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Title	Choosing parameter values for a geometric CUSUM chart for detecting an upward shift in a proportion
Authors	Bourke, Patrick D.
Publication date	2018-04-04
Original Citation	Bourke, P. D. (2018) Choosing parameter values for a geometric CUSUM chart for detecting an upward shift in a proportion, Quality Engineering, 30 (4), pp. 621-634, doi: 10.1080/08982112.2018.1443341
Type of publication	Article (peer-reviewed)
Link to publisher's version	https://www.tandfonline.com/doi/abs/10.1080/08982112.2018.1443341 - 10.1080/08982112.2018.1443341
Rights	© 2018 Informa UK Limited. This is an Accepted Manuscript of an article published by Taylor & Francis in Quality Engineering on 04 April 2018, available online: http://www.tandfonline.com/10.1080/08982112.2018.1443341
Download date	2024-04-25 09:33:50
Item downloaded from	https://hdl.handle.net/10468/9764

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Choosing Parameter Values for a Geometric CUSUM Chart for Detecting an Upward Shift in a Proportion

by

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February 2018

ABSTRACT

This paper considers the task of providing practical advice to quality engineers and practitioners on the choice of values for the two parameters of a geometric CUSUM control chart. This CUSUM chart was developed to enable the detection of sudden shifts from an acceptable level for a proportion (p) such as fraction nonconforming. In the first part of the paper, tables are presented listing recommended parameter-choices for each of 18 in-control levels for p in the range (0.04 to 0.001) for detection of each of five sizes of upward shift. In the second part of the paper, some empirical relationships among the parameter-values are identified. These relationships are used with interpolation to design geometric CUSUM schemes for any in-control level (p_a) of p within the range covered by the tables, and with extrapolation for levels of p_a as low as 0.0001. Because of the equivalence between a geometric CUSUM and a Bernoulli CUSUM, the tables and the proposed methods of interpolation and extrapolation also provide assistance in the design of a Bernoulli CUSUM chart.

Key-words: Statistical Process Control; Process Monitoring; Cumulative Sum Scheme; Bernoulli CUSUM; Exponential CUSUM.

1. Introduction

There are many circumstances where it is desired to monitor a proportion associated with a stream of discrete items that are generated from a repetitive process. In the context of Quality Engineering for industry, the most familiar instance of this is the task of monitoring the fraction of items found to be nonconforming for a manufacturing process. In many situations the volume of items produced can be quite large, and despite this it is often the case that 100% inspection of the items in order of production is both necessary and feasible. When such inspection is underway, a nonconforming item will occasionally be found even when the process is operating at an acceptable level (p_a) for the proportion (p). We consider those situations where it is feasible to maintain an ordered record of the number (X_i) of conforming items that are found between successive nonconforming items, and the term conforming run-length (CRL) has been used to refer to the X_i value. As each item is produced and found to be conforming, the current X_i value is incremented by one. If the item is found to be nonconforming, we record the most recent X_i, and then set X_{i+1} to zero to start a new count. It is also assumed that there is little delay between the production of the item and the corresponding updating of the X_i value. (It should be noted that each nonconforming item found does not contribute to the X_i value.) The sequence of X_i values can be used as the data for a monitoring scheme: if there appears to be a reduction in value for these $\{X_i\}$ this provides an indication that the process-proportion may have increased. On the other hand, an increase in the X_i values suggests that the proportion may have shifted downward.

There are a number of ways of using the ordered $\{X_i\}$ data, and for a discussion of these and other related methods one may refer to Szarka and Woodall (2011). One of the recommendations of these researchers is that either a geometric CUSUM chart or a Bernoulli CUSUM chart be used for detection of a shift of specified size for a proportion. It is over 25 years since the first investigation of the geometric CUSUM chart, and since then there has been continuing interest in this chart, and it is included in many comparative studies of control-chart performance, particularly for very small values of a proportion. But it is unclear whether this chart has been used much by many practitioners of statistical process control. It seems probable that a reason for its likely under-use is the complexity of choosing parameter-values for this CUSUM chart. This paper seeks to aid the task of finding suitable values for these parameters. The results presented here can help to bridge the gap between developments in the methodology of statistical process monitoring and current practice. The importance of addressing such gaps has been highlighted in a recent paper (Woodall, 2017).

The paper is organized as follows. In the next Section we provide a brief introduction to the geometric CUSUM chart and its parameters k and h. In Section 3, we consider measures of performance for the

geometric CUSUM, and review the recommended design procedure for this CUSUM chart. In Section 4, we consider the choice of the parameter k, and describe an investigation to aid this choice. In Section 5, five tables are presented listing recommended parameter choices for CUSUM schemes to detect five specific shift-sizes. In Section 6, the performance of a geometric CUSUM for detecting a range of sizes of upward shift is considered. In Section 7, a series of tables is given which demonstrate some simple empirical relationships among the values of parameters k, h and p_a, in particular the stability of the ratio h/k. Using estimates of this ratio, sixteen interpolations are given in Section 8 to find CUSUM schemes for values of p_a within the range (0.04 to 0.001). The usefulness of the ratio h/k in extrapolations to find CUSUM schemes for p_a values below the range (0.04 to 0.001) is demonstrated in Section 9 for the relatively low value $p_a = 0.0001$. Section 10 provides a summary of the paper, and indications of areas of further research.

2. The Geometric CUSUM Chart

The proposal to form a CUSUM scheme using the ordered sequence of CRL values was made in Bourke (1991), and this CUSUM has come to be known as the geometric CUSUM. The CUSUM (G_i) is as follows:

$$G_i = \max (0, k - X_i + G_{i-1}) \qquad i = 1, 2 \ldots \tag{2.1}$$

In order to start the CUSUM, an initial value (G_0) must be chosen. This is often taken to be zero, but a positive value (termed a head-start level) may also be used. The quantity k is known as the *reference value* of the CUSUM and is one of the two integer-valued parameters of the scheme. The second parameter is denoted by h, and is often referred to as the *decision interval* value. As long as the process-proportion (p) remains at an acceptable level p_a (or below), then we expect the computed value arising from equation (2.1) to rarely rise much above zero. With p at the acceptable level, if G_i were to reach or exceed h, we say that the CUSUM has signalled, and an investigation should indicate that this is a false signal. In designing a geometric CUSUM scheme, we wish to limit the frequency of false signals, but also to quickly signal sizable upward shifts from the acceptable level p_a. It is assumed in this paper that we know p_a and also p_r, the minimum size of shifted proportion that should be detected, so that p_r is a rejectable level for p. (Some researchers prefer the notation p_0 and p_1, rather than p_a and p_r.) We shall also consider what may be done when there is uncertainty about the level of p_r that should be used.

Following the first paper on the geometric CUSUM, some researchers (e.g. Chang and Gan (2001), Szarka and Woodall (2012)) have preferred to use a slightly different formulation, where the geometric count value {Y_i, say} is defined as the number of items observed until a nonconforming item is found, so

that the count Y_i would then include the nonconforming item. Clearly, $Y_i = X_i + 1$. If one uses the $\{Y_i\}$ as the observed data, the relevant value of k will be one more than the value when one uses $\{X_i\}$ as the observed data. In this paper, we shall use $\{X_i\}$ as the observed data.

3. Measures of Performance for a Geometric CUSUM Chart

The measure average number of observations until a signal (ANOS) was recommended in Reynolds and Stoumbos (1999) and also in Szarka and Woodall (2011), which provides a discussion of various performance-measures. The ANOS measure is particularly relevant when one wishes to make performance-comparisons for competing monitoring schemes. In this paper, the measure average number of nonconformities until a signal (ANNS) is introduced, and will be used throughout the paper. This measure is defined as follows:

ANNS(p) = E[Number of CRLs until the CUSUM signals, following a shift from p_a to p] (3.1)

We note that ANNS(p) equals p multiplied by ANOS(p).

In setting up a geometric CUSUM chart (or other monitoring scheme), one would like the frequency of false signals to be low, while achieving speedy detection following the occurrence of a sizable shift. We can express these aims in terms of the chosen measure (ANNS in this paper) by seeking parameter-values such that ANNS(p_a) is at least some chosen high level, and so that ANNS(p_r) is as small as possible. It is important for any user of a monitoring scheme to be aware that there is an interplay between ANNS(p_r) and the level chosen for ANNS(p_a), and this is discussed further in Section S1 of the supplementary material available online.

It is our view that ANNS(p) can be a more suitable performance-measure for a user of a geometric CUSUM chart. Consider a user who wishes to monitor a process for which $p_a = 0.001$, with the intention of stopping the process as soon as possible following a shift in p to 0.003 or above. If one is using ANOS as a performance-measure, the quality engineer is asked to think about (and then specify) a minimum level for the average number of items observed after a random starting point until the next false signal from the CUSUM. This specified level can be quite large, e.g 50,000, 100,000 or even 200,000. On the other hand, if one is using ANNS as a performance-measure the engineer needs to choose an acceptable minimum level for the average number of nonconforming items found after a random starting point until the next false signal, and suitable levels might be 25, 50, 100, etc. We believe that these lower numbers can make this an easier specification for the engineer to make. It will be seen in Section 7 that a significant benefit of the ANNS measure is that its use here has enabled the identification of the stability of the ratio h/k for a range of geometric CUSUM schemes for which ANNS(p_a) is fixed, and the shift-size (as a multiple of p_a) is fixed.

