  $P_i$  is Hermitian, with  $\|P_i\| = 1$ . It follows by the theorem of Lancaster and Farahat ([11]) that

$$\|(\lambda - T)^{-1}\| = \max_i \{ \|(\lambda - T)^{-1} P_i\| \}.$$

$\square$

**Theorem 3.18.** *Let  $A$  be a Banach algebra and  $\{p_i\}_{i=1}^n$  be a finite family of idempotent elements of  $A$  such that  $p_i p_j = 0$  if  $i \neq j$ . Suppose  $\exists f : (\mathbb{R}^+)^n \rightarrow \mathbb{R}$  such that*

$$\|a\| = f(\|p_1 a\|, \dots, \|p_n a\|) \quad \forall a \in A.$$

*Then each  $p_i$  is Hermitian and*

$$f(\lambda_1, \dots, \lambda_n) = \max \{\lambda_1, \dots, \lambda_n\}, \quad \lambda_i \in \mathbb{R}^+.$$

*Further, if  $a \in A$  commutes with each  $p_i$ , then*

$$\Lambda_\varepsilon(A, a) = \bigcup_{i=1}^n \Lambda_\varepsilon(p_i A p_i, p_i a).$$

*Proof.* For  $i = 1$ , say, and  $t \in \mathbb{R}$ ,

$$\begin{aligned} \|e^{itp_1}\| &= \|1 + p_1(e^{it} - a)\| \\ &= f(\|(p_1 e^{it})\|, \dots, \|p_n\|) \\ &= f(\|p_1\|, \dots, \|p_n\|) \\ &= \|1\| \\ &= 1. \end{aligned}$$

Similarly, each  $p_i$  is Hermitian and  $\|p_i\| = 1$ . Next, let  $x = \sum_{i=1}^n \lambda_i p_i$ ,  $\lambda_i > 0$ . Then  $p_i x = \lambda_i p_i \quad \forall i$ . Then

$$\begin{aligned} \lambda_i &= \|\lambda_i p_i\| \\ &= \|p_i x\| \\ &\leq \|x\| \\ &= \left\| \sum_{i=1}^n \lambda_i p_i \right\| \\ &= f(\|\lambda_1 p_1\|, \dots, \|\lambda_n p_n\|) \\ &= f(\lambda_1, \dots, \lambda_n) \quad \forall i. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(\lambda_1, \dots, \lambda_n) &= \left\| \sum_{i=1}^n \lambda_i p_i \right\| \\ &= r\left(\sum_{i=1}^n \lambda_i p_i\right) \\ &\leq \max_i \{\lambda_i\}, \end{aligned}$$

the last inequality following because  $\sigma\left(\sum_{i=1}^n \lambda_i p_i\right) \subseteq \{\lambda_1, \dots, \lambda_n\}$ , due to the fact that  $p_i p_j = 0$ ,  $i \neq j$ . Hence  $\|(\lambda - a)^{-1}\| = \max_i \|p_i (\lambda - a)^{-1}\|$  and thus  $\Lambda_\varepsilon(A, a) =$

$$\bigcup_{i=1}^n \Lambda_\varepsilon(p_i A p_i, p_i a). \quad \square$$

## 4. THE INFINITE CASE

We now examine the above results for an arbitrary family of idempotents  $\{p_\alpha\}_{\alpha \in I}$  in a Banach algebra. We first consider the case  $A = B(H)$  for a Hilbert space  $H$ . The following theorem is elementary and has been discussed in Theorem 2.3 of [4]. We provide the proof here for the convenience of the reader.

**Theorem 4.1.** *Let  $H$  be a Hilbert space and  $\{P_n\}_{n \in \mathbb{N}}$  be a family of self-adjoint idempotent operators in  $A = B(H)$ , such that  $(\sum_{n \in \mathbb{N}} P_n)h = h \forall h \in H$ , that is, the convergence of the series is in the strong operator topology. Suppose  $P_n P_m = 0$  if  $n \neq m$ . Then  $H$  can be written as the direct sum  $\bigoplus_{n \in \mathbb{N}} P_n H$ . Let  $T \in B(H)$  be an operator that commutes with each  $P_n$ . Let  $H_n = P_n H$  and  $T_n = TP_n \upharpoonright_{H_n} \in B(H_n)$ . Then*

$$\sigma(A, T) = \bigcup_{n \in \mathbb{N}} \sigma(P_n A P_n, T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty\}.$$

