Benjamin McKay

Concrete Algebra

With a View Toward Abstract Algebra

March 11, 2022


365

Preface

With my full philosophical rucksack I can only climb slowly up the mountain of mathematics.

— Ludwig Wittgenstein
Culture and Value

These notes are from lectures given in 2015 at University College Cork. They aim to explain the most concrete and fundamental aspects of algebra, in particular the algebra of the integers and of polynomial functions of a single variable, grounded by proofs using mathematical induction. It is impossible to learn mathematics by reading a book like you would read a novel; you have to work through exercises and calculate out examples. You should try all of the problems. More importantly, since the purpose of this class is to give you a deeper feeling for elementary mathematics, rather than rushing into advanced mathematics, you should reflect about how the simple ideas in this book reshape your vision of algebra. Consider how you can use your new perspective on elementary mathematics to help you some day guide other students, especially children, with surer footing than the teachers who guided you.

The temperature of Heaven can be rather accurately computed. Our authority is Isaiah 30:26, "Moreover, the light of the Moon shall be as the light of the Sun and the light of the Sun shall be sevenfold, as the light of seven days." Thus Heaven receives from the Moon as much radiation as we do from the Sun, and in addition $7 \times 7 = 49$ *times as much as the Earth does from the Sun, or 50 times in all. The light we receive from the Moon is one* $1/10\,000$ *of the light we receive from the Sun, so we can ignore that. . . . The radiation falling on Heaven will heat it to the point where the heat lost by radiation is just equal to the heat received by radiation, i.e., Heaven loses 50 times as much heat as the Earth by radiation. Using the Stefan-Boltzmann law for radiation,* (H/E) *temperature of the earth* $(\sim 300\,\mathrm{K})$*, gives* H *as* $798\,\mathrm{K}$ $(525\,^\circ\mathrm{C})$*. The exact temperature of Hell cannot be computed. . . . [However] Revelations 21:8 says "But the fearful, and unbelieving . . . shall have their part in the lake which burneth with fire and brimstone." A lake of molten brimstone means that its temperature must be at or below the boiling point,* $444.6\,^\circ\mathrm{C}$*. We have, then, that Heaven, at* $525\,^\circ\mathrm{C}$ *is hotter than Hell at* $445\,^\circ\mathrm{C}$*.*

— Applied Optics , vol. 11, A14, 1972

In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics.

— Hermann Weyl
Invariants, Duke Mathematical Journal 5, 1939, 489–502

— *and so who are you, after all?*
— *I am part of the power which forever wills evil and forever works good.*

— Goethe
Faust

This Book is not to be doubted.

— Quran , 2:1/2:6-2:10 *The Cow*

Contents

Chapter 1

The integers

God made the integers; all else is the work of man.

— Leopold Kronecker

Notation

We will write numbers using notation like 1 234 567.123 45, using a decimal point . at the last integer digit, and using thin spaces to separate out every 3 digits before or after the decimal point. You might prefer 1,234,567·123,45 or 1,234,567.123,45, which are also fine. We reserve the · symbol for multiplication, writing $2 \cdot 3 = 6$ rather than $2 \times 3 = 6$.

The laws of integer arithmetic

We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.
Through the unknown, remembered gate
When the last of earth left to discover
Is that which was the beginning;
At the source of the longest river
The voice of the hidden waterfall
And the children in the apple-tree
Not known, because not looked for
But heard, half-heard, in the stillness
Between two waves of the sea.

— T.S. Eliot
Little Gidding, Four Quartets

The integers are the numbers $\ldots, -2, -1, 0, 1, 2, \ldots$. Let us distill their essential properties, using only the concepts of addition and multiplication.

Addition laws:

a. The associative law: For any integers a, b, c: $(a + b) + c = a + (b + c)$.

b. The identity law: There is an integer 0 so that for any integer a: $a+0=a$.

c. The existence of negatives: for any integer a: there is an integer b (denoted by the symbol $-a$) so that $a+b=0$.

d. The commutative law: For any integers a, b: $a+b=b+a$.

Multiplication laws:

a. The associative law: For any integers a, b, c: $(ab)c=a(bc)$.

b. The identity law: There is an integer 1 so that for any integer a: $a1=a$.

c. The zero divisors law: For any integers a, b: if $ab=0$ then $a=0$ or $b=0$.

d. The commutative law: For any integers a, b: $ab=ba$.

The distributive law:

a. For any integers a, b, c: $a(b+c)=ab+ac$.

Sign laws:

Certain integers are called *positive*.

a. The succession law: An integer b is positive just when either $b=1$ or $b=c+1$ for a positive integer c.

b. Determinacy of sign: Every integer a has precisely one of the following properties: a is positive, $a=0$, or $-a$ is positive.

We write $a<b$ to mean that there is a positive integer c so that $b=a+c$.

The law of well ordering:

a. Any nonempty collection of positive integers has a least element; that is to say, an element a so that every element b satisfies $a<b$ or $a=b$.

All of the other arithmetic laws we are familiar with can be derived from these. For example, the associative law for addition, applied twice, shows that $(a+b)+(c+d)=a+(b+(c+d))$, and so on, so that we can add up any finite sequence of integers, in any order, and get the same result, which we write in this case as $a+b+c+d$. A similar story holds for multiplication.

Of course, we write $1+1$ as 2, and $1+1+1$ as 3 and so on. Write $a>b$ to mean $b<a$. Write $a\leq b$ to mean $a<b$ or $a=b$. Write $a\geq b$ to mean $b\leq a$. Write $|a|$ to mean a, if $a\geq 0$, and to mean $-a$ otherwise, and call it the *absolute value* of a. An integer a is *negative* if $-a$ is positive.

To understand mathematics, you have to solve a large number of problems.

> *I prayed for twenty years but received no answer until I prayed with my legs.*
>
> — Frederick Douglass, statesman who escaped slavery

1.1 For each equation below, what law above justifies it?

a. $7(3+1) = 7 \cdot 3 + 7 \cdot 1$

b. $4(9 \cdot 2) = (4 \cdot 9)2$

c. $2 \cdot 3 = 3 \cdot 2$

1.2 Use the laws above to prove that for any integers a, b, c: $(a+b)c = ac + bc$.

1.3 Use the laws above to prove that $0 + 0 = 0$.

1.4 Use the laws above to prove that $0 \cdot 0 = 0$.

1.5 Use the laws above to prove that, for any integer a: $a \cdot 0 = 0$.

1.6 Use the laws above to prove that, for any integer a: there is exactly one integer b so that $a + b = 0$; of course, we call this b by the name $-a$.

1.7 Use the laws above to prove that, for any integer a: $(-1)a = -a$.

1.8 Use the laws above to prove that $(-1)(-1) = 1$.

1.9 Use the laws above to prove that, for any integers b, c: $|bc| = |b||c|$.

1.10 Our laws ensure that there is an integer 0 so that $a + 0 = 0 + a = a$ for any integer a. Could there be two different integers, say p and q, so that $a + p = p + a = a$ for any integer a, and also so that $a + q = q + a = a$ for any integer a? (Roughly speaking, we are asking if there is more than one integer which can "play the role" of zero.)

1.11 Our laws ensure that there is an integer 1 so that $a \cdot 1 = 1 \cdot a = a$ for any integer a. Could there be two different integers, say p and q, so that $ap = pa = a$ for any integer a, and also so that $aq = qa = a$ for any integer a?

Theorem 1.1 (The equality cancellation law for addition)**.** *Suppose that a, b and c are integers and that $a + c = b + c$. Then $a = b$.*

Proof. By the existence of negatives, there is a integer $-c$ so that $c + (-c) = 0$. Clearly $(a + c) + (-c) = (b + c) + (-c)$. Apply the associative law for addition: $a + (c + (-c)) = b + (c + (-c))$, so $a + 0 = b + 0$, so $a = b$. □

1.12 Prove that $-(b + c) = (-b) + (-c)$ for any integers b, c.

1.13 A *predecessor* of an integer b is an integer c so that $b = c + 1$. Prove that every integer has a unique predecessor.

We haven't mentioned subtraction yet.

1.14 Suppose that a and b are integers. Prove that there is a unique integer c so that $a = b + c$. Of course, from now on we write this integer c as $a - b$.

1.15 Prove that, for any integers a, b, $a - b = a + (-b)$.

1.16 Prove that subtraction *distributes over multiplication*: for any integers a, b, c, $a(b - c) = ab - ac$.

1.17 Prove that any two integers b and c satisfy just precisely one of the conditions $b > c$, $b = c$, $b < c$.

Theorem 1.2. *The equality cancellation law for multiplication: for an integers a, b, c if $ab = ac$ and $a \neq 0$ then $b = c$.*

Proof. If $ab = ac$ then $ab - ac = ac - ac = 0$. But $ab - ac = a(b - c)$ by the distributive law. So $a(b - c) = 0$. By the zero divisors law, $a = 0$ or $b = c$. □

1.18 We know how to add and multiply 2×2 matrices with integer entries. Of the various laws of addition and multiplication and signs for integers, which hold true also for such matrices?

1.19 Use the laws above to prove that the sum of any two positive integers is positive.

1.20 Use the laws above to prove that the product of any two positive integers is positive.

1.21 Use the laws above to prove that the product of any two integers is positive just when (1) both are positive or (2) both are negative.

1.22 Use the laws above to prove that the product of any two negative integers is positive.

1.23 Prove the inequality cancellation law for addition: For any integers a, b, c: if $a + c < b + c$ then $a < b$.

1.24 Prove the inequality cancellation law for multiplication: For any integers a, b, c: if $a < b$ and if $0 < c$ then $ac < bc$.

1.25 Suppose that S is a set of integers. A *lower bound* on S is an integer b so that $b \leq c$ for every integer c from S; if S has a lower bound, S is *bounded from below*. Prove that a nonempty set of integers bounded from below contains a least element.

Division of integers

> *Can you do Division? Divide a loaf by a knife—what's the answer to that?*
>
> — Lewis Carroll
> Through the Looking Glass

We haven't mentioned division yet. *Danger:* although 2 and 3 are integers,

$$\frac{3}{2} = 1.5$$

is *not* an integer.

1.26 Suppose that a and b are integers and that $b \neq 0$. Prove from the laws above that there is *at most* one integer c so that $a = bc$. Of course, from now on we write this integer c as $\frac{a}{b}$ or a/b.

1.27 *Danger:* why *can't* we divide by zero?

We already noted that $3/2$ is *not* an integer. At the moment, we are trying to work with integers only. An integer b *divides* an integer c if c/b is an integer; we also say that b is a *divisor* of c.

1.28 Explain why every integer divides into zero.

1.29 Prove that, for any two integers b and c, the following are equivalent:

a. b divides c,

b. $-b$ divides c,

c. b divides $-c$,

d. $-b$ divides $-c$,

e. $|b|$ divides $|c|$.

Proposition 1.3. *Take any two integers b and c. If b divides c, then $|b| < |c|$ or $b = c$ or $b = -c$.*

Proof. By the solution of the last problem, we can assume that b and c are positive. If $b = c$ the proposition holds, so suppose that $b > c$; write $b = c+k$ for some $k > 0$. Since b divides c, say $c = qb$ for some integer q. Since $b > 0$ and $c > 0$, $q > 0$ by problem 1.21 on the facing page. If $q = 1$ then $c = b$, so the proposition holds. If $q > 1$, then $q = n+1$ for some positive integer n. But then $c = qb = (n+1)(c+k) = nc+c+nk+k$. Subtract c from both sides: $0 = nc+nk+k$, a sum of positive integers, hence positive (by problem 1.20 on the preceding page) a contradiction. □

Theorem 1.4 (Euclid)**.** *Suppose that b and c are integers and $c \neq 0$. Then there are unique integers q and r (the* quotient *and* remainder*) so that $b = qc + r$ and so that $0 \leq r < |c|$.*

Proof. To make our notation a little easier, we can assume (by perhaps changing the signs of c and q in our statement above) that $c > 0$.

Consider all integers of the form $b - qc$, for various integers q. If we were to take $q = -|b|$, then $b - qc = b + |b| + (c-1)|b| \geq 0$. By the law of well ordering, since there is an integer of the form $b - qc \geq 0$, there is a smallest integer of the form $b - qc \geq 0$; call it r. If $r \geq c$, we can replace $r = b - qc$ by $r - c = b - (q+1)c$, and so r was not smallest.

We have proven that we can find a quotient q and remainder r. We need to show that they are unique. Suppose that there is some other choice of integers Q and R so that $b = Qc + R$ and $0 \leq R < c$. Taking the difference between $b = qc + r$ and $b = Qc + R$, we find $0 = (Q - q)c + (R - r)$. In particular, c divides into $R - r$. Switching the labels as to which ones are q, r and which are Q, R, we can assume that $R \geq r$. So then $0 \leq r \leq R < c$, so $R - r < c$. By proposition 1.3 on the preceding page, since $0 \leq R - r < c$ and c divides into $R - r$, we must have $R - r = 0$, so $r = R$. Plug into $0 = (Q - q)c + (R - r)$ to see that $Q = q$. □

How do we actually find quotient and remainder, by hand? Long division:

$$\begin{array}{r} 14 \\ 17\overline{)249} \\ \underline{17} \\ 79 \\ \underline{68} \\ 11 \end{array}$$

So if we start with $b = 249$ and $c = 17$, we carry out long division to find the quotient $q = 14$ and the remainder $r = 11$.

1.30 Find the quotient and remainder for b, c equal to:

a. $-180, 9$

b. $-169, 11$

c. $-982, -11$

d. $279, -11$

e. $247, -27$

The greatest common divisor

A *common divisor* of some integers is an integer which divides them all. A *greatest common divisor* is a common divisor $d \geq 0$ so that every common divisor divides d. We will see that every set of integers has a unique greatest common divisor.

1.31 Prove that if a set of integers has a greatest common divisor, it has only one.

Denote the greatest common divisor of integers $m_1, m_2, \ldots, m_n$ as

$$\gcd\{m_1, m_2, \ldots, m_n\}.$$

Note that $\gcd\{0\} = 0$ and also that $\gcd\{\} = 0$ by our definition. If the greatest common divisor of two integers is 1, they are *coprime*.