3.1 Evaluation of ANNS(p): Zero-state and Steady-state Evaluations

In early investigations of the geometric CUSUM chart, evaluations of measures such as ANOS were based on an assumption about the stream of CRL data from the process, and on an assumption about the operation of the associated CUSUM chart. It was assumed that the CUSUM was started directly after a nonconforming item (with an initial CUSUM value of zero or some positive level), and was then continued until the CUSUM signalled. For the stream of CRL data generated by the process, it was assumed that the level of the proportion p remained constant, and in evaluating ANOS two levels for p were usually considered: the acceptable level p_a, and the minimum rejectable level p_r. These evaluations of ANOS are known as zero-state or initial-state evaluations. In such evaluations, there is no consideration of the way in which a shift from p_a to p_r might occur. In subsequent research, consideration was given to two alternative assumptions relating to the occurrence of a shift:

A1: The sudden shift in the value of the proportion from p_a to p_r can occur at any item, conforming or nonconforming. This has been termed the random-shift assumption.

A2. The shift in the value of p can only occur directly after the occurrence of a nonconforming item. This has been termed the fixed-shift assumption.

In preparing the tables reported in Section 5 of this paper, we have chosen to make the random-shift assumption, and we also assume that the CUSUM has been in operation for a lengthy period with $p = p_a$ prior to the shift, so that the non-signalling CUSUM values at the moment of shift will have reached a steady-state distribution. (Following any false signal, the CUSUM is re-set to zero.) It is noted (on page 58) in Szarka and Woodall (2012) that the random-shift assumption is more realistic. In the latter paper it is also stated "Some authors who have studied the geometric CUSUM chart have misleadingly, in our opinion, applied the fixed-shift model..." The analysis for the evaluation of ANNS for the case of the random-shift assumption (in conjunction with the CUSUM values having reached a steady-state distribution prior to the shift) is given in Bourke (2001) and is briefly reviewed in the Appendix. The use of simulation is another option, and such evaluations have been reported in Szarka and Woodall (2012).

Tables to aid the design of geometric CUSUM charts were provided in Bourke (1991) and in Chang and Gan (2001), but in both cases only zero-state evaluations were presented. In Table 3 of Chang and Gan (2001) the parameters of geometric CUSUM schemes were provided for seven levels of p_a in the range 0.0005 to 0.00001, and for four sizes of upward shift. In their table, the levels used for ANNS(p_a) ranged from 4 to 15 for $p_a = 0.0005$, and from 3 to 8 for $p_a = 0.00001$. (Their Table 3 refers to in-control ANIS, which is the same as zero-state ANOS(p_a).) The use of a table based on zero-state evaluations for situations where random shifts may occur can lead to misleading conclusions. As an illustration of this, we consider one of the CUSUM schemes listed in Table 3 of Chang and Gan (2001) for detection of a shift from $p_a =$

0.0002 to $p_r = 0.0006$. They recommend a scheme with parameter values $k = 2746$, $h = 2705$. We have evaluated zero-state ANOS(p_a) for this scheme, and we can confirm the reported value of 50000. We also found the steady-state ANNS(p_r) value (at 3.625) under the random-shift assumption for this scheme to be about 22% above the zero-state ANNS(p_r) value (2.971). Another scheme from Table 3 of Chang and Gan (2001) is for detection of a shift from $p_a = 0.0001$ to $p_r = 0.0002$, and the recommended parameter values are $k = 6931$, $h = 8267$. This time the steady-state ANNS(p_r) value is about 14% above the corresponding zero-state value. Thus, a reliance on zero-state evaluations of ANNS(p_r) can be misleading if shifts in the level of p can occur randomly at any item.

4. Choosing a Value for Parameter *k*

In Bourke (1991), the following formula, derived from the sequential probability ratio (SPR) analysis for detecting a shift from p_a to p_r, was proposed for choosing a value for k.

$$k = \{\ln(p_r / p_a)\}/\{\ln[(1- p_a)/(1- p_r)]\} \qquad (4.1)$$

There is a theoretical result in Moustakides (1986) which indicates that a CUSUM scheme that uses a k value arising from an SPR analysis is an optimal detector of a shift. Lucas (1994) has commented that the Moustakides paper is so abstract that few can read it. In Yaschin (1993) there is a more accessible discussion of this result, and it is stated (page 47) that this optimality result relates to "worst-case ARL", which in the present context is zero-state ANNS. We have carried out computations to verify that if one is using zero-state ANNS(p), the use of a k value from equation (4.1) is optimal. In preparing this paper, an extensive computational investigation was conducted to see how well this formula works when one is using steady-state evaluation of ANNS(p) together with the random-shift assumption.

In this investigation we searched for the value for k that would give the lowest value for ANNS(p_r) for a series of geometric CUSUM schemes for which ANNS(p_a) $\geq$ a specified level, and this was repeated for a range of shift-sizes. Five levels for ANNS(p_a) were considered, and these were: 25, 50, 100, 200 and 300. For each of these levels, five sizes of shift from p_a to p_r were considered, and these were:

$p_r = 1.5\times p_a$ $\quad$ $p_r = 2\times p_a$ $\quad$ $p_r = 3\times p_a$ $\quad$ $p_r = 5\times p_a$ $\quad$ $p_r = 7\times p_a$.

To describe this investigation, one of the cases that were considered is now presented:

For $p_a = 0.002$, ANNS(p_a) = 100, and $p_r = 0.004$, we searched for a range of (k, h) parameter choices for which ANNS(p_a) closely exceeded 100. For each of these CUSUM schemes, we evaluated ANNS(p_r), and identified the value of k for which ANNS(p_r) was minimum. The best choice of k was found to be 379 (and the value of h was 1701). The value of ANNS(p_r) for this scheme was 12.16, which is about 1.2% below

the ANNS(p_r) value for the scheme (k = 346, h = 1333) where the value of k was determined using equation (4.1). For each of these CUSUM schemes, the value of ANNS(p_a) barely exceeds 100.

In all cases considered, it was found that the minimum level for ANNS(p_r) occurred when k was larger than the value arising from equation (4.1). The percentage by which the best choice for k exceeded the k value from equation (4.1) was evaluated for each of the 25 combinations of ANNS(p_a) level and shift-size. These percentages were determined for two levels of p_a (0.002 and 0.005). It was found that the main influence on these percentages is the level of ANNS(p_a). While there is also a dependence on the size of shift, the dependence on p_a is slight. The recommended percentages are given in Table 1. It must also be remarked that in all the cases considered the minima are relatively flat, and the reduction in ANNS(p_r) value arising from using the recommended k value rather than that from equation (4.1) can range from about 1.8% (for schemes with ANNS(p_a) = 25, for detection of small shifts) through about 1% (for schemes with ANNS(p_a) at 50 or 100, for detection of moderate shifts). For higher levels of ANNS(p_a) and detection of large shifts, the corresponding reduction in ANNS(p_r) is about 0.3%. A limited earlier study on the best choice of value for the parameter k was reported in Bourke (2001), which provides a graphical illustration.

TABLE 1: Recommended Value of k: Percentage Increase over the Value given by the Sequential Probability Ratio Formula.

	Shift-size to be detected				
	$1.5\times p_a$	$2\times p_a$	$3\times p_a$	$5\times p_a$	$7\times p_a$
ANNS(p_a) = 25	15.9	17.1	17.7	17.7	17.7
ANNS(p_a) = 50	11.2	12.6	13.2	13.2	13.2
ANNS(p_a) = 100	8.6	9.6	10.1	10.1	10.1
ANNS(p_a) = 200	6.5	7.3	7.7	8.2	8.4
ANNS(p_a) = 300	5.5	6.2	6.7	7.1	7.3

5. Tables of k, h values for a Geometric CUSUM Chart

In order to provide assistance to potential users of geometric CUSUM charts, tables of recommended k, h values have been developed. The structure and range of these tables is now described.

Table 2 provides k, h values for detecting a shift from p_a to $p_r = 1.5\times p_a$ for schemes with each of the following levels for ANNS(p_a): 25, 50, 100, 200, 300, across a range of values of p_a from 0.04 to 0.001. Table 3 provides k, h values for detecting a shift from p_a to $p_r = 2\times p_a$ for schemes with each of the following levels for ANNS(p_a): 25, 50, 100, 200, 300, across a range of values of p_a from 0.04 to 0.001.

Table 4 provides k, h values for detecting a shift from p_a to $p_r = 3\times p_a$ for schemes with each of the following levels for ANNS(p_a): 25, 50, 100, 200, 300, across a range of values of p_a from 0.04 to 0.001.
Table 5 provides k, h values for detecting a shift from p_a to $p_r = 5\times p_a$ for schemes with each of the following levels for ANNS(p_a): 25, 50, 100, 200, 300 across a range of values of p_a from 0.04 to 0.001.
Table 6 provides k, h values for detecting a shift from p_a to $p_r = 7\times p_a$ for schemes with each of the following levels for ANNS(p_a): 25, 50, 100, 200, 300, across a range of values of p_a from 0.04 to 0.001.
In these tables, some combinations of ANNS(p_a) level and shift-level are likely to be of lesser interest: such as shifts of size $1.5\times p_a$ with ANNS(p_a) ≥ 100, and shifts of size $2\times p_a$ with ANNS(p_a) = 300, for which the values of ANNS(p_r) may be considered to be too large by some users.