*Proof.* We recall that for an operator  $T = \bigoplus_n T_n$  on a direct sum of Hilbert spaces,  $\|T\| = \sup_n \|T_n\|$  (See Exercise II.1.12 of [5]). Suppose  $\lambda - T$  is invertible in  $A$ . Then clearly,  $\lambda P_n - T_n$  is invertible in  $P_n A P_n \forall n$  with inverses  $P_n(\lambda - T)^{-1}$ , and  $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \|(\lambda - T)^{-1}\| < \infty$ . Conversely, suppose  $(\lambda P_n - T_n)$  is invertible in  $P_n A P_n$  for all  $n$  with inverses  $(\lambda P_n - T_n)^{-1}$ , and  $\sup_n \|(\lambda P_n - T_n)^{-1}\| < \infty$ , then  $\bigoplus_n (\lambda P_n - T_n)^{-1}$  is the inverse of  $\lambda - T$  in  $A$ .  $\square$

We now look at  $B(X)$  for a Banach space  $X$ . As in the Hilbert space case, let  $\{P_n\}_{n \in \mathbb{N}}$  be a family of idempotent operators in  $B(X)$ , such that  $(\sum_{n \in \mathbb{N}} P_n)x = x \forall x \in X$  (that is, the series  $\sum_{n \in \mathbb{N}} P_n$  converges to the identity operator in the strong operator topology) and  $P_n P_m = 0$  if  $n \neq m$ . Let  $X_n = P_n X$ , and  $x_n = P_n x$  for  $x \in X$ . For a direct sum of  $X_n$  to make sense, we must specify the subspace of  $\prod_n X_n$  and the norm on it that we intend to use (See page 72 of [5]). Most often, we use either the  $r$ -norm ( $1 \leq r < \infty$ ) or the supremum norm. That is,  $\bigoplus_n X_n = \{(x_n) : \sum_{n=1}^{\infty} \|x_n\|^r < \infty\}$  and  $\|x\| = \|(x_n)\| = (\sum_{n=1}^{\infty} \|x_n\|^r)^{1/r}$ ,  $1 \leq r < \infty$  and  $\bigoplus_n X_n = \{(x_n) : \sup_n \|x_n\| < \infty\}$  and  $\|x\| = \|(x_n)\| = \sup_n \|x_n\|$ . We note that in each of these cases, if  $T_n \in B(X_n)$ , the operator  $T = \bigoplus_n T_n \in B(X)$  iff  $\sup_n \|T_n\| < \infty$ . For  $T \in B(X)$ , let  $T_n = TP_n \upharpoonright_{X_n} \in B(X_n)$ . Examples of such Banach spaces are  $l_p$  spaces or more generally,  $l_p$  direct sums of Banach spaces (See [15]).

The following result is an elementary generalisation of Theorem 4.1 for operators on direct sums of Banach spaces.

**Theorem 4.2.** *Let  $X$  be a Banach space and  $\{P_n\}_{n \in \mathbb{N}}$  be a family of idempotent operators in  $A = B(X)$  satisfying the above conditions. Let  $T \in B(X)$  be an operator that commutes with each  $P_n$ . Suppose the norm of  $x \in X$  is a function  $f$  of  $(\|x_n\|)$  and satisfies the following monotone condition:  $\|x_n\| \leq \|y_n\| \forall n \implies$*

$\|x\| \leq \|y\|$ ,  $x, y \in X$  and  $x_n = P_n x, y_n = P_n y$ . Let  $X_n = P_n X$  and  $T_n = TP_n \upharpoonright_{X_n}$ . Then each  $P_n$  is Hermitian and

$$\sigma(A, T) = \bigcup_{n \in \mathbb{N}} \sigma(P_n A P_n, T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty\}.$$

*Proof.* Let  $x = (x_n) \in X$ . Let  $t \in \mathbb{R}$ . For  $i = 1$ , say

$$\begin{aligned} \|e^{itP_1}x\| &= \|(I + P_1(e^{it} - T))x\| \\ &= f(\|(P_1 e^{it}x\|, \|P_2x\|, \dots)) \\ &= f(\|P_1x\|, \|P_2x\|, \dots) \\ &= \|x\|. \end{aligned}$$