Lemma 1.5. *Take two integers b, c with $c \neq 0$ and compute the quotient q and remainder r, so that $b = qc + r$ and $0 \leq r < |c|$. Then b, c have a greatest common divisor and* $\gcd\{b, c\} = \gcd\{c, r\}$.

Proof. Any divisor of c and r divides the right hand of the equation $b = qc + r$, so it divides the left hand side, and so divides b and c. By the same reasoning, writing the same equation as $r = b - qc$, we see that any divisor of b and c divides c and r. So b, c and c, r have the same divisors. Note that we can pick c to be the smaller in absolute value of b, c. Existence of a greatest common divisor follows by induction since the larger, in absolute value, of b, c is larger than that of c, r. □

This makes very fast work of finding the greatest common divisor by hand. For example, if $b = 249$ and $c = 17$ then we found that $q = 14$ and $r = 11$, so $\gcd\{249, 17\} = \gcd\{17, 11\}$. Repeat the process. It is convenient to write down our first two numbers,

$$249, 17$$

next write the remainder of dividing them,

$$249, 17, 11$$

then write then remainder of dividing the last two on the list:

$$249, 17, 11, 6$$

and repeat

$$249, 17, 11, 6, 5, 1, 0.$$

In more detail, we divide the smaller integer (smaller in absolute value) into that the larger:

$$\begin{array}{r} 14 \\ 17\overline{)249} \\ \underline{17} \\ 79 \\ \underline{68} \\ 11 \end{array}$$

Throw out the larger integer, 249, and replace it by the remainder, 11, and divide again:

$$\begin{array}{r} 1 \\ 11\overline{)17} \\ \underline{11} \\ 6 \end{array}$$

Again we throw out the larger integer (in absolute value), 17, and replace with the remainder, 6, and repeat:

$$\begin{array}{r} 1 \\ 6\overline{)11} \\ \underline{6} \\ 5 \end{array}$$

and again:

$$\begin{array}{r} 1 \\ 5\overline{)6} \\ \underline{5} \\ 1 \end{array}$$

and again:

$$\begin{array}{r} 5 \\ 1\overline{)5} \\ \underline{5} \\ 0 \end{array}$$

The remainder is now zero. The greatest common divisor is therefore 1, the final *nonzero* remainder: $\gcd\{249, 17\} = 1$.

This method to find the greatest common divisor is called the *Euclidean algorithm*.

If the final nonzero remainder is negative, just change its sign to get the greatest common divisor:

$$\gcd\{-4, -2\} = \gcd\{-2, 0\} = \gcd\{-2\} = 2.$$

1.32 Find the greatest common divisor of

a. $4233, 884$

b. $-191, 78$

c. $253, 29$

d. $84, 276$

e. $-92, 876$

f. $147, 637$

g. $266\,664, 877\,769$

1.33* Take a nonempty collection of integers, at least one of which is not zero. Allow yourself to "build new integers" by adding or subtracting the ones you have from one another, repeatedly. Prove that the collection has a greatest common divisor, and it is the smallest positive integer you can "build".

1.34 Show that any collection of integers has a unique greatest common divisor.

1.35 How can you calculate the greatest common divisor of three integers? Four?

The least common multiple

The *least common multiple* of a finite collection of integers $m_1, m_2, \ldots, m_n$ is the smallest positive integer $\ell = \operatorname{lcm}\{m_1, m_2, \ldots, m_n\}$ so that all of the integers $m_1, m_2, \ldots, m_n$ divide ℓ.

Lemma 1.6. *The least common multiple of any two integers b, c (not both zero) is*

$$\operatorname{lcm}\{b,c\} = \frac{|bc|}{\gcd\{b,c\}}.$$

Proof. For simplicity, assume that $b > 0$ and $c > 0$; the cases of $b \leq 0$ or $c \leq 0$ are treated easily by flipping signs as needed and checking what happens when $b = 0$ or when $c = 0$ directly; we leave this to the reader. Let $d := \gcd\{b,c\}$, and factor $b = Bd$ and $c = Cd$. Then B and C are coprime. Write the least common multiple ℓ of b, c as either $\ell = b_1 b$ or as $\ell = c_1 c$, since it is a multiple of both b and c. So then $\ell = b_1 Bd = c_1 Cd$. Cancelling, $b_1 B = c_1 C$. So C divides $b_1 B$, but doesn't divide B, so divides b_1, say $b_1 = b_2 C$, so $\ell = b_1 Bd = b_2 CBd$. So BCd divides ℓ. So $(bc)/d = BCd$ divides ℓ, and is a multiple of b: $BCd = (Bd)C = bC$, and is a multiple of c: $BCd = B(Cd) = Bc$. But ℓ is the least such multiple. □

Sage

Computers are useless. They can only give you answers.

— Pablo Picasso

These lecture notes include optional sections explaining the use of the sage computer algebra system. At the time these notes were written, instructions to install sage on a computer are at www.sagemath.org, but you should be able to try out sage on sagecell.sagemath.org or even create worksheets in sage on cocalc.com over the internet without installing anything on your computer. If we type

```
gcd(1200,1040)
```

and press *shift–enter*, we see the greatest common divisor: 80. Similarly type

```
lcm(1200,1040)
```

and press *shift–enter*, we see the least common multiple: 15600. Sage can carry out complicated arithmetic operations. The multiplication symbol is `*`:

```
12345*13579
```

gives 167632755. Sage uses the expression `2^3` to mean 2^3. You can invent variables in sage by just typing in names for them. For example, ending each line with *enter*, except the last which you end with *shift–enter* (when you want sage to compute results):

```
x=4
y=7
x*y
```

to print 28. The expression `15 % 4` means the remainder of 15 divided by 4, while `15//4` means the quotient of 15 divided by 4. We can calculate the greatest common divisor of several integers as

```
gcd([3800,7600,1900])
```

giving us 1900.

Chapter 2

Mathematical induction

It is sometimes required to prove a theorem which shall be true whenever a certain quantity n which it involves shall be an integer or whole number and the method of proof is usually of the following kind. 1st. The theorem is proved to be true when $n = 1$. 2ndly. It is proved that if the theorem is true when n is a given whole number, it will be true if n is the next greater integer. Hence the theorem is true universally.

— George Boole

So nat'ralists observe, a flea
Has smaller fleas that on him prey;
And these have smaller fleas to bite 'em.
And so proceeds Ad infinitum.

— Jonathan Swift
On Poetry: A Rhapsody

It is often believed that everyone with red hair must have a red haired ancestor. But this principle is not true. We have all seen someone with red hair. According to the principle, she must have a red haired ancestor. By the same principle, he must have a red haired ancestor too, and so on. So the principle predicts an infinite chain of red haired ancestors. But there have only been a finite number of creatures (so the geologists tell us). So some red haired creature had no red haired ancestor.

We will see that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for any positive integer n. First, let's check that this is correct for a few values of n just to be careful. *Danger:* just to be *very* careful, we put $\stackrel{?}{=}$ between any two quantities when we are *checking* to see if they are equal; we are making clear that we don't yet know.

For $n = 1$, we get $1 \stackrel{?}{=} 1(1+1)/2$, which is true because the right hand side is $1(1+1)/2) = 2/2 = 1$.

For $n = 2$, we need to check $1 + 2 \stackrel{?}{=} 2(2+1)/2$. The left hand side of the

equation is 3, while the right hand side is:

$$\frac{2(2+1)}{2} = \frac{2(3)}{2} = 3,$$

So we checked that they are equal.

For $n = 3$, we need to check that $1 + 2 + 3 \stackrel{?}{=} 3(3 + 1)/2$. The left hand side is 6, and you can check that the right hand side is $3(4)/2 = 6$ too. So we checked that they are equal.

But this process will just go on for ever and we will never finish checking all values of n. We want to check *all* values of n, all at once.

Picture a row of dominoes. If we can make the first domino topple, and we can make sure that each domino, if it topples, will make the next domino topple, then they will all topple.

Theorem 2.1 (The Principle of Mathematical Induction)**.** *Take a collection of positive integers. Suppose that* 1 *belongs to this collection. Suppose that, whenever all positive integers less than a given positive integer* n *belong to the collection, then so does the integer* n*. (For example, if* $1, 2, 3, 4, 5$ *belong, then so does* 6*, and so on.) Then the collection consists precisely of all of the positive integers.*

Proof. Let S be the set of positive integers *not* belonging to that collection. If S is empty, then our collection contains all positive integers, so our theorem is correct. But what is S is *not* empty? By the law of well ordering, if S is not empty, then S has a least element, say n. So n is *not* in our collection. But being the least integer not in our collection, all integers $1, 2, \ldots, n - 1$ less than n *are* in our collection. By hypothesis, n is also in our collection, a contradiction to our assumption that S is not empty. □

Let's prove that $1 + 2 + 3 + \cdots + n = n(n + 1)/2$ for any positive integer n. First, note that we have already checked this for $n = 1$ above. Imagine that we have checked our equation $1 + 2 + 3 + \cdots + n = n(n + 1)/2$ for any positive integer n up to, but not including, some integer $n = k$. Now we want to check for $n = k$ whether it is still true: $1 + 2 + 3 + \cdots + n \stackrel{?}{=} n(n + 1)/2$. Since we already know this for $n = k - 1$, we are allowed to write that out, without question marks:

$$1 + 2 + 3 + \cdots + (k - 2) + (k - 1) = (k - 1)(k - 1 + 1)/2.$$

Simplify:

$$1 + 2 + 3 + \cdots + k - 1 = (k - 1)k/2.$$

Now add k to both sides. The left hand side becomes $1 + 2 + 3 + \cdots + k$. The

right hand side becomes $(k-1)k/2 + k$, which we simplify to

$$\begin{aligned}\frac{(k-1)k}{2} + k &= \frac{(k-1)k}{2} + \frac{2k}{2},\\ &= \frac{k^2 - k + 2k}{2},\\ &= \frac{k^2 + k}{2},\\ &= \frac{k(k+1)}{2}.\end{aligned}$$

So we have found that, as we wanted to prove,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

Note that we *used* mathematical induction to prove this: we prove the result for $n = 1$, and then suppose that we have proven it already for all values of n up to some given value, and then show that this will ensure it is still true for the next value.

The general pattern of induction arguments: we start with a statement we want to prove, which contains a variable, say n, representing a positive integer.

a. *The base case:* Prove your statement directly for $n = 1$.
b. *The induction hypothesis:* Assume that the statement is true for all positive integer values less than some value n.
c. *The induction step:* Prove that it is therefore also true for that value of n.

We can use induction in definitions, not just in proofs. We haven't made sense yet of exponents. (Exponents are often called *indices* in Irish secondary schools, but nowhere else in the world to my knowledge).

a. *The base case:* For any integer $a \neq 0$ we define $a^0 := 1$. *Watch:* We can write $:=$ instead of $=$ to mean that this is our *definition* of what a^0 *means*, not an equation we have somehow calculated out. For any integer a (including the possibility of $a = 0$) we also define $a^1 := a$.
b. *The induction hypothesis:* Suppose that we have defined already what
$$a^1, a^2, \ldots, a^b$$
means, for some positive integer b.
c. *The induction step:* We then define a^{b+1} to mean $a^{b+1} := a \cdot a^b$.

For example, by writing out

$$\begin{aligned}a^4 &= a \cdot a^3,\\ &= a \cdot a \cdot a^2,\\ &= a \cdot a \cdot a \cdot a^1,\\ &= \underbrace{a \cdot a \cdot a \cdot a}_{4 \text{ times}}.\end{aligned}$$

Another example:

$$a^3a^2 = (a \cdot a \cdot a)\,(a \cdot a) = a^5.$$

In an expression a^b, the quantity a is the *mantissa* or *base* and b is the *exponent*.

2.1 Use induction to prove that $a^{b+c} = a^b a^c$ for any integer a and for any positive integers b, c.

2.2 Use induction to prove that $\left(a^b\right)^c = a^{bc}$ for any integer a and for any positive integers b, c.

Sometimes we start induction at a value of n which is *not* at $n = 1$.

Let's prove, for any integer $n \geq 2$, that $2^{n+1} < 3^n$.

a. *The base case:* First, we check that this is true for $n = 2$: $2^{2+1} < 3^2$? Simplify to see that this is $8 < 9$, which is clearly true.

b. *The induction hypothesis:* Next, suppose that we know this is true for all values $n = 2, 3, 4, \ldots, k-1$.

c. *The induction step:* We need to check it for $n = k$: $2^{k+1} < 3^k$? How can we relate this to the values $n = 2, 3, 4, \ldots, k-1$?

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k, \\ &< 3 \cdot 3^{k-1} \text{ by assumption,} \\ &= 3^k. \end{aligned}$$

We conclude that $2^{n+1} < 3^n$ for all integers $n \geq 2$.

2.3 All horses are the same colour; we can prove this by induction on the number of horses in a given set.

a. *The base case:* If there's just one horse then it's the same colour as itself, so the base case is trivial.

b. *The induction hypothesis:* Suppose that we have proven that, in any set of at most k horses, all of the horses are the same colour as one another, for any number $k = 1, 2, \ldots, n$.

c. *The induction step:* Assume that there are n horses numbered 1 to n. By the induction hypothesis, horses 1 through $n-1$ are the same color as one another. Similarly, horses 2 through n are the same color. But the middle horses, 2 through $n-1$, can't change color when they're in different groups; these are horses, not chameleons. So horses 1 and n must be the same color as well.

Thus all n horses are the same color. What is wrong with this reasoning?

2.4 We could have assumed much less about the integers than the laws we gave in chapter 1. Using only the laws for addition and induction,

a. Explain how to define multiplication of integers. Prove by induction that your definition uniquely determines the product of any two integers.

b. Use your definition of multiplication to prove the associative law for multiplication.

c. Use your definition of multiplication to prove the equality cancellation law for multiplication.