In all of these tables, the choice of a value for k was made using the conclusions reported in Section 4. In all cases, once the value of k had been chosen, a search was conducted to find the smallest value of h such that ANNS(p_a) was at least the chosen level (i.e. one of 25, 50 100, 200, 300). Some consideration was given to the alternative search of finding the value for h which would give a value for ANNS(p_a) closest to the target (i.e. 25, 50, etc.), but it was deemed preferable to have ANNS(p_a) closely exceeding the target.

The question may be asked: is there advice on the choice of parameter values for shift-sizes other than those listed in the tables? In addressing this question, we consider the detection of a shift from $p_a = 0.005$ to $p_r = 4\times p_a$, with the requirement that ANNS(p_a) should be at least 100. Interpolation between the entries in Table 4 (for shifts to $3\times p_a$) and Table 5 (for shifts to $5\times p_a$) has been considered, but there does not seem to be any obvious method of interpolation. From Tables 4 and 5 we have two possible schemes: (k = 120, h = 383) for detecting shifts to $p_r = 3\times p_a$ and (k = 87, h = 199) for detecting shifts to $p_r = 5\times p_a$. Both schemes give a detection performance (when the shift is to $4\times p_a$) that is quite close to that of the un-tabled scheme (k = 100, h = 260) which was designed to detect a shift to $p_r = 4\times p_a$. The values for ANNS(p_r = 0.02) for these three schemes are 6.24, 6.20 and 6.12, respectively. We note that the scheme for detecting $p_r = 5\times p_a$ does a little better than the scheme for detecting $p_r = 3\times p_a$, when the actual shift is to $p_r = 4\times p_a$. Investigations for other shift-sizes and other levels of ANNS(p_a) show similar results. Arising from this, it is recommended that for detecting shift-sizes that are not tabulated, one should use the closest shift-size that is tabulated. If the shift-size of interest is midway between two tabulated shift-sizes, it is recommended that the scheme for detecting the larger of the two tabulated shifts be used. It is also of interest to consider the performance of a geometric CUSUM chart for detection of a range of shift-sizes, and this is considered in the next Section.

TABLE 2: Proposed values of Geometric CUSUM parameters (k,h) for a range of values of p_a, and for five levels of ANNS(p_a) for detecting a shift from p_a to $p_r = 1.5 \times p_a$

	Values of the in-control proportion p_a (in percent)																	
	4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ANNS(p_a) = 25																	
k	22	30	36	46	51	58	66	77	93	103	116	133	155	187	234	312	469	939
h	77	106	126	165	181	209	235	272	331	363	408	470	545	663	828	1101	1658	3316
ANNS(p_r)	*10.05*	*10.05*	*10.11*	*10.10*	*10.10*	*10.10*	*10.13*	*10.12*	*10.13*	*10.13*	*10.12*	*10.14*	*10.14*	*10.15*	*10.14*	*10.15*	*10.15*	*10.15*
	ANNS(p_a) = 50																	
k	21	29	35	44	49	55	63	74	89	99	112	128	149	179	224	299	450	901
h	97	140	169	211	235	261	298	354	425	472	539	614	709	852	1065	1420	2147	4296
ANNS(p_r)	*14.43*	*14.48*	*14.57*	*14.55*	*14.58*	*14.61*	*14.60*	*14.64*	*14.65*	*14.64*	*14.65*	*14.65*	*14.66*	*14.68*	*14.68*	*14.68*	*14.69*	*14.70*
	ANNS(p_a) = 100																	
k	21	28	34	43	48	54	62	72	87	97	109	125	146	175	219	292	439	880
h	133	171	210	269	301	337	388	442	540	606	676	780	910	1084	1355	1801	2714	5453
ANNS(p_r)	*19.93*	*20.03*	*20.07*	*20.14*	*20.11*	*20.18*	*20.16*	*20.18*	*20.21*	*20.24*	*20.23*	*20.26*	*20.26*	*20.28*	*20.28*	*20.30*	*20.30*	*20.31*
	ANNS(p_a) = 200																	
k	21	28	33	42	47	53	61	71	85	95	107	122	143	172	215	287	431	863
h	174	222	245	321	364	410	477	548	646	732	824	929	1100	1326	1651	2203	3307	6619
ANNS(p_r)	*26.37*	*26.38*	*26.58*	*26.61*	*26.66*	*26.69*	*26.68*	*26.75*	*26.76*	*26.79*	*26.81*	*26.81*	*26.84*	*26.84*	*26.85*	*26.87*	*26.89*	*26.91*
	ANNS(p_a) = 300																	
k	20	27	33	42	46	52	60	70	84	94	106	121	142	170	213	284	427	854
h	165	221	279	367	383	435	512	593	706	804	909	1030	1225	1451	1823	2421	3651	7273
ANNS(p_r)	*30.47*	*30.57*	*30.65*	*30.75*	*30.87*	*30.86*	*30.88*	*30.95*	*30.99*	*31.02*	*31.02*	*31.05*	*31.06*	*31.09*	*31.10*	*31.12*	*31.14*	*31.16*

TABLE 3: Proposed values of Geometric CUSUM parameters (k,h) for a range of values of p_a, and for five levels of ANNS(p_a) for detecting a shift from p_a to $p_r = 2 \times p_a$

	Values of the in-control proportion p_a (in percent)																	
	4	3	2.5	2	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ANNS(p_a) = 25																	
k	19	26	31	39	44	50	57	66	80	89	100	115	134	161	202	269	405	810
h	53	73	84	106	121	139	157	179	219	243	272	315	366	439	552	732	1105	2204
ANNS(p_r)	*7.00*	*7.01*	*6.96*	*7.02*	*6.99*	*7.01*	*7.01*	*7.03*	*7.03*	*7.03*	*7.04*	*7.04*	*7.05*	*7.05*	*7.05*	*7.06*	*7.06*	*7.06*
	ANNS(p_a) = 50																	
k	18	25	30	38	42	48	55	64	77	86	96	110	129	155	194	259	389	779
h	64	92	108	138	150	175	201	231	277	312	342	393	463	556	696	928	1393	2787
ANNS(p_r)	*9.30*	*9.35*	*9.31*	*9.35*	*9.35*	*9.36*	*9.41*	*9.40*	*9.39*	*9.42*	*9.41*	*9.42*	*9.42*	*9.43*	*9.44*	*9.44*	*9.45*	*9.45*
	ANNS(p_a) = 100																	
k	18	24	29	37	41	46	53	62	75	83	94	107	125	151	189	252	379	759
h	83	107	129	168	184	205	238	278	338	371	424	477	557	679	850	1129	1701	3405
ANNS(p_r)	*11.87*	*11.91*	*11.97*	*12.01*	*12.01*	*12.08*	*12.08*	*12.10*	*12.08*	*12.11*	*12.12*	*12.11*	*12.12*	*12.13*	*12.15*	*12.15*	*12.16*	*12.17*
	ANNS(p_a) = 200																	
k	17	24	29	36	40	45	52	61	73	82	92	105	123	148	185	247	371	743
h	88	134	161	194	215	240	281	331	392	448	498	566	666	803	1000	1334	2003	4009
ANNS(p_r)	*14.66*	*14.84*	*14.86*	*14.90*	*14.98*	*14.98*	*14.98*	*15.01*	*15.05*	*15.05*	*15.05*	*15.09*	*15.10*	*15.11*	*15.11*	*15.12*	*15.13*	*15.14*
	ANNS(p_a) = 300																	
k	17	23	28	36	40	45	51	60	73	81	91	104	122	146	183	244	367	735
h	99	133	163	218	240	269	298	354	439	484	541	616	728	864	1086	1443	2175	4353
ANNS(p_r)	*16.53*	*16.64*	*16.64*	*16.75*	*16.73*	*16.80*	*16.82*	*16.85*	*16.86*	*16.86*	*16.89*	*16.89*	*16.91*	*16.92*	*16.95*	*16.96*	*16.97*	*16.98*

TABLE 4: Proposed values of Geometric CUSUM parameters (k,h) for a range of values of p_a, and for five levels of ANNS(p_a) for detecting a shift from p_a to $p_r = 3 \times p_a$