Similarly, each  $e^{itP_i}$  is an isometry whence each  $P_i$  is Hermitian and  $\|P_i\| = 1 \forall i$ . Now,

$$\begin{aligned} \|Tx\| &= \|(T_n x_n)\| \\ &\leq \|(\|T_n\|x_n)\| \\ &\leq \|(\sup_n \|T_n\|x_n)\| \\ &= \|(\sup_n \|T_n\|)(x_n)\| \\ &= \sup_n \|T_n\| \|x\|. \end{aligned}$$

Clearly,  $\|T_n\| \leq \|T\| \forall n$ . Hence  $\|T\| = \sup_n \|T_n\|$ , and the theorem then follows as in Theorem 4.1.  $\square$

We next move on to the pseudospectra decomposition. We recall that some authors (such as in [1] and [17]) have defined the following set as the  $\varepsilon$ -pseudospectrum of  $a$ :

$$\Lambda_\varepsilon^*(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| > \varepsilon^{-1}\}.$$

We first give a decomposition for the  $\varepsilon$ -pseudospectrum defined this way.

**Theorem 4.3.** *Let  $X$  be a Banach space and  $\{P_n\}_{n \in \mathbb{N}}$  be a family of idempotent operators in  $A = B(X)$  as above. Let  $T \in B(X)$  be an operator that commutes with each  $P_n$ . Suppose the norm on  $X$  satisfies the following monotone condition:  $\|x_n\| \leq \|y_n\| \forall n \implies \|x\| \leq \|y\|$ ,  $x, y \in X$  and  $x_n = P_n x, y_n = P_n y$  as before. Let  $X_n = P_n X$  and  $T_n = TP_n \upharpoonright_{X_n}$ . Let  $\varepsilon > 0$ . Then*

$$\Lambda_\varepsilon^*(A, T) = \bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon^*(P_n A P_n, T_n).$$

*Proof.* Let  $\lambda \in \Lambda_\varepsilon^*(A, T)$ . If  $\lambda \in \sigma(A, T)$ , then either  $\lambda \in \bigcup_n \sigma(P_n A P_n, T_n)$  or  $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$ . If  $\lambda \in \sigma(P_{n_0} A P_{n_0}, T_{n_0})$  for some  $n_0$ , then  $\lambda \in \bigcup_n \Lambda_\varepsilon^*(P_n A P_n, T_n)$ . If  $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$ , then  $\exists n_0$  such that  $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}$ , hence  $\lambda \in \bigcup_n \Lambda_\varepsilon^*(P_n A P_n, T_n)$ . If  $\lambda - T$  is invertible and  $\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}$ , then  $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}$  for some  $n_0$ , hence  $\lambda \in \Lambda_\varepsilon^*(P_{n_0} A P_{n_0}, T_{n_0})$ .

For the other inclusion, let  $\lambda \in \bigcup_n \Lambda_\varepsilon^*(P_n A P_n, T_n)$ . If  $\lambda \in \bigcup_n \sigma(P_n A P_n, T_n)$ , then  $\lambda \in \sigma(A, T) \subseteq \Lambda_\varepsilon^*(A, T)$ . If  $\lambda - T_n$  is invertible  $\forall n$  and  $\sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| < \infty$ , then  $\|(\lambda - T)^{-1}\| > \frac{1}{\varepsilon}$ . If  $\sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty$ , then  $\lambda \in \sigma(A, T) \subseteq \Lambda_\varepsilon^*(A, T)$ .  $\square$

Now, let us consider the decomposition of the pseudospectrum defined with the non-strict inequality.

**Theorem 4.4.** *Let  $X$  be a Banach space and  $\{P_n\}_{n \in \mathbb{N}}$  be a family of idempotent operators in  $A = B(X)$  as above. Let  $T \in B(X)$  be an operator that commutes with each  $P_n$ . Suppose the norm on  $X$  satisfies the following monotone condition:  $\|x_n\| \leq \|y_n\| \forall n \implies \|x\| \leq \|y\|$ ,  $x, y \in X$  and  $x_n = P_n x, y_n = P_n y$  as before. Let  $X_n = P_n X$  and  $T_n = T P_n \upharpoonright_{X_n}$ . Let  $\varepsilon > 0$ . Then*

$$\Lambda_\varepsilon(A, T) = \bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(P_n A P_n, T_n) \cup \left\{ \lambda \in \mathbb{C} : \sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \frac{1}{\varepsilon} \right\}.$$

