Lebesgue space bounds for 2-surfaces in Convolution and Decoupling

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I, Conor Meade, certify that this thesis is my own work and I have not obtained a degree in this university or elsewhere on the basis of the work submitted in this thesis.

Conor Meade

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Pre-Introduction

This thesis is a collection of proofs of theorems in the field of Harmonic Analysis. The through-line that connects these theorems is the estimation of L^p norms of operators relating to polynomial 2-surfaces, with the curvature of these surfaces playing a key role in each of the results.

The operators that define these theorems are most generally parameterised by a choice of manifold in $\mathbb{R}^n$. Typically, the cases where the manifold is of dimension 1 or of dimension $n-1$ have been well studied, but intermediate dimensions remain difficult to obtain results for. In this thesis, we aim to extend techniques used for 1-dimensional manifolds to obtain results for some 2-dimensional manifolds.

Within each of these sections, we will discuss the context and background of these problems in greater detail, however this section will provided a brief road map of the layout of this thesis.

These proofs will be presented in the order in which they were derived, starting with some uniform estimates for convolution for complex polynomial curves in $\mathbb{C}^3$.

We then move onto a collaborative result, researched with Jaume de Dios Pont, showing that the complex moment curve exhibits decoupling, in appropriate L^p contexts.

Lastly, we return to the topic of convolution, this time establishing near optimal estimates for a particular well-curved 2-surface, which is not generated by a complex polynomial curve.

Before moving on however, we first take the opportunity to introduce some commonly accepted notation for these kinds of problems. We say that, for positive quantities A and B, we have $A \lesssim B$ if $A \leq CB$, for some constant $C > 0$. Furthermore, we say say $A \sim B$ if $A \lesssim B \lesssim A$. Here, C, the implicit constant, can depend on whatever parameters are appropriate for the problem in question.

Chapter 1

Uniform estimates for Convolution operators associated to complex polynomial curves in $\mathbb{C}^3$

1.1 Statement of theorem

Consider the complex polynomial curve $\Gamma(z) := (P_1(z), P_2(z), P_3(z))$, for complex polynomials P_i.

We let

$$L_\Gamma(z) := \left|\det(\Gamma'(z), \Gamma''(z), \Gamma'''(z))\right|, \tag{1.1}$$

and

$$\lambda_\Gamma := L_\Gamma^{\frac{1}{3}}. \tag{1.2}$$

For $f : \mathbb{C}^3 \to \mathbb{C}$ we define the weighted complex convolution operator associated to Γ as:

$$T_\Gamma f(z) := \int_{\mathbb{C}} f\left(z - \Gamma(w)\right) \lambda_\Gamma(w) dw = \nu * f(z), \tag{1.3}$$

where ν is the measure defined via, for a test function ϕ,

$$\nu(\phi) = \int_{\mathbb{C}} \phi(\Gamma(w))\lambda_\Gamma(w) dw.$$

Note that this measure is affine invariant, in that the measure of ϕ does not change if we replace Γ with $A\Gamma$, for A a complex affine map. Furthermore, thanks to our choice of exponent in definition (1.2), it is also invariant under reparameterisation of Γ.

In this section, we establish a key bound for T_Γ:

Theorem 1.1.1. *Let Γ be a complex polynomial curve in $\mathbb{C}^3$ of degree N, and $f \in L^{2,u}$. Then, for the complex convolution operator associated to Γ, T_Γ, there exists $C = C(N)$ such that:*

$$||T_\Gamma f||_{L^{3,v}} \leq C\,||f||_{L^{2,u}}$$

for all $u \in [0,3]$, $v \in [2,\infty]$, and $u < v$.

To establish this theorem, we will first establish the restricted-weak type estimate at $(p,q) = (2,3)$, and apply the argument from [Sto09] to obtain this range of Lorentz estimates. Utilising this theorem, we can obtain more estimates that are uniform in Γ. This is done by interpolating between the estimate in Theorem 1.1.1 with its dual estimate, using the Marcinkiewicz interpolation theorem. This results in the fact that

$$||T_\Gamma f||_{L^{q_\theta}(\mathbb{C}^3)} \leq C||f||_{L^{p_\theta}(\mathbb{C}^3)},$$

for $(p_\theta, q_\theta) = (\frac{6}{3+\theta}, \frac{6}{2+\theta})$, for $\theta \in (0,1)$, and some constant dependant only on N and θ. Furthermore, if we replace T_Γ with its local analogue, where the region of integration is restricted to some disk D, we can acquire a larger range of non-uniform estimates by interpolating with a trivial $L^\infty \to L^\infty$ estimate, and its dual.

Explicitly, letting $\mathcal{R}$ be the convex hull of the points $(1,1)$, $(0,0)$, $\left(p_0^{-1}, q_0^{-1}\right)$, and $\left(p_1^{-1}, q_1^{-1}\right)$. Then if $(p^{-1}, q^{-1}) \in \mathcal{R}$, then for all complex polynomial curves, Γ, there exists a constant $C = C(N,\Gamma,D,p,q)$, such that for all functions $f \in L^p(\mathbb{C}^3)$:

$$||T_\Gamma f||_{L^q(\mathbb{C}^3)} \leq C||f||_{L^p(\mathbb{C}^3)}.$$

Much work has been done on these kinds of estimates, and they are known when T_Γ is replaced with its real analogue. In the real case, in arbitrary dimension, d, Γ is a real polynomial curve, the integration is over some interval, and the weight function λ_Γ is replaced with $\lambda_\Gamma(t) := L_\Gamma^{\frac{2}{d(d+1)}}(t)$. Furthermore, our constant C may depend on the dimension we are in for these estimates to hold.

Usually, these estimates have been for a particular family of polynomial curves, or in a particular dimension. For example, in 1998, in [Chr98], Christ established this result for the moment curve in any dimension. That is to say he established it for $\Gamma(t) = (t, t^2 \cdots t^n)$ in $\mathbb{R}^n$. It should be noted that for curves like this, we have $\lambda_\Gamma(t) = n!$, which simplifies the problem. This can be considered the most non-degenerate case, as all derivatives of Γ are linearly independent. Later papers introduced the weight λ_Γ in order to deal with problematic singular points of Γ, where the derivatives become linearly dependent. At such points, λ_Γ vanishes, allowing us to mitigate the effect of such points during the integration.

Further estimates were established for other families of curves, but it was in 2002, in [Obe02a], that Oberlin established the real estimate for general polynomial curves in two dimensions, and in three dimensions, established it for curves of the form $\gamma(t) = (t, P_1(t), P_2(t))$. In [DLW09], this three dimensional case was extended to general polynomial curves, and in [Sto16a], this was extended to higher dimensions.

The first such estimates for surfaces instead of curves can be found in [DG91], in which Drury and Guo proved a class of estimates of this type, but instead of convolving with a measure arising from a polynomial curve, their estimates were pertaining to the case where Γ was instead a two dimensional surface in $\mathbb{R}^4$, of the form $(x, y, \phi_1(x, y), \phi_2(x, y))$, with certain conditions on ϕ_1 and ϕ_2.

In [CH18], Chung and Ham established similar results, considering surfaces that could be represented as complex curves in $\mathbb{C}^d$. They established these bounds for curves of the form $\left(z, z^2 \cdots, z^{d-1}, z^N\right)$ in $\mathbb{C}^d$, and $(z, z^2, \phi(z))$ in $\mathbb{C}^3$, for analytic ϕ. They do this by using a more powerful analogue to the Lemma (1.3.1) which appears in this section, which is established in the complex Fourier restriction paper [BH14], involving a lower bound in terms of the arithmetic means of $\{L_\Gamma(z_i)\}$, as opposed to the geometric mean. This method suffices for the complex curves in question in [CH18], but in general does not hold. In the case of Theorem 1.1.1, we do not utilise a lower bound in terms of the arithmetic mean, but instead adapt the geometric mean bound deriving from [DW10] into the complex case, and achieve the overall estimate using that.

Note that while the two-dimensional estimate has not been established for general complex polynomial curves, these estimates can be derived from the procedure described in our proof of Theorem 1.1.1, with simple modifications.

The explicit description of this is omitted to aid the presentation of the three dimensional case.

1.2 Decomposition of $\mathbb{C}$

To achieve the desired results, we first partition $\mathbb{C}$ into various convex sets. This is done using two decompositions as described in [DW10]. In [DW10], these decompositions are described in relation to $\mathbb{R}$, but can be extended to $\mathbb{C}$. The first of these decompositions will be referred to as D1.

D1: Given a polynomial $Q(z)$, we can decompose $\mathbb{C}$ into a bounded number of convex sets, B_i. In each of these B_i, we have $|Q(z)| \sim c(Q)\,|z-b_i|^{k_i}$, for b_i a root of $Q(z)$, and k_i some positive integer. We will frequently omit the subscripts of k and b for convenience.

To see how this decomposition works, we first write $Q(z) = A\prod_{j=1}^{d'}(z-\eta_j)^{\alpha_j}$, with d' being the number of distinct roots of Q. We now decompose $\mathbb{C}$ into a union of d' convex sets, given by the Voronoi diagram associated to $\{\eta_1, \eta_2, \cdots, \eta_{d'}\}$. In these sets, we will have our complex centers b be the root associated with this set. Formally, these sets are denoted $S(\eta_j) = \{z \in \mathbb{C} \mid |z-\eta_j| \leq |z-\eta_i| \text{ for all } i \neq j\}$.

Each of these sets is further divided into finitely many sectors, centered on b. For $n \geq 1$, denote $\Delta^n_\varepsilon(\eta_j) = \left\{z - \eta_j = re^{i\theta} \in S(\eta_j) | (n-1)\varepsilon \leq \theta \leq n\varepsilon\right\}$. Here, ε is chosen such that $\frac{2\pi}{\varepsilon}$ is an integer. Specifically, we need to take $\varepsilon \leq \frac{\pi}{4}$, which is necessary to decompose the intersection of some annuli and these sectors into convex sets. While this is sufficient for D1 to function as described, we will see that we also need to choose ε to be dependent on d, the degree of our polynomial curve. Thus the choice of ε is currently not specified.

Fixing one such $\Delta^n_\varepsilon(\eta_j) = \Delta(\eta_j)$, we relabel $\eta_j = \hat{\eta}_1$, and the remaining roots as $\hat{\eta}_k$ so that we have $|\hat{\eta}_1 - \hat{\eta}_2| \leq |\hat{\eta}_1 - \hat{\eta}_3| \leq \cdots \leq |\hat{\eta}_1 - \hat{\eta}_{d'}|$. That is to say we order them according to their distance from η_j We let, for $k \geq 2$:

$$T^n_{k,j} = \left\{z \in \Delta(\hat{\eta}_1) \cap S(\hat{\eta}_1) \;\Big|\; |z-\hat{\eta}_1| \leq \frac{1}{2}\,|\hat{\eta}_1 - \hat{\eta}_k|\right\}. \tag{1.4}$$

Labelling $T_{1,j} = \varnothing$, and $T_{d'+1,j} = \Delta(\hat{\eta}_1)$, we see that $T_{1,j} \subseteq T_{2,j} \subseteq \cdots \subseteq T_{d',j} \subseteq T_{d'+1,j} = \Delta(\hat{\eta}_1)$. Also note that if $z \in T^n_{k,j}$, then $\frac{1}{2}\,|\hat{\eta}_l - \hat{\eta}_1| \leq |z - \hat{\eta}_l| \leq \frac{3}{2}\,|\hat{\eta}_l - \hat{\eta}_1|$, for all $l \geq k$, and if $z \notin T^n_{i,j}$, then

$|z-\hat{\eta}_1| < |z-\eta_l| < 3\,|z-\eta_1|$ for all $k \leq i$.

Letting, for $1 \leq i \leq d'$, $I^n_{i,j} = T^n_{i+1,j} \backslash T^n_{i,j}$, we can see that $\Delta^n_\varepsilon(\eta_j) = \bigcup_{k=1}^{d'} I^n_{k,j}$, and on any fixed $I_{k,j}$, we have:

$$|Q(z)| = |A| \prod_{l=1}^{d'} |z-\eta_l|^{\alpha_l} \sim |A|\,|z-\hat{\eta}_1|^{\hat{\alpha}_1+\hat{\alpha}_2+\cdots+\hat{\alpha}_k} \prod_{l=k+1}^{d'} |\hat{\eta}_1-\hat{\eta}_l|^{\hat{\alpha}_l}\,,$$

Note here that $\hat{\alpha}_j$ denotes the multiplicity of $\hat{\eta}_j$.

So indeed, writing $\mathbb{C} = \bigcup_{l=1}^{\frac{2\pi}{\varepsilon}d'^2} I_l = \bigcup_{j=1}^{d'} \bigcup_{n=1}^{\frac{2\pi}{\varepsilon}} \bigcup_{i=1}^{d'} I^n_{i,j}$, we have on $B_l = I^n_{i,j}$ that:

$$|Q(z)| \sim c(Q)\,|z-b_l|^{k_l}\,, \tag{1.5}$$

where $b_l = b_{i,j,n} = \eta_j$, and $k_l = k_{i,j,n} = \hat{\alpha}_1 + \hat{\alpha}_2 \cdots + \hat{\alpha}_i$.

Note that these sets are intersections of some convex sets, and annuli centred at b_l, and therefore are not necessarily convex. This can be remedied by "convexifying" our annuli sectors. This is done on any of these annuli sectors by constructing that tangent to the annuli which is perpendicular to the lower ray of the sector. Each annuli sector in the decomposition is now replaced with the trapezoid which has sides given by these tangents, and the two rays of the sector. More precisely, we replace the set

$$I^n_{k,j} = \left\{ z \in \Delta^n_\varepsilon(\eta_j) \cap S(\eta_j) \;\middle|\; |z-b| \in \left[\frac{1}{2}\,|\eta_j-\hat{\eta}_i|\,, \frac{1}{2}\,|\eta_j-\hat{\eta}_{i+1}|\right] \right\},$$

with the set

$$\tilde{I}^n_{k,j} = \left\{ z \in \Delta^n_\varepsilon(\eta_j) \cap S(\eta_j) \;\middle|\; \mathrm{Re}(e^{-(n-1)\varepsilon}(z-b)) \in \left[\frac{1}{2}\,|\eta_j-\hat{\eta}_i|\,, \frac{1}{2}\,|\eta_j-\hat{\eta}_{i+1}|\right] \right\},$$

where the relabeling of the $\hat{\eta}_i$ are done relative to η_j. Note that this set maintains the property that for all $z \in \tilde{I}^n_{k,j}$, (1.5) holds. This is because $\tilde{I}^n_{k,j}$ is contained in the annulus sector

$$J^n_{k,j} = \left\{ z \in \Delta^n_\varepsilon(\eta_j) \;\middle|\; |z-b| \in \left[\frac{1}{2}\,|\eta_j-\hat{\eta}_i|\,, \frac{\sqrt{1+(\tan\varepsilon)^2}}{2}\,|\eta_j-\hat{\eta}_{i+1}|\right] \right\},$$

for which (1.5) holds for the same reasoning as for $I^n_{k,j}$, with slightly different constants. Furthermore, notice that $\tilde{I}^n_{k,j}$ are all trapezoids, with the exception of the sets $\tilde{I}^n_{k,1}$, which are right triangles. In either case, each $\tilde{I}^n_{k,j}$ is convex.

For the next decomposition, referred to as D2, we require a polynomial $Q(z)$, and a centre, which is some complex number b.

D2: Given a polynomial $Q(z)$, and complex number b, and some convex set J, then J can be decomposed into disjoint convex setsf two types, either dyadic or gap. On dyadic sets, $|z-b| \sim c_1(Q)$, and on gap sets, $|Q(z)| \sim c_2(Q)\,|z-b|^k$, for some positive integer k. Also, for $Q(z+b) = \Sigma c_i z^i$, if $c_j = 0$, then there are no gap sets such that $|Q(z)| \sim c_2(Q)\,|z-b|^j$.

Note that, if we label the collection of dyadic sets as $\{D_j\}$, and the gaps as $\{G_j\}$we have:

- $D_j = \{z \in \mathbb{C} | A^{-1}\,|z_j| < |z| < A\,|z_j|\}$,
- $G_j = \{z \in \mathbb{C} | A\,|z_j| < |z| < A^{-1}\,|z_{j+1}|\}$,

for $A \in \mathbb{R}^+$, and z_j being the roots of $Q(z+b)$, ordered by magnitude.

D2 appears in [DW10], and describes a decomposition of $\mathbb{R}$. However, if we restrict ourselves to sectors of the form $\Delta^n_\varepsilon(\eta_j)$, where η_j are the roots of Q, and Voronoi sets $S(\eta_j)$, then the method can be extended to the complex case. Note again however that this will give us disjoint annuli in the complex case, so we repeat the thickening of these annuli, and converting them into convex sets as in D1.

While not strictly necessary, we may assume $C(Q) = c_1(Q) = c_2(Q) = 1$. This is due to our particular application of this decomposition. We will clarify this further at the start of section 1.3, once the application has been further expounded on.

We also make use of the integral representation of the Jacobian in [DW10]. This states that, in 3-dimensions, for $L_3(z) = L_\Gamma(z)$, $L_2(z) = \det\begin{pmatrix} P_1'(z) & P_1''(z) \\ P_2'(z) & P_2''(z) \end{pmatrix}$, and $L_1(z) = P_1'(z)$, then if we consider the mapping $\Phi_\Gamma(z_1, z_2, z_3) = \Gamma(z_1) + \Gamma(z_2) + \Gamma(z_3)$, we have:

$$J_{\Phi_\Gamma}(z_1, z_2, z_3) = \prod_{s=1}^{3} L_1(z_s) \int_{z_1}^{z_2} \int_{z_2}^{z_3} \prod_{s=1}^{2} \frac{L_2(w_s)}{L_1(w_s)^2} \int_{w_1}^{w_2} \frac{L_1(y)L_3(y)}{L_2(y)^2} dy dw_2 dw_1, \quad (1.6)$$

Where J_{Φ_Γ} is the real Jacobian of the mapping Φ_Γ, and the complex numbers appearing in the bounds of an integral represent that the integral in question is taken over the straight line joining those complex numbers. So in this case, w_1 is on the line joining z_1 to z_2, that w_2 is on the line joining z_2 to z_3, and y is on the line joining w_1 to w_2. Explicitly,

$$\int_{z_1}^{z_2} f(w)dw := (z_2 - z_1)\int_0^1 f(z_1 + (z_2 - z_1)t)dt.$$

While this representation was only derived in $\mathbb{R}$, the derivation for the complex Jacobian follows from an identical argument.

From here, we use D1 and D2 to decompose $\mathbb{C}$ into convex sets so that L_1, L_2, and L_3 can be approximated by either centered monomials or constants. This decomposition results in a bounded number of sets, which are divided into four different types, indexed by bitstrings of length 2. We index the degrees of the monomials that approximate the functions L_i similarly. The degree associated to L_3 is indexed by a zero-length bitstring, and is denoted k. L_1 exponents are indexed by a bitstring of length one, given by the first digit of the bitstring associated to the set we're on. So $|L_1(z)| \sim |z-b|^{k_0}$ on T_{00} and T_{01} sets, and $|L_1(z)| \sim |z-b|^{k_1}$ on T_{10} and T_{11} sets. Lastly the degree associated to L_2 is indexed by a bitstring of length 2, being identical to the bitstring of the set we're on. So for example, $|L_2(z)| \sim |z-b|^{k_{10}}$ on T_{10} sets.

First, we must do a minor decomposition to prepare for a step in lemma 1.2.3. For each $L_i(z) = A_i \prod_{j=1}^{d_i'}(z-b_{i,j})^{\alpha_{i,j}}$, we decompose $C = \bigcup_{j=1}^{d_i'} S(b_{i,j})$, where these sets are as defined in the description of $D1$. That is to say they are the Voronoi sets of the roots of $L_i(z)$.

Now, we decompose each $S(b_{i,j})$ into sectors of aperture ε, centered at $b_{i,j}$. To each of these sets of the form $S(b_{i,j} \bigcap \Delta_\varepsilon^n(b_{i,j})$, we then apply D_2 with respect to L_i and $b_{i,j}$. This gives 3 decompositions of $\mathbb{C}$, which we then overlay on top of each other, to obtain a decomposition of $\mathbb{C} = \bigcup B$, where each B is convex, and for each L_i, is contained in a gap or dyadic annulus of L_i, which is centered at the nearest root of L_i.

We are now ready to proceed with the more standard decomposition.

We use D1 with respect to L_3, so that we get a finite number of convex sets such that $|L_3| \sim |z-b|^k$ on each of the sets from this decomposition, with k and b depending on the sets. Next, we divide this family of sets into two types of sets, referred to as T_0 or T_1 sets. To do this, we first apply D2 to each set already obtained with respect to our centre from the first decomposition, and the polynomial L_1.

T_0 sets are those sets are gap annuli, on which $|L_1(z)| \sim |z-b|^{k_0}$, and $|L_3(z)| \sim |z-b|^k$.

To get the T_1 sets, we further decompose the dyadic annuli. This is done by applying D1 with respect to L_1 on each of these annuli, which results in a family of sets on which $|L_1(z)| \sim |z-b'|^{k_1}$, and $|L_3(z)| \sim 1$.

We apply D2 again to the T_0 and T_1 sets, but this time, with respect to L_2, and the centre b, and b' respectively. This divides the T_0 sets into two new families of sets, referred to as T_{00} sets and T_{01} sets.

T_{00} sets are those on which $|L_2(z)| \sim |z-b|^{k_{00}}$, while we still have $|L_1(z)| \sim |z-b|^{k_0}$, and $|L_3(z)| \sim |z-b|^k$.

T_{01} sets are again obtained by decomposing the dyadic annuli from the last D2 decomposition. We apply D1 to these annuli to obtain a family of convex sets on which $|L_3(z)| \sim |L_1(z)| \sim 1$, as $|z-b| \sim 1$, and $|L_2(z)| \sim |z-b''|^{k_{01}}$.

We decompose the T_1 sets precisely the same way to get families of sets named T_{10} and T_{11}.

On T_{10} sets, we have $|L_3| \sim 1$, $|L_1| \sim |z-b'|^{k_1}$ and $|L_2(z)| \sim |z-b'|^{k_{10}}$.

On T_{11} sets, we have $|L_1(z)| \sim |L_3(z)| \sim 1$, and $|L_2(z)| \sim |z-b''|^{k_{11}}$.

Note that the various b and k values depend on the specific set within these families that they belong to, as opposed to every set of type T_{01} having the same k_{01} value for example. The utility of dividing these sets into these types is that behaviour of the integrand that appears in (1.6) can be divided into four cases corresponding to these types. This will be shown at the end of the section, in (1.11) and the proceeding table.

Furthermore, we shall write b' and b'' as just b for simplicity for the rest of this discussion. Note to make use of these estimates, we need the following lemma.

Lemma 1.2.1. *For all z_1, z_2, z_3 in one of the sets in our decomposition, we have* *that*

$$|J_{\Phi_\Gamma}(z_1,z_2,z_3)| \sim \prod_{s=1}^{3}|L_1(z_s)|\left|\int_{z_1}^{z_2}\left|\int_{z_2}^{z_3}\prod_{s=1}^{2}\frac{|L_2(w_s)|}{|L_1(w_s)|^2}\left|\int_{w_1}^{w_2}\frac{|L_1(y)||L_3(y)|}{|L_2(y)|^2}dy\right|dw_2\right|dw_1\right|.$$

To see this, we first introduce the notion of a sector contained function.

Definition 1.2.2. *An ε-sector contained function of (B,b), for $B\subset\mathbb{C}$ and $b\in\mathbb{C}$, is a function f such that for all $z\in B$, $f(z-b)\in\Delta^n_\varepsilon(0)$, with notation as in the decomposition D1.*

We will occasionally omit the reference to the set (B,b) when we refer to functions that satisfy this definition, understanding that a ε-sector contained function refers to a ε-sector contained function of (B,b), where B is any set in our decomposition, and b is centre associated to B.

Note that the reason we introduce the notion of these functions is because for f, an ε-sector contained function, we have $|\int_u^v f(z-b)dz| \sim |\int_u^v |f(z-b)|\,dz|$ for u,v in any set of our decomposition, for $\varepsilon<\frac{\pi}{4}$.

To see this, note that we immediately have that $|\int_u^v f(z-b)dz| \le |\int_u^v |f(z-b)|\,dz|$, as

$$\begin{aligned}\left|\int_u^v f(z-b)dz\right| &= \left|(v-u)\int_0^1 f(z(t)-b)dt\right|\\ &\le \left|(v-u)\int_0^1 |f(z(t)-b)|\,dt\right| \\ &= \left|\int_u^v |f(z-b)|\,dz\right|.\end{aligned} \tag{1.7}$$

To establish the second inequality, notice that by factoring out some complex number of unit length, we can assume that $f(z-b)$ is contained in a sector which has one ray in the direction of the positive real axis, and the second ray being given by the line containing the complex numbers with argument ε.

We therefore have:

$$\begin{aligned}\left|\int_u^v f(z-b)dz\right| &= \left|(v-u)\int_0^1 \mathrm{Re}(f(z(t)-b)) + i\mathrm{Im}(f(z(t)-b))dt\right|,\\ &\ge \left|(v-u)\int_0^1 \mathrm{Re}(f(z(t)-b))dt\right|,\\ &\sim \left|\int_u^v |f(z-b)|\,dz\right|.\end{aligned}$$

Therefore, to establish our lemma, we would like to show that each integrand is an ε-sector contained function. However, because the argument of $(w_2 - w_1)$ cannot be controlled, this is not quite true, so we instead show that the integrand is the product of some ε contained function, and $(w_2 - w_1)$, which we will see is sufficient to establish our lemma.

Lemma 1.2.3. *Let B be a set from our decomposition. There exists a further decomposition of B such that each L_i is an ε sector contained function for each set in this decomposition.*

Proof of Lemma 1.2.3: In [BH14], it was shown in step 2 of lemma 4.2 that for a polynomial $P(z) = Q(z-b) = \prod_{j=1}^{d}(z-\eta_j)$, if we are in the j-th dyadic or gap annulus associated to $P(z)$ and b, as in the terminology of D2, then we have, by factoring $P(z-b)$ as

$$P(z-b) = g_j(z)(-1)^{d-j}z^j \prod_{l=j+1}^{d} \eta_l,$$

that on the j-th gap or dyadic annulus, we can decompose these sets into a bounded number of sets so that $g_j(z)$ is contained in a sector of aperture less that b_0, for arbitrary b_0. Here, the number of sets are dependant on b_0. As $(-1)^{d-j}z^j \prod_{l=j+1}^{d} \eta_l$ will be contained in a sector of aperture $j\varepsilon$, we have that $P(z)$ will be in a sector of aperture at most $j\varepsilon + b_0$ for z in any of our sets. Therefore, in our initial decomposition, we must ensure that we choose ε and b_0 such that $d\varepsilon + b_0 < \hat{\varepsilon}$ and our result holds. $\square$

Notice that thanks to our preliminary decomposition, we know that each set is entirely contained within a Dyadic or Gap annulus of the nearest root for each L_i simultaneously, which is required for their result.

A similar result immediately follows for rational functions whose numerator and denominator satisfy the conditions of lemma 1.2.3.

Corollary 1.2.4. *Let $Q(z) = \frac{P_1(z)}{P_2(z)}$ for polynomials P_1 and P_2 with degrees d_1 and d_2. Then if B_1 and B_2 are two sets arising from the decomposition in lemma 1.2.3 for a common centre b, being applied to P_1 and P_2 respectively, then Q is a $(d_1 + d_2)\hat{\varepsilon}$ sector contained function of $B_1 \cap B_2$.*

So therefore, for any set in our decomposition, we can decompose it further to ensure each of our L_i are $(d_i\varepsilon + b_{0,i})$-sector contained functions, for d_i the degree of L_i. Therefore, we have that the rational function $\frac{L_1 L_3}{L_2^2}$ is a $4(d+1)\varepsilon$-sector contained function, and $\frac{L_2}{L_1^2}$ is a $3(d+1)\varepsilon$-sector contained function where d is the maximum of the d_i values.

Lastly, we wish to observe that $\int_{w_1}^{w_2} f(y)dy$ is the product of an element of a ε-sector and $(w_2 - w_1)$.