	Values of the in-control proportion p_a (in percent)																	
	4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ANNS(p_a) = 25																	
k	15	20	25	31	35	39	45	53	63	71	80	91	106	128	160	214	322	645
h	31	40	52	63	72	79	91	108	126	144	162	182	211	257	320	428	644	1289
ANNS(p_r)	*4.89*	*4.91*	*4.95*	*4.97*	*4.98*	*4.99*	*4.97*	*4.99*	*4.99*	*4.99*	*4.99*	*4.98*	*4.99*	*5.00*	*5.01*	*5.01*	*5.01*	*5.01*
	ANNS(p_a) = 50																	
k	14	19	24	30	33	38	43	51	61	68	76	88	102	123	154	206	310	621
h	36	49	64	79	85	100	112	134	159	177	196	230	263	319	398	533	803	1608
ANNS(p_r)	*6.10*	*6.22*	*6.20*	*6.23*	*6.22*	*6.23*	*6.27*	*6.26*	*6.28*	*6.27*	*6.28*	*6.29*	*6.28*	*6.30*	*6.29*	*6.30*	*6.30*	*6.31*
	ANNS(p_a) = 100																	
k	14	19	23	29	32	37	42	49	59	66	74	85	100	120	150	201	301	604
h	46	62	74	93	101	120	134	156	187	211	235	270	320	383	478	642	956	1922
ANNS(p_r)	*7.50*	*7.58*	*7.55*	*7.59*	*7.59*	*7.61*	*7.60*	*7.64*	*7.64*	*7.67*	*7.67*	*7.66*	*7.67*	*7.68*	*7.69*	*7.69*	*7.69*	*7.70*
	ANNS(p_a) = 200																	
k	14	19	22	28	32	36	41	48	58	65	73	83	97	117	147	196	295	590
h	56	75	82	105	123	138	155	181	219	248	277	312	364	441	556	739	1114	2220
ANNS(p_r)	*8.92*	*8.97*	*9.02*	*9.05*	*9.03*	*9.08*	*9.07*	*9.07*	*9.08*	*9.11*	*9.11*	*9.12*	*9.13*	*9.14*	*9.15*	*9.15*	*9.16*	*9.17*
	ANNS(p_a) = 300																	
k	13	18	22	28	31	35	41	48	57	64	72	83	97	116	145	194	292	585
h	52	73	90	116	127	143	171	200	233	265	297	345	403	478	596	799	1204	2410
ANNS(p_r)	*9.65*	*9.75*	*9.81*	*9.91*	*9.92*	*9.93*	*9.91*	*9.93*	*9.95*	*9.99*	*9.99*	*9.99*	*10.00*	*10.00*	*10.00*	*10.00*	*10.00*	*10.01*

TABLE 5: Proposed values of Geometric CUSUM parameters (k,h) for a range of values of p_a, and for five levels of ANNS(p_a) for detecting a shift from p_a to $p_r = 5 \times p_a$

	Values of the in-control proportion p_a (in percent)																	
	4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ANNS(p_a) = 25																	
k	10	14	18	22	25	28	32	38	46	51	58	66	78	93	117	156	235	472
h	15	21	28	33	38	42	48	57	69	76	87	98	117	138	174	231	348	698
ANNS(p_r)	*3.66*	*3.69*	*3.70*	*3.72*	*3.73*	*3.73*	*3.76*	*3.74*	*3.75*	*3.75*	*3.75*	*3.75*	*3.76*	*3.76*	*3.76*	*3.76*	*3.77*	*3.77*
	ANNS(p_a) = 50																	
k	10	14	17	21	24	27	31	37	44	49	56	64	75	90	112	150	226	454
h	20	27	33	39	46	51	58	70	82	92	106	120	141	169	208	279	420	844
ANNS(p_r)	*4.50*	*4.42*	*4.49*	*4.44*	*4.51*	*4.49*	*4.48*	*4.49*	*4.49*	*4.53*	*4.53*	*4.51*	*4.52*	*4.54*	*4.53*	*4.53*	*4.53*	*4.54*
	ANNS(p_a) = 100																	
k	10	13	16	21	23	26	30	36	43	48	54	62	73	87	109	146	220	442
h	24	30	37	49	53	60	69	84	99	111	124	143	169	199	249	334	503	1011
ANNS(p_r)	*5.10*	*5.22*	*5.25*	*5.22*	*5.26*	*5.29*	*5.28*	*5.28*	*5.28*	*5.30*	*5.29*	*5.32*	*5.32*	*5.31*	*5.32*	*5.33*	*5.33*	*5.33*
	ANNS(p_a) = 200																	
k	10	13	16	20	23	26	30	35	42	47	53	61	71	86	108	144	216	434
h	29	36	44	54	63	71	82	95	114	128	144	166	192	233	293	390	583	1172
ANNS(p_r)	*5.93*	*6.03*	*6.02*	*6.05*	*6.03*	*6.04*	*6.05*	*6.07*	*6.11*	*6.12*	*6.12*	*6.13*	*6.13*	*6.12*	*6.13*	*6.15*	*6.15*	*6.16*
	ANNS(p_a) = 300																	
k	9	13	16	20	23	26	29	35	42	47	53	60	71	85	106	142	214	430
h	26	39	48	59	69	78	85	105	125	140	158	177	211	251	311	417	630	1266
ANNS(p_r)	*6.30*	*6.44*	*6.46*	*6.50*	*6.48*	*6.51*	*6.56*	*6.57*	*6.58*	*6.58*	*6.59*	*6.61*	*6.61*	*6.61*	*6.61*	*6.62*	*6.64*	*6.64*

TABLE 6: Proposed values of Geometric CUSUM parameters (k,h) for a range of values of p_a, and for five levels of ANNS(p_a) for detecting a shift from p_a to $p_r = 7 \times p_a$

	Values of the in-control proportion p_a (in per cent)																	
	4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
	ANNS(p_a) = 25																	
k	8	11	14	18	20	22	26	30	37	41	46	53	62	75	94	126	189	380
h	10	14	18	23	25	27	32	36	45	50	56	64	75	91	114	152	227	456
ANNS(p_r)	*3.16*	*3.25*	*3.25*	*3.25*	*3.24*	*3.27*	*3.25*	*3.25*	*3.25*	*3.27*	*3.28*	*3.27*	*3.28*	*3.28*	*3.28*	*3.28*	*3.28*	*3.28*
	ANNS(p_a) = 50																	
k	8	11	13	17	19	21	25	29	35	39	44	51	60	72	90	121	182	366
h	14	19	21	28	31	34	41	47	56	62	70	82	96	115	143	192	288	580
ANNS(p_r)	*3.78*	*3.86*	*3.78*	*3.83*	*3.83*	*3.87*	*3.86*	*3.87*	*3.86*	*3.85*	*3.86*	*3.88*	*3.87*	*3.88*	*3.88*	*3.88*	*3.88*	*3.89*
	ANNS(p_a) = 100																	
k	7	10	13	16	18	21	24	28	34	38	43	50	58	70	88	118	177	356
h	14	19	26	31	35	41	46	53	65	72	82	96	110	133	167	224	335	674
ANNS(p_r)	*4.42*	*4.32*	*4.44*	*4.45*	*4.48*	*4.45*	*4.43*	*4.42*	*4.47*	*4.45*	*4.47*	*4.47*	*4.46*	*4.48*	*4.48*	*4.48*	*4.49*	*4.50*
	ANNS(p_a) = 200																	
k	7	10	13	16	18	21	24	28	34	38	43	49	57	69	86	116	174	350
h	16	23	31	37	41	49	56	64	78	87	99	112	129	157	195	264	394	792
ANNS(p_r)	*4.85*	*4.94*	*4.98*	*5.02*	*4.98*	*5.01*	*5.05*	*5.02*	*5.04*	*5.04*	*5.07*	*5.07*	*5.06*	*5.08*	*5.10*	*5.10*	*5.10*	*5.10*
	ANNS(p_a) = 300																	
k	7	10	12	16	18	20	23	28	33	37	42	48	57	68	86	115	173	347
h	18	25	30	40	45	50	57	71	81	92	104	119	142	168	213	285	428	856
ANNS(p_r)	*5.27*	*5.24*	*5.37*	*5.32*	*5.33*	*5.41*	*5.41*	*5.43*	*5.39*	*5.44*	*5.42*	*5.45*	*5.44*	*5.45*	*5.45*	*5.47*	*5.47*	*5.47*

6. The Performance of a Geometric CUSUM Chart for Detecting a Range of Shift-sizes

The Tables presented in Section 5 are useful when one has reliable information on the quantities p_a and p_r. But there may be situations where one would prefer not to specify a single value (p_r) for the size of shift to be detected. For such situations, we will assume that it is possible to specify an interval of possible values for p_r, and our intention is to find parameters for a geometric CUSUM scheme that will provide good detection performance for shifts from p_a to any level in this interval.

It is accepted (e.g. Hawkins and Olwell (1998) page 139) that most CUSUMS are robust in the sense that they work quite well for detecting a moderate range of shift-sizes. In order to investigate this for a geometric CUSUM chart, we considered the comparative performance of ten CUSUM schemes each designed to detect a specified shift, for the case of $p_a = 0.004$, with all ten schemes having the in-control level ANNS(p_a) = 50. Thus, ten CUSUM schemes were found for detecting ten specified shift-sizes ranging from $1.5\times p_a$ to $7\times p_a$. The ten schemes and their parameters are listed in Table 7. This Table provides a tabulation of the percentage by which the ANNS(p) value for a particular scheme exceeds the lowest of the ANNS(p) values for the ten schemes. By way of illustration, the value 5 in the row for $p = 0.008$ for the fourth scheme (a scheme designed to detect shifts of size $3\times p_a$) gives the rounded percentage by which ANNS($p = 0.008$) for the latter scheme exceeds the ANNS($p = 0.008$) for the second scheme (which is the scheme designed to detect shifts from $p_a = 0.004$ to $p_r = 0.008$).

It would seem useful to identify a range of shift-sizes for which each of the ten schemes has a detection performance that is not too far away from that of the best of the ten schemes, and these ranges are indicated in the boxed sections of Table 7. (The boxed section for a particular scheme identifies the levels of p for which ANNS(p) for that scheme is within 5% of the lowest ANNS(p) for these ten schemes.) Here are some observations on detection performance of the various schemes considered:

(i) For the fourth scheme (designed for detection of a shift from $p_a = 0.004$ to $p_r = 3\times p_a$), it can be seen that this scheme works quite well for detecting shifts in the range $p_r = 2\times p_a$ to $p_r = 5\times p_a$.