*Proof.* Let  $\lambda \in \Lambda_\varepsilon(A, T)$ . If  $\lambda \in \sigma(A, T)$ , then either  $\lambda \in \bigcup_n \sigma(P_n A P_n, T_n)$  or  $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$ . If  $\lambda \in \sigma(P_{n_0} A P_{n_0}, T_{n_0})$  for some  $n_0$ , then  $\lambda \in \bigcup_n \Lambda_\varepsilon(P_n A P_n, T_n)$ . If  $\sup_n \|(\lambda P_n - T_n)^{-1}\| = \infty$ , then  $\exists n_0$  such that  $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| \geq \frac{1}{\varepsilon}$ , hence  $\lambda \in \bigcup_n \Lambda_\varepsilon(P_n A P_n, T_n)$ . If  $\lambda - T$  is invertible and  $\|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon}$ , then either  $\|(\lambda P_{n_0} - T_{n_0})^{-1}\| > \frac{1}{\varepsilon}$  for some  $n_0$  or  $\|(\lambda - T)^{-1}\| = \frac{1}{\varepsilon}$ .

For the other inclusion, let  $\lambda \in \bigcup_n \Lambda_\varepsilon(P_n A P_n, T_n)$ . If  $\lambda \in \bigcup_n \sigma(P_n A P_n, T_n)$ , then  $\lambda \in \sigma(A, T) \subseteq \Lambda_\varepsilon(A, T)$ . If  $\lambda - T_n$  is invertible  $\forall n$  and  $\sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| < \infty$ , then  $\|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon}$ . If  $\sup_{n \in \mathbb{N}} \|(\lambda P_n - T_n)^{-1}\| = \infty$ , then  $\lambda \in \sigma(A, T) \subseteq \Lambda_\varepsilon(A, T)$ .  $\square$

*Remark 4.5.* In general,  $\Lambda_\varepsilon(a)$  is not the closure of  $\Lambda_\varepsilon^*(a)$ . However, this is true in many cases. See [13] and [14]. In such cases, the above decomposition can be modified to:

$$\Lambda_\varepsilon(A, T) = \overline{\bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(P_n A P_n, T_n)}.$$

This follows because

$$\begin{aligned} \Lambda_\varepsilon(A, T) &= \overline{\Lambda_\varepsilon^*(A, T)} \\ &= \overline{\bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon^*(P_n A P_n, T_n)} \\ &\subseteq \overline{\bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(P_n A P_n, T_n)}. \end{aligned}$$

Already, since  $\bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(P_n A P_n, T_n) \subseteq \Lambda_\varepsilon(A, T)$ , and since  $\Lambda_\varepsilon(A, T)$  is closed, we have

$$\overline{\bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(P_n A P_n, T_n)} \subseteq \Lambda_\varepsilon(A, T).$$

Theorems 4.2, 4.3 and 4.4 are also true if the countable family  $\{P_n\}_{n \in \mathbb{N}}$  is replaced by an arbitrary family  $\{P_\alpha\}_{\alpha \in I}$ . We write it here for the sake of completeness. The proof is identical to the countable case.

**Theorem 4.6.** *Let  $X$  be a Banach space and  $\{P_\alpha\}_{\alpha \in I}$  be a family of idempotent operators in  $A = B(X)$  such that  $(\sum_{\alpha \in I} P_\alpha)x = x \forall x \in X$ . Let  $T \in B(X)$  be an operator that commutes with each  $P_\alpha$ . Suppose the norm on  $X$  satisfies the following monotone condition:  $\|x_\alpha\| \leq \|y_\alpha\| \forall \alpha \in I \implies \|x\| \leq \|y\|$ ,  $x, y \in X$  and  $x_\alpha = P_\alpha x, y_\alpha = P_\alpha y$ . Let  $X_\alpha = P_\alpha X$  and  $T_\alpha = TP_\alpha \upharpoonright_{X_\alpha}$ . Then each  $P_\alpha$  is Hermitian and*

$$\sigma(A, T) = \bigcup_{\alpha \in I} \sigma(P_\alpha A P_\alpha, T_\alpha) \cup \{\lambda \in \mathbb{C} : \sup_{\alpha \in I} \|(\lambda P_\alpha - T_\alpha)^{-1}\| = \infty\}.$$