Lemma 1.2.5. *Suppose $f(z)$ is a ε-sector contained function. Then there exists a ε-sector contained functon, $F(z)$, such that* $\int_{w_1}^{w_2} f(y)dy{=}(w_2 - w_1)(\xi(w_1, w_2) + i\eta(w_1, w_2))$*, with $\eta < \varepsilon\xi$.*

Proof of Lemma 1.2.5:

Letting $\int_{w_1}^{w_2} f(y)dy = (w_2 - w_1)\text{lim}_{n\to\infty}\frac{1}{n}\Sigma_{j=1}^{n}f(y(\frac{j}{n}))$, for $y(t) = w_1 + (w_2 - w_1)t$. Now, as $f(y)$ is in a sector for all y on this line segment, as B is convex, each $\frac{1}{n}\Sigma_{j=1}^{n}f(y(\frac{j}{n}))$ is in the sector, so therefore the limit of this sequence is in the sector. □

We can now proceed with establishing the main lemma of this section.

Proof of Lemma 1.2.1: Now note that we can write:

$$\begin{aligned} J_{\Phi_\Gamma}(z_1, z_2, z_3) &= \prod_{s=1}^{3} L_1(z_s) \int_{z_1}^{z_2} \int_{z_2}^{z_3} \prod_{s=1}^{2} \frac{L_2(w_s)}{L_1(w_s)^2} \int_{w_1}^{w_2} \frac{L_1(y)L_3(y)}{L_2(y)^2} dy dw_2 dw_1, \\ &= \left(\prod_{s=1}^{3} L_1(z_s)\right)(z_2 - z_1)(z_3 - z_2)I. \end{aligned}$$

where:

$$I := \int_0^1 \int_0^1 \prod_{s=1}^{2} \frac{L_2(w_s)}{L_1(w_s)^2}(w_2 - w_1) \int_0^1 \frac{L_1(y)L_3(y)}{L_2(y)^2} dt_3 dt_2 dt_1. \tag{1.8}$$

Here, note that $w_1 = z_1 + (z_2 - z_1)t_1$, $w_2 = z_2 + (z_3 - z_2)t_2$, and $y = w_1 + (w_2 - w_1)t_3$.

We now wish to bound the size of the integral I. Let $\alpha(t_1, t_2)$ and $\beta(t_1, t_2)$ be real valued functions such that $(w_2 - w_1) = \alpha + i\beta$. Note that each L_i, after our decomposition, are sector-contained functions, and the integral along the line from w_1 to w_2 of a sector contained function is an element of a sector contained multiplied by $(w_2 - w_1)$. Therefore, the above integral can be written as:

$$I = \int_0^1 \int_0^1 (\alpha(t_1, t_2) + i\beta(t_1, t_2))\, g(t_1, t_2) dt_2 dt_1,$$

where g is a $(7(d+1))\varepsilon$-sector contained function. That is to say $g(t_1, t_2) = \xi(t_1, t_2) + i\eta(t_1, t_2)$, for real functions ξ and η with

$|g(t_1,t_2)| \sim \xi(t_1,t_2)$. Note that we make the following assumptions about the points z_1, z_2, z_3. These assumptions can be made without loss of generality via renaming these points for the first two, and factoring out a unit complex number for the last. First, we assume that in the triangle z_1, z_2, z_3, the angle at z_2, which we label θ, is the largest in the triangle. Next, we assume $|z_3 - z_2| \geq |z_2 - z_1|$. Lastly we assume $z_2 - z_1$ is a positive real number.

Note that this immediately gives that β is single signed, and $\theta \in [\frac{\pi}{3}, \pi]$. We compute our estimate by splitting into cases when θ is acute, and when it is obtuse. We also define here the integral quantity

$$G := \int_0^1 \int_0^1 |\alpha(t_1,t_2) + i\beta(t_1,t_2)|\, |g(t_1,t_2)|\, dt_2 dt_1.$$

The aim in the following sections is to show $|I| \gtrsim G$.

Case i) If $\theta \in [\dfrac{\pi}{3}, \dfrac{\pi}{2}]$:

First, note that we have

$$\begin{aligned}\xi \sim |g(t_1,t_2)| &= \left| \prod_{s=1}^{2} \frac{L_2(w_s(t_s))}{L_1(w_s(t_s))^2} \int_0^1 \frac{L_1(y(t_3))L_3(y(t_3))}{L_2(y(t_3))^2} dt_3 \right|, \\ &= \prod_{s=1}^{2} \frac{|L_2(w_s(t_s))|}{|L_1(w_s(t_s))^2|} \left| \int_0^1 \frac{L_1(y(t_3))L_3(y(t_3))}{L_2(y(t_3))^2} dt_3 \right|.\end{aligned}$$

Now, as the integrand here is a sector contained function, we can move the absolute value signs inside the integral, yielding

$$\begin{aligned}|g(t_1,t_2)| &\sim \prod_{s=1}^{2} \frac{|L_2(w_s(t_s))|}{|L_1(w_s(t_s))^2|} \left| \int_0^1 \frac{|L_1(y(t_3))||L_3(y(t_3))|}{|L_2(y(t_3))^2|} dt_3 \right| \\ &\sim \prod_{s=1}^{2} |w_s(t_s)|^{\sigma_2} \left| \int_0^1 |y(t_3)|^{\sigma_3} dt_3 \right|.\end{aligned} \tag{1.9}$$

Now, we let $t_2(t_1)$ be the smallest value of t_2 for which the complex number $(w_2 - w_1)$ has imaginary part greater than half the absolute value of its real part. That is to say, $t_2(t_1)$ is the smallest value such that $\beta > \dfrac{|\alpha|}{2}$. Note that by our assumptions on θ and the relative lengths of our line segments, we have that $t_2(t_1) < \frac{1}{2}$ for all t_1.

From here we will proceed to estimate the t_1 integrand, by considering

$$\begin{aligned}
G(t_1) &:= \int_{t_2(t_1)}^{1} |w_2 - w_1| \prod_{s=1}^{2} |w_s(t_s)|^{\sigma_2} \left| \int_0^1 |y(t_3)|^{\sigma_3} dt_3 \right| dt_2, \\
&= \int_{2t_2(t_1)}^{1} |w_2 - w_1| \prod_{s=1}^{2} |w_s(t_s)|^{\sigma_2} \left| \int_0^1 |y(t_3)|^{\sigma_3} dt_3 \right| dt_2 \\
&\quad + \int_{t_2(t_1)}^{2t_2(t_1)} |w_2 - w_1| \prod_{s=1}^{2} |w_s(t_s)|^{\sigma_2} \left| \int_0^1 |y(t_3)|^{\sigma_3} dt_3 \right| dt_2, \\
&:= G_1 + G_2.
\end{aligned}$$

Now note that $L = L(t_1) = z_2 - w_1$ is a positive real number, and for all the t_2 in the G_2 integral, we have $L \leq |w_2 - w_1| \leq 2L$, and also that this relation holds for $t_2 < t_2(t_1)$. Furthermore, we also note that the triangle with vertices $z_1, z_2, w_2(2t_2(0))$ is contained in the circle centred at z_2, with radius $\varepsilon|z_2|$. This can be seen by considering the case with z_1, z_2 are positive real numbers, z_3 is on the other ray of our sector, $\theta = \frac{\pi}{2}$, and $|z_2 - z_1| = |z_3 - z_2|$, as this arrangement maximises the distance from z_2 and $w_2(2t_2(0))$. Note for all points z in this circle, we have $|z| \sim |z_2|$. We therefore have:

$$\begin{aligned}
G_2 &\sim \int_{t_2(t_1)}^{2t_2(t_1)} |w_2 - w_1||z_2|^{2\sigma_2+\sigma_3} \left| \int_0^1 dt_3 \right| dt_2, \\
&\sim \int_0^{t_2(t_1)} |w_2 - w_1||z_2|^{2\sigma_2+\sigma_3} \left| \int_0^1 dt_3 \right| dt_2, \\
&\sim \int_0^{t_2(t_1)} |w_2 - w_1| \prod_{s=1}^{2} |w_s(t_s)|^{\sigma_2} \left| \int_0^1 |y(t_3)|^{\sigma_3} dt_3 \right| dt_2, \\
&:= G_3.
\end{aligned}$$

Thus, we can write:

$$\begin{aligned}
G(t_1) &\sim G_1 + G_2 + G_3, \\
&= \int_0^1 |w_2 - w_1| \prod_{s=1}^{2} |w_s(t_s)|^{\sigma_2} \left| \int_0^1 |y(t_3)|^{\sigma_3} dt_3 \right| dt_2.
\end{aligned}$$

Integrating both sides with respect to t_1 as t_1 goes from 0 to 1 yields

$$\begin{aligned}
&\iint_{|\beta| \geq \frac{|\alpha|}{2}} |\alpha(t_1,t_2) + i\beta(t_1,t_2)|\, |g(t_1,t_2)| \gtrsim \\
&\qquad \int_0^1 \int_0^1 |w_2(t_2) - w_1(t_1)| \prod_{s=1}^{2} |w_s(t_s)|^{\sigma_2} \left| \int_0^1 |y(t_3)|^{\sigma_3} dt_3 \right| dt_2 dt_1.
\end{aligned}$$

Now, note that the left hand side of this inequality is similar in magnitude to

$$\iint_{|\beta|\geq \frac{|\alpha|}{2}} |\beta|\, \xi,$$

and the right hand side is precisely G. We therefore have that:

$$\begin{aligned}
|I| &\geq \left|\mathrm{Im}\left(\int_0^1\int_0^1 (\alpha+i\beta)g(t_1,t_2)dt_2dt_1\right)\right|,\\
&= \left|\int_0^1\int_0^1 (\xi\beta+\eta\alpha)dt_2dt_1\right|,\\
&\geq \int_0^1\int_0^1 \xi\,|\beta| - |\eta\alpha|\, dt_2dt_1,\\
&\geq \int_0^1\int_0^1 \xi(|\beta| - \varepsilon\,|\alpha|)dt_2dt_1,\\
&\gtrsim \iint_{|\beta|>\frac{|\alpha|}{2}} |\beta|\,\xi dt_2dt_1 - \varepsilon G,\\
&\gtrsim G,
\end{aligned}$$

which establishes the lemma.

Case ii) $\theta \in [\frac{\pi}{2}, \pi]$:

This case is simpler, as now we always have $\alpha > 0$. Using the same notation as above, we have

$$\begin{aligned}
|I| &= \left|\int_0^1\int_0^1 (\alpha+i\beta)(\xi+i\eta)dt_2dt_1\right|,\\
&= \left|\int_0^1\int_0^1 \alpha\xi - \beta\eta + i(\beta\xi+\alpha\eta)dt_2dt_1\right|.
\end{aligned} \tag{1.10}$$

Note that we also have, as β is single signed, and α is positive,

$$\int_0^1\int_0^1 \alpha\xi dt_2dt_1 + \left|\int_0^1\int_0^1 \beta\xi dt_2dt_1\right| = \left|\int_0^1\int_0^1 (\alpha+|\beta|)\xi dt_2dt_1\right|,$$

$$\sim \left|\int_0^1\int_0^1 |w_2(t_2)-w_1(t_1)||g(t_1,t_2)|dt_2dt_1\right| = G.$$

If $\left|\int_0^1\int_0^1 \beta\xi dt_2dt_1\right| \geq \int_0^1\int_0^1 \alpha\xi dt_2dt_1$, then we have:

$$|\mathrm{Im}(I)| = \left|\int_0^1\int_0^1 \beta\xi+\alpha\eta dt_2dt_1\right| \geq \left|\int_0^1\int_0^1 \beta\xi dt_2dt_1\right| - \varepsilon\int_0^1\int_0^1 \alpha\xi dt_2dt_1$$

$$\sim \left|\int_0^1\int_0^1 \beta\xi dt_2dt_1\right| \gtrsim G.$$

While if $\int_0^1\int_0^1 \alpha\xi dt_2dt_1 \geq \left|\int_0^1\int_0^1 \beta\xi dt_2dt_1\right|$, then by the same line of reasoning we have

$$|\mathrm{Re}(I)| = \left|\int_0^1\int_0^1 \alpha\xi - \beta\eta dt_2dt_1\right| \gtrsim G.$$

So indeed this gives us in either case that

$$|I| \gtrsim |\int_0^1\int_0^1 |w_2-w_1|\prod_{s=1}^{2}|w_s(t_s)|^{\sigma_2}\left|\int_0^1 |y(t_3)|^{\sigma_3}dt_3\right| dt_2dt_1|.$$

as again we can move the absolute value around the inner integral into it, as the integrand is a sector contained function.

This completes the lemma, as in all cases, we indeed obtain:

$$|J_{\Phi_\Gamma}(z_1,z_2,z_3)| \sim \prod_{s=1}^{3}|L_1(z_s)|\left|\int_{z_1}^{z_2}\left|\int_{z_2}^{z_3}\prod_{s=1}^{2}\frac{|L_2(w_s)|}{|L_1(w_s)^2|}\left|\int_{w_1}^{w_2}\frac{|L_1(y)||L_3(y)|}{|L_2(y)^2|}dy\right|dw_2\right|dw_1\right|.$$

Note that we can place absolute values around the w_2 integral as it is now clearly an integral of a real, positive integrand, and so we are done. $\square$

So therefore, we have, on any set in our decomposition:

$$|J_{\Phi_\Gamma}(z_1,z_2,z_3)| \sim \prod_{s=1}^{3}|z_s-b|^{\sigma_1}\left|\int_{z_1}^{z_2}\left|\int_{z_2}^{z_3}\prod_{s=1}^{2}|w_s-b|^{\sigma_2}\left|\int_{w_1}^{w_2}|y-b|^{\sigma_3}\,dy\right|dw_2\right|dw_1\right|, \tag{1.11}$$

for some values of $\sigma_1,\sigma_2,\sigma_3$, depending on the type of set we are on. The explicit values of each σ_i are given here:

	σ_1	σ_2	σ_3
T_{00}	k_0	$k_{00}-2k_0$	$k+k_0-2k_{00}$
T_{01}	0	k_{01}	$-2k_{01}$
T_{10}	k_1	$k_{10}-2k_1$	k_1-2k_{10}
T_{11}	0	k_{11}	$-2k_{11}$

1.3 The geometric inequality

In this section, we establish an important inequality which shall be used in both of our main theorems. This lemma is:

Lemma 1.3.1. *For $|J_{\Phi_\Gamma}|$ yielding $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ values as above after the described decomposition*

$$|J_{\Phi_\Gamma}(z_1, z_2, z_3)| \gtrsim \prod_{i=1}^{3} |L_\Gamma(z_i)|^{\frac{1}{3}} \prod_{1 \leq i < j \leq 3} |z_j - z_i|, \tag{1.12}$$

for $\sigma_3 \neq -1$, and $\sigma_2 + \frac{\sigma_3}{2} < -2$ or ≥ 0.

Note this lemma justifies the assumption that $c(Q) = c_1(Q) = c_2(Q) = 1$, which we made in the previous section. One can verify this using the fact that $|L_i(z)| \sim C(L_i)\, |z - b(i)|^{\sigma_i}$, for $b(i)$ the appropriate centre on this set, and the integral representation of the Jacobian. If one applies this to either side of the above inequality, we see that $C(L_1)$, $C(L_2)$, and $C(L_3)$ appear with equal exponents on either side of this inequality, and therefore are irrelevant to this proof.

Furthermore, we can meet the conditions of this lemma, that is to say we can avoid certain values of σ, by applying an affine transformation to our curve Γ before applying the decomposition $D2$. This is done to ensures that we avoid certain σ values. We omit the details, which are exactly the same as in the real case, considered in [DW10]. Specifically, we ensure there is no k_{10} value such that $k_{10} = \frac{-1-k_1}{2}$ on T_{10} intervals, and no k_{00} value such that $k_{00} = \frac{k+k_0+1}{2}$ on T_{00} intervals. This ensures $\sigma_3 \neq -1$. Similarly, we avoid $k_1 = 1$ on T_{10} and any k_0 values on T_{00} such that $k_0 = \frac{k+i}{3}$ for $i = 1, 2, 3, 4$. Note that as k_0 is an integer, this involves excluding at most 2 values of k_0.

Note that since the conception of this project, this inequality, and its higher dimensional analogues, have been established by J. de Dios Pont in [dDP20]. The methods of that proof differ significantly from those presented here. The cited paper uses an inductive argument to effectively reduce this lemma to the case where Γ is a generalized moment curve, via establishing a pseudo-continuity between Γ and L_Γ. Note that this continuity allows one to avoid a great deal of the messier computations that are to follow.

This limited version of the lemma presented here is included for completeness, while avoiding some of the more expansive arguments needed to establish the higher dimensional cases, which allows us to work with an explicit decomposition of $\mathbb{C}$ on which the lemma holds, rather than using the non-constructive methods in [dDP20].

Proof of Lemma 1.3.1: To establish (1.13), we first seek to show that:

$$|J_{\Phi_\Gamma}(z_1, z_2, z_3)| \gtrsim \prod_{i=1}^{3} |z_i - b|^{\sigma_1 + \frac{2\sigma_2}{3} + \frac{\sigma_3}{3}} \left| \int_{z_1}^{z_2} \int_{z_2}^{z_3} \left| \int_{w_1}^{w_2} dy \right| dw_1 dw_2 \right|. \tag{1.13}$$

Note here that we have $|z_i - b|^{\sigma_1 + \frac{2\sigma_2}{3} + \frac{\sigma_3}{3}} \sim |L_\Gamma(z_i)|^{\frac{1}{3}}$.

Firstly we wish to see that:

$$I_1 := \left| \int_{w_1}^{w_2} |y - b|^{\sigma_3} \, dy \right| \gtrsim |w_2 - b|^{\frac{\sigma_3}{2}} |w_1 - b|^{\frac{\sigma_3}{2}} \left| \int_{w_1}^{w_2} dy \right|. \tag{1.14}$$

In proving (1.14), we may assume $|w_2| > |w_1|$, as if not, we can notice that $\left|\int_{w_1}^{w_2} |y-b|^{\sigma_3} \, dy\right| = \left|-\int_{w_2}^{w_1} |y-b|^{\sigma_3} \, dy\right| = \left|\int_{w_2}^{w_1} |y-b|^{\sigma_3} \, dy\right|$.

So we may relabel to ensure that the larger of the two occurs at the upper bound of the integral.

Also notice that if $|w_1 - b| < |w_2 - b| < 9 |w_1 - b|$, we have $|w_1 - b| \sim |y - b| \sim |w_2 - b|$ throughout this line integral, and our result immediately follows. We therefore only have to deal with the case in which $9 |w_1 - b| < |w_2 - b|$.

Let:

- $L := \left\{ y \mid \frac{1}{2} |w_1 - b| < |y - b| < 2 |w_1 - b| \right\}$.
- $U := \left\{ y \mid \frac{1}{3} |w_2 - b| < |y - b| < \frac{1}{2} |w_2 - b| \right\}$.

Note that after parameterizing our line integral, we can see that it is the integral of a purely positive quantity over a real interval, so we can delete portions of the line segment and ensure that this results in a lower bound.

So

$$\begin{aligned} I_1 &\geq \left| \left[\int_L + \int_U \right] |y - b|^{\sigma_3} dy \right| \\ &\sim |w_1 - b|^{\sigma_3 + 1} + |w_2 - b|^{\sigma_3 + 1} \\ &= |w_2 - b| \left(|w_2 - b|^{\sigma_3} + \frac{|w_1 - b|^{\sigma_3 + 1}}{|w_2 - b|} \right), \end{aligned}$$

since the length of L and U are $\sim |w_1 - b|$ and $|w_2 - b|$ respectively.

Note that $|w_2 - b| \sim |w_2 - w_1| = \left|\int_{w_1}^{w_2} dy\right|$, as $9 |w_1 - b| < |w_2 - b|$.

So $I_1 \gtrsim \left|\int_{w_1}^{w_2} dy\right| \max\left(|w_2-b|^{\sigma_3}, \frac{|w_1-b|^{\sigma_3+1}}{|w_2-b|}\right)$.

If σ_3 is positive, we have

$$I_1 \gtrsim \left|\int_{w_1}^{w_2} dy\right| |w_2-b|^{\sigma_3} > \left|\int_{w_1}^{w_2} dy\right| |w_2-b|^{\frac{\sigma_3}{2}} |w_1-b|^{\frac{\sigma_3}{2}}.$$

If σ_3 is negative, then, as σ_3 is an integer which is not equal to -1, σ_3+1 is negative also, so

$$I_1 \gtrsim \left|\int_{w_1}^{w_2} dy\right| \frac{|w_1-b|^{\sigma_3+1}}{|w_2-b|} > \left|\int_{w_1}^{w_2} dy\right| |w_2-b|^{\frac{\sigma_3}{2}} |w_1-b|^{\frac{\sigma_3}{2}}.$$

As this holds for all $\sigma_3 \neq -1$, we are done. With this, we can conclude that

$$|J_{\Phi_\Gamma}(z_1,z_2,z_3)| \gtrsim \prod_{s=1}^{3} |z_s-b|^{\sigma_1} \left|\int_{z_1}^{z_2}\int_{z_2}^{z_3} \prod_{s=1}^{2} |w_s-b|^{\sigma_2+\frac{\sigma_3}{2}} \left|\int_{w_1}^{w_2} dy\right| dw_2 dw_1\right|.$$

We now seek to establish a similar inequality for

$$I_2 = \left|\int_{z_2}^{z_3} |w_2-b|^{\sigma_2+\frac{\sigma_3}{2}} \left|\int_{w_1}^{w_2} dy\right| dw_2\right|.$$

We want to show that $I_2 \gtrsim \max\left(|z_3-b|^{\sigma_2+\frac{\sigma_3}{2}}, \frac{|z_2-b|^{\sigma_2+\frac{\sigma_3}{2}+2}}{|z_3-b|^2}\right) \left|\int_{z_2}^{z_3} \left|\int_{w_1}^{w_2} dy\right| dw_2\right|$, for $\sigma_2+\frac{\sigma_3}{2} < -2$ or ≥ 0.

We assume that $|z_2-b| < |z_3-b|$, and relabel if we are not in such a case. We also notice that the inequality is immediate if we are in the case that $|z_2-b| < |z_3-b| < 9\,|z_2-b|$, so we therefore assume that $9\,|z_2-b| < |z_3-b|$, and again restrict the integral to L and U, which now represent the sets:

- $L := \{y \mid 2\,|z_2-b| < |y-b| < 3\,|z_2-b|\}$.
- $U := \left\{y \mid \frac{1}{3}\,|z_3-b| < |y-b| < \frac{1}{2}\,|z_3-b|\right\}$.

So we have $I_2 \geq \left|\left[\int_L + \int_U\right] |w_2-b|^{\sigma_2+\frac{\sigma_3}{2}} |w_2-w_1|\, dw_2\right|$.

Identically to above, we have $|w_2-w_1| \sim |w_2-b|$, so we can write:

$$
\begin{aligned}
I_2 &\gtrsim \left| \left[\int_L + \int_U \right] |w_2 - b|^{\sigma_2 + \frac{\sigma_3}{2} + 1} \, dw_2 \right| \\
&\sim |z_2 - b|^{\sigma_2 + \frac{\sigma_3}{2} + 2} + |z_3 - b|^{\sigma_2 + \frac{\sigma_3}{2} + 2} \\
&= |z_3 - b|^2 \left(\frac{|z_2 - b|^{\sigma_2 + \frac{\sigma_3}{2} + 2}}{|z_3 - b|^2} + |z_3 - b|^{\sigma_2 + \frac{\sigma_3}{2}} \right) \\
&\gtrsim \max \left(|z_3 - b|^{\sigma_2 + \frac{\sigma_3}{2}}, \frac{|z_2 - b|^{\sigma_2 + \frac{\sigma_3}{2} + 2}}{|z_3 - b|^2} \right) \left| \int_{z_2}^{z_3} \left| \int_{w_1}^{w_2} dy \right| dw_2 \right| .
\end{aligned}
$$

Now, if $\sigma_2 + \frac{\sigma_3}{2} > 0$, we obtain:

$$
\begin{aligned}
I_2 &= |z_3 - b|^{\sigma_2 + \frac{\sigma_3}{2}} \left| \int_{z_2}^{z_3} \left| \int_{w_1}^{w_2} dy \right| dw_2 \right| \\
&\geq |z_3 - b|^{\frac{2\sigma_2}{3} + \frac{\sigma_3}{3}} |z_2 - b|^{\frac{\sigma_2}{3} + \frac{\sigma_3}{6}} \left| \int_{z_2}^{z_3} \left| \int_{w_1}^{w_2} dy \right| dw_2 \right| .
\end{aligned}
$$

if instead, $\sigma_2 + \frac{\sigma_3}{2} < -2$, we see that

$$
\begin{aligned}
I_2 &\gtrsim \frac{|z_2 - b|^{\sigma_2 + \frac{\sigma_3}{2} + 2}}{|z_3 - b|^2} \left| \int_{z_2}^{z_3} |w_2 - w_1| \, dw_2 \right| \\
&> |z_3 - b|^{\frac{2\sigma_2}{3} + \frac{\sigma_3}{3}} |z_2 - b|^{\frac{\sigma_2}{3} + \frac{\sigma_3}{6}} \left| \int_{z_2}^{z_3} \left| \int_{w_1}^{w_2} dy \right| dw_2 \right| ,
\end{aligned}
$$

as $\sigma_2 + \frac{\sigma_3}{2} + 2 < 0$

A similar analysis yields that for

$$
I_3 = \left| \int_{z_1}^{z_2} |w_1 - b|^{\sigma_2 + \frac{\sigma_3}{2}} \left| \int_{w_1}^{w_2} dy \right| dw_1 \right| ,
$$

we have

$$
I_3 \gtrsim |z_1 - b|^{\frac{2\sigma_2}{3} + \frac{\sigma_3}{3}} |z_2 - b|^{\frac{\sigma_2}{3} + \frac{\sigma_3}{6}} \left| \int_{z_2}^{z_3} \left| \int_{w_1}^{w_2} dy \right| dw_2 \right| ,
$$

for the same constraints on our σ values as previously outlined. So we therefore indeed obtain the desired relation, (1.13).

From here we seek to resolve the triple integral which remains in (1.13), and establish that:

$$
\left| \int_{z_1}^{z_2} \left| \int_{z_2}^{z_3} \left| \int_{w_1}^{w_2} dy \right| dw_2 \right| dw_1 \right| \gtrsim |z_3 - z_1| \, |z_3 - z_2| \, |z_2 - z_1| \, . \tag{1.15}
$$

Clearly the inner most integral is given by $\left|\int_{w_1}^{w_2} dy\right| = |w_2 - w_1|$.

Consider $J = J(w_1) = \left|\int_{z_2}^{z_3} |w_2 - w_1|\, dw_2\right|$. If $z_3 - z_2 = |z_3 - z_2|\, e^{i\phi}$, and $t = e^{-i\phi}(w_2 - z_2) \in \mathbb{R}$, we have

$$J = \int_0^{|z_3-z_2|} |t - w_1'| = \int_0^{|z_3-z_2|} \sqrt{(t - \text{Re}(w_1'))^2 + \text{Im}(w_1')^2}\, dt := \int_0^{|z_3-z_2|} g_1(t)\, dt,$$

where $w_1' = e^{-i\phi}(w_1 - z_2)$.

To bound this from below, we introduce a new function $f_1(t) = \left|t - \widehat{w_1'}(t)\right|$, where

$$\widehat{w_1'}(t) = \begin{cases} w_1' - i\dfrac{\text{Im}(w_1')}{\text{Re}(w_1')}t & \text{if } t \leq \text{Re}(w_1') \\ w_1' - i\text{Im}(w_1')\dfrac{|z_3 - z_2| - t}{|z_3 - z_2| - \text{Re}(w_1')}, & t > \text{Re}(w_1') \end{cases}.$$

Note that, for all values of $t \in [0, |z_3 - z_2|]$, we have that $\text{Re}(w_1') = \text{Re}(\widehat{w_1'}(t))$, and that $|\text{Im}(w_1'(t))| > |\text{Im}(\widehat{w_1'}(t))|$. Also, $\widehat{w_1'}(0) = \widehat{w_1'}(|z_3 - z_2|) = w_1'$.