(ii) For the eighth scheme (designed for detection of a shift from $p_a = 0.004$ to $p_r = 5\times p_a$) the detection performance is good for shifts in the range $3\times p_a$ to $10\times p_a$, but is poor for smaller shifts.

(iii) If one wants to have good performance for detection of smaller shifts, the first two schemes are of interest: the performance of the first scheme is acceptable up to $2.25\times p_a$ but is comparatively poor for detecting larger shifts. The second scheme (designed for detection of a shift from $p_a = 0.004$ to $p_r = 2\times p_a$) does quite well for shifts in the range $1.5\times p_a$ up to $3\times p_a$ and even $3.25\times p_a$.

A corresponding investigation for the case of $p_a = 0.002$, with ANNS(p_a) = 100, led to broadly similar results to those presented in Table 7.

TABLE 7 Comparisons of Ten CUSUM Charts for Detecting Shifts from $p_a = 0.004$ to $p_r = m \times p_a$
The charts are designed to detect shifts for values of m = 1.5, 2.0, 2.5...,4.5, 5.0, 6.0, 7.0 subject to ANNS(p_a) ≥ 50.
For each value of p above 0.004, the table-entries give the rounded percentages by which ANNS(p) for a chart exceeds the lowest value of ANNS(p) for all ten charts.
(The row for p = 0.004 contains the ANNS(p_a) values.)

Values of m		1.5	2	2.5	3	3.5	4	4.5	5	6	7
CUSUM parameters	*k*	224	194	171	154	141	129	120	112	100	90
	h	1065	696	506	398	330	275	237	208	171	143
p											
0.004		**50.1**	**50.1**	**50.1**	**50.0**	**50.0**	**50.3**	**50.3**	**50.1**	**50.6**	**50.8**
0.005		0	6	12	16	21	25	28	30	36	41
0.006		0	3	9	15	20	26	30	34	43	50
0.007		0	0	4	8	13	19	23	27	37	45
0.008		3	0	2	5	9	14	17	22	31	39
0.009		5	0	0	2	5	9	12	16	24	32
0.010		8	2	0	1	3	6	9	12	19	26
0.011		11	3	0	0	1	3	6	8	14	21
0.012		14	4	1	0	1	2	4	6	11	17
0.013		16	6	1	0	0	1	2	4	8	13
0.014		19	8	3	0	0	1	1	3	7	11
0.015		21	9	3	1	0	0	1	2	5	9
0.016		23	10	4	1	0	0	0	1	4	7
0.017		25	12	5	2	1	0	0	0	3	5
0.018		27	13	6	3	1	0	0	0	2	4
0.019		29	15	7	4	1	0	0	0	1	3
0.020		31	16	8	4	2	1	0	0	1	2
0.024		37	21	12	7	4	2	1	0	0	1
0.028		42	25	16	10	6	4	3	2	0	0
0.032		47	28	19	12	8	6	5	3	0	0
0.036		50	31	22	14	10	7	6	4	1	0
0.040		53	33	24	15	11	9	8	5	2	0

A useful conclusion from this investigation is that if one has identified an interval of values for the shift-size to be detected, one should *not* take the mid-point of this interval as the level to use for p_r in choosing a value for the parameter k, but rather a value about one third of the way from the lower end of the interval. It seems likely that Tables 3,4 and 5 (for detecting shifts to $2 \times p_a$, $3 \times p_a$ and $5 \times p_a$) will be of most interest. An

approach to the detection of a broader range of shift-sizes is to apply two separate geometric CUSUM schemes (termed a double geometric CUSUM chart) to the same stream of CRL data. This has been investigated by the author, and it is expected that it will be possible to make use of the tables of CUSUM parameter-values reported here. The disadvantage of a double CUSUM chart is its greater complexity both in design and operation.

7. Simple Empirical Relationships among h, k, and p_a

It is useful to consider the quantities ($h \times p_a$), and (h/k) for the CUSUM schemes listed in Tables 2,3,4, 5, and 6. Values of these quantities are presented in Tables 8, 9, 10, 11 and 12. It is noticeable that there is a degree of stability in these values across the range of p_a values in the tables. (In fact, the stability of these quantities was used in guiding the searches to find many of the h values in Tables 2 to 6.) There is even less variability in these quantities for values of p_a below 0.01. It is also noticeable that values of the ratio (h/k) have less variability than the ($h \times p_a$) values. The possibility of making use of the (h/k) ratios to find the parameters of geometric CUSUM schemes for values of p_a other than those listed in Tables 2, 3, 4, 5, and 6 has been investigated. Two separate investigations have been conducted, the first for values of p_a within the range 0.001 to 0.04 (in Section 8), and the second for a value considerably below 0.001 (in Section 9).

TABLE 8: Tabulated Illustrations of the Stability of each of ($h \times p_a$) and (h/k) for Geometric CUSUM schemes for Detecting a Shift from p_a to $1.5 \times p_a$

Values of the in-control proportion p_a (in percent)																		
4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	
ANNS(p_a) = 25					Row 1: values of $h \times p_a$			Row 2: values of h/k										
3.080	3.180	3.150	3.300	3.258	3.344	3.290	3.264	3.310	3.267	3.264	3.290	3.270	3.315	3.312	3.303	3.316	3.316	*Extrapolation*
3.500	3.533	3.500	3.587	3.549	3.603	3.561	3.532	3.559	3.524	3.517	3.534	3.516	3.545	3.538	3.529	3.535	3.531	3.5308
ANNS(p_a) = 50					Row 1: values of $h \times p_a$			Row 2: values of h/k										
3.880	4.200	4.225	4.220	4.230	4.176	4.172	4.248	4.250	4.248	4.312	4.298	4.254	4.260	4.260	4.260	4.294	4.296	*Extrapolation*
4.619	4.828	4.829	4.795	4.796	4.745	4.730	4.784	4.775	4.768	4.813	4.797	4.758	4.760	4.754	4.749	4.771	4.768	4.7681
ANNS(p_a) = 100					Row 1: values of $h \times p_a$			Row 2: values of h/k										
5.320	5.130	5.250	5.380	5.418	5.392	5.432	5.304	5.400	5.454	5.408	5.460	5.460	5.420	5.420	5.403	5.428	5.453	*Extrapolation*
6.333	6.107	6.176	6.256	6.271	6.241	6.258	6.139	6.207	6.247	6.202	6.240	6.233	6.194	6.187	6.168	6.182	6.197	6.1992
ANNS(p_a) = 200					Row 1: values of $h \times p_a$			Row 2: values of h/k										
6.960	6.660	6.125	6.420	6.552	6.560	6.678	6.576	6.460	6.588	6.592	6.503	6.600	6.630	6.604	6.609	6.614	6.619	*Extrapolation*
8.286	7.929	7.424	7.643	7.745	7.736	7.820	7.718	7.600	7.705	7.701	7.615	7.692	7.709	7.679	7.676	7.673	7.670	7.6818
ANNS(p_a) = 300					Row 1: values of $h \times p_a$			Row 2: values of h/k										
6.600	6.630	6.975	7.340	6.894	6.960	7.168	7.116	7.060	7.236	7.272	7.210	7.350	7.255	7.292	7.263	7.302	7.273	*Extrapolation*
8.250	8.185	8.455	8.738	8.326	8.365	8.533	8.471	8.405	8.553	8.575	8.512	8.627	8.535	8.559	8.525	8.550	8.516	8.5435

TABLE 9: Tabulated Illustrations of the Stability of each of ($h \times p_a$) and (h/k) for Geometric CUSUM schemes for Detecting a Shift from p_a to $2 \times p_a$

Values of the in-control proportion p_a (in percent)

4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	
ANNS(p_a) = 25					Row 1: values of $h \times p_a$			Row 2: values of h/k										
2.120	2.190	2.100	2.120	2.178	2.224	2.198	2.148	2.190	2.187	2.176	2.205	2.196	2.195	2.208	2.196	2.210	2.204	*Extrapolation*
2.789	2.808	2.710	2.718	2.750	2.780	2.754	2.712	2.738	2.730	2.720	2.739	2.731	2.727	2.733	2.721	2.728	2.721	2.7269
ANNS(p_a) = 50					Row 1: values of $h \times p_a$			Row 2: values of h/k										
2.560	2.760	2.700	2.760	2.700	2.800	2.814	2.772	2.770	2.808	2.736	2.751	2.778	2.780	2.784	2.784	2.786	2.787	*Extrapolation*
3.556	3.680	3.600	3.632	3.571	3.646	3.655	3.609	3.597	3.628	3.563	3.573	3.589	3.587	3.588	3.583	3.581	3.578	3.5815
ANNS(p_a) = 100					Row 1: values of $h \times p_a$			Row 2: values of h/k										
3.320	3.210	3.225	3.360	3.312	3.280	3.332	3.336	3.380	3.339	3.392	3.339	3.342	3.395	3.400	3.387	3.402	3.405	*Extrapolation*
4.611	4.458	4.448	4.541	4.488	4.457	4.491	4.484	4.507	4.470	4.511	4.458	4.456	4.497	4.497	4.480	4.488	4.486	4.4844
ANNS(p_a) = 200					Row 1: values of $h \times p_a$			Row 2: values of h/k										
3.520	4.020	4.025	3.880	3.870	3.840	3.934	3.972	3.920	4.032	3.984	3.962	3.996	4.015	4.000	4.002	4.006	4.009	*Extrapolation*
5.176	5.583	5.552	5.389	5.375	5.333	5.404	5.426	5.370	5.463	5.413	5.390	5.415	5.426	5.405	5.401	5.399	5.396	5.4048
ANNS(p_a) = 300					Row 1: values of $h \times p_a$			Row 2: values of h/k										
3.960	3.990	4.075	4.360	4.320	4.304	4.172	4.248	4.390	4.356	4.328	4.312	4.368	4.320	4.344	4.329	4.350	4.353	*Extrapolation*
5.824	5.783	5.821	6.056	6.000	5.978	5.843	5.900	6.014	5.975	5.945	5.923	5.967	5.918	5.934	5.914	5.926	5.922	5.9282