Further, if  $\varepsilon > 0$ , then

$$\Lambda_\varepsilon^*(A, T) = \bigcup_{\alpha \in I} \Lambda_\varepsilon^*(P_\alpha A P_\alpha, T_\alpha),$$

and

$$\Lambda_\varepsilon(A, T) = \bigcup_{\alpha \in I} \Lambda_\varepsilon(P_\alpha A P_\alpha, T_\alpha) \cup \{\lambda \in \mathbb{C} : \sup_{\alpha \in I} \|(\lambda P_\alpha - T_\alpha)^{-1}\| = \frac{1}{\varepsilon}\}.$$

*Remark 4.7.* In the case of infinite direct sums, we must assume that the convergence of the infinite series  $(\sum_{n=1}^{\infty} P_n = 1)$  is in the strong operator topology, since

a sequence of idempotents cannot converge to 0, and  $\sum_{n=1}^{\infty} P_n$  convergent implies  $\|P_n\| \rightarrow 0$ . Since there is no natural notion of this convergence in the case of an abstract Banach algebra, we do not have the equivalent of Theorem 3.18 for the case of infinitely many idempotents.

*Remark 4.8.* In the case of finite direct sums, it is sufficient to assume that the norm on  $X = \bigoplus_{i=1}^n X_i$  is an absolute norm (that is,  $\|x_n\| = \|y_n\| \forall n \implies \|x\| = \|y\|$ ), since every absolute norm on a finite direct sum is monotonic. This is not clear for an infinite direct sum and hence we have assumed the monotone condition on the norm in this case.

*Remark 4.9.* We also observe that if we have an uncountable family of idempotent operators  $\{P_\alpha\}_{\alpha \in I}$  satisfying  $(\sum_\alpha P_\alpha)x = x \forall x \in X$ , then for each  $x \in X$ , all but countably many of the  $P_\alpha x$  must be 0 (see Corollary 5.28 in [10]). Thus, in the case of a separable Banach space, we obtain  $P_\alpha = 0$  for all but countably many  $\alpha$  and it suffices to consider the countable case.

We give some examples to illustrate the above theorems.

**Example 4.10.** Let  $X_n$  be Banach spaces and  $X = \bigoplus_n X_n$  with a suitable monotonic norm defined on the direct sum. Let  $T_n = \frac{1}{n} I_n \in B(X_n)$ , and  $T = \bigoplus_n T_n \in$



$B(X)$ . Then  $\sigma(T_n) = \{\frac{1}{n}\} \forall n$ . Then it can be seen that  $\|(\lambda I_n - T_n)^{-1}\| = \frac{1}{|\lambda - \frac{1}{n}|}$  and hence  $\sigma(T) = \bigcup_n \{\frac{1}{n}\} \cup \{0\}$ , and  $\Lambda_\varepsilon(T_n) = D(\frac{1}{n}; \varepsilon) \forall n$ . For  $\lambda \notin \sigma(T)$ ,

$$\sup_n \|(\lambda - T_n)^{-1}\| = \sup_n \frac{1}{|\lambda - \frac{1}{n}|} = \begin{cases} \frac{1}{|\lambda|}, & \text{Re } \lambda < 0 \\ \frac{1}{|\lambda-1|}, & \text{Re } \lambda > 1 \\ \frac{1}{|\lambda - \frac{1}{n_0}|} & \text{for some } n_0 \in \mathbb{N}, 0 < \text{Re } \lambda < 1. \end{cases}$$

Hence

$$\Lambda_\varepsilon(T) = \bigcup_{n \in \mathbb{N}} D(\frac{1}{n}; \varepsilon) \cup D(0; \varepsilon) = \bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n \in \mathbb{N}} \|(\lambda I_n - T_n)^{-1}\| = \frac{1}{\varepsilon}\}.$$

The following example is based on the example given in Problem 98 in [6].