Clearly $g_1(t) \geq f_1(t)$ for all t in the interval $[0, |z_3 - z_2|]$, as t is real, and $\widehat{w_1'}(t)$ is w_1' with a smaller imaginary part. A computation shows that $f_1(t)$ is given by :

$$f_1(t) = \begin{cases} \dfrac{|z_2 - w_1|}{\text{Re}(w_1')}(\text{Re}(w_1') - t) & \text{if } t \leq \text{Re}(w_1') \\ \dfrac{|z_3 - w_1|}{|z_3 - z_2| - \text{Re}(w_1')}(t - \text{Re}(w_1')) & t > \text{Re}(w_1') \end{cases}.$$

So $J = \int_0^{|z_3-z_2|} g_1(t)\, dt > \int_0^{|z_3-z_2|} f_1(t)\, dt$. Note that $g_1(t)$ increases as $|t - \text{Re}(w_1')|$ increases, $g_1(0) = |z_2 - w_1| = f_1(0)$ and $g_1(|z_3 - z_2|) = |z_3 - w_1| = f_1(|z_3 - z_2|)$.

Therefore if $\text{Re}(w_1') < \frac{|z_3-z_2|}{2}$, we have $|z_3 - w_1| > |z_2 - w_1|$.

Alternatively if $\text{Re}(w_1') > \frac{|z_3-z_2|}{2}$, then we instead get $|z_2 - w_1| > |z_3 - w_1|$.

Now, suppose $\text{Re}(w_1') \leq 0$. Then $|z_3 - w_1| > |z_2 - w_1|$, so:

$$\begin{aligned}
J &> \int_0^{|z_3-z_2|} f_1(t)\,dt \\
&= \int_0^{|z_3-z_2|} \frac{|z_3-w_1|}{|z_3-z_2|-\mathrm{Re}(w_1')}(t-\mathrm{Re}(w_1'))\,dt \\
&= \frac{|z_3-w_1|}{|z_3-z_2|-\mathrm{Re}(w_1')}\left[\frac{t(t-\mathrm{Re}(w_1'))}{2}\right]_0^{|z_3-z_2|} \\
&\sim |z_3-z_2|\,|z_3-w_1|\,.
\end{aligned}$$

A symmetric argument gives that if $\mathrm{Re}(w_1') > |z_3-z_2|$ then $|z_2-w_1| > |z_3-w_1|$, and $J \gtrsim |z_3-z_2|\,|z_2-w_1|$.

Lastly, if $\mathrm{Re}(w_1') \in [0, |z_3-z_2|]$, then:

$$\begin{aligned}
J &> \int_0^{\mathrm{Re}(w_1')} \frac{|z_2-w_1|}{\mathrm{Re}(w_1')}(\mathrm{Re}(w_1')-t)\,dt + \int_{\mathrm{Re}(w_1')}^{|z_3-z_2|} \frac{|z_3-w_1|}{|z_3-z_2|-\mathrm{Re}(w_1')}(t-\mathrm{Re}(w_1'))\,dt \\
&= \frac{1}{2}\left(|z_2-w_1|\,\mathrm{Re}(w_1') + |z_3-w_1|\left(|z_3-z_2|-\mathrm{Re}(w_1')\right)\right) \\
&\sim |z_3-z_2|\max\left(|z_3-w_1|\,,|z_2-w_1|\right).
\end{aligned}$$

So indeed $J \gtrsim |z_3-z_2|\max(|z_3-w_1|\,,|z_2-w_1|)$ in each of these cases. Applying this to the triple integral on the left of (1.15), we get

$$\left|\int_{z_1}^{z_2}\int_{z_2}^{z_3} |w_2-w_1|\,dw_2dw_1\right| \gtrsim |z_3-z_2|\left|\int_{z_1}^{z_2} |z_3-w_1|\,dw_1\right|.$$

As we can go through an identical procedure for $J' := \int_{z_1}^{z_2} |z_3-w_1|\,dw_1$, this indeed establishes (1.15). This in conjunction with (1.13) yields the proof of Lemma 1.3.1. □

1.4 Application to the convolution operator

We will now use the work from sections 2 and 3 to establish Theorem 1.1.1. Before that however, we must use them to establish the restricted-weak type estimate at $(2,3)$.

Lemma 1.4.1. *T_Γ is of restricted weak-type at $(2,3)$. That is to say for measurable subsets of $\mathbb{C}^3$, E and F, there exists $C = C(N)$ such that*

$$\langle T_\Gamma \chi_E, \chi_F\rangle \le C\,|E|^{\frac{1}{2}}\,|F|^{\frac{2}{3}}\,.$$

Before proving this lemma, we first reiterate why this lemma implies that we obtain uniform estimates for T_Γ along the "critical line" of the convolution problem, not including its endpoints, and non-uniform estimates in the rest of the region $\mathcal{R}$. To see this, first notice that we need only establish that the operator T is of restricted weak-type at $(p, q) = (2, 3)$, as the operator is "almost" self dual, we can use duality properties to conclude that we also have the bound corresponding to $(p, q) = (\frac{3}{2}, 2)$. Furthermore, following from the Marcinkiewicz interpolation theorem, this lemma establishes the strong type estimate for all $(p_\theta, q_\theta) = (\frac{6}{3+\theta}, \frac{6}{2+\theta})$, for $\theta \in (0, 1)$. Pairing this with the fact that the local bound at $(p, q) = (\infty, \infty)$ follows trivially from the fact that $T_\Gamma f$ is bounded whenever f is bounded, we see that, with this lemma, and utilising duality properties and interpolation, we acquire strong type estimates for all $(p, q) \in \mathcal{R}$, except at (p_0, q_0), and (p_1, q_1), where we have restricted weak type estimates.

Note also, the bound we will establish at $(2, 3)$ is uniform, as opposed to the trivial bound at $(0, 0)$. Therefore, in the Riesz diagram, we have uniform estimates along the line segment joining the vertices $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{2}\right)$, and local estimates, with implicit constants dependent on the disk and the curve, at every other (p, q) pair in $\mathcal{R}$.

Proof of Lemma 1.4.1:

Throughout this proof, we will denote by $d\sigma(z)$ the measure $|z|^{\frac{k}{3}}\, dz$, where k is such that $\lambda_\Gamma(z) \sim |z - b|^{\frac{k}{3}}$. We also denote two quantities, α and β, where, for measurable sets E and F, they are given by the equations:

$$\alpha\, |F| = \langle T_\Gamma \chi_{_E}, \chi_{_F} \rangle = \beta\, |E|\,.$$

Here, α and β can be interpreted as the average value of $T_\Gamma \chi_E$ on F, and the average value of $T_\Gamma^* \chi_F$ on E respectively.

Furthermore, we let $\gamma = \max(\alpha, \beta)$, $\nu = \frac{3}{k+6}$, and denote the ball $B_x := \{z \in \mathbb{C} \mid |z| < c_\nu x^\nu\}$, for $c_\nu = (8\pi)^{-\nu}$.

An easy computation shows:

$$\begin{aligned}\sigma(B_x) &= \int_{B_x} |z|^{\frac{k}{3}}\, dz \\ &= 2\pi \int_0^{(8\pi)^{-\nu}x^{\nu}} r^{\frac{k+3}{3}} dr \\ &= \frac{x\nu}{4} \\ &\le \frac{x}{8}.\end{aligned}$$

Notice that if we replace T_Γ with $\widetilde{T}f(z) := \int_{\Delta\setminus B_\gamma} f(z-\Gamma(w))d\sigma(w)$, and if $\gamma = \alpha$ we can notice that:

$$\begin{aligned}\langle \widetilde{T}\chi_E, \chi_F\rangle &\ge \langle \widetilde{T}_\Gamma \chi_E, \chi_F\rangle - |F|\,\frac{\alpha}{2} \\ &= \frac{\alpha}{2}\,|F| = \frac{\beta}{2}\,|E|,\end{aligned}$$

Similarly, if $\gamma = \beta$:

$$\begin{aligned}\langle \widetilde{T}\chi_E, \chi_F\rangle &= \langle \chi_E, \widetilde{T}^*\chi_F\rangle \\ &\ge \langle \chi_E, T_\Gamma^*\chi_F\rangle - |E|\,\frac{\beta}{2} \\ &= \frac{\beta}{2}\,|E| \ge \frac{\alpha}{2}\,|F|,\end{aligned}$$

Therefore, we can introduce the refined set $E_1 := \left\{y \in E \,|\, \widetilde{T}^*\chi_F(y) \ge \frac{\beta}{4}\right\}$ and have it be non empty.

Note that:

$$\langle \widetilde{T}\chi_{E_1}, \chi_F\rangle = \langle \widetilde{T}\chi_E, \chi_F\rangle - \langle \widetilde{T}\chi_{E\setminus E_1}, \chi_F\rangle \ge \alpha\,|F| - \frac{\beta}{2}\,|E| = \frac{\alpha}{2}\,|F|\,.$$

That is to say that the average value of $\widetilde{T}\chi_{E_1}$ on F is greater than $\frac{\alpha}{2}$. We therefore can introduce the set $F_1 := \left\{x \in F \mid \widetilde{T}\chi_{E_1}(x) \ge \frac{\alpha}{4}\right\}$, and this will also be non-empty. Now, if we take $z_0 \in F_1$, we can define the set

$$P_1 = P_{1,z_0} := \{z_1 \in \Delta_\varepsilon^n \setminus B_\gamma \,|\, z_0 - \Gamma(z) \in E_1\}\,.$$

Notice that $\sigma(P_1) = \int_{\Delta_\varepsilon^n\setminus B_\gamma} \chi_{E_1}(z_0 - \Gamma(z_1))\,|z|^{\frac{k}{3}}\,dz = \widetilde{T}_\Gamma\chi_{E_1}(z_0) \gtrsim \alpha$. Now to each $z_1 \in P_1$ associate a set:

$$P_2 = P_{2,z_0,z_1} = \{z_2 \in \Delta_\varepsilon^n \setminus B_\gamma \,|\, z_0 - \Gamma(z_1) + \Gamma(z_2) \in F\}\,,$$

which has the property $\sigma(P_2) \sim \widetilde{T}_\Gamma^* \chi_{F_1}(z_0 - \Gamma(z_1)) \gtrsim \beta$, from the construction of P_1.

Lastly, letting:

$$P_3 = P_{2,z_0,z_1,z_2} = \{z_3 \in \Delta_\varepsilon^n \setminus B_\gamma \,|\, z_0 - \Gamma(z_1) + \Gamma(z_2) - \Gamma(z_3) \in E\},$$

We again see that $\sigma(P_3) \gtrsim \alpha$, by the same logic as for P_1.

Now, letting $H(z_1, z_2, z_3) := -\Gamma(z_1) + \Gamma(z_2) - \Gamma(z_3)$, and $\Phi = P_1 \times P_2 \times P_3$, we can notice $H(\Phi) \subset E$. Thus, we obtain the lower bound for $|E|$ via:

$$\begin{aligned}
|E| &\gtrsim \iiint_\Phi |J_\mathbb{R} H(z_1, z_2, z_3)| \, dz_3 dz_2 dz_1 \\
&= \iiint_\Phi |J_\mathbb{C} H(z_1, z_2, z_3)|^2 \, dz_3 dz_2 dz_1 \\
&\gtrsim \iiint_\Phi \prod_{i=1}^3 |z_i|^{\frac{2k}{3}} \prod_{1 \le i < j \le 3} |z_i - z_j|^2 \, dz_3 dz_2 dz_1 \\
&= \int_{P_1} \int_{P_2} \int_{P_3} \prod_{i=1}^3 |z_i|^{\frac{k}{3}} \prod_{1 \le i < j \le 3} |z_i - z_j|^2 \, d\sigma(z_3) d\sigma(z_2) d\sigma(z_1).
\end{aligned}$$

Also notice that we have the property that on P_i, $|z_i| \ge c_\nu \gamma^\nu$. From here, we aim to acquire sets $P_i' \subset P_i$, such that $\sigma(P_i') \gtrsim \alpha$ for $i = 1, 3$, or $\gtrsim \beta$ for $i = 2$, but we also want to establish lower bounds for the separations of the variables.

Specifically, we wish to construct, for $i = 1, 2$, P_i', for $z_i \in P_i'$

- $|z_3 - z_i| \gtrsim \alpha^{\frac{1}{2}} |z_i|^{-\frac{k}{6}}$,
- $|z_2 - z_1| \gtrsim \beta^{\frac{1}{2}} |z_1|^{-\frac{k}{6}}$.

To establish these bounds, we can notice the condition that the inequality $|z| \gtrsim x^\nu$ implies $|z| \gtrsim x^{\frac{1}{2}} |z|^{-\frac{k}{6}}$, which is obtained by raising either side of the inequality to $\frac{1}{2\nu}$, and multiplying across by $|z|^{-\frac{k}{6}}$.

In the first case, we consider the balls $B_x(w) := \left\{z \in \mathbb{C} \,|\, |z - w| < c_0 x^{\frac{1}{2}} |w|^{-\frac{k}{6}}\right\}$ for some c_0 to be chosen. The aim here will be to remove $B_\alpha(z_i)$ from P_3, so we would like to show that its σ-measure is a small fraction of α.

Now, if $z \in B_\alpha(z_i)$, we have $|z - z_i| < c_0 \alpha^{\frac{1}{2}} |z_i|^{-\frac{k}{6}} < c_0 |z_i|$, which implies

$|z| \lesssim c_0 |z_i|$. We can now estimate $\sigma(B_\alpha(z_i))$ via:

$$\begin{aligned}\sigma(B_\alpha(z_i)) &= \int_{B_\alpha(z_i)} |z|^{\frac{k}{3}}\, dz \\ &\lesssim c_0^2 \alpha |z_i|^{-\frac{k}{3}} \times c_0^{\frac{k}{3}} |z_i|^{\frac{k}{3}} \\ &\leq c_0^{2+\frac{k}{3}} \alpha.\end{aligned}$$

So for sufficiently small c_0, $B_\alpha(z_i)$, can be removed from P_3 to ensure the first bullet point holds.

The proof of the second bullet point similarly holds via the same argument, by replacing α with β, z_3 with z_2, and P_3 with P_2.

Finally, we need to equally redistribute the exponent of the monomial on the right hand side of this inequality. Specifically, we wish to show that $|z_3 - z_i| \gtrsim \alpha^{\frac{1}{2}} |z_3 z_i|^{\frac{-k}{12}}$, and $|z_2 - z_1| \gtrsim \beta^{\frac{1}{2}} |z_2 z_1|^{\frac{-k}{12}}$.

Clearly if $|z_i| \sim |z_j|$ these follow immediately, so we need only show it in the case $2|z_3| < |z_1|$, or $2|z_1| < |z_3|$, and similarly for the inequality involving z_2 and z_1.

To see this, notice that the separation ensures

$$|z_3 - z_i| \gtrsim |z_3| = |z_3|^{1+\frac{k}{12}} |z_i|^{\frac{k}{12}} |z_i z_3|^{-\frac{k}{12}} \gtrsim \gamma^{\nu(1+\frac{k}{6})} |z_i z_3|^{-\frac{k}{12}} \geq \alpha^{\frac{1}{2}} |z_i z_3|^{-\frac{k}{12}},$$

so we are done. An identical argument works for $|z_2 - z_1|$. Now we can return to our lower bound for $|E|$, with these new estimates.

$$\begin{aligned}|E| &\gtrsim \int_{P_1'} \int_{P_2'} \int_{P_3'} \prod_{i=1}^{3} |z_i|^{\frac{k}{3}} \prod_{1 \leq i < j \leq 3} |z_i - z_j|^2 \, d\sigma(z_3) d\sigma(z_2) d\sigma(z_1) \\ &\gtrsim \alpha^2 \beta \int_{P_1'} \int_{P_2'} \int_{P_3'} \prod_{i=1}^{3} |z_i|^{\frac{k}{3}} \left(|z_1 z_2| |z_2 z_3| |z_1 z_3|\right)^{-\frac{k}{6}} d\sigma(z_3) d\sigma(z_2) d\sigma(z_1) \\ &= \alpha^2 \beta \sigma(P_1') \sigma(P_2') \sigma(P_3') \\ &\gtrsim \alpha^4 \beta^2,\end{aligned}$$

and after substituting in $\alpha = \frac{\langle T_\Gamma \chi_E, \chi_F \rangle}{|F|}$, and $\beta = \frac{\langle T_\Gamma \chi_E, \chi_F \rangle}{|E|}$, we see this implies Lemma 1.4.1. □

Now, we want to follow up from this proof and establish Theorem 1.1.1. The method demonstrated here follows almost identically the method described in

[CH18], which in turn was based on the work done in the real case of [Sto09] To establish this theorem, we need only establish the following two lemmas, which via the reasoning in the appendix from [Sto09], imply the theorem in question. Note that the argument presented in that appendix relies upon the restricted weak type estimate already established, so we could not have simply started with an attempt to establish this range of Lorentz estimates.

Lemma 1.4.2. *Let E_1, E_2, $F \subset \mathbb{C}^3$ be measurable sets with finite Lebesgue measure. If we have α_1 and α_2 such that $T_\Gamma \chi_{E_i}(x) \geq \alpha_i$ for $x \in F$, and $\alpha_2 \geq \alpha_1$, then we have the following lower bound for the measure of E_2:*

$$|E_2| \gtrsim \alpha_1^{2-2\nu} \alpha_2^{2+2\nu} \beta^2,$$

for $\nu = \frac{3}{k+6}$, and $\beta = \alpha_1 \frac{|F|}{|E_1|}$.

Lemma 1.4.3. *Let F_1, F_2, $E \subset \mathbb{C}^3$ be measurable sets with finite measure. If we have β_1 and β_2 such that $T^*_\Gamma \chi_{F_i}(y) \geq \beta_i$ for $y \in E$, and $\beta_2 \geq \beta_1$, then we have the following lower bound for the measure of F_2:*

$$|F_2| \gtrsim \alpha^3 \beta_1 \beta_2^2,$$

for $\alpha = \beta_1 \frac{|E|}{|F_1|}$.

For both of these proofs we will use the notation from Lemma 1.4.1, such as the measure σ being given by $d\sigma(z) = |z|^{\frac{k}{3}}\, dz$.

Proof of Lemma 1.4.2:

First, we wish to construct 3 sets, P_1, P_2, $P_3 \subset \Delta^n_\varepsilon$, with the following properties:

- $\sigma(P_1) \gtrsim \alpha_1$, and if $z_1 \in P_1$, then $|z_1| \geq c_\nu \max(\alpha_1, \beta)^\nu$,
- $\sigma(P_2) \gtrsim \beta$, and if $z_2 \in P_2$, then $|z_2| \geq c_\nu \max(\alpha_1, \beta)^\nu$,
- $\sigma(P_3) \gtrsim \alpha_2$, and if $z_3 \in P_3$, then $|z_3| \geq c'_\nu \alpha_2^\nu$.

To ease notation, we define $\gamma := \max(\alpha_1, \beta)$. Furthermore, we denote, as before, B_x to be the open ball of radius $\left(\frac{x}{8\pi}\right)^\nu$.

We therefore again have:$\sigma(B_x) = \int_{B_x} |z|^{\frac{k}{3}}\, dz \leq \frac{x}{2}$.

Now, letting $\widetilde{T}f(z) := \int_{\Delta\setminus B_\gamma} f(z-\Gamma(w))d\sigma(w)$, notice that:

$$\begin{aligned}\langle \widetilde{T}\chi_{E_1}, \chi_F\rangle &\geq \langle T_\Gamma \chi_{E_1}, \chi_F\rangle - |F|\frac{\gamma}{2}\\ &\geq \frac{\alpha_1}{2}|F| = \frac{\beta}{2}|E_1|,\end{aligned}$$

irrespective of which value γ takes on.

This shows that the set $F^1 := \left\{x \in F \,|\, \widetilde{T}\chi_{E_1}(x) \geq \frac{\alpha_1}{4}\right\}$ is non-empty. Furthermore:

$$\begin{aligned}\langle \chi_{E_1}, \widetilde{T}^*\chi_{F^1}\rangle &= \langle \chi_{E_1}, \widetilde{T}^*\chi_{F^1}\rangle - \langle \widetilde{T}\chi_{E_1\setminus E_1^1}, \chi_{F^1}\rangle\\ &\geq \frac{\alpha_1}{2}|F| - \frac{\alpha_1}{4}|F|\\ &= \frac{\beta}{4}|E_1| = \frac{\alpha_1}{4}|F|.\end{aligned}$$

Meaning we can define the non-empty set $E_1^1 := \left\{y \in E_1 \,|\, \widetilde{T}^*\chi_{F^1}(y) \geq \frac{\beta}{8}\right\}$, and by a similar argument, a non-empty set $F^2 := \left\{x \in F^1 \,|\, \widetilde{T}\chi_{E_1^1}(x) \geq \frac{\alpha_1}{16}\right\}$.

We are now ready to define P_1, P_2, P_3. First fix $x_0 \in F^2$, and define

$$\begin{aligned}P_1 &:= \left\{z_1 \in \Delta_\varepsilon^n \setminus B_\gamma \,|\, x_0 - \Gamma(z_1) \in E_1^1\right\},\\ P_2 = P_{2,z_1} &:= \left\{z_2 \in \Delta_\varepsilon^n \setminus B_\gamma \,|\, x_0 - \Gamma(z_1) + \Gamma(z_2) \in F^1\right\},\\ P_3 = P_{3,z_1,z_2} &:= \left\{z_3 \in \Delta_\varepsilon^n \setminus B_{\alpha_2} \,|\, x_0 - \Gamma(z_1) + \Gamma(z_2) - \Gamma(z_3) \in E_2\right\},\end{aligned}$$

where it is understood that $z_i \in P_i$ when we are defining P_j, for $j > i$.

Notice that we have $\sigma(P_1) = \widetilde{T}^*\chi_{F^1}(x_0) \geq \frac{\alpha_1}{16}$. Similarly $\sigma(P_2) = \widetilde{T}\chi_{E_1^1}(x_0 + \Gamma(z_1)) \geq \frac{\beta}{8}$.

For P_3, notice that:

$$\begin{aligned}\sigma(P_3) &= \int_{\Delta_\varepsilon^n\setminus B_{\alpha_2}} \chi_{E_2}(x_0 - \Gamma(z_1) + \Gamma(z_2) - \Gamma(z_3))d\sigma(z_3)\\ &\geq \int_{\Delta_\varepsilon^n} \chi_{E_2}(x_0 - \Gamma(z_1) + \Gamma(z_2) - \Gamma(z_3))d\sigma(z_3) - \sigma(B_{\alpha_2})\\ &\geq \frac{\alpha_2}{2}.\end{aligned}$$

Thus, as $P_1,\ P_2,$ and P_3 are contained within sectors with the appropriate balls removed from them, these sets indeed satisfy the conditions we initially laid out.

From here, we must now construct subsets $P_i' \subset P_i$, for each i, satisfying the following separation conditions:

$$\cdot \text{ if } z_2 \in P_2', \text{ and } z_1 \in P_1', \text{ then } |z_2 - z_1| \geq c\beta^{\frac{1}{2}} |z_1|^{-\frac{k}{6}}, \tag{1.16}$$

$\cdot$ if $z_3 \in P_3'$, and $z_i \in P_i'$ for $i \neq 3$ then:

$$- \text{ if } |z_i| \leq \frac{1}{2} c_\nu \alpha_2^\nu, \text{ then } |z_3 - z_i| \geq c\alpha_2^{\frac{1}{2}} |z_3|^{-\frac{k}{6}}. \tag{1.17}$$

$$- \text{ if } |z_i| \geq \frac{1}{2} c_\nu \alpha_2^\nu, \text{ then } |z_3 - z_i| \geq c\alpha_2^{\frac{1}{2}} |z_i|^{-\frac{k}{6}}. \tag{1.18}$$

where c_ν is as in the previous section, and c is some small constant. We also need to retain the property than $\sigma(P_1') \gtrsim \alpha_1$, $\sigma(P_2') \gtrsim \beta$, and $\sigma(P_3') \gtrsim \alpha_2$.

To show the cases in the second bullet point, we denote $B_x(w) := \left\{ z \in \Delta \mid |z - w| < c_0 x^{\frac{1}{2}} |w|^{-\frac{k}{6}} \right\}$, with c_0 to be specified. Also we must note that, for any α, the condition $|z| \geq c_\nu \alpha^\nu$ is equivalent to $|z| \geq c_\nu^{\frac{1}{2\nu}} \alpha^{\frac{1}{2}} |z|^{-\frac{k}{6}}$, which is acquired by raising each side of the inequality to the power $\frac{1}{2\nu}$.

We now verify the simpler cases of (1.17) and (1.18).

For (1.17), if $|z_i| \leq \frac{1}{2} c_\nu \alpha_2^\nu$, then $|z_3 - z_i| \geq c\alpha_2^{\frac{1}{2}} |z_i|^{-\frac{k}{6}}$, we can see this immediately from the fact that no element of B_{α_2} was in P_3, and $z_i \in B_{\frac{\alpha_2}{2}}$ which implies the separation required such that:

$$|z_3 - z_i| \geq \frac{|z_3|}{2} \geq \frac{1}{2} c_\nu^{\frac{1}{2\nu}} \alpha^{\frac{1}{2}} |z_3|^{-\frac{k}{6}}.$$

If instead, we are in case (1.18), we have that if $|z_i| \geq \frac{1}{2} c_\nu \alpha_2^\nu$, then $|z_3 - z_i| \geq c\alpha_2^{\frac{1}{2}} |z_i|^{-\frac{k}{6}}$:

In this case, we must first consider the sigma-measure of $B_{\alpha_2}(z_i)$, and show that it is a small multiple of α_2, as we will wish to remove it from P_3, and ensure separation between the variables.

We will need to utilise the fact that if $z \in B_{\alpha_2}(z_i)$, it follows that:

$$|z - z_i| < c_0 \alpha_2^{\frac{1}{2}} |z_i|^{-\frac{k}{6}} < 2c_0 c_\nu |z_i|.$$

which implies $|z| < \left(1 + 2c_0 c_\nu^{-\frac{1}{2\nu}}\right) |z_i|$. From here, we can bound the size of

$B_{\alpha_2}(z_i)$ via:

$$\begin{aligned}\sigma(B_{\alpha_2}(z_i)) &= \int_{B_{\alpha_2}(z_i)} |z|^{\frac{k}{3}}\, dz \\ &\leq \pi \left(c_0\alpha_2^{\frac{1}{2}} |z_i|^{-\frac{k}{6}}\right)^2 \times \left|\left(1+2c_0c_\nu^{-\frac{1}{2\nu}}\right)|z_i|\right|^{\frac{k}{3}} \\ &= \pi c_0^2 \left(1+2c_0c_\nu^{-\frac{1}{2\nu}}\right)^{\frac{k}{3}} \alpha_2 = c_1\alpha_2.\end{aligned}$$

Thus, we can choose c_0 sufficiently small to ensure $\sigma(B_{\alpha_2}(z_i)) \geq \frac{\alpha_2}{4}$, and have the set $P_3' = P_3 \setminus B_{\alpha_2}(z_i)$ satisfy the condition that $\sigma(P_3') \gtrsim \alpha_2$.

The first case, (1.16) follows in an identical argument to this, by replacing α_2 with α_1. As both P_1 and P_2 contain no elements of B_γ, we know $|z_1| \geq c_\nu \frac{1}{2}\gamma^\nu \geq c_\nu \frac{1}{2}\beta^\nu$. Thus, we can ensure that, for some sufficiently small value of c_0, setting $P_2' = P_2 \setminus B_\beta(z_1)$, yields $\sigma(P_2) \gtrsim \beta$. Lastly, leaving $P_1' = P_1$, we have our 3 further refined sets.

We need one final improvement on these separations, that for $z_i \in P_i'$, we have:

$$|z_2 - z_1| \gtrsim \beta^{\frac{1}{2}} |z_2z_1|^{-\frac{k}{12}}, \tag{1.19}$$

$$|z_3 - z_i| \gtrsim \alpha_2^{\frac{1+2\nu}{4}} \alpha_1^{\frac{1-2\nu}{4}} |z_3z_i|^{-\frac{k}{12}}. \tag{1.20}$$

Notice that if $\frac{|z_1|}{2} < |z_2| < 2|z_1|$, the first estimate follows from the fact that $|z_2 - z_1| \geq c\beta^{\frac{1}{2}} |z_1|^{-\frac{k}{6}} \sim \beta^{\frac{1}{2}} |z_2z_1|^{-\frac{k}{12}}$.

Similarly the second estimate follows in the case that $\frac{|z_i|}{2} < |z_3| < 2|z_i|$, as either $|z_3 - z_i| \geq c\alpha_2^{\frac{1}{2}} |z_i|^{-\frac{k}{6}}$, or $|z_3 - z_i| \geq c\alpha_2^{\frac{1}{2}} |z_3|^{-\frac{k}{6}}$.