TABLE 10: Tabulated Illustrations of the Stability of each of ($h \times p_a$) and (h/k) for Geometric CUSUM schemes for Detecting a Shift from p_a to $3 \times p_a$

Values of the in-control proportion p_a (in percent)

4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	
ANNS(p_a) = 25					Row 1: values of $h \times p_a$			Row 2: values of h/k										
1.240	1.200	1.300	1.260	1.296	1.264	1.274	1.296	1.260	1.296	1.296	1.274	1.266	1.285	1.280	1.284	1.288	1.289	*Extrapolation*
2.067	2.000	2.080	2.032	2.057	2.026	2.022	2.038	2.000	2.028	2.025	2.000	1.991	2.008	2.000	2.000	2.000	1.998	2.0010
ANNS(p_a) = 50					Row 1: values of $h \times p_a$			Row 2: values of h/k										
1.440	1.470	1.600	1.580	1.530	1.600	1.568	1.608	1.590	1.593	1.568	1.610	1.578	1.595	1.592	1.599	1.606	1.608	*Extrapolation*
2.571	2.579	2.667	2.633	2.576	2.632	2.605	2.627	2.607	2.603	2.579	2.614	2.578	2.593	2.584	2.587	2.590	2.589	2.5873
ANNS(p_a) = 100					Row 1: values of $h \times p_a$			Row 2: values of h/k										
1.840	1.860	1.850	1.860	1.818	1.920	1.876	1.872	1.870	1.899	1.880	1.890	1.920	1.915	1.912	1.926	1.912	1.922	*Extrapolation*
3.286	3.263	3.217	3.207	3.156	3.243	3.190	3.184	3.169	3.197	3.176	3.176	3.200	3.192	3.187	3.194	3.176	3.182	3.1845
ANNS(p_a) = 200					Row 1: values of $h \times p_a$			Row 2: values of h/k										
2.240	2.250	2.050	2.100	2.214	2.208	2.170	2.172	2.190	2.232	2.216	2.184	2.184	2.205	2.224	2.217	2.228	2.220	*Extrapolation*
4.000	3.947	3.727	3.750	3.844	3.833	3.780	3.771	3.776	3.815	3.795	3.759	3.753	3.769	3.782	3.770	3.776	3.763	3.7700
ANNS(p_a) = 300					Row 1: values of $h \times p_a$			Row 2: values of h/k										
2.080	2.190	2.250	2.320	2.286	2.288	2.394	2.400	2.330	2.385	2.376	2.415	2.418	2.390	2.384	2.397	2.408	2.410	*Extrapolation*
4.000	4.056	4.091	4.143	4.097	4.086	4.171	4.167	4.088	4.141	4.125	4.157	4.155	4.121	4.110	4.119	4.123	4.120	4.1270

TABLE 11: Tabulated Illustrations of the Stability of each of ($h \times p_a$) and (h/k) for Geometric CUSUM schemes for Detecting a Shift from p_a to $5 \times p_a$

	Values of the in-control proportion p_a (in percent)																	
	4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
ANNS(p_a) = 25	Row 1: values of $h \times p_a$																	
	0.600	0.630	0.700	0.660	0.684	0.672	0.672	0.684	0.690	0.684	0.696	0.686	0.702	0.690	0.696	0.693	0.696	0.698
	Row 2: values of h/k																	
	1.500	1.500	1.556	1.500	1.520	1.500	1.500	1.500	1.500	1.490	1.500	1.485	1.500	1.484	1.487	1.481	1.481	1.479
	Extrapolation																	
	1.4835																	
ANNS(p_a) = 50	Row 1: values of $h \times p_a$																	
	0.800	0.810	0.825	0.780	0.828	0.816	0.812	0.840	0.820	0.828	0.848	0.840	0.846	0.845	0.832	0.837	0.840	0.844
	Row 2: values of h/k																	
	2.000	1.929	1.941	1.857	1.917	1.889	1.871	1.892	1.864	1.878	1.893	1.875	1.880	1.878	1.856	1.860	1.858	1.859
	Extrapolation																	
	1.8684																	
ANNS(p_a) = 100	Row 1: values of $h \times p_a$																	
	0.960	0.900	0.925	0.980	0.954	0.960	0.966	1.008	0.990	0.999	0.992	1.001	1.014	0.995	0.996	1.002	1.006	1.011
	Row 2: values of h/k																	
	2.400	2.308	2.313	2.333	2.304	2.308	2.300	2.333	2.302	2.313	2.296	2.306	2.315	2.287	2.284	2.288	2.286	2.287
	Extrapolation																	
	2.2919																	
ANNS(p_a) = 200	Row 1: values of $h \times p_a$																	
	1.160	1.080	1.100	1.080	1.134	1.136	1.148	1.140	1.140	1.152	1.152	1.162	1.152	1.165	1.172	1.170	1.166	1.172
	Row 2: values of h/k																	
	2.900	2.769	2.750	2.700	2.739	2.731	2.733	2.714	2.714	2.723	2.717	2.721	2.704	2.709	2.713	2.708	2.699	2.700
	Extrapolation																	
	2.7087																	
ANNS(p_a) = 300	Row 1: values of $h \times p_a$																	
	1.040	1.170	1.200	1.180	1.242	1.248	1.190	1.260	1.250	1.260	1.264	1.239	1.266	1.255	1.244	1.251	1.260	1.266
	Row 2: values of h/k																	
	2.889	3.000	3.000	2.950	3.000	3.000	2.931	3.000	2.976	2.979	2.981	2.950	2.972	2.953	2.934	2.937	2.944	2.944
	Extrapolation																	
	2.9499																	

TABLE 12: Tabulated Illustrations of the Stability of each of ($h \times p_a$) and (h/k) for Geometric CUSUM schemes for Detecting a Shift from p_a to $7 \times p_a$

Values of the in-control proportion p_a (in percent)																		
4.0	3.0	2.5	2.0	1.8	1.6	1.4	1.2	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	
ANNS(p_a) = 25					Row 1: values of $h \times p_a$			Row 2: values of h/k										
0.400	0.420	0.450	0.460	0.450	0.432	0.448	0.432	0.450	0.450	0.448	0.448	0.450	0.455	0.456	0.456	0.454	0.456	*Extrapolation*
1.250	1.273	1.286	1.278	1.250	1.227	1.231	1.200	1.216	1.220	1.217	1.208	1.210	1.213	1.213	1.206	1.201	1.200	1.2085
ANNS(p_a) = 50					Row 1: values of $h \times p_a$			Row 2: values of h/k										
0.560	0.570	0.525	0.560	0.558	0.544	0.574	0.564	0.560	0.558	0.560	0.574	0.576	0.575	0.572	0.576	0.576	0.580	*Extrapolation*
1.750	1.727	1.615	1.647	1.632	1.619	1.640	1.621	1.600	1.590	1.591	1.608	1.600	1.597	1.589	1.587	1.582	1.585	1.5914
ANNS(p_a) = 100					Row 1: values of $h \times p_a$			Row 2: values of h/k										
0.560	0.570	0.650	0.620	0.630	0.656	0.644	0.636	0.650	0.648	0.656	0.672	0.660	0.665	0.668	0.672	0.670	0.674	*Extrapolation*
2.000	1.900	2.000	1.938	1.944	1.952	1.917	1.893	1.912	1.895	1.907	1.920	1.897	1.900	1.898	1.898	1.893	1.893	1.8988
ANNS(p_a) = 200					Row 1: values of $h \times p_a$			Row 2: values of h/k										
0.640	0.690	0.775	0.740	0.738	0.784	0.784	0.768	0.780	0.783	0.792	0.784	0.774	0.785	0.780	0.792	0.788	0.792	*Extrapolation*
2.286	2.300	2.385	2.313	2.278	2.333	2.333	2.286	2.294	2.289	2.302	2.286	2.263	2.275	2.267	2.276	2.264	2.263	2.2720
ANNS(p_a) = 300					Row 1: values of $h \times p_a$			Row 2: values of h/k										
0.720	0.750	0.750	0.800	0.810	0.800	0.798	0.852	0.810	0.828	0.832	0.833	0.852	0.840	0.852	0.855	0.856	0.856	*Extrapolation*
2.571	2.500	2.500	2.500	2.500	2.500	2.478	2.536	2.455	2.486	2.476	2.479	2.491	2.471	2.477	2.478	2.474	2.467	2.4758

8. Finding Geometric CUSUM Schemes by Interpolation for p_a in the Range 0.001 to 0.04

In this Section, it is shown that one may use interpolation to find geometric CUSUM schemes for any p_a within the range 0.001 to 0.04. The CUSUM schemes that may be found by the following method are limited to the set of ANNS(p_a) values (i.e. 25, 50, 100, 200 and 300) and also the specified shift-sizes ($1.5\times p_a$, $2\times p_a$, $3\times p_a$, $5\times p_a$ and $7\times p_a$) that were used in searching for the schemes listed in Tables 2 to 6. It was decided to use the stability of (h/k) in the interpolations.