2.5 Suppose that x, b are positive integers. Prove that x can be written as

$$x = a_0 + a_1 b + a_2 b^2 + \cdots + a_k b^k$$

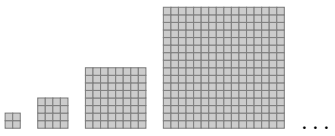
for unique integers $a_0, a_1, \ldots, a_k$ with $0 \leq a_i < b$. Hint: take quotient and remainder, and apply induction. (The sequence $a_k, a_{k-1}, \ldots, a_1, a_0$ is the sequence of digits of x in base b notation.)

2.6 For any positive integer n, prove that there is a unique largest integer k for which $k^2 \leq n$; call it $s(n)$. Suppose that S is a collection of positive integers containing 1, and that, for any positive integer n, if S contains all integers $1, 2, \ldots, s(n)$ then it also contains n. What is S?

2.7 Prove that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

2.8 Picture a 2×2 grid, a 4×4 grid, an 8×8 grid, a 16×16 grid, and so on.



Fix a positive integer n. Show that it is possible to tile any $2^n \times 2^n$ grid, but with exactly one square removed, using 'L'-shaped tiles of three squares: .

Recall that $\binom{n}{k}$ means the binomial coefficient, i.e. the coefficient of x^k (or of x^{n-k}) in the expansion of $(1+x)^n$ in powers of x:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

2.9 Prove that the binomial coefficients satisfy $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$. Explain how this gives Pascal's triangle:

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & 1 & & & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & & & & & & & & \\
 & & & & & & & 1 & & 2 & & 1 & & & & & & & & \\
 & & & & & & 1 & & 3 & & 3 & & 1 & & & & & & & \\
 & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & & \\
 & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & & \\
 & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & & \\
 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & & \\
 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 & & \\
1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1 \\
1 & & 10 & & 45 & & 120 & & 210 & & 252 & & 210 & & 120 & & 45 & & 10 & & 1 \\
1 & & 11 & & 55 & & 165 & & 330 & & 462 & & 462 & & 330 & & 165 & & 55 & & 11 & & 1 \\
1 & & 12 & & 66 & & 220 & & 495 & & 792 & & 924 & & 792 & & 495 & & 220 & & 66 & & 12 & & 1 \\
1 & & 13 & & 78 & & 286 & & 715 & & 1287 & & 1716 & & 1716 & & 1287 & & 715 & & 286 & & 78 & & 13 & & 1 \\
1 & & 14 & & 91 & & 364 & & 1001 & & 2002 & & 3003 & & 3432 & & 3003 & & 2002 & & 1001 & & 364 & & 91 & & 14 & & 1 \\
1 & & 15 & & 105 & & 455 & & 1365 & & 3003 & & 5005 & & 6435 & & 6435 & & 5005 & & 3003 & & 1365 & & 455 & & 105 & & 15 & & 1
\end{array}
$$

2.10 For each positive integer n, let

$$p_n(x) \coloneqq 1 + x + \cdots + x^{n-1} = \sum_{k=0}^{n-1} x^k.$$

Prove that

$$p_n(1+x) = \sum_{k=0}^{n-1} \binom{n}{k+1} x^k.$$

Sage

> *A computer once beat me at chess, but it was no match for me at kick boxing.*
>
> — Emo Philips

To define a function in sage, you type `def`, to mean *define*, like:

```
def f(n):
    return 2*n+1
```

and press *shift–enter*. For any input n, it will return a value $f(n) = 2n + 1$. Careful about the `*` for multiplication. Use your function, for example, by typing

```
f(3)
```

and press *shift–enter* to get 7.

A function can depend on several inputs:

```
def g(x,y):
    return x*y+2
```

A *recursive function* is one whose value for some input depends on its value for other inputs. For example, the sum of the integers from 1 to n is:

```
def sum_of_integers(n):
    if n<=0:
        return 0
    else:
        return n+sum_of_integers(n-1)
```

giving us `sum_of_integers(5)` $= 15$.

2.11 The *Fibonnaci sequence* is $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 3$. Write out a function `F(n)` in sage to compute the Fibonacci sequence.

Lets check that $2^{n+1} < 3^n$ for some values of n, using a loop:

```
for n in [0..10]:
    print(n,3^n-2^(n+1))
```

This will try successively plugging in each value of n starting at $n = 0$ and going up to $n = 1$, $n = 2$, and so on up to $n = 10$, and print out the value of n and the value of $3^n - 2^{n+1}$. It prints out:

```
(0, -1)
(1, -1)
(2, 1)
(3, 11)
(4, 49)
(5, 179)
(6, 601)
(7, 1931)
(8, 6049)
(9, 18659)
(10, 57001)
```

giving the value of n and the value of the difference $3^n - 2^{n+1}$, which we can readily see is positive once $n \geq 2$. Tricks like this help us to check our induction proofs.

If we write `a=a+1` in sage, this means that the new value of the variable `a` is equal to one more than the old value that `a` had previously. Unlike mathematical variables, sage variables can change value over time. In sage the expression `a=b!` means $a \neq b$.

Although sage knows how to calculate the greatest common divisor of some integers, we can define own greatest common divisor function:

```
def my_gcd(a, b):
    if abs(a)<abs(b):
        b,a=a,b
    while b!=0:
        a,b=b,a%b
    return abs(a)
```

We could also write a greatest divisor function that uses repeated subtraction, avoiding the quotient operation %:

```
def gcd_by_subtraction(a, b):
    if a<0:
        a=-a
    if b<0:
        b=-b
    if a==0:
        return b
    if b==0:
        return a
    while a!=b:
        if a>b:
            a=a-b
        else:
            b=b-a
    return a
```

Lists in sage

Sage manipulates lists. A list is a finite sequence of objects written down next to one another. The list `L=[4,4,7,9]` is a list called L which consists of four numbers: $4, 4, 7$ and 9, in that order. Note that the same entry can appear any number of times; in this example, the number 4 appears twice. You can think of a list as like a vector in $\mathbb{R}^n$, a list of numbers. When you "add" lists they concatenate: `[6]+[2]` yields $[6, 2]$. Sage uses notation `L[0]`, `L[1]`, and so on, to retrieve the elements of a list, instead of the usual vector notation. *Warning:* the element `L[1]` is *not* the first element. The length of a list L is `len(L)`. To create a list $[0, 1, 2, 3, 4, 5]$ of successive integer, type `list(range(0,6))`. Note that the number 6 here tells you to go up to, but not include, 6. So a L list of 6 elements has elements $L_0, L_1, \ldots, L_5$, denoted `L[0]`, `L[1]`, ..., `L[5]` in sage. To retrieve the list of elements of L from L_2 to L_4, type `L[2:5]`; again strangely this means up to but not including L_5.

We create a list with 10 entries, print it:

```
L=list(range(0,10))
print(L)
```

yielding $[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]$ and then print its reversal:

```
print(L.reverse())
```

yielding None.

We can construct a list of elements of the Fibonacci sequence:

```
def fibonacci_list(n):
    L=[1,1]
    for i in range(2,n):
        L=L+[L[i-1]+L[i-2]]
    return L
```

so that `fibonacci_list(10)` yields $[1, 1, 2, 3, 5, 8, 13, 21, 34, 55]$, a list of the first ten numbers in the Fibonacci sequence. The code works by first creating a list `L=[1,1]` with the first two entries in it, and then successively building up the list L by concatenating to it a list with one more element, the sum of the two previous list entries, for all values $i = 2, 3, \ldots, n$.

Chapter 3

Greatest common divisors

The laws of nature are but the mathematical thoughts of God.

— Euclid

Euclid alone has looked on Beauty bare.
Let all who prate of Beauty hold their peace,
And lay them prone upon the earth and cease
To ponder on themselves, the while they stare
At nothing, intricately drawn nowhere
In shapes of shifting lineage; let geese
Gabble and hiss, but heroes seek release
From dusty bondage into luminous air.
O blinding hour, O holy, terrible day,
When first the shaft into his vision shone
Of light anatomized! Euclid alone
Has looked on Beauty bare. Fortunate they
Who, though once only and then but far away,
Have heard her massive sandal set on stone.

— Edna St. Vincent Millay
Euclid Alone

The extended Euclidean algorithm

Theorem 3.1 (Bézout)**.** *For any two integers b, c, there are integers s, t so that $sb + tc = \gcd\{b, c\}$.*

These *Bézout coefficients* s, t play an essential role in advanced arithmetic, as we will see.

To find the Bézout coefficients of $12, 8$, write out a matrix

$$\begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & 8 \end{pmatrix}.$$

Repeatedly add some integer multiple of one row to the other, to try to make the bigger number in the last column (bigger by absolute value) get smaller

(by absolute value). In this case, we can subtract row 2 from row 1, to get rid of as much of the 12 as possible. All operations are carried out a whole row at a time:

$$\begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & 8 \end{pmatrix}$$

Now the bigger number (by absolute value) in the last column is the 8. We subtract as large an integer multiple of row 1 from row 2, in this case subtract 2 (row 1) from row 2, to kill as much of the 8 as possible:

$$\begin{pmatrix} 1 & -1 & 4 \\ -2 & 3 & 0 \end{pmatrix}$$

Once we get a zero in the last column, the other entry in the last column is the greatest common divisor, and the entries in the same row as the greatest common divisor are the Bézout coefficients. In our example, the Bézout coefficients of $12, 8$ are $1, -1$ and the greatest common divisor is 4. This trick to calculate Bézout coefficients is the *extended Euclidean algorithm.*

Again, we have to be a little bit careful about minus signs. Find Bézout coefficients of $-8, -4$:

$$\begin{pmatrix} 1 & 0 & -8 \\ 0 & 1 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -4 \end{pmatrix},$$

the process stops here, and where we expect to get an equation $sb + tc = \gcd\{b, c\}$, instead we get

$$0(-8) + 1(-4) = -4,$$

which has the wrong sign, a negative greatest common divisor. So if the answer pops out a negative for the greatest common divisor, we change signs across the whole row: $s = 0, t = -1, \gcd\{b, c\} = 4$.

Theorem 3.2. *The extended Euclidean algorithm calculates Bézout coefficients. In particular, Bézout coefficients s, t exist for any integers b, c with $c \neq 0$.*

Proof. At each step in the extended Euclidean algorithm, the third column proceeds exactly by the Euclidean algorithm, replacing the larger number (in absolute value) with the remainder by the division, except for perhaps a minus sign. So at the last step, the last nonzero number in the third column is the greatest common divisor, except for perhaps a minus sign.

Our extended Euclidean algorithm starts with

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix},$$

a matrix which satisfies

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \end{pmatrix} \begin{pmatrix} b \\ c \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

It is easy to check that if some 2×3 matrix M satisfies

$$M \begin{pmatrix} b \\ c \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then so does the matrix you get by adding any multiple of one row of M to the other row.

Eventually we get a zero in the final column, say

$$\begin{pmatrix} s & t & d \\ S & T & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} s & t & 0 \\ S & T & D \end{pmatrix}.$$

It doesn't matter which since, at any step, we could always swap the two rows without changing the steps otherwise. So suppose we end up at

$$\begin{pmatrix} s & t & d \\ S & T & 0 \end{pmatrix}.$$

But

$$\begin{pmatrix} s & t & d \\ S & T & 0 \end{pmatrix} \begin{pmatrix} b \\ c \\ -1 \end{pmatrix} = \begin{pmatrix} sb + tc - d \\ 0 \end{pmatrix}.$$

This has to be zero, so: $sb + tc = d$. □

Proposition 3.3. *Take two integers b, c, not both zero. The number $\gcd\{b, c\}$ is precisely the smallest positive integer which can be written as $sb + tc$ for some integers s, t.*

Proof. Let $d := \gcd\{b, c\}$. Taking Bézout coefficients, d is a positive integer of that form. Since d divides into b and c it divides into $sb + tc$ for all integers s, t, so d divides into all positive integers of that form, so there is no smaller one than d by proposition 1.3 on page 5. □

3.1 Find Bézout coefficients and greatest common divisors of

a. $2468, 180$

b. $79, -22$

c. $45, 16$

d. $-1000, 2002$

3.2 How would you find the greatest common divisor of several integers? The Bézout coefficients?

Sage

> *Computer science is no more about computers than astronomy is about telescopes.*
>
> — Edsger Dijkstra

The extended Euclidean algorithm is built in to sage: `xgcd(12,8)` returns $(4, 1, -1)$, the greatest common divisor and the Bézout coefficients, so that $4 = 1 \cdot 12 + (-1) \cdot 8$. We can write our own function to compute Bézout coefficients:

```
def bez(b,c):
    p,q,r=1,0,b
    s,t,u=0,1,c
    while u!=0:
        if abs(r)>abs(u):
            p,q,r,s,t,u=s,t,u,p,q,r
        Q=u//r
        s,t,u=s-Q*p,t-Q*q,u-Q*r
    return r,p,q
```

so that `bez(45,210)` returns a triple (g, s, t) where g is the greatest commmon divisor of $45, 210$, while s, t are the Bézout coefficients.

Chapter 4

Prime numbers

I think prime numbers are like life. They are very logical but you could never work out the rules, even if you spent all your time thinking about them.

— Mark Haddon
The Curious Incident of the Dog in the Night-Time

Prime numbers

A positive integer $p \geq 2$ is *prime* if the only positive integers which divide p are 1 and p.

4.1 Prove that 2 is prime and that 3 is prime and that 4 is not prime.

Lemma 4.1. *If b, c, d are integers and d divides bc and d is coprime to b then d divides c.*

Proof. Since d divides the product bc, we can write the product as $bc = qd$. Since d and b are coprime, their greatest common divisor is 1. So there are Bézout coefficients s, t for b, d, so $sb + td = 1$. So then $sbc + tdc = c$, i.e. $sqd + tdc = c$, or $d(sq + dc) = c$, i.e. d divides c. □

Corollary 4.2. *A prime number divides a product bc just when it divides one of the factors b or c.*

An expression of a positive integer as a product of primes, such as $12 = 2 \cdot 2 \cdot 3$, written down in increasing order, is a *prime factorization*.