Therefore, we need only consider the case where there is sufficient separation of the variables.

First, for (1.19), assume $2|z_1| < |z_2|$, or $2|z_2| < |z_1|$. In either case, again using the fact that we have removed B_γ from P_1' and P_2', we have:

$$\begin{aligned}|z_2 - z_1| &\gtrsim |z_2| \\ &= |z_2|^{1+\frac{k}{12}} |z_1|^{\frac{k}{12}} |z_1z_2|^{-\frac{k}{12}} \\ &\geq (c_\nu\gamma^\nu)^{1+\frac{k}{12}} (c_\nu\gamma^\nu)^{\frac{k}{12}} |z_1z_2|^{-\frac{k}{12}} \\ &\geq \beta^{\frac{1}{2}} |z_1z_2|^{-\frac{k}{12}}.\end{aligned}$$

For (1.20), that is to establish that $|z_3 - z_i| \gtrsim \alpha_2^{\frac{1+2\nu}{4}} \alpha_1^{\frac{1-2\nu}{4}} |z_3 z_i|^{-\frac{k}{12}}$, again, we only need to deal with the case when $2|z_i| < |z_3|$ or $2|z_3| < |z_i|$. Using the same logic as last time, except utilising the fact that $|z_3| \gtrsim \alpha_2^\nu$ as well, we have:

$$\begin{aligned}
|z_3 - z_i| &\gtrsim |z_3| \\
&\geq (c_\nu \alpha_2^\nu)^{1+\frac{k}{12}} (c_\nu \gamma^\nu)^{\frac{k}{12}} |z_1 z_2|^{-\frac{k}{12}} \\
&\geq \alpha_2^{\frac{1+2\nu}{4}} \alpha_1^{\frac{1-2\nu}{4}} |z_1 z_2|^{-\frac{k}{12}}.
\end{aligned}$$

With this, we can now finish the proof of this lemma, as, recalling that the image of $\Phi = P_1' \times P_2' \times P_3'$ under the mapping $(z_1, z_2, z_3) \to H(z_1, z_2, z_3) = x_0 - \Gamma(z_1) + \Gamma(z_2) - \Gamma(z_3)$ is contained in E_2, we have:

$$\begin{aligned}
|E_2| &\gtrsim \iiint_\Phi |J_\mathbb{R} H(z_1, z_2, z_3)| \, dz_1 dz_2 dz_3 \\
&= \iiint_\Phi |J_\mathbb{C} H(z_1, z_2, z_3)|^2 \, dz_1 dz_2 dz_3 \\
&\gtrsim \iiint_\Phi \prod_{i=1}^3 |z_i|^{\frac{2k}{3}} \prod_{1 \leq i < j \leq 3} |z_i - z_j|^2 \, dz_1 dz_2 dz_3 \\
&\gtrsim \beta \alpha_1^{1-2\nu} \alpha_2^{1+2\nu} \iiint_\Phi \prod_{i=1}^3 |z_i|^{\frac{2k}{3}} |z_1 z_2 z_3|^{-\frac{k}{3}} \, dz_1 dz_2 dz_3 \\
&\gtrsim \beta \alpha_1^{1-2\nu} \alpha_2^{1+2\nu} \sigma(P_1') \sigma(P_2') \sigma(P_3') \\
&= \beta^2 \alpha_1^{2-2\nu} \alpha_2^{2+2\nu},
\end{aligned}$$

so we are done. □

Proof of Lemma 1.4.3:

To prove the second lemma, we initially follow the same process as with the previous lemma, except with β_i replacing α_i, α replacing β, F_i replacing E_i, and E replacing F. This results in a collection of 3 sets, Q_1',Q_2' and Q_3', such that, for $z_i \in Q_i'$, and $\nu = \frac{3}{k+6}$, and $\gamma = \max(\beta_1, \alpha)$:

- $\sigma(Q_1') \gtrsim \beta_1$, and $|z_1| \geq c_\nu \gamma^\nu$,
- $\sigma(Q_2') \gtrsim \alpha$ and $|z_2| \geq c_\nu \gamma^\nu$,
- $\sigma(Q_3') \gtrsim \beta_2$ and $|z_3| \geq c_\nu \beta_2^\nu$.

As well as the following separations of our variables:

- $|z_2 - z_1| \gtrsim \alpha^{\frac{1}{2}} |z_2 z_1|^{-\frac{k}{12}}$,

- if $|z_i| < \frac{1}{2}c_\nu\beta_2^\nu$, then $|z_3 - z_i| \gtrsim \beta_2^{\frac{1}{2}} |z_3|^{-\frac{k}{6}}$,
- if $|z_i| \geq \frac{1}{2}c_\nu\beta_2^\nu$, then $|z_3 - z_i| \gtrsim \beta_2^{\frac{1}{2}} |z_i|^{-\frac{k}{6}}$.

If we continued following the previous proof from here we would obtain $|F_2| \gtrsim \beta_1^{2-2\nu}\beta_2^{2+2\nu}\alpha^2$. However, this would not establish our lemma.

We can however note that this is indeed sufficient if $\beta \gtrsim \alpha_1$, say $2\beta > \alpha_1$, as we now have $|F_2| \gtrsim \beta_1^{2-2\nu}\beta_2^{2+2\nu}\alpha^2 \gtrsim \beta_1^{2-2\nu}\beta_2^{2+2\nu}\alpha^2 \left(\frac{\alpha}{\beta_1}\right)\left(\frac{\beta_1}{\beta_2}\right)^{2\nu} = \beta_1\beta_2^2\alpha^3$.

Therefore, we need only establish Lemma 1.4.3 when $2\beta_1 \leq \alpha$. To deal with this case we define $B_{\delta,\alpha}(z_i) = \left\{z \in \Delta \mid |z - z_i| < \delta\alpha^{\frac{1}{2}} |z_1z_2|^{-\frac{k}{12}}\right\}$. If we set $\delta = \min\left\{\frac{c}{3}, \frac{c_\nu^{\frac{k}{6}+1}}{2}\right\}$, where c is the implicit constant in the inequality $|z_2 - z_1| \gtrsim \alpha^{\frac{1}{2}} |z_2z_1|^{-\frac{k}{12}}$, we ensure the balls $B_{\delta,\alpha}(z_1)$, and $B_{\delta,\alpha}(z_2)$ are disjoint. We prove this lemma by considering whether z_3 is in one of these balls, or if it is in neither.

First, if $z_3 \in B_{\delta,\alpha}(z_1)$, and therefore $z_3 \notin B_{\delta,\alpha}(z_2)$, we have the estimate $|z_3 - z_1| < \delta\alpha^{\frac{1}{2}} |z_1z_2|^{-\frac{k}{12}} < \delta\alpha^{\frac{1}{2}} |c_\nu\alpha^\nu|^{-\frac{k}{6}} < \delta c_\nu^{-\frac{k}{6}}\alpha^\nu < \delta c_\nu^{-\frac{k}{6}-1} |z_1| \leq \frac{|z_1|}{2}$.

We can therefore conclude $|z_3| \sim |z_1|$.

As $z_3 \notin B_{\delta,\alpha}(z_2)$, we have $|z_3 - z_2| > \delta\alpha^{\frac{1}{2}} |z_1z_2|^{-\frac{k}{12}} \sim \alpha^{\frac{1}{2}} |z_3z_2|^{-\frac{k}{12}}$.

Furthermore, from the properties of the Q_i', due to similarity in size of z_1 and z_3 we have that $|z_3 - z_1| \gtrsim \beta_2^{\frac{1}{2}} |z_1z_3|^{-\frac{k}{12}}$. We can therefore substitute these inequality into our lower bound for the measure of F_2 to get:

$$
\begin{aligned}
|F_2| &\gtrsim \iiint_\Phi \prod_{i=1}^{3} |z_i|^{\frac{2k}{3}} \prod_{1\leq i<j\leq 3} |z_j - z_i|^2 \, dz_3dz_2dz_1 \\
&\gtrsim \beta_2\alpha^2 \iiint_\Phi \prod_{i=1}^{3} |z_i|^{\frac{2k}{3}} |z_1z_2z_3|^{-\frac{k}{3}} \, dz_3dz_2dz_1 \\
&\gtrsim \beta_1\beta_2^2\alpha^3,
\end{aligned}
$$

which establishes the lemma. An identical argument yields this result for $z_3 \in B_{\delta,\alpha}(z_2)$.

We now only have to deal with the case where $z_3 \notin B_{\delta,\alpha}(z_i)$ for $i = 1, 2$. In this situation, we need to further divide this case, based on the relative sizes of z_1 and z_3.

If $|z_3| < |z_1| < 2|z_3|$, or $|z_1| < |z_3| < 2|z_1|$, then we have that

$|z_3 - z_2| \gtrsim \alpha^{\frac{1}{2}} |z_3 z_2|^{-\frac{k}{12}}$, as $|z_1| \sim |z_3|$ and $z_3 \notin B_{\delta,\alpha}(z_2)$. Similarly, from the initial list of properties of our Q_i, and as $|z_1| \sim |z_3|$, we have $|z_3 - z_1| \gtrsim \beta_2^{\frac{1}{2}} |z_3 z_1|^{-\frac{k}{12}}$.

This is enough again to immediately observe $|F_2| \gtrsim \alpha^2 \beta_2 \sigma(Q_1')\sigma(Q_2')\sigma(Q_3') \gtrsim \alpha^3 \beta_2^2 \beta_1$, by the standard argument used up to here.

The final cases are $|z_3| > 2|z_1|$ or $|z_1| > 2|z_3|$. If this is the case, using the fact that $|z_3 - z_1| \gtrsim |z_3|$, that $z_3 \notin B_{\delta,\alpha}(z_2)$, and that $|z_3| \gtrsim \beta_2^{\nu}$, we obtain obtain:

$$\begin{aligned}
|F_2| &\gtrsim \iiint_\Phi \prod_{i=1}^{3} |z_i|^{\frac{2k}{3}} \prod_{1\leq i<j\leq 3} |z_j - z_i|^2 \, dz_3 dz_2 dz_1 \\
&\gtrsim \alpha^2 \iiint_\Phi \prod_{i=1}^{3} |z_i|^{\frac{2k}{3}} |z_1 z_2|^{-\frac{k}{3}} |z_3|^2 \, dz_3 dz_2 dz_1 \\
&\gtrsim \alpha^2 \iiint_\Phi \prod_{i=1}^{3} |z_i|^{\frac{k}{3}} |z_3|^{\frac{1}{\nu}} \, dz_3 dz_2 dz_1 \\
&\gtrsim \alpha^2 \beta_2 \sigma(Q_1')\sigma(Q_2')\sigma(Q_3') \\
&\gtrsim \alpha^3 \beta_2^2 \beta_1.
\end{aligned}$$

Therefore, the lemma has been established in all cases, so we are done. □

Chapter 2

Decoupling of the complex moment curve

2.1 Introduction

Let Γ denote the complex moment curve in $\mathbb{C}^d$. That is to say

$$\Gamma(z) = \Gamma_d(z) := \left(z, z^2, \cdots, z^d\right).$$

Letting $\mathcal{K}^d$ be the complex unit square, by which we mean unit $2d$-square in $\mathbb{R}^{2d} \simeq \mathbb{C}^d$, centered at the origin, regarded as a set of complex numbers, and defining the following $d \times d$ matrices,

$$L_\Gamma(z) := \begin{pmatrix} \Gamma'(z) & \Gamma''(z) & \cdots & \Gamma^{(d)}(z) \end{pmatrix} \quad \text{and} \quad \Lambda_\delta := \text{diag}\left(\delta, \delta^2, \cdots, \delta^d\right),$$

we say, given a square $Q \subset \mathbb{C}$, with center c_Q, and side length l, we define the parallelepiped associated to Q, and adapted to Γ to be

$$\mathcal{U}_Q := L_\Gamma(c_Q)\Lambda_l\mathcal{K}^d + \Gamma(c_Q),$$

Roughly speaking, this can be thought of as a parallelepiped approximating a δ-neighbourhood of Γ over Q.

For $\delta \in [0, 1]$, and a square, Q, with length l, we define the δ-partion of Q as $P(Q, \delta)$, which is the division of Q into squares of length $l \times 2^{-n}$, where $n \in \mathbb{N}$ is chosen such that $\frac{\delta}{2} < l \times 2^{-n} < \delta$. We also refer to $P(\mathcal{K}^1, 1 \to \delta)$ as simply $P(\delta)$.

Lastly we introduce the notion of the $\ell^2 L^p$ decoupling constant. Specifically, we are interested in the case $p = p_d = d(d+1)$.

Definition 2.1.1. *Let the $\ell^2 L^{p_d}$ decoupling constant of the d dimensional moment curve, denoted as $\mathcal{D}(\delta)$, be the smallest number satisfying the inequality*

$$\left|\left|\sum_{Q\in P(\delta)} f_Q\right|\right|_{L^{p_d}} \leq \mathcal{D}(\delta)\sqrt{\sum_{Q\in P(\delta)} ||f_Q||^2_{L^{p_d}}} \tag{2.1}$$

for any collection of functions $\{f_Q\}_{Q\in P(\delta)}$ each satisfying supp $\widehat{f_Q} \subset \mathcal{U}_Q$.

We generalise the quantity on the right hand side with the notation

$$\sqrt{\sum_{a\in A} z_j^2} := ||z_a||_{\ell^2_{a\in A}},$$

so we can rewrite the above as

$$\left|\left|\sum_{Q\in P(\delta)} f_Q\right|\right|_{L^{p_d}} \leq \mathcal{D}(\delta)\left|\left|||f_Q||_{L^{p_d}}\right|\right|_{\ell^2_{Q\in P(\delta)}}.$$

Our main theorem in this section is the following bound:

Theorem 2.1.2. *For all $\varepsilon > 0$, there exists a constant, $C_{d,\varepsilon}$, such that*

$$\mathcal{D}(\delta) \leq C_{d,\varepsilon}\delta^{-\varepsilon}. \tag{2.2}$$

for all $\delta \in (0,1)$.

Now to obtain this we introduce a second decoupling constant, referred to as the mollified decoupling constant.

Definition 2.1.3. *Let the $\ell^2 L^{p_d}$ mollified decoupling constant of the d dimensional moment curve, denoted as $\widetilde{\mathcal{D}}(\delta)$, be the smallest number satisfying the inequality*

$$\left|\left|\sum_{Q\in P(\delta)} f_Q\right|\right|_{L^{p_d}(\omega dz)} \leq \widetilde{\mathcal{D}}(\delta)\left|\left|||f_Q||_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{Q\in P(\delta)}} \tag{2.3}$$

for any collection of functions $\{f_Q\}_{Q\in P(\delta)}$ each satisfying supp$\widehat{f_Q} \subset \mathcal{U}_Q$, and any weight function ω, with supp $\widehat{\omega} \subset B(0, \frac{\delta^d}{10})$.

Now in the proof of our theorem we will work on bounding $\widetilde{\mathcal{D}}(\delta)$. Note that this is sufficient, as $\mathcal{D}(\delta) \lesssim \widetilde{\mathcal{D}}(\delta)$, which can be seen by $\omega = 1$.

If $\{U_i\}_{i=1}^n$ are a collection of subsets of $\mathbb{C}^d$, and $\mathcal{D}(\{U_i\}_{i=1}^n)$ is similarly defined to 2.1, with U_i playing the role of $\mathcal{U}_Q$, then, if estimates similar to 2.1.2 hold, that is $\mathcal{D}(\{U_i\}_{i=1}^n) \lesssim n^\varepsilon$, we say $\{U_i\}_{i=1}^n$ exhibits genuine decoupling. Due to the iterative nature of the techniques used, this is a natural bound to aim for, although $\mathcal{O}(1)$ bounds do exist, such as if we were working with p_d replaced with 2.

Decoupling inequalities, such as theorem 2.1.2, were first introduced in [Wol00], in which he worked with inequalities related to ℓ^q summations instead of just ℓ^2.

Furthermore, with regards to the L^p norm, taking $p = d(d+1)$ is motivated by considering the case where $\hat{f}_Q$ are bump functions adapted to $\mathcal{U}_Q$. Modern decoupling arguments have been inspired by the efficient congruencing arguments from [Woo19], hinting at the relationship between harmonic analysis and number theory. Particularly, this section follows the arguments presented in [GLYZK21], which presents a proof of genuine decoupling of the real moment curve utilising these modern arguments, though the result was first established in [BDG16].

Some notes on this section:

- If $\mathcal{U}$ is a parallelepiped, or square, we write $C\mathcal{U}$ for the similar parallelepiped with the same centre as $\mathcal{U}$, but with sides length scaled by a factor of C.
- For any square Q in this section, we let c_Q denote the center of Q.
- If we have $\{f_{Q'}\}_{Q' \in P(Q,\delta)}$ such that $\mathrm{supp}\widehat{f_{Q'}} \subset \mathcal{U}_{Q'}$, then we write $f_Q := \sum_{Q' \in P(Q,\delta)} f_{Q'}$. Explicitly, we $P(Q, \delta)$ can be taken to be a collection of squares whose side lengths are 2^{-n}, such that $2^{-n} < \delta \leq 2^{-n+1}$.
- Note that when we refer to the Fourier transform of a complex valued function throughout this paper, we interpret this as $\widehat{f}(\xi) := \int_{\mathbb{R}^{2d}} e^{2\pi i (x+y)\cdot\xi} f(x+iy)dxdy$.
- To describe decoupling at different scales, if $Q \in P(\delta_1)$, and $\delta_2 < \delta_1$, then $\widetilde{\mathcal{D}}(Q, \delta_1 \to \delta_2)$ is the smallest real number satisfying

$$\left|\left|\sum_{L \in P(Q,\delta_2)} f_L\right|\right|_{L^{p_d}(\omega dz)} \leq \widetilde{\mathcal{D}}(Q, \delta_1 \to \delta_2) \left|\left| ||f_L||_{L^{p_d}(\omega dz)} \right|\right|_{\ell^2_{L \in P(Q,\delta_2)}} \tag{2.4}$$

for any collection of functions $\{f_L\}_{L\in P(Q,\delta_2)}$ each satisfying $\text{supp}\widehat{f_L} \subset \mathcal{U}_L$, and any weight function ω, with supp $\widehat{\omega} \subset B(0, \frac{\delta_2^d}{10})$. We will refer to this as "decoupling from scale δ_1 to scale δ_2".

Note that in this notation, we have

$$\widetilde{\mathcal{D}}(\delta) = \widetilde{\mathcal{D}}(\mathcal{K}, 1 \to \delta).$$

Furthermore, when working with the moment curve, this is independent of Q, so we will often simply denote this generalised decoupling constant $\widetilde{\mathcal{D}}(\delta_1 \to \delta_2)$. This follows form the

- Whenever we use notation such as above, for any kind of decoupling constants in this paper, we will assume the δ values make sense. So if a lemma talks about $\widetilde{\mathcal{D}}(\delta_1 \to \delta_2)$ for example, we assume this means $\delta_1 \geq \delta_2$.

2.2 Bilinear Decoupling

In order to bound obtain our estimate, we introduce the notion of bilinear decoupling, which let's us describe the simultaneous decoupling of two squares at a scale ~ 1 to scale δ. Then, using the affine invariance of the problem, we will derive its behaviour when simultaneously decoupling between any two scales.

Definition 2.2.1. *For $\delta \in (0, \frac{1}{4})$, let the mollified $\ell^2 L^{p_d}$ bilinear decoupling constant of the d dimensional moment curve, denoted as $\widetilde{\mathcal{B}}(\delta)$, be the smallest number satisfying the inequality*

$$\int_{\mathbb{C}^d} |f_Q|^{\frac{p_d}{2}} |f_{Q'}|^{\frac{p_d}{2}} \omega \leq \widetilde{\mathcal{B}}(\delta)^{p_d} \Big|\Big| ||f_L||_{L^{p_d}(\omega dz)} \Big|\Big|^{\frac{p_d}{2}}_{\ell^2_{L\in P(Q,\delta)}} \Big|\Big| ||f_{L'}||_{L^{p_d}(\omega dz)} \Big|\Big|^{\frac{p_d}{2}}_{\ell^2_{L'\in P(Q',\delta)}} \tag{2.5}$$

for all pairs of squares, $(Q, Q') \in \mathcal{P}(\frac{1}{4})^2$, such that dist$(Q, Q') \geq \frac{1}{4}$, all weight functions ω, with supp $\hat{\omega} \subset B(0, \frac{\delta^d}{10})$, and collections of functions $\{f_L\}_{L\in P(Q,\delta)\cup P(Q',\delta)}$ such that $\text{supp}\widehat{f_L} \subset \mathcal{U}_L$, and $\text{supp}\widehat{f_{Q'}} \subset \mathcal{U}_{Q'}$.

Again, we generalise this notion, by replacing $\frac{1}{4}$ with δ_1[1], and δ with δ_2, then we define a new constant, which we refer to as $\widetilde{\mathcal{B}}(\delta_1 \to \delta_2)$.

We want to use these quantities to bound $\widetilde{\mathcal{D}}(\delta)$, but to do so, we need to

[1]in both instances.

establish the following affine rescaling results.

Lemma 2.2.2. *Suppose $Q \in P(\delta_1)$. Then, for any $\delta_2 \in (0, \delta_1)$, we have that*

$$\widetilde{\mathcal{D}}(Q, \delta_1 \to \delta_2) \leq \widetilde{\mathcal{D}}\left(\mathcal{K}, 1 \to \frac{\delta_2}{\delta_1}\right).$$

Similarly,

$$\widetilde{\mathcal{B}}(\delta_1 \to \delta_2) \lesssim \widetilde{\mathcal{B}}\left(\frac{1}{4} \to \frac{\delta_2}{4\delta_1}\right).$$

Proof:

Note, this argument relies on the existence of an affine map, A_Q which fixes the complex moment curve, and satisfies $A_Q\Gamma(\mathcal{K}) = \Gamma(Q)$. Explicitly, this map is given by

$$A_Q z := \Gamma(c_Q) + \sum_{j=1}^{d} \frac{\delta_1^j z_j}{j!} \Gamma^{(j)}(c_Q).$$

For convenience, we rewrite this as $A_Q z = \Gamma(c_Q) + B_Q z$.

Now, for each $Q \in P(\delta_1)$, and $L \in P\left(\frac{\delta_2}{\delta_1}\right)$, we denote $L^Q = c_Q + \delta_1 L$. This means that $\{L^Q\}_{L \in P\left(\frac{\delta_2}{\delta_1}\right)} = P(Q, \delta_2)$.

Now, given a collection of functions $\{f_{L^Q}\}_{L \in P\left(\frac{\delta_2}{\delta_1}\right)}$ satisfying $\text{supp}\widehat{f_{L^Q}} \subset \mathcal{U}_{L^Q}$, we associate to each of these, the function g_L as:

$$g_L(z) := \det\left(B_Q^{-1}\right) e^{2\pi i z \cdot B_Q^{-1}\Gamma(c_Q)} f_{L^Q}(B_Q^{-T} z)).$$

Now, if we compute the Fourier transform of g_L, we have

$$\begin{aligned}
\widehat{g_L}(\boldsymbol{\xi}) &= \det\left(B_Q\right) \int e^{2\pi i z \cdot \left(\xi + B_Q^{-1}\Gamma(c_Q)\right)} f_{L^Q}(B_Q^{-T} z) dz \\
&= \int e^{2\pi i B_Q^T u \cdot \left(\xi + B_Q^{-1}\Gamma(c_Q)\right)} f_{L^Q}\left(u\right) du \\
&= \widehat{f_{L^Q}}(A_Q \xi).
\end{aligned}$$

So therefore, $\text{supp}\widehat{g_L} = A_Q^{-1}\mathcal{U}_{L^Q} = \mathcal{U}_L$.

Therefore, $\{g_L\}_{L \in P\left(\frac{\delta_2}{\delta_1}\right)}$ is a collection of functions with appropriate Fourier support to decouple, so we have

$$\left\| \sum_{L\in P\left(\frac{\delta_2}{\delta_1}\right)} g_L \right\|_{L^{p_d}(\omega^\circ dz)} \leq \widetilde{\mathcal{D}}\left(\frac{\delta_2}{\delta_1}\right) \left|\left|\left|\left|g_L\right|\right|_{L^{p_d}}\right|\right|_{\ell^2_{L\in P\left(\frac{\delta_2}{\delta_1}\right)}}.$$

for any weight function ω° which has its Fourier support in $B(0, \frac{\delta_2^d}{10\delta_1^d})$

Now, for any ω with its Fourier support in $B(0, \frac{\delta_2^d}{10})$, we can take $\omega^\circ(z) = \omega(B_Q^{-T}z)$. Note that this function has its Fourier support in $B_Q^T B(0, \frac{\delta_2^d}{10})$, which is indeed a subset of $CB(0, \frac{\delta_2^d}{10\delta_1^d})$, for some $C \sim 1$. We briefly discuss in the appendix of this section why this is not a cause for concern, and can proceed as if this C was present.

Doing a change of variables within the L^{p_d} norms, and canceling the exponential terms indeed recovers the expression

$$\left\| \sum_{L\in P\left(\frac{\delta_2}{\delta_1}\right)} f_{L^Q} \right\|_{L^{p_d}(\omega dz)} \leq \widetilde{\mathcal{D}}\left(\frac{\delta_2}{\delta_1}\right) \left|\left|\left|\left|f_{L^Q}\right|\right|_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{L\in P\left(\frac{\delta_2}{\delta_1}\right)}},$$

which as previously established, is equivalent to

$$\left\| \sum_{\tilde{L}\in P(Q,\delta_2)} f_{\tilde{L}} \right\|_{L^{p_d}(\omega dz)} \leq \widetilde{\mathcal{D}}\left(\frac{\delta_2}{\delta_1}\right) \left|\left|\left|\left|f_{\tilde{L}}\right|\right|_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{\tilde{L}\in P(Q,\delta_2)}}.$$

As this is precisely the decoupling equation corresponding to decoupling Q from scale δ_1 to δ_2, and it holds for any appropriate ω, we are done.

For the second result, we proceed almost identically. However, due to technical differences in the definitions, we first fix two squares M and $M' \in P(\frac{1}{4})$.

Now for each pair $Q, Q' \in P(\delta_1)$, and $L \in P\left(M, \frac{\delta_2}{4\delta_1}\right)$, we construct as before L^Q, and functions g_L which have their Fourier support in $\mathcal{U}_L$. We define $g_M = \sum_{L\in P\left(M,\frac{\delta_2}{4\delta_1}\right)} g_L,$.

For the $g_{L'}$ however, we define them slightly differently, via

$$g'_L(z) := \det\left(B_{Q'}^{-1}\right) e^{2\pi i z\cdot B_{Q'}^{-1}\Gamma(c_Q)} f_{L^{Q'}}(L_\Gamma^{-T}(c_Q)L_\Gamma^T(c_{Q'})B_{Q'}^{-T}z).$$

Note that as $L_\Gamma^{-T}(c_{Q'})L_\Gamma^T(c_{Q'})$ is a matrix with bounded entries, and determinant 1, this won't change the Fourier support very much, instead ensuring that the Fourier support lies in $C\mathcal{U}_L$, for some $C \sim 1$.

We will see that this is sufficient to establish decoupling inequalities, up to a constant, so we obtain:

$$\int |g_M|^{\frac{p_d}{2}} |g_{M'}|^{\frac{p_d}{2}} \omega^\circ \lesssim \widetilde{\mathcal{B}}(\frac{\delta_2}{4\delta_1})^{p_d} \left\| \left\| g_L (\omega^\circ)^{\frac{1}{p_d}} \right\|_{L^{p_d}} \right\|_{\ell^2_{L\in P\left(M, \frac{\delta_2}{4\delta_1}\right)}}^{\frac{p_d}{2}} \left\| \left\| g_{L'} (\omega^\circ)^{\frac{1}{p_d}} \right\|_{L^{p_d}} \right\|_{\ell^2_{L'\in P\left(M', \frac{\delta_2}{4\delta_1}\right)}}^{\frac{p_d}{2}},$$

for any ω° with Fourier support in $B(0, \frac{\delta_2^d}{10\delta_1^d})$.

We then, for any ω with Fourier support in $B(0, \frac{\delta_2^d}{10})$, can again take $\omega^\circ(z) = \omega(B_Q^{-T} z)$.