**Example 4.11.** Let  $X = l^1(\mathbb{N})$  and let  $T \in B(X)$  be the weighted unilateral shift operator with weights given by the sequence  $\{1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots\}$ .  $l^1$  can be expressed as the direct sum

$$\mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \dots,$$

where each  $\mathbb{C}^n$  is endowed with the 1-norm and the direct sum is also endowed with the 1-norm. Then  $T$  can be expressed as the direct sum

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \dots$$

Let  $T_n \in B(\mathbb{C}^{n+1})$  be the  $n$ -th term in the direct sum. Then the following can be computed:  $\sigma(T_n) = \{0\} \forall n$ , and  $\|(\lambda - T_n)^{-1}\| = \sum_{k=1}^{n+1} \frac{1}{|\lambda|^k} \forall n$ . Hence  $\sigma(T) = D(0; 1)$  and

$$\Lambda_\varepsilon(T_n) = \{\lambda \in \mathbb{C} : |\lambda|^{n+1} - \varepsilon(|\lambda|^n + \dots + |\lambda| + 1) \leq 0\} = \{\lambda \in \mathbb{C} : |\lambda|^{n+1} \leq \varepsilon \left( \frac{|\lambda|^{n+1} - 1}{|\lambda| - 1} \right)\}.$$

We also see that  $\|(\lambda - T)^{-1}\| = \frac{1}{|\lambda| - 1}$  for  $|\lambda| > 1$ , and hence  $\Lambda_\varepsilon(T) = D(0; 1 + \varepsilon)$ . For  $|\lambda| > 1$ ,

$$\Lambda_\varepsilon(T_n) = \{\lambda \in \mathbb{C} : |\lambda|^{n+1} \leq \varepsilon \left( \frac{|\lambda|^{n+1} - 1}{|\lambda| - 1} \right)\} = \{\lambda \in \mathbb{C} : |\lambda|^{n+2} - (1 + \varepsilon)|\lambda|^{n+1} + \varepsilon \leq 0\}.$$

Now, if  $|\lambda| \geq 1 + \varepsilon$ , then  $|\lambda|^{n+2} - (1 + \varepsilon)|\lambda|^{n+1} + \varepsilon \geq |\lambda|^{n+2} - |\lambda|^{n+1} + \varepsilon = \varepsilon > 0 \forall n$ . Hence for each  $n$ ,  $\Lambda_\varepsilon(T_n) \subsetneq D(0; 1 + \varepsilon)$ . Thus we get

$$\Lambda_\varepsilon(T) = \bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(T_n) \cup \{\lambda \in \mathbb{C} : \sup_{n \in \mathbb{N}} \|(\lambda I_n - T_n)^{-1}\| = \frac{1}{\varepsilon}\}.$$

In fact, in this case, it is true that

$$\Lambda_\varepsilon(T) = \overline{\bigcup_{n \in \mathbb{N}} \Lambda_\varepsilon(T_n)}.$$

This is because the space  $l^1(\mathbb{N})$  is complex uniformly convex, hence  $\Lambda_\varepsilon(T) = \overline{\Lambda_\varepsilon^*(T)}$  (see [13] and [14]).

*Remark 4.12.* Let  $u$  be a non-trivial idempotent in a Banach algebra  $A$ . Any  $x \in A$  can be represented as

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_u$$

where  $a = uxu, b = ux(1 - u), c = (1 - u)xu$  and  $d = (1 - u)x(1 - u)$ . The representation is in diagonal form precisely when  $ux = xu$ .

In [8], the author states that if an element  $x$  of a Banach algebra has the diagonal representation

$$x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}_u$$

with respect to the idempotent  $u$ , then  $\|x\|$  can be defined as  $\max\{\|a\|, \|b\|\}$ . However, this norm is not equal to the original norm on  $A$  in general.

Consider Example 3.9. Here,  $A = (\mathbb{C}^{2 \times 2}, \|\cdot\|_1)$  and  $u = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ . It can be seen that for  $a$  as in the example and  $\lambda = i$ ,  $\|(\lambda - a)^{-1}\| = 1$  which is strictly greater than the maximum of  $\|u(\lambda - a)^{-1}\|$  and  $\|(1 - u)(\lambda - a)^{-1}\|$  both of which are equal to  $\frac{1}{\sqrt{2}}$ .

The author of [8] discusses the the  $(p, q)$ -generalised outer inverse of  $a \in A$  and further defines the  $(p, q) - \varepsilon$  pseudospectrum of  $a \in A$ .

Let  $a \in A$  and  $p, q \in A$  be idempotent elements. An element  $b \in A$  satisfying  $bab = b, ba = p$  and  $1 - ab = q$  will be called the  $(p, q)$ -generalised outer inverse of  $a$  and denoted by  $a_{p,q}^{(2)}$ . The set of elements with  $(p, q)$ -outer generalised inverses is denoted by  $A_{p,q}^{(2)}$ .