Theorem 4.3. *Every positive integer n has a unique prime factorization.*

Proof. Danger: if we multiply together a collection of integers, we must insist that there can only be finitely many numbers in the collection, to make sense out of the multiplication.

More danger: an *empty* collection is always allowed in mathematics. We will simply say that if we have *no* numbers at all, then the product of those numbers (the product of no numbers at all) is defined to mean 1. This will be the right definition to use to make our theorems have the simplest expression. In particular, the integer 1 has the prime factorization consist of *no* primes.

First, let's show that there is a prime factorization, and then we will see that it is unique. It is clear that $1, 2, 3, \ldots, 12$ have prime factorizations, as in our table above.

$$
\begin{aligned}
2 &= 2,\\
3 &= 3,\\
4 &= 2 \cdot 2,\\
5 &= 5,\\
6 &= 2 \cdot 3,\\
7 &= 7,\\
8 &= 2 \cdot 2 \cdot 2,\\
9 &= 3 \cdot 3,\\
10 &= 2 \cdot 5,\\
11 &= 11,\\
12 &= 2 \cdot 2 \cdot 3,\\
&\vdots\\
2395800 &= 2^3 \cdot 3^2 \cdot 5^2 \cdot 11^3,\\
&\vdots
\end{aligned}
$$

Suppose that all integers $1, 2, \ldots, n-1$ have prime factorizations. If n does not factor into smaller factors, then n is prime and $n = n$ is a factorization. Suppose that n factors, say into a product $n = bc$ of positive integers, neither equal to 1. Write down a prime factorization for b and then next to it one for c, giving a prime factorization for $n = bc$. So every positive integer has at least one prime factorization.

Let p be the smallest integer so that $p \geq 2$ and p divides n. Clearly p is prime. Since p divides the product of the primes in any prime factorization of n, p divides one of the primes in the factorization. But then p must equal one of these primes, and must be the smallest prime in the factorization. This determines the first prime in the factorization. So write $n = pn_1$ for a unique integer n_1, and by induction we can assume that n_1 has a unique prime factorization, and so n does as well. □

Theorem 4.4 (Euclid)**.** *There are infinitely many prime numbers.*

Proof. Write down a list of finitely many primes, say $p_1, p_2, \ldots, p_n$. Let $b := p_1 p_2 \ldots p_n$ and let $c := b + 1$. Then clearly b, c are coprime. So the prime decomposition of c has none of the primes $p_1, p_2, \ldots, p_n$ in it, and so must have some other primes distinct from these. □

4.2 Write down the positive integers in a long list starting at 2:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \ldots$$

Strike out all multiples of 2, except 2 itself:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \ldots$$

Skip on to the next number which isn't struck out: 3. Strike out all multiples of 3, except 3 itself:

$$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \ldots$$

Skip on to the next number which isn't struck out, and strike out all of its multiples except the number itself. Continue in this way forever. Prove that the remaining numbers, those which do *not* get struck out, are precisely the prime numbers. Use this to write down all prime numbers less than 120.

Greatest common divisors

We speed up calculation of the greatest common divisor by factoring out any obvious common factors: $\gcd\{46, 12\} = \gcd\{2 \cdot 23, 2 \cdot 6\} = 2\gcd\{23, 6\} = 2$. For example, this works well if both numbers are even, or both are multiples of ten, or both are multiples of 5, or even if both are multiples of 3.

4.3 Prove that an integer is a multiple of 3 just when its digits sum up to a multiple of 3. Use this to see if 12345 is a multiple of 3. Use this to find $\gcd\{12345, 123456\}$.

4.4 An *even integer* is a multiple of 2; an *odd integer* is *not* a multiple of 2. Prove that if b, c are positive integers and

a. b and c are both even then $\gcd\{b, c\} = 2\gcd\{b/2, c/2\}$,

b. b is even and c is odd, then $\gcd\{b, c\} = \gcd\{b/2, c\}$,

c. b is odd and c is even, then $\gcd\{b, c\} = \gcd\{b, c/2\}$,

d. b and c are both odd, then $\gcd\{b, c\} = \gcd\{b, b - c\}$.

Use this to find the greatest common divisor of $4864, 3458$ without using long division. Answer: $\gcd\{4864, 3458\} = 19$.

4.5 Prove that any integer divides some integers just when it divides their greatest common divisor.

4.6 Prove that, for any positive integer m and integers b, c, not both zero, $m \gcd\{b, c\} = \gcd\{mb, mc\}$.

Sage

> *Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.*
>
> — Leonhard Euler

Sage can factor numbers into primes: `factor(1386)` gives $2 \cdot 3^2 \cdot 7 \cdot 11$. Sage can find the next prime number larger than a given number: `next_prime(2005)`= 2011. You can test if the number 4 is prime with `is_prime(4)`, which returns `False`. To list the primes between 14 and 100, type `prime_range(14,100)` to see

```
[17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89,
97]
```

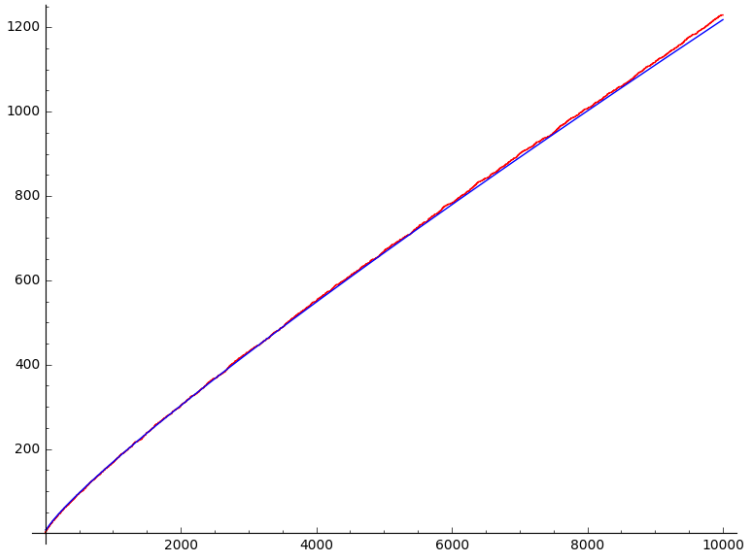
The number of primes less than or equal to a real number x is traditionally denoted $\pi(x)$ (which has nothing to do with the π of trigonometry). In sage, $\pi(x)$ is denoted `prime_pi(x)`. For example, $\pi(10^6) = 78498$ is calculated as `prime_pi(10^6)`. The prime number theorem says that $\pi(x)$ gets more closely approximated by

$$\frac{x}{-1 + \log x}$$

as x gets large. We can plot $\pi(x)$, and compare it to that approximation.

```
p=plot(prime_pi, 0, 10000, rgbcolor='red')
q=plot(x/(-1+log(x)), 5, 10000, rgbcolor='blue')
show(p+q)
```

which displays:



4.7 Write code in sage to find the greatest common divisor of two integers, using the ideas of problem 4.4 on page 26.

Chapter 5

Modular arithmetic

Mathematics is the queen of the sciences and number theory is the queen of mathematics.

— Carl Friedrich Gauss

Definition

If we divide integers by 7, the possible remainders are

$$0, 1, 2, 3, 4, 5, 6.$$

For example, $65 = 9 \cdot 7 + 2$, so the remainder is 2. Two integers are *congruent mod* 7 if they have the same remainder modulo 7. So 65 and 2 are congruent mod 7, because $65 = 9 \cdot 7 + 2$ and $2 = 0 \cdot 7 + 2$. We denote congruence modulo 7 as

$$65 \equiv 2 \pmod 7.$$

Sometimes we will allow ourselves a sloppy notation, where we write the remainder of 65 modulo 7 as $\overline{65}$. This is sloppy because the notation doesn't remind us that we are working out remainders modulo 7. If we change 7 to some other number, we could get confused by this notation. We will often compute remainders modulo some chosen number, say m, instead of 7.

If we add multiples of 7 to an integer, we don't change its remainder modulo 7:

$$\overline{65 + 7} = \overline{65}.$$

Similarly,

$$\overline{65 - 7} = \overline{65}.$$

If we add, or multiply, some numbers, what happens to their remainders?

Theorem 5.1. *Take a positive integer m and integers a, A, b, B. If $a \equiv A \pmod m$ and $b \equiv B \pmod m$ then*

$$\begin{aligned} a + b &\equiv A + B \pmod m, \\ a - b &\equiv A - B \pmod m, \\ \text{and } ab &\equiv AB \pmod m. \end{aligned}$$

The bar notation is more elegant. If we agree that $\bar{a}$ means remainder of a modulo the fixed choice of integer m, we can write this as: if

$$\bar{a} = \bar{A} \text{ and } \bar{b} = \bar{B}$$

then

$$\begin{aligned} \overline{a+b} &= \overline{A+B}, \\ \overline{a-b} &= \overline{A-B}, \text{ and} \\ \overline{ab} &= \overline{AB}. \end{aligned}$$

Note that $9 \equiv 2 \pmod 7$ and $4 \equiv 11 \pmod 7$, so our theorem tells us that $9 \cdot 4 \equiv 2 \cdot 11 \pmod 7$. Let's check this. The left hand side:

$$\begin{aligned} 9 \cdot 4 &= 36, \\ &= 5(7) + 1, \end{aligned}$$

so that $9 \cdot 4 \equiv 1 \pmod 7$. The right hand side:

$$\begin{aligned} 2 \cdot 11 &= 22, \\ &= 3(7) + 1, \end{aligned}$$

so that $2 \cdot 11 \equiv 1 \pmod 7$.

So it works in this example. Let's prove that it always works.

Proof. Since $a - A$ is a multiple of m, as is $b - B$, note that

$$(a+b) - (A+B) = (a-A) + (b-B)$$

is also a multiple of m, so addition works. In the same way

$$ab - AB = (a-A)b + A(b-B)$$

is a multiple of m. □

5.1 Prove by induction that every "perfect square", i.e. integer of the form n^2, has remainder $0, 1$ or 4 modulo 8.

5.2 Take an integer like 243098 and write out the sum of its digits $2+4+3+0+9+8$. Explain why every integer is congruent to the sum of its digits, modulo 9.

5.3 Take integers a, b, m with $m \neq 0$. Let $d := \gcd\{a, b, m\}$. Prove that $a \equiv b \pmod m$ just when $a/d \equiv b/d \pmod{m/d}$.

Arithmetic of remainders

We now *define* an addition law on the numbers $0, 1, 2, 3, 4, 5, 6$ by declaring that when we add these numbers, we then take remainder modulo 7. This is *not* usual addition. To make this clear, we write the remainders with bars over them, always. For example, we are saying that in this addition law $\bar{3} + \bar{5}$ means $\overline{3+5} = \bar{8} = \bar{1}$,

since we are working modulo 7. We adopt the same rule for subtraction, and for multiplication. For example, modulo 13,

$$\begin{aligned}\left(\overline{7}+\overline{9}\right)\left(\overline{11}+\overline{6}\right) &= \overline{16}\cdot\overline{17},\\ &= \overline{13+3}\cdot\overline{13+4},\\ &= \overline{3}\cdot\overline{4},\\ &= \overline{12}.\end{aligned}$$

If we are daring, we might just drop all of the bars, and state clearly that we are calculating modulo some integer. In our daring notation, modulo 17,

$$\begin{aligned}16\cdot 29-7\cdot 5 &= 16\cdot 12-7\cdot 5,\\ &= 192-35,\\ &= (11\cdot 17+5)-(2\cdot 17+1),\\ &= 5-1,\\ &= 4.\end{aligned}$$

5.4 Expand and simplify $\overline{5}\cdot\overline{2}\cdot\left(\overline{6}-\overline{9}\right)$ modulo 7.

The addition and multiplication tables for remainder modulo 5:

+	0	1	2	3	4
0	1	2	3	4	0
1	2	3	4	0	1
2	3	4	0	1	2
3	4	0	1	2	3
4	0	1	2	3	4

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

5.5 Describe laws of modular arithmetic, imitating the laws of integer arithmetic. If we work modulo 4, explain why the zero divisors law fails. Why are there no sign laws?

5.6 Compute the remainder when dividing 19 into 37^{200}.

5.7 Compute the last two digits of 9^{2000}.

5.8 Prove that the equation $a^2+b^2=3c^2$ has no solutions in nonzero integers a, b and c. Hint: start by proving that modulo 4, $a^2=0$ or 1. Then consider the equation modulo 4; show that a, b and c are divisible by 2. Then each of a^2, b^2 and c^2 has a factor of 4. Divide through by 4 to show that there would be a smaller set of solutions to the original equation. Apply induction.

To carry a remainder to a huge power, say 2^{2005} modulo 13, we can build up the power out of smaller ones. For example, $2^2=4$ modulo 13, and therefore

modulo 13,

$$\begin{aligned} 2^4 &= \left(2^2\right)^2, \\ &= 4^2, \\ &= 16, \\ &= 3. \end{aligned}$$

Keeping track of these partial results as we go, modulo 13,

$$\begin{aligned} 2^8 &= \left(2^4\right)^2, \\ &= 3^2, \\ &= 9. \end{aligned}$$

We get higher and higher powers of 2: modulo 13,

k	2^k	$2^{2^k} \bmod 13$
0	1	2
1	2	4
2	4	3
3	8	9
4	16	$9^2 = 81 = 3$
5	32	$3^2 = 9$
6	64	$9^2 = 3$
7	128	$3^2 = 9$
8	256	$9^2 = 3$
9	512	$3^2 = 9$
10	1024	$9^2 = 3$
11	2048	

The last row gets into 2^{2048}, too large to be relevant to our problem. We now want to write out exponent 2005 as a sum of powers of 2, by first dividing in 1024:

$$2005 = 1024 + 981$$

and then dividing in the next power of 2 we can fit into the remainder,

$$\begin{aligned} &= 1024 + 512 + 469, \\ &= 1024 + 512 + 256 + 128 + 64 + 16 + 4 + 1. \end{aligned}$$

Then we can compute out modulo 13:

$$\begin{aligned} 2^{2005} &= 2^{1024+512+256+128+64+16+4+1}, \\ &= 2^{1024}2^{512}2^{256}2^{128}2^{64}2^{16}2^{4}2^{1}, \\ &= 3 \cdot 9 \cdot 3 \cdot 9 \cdot 3 \cdot 3 \cdot 3 \cdot 2, \\ &= (3 \cdot 9)^3 \cdot 2, \\ &= 27^2 \cdot 2, \\ &= 1^2 \cdot 2, \\ &= 2. \end{aligned}$$

5.9 Compute 2^{100} modulo 125.

5.10 Use modular arithmetic to prove that the integer coefficient polynomial

$$b(x) = x^4 - 2x - 7$$

has no integer roots.