Now expanding the left hand side here and applying the same substitution, we get

$$\begin{aligned} &= \det{(B_Q)}^{\frac{p_d}{2}-1} \det{(B_{Q'})}^{\frac{p_d}{2}} \\ &\times \int \left| \sum_{\tilde{L}\in P(Q,\delta_2)} f_{\tilde{L}}(u) \right|^{\frac{p_d}{2}} \left| \sum_{\tilde{L'}\in P(Q',\delta_2)} f_{\tilde{L}'}(L_\Gamma^{-T}(c_Q)L_\Gamma^T(c_{Q'})B_{Q'}^{-T}B_Q^T u) \right|^{\frac{p_d}{2}} \omega^\circ(B_Q^T u) du \\ &= \det{(B_Q)}^{\frac{p_d}{2}-1} \det{(B_{Q'})}^{\frac{p_d}{2}} \int \left| \sum_{\tilde{L}\in P(Q,\delta_2)} f_{\tilde{L}}(u) \right|^{\frac{p_d}{2}} \left| \sum_{\tilde{L}'\in P(Q',\delta_2)} f_{\tilde{L}'}(u) \right|^{\frac{p_d}{2}} \omega(u) du \end{aligned}$$

Then, proceeding with identical substitutions on the right side allows us to cancel the determinants, and establish our inequality. □

This rescaling lemma allows us to establish an upper bound of $\widetilde{\mathcal{D}}$ in terms of $\widetilde{\mathcal{B}}$.

Lemma 2.2.3. *For all $N \in \mathbb{N}$ with $N \geq 2$, we have that*

$$\widetilde{\mathcal{D}}_d(2^{-N}) \lesssim \left(1 + \sum_{n=2}^{N} \widetilde{\mathcal{B}}\left(2^{-n} \to 2^{-N}\right)\right)^{\frac{1}{2}}$$

Proof: To see this, let f_Q have Fourier support inside $\mathcal{U}_Q$ for all $Q \in P(2^{-N})$.

We then cover the set $\mathcal{K}^2$ into pairs of squares we can simultaneously decouple. Denoting $\mathcal{W}_1 := \emptyset$, and for $1 < n \leq N$, denote

$$\mathcal{W}_n := \{(Q_1, Q_2) \in P(2^{-n})^2 \,|\, Q_2 = Q_1 \pm n_1 2^{-n} \pm i n_2 2^{-n} \,, n_i \in \{2,3\}\}.$$

These sets cover the complex square $\mathcal{K}^2$ almost everywhere, except for some complex squares centered along its diagonal, which each have side length of 2^{-N}. We denote this set as

$$\widetilde{\mathcal{W}}_N = \{(Q_1, Q_2) \in P(2^{-N}) \,|\, \text{dist}(Q_1, Q_2) = 0\}\}.$$

We label this decomposition as $\mathcal{W}^N := \left(\bigcup_{n=1}^N \mathcal{W}_n\right) \cup \widetilde{\mathcal{W}}_N$. Now taking $\{f_Q\}_{Q \in P(2^{-N})}$ and ω satisfying the conditions in (2.3), we rewrite

$$\begin{aligned}
\left|\left|\sum_{Q \in P(2^{-N})} f_Q\right|\right|_{L^{p_d}(\omega dz)} &= \left|\left|\sum_{(Q,Q') \in \mathcal{W}^N} f_Q \overline{f_{Q'}} \omega^{\frac{2}{p_d}}\right|\right|_{L^{\frac{p_d}{2}}}^{\frac{1}{2}} \\
&\leq \left(\sum_{(Q,Q') \in \mathcal{W}^N} \left|\left|f_Q \overline{f_{Q'}} \omega^{\frac{2}{p_d}}\right|\right|_{L^{\frac{p_d}{2}}}\right)^{\frac{1}{2}} \\
&= \left(\sum_{(Q,Q') \in \widetilde{\mathcal{W}}_N} \left|\left|f_Q \overline{f_{Q'}} \omega^{\frac{2}{p_d}}\right|\right|_{L^{\frac{p_d}{2}}} + \sum_{n=2}^N \sum_{(Q,Q') \in \mathcal{W}_n} \left|\left|f_Q \overline{f_{Q'}} \omega^{\frac{2}{p_d}}\right|\right|_{L^{\frac{p_d}{2}}}\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.6}$$

For the first sum, we can estimate it as:

$$\begin{aligned}
\sum_{(Q,Q') \in \widetilde{\mathcal{W}}_N} \left|\left|f_Q \overline{f_{Q'}} \omega^{\frac{2}{p_d}}\right|\right|_{L^{\frac{p_d}{2}}} &\leq \sum_{(Q,Q') \in \widetilde{\mathcal{W}}_N} ||f_Q||_{L^{p_d}(\omega dz)} \, ||f_{Q'}||_{L^{p_d}(\omega dz)} \\
&\leq \sum_{(Q,Q') \in \widetilde{\mathcal{W}}_N} \left(||f_Q||^2_{L^{p_d}(\omega dz)} + ||f_{Q'}||^2_{L^{p_d}(\omega dz)}\right) \\
&\leq \sum_{Q \in P(2^{-N})} 18 \, ||f_Q||^2_{L^{p_d}(\omega dz)}
\end{aligned}$$

As each square $Q \in P(2^{-N})$ has distance zero from at most 9 other squares in $P(2^{-N})$, and hence it appears in the above sum 9 times in the role of Q, and another 9 times as Q'.

For the second term, we use the distance between the squares to apply lemma

2.2.2 to get, for $(Q, Q') \in \mathcal{W}_n$, the estimate:

$$\begin{aligned}
&\left|\left|f_Q\overline{f_{Q'}}\right|\right|_{L^{\frac{p_d}{2}}(\omega dz)} \\
&\lesssim \widetilde{\mathcal{B}}(2^{-n} \to 2^{-N})^2 \left|\left|||f_L||_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{L\in P(Q,2^{-N})}} \left|\left|||f'_L||_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{L'\in P(Q',2^{-N})}} \\
&\lesssim \widetilde{\mathcal{B}}(2^{-n} \to 2^{-N})^2 \left(\left|\left|||f_L||_{L^{p_d}(\omega dz)}\right|\right|^2_{\ell^2_{L\in P(Q,2^{-N})}} + \left|\left|||f'_L||_{L^{p_d}(\omega dz)}\right|\right|^2_{\ell^2_{L'\in P(Q',2^{-N})}}\right)
\end{aligned}$$

We therefore have

$$\begin{aligned}
&\left|\left|f_Q\overline{f_{Q'}}\right|\right|_{L^{\frac{p_d}{2}}(\omega dz)} \\
&\lesssim \widetilde{\mathcal{B}}(2^{-n} \to 2^{-N})^2 \left|\left|||f_L||_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{L\in P(Q,2^{-N})}} \left|\left|||f'_L||_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{L'\in P(Q',2^{-N})}} \\
&\lesssim \widetilde{\mathcal{B}}(2^{-n} \to 2^{-N})^2 \left(\left|\left|||f_L||_{L^{p_d}(\omega dz)}\right|\right|^2_{\ell^2_{L\in P(Q,2^{-N})}} + \left|\left|||f'_L||_{L^{p_d}(\omega dz)}\right|\right|^2_{\ell^2_{L'\in P(Q',2^{-N})}}\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\sum_{(Q,Q')\in\mathcal{W}_n} \left|\left|f_Q\overline{f_{Q'}}\right|\right|_{L^{\frac{p_d}{2}}(\omega dz)} &\leq \sum_{Q\in P(2^{-n})} 80\widetilde{\mathcal{B}}(2^{-n} \to 2^{-N})^2 \left|\left|||f_L||_{L^{p_d}(\omega dz)}\right|\right|^2_{\ell^2_{L\in P(Q,2^{-N})}} \\
&= 80\widetilde{\mathcal{B}}(2^{-n} \to 2^{-N})^2 \left|\left|||f_L||_{L^{p_d}(\omega dz)}\right|\right|^2_{\ell^2_{L\in P(2^{-N})}},
\end{aligned}$$

as any Q appears in at most 80 of these (Q, Q') pairs. Furthermore, notice the final equality comes from the fact that the ℓ^2 norm is over a decomposition of Q of resolution 2^{-N}, and as we sum over these Q, whose union is $\mathcal{K}$, we are indeed summing over $P(2^{-N})$.

So we have, via inserting these estimates into 2.6:

$$\left|\left|\sum_{Q\in P(2^{-N})} f_Q\right|\right|_{L^{p_d}(\omega dz)} \lesssim \left(1 + \sum_{n=2}^{N} \widetilde{\mathcal{B}}(2^{-n} \to 2^{-N})^2\right)^{\frac{1}{2}} \left|\left|||f_Q||_{L^{p_d}(\omega dz)}\right|\right|_{\ell^2_{Q\in P(2^{-N})}}$$

So the conclusion of lemma 2.2.3 does indeed hold. $\square$

Note that when we have to use this lemma in the final section however, it is best to regard it as

$$\mathcal{D}(2^{-N}) \lesssim \Big(1 + \sum_{n=2}^{N} \widetilde{\mathcal{B}}(2^{-N+n-2})^2\Big)^{\frac{1}{2}},$$

via lemma 2.2.2.

2.3 The Asymmetric Decoupling Constants

The last decoupling constants we need to utilise are defined in relation to the simultaneous decoupling of sets of different scales, with non-symmetric exponents.

Definition 2.3.1. *For any integer $l \in \{0, 1, \cdots, d-1\}$, we define the $\ell^2 L^{p_d}$ l-asymmetric decoupling constant of the d dimensional moment curve, denoted as $\widetilde{\mathcal{B}}_l(\delta_1 \to \delta, \delta_2 \to \delta)$, to be the smallest number satisfying the inequality:*

$$\begin{aligned}
&\int_{\mathbb{C}^d} |f_Q|^{p_l} \, |f_{Q'}|^{p_d - p_l} \, \omega \\
&\quad \leq \widetilde{\mathcal{B}}_l(\delta_1 \to \delta, \delta_2 \to \delta)^{p_d} \Big|\Big| ||f_L||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_l}_{\ell^2_{L \in P(Q,\delta)}} \Big|\Big| ||f_{L'}||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_d - p_l}_{\ell^2_{L' \in P(Q',\delta)}}
\end{aligned} \tag{2.7}$$

For all squares $Q \in \mathcal{P}(\delta_1)$, and $Q' \in \mathcal{P}(\delta_2)$ such that dist$(Q, Q') \geq \frac{1}{4}$, all weight functions ω, with supp $\omega \subset B(0, \frac{\delta^d}{10})$, and all collections of functions $\{f_L\}_{L \in P(Q,\delta) \cup P(Q',\delta)}$ such that supp$\widehat{f_L} \subset \mathcal{U}_L$.

Lemma 2.3.2. *For every $l \in \{0, 1, \cdots, d-1\}$, and $\delta \in (0, \frac{1}{4})$ we have the estimate:*

$$\widetilde{\mathcal{B}}(\delta) \lesssim \delta_1^{-\frac{2p_l}{p_d}} \delta_2^{-\frac{2(p_d - p_l)}{p_d}} \widetilde{\mathcal{B}}_l(\delta_1 \to \delta, \delta_2 \to \delta),$$

Proof of 2.3.2: First, take Q and $Q' \in \mathcal{P}(\frac{1}{4})$, with dist$(Q, Q') \geq \frac{1}{4}$, so that they satisfy the conditions for (2.5) to hold. Now suppose we have $\{f_L\}_{L \in P(Q,\delta) \cup P(Q',\delta)}$ with supp$\widehat{f_L} \subset \mathcal{U}_L$. First, notice we have

$$\begin{aligned}
\int_{\mathbb{C}^d} |f_Q|^{\frac{p_d}{2}} \, |f_{Q'}|^{\frac{p_d}{2}} \, \omega &= \Big|\Big| f_Q^{\frac{p_l}{2}} f_{Q'}^{\frac{p_d - p_l}{2}} \omega^{\frac{1}{2}} f_Q^{\frac{p_d - p_l}{2}} f_{Q'}^{\frac{p_l}{2}} \omega^{\frac{1}{2}} \Big|\Big|_{L^1} \\
&\leq \Big(\int_{\mathbb{C}^d} |f_Q|^{p_l} \, |f_{Q'}|^{p_d - p_l} \, \omega\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{C}^d} |f_Q|^{p_l} \, |f_{Q'}|^{p_d - p_l} \, \omega\Big)^{\frac{1}{2}}
\end{aligned}$$

by Hölder's inequality. First note that

$$\int_{\mathbb{C}^d} |f_Q|^{p_l} |f_{Q'}|^{p_d-p_l} \omega \leq \int_{\mathbb{C}^d} \Big(\sum_{L \in P(Q,\delta_1)} |f_L| \Big)^{p_l} \Big(\sum_{L' \in P(Q',\delta_2)} |f_{L'}| \Big)^{p_d-p_l} \omega.$$

Next, as $\left(\sum_{i=1}^n a_i\right)^m \leq n^{m-1} \sum_{i=1}^n a_i^m$ for any collection of positive real numbers a_i, we can bound the above as:

$$\begin{aligned}
&\lesssim \delta_1^{-2(p_l-1)} \delta_2^{-2(p_d-p_l-1)} \sum_{L \in P(Q,\delta_1)} \sum_{L' \in P(Q',\delta_2)} \int_{\mathbb{C}^d} |f_L|^{p_l} |f_{L'}|^{p_d-p_l} \omega \\
&\lesssim \delta_1^{-2(p_l-1)} \delta_2^{-2(p_d-p_l-1)} \widetilde{\mathcal{B}}_l((\delta_1 \to \delta), (\delta_2 \to \delta))^{p_d} \\
&\quad \times \sum_{L \in P(Q,\delta_1)} \sum_{L' \in P(Q',\delta^b)} \Big|\Big| ||f_K||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_l}_{\ell^2_{K \in P(L,\delta)}} \Big|\Big| ||f_{K'}||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_d-p_l}_{\ell^2_{K' \in P(L',\delta)}} \\
&\leq \delta_1^{-2(p_l-1)} \delta_2^{-2(p_d-p_l-1)} \widetilde{\mathcal{B}}_l((\delta_1 \to \delta), (\delta_2 \to \delta))^{p_d} \\
&\quad \times \Big|\Big| ||f_K||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_l}_{\ell^2_{K \in P(Q,\delta)}} \Big|\Big| ||f_{K'}||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_d-p_l}_{\ell^2_{K' \in P(Q',\delta)}}
\end{aligned}$$

And we get an identical estimate for the other bracket, via decomposing Q as $P(Q,\delta_2)$, and Q' as $P(Q',\delta_1)$, and arguing as above, such that:

$$\begin{aligned}
&\int_{\mathbb{C}^d} |f_Q|^{p_d-p_l} |f_{Q'}|^{p_l} \omega \\
&\quad \leq \delta_1^{-2(p_d-p_l)} \delta_2^{-2p_l} \widetilde{\mathcal{B}}_l((\delta_1 \to \delta), (\delta_2 \to \delta))^{p_d} \\
&\qquad \times \Big|\Big| ||f_K||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_d-p_l}_{\ell^2_{K \in P(Q,\delta)}} \Big|\Big| ||f_{K'}||_{L^{p_d}(\omega dz)} \Big|\Big|^{p_l}_{\ell^2_{K' \in P(Q',\delta)}}.
\end{aligned}$$

Inserting these into our initial estimate, we get that:

$$\begin{aligned}
&\int_{\mathbb{C}^d} |f_Q|^{\frac{p_d}{2}} |f_{Q'}|^{\frac{p_d}{2}} \omega \\
&\quad \leq \delta_1^{-2p_l} \delta_2^{-2(p_d-p_l)} \widetilde{\mathcal{B}}_l((\delta_1 \to \delta), (\delta_2 \to \delta))^{p_d} \\
&\qquad \times \Big|\Big| ||f_K||_{L^{p_d}(\omega dz)} \Big|\Big|^{\frac{p_d}{2}}_{\ell^2_{K \in P(Q,\delta)}} \Big|\Big| ||f_{K'}||_{L^{p_d}(\omega dz)} \Big|\Big|^{\frac{p_d}{2}}_{\ell^2_{K' \in P(Q',\delta)}}.
\end{aligned}$$

So indeed, the Lemma holds. $\square$

Lemma 2.3.3. *Let $\gamma : \mathbb{C} \to \mathbb{C}^l$ be a polynomial curve, with $\gamma^{(k)}(z) \lesssim 1$ for all*

$k \in \mathbb{N}$, *and* $\det(L_\gamma(z)) \gtrsim 1$.[2]

Suppose also that 2.1.2 has been established in l *dimensions for the complex moment curve.*

Then we also have $\mathcal{D}_\gamma(\delta) \lesssim \delta^{-\varepsilon}$, *where* $\mathcal{D}_\gamma(\delta)$ *is the decoupling constant for* $\{\mathcal{U}_{Q,\gamma}\}_{Q\in P(\delta)}$, *and* $\mathcal{U}_{Q,\gamma} := L_\gamma(c_Q)\Lambda_\delta \mathcal{K}^l + \gamma(c_Q)$

Proof: We wish to show, for any $\{f_Q\}_{Q\in P(\delta)}$ with supp$\widehat{f_Q} \subset \mathcal{U}_{Q,\gamma}$, and ω with Fourier support in $B(0, \frac{\delta^d}{10})$

$$\left|\left|\sum_{Q\in P(\delta)} f_Q\right|\right|_{L^{p_l}(\omega dz)} \lesssim \delta^{-\varepsilon} \left|\left|||f_Q||_{L^{p_l}(\omega dz)}\right|\right|_{\ell^2_{Q\in P(\delta)}}.$$

To establish this, we instead prove the following sufficient statement.

Suppose $\Delta > \delta^{\frac{l}{l+1}}$, and $L \in P(\Delta)$. Let $f_L = \sum_{Q\in P(L,\delta)} f_Q$, and similarly for each $L' \in P(L, \Delta^{\frac{l+1}{l}})$, we let $f_{L'} = \sum_{Q\in P(L',\delta)} f_Q$. Then

$$||f_L||_{L^{p_l}(\omega dz)} \lesssim \Delta^{-\varepsilon} \left|\left|||f_{L'}||_{L^{p_l}(\omega dz)}\right|\right|_{\ell^2_{L'\in P(L,\Delta^{\frac{l+1}{l}})}},$$

or, in other terms,

$$\mathcal{D}_\gamma(\Delta \to \Delta^{\frac{l+1}{l}}) \lesssim \Delta^{-\varepsilon}. \tag{2.8}$$

To briefly address why this reduction is sufficient, note that

$$\left|\left|\sum_{Q\in P(\delta)} f_Q\right|\right|_{L^{p_l}(\omega dz)} = \left|\left|\sum_{L\in P(\Delta)} f_L\right|\right|_{L^{p_l}(\omega dz)} \leq \Delta^{-1} \left|\left|||f_L||_{L^{p_l}(\omega dz)}\right|\right|_{\ell^2_{L\in P(\Delta)}}$$

via a trivial decoupling estimate, for any Δ. Now, take $\Delta = \delta^{\frac{l^A}{(l+1)^A}}$, for some large integer A to be determined. Now, we can apply (2.8) to each $||f_L||_{L^{p_l}(\omega dz)}$, to obtain

$$\left|\left|\sum_{Q\in P(\delta)} f_Q\right|\right|_{L^{p_l}(\omega dz)} \lesssim \Delta^{-1}\Delta^{-\varepsilon} \left|\left|||f_{L'}||_{L^{p_l}(\omega dz)}\right|\right|_{\ell^2_{L'\in P(\Delta^{\frac{l+1}{l}})}}.$$

Applying this A times will result in us decoupling down from to scale $\Delta^{\frac{(l+1)^A}{l^A}} = \delta$, giving us:

[2]Here, we define L_γ identically to L_Γ, except it is instead a $l \times l$ matrix.

$$\left\| \sum_{Q \in P(\delta)} f_Q \right\|_{L^{p_l}(\omega dz)} \lesssim \Delta^{-1} \Delta^{-\varepsilon} \Delta^{-\varepsilon \frac{l+1}{l}} \cdots \Delta^{-\varepsilon \frac{(l+1)^{A-1}}{l^{A-1}}} \left\| ||f_Q||_{L^{p_l}(\omega dz)} \right\|_{\ell^2_{Q \in P(\delta)}},$$

$$\begin{aligned} \left\| \sum_{Q \in P(\delta)} f_Q \right\|_{L^{p_l}(\omega dz)} &\lesssim \Delta^{-1} \Delta^{-\varepsilon} \Delta^{-\varepsilon \frac{l+1}{l}} \cdots \Delta^{-\varepsilon \frac{(l+1)^{A-1}}{l^{A-1}}} \left\| ||f_Q||_{L^{p_l}(\omega dz)} \right\|_{\ell^2_{Q \in P(\delta)}}, \\ &= \delta^{-\frac{l^A}{(l+1)^A} - \varepsilon \frac{l^A}{(l+1)^A} - \varepsilon \frac{l^{A-1}}{(l+1)^{A-1}} \cdots - \varepsilon \frac{l}{l+1}} \left\| ||f_Q||_{L^{p_l}(\omega dz)} \right\|_{\ell^2_{Q \in P(\delta)}} \\ &= \delta^{-\frac{l}{l+1}^A - \varepsilon l (1 - (\frac{l}{l+1})^A)} \left\| ||f_Q||_{L^{p_l}(\omega dz)} \right\|_{\ell^2_{Q \in P(\delta)}} \end{aligned}$$

which shows it to be sufficient by choosing A large enough.

Now, to establish this result, notice that

$$\begin{aligned} \gamma(z) &= \gamma(c_L) + \gamma'(c_L)(z - c_L) + \cdots \gamma^{(l)}(c_L)(z - c_L)^l + \mathcal{O}(z^{l+1}) \\ &= L_\gamma(c_L)(\Gamma_l(z) - c_L) + \mathcal{O}((z - c_L)^{l+1}). \end{aligned}$$

Now, by the hypotheses on γ, $L_\gamma(c_L)(\Gamma_l(z - c_L))$ is a non-singular, affine transformation. Now this means, for each $L' \in P(L, \Delta^{\frac{l+1}{l}})$, we have $\text{supp} f'_L \subset \mathcal{U}_{L',\gamma} \subset C\mathcal{U}_{L'}$. This follows as the shortest side of $\mathcal{U}_{L',\gamma}$ is $\sim \Delta^{\frac{l+1}{l}^l} = \Delta^{l+1}$.

It therefore follows that

$$\mathcal{D}_\gamma(\Delta \to \Delta^{\frac{l+1}{l}}) \lesssim \mathcal{D}(\Delta \to \Delta^{\frac{l+1}{l}}) \leq \mathcal{D}(1 \to \Delta^{\frac{1}{l}}) \lesssim \Delta^{-\frac{\varepsilon}{l}},$$

which is sufficient to establish our result. □

We have now sufficient tools to establish the key estimate required to prove (2.1.2).

Lemma 2.3.4. *Let l be a postive integer less than d, and let $\delta < \delta_1$ and δ_2, and furthermore let $\delta_1 > \delta_2^{\frac{d-l+1}{l}}$. Then*

$$\widetilde{B}_l(\delta_1 \to \delta, \delta_2 \to \delta) \lesssim \widetilde{\mathcal{D}}_{l,\mathit{Proj}_{d-l}\Gamma}(\delta_1 \to \delta_2^{\frac{d-l+1}{l}})^{\frac{p_d}{p_l}} \widetilde{B}_l(\delta_2^{\frac{d-l+1}{l}} \to \delta, \delta_2 \to \delta),$$

where $\mathit{Proj}_{d-l}\Gamma$ refers to the projection of Γ onto the last l of its dimensions. That *is to say Proj*$_{d-l}\Gamma = (z^{d-l+1}, z^{d-l+2}, \cdots, z^d)$

Proof: Let Q and Q' be given such that $d(Q, Q') \geq \frac{1}{4}$, in $P(\delta_1)$ and $P(\delta_2)$ respectively, as well as functions f_L with Fourier support in $\mathcal{U}_L$ for all L in $P(Q,\delta)$ and $P(Q',\delta)$. Without loss of generality, via an affine transformation, we may assume Q' is centered at the origin, and Q is distance ~ 1 from the origin.

Now for any weight ω with its Fourier support in $B(0, \frac{\delta^d}{10})$, we let $\omega^\circ = |f_{Q'}|^{p_d - p_l}\,\omega$. Note that the Fourier support ω° is therefore contained in $CB(0, \frac{\delta_2}{10})$ for some $C \sim 1$.

This follows from the fact that $\widehat{\omega^\circ} = \widehat{\omega} * \underbrace{|\widehat{f_{Q'}}| * |\widehat{f_{Q'}}| * \cdots * |\widehat{f_{Q'}}|}_{p_d - p_l \text{ times}}$.

Since, for any f and g for which $f * g$ is defined, $\text{supp}(f * g) \subset \text{supp}f + \text{supp}g$, we therefore have

$$\text{supp}\,\widehat{\omega^\circ} \subset B(0, \frac{\delta^d}{10}) + (p_d - p_l)\,\mathcal{U}_{Q'} \subset C\mathcal{U}_{Q'}$$

We therefore want to estimate

$$\int |f_Q|^{p_l}\,|f_{Q'}|^{p_d - p_l}\,\omega = \int |f_Q|^{p_l}\,\omega^\circ$$

Note, here we utilise lower dimension decoupling to accommodate the exponent p_l. This can be done by using lemma 2.4.6 from the appendix.

We can therefore project each $\mathcal{U}_Q$ down to the first $d-l$ dimensions and utilise our lower dimensional mollified decoupling constant. As Q is far from the origin, the Fourier support of the projections will be contained in a parallelipiped similar to $\mathcal{U}_{Q,\text{Proj}_{d-l}(\Gamma)}$, as long as $\delta_1^l > \delta_2^{d-l+1}$. Furthermore, under this condition, we can decouple this parallelepiped into smaller parallelepipeds of scale $\delta_2^{\frac{d-l+1}{l}}$. Furthermore, when we do this projection, the Fourier Support ω° is contained within a ball of radius $\sim \delta_2^{d-l+1}$, as this is the smallest side length of this projection of $\mathcal{U}_{Q'}$. This means it's an appropriate weight to decouple down to scale $\delta_2^{\frac{d-l+1}{l}}$, when decoupling in l dimensions.