The  $(p, q) - \varepsilon$  pseudospectrum of  $a \in A$  is defined as

$$\Lambda_{(p,q)-\varepsilon}(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A_{p,q}^{(2)} \text{ or } \|(\lambda - a)_{p,q}^{(2)}\| \geq \frac{1}{\varepsilon}\}.$$

It is shown that for  $u \in A$ , an idempotent and  $a \in A$  such that  $ua = au$ , if  $p_1, q_1$  idempotents in  $uAu$  and  $p_2, q_2$  are idempotents in  $(1 - u)A(1 - u)$  and  $p = p_1 + p_2, q = q_1 + q_2$ ,

$$\Lambda_{(p,q)-\varepsilon}(A, a) = \Lambda_{(p_1,q_1)-\varepsilon}(uAu, ua) \cup \Lambda_{(p_2,q_2)-\varepsilon}((1 - u)A(1 - u), (1 - u)a). \quad (4.1)$$

However, this is not true in general. This decomposition occurs if and only if

$$\|(\lambda - a)_{p,q}^{(2)}\| = \max \{ \|(\lambda u - ua)_{p_1,q_1}^{(2)}\|, \|(\lambda(1 - u) - (1 - u)a)_{p_2,q_2}^{(2)}\| \}.$$

For instance, if the hypotheses of Theorem 3.13 or Theorem 3.14 hold, then we obtain the decomposition (4.1).

#### ACKNOWLEDGEMENT

The first author thanks the Council of Scientific and Industrial Research (CSIR), India for the financial support (File No: 09/084(0647)/2013-EMR-I).

## REFERENCES

1. J. Alaminos, J. Extremera and A. R. Villena, *Approximately spectrum-preserving maps*, J. Funct. Anal. **261** (2011), no. 1, 233–266.
2. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, 1973.
3. F. F. Bonsall and J. Duncan, *Numerical ranges II* (Vol. 2), Cambridge University Press, 1973.
4. E. O. Cevik and Z. I. Ismailov, *Spectrum of the direct sum of operators*, Electron. J. Differential Equations (2012), no. 210, 1–8.
5. J. B. Conway, *A Course in Functional Analysis* (Vol. 96), Springer Science and Business Media, 2013.
6. P. R. Halmos, *A Hilbert Space Problem Book*, second edition, Springer-Verlag, New York, 1982.
7. P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Springer-Verlag, Berlin, 1993.
8. M. Z. Kolundžija, *(p,q)- outer generalized inverse of block matrices in Banach algebras*, Banach J. Math. Anal. **8** (2014), no. 1, 98–108.
9. A. Krishnan and S. H. Kulkarni, *Pseudospectrum of an element of a Banach algebra*, Oper Matrices **11** (2017), no. 1, 263–287.
10. C. S. Kubrusly, *Elements of Operator Theory*, Birkhuser, Boston, 2001.
11. P. Lancaster and H. K. Farahat, *Norms on direct sums and tensor products*, Math. Comp. **26** (1972), no. 118, 401–414.
12. C. R. Putnam, *Operators satisfying a  $G_1$  condition*, Pacific J. Math. **84** (1979), no. 2, 413–426.
13. E. Shargorodsky, *On the level sets of the resolvent norm of a linear operator*, Bull. London Math. Soc. **40** (2008), no. 3, 493–504.
14. E. Shargorodsky, *On the definition of pseudospectra*, Bull. London Math. Soc. **41** (2009), no. 3, 524–534.
15. S. Shkarin, *Norm attaining operators and pseudospectrum*, Integral Equations Operator Theory **64** (2009), no. 1, 115–136.
16. A. M. Sinclair, *The norm of a Hermitian element in a Banach algebra*, Proc. Amer. Math. Soc. **28** (1971), 446–450.
17. L. N. Trefethen and M. Embree, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*, Princeton Univ. Press, Princeton, NJ, 2005.

<sup>1</sup>DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY MADRAS, CHENNAI 600036, INDIA.

*E-mail address:* [arundhathi.krishnan@gmail.com](mailto:arundhathi.krishnan@gmail.com)

<sup>2</sup>DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY MADRAS, CHENNAI 600036, INDIA.

*E-mail address:* [shk@iitm.ac.in](mailto:shk@iitm.ac.in)