Reciprocals

Every nonzero rational number b/c has a reciprocal: c/b. Since we now have modular arithmetic defined, we want to know which remainders have "reciprocals". Working modulo some positive integer, say that a remainder x has a reciprocal $y = x^{-1}$ if $xy = 1$. (It seems just a little too weird to write it as $y = 1/x$, but you can if you like.) Reciprocals are also called *multiplicative inverses*.

Modulo 7

$$\begin{aligned}
\bar{1} \cdot \bar{1} &= \bar{1}, \\
\bar{2} \cdot \bar{4} &= \bar{1}, \\
\bar{3} \cdot \bar{5} &= \bar{1}, \\
\bar{4} \cdot \bar{2} &= \bar{1}, \\
\bar{5} \cdot \bar{3} &= \bar{1}, \\
\bar{6} \cdot \bar{6} &= \bar{1}.
\end{aligned}$$

So in this weird type of arithmetic, we can allow ourselves the freedom to write these equations as identifying a reciprocal.

$$\begin{aligned}
\bar{1}^{-1} &= \bar{1}, \\
\bar{2}^{-1} &= \bar{4}, \\
\bar{3}^{-1} &= \bar{5}, \\
\bar{4}^{-1} &= \bar{2}, \\
\bar{5}^{-1} &= \bar{3}, \\
\bar{6}^{-1} &= \bar{6}.
\end{aligned}$$

A remainder that has a reciprocal is a *unit*.

5.11 *Danger:* If we work modulo 4, then prove that $\bar{2}$ has *no* reciprocal. Hint: $\bar{2}^2 = 0$.

5.12 Prove that, modulo any integer m, $(m-1)^{-1} = m - 1$.

Theorem 5.2. *Take a positive integer m and a remainder $\bar{r}$ modulo m. Take the Bézout coefficients of r and m: $sr + tm = d$, so that d is the greatest common divisor of r and m.*

If $d = 1$ then $\bar{r}^{-1} = \bar{s}$ modulo m. If $d \neq 1$ then $\bar{r}^{-1}$ does not exist. In particular, in the remainders modulo m, a remainder $\bar{r}$ is a unit just when r, m are coprime integers.

Proof. If r, m are coprime integers, so their greatest common divisor is 1, then write Bézout coefficients $sr + tm = 1$, and quotient by m:

$$\bar{s}\bar{r} = \bar{1}.$$

On the other hand, if

$$\bar{s}\bar{r} = \bar{1},$$

then sr is congruent to 1 modulo m, i.e. there is some quotient q so that $sr = qm + 1$, so $sr - qm = 1$, giving Bézout coefficients $s = s, t = -q$, so the greatest common divisor of r, m is 1. □

Working modulo 163, let's compute 14^{-1}. First we carry out the long division

$$\begin{array}{r} 11 \\ 14\overline{)\,163} \\ \underline{14} \\ 23 \\ \underline{14} \\ 9 \end{array}$$

Now let's start looking for Bézout coefficients, by writing out matrix:

$$\begin{pmatrix} 1 & 0 & 14 \\ 0 & 1 & 163 \end{pmatrix}$$

and then add $-11 \cdot$ row 1 to row 2:

$$\begin{pmatrix} 1 & 0 & 14 \\ -11 & 1 & 9 \end{pmatrix}.$$

Add $-$row 2 to row 1:

$$\begin{pmatrix} 12 & -1 & 5 \\ -11 & 1 & 9 \end{pmatrix}.$$

Add $-$row 1 to row 2:

$$\begin{pmatrix} 12 & -1 & 5 \\ -23 & 2 & 4 \end{pmatrix}.$$

Add $-$row 2 to row 1:

$$\begin{pmatrix} 35 & -3 & 1 \\ -23 & 2 & 4 \end{pmatrix}.$$

Add $-4 \cdot$ row 1 to row 2:

$$\begin{pmatrix} 35 & -3 & 1 \\ -163 & 14 & 0 \end{pmatrix}.$$

Summing it all up: $35 \cdot 14 + (-3) \cdot 163 = 1$. Quotient out by 163: modulo 163, $35 \cdot 14 = 1$, so modulo 163, $14^{-1} = 35$.

5.13 Use this method to find reciprocals:

a. 13^{-1} modulo 59

b. 10^{-1} modulo 11

c. 2^{-1} modulo 193.

d. 6003722857^{-1} modulo 77695236973.

5.14 Suppose that b, c are remainders modulo a prime. Prove that $bc = 0$ just when either $b = 0$ or $c = 0$.

5.15 Suppose that p is a prime number and n is an integer with $n < p$. Explain why, modulo p, the numbers $0, n, 2n, 3n, \ldots, (p-1)n$ consist in precisely the remainders $0, 1, 2, \ldots, p-1$, in some order. (Hint: use the reciprocal of n.) Next, since every nonzero remainder has a reciprocal remainder, explain why the product of the nonzero remainders is 1. Use this to explain why

$$n^{p-1} \equiv 1 \pmod{p}.$$

Finally, explain why, for any integer k,

$$k^p \equiv k \pmod{p}.$$

Chapter 6

The Chinese remainder theorem

An old woman goes to market and a horse steps on her basket and crushes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same happened when she picked them out three, four, five, and six at a time, but when she took them seven at a time, there were none left. What is the smallest number of eggs she could have had?

— Brahmagupta (580CE–670CE)
Brahma-Sphuta-Siddhanta (Brahma's Correct System)

The Chinese remainder theorem

Take a look at some numbers and their remainders modulo 3 and 5:

n	$n \bmod 3$	$n \bmod 5$
0	0	0
1	1	1
2	2	2
3	0	3
4	1	4
5	2	0
6	0	1
7	1	2
8	2	3
9	0	4
10	1	0
11	2	1
12	0	2
13	1	3
14	2	4
15	0	0

Remainders modulo 3 repeat every 3, and remainders modulo 5 repeat every 5, but the pair of remainders modulo 3 and 5 together repeat every 15.

How can you find an integer x so that

$$\begin{aligned} x &\equiv 1 \pmod 3, \\ x &\equiv 2 \pmod 4, \\ x &\equiv 1 \pmod 7? \end{aligned}$$

A recipe: how to find an unknown integer x given only the knowledge of its remainders modulo various integers. Suppose we know x has remainder r_1 modulo m_1, r_2 modulo m_2, $\dots r_n$ modulo m_n. Let

$$m := m_1 m_2 \dots m_n.$$

For each i, let

$$u_i := \frac{m}{m_i} = m_1 m_2 \dots m_{i-1} \cancel{m_i} m_{i+1} m_{i+2} \dots m_n,$$

dropping the m_i factor. Each u_i has some reciprocal modulo m_i, given as the remainder of some integer v_i. Let

$$x := r_1 u_1 v_1 + r_2 u_2 v_2 + \dots + r_n u_n v_n.$$

We can simplify this a little: add or subtract multiples of m until x is in the range $0 \le x \le m - 1$.

Theorem 6.1. *Take some positive integers $m_1, m_2, \dots, m_n$, coprime to one another. The recipe above works: it finds an integer x with required remainder, and x is unique modulo*

$$m_1 m_2 \dots m_n.$$

Proof. All of $m_1, m_2, \dots, m_n$ divide u_i, except m_i. So if $j \neq i$ then modulo m_j, $u_i = 0$. All of the other m_j are coprime to m_i, so their product is coprime to m_i, so has a reciprocal modulo m_i. Then modulo m_i, $x = r_i$. So the recipe gives the required integer x. If there are two, then their difference vanishes modulo all of the m_i, so is a multiple of every one of the m_i, which are coprime, so is a multiple of their product m. □

Let's find an integer x so that

$$\begin{aligned} x &\equiv 1 \pmod 3, \\ x &\equiv 2 \pmod 4, \\ x &\equiv 1 \pmod 7. \end{aligned}$$

So in this problem we have to work modulo $(m_1, m_2, m_3) = (3, 4, 7)$, and get remainders $(r_1, r_2, r_3) = (1, 2, 1)$. First, no matter what the remainders, we have to work out the reciprocal mod each m_i of the product of all of the other

m_j. So let's reduce these products down to their remainders:

$$\begin{aligned} u_1 &= \cancel{3} \cdot 4 \cdot 7 = 28, \\ u_2 &= 3 \cdot \cancel{4} \cdot 7 = 21, \\ u_3 &= 3 \cdot 4 \cdot \cancel{7} = 12. \end{aligned}$$

Reduce to remainders:

$$\begin{aligned} u_1 &= 9 \cdot 3 + 1 \equiv 1 \pmod 3, \\ u_2 &= 5 \cdot 4 + 1 \equiv 1 \pmod 4, \\ u_3 &= 1 \cdot 7 + 5 \equiv 5 \pmod 7. \end{aligned}$$

We need the reciprocals of these, which, to save ink, we just write down for you without writing out the calculations:

$$\begin{aligned} v_1 &\equiv 1^{-1} \equiv 1 \pmod 3, \\ v_2 &\equiv 1^{-1} \equiv 1 \pmod 4, \\ v_3 &\equiv 5^{-1} \equiv 3 \pmod 7. \end{aligned}$$

(You can easily check those.) Compute out the products $u_i v_i$:

$$\begin{aligned} u_1 v_1 &= 28 \cdot 1 = 28, \\ u_2 v_2 &= 21 \cdot 1 = 21, \\ u_3 v_3 &= 12 \cdot 3 = 36. \end{aligned}$$

It is only at this final stage that we use the remainders: add up remainder times product:

$$\begin{aligned} x &= r_1 u_1 v_1 + r_2 u_2 v_2 + r_3 u_3 v_3, \\ &= 1 \cdot 28 + 2 \cdot 21 + 1 \cdot 36, \\ &= 106. \end{aligned}$$

We can now check to be sure:

$$\begin{aligned} 106 &\equiv 1 \pmod 3, \\ 106 &\equiv 2 \pmod 4, \\ 106 &\equiv 1 \pmod 7. \end{aligned}$$

The Chinese remainder theorem tells us also that 106 is the unique solution modulo $3 \cdot 4 \cdot 7 = 84$. But then $106 - 84 = 22$ is also a solution, the smallest positive solution.

6.1 Solve the problem about eggs. Hint: ignore the information about eggs taken out six at a time.

6.2 Use the Chinese remainder theorem to determine the smallest number of soldiers possible in Han Xin's army if the following facts are true. When they parade in rows

of three soldiers, two soldiers will be left. When they parade in rows of five, three will be left, and in rows of seven, two will be left.

6.3 Use the Chinese remainder theorem to determine the smallest number of soldiers possible in Han Xin's army if the following facts are true. When they parade in rows of three soldiers, one soldier will be left. When they parade in rows of seven, two will be left, and in rows of 19, three will be left.

6.4 Use the Chinese remainder theorem to find the smallest positive integer x so that

$$\begin{aligned} x &= 2 \pmod{22}, \\ x &= 8 \pmod{39}. \end{aligned}$$

6.5 Use the Chinese remainder theorem to find the smallest positive integer x so that

$$\begin{aligned} x &= 3 \pmod{4}, \\ x &= 4 \pmod{5}, \\ x &= 0 \pmod{17}. \end{aligned}$$

Bird's eye view of the Chinese remainder theorem

Given some integers $m_1, m_2, \ldots, m_n$, we consider sequences $(b_1, b_2, \ldots, b_n)$ consisting of remainders: b_1 a remainder modulo m_1, and so on. Add sequences of remainders in the obvious way:

$$(b_1, b_2, \ldots, b_n) + (c_1, c_2, \ldots, c_n) = (b_1 + c_1, b_2 + c_2, \ldots, b_n + c_n)\,.$$

Similarly, we can subtract and multiply sequences of remainders:

$$(b_1, b_2, \ldots, b_n)\,(c_1, c_2, \ldots, c_n) = (b_1 c_1, b_2 c_2, \ldots, b_n c_n)\,,$$

by multiplying remainders as usual, modulo the various $m_1, m_2, \ldots, m_n$.

Modulo $(3, 5)$, we multiply

$$\begin{aligned} (2,4)(3,2) &= (2 \cdot 3, 4 \cdot 2), \\ &= (6, 8), \\ &= (0, 3). \end{aligned}$$

Let $m := m_1 m_2 \ldots m_n$. To each remainder modulo m, say b, associate its remainder b_1 modulo m_1, b_2 modulo m_2, and so on. Associate the sequence $\vec{b} := (b_1, b_2, \ldots, b_n)$ of all of those remainders. In this way we make a map taking each remainder b modulo m to its sequence $\vec{b}$ of remainders modulo all of the various m_i. Moreover, $\overrightarrow{b + c} = \vec{b} + \vec{c}$ and $\overrightarrow{b - c} = \vec{b} - \vec{c}$ and $\overrightarrow{bc} = \vec{b}\vec{c}$, since each of these works when we take remainder modulo anything.