More explicitly, letting $\delta_2^{\frac{d-l+1}{l}} = \delta_2'$, our functions f_Q are split up into $\{f_K\}_{K \in P(Q,\delta_2')}$ so that they have their Fourier support in $\{\mathcal{U}_{K,\Gamma}\}_{K \in P(Q,\delta_2')}$. As $\text{Proj}_{d-l}\mathcal{U}_{K,\Gamma} \subset C\mathcal{U}_{K,\text{Proj}_{d-l}\Gamma} \times \mathbb{C}^{d-l}$, we can therefore decouple them with the decoupling constant

$$\widetilde{\mathcal{D}}(\{\mathcal{U}_{K,\mathrm{Proj}_{d-l}\Gamma} \times \mathbb{C}^{d-l}\}) = \widetilde{\mathcal{D}}(\{\mathcal{U}_{K,\mathrm{Proj}_{d-l}\Gamma}\}) = \widetilde{\mathcal{D}}_{l,\mathrm{Proj}_{d-l}\Gamma}(\delta_1 \to \delta_2').$$

$$\begin{aligned}\int |f_Q|^{p_l}\,\omega^\circ &\lesssim \widetilde{\mathcal{D}}(\{\mathcal{U}_{K,\mathrm{Proj}_{d-l}\Gamma} \times \mathbb{C}^{d-l}\})^{p_l} \left|\left| ||f_K||_{L^{p_l}(\omega^\circ dz)} \right|\right|^{p_l}_{\ell^2_{K\in P(Q,\delta_2')}} \\ &\leq \widetilde{\mathcal{D}}_l(\delta_1 \to \delta_2')^{p_l} \left|\left| ||f_K||_{L^{p_l}(\omega^\circ dz)} \right|\right|^{p_l}_{\ell^2_{K\in P(Q,\delta_2')}}\end{aligned}$$

To conclude, we simply need to expand the $||f_K||_{L^{p_l}(\omega^\circ dz)}$, and decouple them asymmetrically.

$$\begin{aligned}\left|\left| ||f_K||_{L^{p_l}(\omega^\circ dz)} \right|\right|^{p_l}_{\ell^2_{K\in P(Q,\delta_2')}} &= \left(\sum_{K\in P(Q,\delta_2')} ||f_K||^2_{L^{p_l}(\omega^\circ dz)} \right)^{\frac{p_l}{2}} \\ &= \left(\sum_{K\in P(Q,\delta_2')} \left|\left| f_K\, |f_{Q'}|^{\frac{p_d-p_l}{p_l}} \right|\right|^2_{L^{p_l}(\omega dz)} \right)^{\frac{p_l}{2}} \\ &= \left(\sum_{K\in P(Q,\delta_2')} \left(\int |f_K|^{p_l}\, |f_{Q'}|^{p_d-p_l}\,\omega \right)^2 \right)^{\frac{p_l}{2}} \\ &\lesssim \widetilde{B}_l(\delta_2' \to \delta, \delta_2 \to \delta) \\ &\quad \times ||||f_L||_{L^{p_d}}||_{L\in P(Q,\delta)}\, ||||f_{L'}||_{L^{p_d}}||_{L'\in P(Q',\delta)}\end{aligned}$$

which gives the desired result when inserted into our previous estimate. □

Lemma 2.3.5. *For any positive integer l such that $l < d$, then for $\delta_1, \delta_2, \delta \in (0,1)$, with $\delta < \delta_1,\ \delta_2$ we have that:*

$$\widetilde{\mathcal{B}}_l(\delta_1 \to \delta, \delta_2 \to \delta) \leq \widetilde{\mathcal{B}}_{l-1}(\delta_1 \to \delta, \delta_2 \to \delta)^{\frac{d-l}{d-l+1}} \widetilde{\mathcal{B}}_{d-l}(\delta_2 \to \delta, \delta_1 \to \delta)^{\frac{1}{d-l+1}}.$$

Proof: This follows simply from an application of Hölder's inequality, if we note

$$\begin{aligned}
\int |f_Q|^{p_l}\,|f_{Q'}|^{p_d-p_l}\,\omega &= \int \left|f_Q\omega^{\frac{1}{p_d}}\right|^{p_l}\left|f_{Q'}\omega^{\frac{1}{p_d}}\right|^{p_d-p_l},\\
&= \int \left|f_Q\omega^{\frac{1}{p_d}}\right|^{(1-\frac{1}{d-l+1})p_{l-1}+\frac{p_d-p_{d-l}}{d-l+1}}\\
&\qquad\times\left|f_{Q'}\omega^{\frac{1}{p_d}}\right|^{(1-\frac{1}{d-l+1})(p_d-p_{l-1})+\frac{p_{d-l}}{d-l+1}},\\
&= \int \left|f_Q^{p_{l-1}}\omega^{\frac{p_{l-1}}{p_d}}f_{Q'}^{(p_d-p_{l-1})}\omega^{\frac{(p_d-p_{l-1})}{p_d}}\right|^{(1-\frac{1}{d-l+1})}\\
&\qquad\times\left|f_Q^{p_d-p_{d-l}}\omega^{\frac{p_d-p_{d-l}}{p_d}}f_{Q'}^{p_{d-l}}\omega^{\frac{p_{d-l}}{p_d}}\right|^{\frac{1}{d-l+1}},\\
&\le \left(\int |f_Q|^{p_{l-1}}\,|f_{Q'}|^{p_d-p_{l-1}}\,\omega\right)^{(1-\frac{1}{d-l+1})}\\
&\qquad\times\left(\int \left|f_Q^{p_d-p_{d-l}}\right|\left|f_{Q'}^{p_{d-l}}\right|\omega\right)^{\frac{1}{d-l+1}},
\end{aligned}$$

which is sufficient to show our result. □

Corollary 2.3.6. *Let l be a positive integer less than d, and assume Theorem 2.1.2 has been proven in dimension l. Then*

$$\begin{aligned}
\widetilde{\mathcal{B}}_l(\delta_1^{\frac{d-l+1}{l}}\to\delta,\delta_1\to\delta) &\lesssim \delta_1^{-\varepsilon}\widetilde{\mathcal{B}}_{l-1}(\delta_1^{\frac{d-l+2}{l-1}}\to\delta,\delta_1\to\delta)^{\frac{d-l}{d-l+1}}\\
&\qquad\times\widetilde{\mathcal{B}}_{d-l}(\delta_1^{\frac{l+1}{l}\frac{d-l+1}{d-l}}\to\delta,\delta_1^{\frac{d-l+1}{l}}\to\delta)^{\frac{1}{d-l+1}}.
\end{aligned}$$

Proof: To see this, we first apply lemma 2.3.5 to obtain

$$\begin{aligned}
\widetilde{\mathcal{B}}_l(\delta_1^{\frac{d-l+1}{l}}\to\delta,\delta_1\to\delta) &\lesssim \widetilde{\mathcal{B}}_{l-1}(\delta_1^{\frac{d-l+1}{l}}\to\delta,\delta_1\to\delta)^{\frac{d-l}{d-l+1}}\\
&\qquad\times\widetilde{\mathcal{B}}_{d-l}(\delta_1\to\delta,\delta_1^{\frac{d-l+1}{l}}\to\delta)^{\frac{1}{d-l+1}}.
\end{aligned}$$

Now, applying lemma 2.3.4 yields

$$\begin{aligned}
\widetilde{\mathcal{B}}_{l-1}(\delta_1^{\frac{d-l+1}{l}}\to\delta,\delta_1\to\delta)^{\frac{d-l}{d-l+1}} &\lesssim \widetilde{\mathcal{D}}_{l-1,\mathrm{Proj}_{d-l+1}\Gamma}^{\frac{p_d}{p_l}\frac{d-l}{d-l+1}}(\delta_1^{\frac{d-l+1}{l}}\to\delta_1^{\frac{d-l+2}{l-1}})\\
&\qquad\times\widetilde{\mathcal{B}}_{l-1}^{\frac{d-l}{d-l+1}}(\delta_1^{\frac{d-l+2}{l-1}}\to\delta,\delta_1\to\delta),\\
&\lesssim \delta_1^{-\varepsilon}\widetilde{\mathcal{B}}_{l-1}(\delta_1^{\frac{d-l+2}{l-1}}\to\delta,\delta_1\to\delta)^{\frac{d-l}{d-l+1}},
\end{aligned}$$

where the second estimate comes from lemma 2.2.2, lemma 2.3.3, and the hypothesis of this corollary. We do the same for the second term, obtaining

$$\begin{aligned}\widetilde{\mathcal{B}}_{d-l}(\delta_1 \to \delta, \delta_1^{\frac{d-l+1}{l}} \to \delta)^{\frac{1}{d-l+1}} &\lesssim \widetilde{\mathcal{D}}_{d-l,\mathrm{Proj}_l\Gamma}^{\frac{1}{d-l+1}}(\delta_1 \to \delta_1^{\frac{d-l+1}{l}\frac{l+1}{d-l}})^{\frac{p_d}{p_{d-l}}} \\ &\times \widetilde{\mathcal{B}}_{d-l}^{\frac{1}{d-l+1}}(\delta_1^{\frac{d-l+1}{l}\frac{l+1}{d-l}} \to \delta, \delta_1^{\frac{d-l+1}{l}} \to \delta), \\ &\lesssim \delta_1^{-\varepsilon}\widetilde{\mathcal{B}}_{d-l}(\delta_1^{\frac{d-l+1}{l}\frac{l+1}{d-l}} \to \delta, \delta_1^{\frac{d-l+1}{l}} \to \delta)^{\frac{1}{d-l+1}}.\end{aligned}$$

So in this case, if $l = 1$, we regard $\widetilde{\mathcal{B}}_{l-1}(\delta_1^{\frac{d-l+2}{l-1}} \to \delta, \delta_1 \to \delta)$ as $\widetilde{\mathcal{D}}(\delta_1 \to \delta)$, and do not concern ourselves with the singularity in the exponents of the first scale.[3] □

Proof of Theorem 2.1.2:

We establish this proof via induction on the dimension d. Clearly, when $d = 1$, as $p_1 = 2$, the result holds even without the $\delta^{-\varepsilon}$ factor, as a consequence of Plancherel's theorem, and the disjoint Fourier support of our functions.

We therefore assume $d \geq 2$, and assume we have our result in all lower dimensions. First, let η be the smallest exponent for which (2.2) holds.[4]

Define $A_l(\delta_1)$ be the infimum of the set of exponents, A, such that

$$\widetilde{\mathcal{B}}_l(\delta_1^{\frac{d-l+1}{l}} \to \delta, \delta_1 \to \delta) \lesssim \delta^{-A}.$$

It follows that, letting $\delta_1 = \delta^a$, $A_0(\delta_1) = (1-a)\eta$.

We can now use our previously established result, corollary 2.3.6, to see that

$$A_l(\delta_1) \leq \frac{1}{d-l+1}A_{d-l}(\delta_1^{\frac{d-l+1}{l}}) + \frac{d-l}{d-l+1}A_{l-1}(\delta_1).$$

Next, defining

$$A_l = \liminf_{a\to 0} \frac{\eta - A_l(\delta^a)}{a}$$

[3]We always have that that $\widetilde{\mathcal{B}}_0(\delta_1 \to \delta, \delta_2 \to \delta) = \widetilde{\mathcal{D}}(\delta_2 \to \delta)$.
[4]in place of ε.

We use this inequality to get the estimate:

$$\begin{aligned}
A_l &= \liminf_{a\to 0} \frac{\eta_d - A_l(\delta^a)}{a} \\
&\geq \liminf_{a\to 0} \frac{\eta - \frac{1}{d-l+1}A_{d-l}(\delta^{a\frac{d-l+1}{l}}) - \frac{d-l}{d-l+1}A_{l-1}(\delta^a)}{a} \\
&= \liminf_{a\to 0} \frac{\eta - \frac{1}{d-l+1}A_{d-l}(\delta^{a\frac{d-l+1}{l}}) - \frac{d-l}{d-l+1}A_{l-1}(\delta^a)}{a} \\
&\qquad\qquad + \frac{\eta}{(d-l+1)a} - \frac{\eta}{(d-l+1)a} \\
&= \liminf_{a\to 0} \frac{d-l}{d-l+1}\left(\frac{\eta - A_{l-1}(\delta^a)}{a}\right) + \frac{1}{l}\left(\frac{\eta - A_{d-l}(\delta^{\frac{d-l+1}{l}a})}{\frac{d-l+1}{l}a}\right) \\
&\geq \frac{d-l}{d-l+1}A_{l-1} + \frac{1}{l}A_{d-l}
\end{aligned}$$

We can then sum the inequalities for each l such that $1 \leq l \leq d-1$. This gives us that

$$\begin{aligned}
\sum_{l=1}^{d-1} A_l &\geq \sum_{l=1}^{d-1} \frac{d-l}{d-l+1}A_{l-1} + \sum_{l=1}^{d-1}\frac{1}{l}A_{d-l} \\
&= \frac{d-1}{d}A_0 + \sum_{l=2}^{d-1}\frac{d-l}{d-l+1}A_{l-1} + \sum_{m=2}^{d}\frac{1}{d-m+1}A_{m-1} \\
&= \frac{d-1}{d}A_0 + \sum_{l=1}^{d-1}A_l.
\end{aligned}$$

We therefore conclude that $0 \geq \frac{d-1}{d}A_0 = \frac{d-1}{d}\eta$, which means $\eta = 0$ for all d, which completes the proof.

Note, to justify the previous summing up of inequalities, we need A_l to be finite. To see why this holds, note that

$$\begin{aligned}
\left|\left||f_I|^{p_l}\,|f_{I'}|^{p_d-p_l}\right|\right|_{L^1(\omega dz)} &\leq ||f_I^{p_l}||_{L^{\frac{p_d}{p_l}}(\omega dz)}\, ||f_{I'}^{p_l}||_{L^{\frac{p_d-p_l}{p_d}}(\omega dz)} \\
&= ||f_I||_{L^{p_d}(\omega dz)}\, ||f_{I'}||_{L^{p_d}(\omega dz)}\,.
\end{aligned}$$

This immediately implies

$$\widetilde{\mathcal{B}_l}(\delta_1^{\frac{d-l+1}{l}} \to \delta, \delta+1 \to \delta) \leq \widetilde{\mathcal{D}}(\delta_1^{\frac{d-l+1}{l}} \to \delta)\widetilde{\mathcal{D}}(\delta_1 \to \delta),$$

which give the inequalities, by applying lemma 2.2.2, and letting $\delta_1 = \delta^a$

$$A_l(\delta_1) \le \eta(1 - \frac{a(d-l+1)}{l})\frac{p_l}{p_d} + \eta(1-a)\frac{p_d - p_l}{p_l}$$

$$\begin{aligned} A_l(\delta_1) &\le \eta(1 - \frac{a(d-l+1)}{l})\frac{p_l}{p_d} + \eta(1-a)\frac{p_d - p_l}{p_l} \\ &\le \eta - aC_d, \end{aligned}$$

For some constant C_d. Pairing this with lemmas 2.2.3 and 2.3.2, which themselves imply

$$\eta \le a(\frac{d-l+1}{l}\frac{p_l}{p_d} + \frac{p_d - p_l}{p_d}) + A_l(\delta_1).$$

These two estimates immediately imply that both η and $A_l(\delta_1)$ are $\lesssim 1$ □

2.4 Appendix

In this section, we will briefly address some general facts about decoupling constants that were used in this section. These facts hold for the bilinear, and asymmetric bilinear decoupling constants as well, with appropriate modifications.

We start with defining a more general notion of a decoupling constant.

Definition 2.4.1. *Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a collection of open sets in $\mathbb{R}^d$. The L^p decoupling constant associated to this collection, denoted $\mathcal{D}_p(\mathcal{U})$, is the smallest number satisfying*

$$\left|\left|\sum_{i=1}^n f_i\right|\right|_{L^p} \le \mathcal{D}_p(\mathcal{U})\, ||||f_i||_{L^p}||_{\ell^2_{i\in\{1,2\cdots n\}}}$$

for any collection of functions $\{f_i\}_{i=1}^n$ satisfying supp $\hat{f}_i \subset U_i$.

Notice in this notation, $\mathcal{D}_p(\delta) = \mathcal{D}_p(\{\mathcal{U}_Q\}_{Q\in P(\delta)})$. When it is relevant, we introduce the general mollified decoupling constant as

Definition 2.4.2. *Let $\mathcal{U} = \{U_i\}_{i=1}^n$ be a collection of open sets in $\mathbb{R}^d$. Let also Ω be a set in $\mathbb{R}^d$. The mollified L^p decoupling constant associated to this collection and set, denoted $\mathcal{D}_p(\mathcal{U}, \Omega)$, is the smallest number satisfying*

$$\left|\left|\sum_{i=1}^{n} f_i\right|\right|_{L^p(\omega dz)} \leq \mathcal{D}_p(\mathcal{U}) \left|\left| ||f_i||_{L^p(\omega dz)} \right|\right|_{\ell^2_{i\in\{1,2\cdots n\}}}$$

for any collection of functions $\{f_i\}_{i=1}^n$ satisfying supp $\hat{f}_i \subset U_i$, and weight function ω, satisfying $\omega \subset \Omega$.

Our first lemma is about the affine-invariance of decoupling constants.

Lemma 2.4.3. *Let L be an invertible affine transformation from $\mathbb{R}^d$ to $\mathbb{R}^d$. Then $\mathcal{D}_p(\{U_i\}_{i=1}^n) = \mathcal{D}_p(\{LU_i\}_{i=1}^n)$.*

Proof: The idea for this proof is that if $\{f_i\}$ is a collection of functions such that each f_i has its Fourier support in U_i, then, for $Lx = Ax + b$, we can then construct a new collection of functions, $\{g_i\}$, by setting

$$g_i(x) = e^{-2i\pi(x\cdot b)} f_i(A^T x),$$

which satisfy supp $\hat{g}_i \subset LU_i$. Furthermore, computation yields $||g_i||_{L^p} = \frac{1}{|\det A|^p} ||f_i||_{L^p}$, as well as $||\sum_{i=1}^n g_i||_{L^p} = \frac{1}{|\det A|^p} ||\sum_{i=1}^n f_i||_{L^p}$ so we therefore have, by the definition,

$$\left|\left|\sum_{i=1}^{n} g_i\right|\right|_{L^p} \leq \mathcal{D}_p(\{LU_i\}_{i=1}^n) \, ||||g_i||_{L^p}||_{\ell^2_{i\in\{1,2\cdots n\}}},$$

which, by substituting in the above identities gives

$$\left|\left|\sum_{i=1}^{n} f_i\right|\right|_{L^p} \leq \mathcal{D}_p(\{LU_i\}_{i=1}^n) \left|\left|\left|\left| f_{U_i'} \right|\right|_{L^p}\right|\right|_{\ell^2_{i\in\{1,2\cdots n\}}}$$

We therefore have $\mathcal{D}_p(\{U_i\}_{i=1}^n) \leq \mathcal{D}_p(\{LU_i\}_{i=1}^n)$.

This is sufficient to show our result, as can be seen either by computing a similar function with respect to L^{-1}, or simply by relabeling $LU_i = V_i$, and therefore having $U_i = L^{-1}V_i$.

We then have $\mathcal{D}_p(\{L^{-1}V_i\}_{i=1}^n) \leq \mathcal{D}_p(\{V_i\}_{i=1}^n)$, or equivalently

$$\mathcal{D}_p(\{U_i\}_{i=1}^n) \leq \mathcal{D}_p(\{LU_i\}_{i=1}^n).$$

$\square$

Lemma 2.4.4. *Let $\mathcal{U}$ and $\mathcal{U}'$ be non-empty collections of non-empty subsets of $\mathbb{R}^d$, such that $\mathcal{U} \bigcup \mathcal{U}'$ is made up of pair-wise disjoint subsets.*

Then $\mathcal{D}_p(\mathcal{U} \bigcup \mathcal{U}') \leq \sqrt{\mathcal{D}_p(\mathcal{U})^2 + \mathcal{D}_p(\mathcal{U}')^2}$.

Let $\mathcal{U} = \{U_i\}_{i=1}^n$ and $\mathcal{U}' = \{U_j'\}_{j=1}^m$. Given collections $\{f_{U_i}\}$ and $\{f_{U_j'}\}$, we have

$$\begin{aligned} \left|\left|\sum_{i=1}^n f_{U_i} + \sum_{j=1}^m f_{U_j'}\right|\right|_{L^p} &\leq \left|\left|\sum_{i=1}^n f_{U_i}\right|\right|_{L^p} + \left|\left|\sum_{j=1}^m f_{U_j'}\right|\right|_{L^p} \\ &\leq \mathcal{D}_p(\mathcal{U}) \, || ||f_{U_i}||_{L^p} ||_{\ell^2_{i\in\{1,2\cdots n\}}} + \mathcal{D}_p(\mathcal{U}') \left|\left| \left|\left|f_{U_j'}\right|\right|_{L^p} \right|\right|_{\ell^2_{j\in\{1,2\cdots m\}}} \\ &\leq \sqrt{\mathcal{D}_p(\mathcal{U})^2 + \mathcal{D}_p(\mathcal{U}')^2} \sqrt{\sum_{i=1}^n ||f_{U_i}||_{L^p}^2 + \sum_{j=1}^m \left|\left|f_{U_j'}\right|\right|_{L^p}^2}, \end{aligned}$$

using the Cauchy-Schwarz inequality. □

These lemmas are required to establish the following corollary, which we use frequently in our main proof.

Corollary 2.4.5. *Suppose, for a collection of sets $\{V_i\}_{i=1}^n$, there exists a cube Q, with translates $Q_i = Q + b_i$ such that $Q_i \subset V_i \subset CQ_i$, for some real number $C \sim 1$. Then*

$$\mathcal{D}_p(\{V_i\}_{i=1}^n) \sim \mathcal{D}_p(\{Q_i\}_{i=1}^n).$$

Proof:

The fact that $\mathcal{D}_p(\{Q_i\}_{i=1}^n) \leq \mathcal{D}_p(\{V_i\}_{i=1}^n)$ is trivial, as any f_{Q_i} with Fourier support in Q_i has its Fourier support in V_i, and can therefore be decoupled with respect to $\mathcal{D}_p(\{V_i\}_{i=1}^n)$.

For the other direction, note that by the same reasoning we clearly have $\mathcal{D}_p(\{V_i\}_{i=1}^n) \leq \mathcal{D}_p(\{CQ_i\}_{i=1}^n)$. We then seek to show $\mathcal{D}_p(\{CQ_i\}_{i=1}^n) \lesssim \mathcal{D}_p(\{Q_i\}_{i=1}^n)$.

To do this, we cover each CQ_i with copies of Q, translated along the standard directions. So if the cube has length L, we cover CQ_i with the collection

$$\mathcal{Q}_i = \{Q_i + L\sum_{k=1}^d \delta_k e_k \mid \delta \in \mathbb{Z}^d, \quad \delta_k \leq \lceil C \rceil\}.$$

We will index these via $\mathcal{Q}_i = \{L_j Q_i\}_{j=1}^m$, where the L_j are the translations used to cover any of the CQ_i. Importantly, we use the same translations of Q_i to cover each CQ_i.

Note that $\mathcal{D}_p(\{CQ_i\}_{i=1}^n) \leq \mathcal{D}_p(\{L_jQ_i\}_{i=1,j=1}^{i=n,j=m})$, which can be seen by taking any collection of $\{f_{CQ_i}\}$ for f_{CQ_i} with Fourier support in CQ_i, and for each $L_jQ_i \in \mathcal{Q}_i$, letting $f_{L_jQ_i}$ be defined via $\widehat{f_{L_jQ_i}} = \chi_{L_jQ_i}\widehat{f_{CQ_i}}$.

Then, letting $\mathcal{D}_p(\{L_jQ_i\}_{i=1,j=1}^{i=n,j=m}) = \mathcal{D}_p(\bigcup_{j=1}^m \{L_jQ_i\}_{i=1}^{i=n})$, we have

$$\mathcal{D}_p(\bigcup_{j=1}^{m}\{L_jQ_i\}_{i=1}^{i=n}) \leq \sqrt{\sum_{j=1}^{m}\mathcal{D}_p(\{L_jQ_i\}_{i=1}^{i=n})^2} = \sqrt{m}\mathcal{D}_p(\{Q_i\}_{i=1}^{i=n}),$$

which proves our result, as $m \sim 1$. □

Note that this lemma verifies $\mathcal{D}_p(\delta) \sim \widetilde{\mathcal{D}_p}(\delta)$. To see this, recall that we noted $\mathcal{D}_p(\delta) \lesssim \widetilde{\mathcal{D}_p}(\delta)$ in the introduction of this section.

To see the other direction, note that if f_Q has Fourier support in $\mathcal{U}_Q$, and ω has Fourier support in $B(0, \frac{\delta^d}{10})$, then the functions $f_Q^{p_d}\omega$ has its Fourier support in $C\mathcal{U}_Q$ for some $C \sim 1$. That means we have

$$\int |f_Q|^{p_d}\,\omega \leq \mathcal{D}_p(\{C\mathcal{U}_Q\}_{Q\in P(\delta)})^{p_d} \left|\left|\left|\left|f_Q\omega^{\frac{1}{p_d}}\right|\right|_{p_d}\right|\right|_{\ell^2_{Q\in P(\delta)}}^{p_d}.$$

And as $\mathcal{D}_p(\{C\mathcal{U}_Q\}_{Q\in P(\delta)}) \sim \mathcal{D}_p(\{\mathcal{U}_Q\}_{Q\in P(\delta)}) = \mathcal{D}_p(\delta)$, we have $\tilde{\mathcal{D}_p}(\delta) \lesssim \mathcal{D}_p(\delta)$.

A similar argument shows that we obtain the same constant if we work with ω whose Fourier support is in $CB(0, \frac{\delta^d}{10})$, for some $C \sim 1$ instead.

Lastly, we establish the "adding dimensions" lemma, which was necessary to establish lemma 2.3.3.

Lemma 2.4.6. *Let $\{U_i\}_{i=1}^n$ be a collection of non-empty sets in $\mathbb{R}^{d_1}$, then for any positive integer d_2, and $p \geq 2$, we have*

$$\mathcal{D}_p(\{U_i\}_{i=1}^n, \Omega) = \mathcal{D}_p(\{U_i \times \mathbb{R}^{d_2}\}_{i=1}^n, \Omega \times \mathbb{R}^{d_2}).$$

Proof: To establish some notation, we will refer to $U_i \times \mathbb{R}^{d_2}$ as U_i'. We also label $\Omega \times \mathbb{R}^{d_2}$ as Ω'. We also let $\mathcal{F}_d$ represent the d dimensional Fourier transform.

First, we show

$$\mathcal{D}_p(\{U_i\}_{i=1}^n, \Omega) \leq \mathcal{D}_p(\{U_i'\}_{i=1}^n, \Omega'),$$

To see this, given $\{f_{U_i}\}$ with Fourier support in U_i, and ω with Fourier support in Ω, we define, for $x \in \mathbb{R}^{d_1}$, and $x' \in \mathbb{R}^{d_2}$,

$$f_{U_i'}(x,x') := g(x')f_{U_i}(x),$$

and

$$\omega'(x,x') := h(x')\omega(x),$$

for some positive function g such that, $g \in L^p(\mathbb{R}^{d_2})$, $gf_{U_i} \in L^p(\mathbb{R}^{d_1+d_2})$ for all i, and that $\hat{g}$ is well defined. Similarly, take $h : \mathbb{R}^{d_2} \to \mathbb{R}^+$ such that g is also in $L^p(\mathbb{R}^{d_2}, hdx')$, and $gf_{U_i'} \in L^p(\mathbb{R}^{d_1+d_2}, hdx'dx)$

Then, taking the $d_1 + d_2$ dimensional Fourier transform of $f_{U_i'}$ yields

$$\mathcal{F}_{d_1+d_2}f_{U_i'}(\xi_1, \cdots, \xi_{d_1+d_2}) = \mathcal{F}_{d_1}f(\xi_1, \cdots, \xi_{d_1})\mathcal{F}_{d_2}g(\xi_{d_1+1}, \cdots \xi_{d_1+d_2}),$$

which clearly has its support in $U_i \times \mathbb{R}^{d_2}$. Similarly,

$$\mathcal{F}_{d_1+d_2}\omega'(\xi_1, \cdots, \xi_{d_1+d_2}) = \mathcal{F}_{d_1}\omega(\xi_1, \cdots, \xi_{d_1})\mathcal{F}_{d_2}h(\xi_{d_1+1}, \cdots \xi_{d_1+d_2}),$$

shows ω' has its Fourier support in Ω'.

We therefore have the following line of reasoning

$$\left|\left|\sum_{i=1}^{n} g(x')f_{U_i}(x)\right|\right|_{L^p(\mathbb{R}^{d_1+d_2}(\omega' dx dx')}$$
$$\leq \mathcal{D}_p(\{U_i'\})_{i=1}^n \left|\left|\left|\left|gf_{U_i'}\right|\right|_{L^p(\mathbb{R}^{d_1+d_2})(\omega' dx dx')}\right|\right|_{\ell^2_{i\in\{1,\cdots,n\}}}$$

$$||g||_{L^p(\mathbb{R}^{d_2}(hdx))} \left|\left|\sum_{i=1}^{n} f_{U_i}(x)\right|\right|_{L^p(\mathbb{R}^{d_1})(\omega dx)}$$
$$\leq \mathcal{D}_p(\{U_i'\})_{i=1}^n \, ||g||_{L^p(\mathbb{R}^{d_2})(hdx)} \left|\left|\left|\left|f_{U_i'}\right|\right|_{L^p(\mathbb{R}^{d_1})(\omega dx)}\right|\right|_{\ell^2_{i\in\{1,\cdots,n\}}}$$
$$\left|\left|\sum_{i=1}^{n} f_{U_i}(x)\right|\right|_{L^p(\mathbb{R}^{d_1})(\omega dx)} \leq \mathcal{D}_p(\{U_i'\})_{i=1}^n \left|\left|\left|\left|f_{U_i'}\right|\right|_{L^p(\mathbb{R}^{d_1})(\omega dx)}\right|\right|_{\ell^2_{i\in\{1,\cdots,n\}}},$$

which establishes

$$\mathcal{D}_p(\{U_i\}_{i=1}^n, \Omega) \leq \mathcal{D}_p(\{U_i'\}_{i=1}^n, \Omega').$$

Now, it remains to show

$$\mathcal{D}_p(\{U_i\}_{i=1}^n, \Omega) \geq \mathcal{D}_p(\{U_i'\}_{i=1}^n, \Omega').$$

Suppose $f_{U_i'}$ has its Fourier support in U_i', and ω' has its Fourier support in Ω'.