The results of this investigation are given in Table 13. Section A of Table 13 shows the combinations of ANNS(p_a) levels, shift-sizes and values of p_a for each of which we wished to find a geometric CUSUM scheme. Thus, the intention was to find the parameters of 16 geometric CUSUM schemes by interpolation among the schemes listed in Tables 2 to 6.

We will describe the interpolation procedure by presenting the following application:

Example: As part of the manufacturing process for solid-state drives (SSDs) each SSD is given a functionality test, and is declared conforming or nonconforming. The result of each test is available shortly after testing, and it is feasible to maintain the test results as an ordered stream. Based on the most recent experience of the process, the quality engineer with responsibility for the process has concluded that the in-control fraction nonconforming (p_a) is about 0.0025. The quality engineer wishes to find the parameters of a geometric CUSUM chart so that upward shifts to $p_r = 3\times p_a$ and above can be detected quickly. In quantifying an acceptable rate for false-signals, it has been decided to have a target-level of 100 for ANNS(p_a).

Step 1: In finding the value of k, we begin by using equation (4.1). The resulting value (218.62) is then increased by the appropriate percentage from Table 1 (in this case 10.1) taking account of the results of the investigation in Section 4. We round this to the nearest integer, giving the value of 241 for k.

Step 2: We interpolate the value of the ratio (h/k) by using 6 values (or 4 or even 2, if 6 are unavailable) from the relevant section of Table 10. The 6 values are chosen so that 3 values are on either side of the target p_a value. In this case (with the target $p_a = 0.0025$), 4 values were used: 3.187, 3.194, 3.176, 3.182. We then multiply the average of these 4 values by the k value from step 1, and round to the nearest integer to get the value of parameter h, which turns out to be 768. Thus the parameters of the geometric CUSUM chart for this application are $k = 241$ and $h = 768$. In starting this CUSUM, one could take h/2 as the initial value of the CUSUM, and if the proportion is already shifted from $p_a = 0.0025$ then this will speed up the occurrence of a signal from the CUSUM chart. But if the proportion has not shifted, this head-start value for the CUSUM will have a negligible effect in producing a false alarm.

TABLE 13: Results of Interpolation to find Parameters of Geometric CUSUM Schemes

Section A: Specifications of each of 16 CUSUM Schemes (*each entry in the table is the level of* p_a)

	Shift-size to be detected			
	$2\times p_a$	$3\times p_a$	$5\times p_a$	$7\times p_a$
ANNS(p_a) = 25	0.017	0.0078	0.0035	0.013
ANNS(p_a) = 50	0.0075	0.023	0.0078	0.035
ANNS(p_a) = 100	0.028	0.0025	0.017	0.0086
ANNS(p_a) = 200	0.0052	0.013	0.028	0.0053

Section B: Parameter-values (*k, h*) *found by interpolation for each specification in Section A*

	Shift-size to be detected			
	$2\times p_a$	$3\times p_a$	$5\times p_a$	$7\times p_a$
ANNS(p_a) = 25	47, 129	82, 165	134, 199	28, 34
ANNS(p_a) = 50	103, 370	26, 68	57, 107	9, 16
ANNS(p_a) = 100	26, 117	241, 768	25, 58	40, 76
ANNS(p_a) = 200	142, 768	44, 167	14, 39	65, 148

Section C: Evaluations of ANNS(p_a) *for each CUSUM scheme in Section B*

	Shift-size to be detected			
	$2\times p_a$	$3\times p_a$	$5\times p_a$	$7\times p_a$
ANNS(p_a) = 25	24.5	24.9	25.1	25.2
ANNS(p_a) = 50	50.0	49.4	51.0	66.8
ANNS(p_a) = 100	99.5	100.3	99.7	100.7
ANNS(p_a) = 200	201.0	211.3	231.3	202.7

It is hoped that the resulting geometric CUSUM scheme with parameters $k = 241$, and $h = 768$ will yield a value for ANNS(p_a) that is close to 100, but exceeds it. Using the evaluation procedure presented in the Appendix, we found that ANNS(p_a) = 100.3 for this scheme. This is an encouraging result, but other checks are necessary: further results from this interpolation procedure are presented in Table 13.
In most cases, the procedure leads to (k, h) parameter values that come close to yielding the desired ANNS(p_a) specification.

There are two exceptions to this in Table 13, both associated with relatively large values of p_a. The scheme ($k = 9$, $h = 16$) for $p_a = 0.035$ and $p_r = 7 \times p_a$, resulted in ANNS(p_a) = 66.8, which is considerably larger than the ANNS(p_a) specification of 50. This scheme has relatively small values for k and h, and such values arise for larger values of p_a (and also for detection of larger shifts). CUSUM specifications that lead to small values of k and h have the consequence that ANNS(p_a) is less likely to fall close to its target, as in this case. It is noted that in interpolating for this scheme ($k = 9$, $h = 16$), the value (0.035) for p_a has only one value (i.e. $p_a = 0.04$) above it in Table 12. So, in this case the interpolation for the ratio (h/k) was based on only 2 values (1.750, 1.727) rather than on 4 or 6 values. A direct search (using exact steady-state evaluation of ANNS) yielded the scheme ($k = 9$, $h = 15$) for which ANNS(p_a) = 54.0.

It is suggested that the interpolation procedure (and also the extrapolation considered in Section 9) could be conveniently implemented using a spreadsheet system containing the (k,h) values in Tables 2 to 6. A spreadsheet version of these tables is provided in the supplementary material available online.

9. Finding Geometric CUSUM Schemes by Extrapolation

In this Section, we investigate the possibility of using the stability of the (h/k) ratio to find CUSUM schemes for a value of p_a (0.0001) much lower than 0.001. As for the case of interpolation considered earlier, there are two steps in the procedure. The first step (finding a value for k) is carried out in the same manner as described earlier. In the second step (finding a value for h) a simple extrapolation procedure is used. By inspecting the entries for the ratio (h/k) in Tables 8 to 12, it may be noted that this ratio becomes more stable for the lower values of p_a, and this is because the geometric CUSUM is tending to converge toward the exponential CUSUM. (There is further consideration of this in Section S2 of the supplementary material available online),

For each of the extrapolations, it was decided to use the set of 8 values for (h, k) arising from the p_a values 0.008, 0.007,...,0.001. The largest and smallest of the 8 ratios (h/k) were discarded and the average of the 6 remaining ratios was evaluated in each case. (These averages are given under the heading *Extrapolation* in Tables 8 to 12, on the right-hand side.) The value of k (from step 1) was multiplied by the relevant average, and the resulting value was rounded to the nearest integer to provide the value for h.

These extrapolations were carried out for a single value of p_a (i.e. 0.0001) and for four levels of ANNS(p_a), and four sizes of shift above p_a. The 16 schemes found using this extrapolation procedure are presented in Table 14. One would hope that these schemes come close to achieving the specified levels for ANNS(p_a). Unlike the interpolations in Section 8, it was not possible to get exact evaluations of ANNS(p_a) because the values of h for these 16 schemes are too large to allow the matrix inversions referred to in the Appendix. It was necessary to use simulation to evaluate ANNS(p_a), and this was done using five million repetitions for each of the 16 schemes. The resulting values for ANNS(p_a) are presented in Section B of Table 14 showing that the ANNS(p_a) specifications have been met quite well. This is encouraging and indicates that the stability of the (h/k) ratio may be relied upon to find geometric CUSUM schemes for a level of p_a which is considerably lower than the levels for which exact ANNS evaluations are possible.

TABLE 14: Results of Extrapolation to find Parameters of Geometric CUSUM Schemes

Section A: Parameter-values (k, h) *found by extrapolation for four levels of ANNS*(p_a) *and four shift-sizes. All 16 CUSUM schemes are for* $p_a = 0.0001$.

	Shift-size to be detected			
	$1.5\times p_a$	$2\times p_a$	$3\times p_a$	$5\times p_a$
ANNS(p_a) = 25	9398, 33182	8116, 22131	6464, 12934	4734, 7023
ANNS(p_a) = 50	9016, 42989	7804, 27950	6217, 16085	4553, 8507
ANNS(p_a) = 100	8806, 54590	7596, 34064	6047, 19257	4429, 10151
ANNS(p_a) = 200	8635, 66332	7436, 40190	5915, 22300	4352, 11788

Section B: Simulation evaluations of ANNS(p_a) *for each CUSUM scheme in Section A*

	Shift-size to be detected			
	$1.5\times p_a$	$2\times p_a$	$3\times p_a$	$5\times p_a$
ANNS(p_a) = 25	25.0	25.1	25.1	25.2
ANNS(p_a) = 50	50.1	50.0	50.1	50.8
ANNS(p_a) = 100	100.4	100.3	100.7	100.9
ANNS(p_a) = 200	201.9	201.8	201.0	203.5

It should be pointed out that, despite the fact that the geometric CUSUM chart is often listed as a method for monitoring "high quality processes" this chart is also very efficient for monitoring proportions that are not small. As might be expected, an extrapolation procedure for values of p_a above 0.04 does not work well. For example, for $p_a = 0.10$ and $p_r = 0.20$, where we would like ANNS(p_a) to be at least 100, one finds by extrapolation that the parameters turn out to be ($k = 6$, $h = 27$) with ANNS(p_a) = 129.8. Using exact evaluations, the scheme (6,25) is found for which ANNS(p_a) = 101.6. Other attempts at extrapolation are more inaccurate, so that we cannot recommend extrapolation for p_a on the upper side of the range (0.04 to 0.001).