Take m_1, m_2, m_3 to be $3, 4, 7$. Then $m = 3 \cdot 4 \cdot 7 = 84$. If $b = 8$ modulo 84,

then

$$\begin{aligned} b_1 &= 8 \mod 3, \\ &= 2, \\ b_2 &= 8 \mod 4, \\ &= 0, \\ b_3 &= 8 \mod 7, \\ &= 1, \\ \vec{b} &= (b_1, b_2, b_3), \\ &= (2, 0, 1) \mod (3, 4, 7). \end{aligned}$$

Corollary 6.2. *Take some positive integers $m_1, m_2, \ldots, m_n$, so that any two of them are coprime. Let $m := m_1 m_2 \ldots m_n$. The map taking b to $\vec{b}$, from remainders modulo m to sequences of remainders modulo $m_1, m_2, \ldots, m_n$, is one-to-one and onto, identifies sums with sums, products with products, differences with differences, units with sequences of units.*

Euler's totient function

Euler's totient function ϕ assigns to each positive integer $m = 2, 3, \ldots$ the number of all remainders modulo m which are units (in other words, coprime to m) [in other words, which have reciprocals]. It is convenient to *define* $\phi(1) := 1$.

6.6 Explain by examining the remainders that the first few values of ϕ are

m	$\phi(m)$
1	1
2	1
3	2
4	2
5	4
6	2
7	6

6.7 Prove that a positive integer $m \geq 2$ is prime just when $\phi(m) = m - 1$.

Theorem 6.3. *Suppose that $m \geq 2$ is an integer with prime factorizaton*

$$m = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n},$$

so that $p_1, p_2, \ldots, p_n$ are prime numbers and $a_1, a_2, \ldots, a_n$ are positive integers. Then

$$\phi(m) = \left(p_1^{a_1} - p_1^{a_1 - 1}\right)\left(p_2^{a_2} - p_2^{a_2 - 1}\right) \ldots \left(p_n^{a_n} - p_n^{a_n - 1}\right).$$

Proof. If m is prime, this follows from problem 6.7.

Suppose that m has just one prime factor, or in other words that $m = p^a$ for some prime number p and integer a. It is tricky to count the remainders coprime to m, but

easier to count those not coprime, i.e. those which have a factor of p. Clearly these are the multiples of p between 0 and $p^a - p$, so the numbers pj for $0 \leq j \leq p^{a-1} - 1$. So there are p^{a-1} such remainders. We take these out and we are left with $p^a - p^{a-1}$ remainders left, i.e. coprime.

If b, c are coprime integers ≥ 2, then corollary 6.2 on the preceding page maps units modulo bc to pairs of a unit modulo b and a unit modulo c, and is one-to-one and onto. Therefore counting units: $\phi(bc) = \phi(b)\phi(c)$. □

Theorem 6.4 (Euler)**.** *For any positive integer m and any integer b coprime to m,*

$$b^{\phi(m)} \equiv 1 \pmod{m}.$$

Proof. Working with remainders modulo m, we have to prove that for any unit remainder b, $b^{\phi(m)} = 1$ modulo m.

Let U be the set of all units modulo m, so U is a subset of the remainders $0, 1, 2, \ldots, m-1$. The product of units is a unit, since it has a reciprocal (the product of the reciprocals). Therefore the map

$$u \in U \mapsto bu \in U$$

is defined. It has an inverse:

$$u \in U \mapsto b^{-1}u \in U,$$

where b^{-1} is the reciprocal of b. Writing out the elements of U, say as

$$u_1, u_2, \ldots, u_q$$

note that $q = \phi(m)$. Then multiplying by b scrambles these units into a different order:

$$bu_1, bu_2, \ldots, bu_q.$$

But multiplying by b just scrambles the order of the roots, so if we multiply them all, and then scramble them back into order:

$$(bu_1)(bu_2)\ldots(bu_q) = u_1u_2\ldots u_q.$$

Divide every unit $u_1, u_2, \ldots, u_q$ out of boths sides to find $b^q = 1$. □

6.8 Take an integer $m \geq 2$. Suppose that b is a unit in the remainders modulo m. Prove that the reciprocal of b is $b^{\phi(m)-1}$.

Euler's theorem is important because we can use it to calculate quickly modulo prime numbers (and sometimes even modulo numbers which are not prime).

Modulo 19, let's find $123456789^{987654321}$. First, the base of this expression is $123456789 = 6497725 \cdot 19 + 14$ So modulo 19:

$$123456789^{987654321} = 14^{987654321}.$$

That helps with the base, but the exponent is still large. According to Euler's theorem, since 19 is prime, modulo 19:

$$b^{19-1} = 1$$

for any remainder b. In particular, modulo 19,

$$14^{18} = 1.$$

So every time we get rid of 18 copies of 14 multiplied together, we don't change our result. Divide 18 into the exponent:

$$987654321 = 54869684 \cdot 18 + 9.$$

So then modulo 19:

$$14^{987654321} = 14^9.$$

We leave the reader to check that $14^9 = 18$ modulo 19, so that finally, modulo 19,

$$123456789^{987654321} = 18.$$

6.9 By hand, using Euler's totient function, compute 127^{162} modulo 120.

Lemma 6.5. *For any prime number p and integers b and k, $b^{1+k(p-1)} = b$ modulo p.*

Proof. If b *is* a multiple of p then $b = 0$ modulo p so both sides are zero. If b is not a multiple of p then $b^{p-1} = 1$ modulo p, by Euler's theorem. Take both sides to the power k and multiply by b to get the result. □

Theorem 6.6. *Suppose that $m = p_1 p_2 \dots p_n$ is a product of distinct prime numbers. Then for any integers b and k with $k \geq 0$,*

$$b^{1+k\phi(m)} \equiv b \pmod{m}.$$

Proof. By lemma 6.5, the result is true if m is prime. So if we take two prime numbers p and q, then $b^{1+k(p-1)(q-1)} = b$ modulo p, but also modulo q, and therefore modulo pq. The same trick works if we start throwing in more distinct prime factors into m. □

6.10 Give an example of integers b and m with $m \geq 2$ for which $b^{1+\phi(m)} \neq b$ modulo m.

Sage

In Sage, the quotient of 71 modulo 13 is `mod(71,13)`. The tricky bit: it returns a "remainder modulo 13", so if we write

```
a=mod(71,13)
```

this will define a to be a "remainder modulo 13". The value of a is then 6, but the value of a^2 is 10, because the result is again calculated modulo 13.

Euler's totient function is

```
euler_phi(777)
```

which yields $\phi(777) = 432$. To find 14^{-1} modulo 19,

```
inverse_mod(14,19)
```

yields $14^{-1} = 15$ modulo 19.

We can write our own version of Euler's totient function, just to see how it might look:

```
def phi(n):
    return prod(p^a-p^(a-1) for (p,a) in factor(n))
```

where `prod` means product, so that `phi(666)` yields $\phi(666) = 216$. The code here uses the function `factor()`, which takes an integer n and returns a list `p=factor(n)` of its prime factors. In our case, the prime factorisation of 666 is $666 = 2 \cdot 3^2 \cdot 37$. The expression `p=factor(n)` when $n = 666$ yields a list `p=[(2,1), (3,2), (37,1)]`, a list of the prime factors together with their powers. To find these values, `p[0]` yields `(2,1)`, `p[1]` yields `(3,2)`, and `p[2]` yields `(37,1)`. The expression `len(p)` gives the length of the list `p`, which is the number of entries in that list, in this case 3. For each entry, we set $b = p_i$ and $e = a_i$ and then multiply the result r by $b^e - b^{e-1}$.

To use the Chinese remainder theorem, suppose we want to find a number x so that x has remainders 1 mod 3 and 2 mod 7,

```
crt([1,2],[3,7])
```

gives you $x = 16$; you first list the two remainders `1,2` and then list the moduli `3,7`.

Another way to work with modular arithmetic in sage: we can create an object which represents the remainder of 9 modulo 17:

```
a=mod(9,17)
a^(-1)
```

yielding 2. As long as all of our remainders are modulo the same number 17, we can do arithmetic directly on them:

```
b=mod(7,17)
a*b
```

yields 12.

Chapter 7

Secret messages

And when at last you find someone to whom you feel you can pour out your soul, you stop in shock at the words you utter— they are so rusty, so ugly, so meaningless and feeble from being kept in the small cramped dark inside you so long.

— Sylvia Plath
The Unabridged Journals of Sylvia Plath

Man is not what he thinks he is, he is what he hides.

— André Malraux

Enigma, Nazi secret code machine

RSA: the Cocks, Rivest, Shamir and Adleman algorithm

Alice wants to send a message to Bob, but she doesn't want Eve to read it. She first writes the message down in a computer, as a collection of 0's and 1's. She can think of these 0's and 1's as binary digits of some large integer x, or as binary digits of some *remainder* x modulo some large integer m. Alice takes her message x and turns it into a secret coded message by sending Bob not the original x, but instead sending him x^d modulo m, for some integer d. This will scramble up the digits of x unrecognizably, if d is chosen "at random". For random enough d, and suitably chosen positive integer m, Bob can unscramble the digits of x^d to find x.

For example, take $m := 55$ and let $d := 17$ and $y := x^{17} \bmod 55$:

x	y	x	y	x	y	x	y
0	0	14	9	28	8	42	37
1	1	15	5	29	39	43	43
2	7	16	36	30	35	44	44
3	53	17	52	31	26	45	45
4	49	18	28	32	32	46	51
5	25	19	24	33	33	47	42
6	41	20	15	34	34	48	38
7	17	21	21	35	40	49	14
8	13	22	22	36	31	50	30
9	4	23	23	37	27	51	6
10	10	24	29	38	3	52	2
11	11	25	20	39	19	53	48
12	12	26	16	40	50	54	54
13	18	27	47	41	46	55	0

If Alice sends Bob the secret message $y = x^{17}$, then Bob decodes it by $x = y^{33}$, as we will see.

Theorem 7.1. *Pick two different prime numbers p and q and let $m := pq$. Recall Euler's totient function $\phi(m)$. Suppose that d and e are positive integers so that $de = 1$ modulo $\phi(m)$. If Alice maps each remainder x to $y = x^d$ modulo m, then Bob can invert this map by taking each remainder y to $x = y^e$ modulo m.*

Proof. We have to prove that $\left(x^d\right)^e = x$ modulo m for all x, i.e. that $x^{de} = x$ modulo m for all x. For x coprime to m, this follows from Euler's theorem, but for other values of x the result is not obvious.

By theorem 6.3 on page 41, $\phi(m) = (p-1)(q-1)$. Since $de = 1$ modulo $\phi(m)$, clearly

$$de - 1 = k(p-1)(q-1)$$

for some integer k. If $k = 0$, then $de = 1$ and the result is clear: $x^1 = x$. Since d and e are positive integers, $de - 1$ is not negative, so $k \geq 0$. So we can assume that $k > 0$. The result follows from theorem 6.6 on page 43. □

7.1 Take each letter in the alphabet, ordered

$$abc \dots zABC \dots Z,$$

and one "blank space" letter _, so $26 + 26 + 1 = 53$ letters in all. Use the rule that each letter is then represented by a number from 100 to 152, starting with $a \mapsto 100$, $b \mapsto 101$, and so on to _ $\mapsto 152$. Then string the digits of these numbers together into a single number. For example, *ab_c* is 100 101 152 102.

a. Write out the message *Hail_Caesar* as a number.

b. Translate the number 127 114 114 111 104 into letters by this encoding.

c. Apply the method above, taking

$$p = 993\,319,$$
$$q = 999\,331,$$
$$d = 13$$

and use a computer to compute the secret code on the number $x = 127\,114\,114\,111\,104$. You should get:

$$y = 202\,002\,770\,451$$

d. Use a computer to check that the associated number e for the given numbers p, q, d in this case is

$$e = 839\,936\,711\,257.$$

How can you use a computer to find this number e, if you only know the numbers p, q, d in this case?

e. Use the process described in theorem 7.1 on the facing page to decode the message

$$y = 660\,968\,731\,660.$$

Warning: you might want to factor e first and then compute the power y^e modulo m by computing one factor of e at a time.

Sage

Sage shines because we can't do any of the computations of this chapter by hand.

Alice

To encode a single letter as a number, sage has a function `ord()`:

```
ord('A')
```

yields 65, giving each letter (and punctuation sign and space, etc.) a different positive integer less than 100. (If Alice wants to use lower case letters, we need to go beyond 100, so let's not do that.) To apply this to an entire string of text,

```
m = "HELLO WORLD"
m = list(map(ord, m))
m
```

yields $[72, 69, 76, 76, 79, 32, 87, 79, 82, 76, 68]$. Alice turns this list of numbers into a single number, by first reversing order:

```
m.reverse()
m
```

yields $[68, 76, 82, 79, 87, 32, 79, 76, 76, 69, 72]$. Then the expression

```
ZZ(m,100)
```

adds up successively these numbers, multiplying by 100 at each step, so that

```
x=ZZ(m,100)
x
```

yields x. This is the integer Alice wants to encode. Alice sets up the values of her primes and exponent:

```
p=993319
q=999331
d=13
m=p*q
f=euler_phi(m)
e=inverse_mod(d,f)
```

The secret code Alice sends is $y = x^d \pmod{m}$. Sage can compute powers in modular arithmetic quickly (using the same tricks we learned previously):

```
y=power_mod(x, d, m)
y
```

yielding 782102149108. This is Alice's encoded message. She can send it to Bob openly: over the phone, by email, or on a billboard.

Bob

Bob decodes by

```
x=power_mod(y, e, m)
x
```

yielding 53800826153. (Alice has secretly whispered e and m to Bob.) Bob turns this number x back into text by using the function `chr()`, which is the inverse of `ord()`:

```
def recover_message(x):
    if x==0:
        return ""
    else:
        return recover_message(x//100)+chr(x%100)
```

Bob applies this:

```
print(recover_message(x))
```

to yield HELLO WORLD.

Chapter 8

Rational, real and complex numbers

A fragment of the Rhind papyrus, 1650BCE, containing ancient Egyptian calculations using rational numbers

There is a remote tribe that worships the number zero. Is nothing sacred?