Then we have

$$\begin{aligned}
\left\|\sum_{i=1}^{n} f_{U_i'}\right\|_{L^p(\mathbb{R}^{d_1+d_2}(\omega' dx dx'))}^{p} &= \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} \left|\sum_{i=1}^{n} f_{U_i'}(x,x')\right|^{p} \omega'(x,x') dx dx' \\
&= \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \left|\sum_{i=1}^{n} f_{U_i'}(x,x')\right|^{p} \omega'(x,x') dx \right)^{\frac{1}{p}\cdot p} dx', \\
&\leq \mathcal{D}_p^p(\{U_i\}_{i=1}^n) \int_{\mathbb{R}^{d_2}} \left(\sum_{i=1}^{n} \left(\int_{\mathbb{R}^{d_1}} \left| f_{U_i'}(x,x')\right|^{p} \omega'(x,x') dx \right)^{\frac{2}{p}} \right)^{\frac{p}{2}} dx', \\
&\leq \mathcal{D}_p^p(\{U_i\}_{i=1}^n) \left(\sum_{i=1}^{n} \left(\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} \left| f_{U_i'}(x,x')\right|^{p} \omega'(x,x') dx dx' \right)^{\frac{2}{p}} \right)^{\frac{1}{2}},
\end{aligned}$$

with the first inequality coming from regarding $\omega'(x, x')$ as a function of only x, and noting the Fourier support of this function is in Ω, and the last line coming from the interchanging of an $L^p\ell^2$ norm with an $\ell^2 L^p$ norm, for $p \geq 2$, and so our result follows. $\square$

Chapter 3

Well-curved Convolution in $\mathbb{R}^6$

3.1 Introduction and statement of main theorem

Consider the 2-surface in $\mathbb{R}^6$, given by

$$\Gamma(u,v) = \begin{bmatrix} u \\ v \\ u^2 \\ v^2 \\ uv \\ u^3+v^3 \end{bmatrix}, \tag{3.1}$$

for $(u,v) \in [0,1]^2$.

We define the convolution operator associated to Γ as, for $f : \mathbb{R}^6 \to \mathbb{R}$, as

$$T_\Gamma f(x) := \int_{[0,1]^2} f\left(x - \Gamma(u,v)\right) dudv, \tag{3.2}$$

In this paper, we establish a key bound for T_Γ:

Theorem 3.1.1. *Let $\mathcal{D}$ be the convex hull of the points $(0,0)$, $(1,1)$, $(\frac{6}{11},\frac{4}{11})$, $(\frac{7}{11},\frac{5}{11})$.*

Then, if (p^{-1},q^{-1}) is in $\mathcal{D} \setminus \{(\frac{6}{11},\frac{4}{11}) + s(\frac{1}{11},\frac{1}{11}) \mid s \in [0,1]\}$, then there exist $C \in \mathbb{R}$ such that

$$||T_\Gamma f||_{L^q} \leq C\, ||f||_{L^p}\,.$$

To establish this theorem, we seek to establish the restricted weak type estimate at a (p, q) arbitrarily close to $(\frac{6}{11}, \frac{4}{11})$ for T_Γ.

Here we say a restricted weak type estimate at a point (p, q) exists, if the inequality

$$\langle T_\Gamma \chi_E, \chi_F \rangle \leq C\,|E|^{\frac{1}{p}}\,|F|^{\frac{1}{q'}}$$

holds for all measurable sets, E and F in $\mathbb{R}^6$.

This is sufficient, as we can use the dual-estimate, paired with the Marcinkiewicz interpolation theorem, and the trivial bound at (∞, ∞), allows us to show (p, q) estimates for all (p, q) with $(p^{-1}, q^{-1}) \in \mathcal{D}$, aside from the non-trivial vertices.

To quickly note that this is an almost optimal range, we will show using counterexample that a restricted weak type estimate can only hold within $\mathcal{D}$. To do this, consider two sets in $\mathbb{R}^6$, B_ε, the ball of radius ε, centered at the origin, and N_ε, the ε-neighbourhood of the surface $\Gamma([0, 1]^2)$. To these two sets, we will consider applying the anisotropic scalings, at scale r given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \longrightarrow \begin{bmatrix} rx_1 \\ rx_2 \\ r^2x_3 \\ r^2x_4 \\ r^2x_5 \\ r^3x_6 \end{bmatrix}$$

We will denote by $B_{\varepsilon,r}$ and $N_{\varepsilon,r}$ the scaled ball and neighbourhood respectively. From here we just consider the inner product of their character functions to see:

$$\begin{aligned} \langle T_\Gamma \chi_{N_{\varepsilon,r}}, \chi_{B_{\varepsilon,r}} \rangle &= \int_{B_{\varepsilon,r}} T\chi_{N_{\varepsilon,r}}(x)dx \\ &\sim |B_{\varepsilon,r}|\, r^2 \\ &= r^{13}\varepsilon^6. \end{aligned}$$

So to have our desired estimates for this family of pairs of sets, we need to have this quantity be bounded by $|N_{\varepsilon,r}|^{\frac{1}{p}}\,|B_\varepsilon|^{\frac{1}{q'}} \sim (r^{11}\varepsilon^4)^{\frac{1}{p}}(r^{11}\varepsilon^6)^{\frac{1}{q'}}$. This gives us, by letting ε and r go to zero independently, the estimates $\frac{3}{q} > \frac{2}{p}$ and $\frac{2}{11} > \frac{1}{p} - \frac{1}{q}$ respectively. Taking the dual estimate for the first, we also get that $1 > \frac{2}{q} - \frac{3}{p}$. These bounds indeed verify that if a restricted weak type estimate holds at (p, q), we must have $(p^{-1}, q^{-1}) \in \mathcal{D}$.

To briefly address why we should consider this the "well-curved" case, we appeal to [Gre19]. This paper relates to the Oberlin curvature condition, which states that a measure μ on $\mathbb{R}^n$ satisfies the Oberlin curvature condition with exponent α when there exists $C \sim 1$ such tha, for every compact, convex subset $K \subset \mathbb{R}^n$, we have that

$$\mu(K) \leq C |K|^{\alpha}.$$

Now, theorem 3.1.1 can be interpreted as an estimate of the form $||\mu * f||_{L^q} \lesssim ||f||_{L^p}$, where μ is a measure supported on the surface $\Gamma(u, v)$. Such estimates can only exist if μ satisfies the Oberlin curvature condition with exponent $\alpha = \frac{1}{p} - \frac{1}{q}$.

Theorem 1 of [Gre19] states that if μ is a positive measure supported on a d dimensional manifold in $\mathbb{R}^n$, then the maximum α for which this condition can hold is $\frac{d}{Q(d,n)}$, where $Q(d, n)$ is a quantity known as the homogeneous dimension of of the manifold. In our case with a 2-surface in $\mathbb{R}^6$, we have $Q(2, 6) = 11$, so the maximum α value is $\frac{2}{11}$.

Therefore, the maximum (p, q) range we can obtain for estimates of this type is bounded by the condition $\frac{1}{p} - \frac{1}{q} = \frac{2}{11}$.

As this is an edge of $\mathcal{D}$, and the other two edges come from a criteria independent of the particular surface Γ, we see that, aside from the $\frac{1}{p} - \frac{1}{q} = \frac{2}{11}$ edge, we obtain the optimal range of (p, q) estimates for Γ.

This motivates us to view Γ as a well-curved surface in $\mathbb{R}^6$.

3.2 Method of Refinements

In this section, we proceed with using the method of refinement, introduced in [Chr98].

Let E and F be non-empty, measurable sets that we wish to show our estimate for. Define α and β via the equations

$$\alpha |F| = \langle T_\Gamma \chi_E, \chi_F \rangle = \beta |E|.$$

Next, if we sought to show that T_Γ is restricted weak type at our critical vertex, we would need to show

$$\langle T_\Gamma \chi_E, \chi_F \rangle \lesssim |E|^{\frac{6}{11}} |F|^{\frac{7}{11}} ,$$

which can be written as

$$\alpha^{\frac{5}{2}}\beta^3 \lesssim |F| \qquad \text{or equivalently,} \qquad \alpha^{\frac{7}{2}}\beta^2 \lesssim |E| .$$

Note that we will not obtain this estimate, but instead obtain a family of estimates that are arbitrarily close to it. We will assume for now that we seek a lower bound for F, though seeking a bound for E also would work. Now, we can obtain this by constructing a map $\Phi : \Omega \to F^* \subset F$. Then, if Φ only ever takes a bounded number of inputs to the same output, we have that

$$|F| \gtrsim \int_\Omega |J_\Phi| ,$$

where J_Φ is the Jacobian of Φ. Now, we define the following sequence of refined subsets of E and F:

$$\begin{aligned} E_1 &= \left\{ y \in E \;\middle|\; T_\Gamma^* \chi_F(y) > \frac{\beta}{2} \right\}, \\ F_1 &= \left\{ x \in F \;\middle|\; T_\Gamma \chi_{E_1}(x) > \frac{\alpha}{4} \right\}, \\ E_2 &= \left\{ y \in E_1 \;\middle|\; T_\Gamma^* \chi_{F_1}(y) > \frac{\beta}{8} \right\}. \end{aligned}$$

Note, as we can interpret β as the average value of $T_\Gamma^* \chi_F$ on E clearly E_1 is non-empty. To see that F_1 is non empty, we compute a lower bound for the average value of $T_\Gamma \chi_{E_1}$ on F:

$$\langle T_\Gamma \chi_{E_1}, \chi_F \rangle = \langle \chi_E, T_\Gamma^* \chi_F \rangle - \langle \chi_{E \setminus E_1}, T_\Gamma^* \chi_F \rangle \geq \beta |E| - \frac{\beta}{2} |E| = \frac{\alpha}{2} |F| .$$

Since the average value of $T_\Gamma \chi_{E_1}$ on F is at least $\frac{\alpha}{2}$, clearly F_1 is non empty. Similarly, as

$$\langle \chi_{E_1}, T_\Gamma^* \chi_{F_1} \rangle = \langle T_\Gamma \chi_{E_1}, \chi_F \rangle - \langle T_\Gamma \chi_{E_1}, \chi_{F \setminus F1} \rangle \geq \frac{\beta}{2} |E| - \frac{\alpha}{4} |F| = \frac{\beta}{4} |E| ,$$

E_2 is non-empty, and we can therefore take $y_0 \in E_2$. Furthermore, if we define the parameterization maps:

$$\Phi_1^*(u_1,v_1) := y_0 + \Gamma(u_1,v_1),$$
$$\Phi_1(u_1,v_1,u_2,v_2) := y_0 + \Gamma(u_1,v_1) - \Gamma(u_2,v_2),$$
$$\Phi = \Phi_2^*(u_1,v_1,u_2,v_2,u_3,v_3) := y_0 + \Gamma(u_1,v_1) - \Gamma(u_2,v_2) + \Gamma(u_3,v_3),$$

these maps pass through our sets for a "typical" amount of (u_i,v_i), where (u_i,v_i) are the last two variables of our map[1]. We can see this as, for $y_0 \in E_2$, if we let

$$P_1 = P_1^{y_0} := \{(u_1,v_1) \,\big|\, \Phi_1^*(u_1,v_1) \in F_1\},$$

we have that

$$|P_1^{y_0}| = T_\Gamma^* \chi_{F_1}(y_0) \gtrsim \beta.$$

Similarly, we define, for each $(u_1,v_1) \in P_1$

$$P_2 = P_2^{y_0,u_1,v_1} := \{(u_2,v_2) \,\big|\, \Phi_1(u_1,v_1,u_2,v_2) \in E\},$$

and we have

$$|P_2| = T_\Gamma(y_0 + \Gamma(u_1,v_1)) \gtrsim \alpha.$$

Lastly, taking a $(u_1,v_1) \in P_1$, and a $(u_2,v_2) \in P_2^{y_0,u_1,v_1}$, the set

$$P_3 = P_3^{y_0,u_1,v_1,u_2,v_2} = \{(u_3,v_3) \,\big|\, \Phi(u_1,v_1,u_2,v_2,u_3,v_3) \in F\},$$

gives

$$|P_3| = T_\Gamma^*(y_0 + \Gamma(u_1,v_1) - \Gamma(u_2,v_2)) \gtrsim \beta.$$

This means, letting $\Omega = P_1^{y_0} \times P_2^{y_0,u_1,v_1} \times P_3^{y_0,u_1,v_1,u_2,v_2}$,

we have $\Phi : \Omega \to F$ which is at most finite to 1 on Ω, and can therefore use the estimate mentioned earlier. For readability, we will refer to $\Omega = P_1 \times P_2 \times P_3$ from here, but it is important to remember that every P_2 space has an associated (u_1,v_1), and every P_3 space has an associated (u_1,v_1), and (u_2,v_2).

Note we can do a similar construction, where we swap the roles of E and F, beginning instead with F_1, and continuing on analogously, to obtain a lower

[1]By this we mean, if we fix all but the last two variables of each of these maps, they represent the integrand of our operator. Therefore the measure of the (u_i,v_i) that cause these maps to land in one of the sets will just be our operator evaluated at a certain value.

bound for E.

From here, the idea is to get our desired bound by acquiring a lower bound for J_Φ, and (continuing to assume we are aiming for a lower bound for $|F|$) use it via:

$$|F| \gtrsim \int_\Omega |J_\Phi| \gtrsim |\Omega| \min_\Omega |J_\Phi| \gtrsim \alpha\beta^2 \min_\Omega |J_\Phi| .$$

Now, this idea cannot immediately work, as we cannot obtain a bound for J_ϕ on generic Ω, so the aim is to replace Ω with $\tilde{\Omega} \subset \Omega$, which still satisfies $\left|\tilde{\Omega}\right| \gtrsim \alpha\beta^2$, but we have removed the areas around which J_Φ vanishes. This can be done by holding u_1, v_1, u_2, v_2 fixed, and only regarding J_Φ as a function on P_3. We then obtain an estimate for the size of the sublevel set of J_Φ, at arbitrary level ε, and then choose $\varepsilon = \varepsilon_0$ so that the sublevel set has a size which is $\lesssim \beta$, with suitably small implicit constant[2]. This ensures that we can remove this sublevel set from $\tilde{\Omega}$, and get that $|J| > \varepsilon_0$ on the new refined set.

More rigorously, if $f : A \subset \mathbb{R}^n \to \mathbb{R}$, we define its sublevel set, at level ε to be:

$$S_\varepsilon(f) := \{x \in A \mid |f(x)| < \varepsilon\}.$$

What we obtain in the next section is:

$$|S_\varepsilon(J_\Phi)| \lesssim \frac{\varepsilon \log_2 \frac{1}{\varepsilon}}{|(u_2 - u_1)^3 + (v_2 - v_1)^3|}.$$

Thus, to obtain a more useful bound for the size of $S_\varepsilon(J_\Phi)$, we want to remove $S_{\tilde{\varepsilon}}((u_2 - u_1)^3 + (v_2 - v_1)^3)$ in P_2. Here, $\tilde{\varepsilon}$ will be chosen so that this sublevel set will have an area which is a small multiple of α.

We will see that this will occur if we take $\tilde{\varepsilon} \lesssim \alpha^{\frac{3}{2}}$. Thus we can, for each $(u_1, v_1) \in P_1$, remove this set, leaving us with an $\tilde{\Omega}$ which is still $\gtrsim \alpha\beta^2$, and for $(u_1, v_1, u_2, v_2, u_3, v_3) \in \tilde{\Omega}$, we have $|(u_2 - u_1)^3 + (v_2 - v_1)^3| \gtrsim \alpha^{\frac{3}{2}}$.

Pairing this with the fact that for any fixed $\delta \in (0, 1)$, $\varepsilon \log_2(\frac{1}{\varepsilon}) \lesssim C_\delta \varepsilon^{1-\delta}$. This gives us that on $\tilde{\Omega}$,

$$|S_\varepsilon(J)| \lesssim C_\delta \frac{\varepsilon^{1-\delta}}{\alpha^{\frac{3}{2}}}.$$

[2]Say $\frac{1}{8}$, which should do for any of the implicit constants in this section.

Now, as we want to remove this set from P_3, we need to choose ε such that $|S_\varepsilon(J)| \lesssim \beta$, which we can do by taking $\varepsilon = \frac{1}{8C_\delta}\left(\alpha^{\frac{3}{2}}\beta\right)^{\frac{1}{1-\delta}}$. Therefore we again remove this set from $\tilde{\Omega}$, to obtain a new set, which will still be referred to as $\tilde{\Omega}$, but on which $J \gtrsim \frac{1}{C_\delta}\left(\alpha^{\frac{3}{2}}\beta\right)^{\frac{1}{1-\delta}}$. This indeed gives us, after using the aforementioned argument, and some algebra, the estimate:

$$\alpha^{1+\frac{3}{2(1-\delta)}}\beta^{2+\frac{1}{1-\delta}} \lesssim |F| \Leftrightarrow \alpha^{2+\frac{3}{2(1-\delta)}}\beta^{1+\frac{1}{1-\delta}} \lesssim |E|\,,$$

or equivalently

$$\langle T_\Gamma \chi_E, \chi_F\rangle \lesssim |E|^{\frac{6-4\delta}{11-6\delta}}\,|F|^{\frac{7-4\delta}{11-6\delta}}\,,$$

which shows a restricted weak type estimate at $(p_\delta, q_\delta) = (\frac{11-6\delta}{6-4\delta}, \frac{11-6\delta}{4-2\delta})$, with an implicit constant depending on δ. Clearly, this point is arbitrarily close to our critical vertex, as $\delta \to 0$. Note that $C_\delta \to \infty$ as $\delta \to 0$, so this method does not establish the result at our critical vertex.

Therefore, if we want to show an estimate corresponding to a point in the interior of $\mathcal{D}$, we can fix a δ so that the point also lies inside $\mathcal{D}_\delta$, the convex hull of $(0,0)$, $(1,1)$, $(p_\delta^{-1}, q_\delta^{-1})$, $(q_\delta'^{-1}, p_\delta'^{-1})$.

Since we have restricted weak type estimates at all the vertices of $\mathcal{D}_\delta$, via interpolation, we obtain full estimates for the interior of $\mathcal{D}_\delta$.

Lastly, we can also obtain the full estimate on two of the edges of $\mathcal{D}$. To see this, note that one of our edges is given by the equation $\frac{1}{q} = \frac{2}{3}\frac{1}{p}$, for $\frac{1}{p} \in [0, \frac{6}{11})$. We therefore must show

$$\alpha^{3p-3}\beta^3 \lesssim |F| \qquad \text{for all } p > \frac{11}{6}.$$

The dual to this edge is given by $\frac{1}{q} = \frac{3}{2}\frac{1}{p} - \frac{1}{2}$, for $\frac{1}{p} \in (\frac{7}{11}, 1]$. This estimate is equivalent to

$$\alpha^2\beta^{\frac{2}{p-1}} \lesssim |F| \qquad \text{for all } p \in [1, \frac{11}{7}).$$

Note that these correspond to dual estimates, so in any fixed case, if we can show it for all p values in their respective intervals, we obtain the other estimate also.

Suppose first that $\beta > \alpha$. In this case, we simply note that we have obtained[3]

$$\begin{aligned} &\alpha^{\frac{5}{2}+\varepsilon}\beta^{3+\varepsilon} \lesssim |F| \\ \implies &\alpha^{\frac{5}{2}+2\varepsilon}\beta^{3} \lesssim |F|. \end{aligned}$$

Note then that, as $p > \frac{11}{6}$, we have $3p-3 > \frac{5}{2}$. Therefore we just choose ε such that $3p-3 > \frac{5}{2}+2\varepsilon$. Then this gives, as $\alpha \lesssim 1$, that

$$\alpha^{3p-3}\beta^3 \lesssim |F|.$$

Now if $\alpha > \beta$, we prove the dual estimate. In this case, we have

$$\begin{aligned} &\alpha^{\frac{5}{2}+\varepsilon}\beta^{3+\varepsilon} \lesssim |F| \\ \implies &\alpha^{2}\beta^{\frac{7}{2}+2\varepsilon} \lesssim |F|. \end{aligned}$$

In the same manner, as $p < \frac{11}{7}$, we have $\frac{2}{p-1} > \frac{7}{2}$, so we again choose ε such that $\frac{7}{2}+2\varepsilon < \frac{2}{p-1}$. Again $\beta \lesssim 1$, so we obtain $\alpha^2\beta^{\frac{2}{p-1}} \lesssim |F|$, proving Theorem 3.1.1.

3.3 Bounding the Sublevel Set

In this section, we shall go into detail about bounding the size of the sublevel set of our Jacobian. This can be done by noticing J has a factorisation of the form $J = Q \times L$, where

$$Q = (u_1-u_2)(u_1-u_3)(u_2-u_3) + (v_1-v_2)(v_1-v_3)(v_2-v_3),$$

and

$$L = (u_1-u_2)(v_2-v_3) - (v_1-v_2)(u_2-u_3)$$

Note, we will do a change of variables via $u_i - u_{i+1} \to u_i$ and $v_i - v_{i+1} \to v_i$, which the following discussion assumes.

An important step in this is to obtain an upper bound for the size of sub-level set of this Jacobian. We defined in the previous section, if $f : A \subset \mathbb{R}^n \to \mathbb{R}$, its

[3] Rewriting our $\frac{1}{1-\delta}$ as $1+\varepsilon$.

sublevel set, at level ε is

$$S_\varepsilon(f) = \{x \in A \mid \ |f(x)| < \varepsilon\}.$$

We also denote the approximate level set as

$$\tilde{S}_\varepsilon(f) = \{x \in A \mid \frac{\varepsilon}{2} < |f(x)| < \varepsilon\}.$$

Note that in our case, for $f = J = Q \times L$,

$$S_\varepsilon(J) = \bigcup_{j \in \mathbb{Z}} \Big(S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L) \Big).$$

So

$$S_\varepsilon(J) = \sum_{j \in \mathbb{Z}} \left| S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L) \right|.$$

We will see it is sufficient to obtain the bounds that

$$\left| S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L) \right| \lesssim \frac{\varepsilon}{|u_1^3 + v_1^3|},$$

for arbitrary j.

As an aside, note that although we could obtain a lower bound for $|J|$ via simply finding an appropriate ε_1 and ε_2 so that we could remove $S_{\varepsilon_1}(Q)\ S_{\varepsilon_2}(L)$ from P_3 to obtain $|J| > \varepsilon_1 \varepsilon_2$. However, it turns out that this method is too coarse to obtain the desired bound for J[4].

Furthermore, via some symmetry in this problem, we can assume $u_1 > 0$, and only concern ourselves with bounding the "upper half" of this set. We will also regard Q and L as functions of (u_2, v_2) only, and imagine (u_1, v_1) as a parameter that determines the behaviour of Q and L.

To obtain our bounds, we need to split this problem into several cases, and to do so, we will introduce a quantity, for any $C \in \mathbb{R}$,

$$h_C := \sqrt{\frac{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 4C)}{4u_1 v_1}}.$$

This quantity is relevant to identifying cases. For example, consider the sets

[4] Roughly speaking, we get $\varepsilon_1 = \alpha\beta$, and $\varepsilon_2 = \alpha^{\frac{1}{2}}\beta$, which gives "too many" α and β.

$\{Q = C\}$, and $\{L = K\}$.

- If $u_1 v_1 > 0$, then these sets intersect if and only if $K \in [-h_C, h_C]$
- If $u_1 v_1 < 0$, and $(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 4C) < 0$, then we have intersections if and only if $K \notin [-h_C, h_C]$
- If $u_1 v_1 < 0$, and $(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 4C) > 0$, then $h_C \notin \mathbb{R}$, and we have intersections for all K.

Effectively, h_C is the critical value of K for which the interactions between these sets change. Geometrically speaking, we have that $L = \pm h_C$ is tangent to $Q = C$, whenever $h_C \in \mathbb{R}$, and when $h_C \notin \mathbb{R}$, no such tangent exists.

Now, our method for obtaining these estimates will be to parameterise these sets in a "nice" manner, and then integrate over the regions of these parameters. We parameterise the sets via

$$\Psi(s, t) = x + \frac{s}{\sqrt{u_1^2 + v_1^2}} (-v_1, u_1) + \frac{t}{\sqrt{u_1^2 + v_1^2}} (u_1, v_1),$$

with the regions these parameters vary in depending on the case we are in. However, we will write them in the form of $s \in I$, and $t \in I_s$. Therefore, as $||\Psi_s \times \Psi_t|| = 1$, our area will be

$$S_\varepsilon(J) = \int_I |I_s| \, ds.$$

It will be convenient to think about this in terms of integrating between $L = a$ and $L = b$. This will give $I = \left[\frac{a}{\sqrt{u_1^2+v_1^2}}, \frac{b}{\sqrt{u_1^2+v_1^2}}\right]$, and that $|I_s|$ will just be the total length of $L = \sqrt{u_1^2 + v_1^2} s$ which lies inside $S_{2^j}(Q)$.

Note that it will be useful to notice that the distances between the intersections, when they exist, of $L = k$ and $Q = C$ are

$$\frac{\sqrt{u_1^2 + v_1^2}}{|u_1^3 + v_1^3|} \sqrt{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 4C) - 4k^2 u_1 v_1}.$$

Lastly, it will be helpful to familiarize ourselves with the following tautologies which occur frequently in these calculations.

- $h_{\pm C} < \varepsilon C^{-1}$ iff $\frac{C}{\sqrt{4u_1 v_1}} \leq \frac{\varepsilon}{\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+4C)}}$

- $h_C\sqrt{\frac{u_1^3+v_1^3\pm 4C}{u_1^3+v_1^3}} = \frac{u_1^3+v_1^3+4C}{\sqrt{4u_1v_1}}$

3.3.1 If $u_1v_1 > 0$

Now we begin with calculations of $\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right|$.

It should be noted that $u_1v_1 > 0$ indicates that $\{Q = C\}$ is an ellipse, and so $S_{2^j}(Q)$ is the interior of the ellipse $\{Q = 2^j\}$ with the interior of $\{Q = -2^j\}$ removed. Of course the interior of $\{Q = -2^j\}$ may be empty, which means $S_{2^j}(Q)$ is either an ellipse, or an elliptical annulus.

Furthermore $\tilde{S}_{\varepsilon 2^{-j}}(L)$, as in all of the following cases, is a pair of straight strips, between the lines $L = \varepsilon 2^{-j}$ and $L = \varepsilon 2^{-j-1}$ and between the lines $L = -\varepsilon 2^{-j-1}$ and $L = -\varepsilon 2^{-j}$. Though, to repeat, we need not consider the second of these strips, by symmetry.

We can calculate $\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right|$ via computing the distance between the the intersections between $L = K$, and $Q = 2^j$, and then integrating this distance in a suitable manner, by letting K go between $\varepsilon 2^{-j-1}$ and $\varepsilon 2^{-j}$. Note, in some cases we will only integrate up to a h_{2^j}, depending on the specific case.

$$
\begin{aligned}
&\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| = \\
&\frac{\sqrt{u_1^2+v_1^2}}{|u_1^3+v_1^3|}\int_{\frac{a}{\sqrt{u_1^2+v_1^2}}}^{\frac{b}{\sqrt{u_1^2+v_1^2}}} \sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2}) - 4(u_1^2+v_1^2)u_1v_1s^2}ds, \\
&= \frac{1}{|u_1^3+v_1^3|}\int_a^b \sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2}) - 4u_1v_1s^2}ds, \\
&= \frac{1}{|u_1^3+v_1^3|}\left(\frac{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2})}{2\sqrt{4u_1v_1}}\sin^{-1}\left(\sqrt{\frac{4u_1v_1}{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2})}}s\right)\right. \\
&\qquad \left.\left. + \frac{s}{2}\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2}) - 4u_1v_1s^2}\right)\right|_{s=a}^{b}, \\
&\sim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(b-a).
\end{aligned}
$$

This estimate will suffice for our purposes, so we will not use the exact value of the area.

We first deal with the case in which $Q = -2^j$ is empty. The condition for this to occur is for $u_1^3+v_1^3 < 2^{j+2}$.

Clearly the first case, in which $\varepsilon 2^{-j-1} > h_{2^j}$, the intersection is empty, and so is trivially bounded.