10. Summary and Discussion

The problem of providing a convenient method for choosing values for the parameters (k, h) of a geometric CUSUM chart has been considered. Investigations have been conducted on the best choice of the parameter k, and it is recommended that a percentage increase be applied to the value of k computed from the sequential probability ratio formula. The size of the percentage increase depends on both the chosen level of the false-signal rate (quantified by ANNS(p_a)), and the size of shift to be detected. Tables of proposed values for k and h have been developed for 18 levels of the in-control proportion (p_a) in the range (0.04 to 0.001), for five sizes of upward shift above p_a, and for five levels of the occurrence-rate of false signals, so that parameter-sets for 450 CUSUM schemes are listed over five tables. Using these tables, some simple empirical relationships among the quantities k, h, and p_a have been identified. The stability of the ratio (h/k) has been used to interpolate a value for h for any value of p_a within the range (0.04 to 0.001). It is also shown that this stability may be used to design geometric CUSUM schemes for values of p_a as low as 0.0001. The reason for the stability of the ratio is that, for quite small values of p_a, the geometric distribution converges toward the exponential distribution. In a supplementary section available online, the relationship between the exponential CUSUM and geometric CUSUM is illustrated by using an extrapolation approach to find the parameters of five exponential CUSUM schemes.

Consideration has also been given to detection of a shift from an acceptable level (p_a) to any level in an interval, and it is recommended that, in finding a value for the parameter k, one should use a value for p_r about one-third of the way from the lower end of this interval.

Although the results presented here relate to the design of a geometric CUSUM chart, these results are also useful for the design of a Bernoulli CUSUM chart, bearing in mind the equivalence between these two CUSUM schemes as pointed out by a number of researchers, and most recently in Szarka and Woodall (2012). The connection between the two sets of CUSUM parameters is as follows:

$$k_B = 1/(k_G+1) \quad \text{and} \quad h_B = (h_G + k_G)/(k_G+1) \qquad (\text{for } k_G \geq 1,\ h_G \geq 2) \qquad (10.1)$$

where (k_B, h_B) are the parameters of a Bernoulli CUSUM scheme, and (k_G, h_G) are the integer-valued parameters of a geometric CUSUM scheme as defined in this paper. Thus, if one wished to use a Bernoulli CUSUM scheme to detect an upward shift for a proportion, one could use the methods presented here to design a geometric CUSUM chart, and then convert the geometric CUSUM parameter-values to the corresponding levels for a Bernoulli CUSUM scheme. However, this is possible only for the five chosen shift-sizes, and for the five specified levels for ANNS(p_a).

The geometric CUSUM chart and the Bernoulli CUSUM chart are also useful for detecting downward shifts in a proportion, and a corresponding paper to aid parameter-choice in this case is in preparation. Progress has been made on the design of a double geometric CUSUM scheme using the tables presented here to broaden the range of shift-sizes that can be efficiently detected.

ACKNOWLEDGMENT

The author acknowledges the helpful comments of the reviewers and the Editor, and their suggestions for improvement.

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References

Bourke P. D. (1991). Detecting a shift in fraction nonconforming using run-length control charts with 100% inspection. *Journal of Quality Technology* 23(3): 225 - 238.

Bourke P.D. (2001). The geometric CUSUM chart with sampling inspection for monitoring fraction defective. *Journal of Applied Statistics* 28(8): 951-972.

Brook, D., and Evans, D.A. (1972). An approach to the probability distribution of CUSUM run-length. *Biometrika* 59(3): 539-549.

Chang, T.C., and Gan, F.F. (2001). Cumulative sum charts for high yield processes. *Statistica Sinica* 11, 791-805.

Crozier, R.B. (1986). A new two-sided cumulative sum quality control scheme. *Technometrics* 28(3): 187-194.

Hawkins, D.M., and Olwell, D.H. (1998). *Cumulative sum charts and charting for quality improvement*. New York: Springer.

Lorden, G., and Eisenberger, I. (1973). Detection of failure rate increases. *Technometrics* 15(1): 167-175.

Lucas, J.M., and Saccucci, M.S. (1990). Exponentially weighted moving average control schemes: properties and enhancements. *Technometrics* 32(1): 1-12.

Lucas, J.M. (1994). Discussion on "Run-length distributions of special-cause control charts for correlated processes." *Technometrics* 36(1): 17-19.

Moustakides, G.V. (1986). Optimal stopping times for detecting changes in distributions. *Annals of Statistics*. 14, 1379-1387.

Reynolds, M.R., Jr. and Stoumbos, Z.G. (1999). A CUSUM chart for monitoring a proportion when inspecting continuously. *Journal of Quality Technology* 31(1): 87-108.

Schuh, A., Woodall, W.H. and Camelio, J.A. (2013). The effect of aggregating data when monitoring a Poisson process. *Journal of Quality Technology* 45(3): 260-272.

Szarka, J.L.,III and Woodall, W.H. (2011). A review and perspective on surveillance of Bernoulli processes. *Quality and Reliability Engineering International* 27(6): 835-752.

Szarka, J.L.,III and Woodall, W.H. (2012). On the equivalence of the Bernoulli and geometric CUSUM charts. *Journal of Quality Technology* 44(1): 54-62.

Woodall, W.H. (2017). Bridging the gap between theory and practice in basic statistical process monitoring. *Quality Engineering* 29(1): 2-15.

Yaschin, E. (1993). Statistical control schemes: methods, applications, and generalizations. *International Statistical Review* 61(1): 41-66.

Appendix: Evaluation of ANNS(p) for a Geometric CUSUM Chart

When a shift in the value of p occurs, it can happen in either of the following ways, each requiring a separate method for evaluating ANNS:

1. A shift can happen immediately after a conforming item (because of the random-shift assumption). This gives rise to a *mixed* or *cross-over* CRL consisting of two parts: the first part from the in-control process with $p = p_a$, and the second from a shifted process with $p = p_r$. We refer to the associated evaluation of ANNS(p) as ANNS(p, with cross-over CRL).
2. A shift can happen immediately after a non-conforming item, so that there is no cross-over CRL. We refer to the associated evaluation of ANNS(p) as ANNS(p, no cross-over CRL).

Taking account of the methodology presented in Bourke (2001) we have

$$\text{ANNS}(p) = (p)\text{ANNS}[p, \text{ no cross-over CRL}] + (1 - p)\text{ANNS}[p, \text{ with cross-over CRL}] \quad \text{(A1)}$$

In order to evaluate each of the ANNS terms on the right-hand side of equation (A1), the first step is to find the steady-state distribution of CUSUM values when $p = p_a$. An important element in finding this steady-state distribution is to build a transition matrix that causes the CUSUM value to be re-set to zero whenever a false signal occurs from the CUSUM. The methodology for doing this has been developed by Crosier (1986) and Lucas and Saccucci (1990). We denote the resulting steady-state distribution as $\mathbf{p}_{ss}$, whose elements give the relative frequencies of the non-signalling CUSUM values 0,1,2,...(h-1) prior to the occurrence of the shift from p_a to p.

If we consider the following equation from Brook and Evans (1972)

$$\boldsymbol{\mu} = (\mathbf{I} - \mathbf{R})^{-1}\mathbf{1} \quad \text{(A2)}$$

where $\mathbf{R}$ is the transition matrix for the non-signalling CUSUM states, it may be seen that (for a geometric CUSUM) the elements of the vector $\boldsymbol{\mu}$ give the average number of CRLs until a signal occurs for a CUSUM starting out from each of the possible initial states 0, 1, 2,..., (h-1). (The structure of $\mathbf{R}$ is given in Bourke (1991) and the value of p that is relevant here is p_r.)

We can now get the first term in equation (A1) as follows:

$$\text{ANNS}[p, \text{ no cross-over CRL}] = \mathbf{p}_{ss}^{T}\boldsymbol{\mu} \quad \text{(A3)}$$

In order to evaluate the second ANNS term on the right-hand side of equation (A1) we need to consider that a revision of the steady-state distribution ($\mathbf{p}_{ss}$) over the non-signalling CUSUM states is necessary because of the cross-over CRL. This revised distribution is as follows

$$\mathbf{r}_{ss} = \mathbf{p}_{ss}\mathbf{K} \quad \text{(A4)}$$

where $\mathbf{K}$ is the transition matrix for the cross-over CRL, and is given in Bourke (2001). We then have

$$\text{ANNS}[p, \text{ with a cross-over CRL}] = [1 - \textstyle\sum(\mathbf{r}_{ss})_i] + (\mathbf{r}_{ss})^{T}(1+\boldsymbol{\mu}) \quad \text{(A5)}$$

The term in square brackets on the right-hand side of equation (A5) is the probability of a signal from the cross-over CRL.