— Les Dawson

Rational numbers

A *rational number* is a ratio of two integers, like

$$\frac{2}{3}, \quad \frac{-7}{9}, \quad \frac{22}{-11}, \quad \frac{0}{2}.$$

We also write them $2/3, -7/9, 22/-11, 0/2$. In writing $\frac{2}{3}$, the number 2 is the *numerator*, and 3 is the *denominator*. We can think of rational numbers two different ways:

a. Geometry: Divide up an interval of length b into c equal pieces. Each piece has length b/c.

b. Algebra: a rational number b/c is just a formal expression we write down, with two integers b and c, in which $c \neq 0$. We agree that any rational number $\frac{b}{c}$ is equal to $\frac{ab}{ac}$ for any integer $a \neq 0$.

Of course, the two ideas give the same objects, but in these notes, the algebra approach is best. For example, we see that $22/11 = 14/7$, because I can factor out:

$$\frac{22}{11} = \frac{11 \cdot 2}{11 \cdot 1},$$

and factor out

$$\frac{14}{7} = \frac{7 \cdot 2}{7 \cdot 1}.$$

Write any rational number $\frac{b}{1}$ as simply b, and in this way see the integers sitting among the rational numbers.

More generally, any rational number b/c can be simplified by dividing b and c by their greatest common divisor, and then changing the sign of b and of c if needed, to ensure that $c > 0$; the resulting rational number is *in lowest terms*.

8.1 Bring these rational numbers to lowest terms: $224/82$, $324/-72$, $-1000/8800$.

Multiply rational numbers by

$$\frac{b}{c}\frac{B}{C} = \frac{bB}{cC}.$$

Similarly, divide a rational number by a *nonzero* rational number as

$$\frac{\frac{b}{c}}{\frac{B}{C}} = \frac{b}{c}\frac{C}{B}.$$

Add rationals with the *same denominator* as:

$$\frac{b}{d} + \frac{c}{d} = \frac{b+c}{d}.$$

The same for subtraction:

$$\frac{b}{d} - \frac{c}{d} = \frac{b-c}{d}.$$

But if the denominators don't match up, we rescale to make them match up:

$$\frac{b}{c} + \frac{B}{C} = \frac{bC}{cC} + \frac{cB}{cC}.$$

8.2 If we replace $\frac{b}{c}$ by $\frac{ab}{ac}$ and we replace $\frac{B}{C}$ by $\frac{AB}{AC}$, for some nonzero integers a, A, prove that the result of computing out $\frac{b}{c} + \frac{B}{C}$, $\frac{b}{c} - \frac{B}{C}$ or $\frac{b}{c}\frac{B}{C}$ only changes by scaling both numerator and denominator by the same nonzero integer. Hence the arithmetic operations don't contradict our agreement that we declare $\frac{b}{c}$ to equal $\frac{ab}{ac}$.

8.3 By hand, showing your work, simplify

$$\frac{2}{3}-\frac{1}{2}, \frac{324}{49}\cdot\frac{392}{81}, \frac{4}{5}+\frac{7}{4}.$$

Lemma 8.1. *The number $\sqrt{2}$ is irrational. (To be more precise, no rational number squares to 2.)*

Proof. If it is rational, say $\sqrt{2}=\frac{b}{c}$, then $c\sqrt{2}=b$, so squaring gives $c^2\cdot 2=b^2$. Every prime factor in b occurs twice in b^2, so the prime factorization of the right hand side has an even number of factors of each prime. But the left hand side has an odd number of factors of 2, since c^2 has an even number, but there is another factor of 2 on the left hand side. This contradicts uniqueness of prime factorization. □

8.4 Suppose that N is a positive integer. Prove that either $\sqrt{N}$ is irrational or N is a *perfect square*, i.e. $N=n^2$ for some integer n.

8.5 Prove that there are no rational numbers x and y satisfying $\sqrt{3}=x+y\sqrt{2}$.

A rational number is *positive* if it is a ratio of positive integers. We write $x>y$ to mean that $x-y$ is positive, and similarly define $x<y$, $x\leq y$, and so on.

8.6 Prove that every positive rational number has a unique expression in the form

$$\frac{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}{q_1^{\beta_1}q_2^{\beta_2}\cdots q_\ell^{\beta_\ell}}$$

where

$$p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_\ell$$

are prime numbers, all different, with $p_1<p_2<\cdots<p_k$ and $q_1<q_2<\cdots<q_\ell$ and

$$\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_\ell$$

are positive integers. For example

$$\frac{234}{16}=\frac{3^2\cdot 13}{2^3}.$$

8.7 Of the various laws of addition and multiplication and signs for integers, which hold true also for rational numbers?

The most important property that rational numbers have, that integers don't, is that we can divide any rational number by any nonzero rational number.

Real numbers

> *Order is an exotic in Ireland. It has been imported from England but will not grow. It suits neither soil nor climate.*
>
> — J.A. Froude

Think of the rational numbers geometrically, so that b/c is the length of each piece when a rod of length b is cut into c equal lengths. Draw an straight line, infinite in both directions, and mark a point 0 on it. Then mark at point 1 at 1 unit of distance from 0, and so on marking all of the integers to form a *number line*. The points of this continuous number line are the *real numbers*. This physical definition doesn't make it possible to prove anything about the real numbers.

The rational numbers are insufficient for algebra: they have no $\sqrt{2}$. Imagine taking all positive rational numbers b/c for which $b^2/c^2 > 2$. Thinking geometrically, drawing a number line:

Our rational numbers sit along the line getting very close to the circled point of the line, which should be the point $\sqrt{2}$. But there is no rational number there. From now on, we assume that the reader is familiar with the real numbers and their basic properties; see [18] for the complete story of real numbers.

8.8 Given numbers a, b, c, x, note that $ax^2 + bx + c = (ax + b)x + c$. But if we plug in and compute, which expression is faster to compute with? Suppose that we compute approximately, and every addition and multiplication produces some error; which expression is more accurate?

8.9 If you add first and then round to some number of decimals, and I round first and then add, how far apart can our results be? How close? What if we subtract instead of adding? Or if we multiply?

8.10 Explain why $1 = .99999\ldots$. How many different decimal expansions can a real number have, and which real numbers have that many?

Complex numbers

> *Entre deux vérités du domaine réel, le chemin le plus facile et le plus court passe bien souvent par le domaine complexe.*
>
> The shortest and easiest path between any two facts about the real domain often passes through the complex domain.
>
> — Paul Painlevé
> Analyse des travaux scientifiques

Just as the rational numbers have a deficiency, no $\sqrt{2}$, so the real numbers have a deficiency: no $\sqrt{-1}$. We can fix this by introducing the complex numbers. Here are two definitions:

a. *Algebra:* We agree to play a game, where we write down algebraic expressions in real numbers and in an abstract variable i, but we treat the symbol i according to the following rule. Whenever we see two copies of i multiplied together (or we see i^2) appear anywhere, we are allowed to wipe them out and replace with -1, and vice versa when we see a -1 we can replace it with an i^2. We force by hand the existence of a $\sqrt{-1}$ by forcing the abstract symbol i to be $\sqrt{-1}$.

b. *Geometry:* We draw the usual x, y axes on the plane. We pretend that each point of the plane represents a single number, which we write as $z = x + yi$ to represent the point (x, y). If we have two points, $z = x + yi$ and $Z = X + Yi$, we add them by the rule:

$$z + Z = (x + X) + (y + Y)i$$

and multiply them by the rule:

$$zZ = (xX - yY) + (xY + yX)i.$$

For example,

$$(2 + 4i)(3 - i) = (2 \cdot 3 + 4 \cdot 1) + (-2 + 4 \cdot 3)i = 10 + 10i.$$

It isn't at all clear that these two games we can play arrive at the same result. Note that the rule for multiplication seems to come out of the air in the geometric theory, but in the algebraic theory it is just what you get by expanding out both sides and insisting that $i^2 = -1$. Algebra is in this respect clearly superior. But the geometric approach makes very precise what sort of objects we are really dealing with, so we will use it as our definition.

Just like with rational numbers, if a complex number $z = x + yi$ is not zero, or in other words x and y are not both zero, then it has a reciprocal

$$\frac{1}{z} = \frac{x - yi}{x^2 + y^2}.$$

8.11 Check that this is a reciprocal. Why is it defined?

Complex conjugation is the operation taking $z = x + yi$ to $\bar{z} := x - iy$.

8.12 Prove that a complex number z is a real number just when $z = \bar{z}$.

8.13 Prove that addition, subtraction, multiplication and division of complex numbers passes through conjugation:

$$\begin{aligned}
\overline{z + w} &= \bar{z} + \bar{w}, \\
\overline{z - w} &= \bar{z} - \bar{w}, \\
\overline{zw} &= \bar{z}\,\bar{w}, \\
\overline{\left(\frac{z}{w}\right)} &= \frac{\bar{z}}{\bar{w}},
\end{aligned}$$

for any complex numbers z and w. Explain by induction why, for any polynomial $p(z)$ with real number coefficients,

$$\overline{p(z)} = p(\bar{z}),$$

for any complex number z.

The *norm* or *modulus* of a complex number $z = x + yi$ is the real number $|z| := \sqrt{x^2 + y^2}$.

8.14 Prove that $z\bar{z} = |z|$ for any complex number z.

Sage

> *Was ist also Wahrheit? Ein bewegliches Heer von Metaphern, Metonymien, Anthropomorphismen, kurz eine Summe von menschlichen Relationen, die, poetisch und rhetorisch gesteigert, übertragen, geschmückt wurden, und die nach langem Gebrauch einem Volke fest, kanonisch und verbindlich dünken: die Wahrheiten sind Illusionen, von denen man vergessen hat, daßsie welche sind, Metaphern, die abgenutzt und sinnlich kraftlos geworden sind, Münzen, die ihr Bild verloren haben und nun als Metall, nicht mehr als Münzen, in Betracht kommen.*
>
> What, then, is truth? A mobile army of metaphors, metonyms, and anthropomorphisms—in short, a sum of human relations, which have been enhanced, transposed, and embellished poetically and rhetorically, and which after long use seem firm, canonical, and obligatory to a people: truths are illusions about which one has forgotten that this is what they are; metaphors which are worn out and without sensuous power; coins which have lost their pictures and now matter only as metal, no longer as coins.
>
> — Friedrich Nietzsche
> Über Wahrheit und Lüge im aussermoralischen Sinne

Sage represents numbers in two different ways: as exact symbolic expressions, or as a finite decimal approximation (often called a *numeric* or *floating point* representation). For example we compute $\frac{7}{2} + \frac{5}{3}$ symbolically as

```
7/2+5/3
```

yielding $\frac{31}{6}$. We can also play with irrational numbers symbolically: represent $\sqrt{27}$ as

```
sqrt(27)
```

which sage simplifies to yield $3\sqrt{3}$.

We will *never* use decimal approximations in this class (please don't), but here is how you can use them if you wish. By writing out a number with a decimal point, we are asking sage to approximate numerically, keeping only some number of decimals accuracy:

```
sqrt(27.0)
```

yields 5.19615242270663. If we want to ask sage to give us a particular number of decimal places accuracy in a decimal approximation of a symbolic expression, say 50 decimals:

```
numerical_approx(pi, digits=50)
```

yields

$$3.1415926535897932384626433832795028841971693993751.$$

To use complex numbers, try

```
z=3+4*i
real_part(z)
```

yielding 3, the real part, while `imag_part(z)` yields 4.

Chapter 9

Polynomials

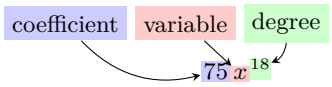
I was x years old in the year x^2.

— Augustus De Morgan (when asked his age)

A *polynomial* in a variable x is an expression like

$$1, x + 1, 5 + 3x^2 - 2x.$$

Let us be more precise.



A *monomial* in a variable x is the product of a number (the *coefficient* of the term) with a nonnegative integer (the *degree*) power of x. Write x^1 as x. Write x^0 as 1, and so write any monomial of degree zero as just its coefficient, and call it a *constant*. If the coefficient of a monomial is 1, write it without a coefficient: $1\,x^{57} = x^{57}$.

A *polynomial* is a finite sum of monomials, its *terms*. We can write the sum in any order, and drop any terms with zero coefficient, without changing the polynomial. If the sum has no terms, write the polynomial as 0. A *constant* is a polynomial which has only a constant term or is 0. Add polynomials by adding coefficients of terms of the same degree:

$$2x^2 + 7x^2 + x = (2 + 7)x^2 + x = 9x^2 + x.$$

Hence we can write any polynomial so that all terms have different degree, say decreasing in degree, or increasing in degree: $1 + x^3 + 5x = 1 + 5x + x^3 = x^3 + 5x + 1$, and drop all terms with zero coefficient. Two polynomials written this way are equal just when they have the same coefficient in every degree. The *degree* is the largest degree of any term. The degree of 0 is defined to be $-\infty$. Multiply monomials like $(7x^2)(5x^3) = (7 \cdot 5)x^{2+3}$, multiplying coefficients and adding degrees. Multiply polynomials by the distributive law, term by term. The coefficients can be integers, rational numbers, real numbers, or complex numbers (in which case, we usually use z as the name of the variable, instead of x). They can even be remainders modulo some integer. As soon as we fix our choice of coefficients, the addition laws, multiplication laws, and distributive law (familiar from chapter 1) follow: write $b(x)$ as a polynomial, instead of an integer b, and so on, and change the word *integer* to *polynomial*. Ignore the sign laws and the law of well ordering.

Let's work modulo 2, in honour of George Boole. Let $p(x) := x^2 + x$. We write $x^2 + x$ to mean $\bar{1}x^2 + \bar{1}x$, i.e. the coefficients are remainders. Note that

$p(x) + p(x) = 0$ because $\bar{2} = \bar{0}$. The polynomial $p(x)$ is not zero, because its coefficients are not zero. *Danger:* we might want to think of this polynomial as a *function* of a variable x, and ask that x also be a remainder modulo 2. But then that function *is* zero, because if we let $x = \bar{0}$ then

$$p\big(\bar{0}\big) = \bar{1} \cdot \bar{0}^2 + \bar{1} \cdot \bar{0} = \bar{0}$$

and if we let $x = \bar{1}$ then

$$p\big(\bar{1}\big) = \bar{1} \cdot \bar{1}^2 + \bar{1} \cdot \bar{1} = \bar{0}$$

modulo 2. So a polynomial is *not* just a function: the function can be zero while the polynomial is not. A polynomial is a purely algebraic object.