In the second case, where $\varepsilon 2^{-j-1} < h_{2^j} < \varepsilon 2^{-j}$, we must "integrate" between $\varepsilon 2^{-j-1}$ and h_{2^j}, and the area of intersection, using our previous estimate, is bounded by:

$$\begin{aligned}
\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| &\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(h_{2^j}-\varepsilon 2^{-j-1}),\\
&< \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}h_{2^j},\\
&= \frac{u_1^3+v_1^3+2^{j+2}}{\sqrt{4u_1v_1}},\\
&\lesssim \frac{2^j}{\sqrt{u_1v_1}},\\
&\lesssim \frac{\varepsilon}{\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2})}},\\
&\lesssim \frac{\varepsilon}{|u_1^3+v_1^3|}.
\end{aligned}$$

Here, we use the fact that in this case $u_1^3+v_1^3 \lesssim 2^j$, and the fact that $h_{2^j} < \varepsilon 2^{-j}$.

Lastly in the third case, we have $h_{2^j} > \varepsilon 2^{-j}$, and we integrate between $\varepsilon 2^{-j-1}$ and $\varepsilon 2^{-j}$. Therefore our area is bounded by:

$$\begin{aligned}
\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| &\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(\varepsilon 2^{-j}-\varepsilon 2^{-j-1}),\\
&= \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}\varepsilon 2^{-j-1}.\\
&\sim \frac{\varepsilon}{|u_1^3+v_1^3|}\frac{\sqrt{(u_1^3+v_1^3+2^{j+2})(u_1^3+v_1^3)}}{2^j}.\\
&\lesssim \frac{\varepsilon}{|u_1^3+v_1^3|}.
\end{aligned}$$

Here again, we simply use the fact that $u_1^3+v_1^3 \lesssim 2^j$.

Therefore, in this case, where $u_1v_1 > 0$, and $u_1^3+v_1^3 < 2^{j+2}$, we have our desired bounds.

Before we deal with the next case, in which we have an elliptical annulus, note that our three cases corresponded to:

- $\varepsilon 2^{-j} > \varepsilon 2^{-j-1} > h_{2^j}$,

- $\varepsilon 2^{-j} > h_{2^j} > \varepsilon 2^{-j-1}$,
- $h_{2^j} > \varepsilon 2^{-j} > \varepsilon 2^{-j-1}$,

which may remind us of a kind-of “Stars and Bars" setup, with $\varepsilon 2^{-j}$ and $\varepsilon 2^{-j-1}$ being our bars, and h_{2^j} being our star. These cases correspond to the setups

$$-|-|\star, \quad -|\star|-, \quad \star|-|-.$$

We now move onto when $Q = -2^j$ is non-empty. This means that $u_1^3 + v_1^3 > 2^{j+2}$, and that $h_{-2^j} \in \mathbb{R}$, which leads to more cases, as it is effectively an additional “star". We can use this to tidily index our different cases. Here, they are:

- $\varepsilon 2^{-j} > \varepsilon 2^{-j-1} > h_{2^j} > h_{-2^j}$ (or $-|-|\star\star$),
- $\varepsilon 2^{-j} > h_{2^j} > \varepsilon 2^{-j-1} > h_{-2^j}$ (or $-|\star|\star$),
- $\varepsilon 2^{-j} > h_{2^j} > h_{-2^j} > \varepsilon 2^{-j-1}$ (or $-|\star\star|-$),
- $h_{2^j} > \varepsilon 2^{-j} > h_{-2^j} > \varepsilon 2^{-j-1}$ (or $\star|\star|-$),
- $h_{2^j} > \varepsilon 2^{-j} > \varepsilon 2^{-j-1} > h_{-2^j}$ (or $\star|-|\star$),
- $h_{2^j} > h_{-2^j} > \varepsilon 2^{-j} > \varepsilon 2^{-j-1}$ (or $\star\star|-|$),

Note that this time, we compute the area via

$$\left|S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| = \left|\{Q < 2^j\} \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| - \left|\{Q < -2^j\} \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right|$$

These areas will, similarly to the earlier case, satisfy

$$\lesssim \sqrt{\frac{u_1^3 + v_1^3 + 2^{j+2}}{u_1^3 + v_1^3}}x - \sqrt{\frac{(u_1^3 + v_1^3 - 2^{j+2})}{(u_1^3 + v_1^3)}}y.$$

This notation is a bit vague, so to be explicit, the x will be evaluated with respect to the outer ellipse, and y will be evaluated with respect to the inner ellipse, (so in the case marked $\star|\star|-$, x would be $\varepsilon 2^{-j} - \varepsilon 2^{-j-1}$, as $h_{2^j} > \varepsilon 2^{-j} > \varepsilon 2^{-j-1}$, and y would be $h_{-2^j} - \varepsilon 2^{-j-1}$, as $\varepsilon 2^{-j} > h_{-2^J} > \varepsilon 2^{-j-1}$). Furthermore, notice in some cases, like $-|\star|\star$, we should take $y = 0$ as there is no area of intersection between our strips, and $\{Q < -2^j\}$. This occurs whenever the furthest right symbol is a star.

Again, the first case, $-|-|\star\star$, corresponds to an empty intersection, and is therefore trivial.

We deal with cases $-|\star|\star$ and $-|\star\star|-$ in the same way, so we group them together. Note that our formula for a bound for each of these says, respectively,

- $\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(h_{2^j}-\varepsilon 2^{-j-1})$, (for $-|\star|\star$)
- $\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(h_{2^j}-\varepsilon 2^{-j-1}) - \sqrt{\frac{(u_1^3+v_1^3-2^{j+2})}{(u_1^3+v_1^3)}}(h_{-2^j}-\varepsilon 2^{-j-1})$ (for $-|\star\star|-$)

Now, in the first of these cases, $\varepsilon 2^{-j-1} > h_{-2^j}$, so this bound is

$$\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(h_{2^j}-h_{-2^j}) \lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}h_{2^j} - \sqrt{\frac{(u_1^3+v_1^3-2^{j+2})}{(u_1^3+v_1^3)}}h_{-2^j}.$$

The second can be rewritten as

$$\begin{aligned}= \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}h_{2^j} - \sqrt{\frac{(u_1^3+v_1^3-2^{j+2})}{(u_1^3+v_1^3)}}h_{-2^j}\\ - \varepsilon 2^{-j-1}\left(\sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}} - \sqrt{\frac{(u_1^3+v_1^3-2^{j+2})}{(u_1^3+v_1^3)}}\right),\end{aligned}$$

which is also clearly $\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}h_{2^j} - \sqrt{\frac{(u_1^3+v_1^3-2^{j+2})}{(u_1^3+v_1^3)}}h_{-2^j}$.

Expanding either this gives a quantity that is

$$\sim \frac{2^j}{\sqrt{u_1v_1}} \lesssim \frac{\varepsilon}{\sqrt{(u_1^3+v_1^3+2^{j+2})(u_1^3+v_1^3)}} < \frac{\varepsilon}{|u_1^3+v_1^3|}.$$

Again, we rely on the fact that $\varepsilon 2^{-j} > h_{2^j}$ to obtain this bound.

Similarly, we group $\star|-|\star$ and $\star|\star|-$ due to similarity. Their respective bounds are:

- $\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(\varepsilon 2^{-j}-\varepsilon 2^{-j-1})$, (for $\star|-|\star$)
- $\lesssim \sqrt{\frac{u_1^3+v_1^3+2^{j+2}}{u_1^3+v_1^3}}(\varepsilon 2^{-j}-\varepsilon 2^{-j-1}) - \sqrt{\frac{(u_1^3+v_1^3-2^{j+2})}{(u_1^3+v_1^3)}}((h_{-2^j}-\varepsilon 2^{-j-1}))$(for $\star|\star|-$)

Now, if $u_1^3 + v_1^3 \sim 2^j$, both of these cases clearly give us an area that is

$$\lesssim \sqrt{\frac{u_1^3 + v_1^3 + 2^{j+2}}{u_1^3 + v_1^3}}\varepsilon 2^{-j} \sim \frac{\varepsilon}{2^j} \sim \frac{\varepsilon}{|u_1^3 + v_1^3|}$$

Now, recalling that we are in the case in which $u_1^3 + v_1^3 \gtrsim 2^j$, this means we now assume $u_1^3 + v_1^3 \gg 2^j$.

Furthermore, in the first of these, we have that $h_{-2^j} \sim \varepsilon 2^{-j} < h_{2^j}$. In the second, we immediately have $h_{-2^j} < \varepsilon 2^{-j} < h_{2^j}$. So in either case, we have $\varepsilon 2^{-j} - \varepsilon 2^{-j-1} \lesssim h_{2^j} - h_{-2^j}$. Therefore our area is bounded by:

$$\begin{aligned}
&\lesssim \sqrt{\frac{u_1^3 + v_1^3 + 2^{j+2}}{u_1^3 + v_1^3}}(\varepsilon 2^{-j} - \varepsilon 2^{-j-1}) \\
&\lesssim \sqrt{\frac{u_1^3 + v_1^3 + 2^{j+2}}{u_1^3 + v_1^3}}(h_{2^j} - h_{-2^j}) \\
&\lesssim \sqrt{\frac{u_1^3 + v_1^3 + 2^{j+2}}{u_1^3 + v_1^3}}h_{2^j} - \sqrt{\frac{(u_1^3 + v_1^3 - 2^{j+2})}{(u_1^3 + v_1^3)}}h_{-2^j} \\
&\sim \frac{2^j}{\sqrt{u_1 v_1}} \\
&\lesssim \frac{\varepsilon}{\sqrt{(u_1^3 + v_1^3)(u_1^3 + v_1^3 - 2^{j+2})}} \sim \frac{\varepsilon}{|u_1^3 + v_1^3|}.
\end{aligned}$$

The second last inequality here follows from $h_{-2^j} < \varepsilon 2^{-j-1}$, and the final step from the fact that $u_1^3 + v_1^3 \gg 2^j$.

Lastly, we deal with $-| - | \star\star$. In this case, our bound is

$$\begin{aligned}
&\lesssim \sqrt{\frac{u_1^3 + v_1^3 + 2^{j+2}}{u_1^3 + v_1^3}}(\varepsilon 2^{-j} - \varepsilon 2^{-j-1}) - \sqrt{\frac{(u_1^3 + v_1^3 - 2^{j+2})}{(u_1^3 + v_1^3)}}(\varepsilon 2^{-j} - \varepsilon 2^{-j-1}) \\
&\sim \frac{\varepsilon 2^{-j}}{\sqrt{u_1^3 + v_1^3}}\left(\frac{2^j}{\sqrt{u_1^3 + v_1^3 + 2^{j+2}} + \sqrt{u_1^3 + v_1^3 - 2^{j+2}}}\right) \\
&< \frac{\varepsilon}{|u_1^3 + v_1^3|}.
\end{aligned}$$

So we have therefore obtained the bound $\left|S_{2^j}(Q) \cap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| \lesssim \frac{\varepsilon}{|u_1^3 + v_1^3|}$ in all cases where $u_1 v_1 > 0$, and so now move on to the case where $u_1 v_1 < 0$.

3.3.2 If $u_1 v_1 < 0$

In this case, u_1 and v_1 have different signs. This means $\{Q = C\}$ is a hyperbola, and that the hyperbola $\{Q = 2^j\}$ and $Q = -2^j$ exist for all j. We will have to split our cases again, depending on the signs of $(u_1^3 + v_1^3 + 2^{j+2})(u_1^3 + v_1^3)$, and $(u_1^3 + v_1^3 - 2^{j+2})(u_1^3 + v_1^3)$. In any of these case, we can assume $u_1^3 + v_1^3 > 0$, so really we only need to consider the sign of $(u_1^3 + v_1^3 - 2^{j+2})$

We will first assume that $(u_1^3 + v_1^3 - 2^{j+2})(u_1^3 + v_1^3) > 0$. In this case, the hyperbolas $\{Q = 2^j\}$ and $\{Q = -2^j\}$ are aligned, in the sense that their vertices are colinear. We can therefore compute the length of $L = \sqrt{u_1^2 + v_1^2}s$ within them, and therefore $|I_s|$, via

$$\begin{aligned}
|I_s| &= \frac{\sqrt{u_1^2+v_1^2}}{u_1^3+v_1^3}\Big(\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2}) - 4s^2u_1v_1(u_1^2+v_1^2)} \\
&\qquad - \sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3-2^{j+2}) - 4s^2u_1v_1(u_1^2+v_1^2)}\Big) \\
&= \frac{2^{j+3}\sqrt{u_1^2+v_1^2}}{\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2}) - 4s^2u_1v_1(u_1^2+v_1^2)} + \sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3-2^{j+2}) - 4s^2u_1v_1(u_1^2+v_1^2)}}
\end{aligned}$$

Furthermore, in this case, we always have $I = [\frac{\varepsilon 2^{-j-1}}{\sqrt{u_1^2+v_1^2}}, \frac{\varepsilon 2^{-j}}{\sqrt{u_1^2+v_1^2}}]$.

Note in the next step, as $(u_1^3 + v_1^3) > 0$, and will throw out the $\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3-2^{j+2}) - 4s^2u_1v_1(u_1^2+v_1^2)}$ term in the denominator of $|I_s|$. If this was not the case, we would throw out the other term, and the argument would work the same, with the addition of some absolute value symbols.

$$\begin{aligned}
\left|S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| &\lesssim 2^j \int_{\varepsilon 2^{-j-1}}^{\varepsilon 2^{-j}} \frac{ds}{\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2}) - 4s^2u_1v_1}}, \\
&= \frac{2^j}{\sqrt{4u_1v_1}}\Bigg[\sinh^{-1}\left(\sqrt{\frac{4u_1v_1}{(u_1^3+v_1^3+2^{j+2})(u_1^3+v_1^3)}}\varepsilon 2^{-j}\right) \\
&\qquad - \sinh^{-1}\left(\sqrt{\frac{4u_1v_1}{(u_1^3+v_1^3+2^{j+2})(u_1^3+v_1^3)}}\varepsilon 2^{-j-1}\right)\Bigg], \\
&< \frac{2^j}{\sqrt{4u_1v_1}}\sinh^{-1}\left(\sqrt{\frac{4u_1v_1}{(u_1^3+v_1^3+2^{j+2})(|u_1^3+v_1^3|)}}\varepsilon 2^{-j}\right), \\
&< \frac{\varepsilon}{|u_1^3+v_1^3|},
\end{aligned}$$

Where we simply use $\sinh^{-1}(x) < x$ for the last step.

Now, in the final case, where $(u_1^3 + v_1^3 - 2^{j+2}) < 0$, the two hyperbolas are not aligned, so the area becomes more complicated. We continue to assume $u_1^3 + v_1^3 > 0$, with the argument working if $u_1^3 + v_1^3 < 0$ in a similar manner to the previous case.

In this case, we again split our cases depending on the relationship between $\varepsilon 2^{-j}$ and h_{-2^j}. These cases are

- $\varepsilon 2^{-j} < h_{-2^j}$,
- $\varepsilon 2^{-j-1} < h_{-2^j} < \varepsilon 2^{-j}$,
- $h_{-2^j} < \varepsilon 2^{-j-1}$.

The important difference in this case is that $|I_s|$ takes on different values depending on where s lies in relation to h_{-2^j}.

In the first case, $|I_s|$ is just the distance between the intersections between of $L = \sqrt{u_1^2 + v_1^2}s$ and $Q = 2^j$, which is again just

$$\frac{\sqrt{u_1^2 + v_1^2}}{|u_1^3 + v_1^3|}\sqrt{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2}) - 4s^2 u_1 v_1 (u_1^2 + v_1^2)}.$$

Note that $(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2}) > 0$. We therefore have:

$$\begin{aligned}\left|S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| &\lesssim \frac{1}{|u_1^3 + v_1^3|}\int_{\varepsilon 2^{-j-1}}^{\varepsilon 2^{-j}} \sqrt{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2}) - 4s^2 u_1 v_1} ds, \\ = \frac{1}{|u_1^3 + v_1^3|}&\Bigg(\frac{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2})}{2\sqrt{4\,|u_1 v_1|}} \sinh^{-1}\sqrt{\frac{4\,|u_1 v_1|}{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2})}}s \\ &\qquad + \frac{s}{2}\sqrt{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2}) - 4u_1 v_1 s^2}\Bigg)\Bigg|_{\varepsilon 2^{-j-1}}^{\varepsilon 2^{-j}}\end{aligned}$$

Now, of this sum, the second term dominates, which can be seen by noting that they are equal at $s = 0$, and comparing their derivatives. Therefore we have

$$\begin{aligned}\left|S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| &\lesssim \frac{s}{2(|u_1^3 + v_1^3|)}\sqrt{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2}) - 4u_1 v_1 s^2}\Bigg)\Bigg|_{\varepsilon 2^{-j-1}}^{\varepsilon 2^{-j}} \\ &< \frac{\varepsilon}{2(|u_1^3 + v_1^3|)}\sqrt{\frac{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2})}{2^{-2j}} - \frac{4u_1 v_1 \varepsilon^2 2^{-2j}}{2^{-2j}}} \\ &\lesssim \frac{\varepsilon}{(u_1^3 + v_1^3)}\sqrt{\frac{(u_1^3 + v_1^3)(u_1^3 + v_1^3 + 2^{j+2})}{2^{-2j}}} \\ &\lesssim \frac{\varepsilon}{|u_1^3 + v_1^3|}.\end{aligned}$$

Note that here we can combine the terms in the square root as $\varepsilon 2^{-j} < h_{-2^j}$ implies

$$4\,|u_1v_1|\,\varepsilon^2 2^{-2j} < \left|(u_1^3+v_1^3)(u_1^3+v_1^3-2^{j+2})\right| < (u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2}).$$

Also, note that we also used the assumption $u_1^3+v_1^3-2^{j+2}<0$ to obtain that the term in the root was $\lesssim 1$.

Next, we look at the case in which $\varepsilon 2^{-j-1} < h_{-2^j} < \varepsilon 2^{-j}$. In this case, as in the previous section, we can compute the area via

$$\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| = \left|\{Q<2^j\}\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| - \left|\{Q<-2^j\}\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right|,$$

which in this case is given by

$$\begin{aligned}
&= \frac{1}{|u_1^3+v_1^3|}\Big(\int_{\varepsilon 2^{-j-1}}^{\varepsilon 2^{-j}} \sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2})-4s^2u_1v_1}ds \\
&\qquad - \int_{h_{-2^j}}^{\varepsilon 2^{-j}} \sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3-2^{j+2})-4s^2u_1v_1ds}\Big) \\
&< \frac{1}{|u_1^3+v_1^3|}\int_{\varepsilon 2^{-j-1}}^{\varepsilon 2^{-j}} \sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2})-4s^2u_1v_1}ds.
\end{aligned}$$

Note that this is now identical to the previous case, except instead of having $\varepsilon 2^{-j} < h_{-2^j}$, we have $\varepsilon 2^{-j} \lesssim h_{-2^j}$, which is sufficient to follow the same argument and obtain

$$\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| \lesssim \frac{\varepsilon}{|u_1^3+v_1^3|}.$$

Lastly, in the case with $h_{-2^j} < \varepsilon 2^{-j-1}$, we have that $|I_s|$ is given by

$$\frac{\sqrt{u_1^2+v_1^2}2^{j+2}}{\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3+2^{j+2})-4u_1v_1s^2}+\sqrt{(u_1^3+v_1^3)(u_1^3+v_1^3-2^{j+2})-4u_1v_1s^2}}.$$

However, note that this quantity is the exact same as that of the case in which $(u_1^3+v_1^3-2^{j+2})>0$, and the exact same argument will yield

$$\left|S_{2^j}(Q)\bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| \lesssim \frac{\varepsilon}{|u_1^3+v_1^3|}.$$

3.3.3 Additional minor details

For this estimate to be sufficient, we need a few more minor considerations. We want to obtain an optimal bound for the sublevel set of $|u_1^3+v_1^3|$ so that we can remove it from P_2. So we are considering
$S_\varepsilon(u_1^3+v_1^3)=\{(u_1,v_1)\mid |u_1^3+v_1^3|<\varepsilon\}$.

To obtain this area, we simply want to consider, and then bound, the integral between the two curves $v_1=(\varepsilon-u_1^3)^{1/3}$ and $v_1=(-\varepsilon-u_1^3)^{1/3}$. Clearly, this area is given by

$$\begin{aligned}
\left|S_\varepsilon(u_1^3+v_1^3)\right| &= \int_{-\infty}^{\infty}(\varepsilon-u_1^3)^{1/3}-(-\varepsilon-u_1^3)^{1/3}du_1\\
&=4\varepsilon\int_0^\infty \frac{du_1}{(\varepsilon-u_1^3)^{2/3}+(-\varepsilon-u_1^3)^{2/3}+(\varepsilon-u_1^3)^{1/3}(-\varepsilon-u_1^3)^{1/3}}\\
&<4\varepsilon[\int_0^{2\varepsilon^{1/3}} \frac{du_1}{(\varepsilon-u_1^3)^{2/3}+(-\varepsilon-u_1^3)^{2/3}+(\varepsilon-u_1^3)^{1/3}(-\varepsilon-u_1^3)^{1/3}}\\
&\qquad+\int_{2\varepsilon^{1/3}}^\infty \frac{du_1}{(u_1^6-\varepsilon^2)^{1/3}}]\\
&<4\varepsilon\left[\frac{2\varepsilon^{\frac{1}{3}}}{\varepsilon^{\frac{2}{3}}}+\int_{2\varepsilon^{1/3}}^\infty\frac{du_1}{u_1^2-\varepsilon^{\frac{2}{3}}}\right]\\
&=\varepsilon^{\frac{2}{3}}(8+\frac{\ln(3)}{2}).
\end{aligned}$$

The other detail to consider is the summation. We have that

- $S_\varepsilon(J)=\sum_{j\in\mathbb{Z}}\left|S_{2^j}(Q)\bigcap\tilde{S}_{\varepsilon2^{-j}}(L)\right|$,
- $\left|S_{2^j}(Q)\bigcap\tilde{S}_{\varepsilon2^{-j}}(L)\right|\lesssim\dfrac{\varepsilon}{|u_1^3+v_1^3|}$.

  but cannot use this estimate in an infinite sum. To avoid this, note that $\Omega\subset[0,1]^6$, and therefore the[5] $(u_i,v_i)\in[-1,1]^2$ for each i. We therefore we only need to consider when these intersections lie inside this square[6].

  To see this gives us a lower bound for j, note that the distance between the origin, and the line $L=\pm\varepsilon2^{-j-1}$ is $\sim\frac{\varepsilon2^{-j-1}}{\sqrt{u_1^2+v_1^2}}\lesssim\varepsilon2^{-j-1}$.

  This means, if $j<\lfloor\log_2(\varepsilon)\rfloor-1$, then $\tilde{S}_{\varepsilon2^{-j}}(L)$ lies entirely outside our square, and therefore, these j values correspond to terms that do not contribute to the sum.

  Now, we cannot obtain an upper bound for j, as the contribution of large

[5] The transformed (u_i,v_i)
[6] in "P_3"

j-terms is never zero. However, if $j > 2$, then $[-1,1]^2 \subset S_{2^j}(Q)$, and therefore the area we contribute in these case is simply

$$\left|S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| = \left|\tilde{S}_{\varepsilon 2^{-j}}(L)\right| \sim \frac{\varepsilon 2^{-j}}{\sqrt{u_1^2 + v_1^2}}.$$

We can use this estimate for the larger j-values, to ensure convergence.

Therefore we have

$$\begin{aligned}
|S_\varepsilon(J)| &= \sum_{j \in \mathbb{Z}} \left|S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| \\
&\sim \sum_{j=\lfloor \log_2(\varepsilon) \rfloor - 1}^{2} \left|S_{2^j}(Q) \bigcap \tilde{S}_{\varepsilon 2^{-j}}(L)\right| + \sum_{j=2}^{\infty} \left|\tilde{S}_{\varepsilon 2^{-j}}(L)\right| \\
&\lesssim (3 - \log_2 \varepsilon) \frac{\varepsilon}{u_1^3 + v_1^3} + \frac{\varepsilon}{\sqrt{u_1^2 + v_1^2}}.
\end{aligned}$$

Now, omitting the details, we obtain stronger bounds in the cases where $\log_2 \varepsilon \sim 1$, or in which $\frac{\varepsilon}{\sqrt{u_1^2+v_1^2}}$ is the dominant term, so we therefore work with the estimate

$$|S_\varepsilon(J)| \lesssim \frac{\varepsilon \log_2(\frac{1}{\varepsilon})}{|u_1^3 + v_1^3|}.$$

Bibliography

[BDG16] Jean Bourgain, Ciprian Demeter, and Larry Guth. Proof of the main conjecture in Vinogradov's Mean Value Theorem for degrees higher than three. *Annals of Mathematics*, 184(2):633–682, 2016.

[BH14] Jong-Guk Bak and Seheon Ham. Restriction of the Fourier transform to some complex curves. *Journal of Mathematical Analysis and Applications*, 409(2):1107–1127, 2014.

[BOS12] Jong-Guk Bak, Daniel M. Oberlin, and Andreas Seeger. Restriction of Fourier transforms to curves: An endpoint estimate with affine arclength measure. *Journal für die reine und angewandte Mathematik*, 2013(682):315, 2012.

[CH18] Hyunuk Chung and Seheon Ham. Convolution estimates for measures on some complex curves. *Annali di Matematica Pura ed Applicata*, 198(3):837–867, 2018.

[Chr85] Michael Christ. On the restriction of the Fourier transform to curves: endpoint results and the degenerate case. *Transactions of the American Mathematical Society*, 287(1):223–238, 1985.

[Chr98] Michael Christ. Convolution, curvature, and combinatorics: a case study. *International Mathematics Research Notices*, 1998(19):1033–1048, 1998.

[CRW98] Anthony Carbery, Fulvio Ricci, and James Wright. Maximal functions and Hilbert transforms associated to polynomials. *Revista Matemática Iberoamericana* , 14(1):117–144, 1998.

[dDP20] Jaume de Dios Pont. A geometric lemma for complex polynomial curves with applications in fourier restriction theory, 2020.

[DG91] Stephen William Drury and Kanghui Guo. Convolution estimates related to surfaces of half the ambient dimension. *Mathematical Proceedings of the Cambridge Philosophical Society*, 110(1):151–159, 1991.

[DLW09] Spyridon Dendrinos, Norberto Laghi, and James Wright. Universal L^p improving for averages along polynomial curves in low dimensions. *Journal of Functional Analysis*, 257(5):1355–1378, 2009.

[DW10] Spyridon Dendrinos and James Wright. Fourier restriction to Polynomial Curves 1: A Geometric Inequality. *American Journal of Mathematics*, 132(4):1031–1076, 2010.

[GLYZK21] Shaoming Guo, Zane Kun Li, Po-Lam Yung, and Pavel Zorin-Kranich. A short proof of ℓ^2 decoupling for the moment curve. *American Journal of Mathematics*, 143(6):1983–1998, 2021.

[Gre19] Phillip Gressman. On the Oberlin affine curvature condition. *Duke Mathematical Journal*, 168(11):2075–2126, 2019.

[Obe02a] Daniel Oberin. Convolution with measures on polynomial curves. *Mathematica Scandinavica*, 90(1):126–138, 2002.

[Obe02b] Daniel Oberin. Some convolution inequalities and their applications. *Transactions of the American Mathematical Society*, 354(6):2541–2556, 2002.

[Sjö74] Per Sjöln. Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in $\mathbb{R}^2$. *Studia Mathematica*, 51(2):169–182, 1974.

[Ste70] Elias Stein. *Singular integrals and differentiability properties of functions*. Princeton University Press, 41 William Street, Princeton, 1970.

[Sto09] Betsy Stovall. Endpoint bounds for a generalized Radon transform. *Journal of the London Mathematical Society*, 80(2):357–374, 2009.

[Sto16a] Betsy Stovall. Endpoint $L^p \to L^q$ bounds for integration along certain polynomial curves. *Journal of Functional Analysis*, 259(12):357–374, 2016.

[Sto16b] Betsy Stovall. Uniform estimates for Fourier restriction to polynomial curves in $\mathbb{R}^d$. *American Journal of Mathematics*, 138(2):449–471, 2016.

[Wol00] Thomas Wolff. Local smoothing type estimates on L^p for large p. *Geometric and Functional Analysis*, 10(5):1237–1288, 2000.

[Woo19] Trevor Wooley. Nested efficient congruencing and relatives of Vinogradov's mean value theorem. *Proceedings of the London Mathematical Society*, 118(4):942–1016, 2019.