9.1 Work modulo 5: for which remainders a can we find a remainder b so that $a = b^2$? In other words, which elements have square roots?

9.2 Give an example of a polynomial $b(x)$ none of whose coefficients are integers, so that, if we set x to any integer value, then $b(x)$ takes an integer value.

9.3 Given numbers a, b, c, x, note that $ax^2 + bx + c = (ax + b)x + c$. But if we plug in and compute, which expression is faster to compute with? How can you similarly speed up the computation of a cubic polynomial function (i.e. of degree 3)?

Quotient and remainder

You are familiar with long division of polynomials. For example, to divide $x - 4$ into $x^4 - 3x + 5$, we have to order the terms in decreasing degrees as usual, and add in zero terms to get all degrees to show up then

$$\begin{array}{rl}
 & x^3{+}4x^2\ {+}16x\ {+}61 \\
\hline
x-4 & x^4{+}0x^3\ {+}0x^2\ {-}3x\ {+}5 \\
 & \underline{x^4{-}4x^3\phantom{\ {+}0x^2\ {-}3x\ {+}5}} \\
 & 4x^3\ {+}0x^2\ {-}3x\ {+}5 \\
 & \underline{4x^3{-}16x^2\phantom{\ {-}3x\ {+}5}} \\
 & \phantom{x^4{+}0x^3}16x^2\ {-}3x\ {+}5 \\
 & \phantom{x^4{+}0x^3}\underline{16x^2{-}64x\phantom{\ {+}5}} \\
 & \phantom{x^4{+}0x^3{+}0x^2}61x\ {+}5 \\
 & \phantom{x^4{+}0x^3{+}0x^2}\underline{61x{-}244} \\
 & \phantom{x^4{+}0x^3{+}0x^2}249
\end{array}$$

The quotient is $x^3 + 4x^2 + 16x + 61$ and the remainder is 249. Summing up, this calculation tells us that

$$x^4 - 3x + 5 = (x - 4)\left(x^3 + 4x^2 + 16x + 61\right) + 249,$$

We stop when the remainder has small *degree* than the expression we want to divide in, in this case when 249 has degree zero, smaller than the degree of $x - 4$.

Rules about coefficients

In this simple problem, we never needed to divide. But in general, to divide $b(x)$ by $c(x)$, we might need to divide coefficients of $b(x)$ by those of $c(x)$ at some stage. Therefore from now on, we will restrict the choice of coefficients to ensure that we can always divide. For example, we can carry out long division of any two polynomials with rational coefficients, and we always end up with a quotient and a remainder, which are themselves polynomials with rational coefficients. The same is true if we work with real number coefficients, or complex number coefficients. But if we work with integer coefficients, the resulting quotient and remainder might only have rational coefficients.

9.4 Determine the greatest common divisor of $b(x) = x^3 + x^2 + x$ and $c(x) = x^3 - x^2 - x - 2$ and the associated Bézout coefficients $s(x), t(x)$.

It is trickier to specify the restrictions on coefficients if they are to be remainders modulo an integer. From now on, we will only work with remainders modulo a prime. The reason is that every nonzero remainder modulo a prime has a reciprocal, so we can always divide coefficients.

9.5 Prove that every nonzero remainder modulo a prime has a reciprocal modulo that prime.

The extended Euclidean algorithm

With these restrictions on coefficients, we can immediately generalize the Euclidean and extended Euclidean algorithms, and compute Bézout coefficients and least common multiples, using the same proofs, so we won't repeat them.

Take the polynomials $b(x) = 3+2x-x^2$ and $c(x) = 12-x-x^2$. We want to find the Bézout coefficients and the greatest common divisor among polynomials over the rational numbers. At each step, force down the degree of one of the polynomials.

$$\begin{pmatrix} 1 & 0 & 3+2x-x^2 \\ 0 & 1 & 12-x-x^2 \end{pmatrix} \text{ add -row 2 to row 1,}$$

$$\begin{pmatrix} 1 & -1 & -9+3x \\ 0 & 1 & 12-x-x^2 \end{pmatrix} \text{ add } (x/3)\text{row 1 to row 2,}$$

$$\begin{pmatrix} 1 & -1 & -9+3x \\ \frac{x}{3} & 1-\frac{x}{3} & 12-4x \end{pmatrix} \text{ add } \frac{4}{3}\text{row 1 to row 2,}$$

$$\begin{pmatrix} 1 & -1 & -9+3x \\ \frac{4}{3}+\frac{x}{3} & -\frac{1}{3}-\frac{x}{3} & 0 \end{pmatrix}$$

The equation of the Bézout coefficients is therefore

$$1\left(3+2x-x^2\right) - 1\left(12-x-x^2\right) = -9+3x = 3(x-3).$$

The greatest common divisor, once we divide away any coefficient in front of the highest term, is $x-3$. A *monic polynomial* is one whose highest degree

term has coefficient 1: the greatest common divisor will always be taken to be a monic polynomial, and then there is a unique greatest common divisor.

Fix a choice of whether we are working with rational, real or complex coefficients or if our coefficients will be remainders modulo a prime. We will say this choice is a choice of *field*, and our polynomials are said to live *over* that field. A polynomial over a chosen field is *irreducible* if it does not split into a product $a(x)b(x)$ of polynomials over the same field, except for having $a(x)$ or $b(x)$ constant. The same proof for prime numbers proves that a polynomial over a field admits a factorization into a product of irreducibles over that field, unique up to reordering the factors and perhaps multiplying any of the factors by a nonzero constant.

Let's working over the remainders modulo 2. Notice that $2x = 0x = 0$, and similarly that $-1 = 1$ so $-x^2 = x^2$. Let's find the Bézout coefficients of $b(x) := x^3 + x$ and $c(x) := x^2$:

$$\begin{pmatrix} 1 & 0 & x^3+x \\ 0 & 1 & x^2 \end{pmatrix}, \text{ add } x(\text{row } 2) \text{ to row } 1,$$

$$\begin{pmatrix} 1 & x & x \\ 0 & 1 & x^2 \end{pmatrix}, \text{ add } x(\text{row } 1) \text{ to row } 2,$$

$$\begin{pmatrix} 1 & x & x \\ x & x^2+1 & 0 \end{pmatrix}.$$

The Bézout coefficients are $1, x$, and the greatest common divisor is x:

$$1\left(x^3+x\right) + x\left(x^2\right) = x.$$

9.6 Find the Bèzout coefficients of $b(x) = x^3 + x$, $c(x) = x^4 + 1$ with coefficients in remainders modulo 3.

Factoring

- What is a multiple root of a polynomial?
- Well, it is when we substitute a number in the polynomial and get zero. Then do it again and again get zero and so k times .... But on the $(k+1)$-st time the zero does not appear.

— A. A. Kirillov
What are Numbers?

Proposition 9.1. *Take any constant c and any polynomial $p(x)$ over any field. Then $p(x)$ has remainder $p(c)$ when divided by $x - c$.*

Proof. Take quotient and remainder: $p(x) = (x - c)q(x) + r(x)$, using the Euclidean algorithm. The degree of the remainder is smaller than the degree of $x - c$, so the remainder is a constant, say r_0. Plug in $x = c$: $p(c) = (c - c)q(c) + r_0 = r_0$. □

A *root* of a polynomial $p(x)$ is a number c in whichever field we work over so that $p(c) = 0$.

Corollary 9.2. *A polynomial $p(x)$ over any field has a root at $x = c$, i.e. $p(c) = 0$, for some constant c in the field, just when the linear polynomial $x - c$ divides into $p(x)$.*

9.7 Over the field of real numbers, or the field of rational numbers, or the field of complex numbers, or the field of remainders modulo any odd prime number: show that every constant d has either two square roots, which we denote as $\pm\sqrt{d}$, or one square root (if $2 = 0$ in our field or if $d = 0$) or has no square roots.

9.8 Work over the field of real numbers, or the field of rational numbers, or the field of complex numbers, or the field of remainders modulo any odd prime number. Show that the solutions of the quadratic equation

$$0 = ax^2 + bx + c$$

(where a, b, c are constants and $a \neq 0$) are precisely the numbers x so that

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{2a^2} - \frac{c}{a}.$$

Prove that the solutions of the quadratic equation, over our field, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

just when the required square roots exist.

9.9 Prove that a quadratic or cubic polynomial (in other words of degree 2 or 3) over any field is reducible just when it has a root.

9.10 Prove that the polynomial $x^2 + x + 1$ is irreducible over the field of remainders modulo 2. By writing down all quadratic polynomials over that field, and factoring all of them but this one, show that this is the only irreducible quadratic polynomial over that field.

9.11 Prove that the polynomial $x^3 + x + 1$ is irreducible over the field of remainders modulo 2.

The polynomial $x^3 + x + 1$ is irreducible over the field of remainders modulo 2, as we saw in the previous problem. But then, if it is reducible over the integers, say $a(x)b(x)$, then quotient out the coefficients modulo 2 to get a factorization $\bar{a}(x)\bar{b}(x)$ over the field of remainders modulo 2. By irreducibility, in such a factorization, $\bar{a}(x)$ or $\bar{b}(x)$ is constant. So $a(x)$ or $b(x)$ has even leading term or is constant. But $a(x)b(x) = x^3 + x + 1$ has leading term 1, odd. Therefore $x^3 + x + 1$ is irreducible over the integers.

Corollary 9.3. *Every polynomial over any field has at most one factorisation into linear factors*

$$a_0(x - r_1) \dots (x - r_n)$$

up to reordering the factors.

Proof. The linear factors are determined by the roots. Divide off one linear factor from each root, and apply induction on the degree of the polynomial to see that the number of times each linear factor appears is uniquely determined. □

9.12 Suppose that $p(x)$ and $q(x)$ are polynomials over some field, both of degree at most n, and that $p(c) = q(c)$ for $n + 1$ different numbers c from the field. Prove that $p(c) = q(c)$ for all numbers c from the field.

9.13 Why are we working over fields? Working modulo 6, consider the polynomial $b(x) = x^2 + 5x$. How many roots does it have among the remainders modulo 6? How many factorizations can you make for it, working modulo 6?

A root c of a polynomial $p(x)$ has *multiplicity* k if $(x - c)^k$ is the highest power of $x - c$ which divides $p(x)$.

Corollary 9.4. *Over any field, the degree of a polynomial is never less than the sum of the multiplicities of its roots, with equality just when the polynomial is a product of linear factors.*

Proof. Degrees add when polynomials multiply, by looking at the leading term. □

Rational roots

A powerful trick to find all rational roots of rational coefficient polynomials:

Lemma 9.5. *If a polynomial $p(x)$ with integer coefficients has a rational root $x = n/d$, with n and d coprime integers, then the numerator n divides the coefficient of the lowest term of $p(x)$, while the denominator d divides the coefficient of the highest term of $p(x)$.*

Proof. Write out

$$p(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0,$$

so that these integers $a_0, a_1, \ldots, a_k$ are the coefficients of $p(x)$. Plug in:

$$0 = a_k (n/d)^k + a_{k-1} (n/d)^{k-1} + \cdots + a_1 (n/d) + a_0.$$

Multiply both sides by d^k:

$$0 = a_k n^k + a_{k-1} n^{k-1} d + \cdots + a_1 n d^{k-1} + a_0 d^k.$$

All terms but the last term contain a factor of n, so n divides into the last term. But n and d are coprime, so n divides into a_0. The same trick backwards: all terms but the first term contain a factor of d, so d divides into the first term. But n and d are coprime, so d divides into a_k. □

The polynomial $p(x) = x^3 - 3x - 1$, if it had a rational root n/d, would need to have n divide into -1, while d would divide into 1. So the only possibilities are $n/d = 1$ or $n/d = -1$. Plug in to see that these are not roots, so $x^3 - 3x - 1$ has no rational roots.

9.14 Using lemma 9.5 on the preceding page, find all rational roots of

a. $x^2 - 5$,

b. $x^2 - 3x - 5$,

c. $x^{1000} - 3x - 5$.

d. $x^2/3 - x - 5/3$,

9.15 Prove that any quadratic equation with odd integer coefficients has no rational roots.

Sage

> *The world of the future will be an even more demanding struggle against the limitations of our intelligence, not a comfortable hammock in which we can lie down to be waited upon by our robot slaves.*
>
> — Norbert Wiener
> The Human Use Of Human Beings: Cybernetics And Society

To construct polynomials in a variable x, we first define the variable, using the strange expression:

```
x = var('x')
```

We can then solve polynomial equations:

```
solve(x^2 + 3*x + 2, x)
```

yielding `[x == -2, x == -1]`. To factor polynomials:

```
x=var('x')
factor(x^2-1)
```

yields `(x + 1)*(x - 1)`. To use two variables x and y:

```
x, y = var('x, y')
solve([x+y==6, x-y==4], x, y)
```

yielding `[[x == 5, y == 1]]`. We can take greatest common divisor:

```
x=var('x')
b=x^3+x^2
c=x^2+x
gcd(b,c)
```

yielding x^2+x. For some purposes, we will need to tell sage which field the coefficients of our polynomials come from. We can get the quotient and remainder in polynomial division by

```
R.<x> = PolynomialRing(QQ)
b=x^3+x^2
c=x^2+x
b.quo_rem(c)
```

yielding $(x, 0)$. To find Bézout coefficients,

```
xgcd(x^4-x^2,x^3-x)
```

yields $\left(x^3 - x, 0, 1\right)$.

The expression `R.<x> = PolynomialRing(QQ)` tells sage to use polynomials in a variable x with coefficients being rational numbers. The set of all rational numbers is called $\mathbb{Q}$, written `QQ` when we type into a computer. To work with coefficients that are integers modulo a prime, for example with integers modulo 5:

```
R.<x> = PolynomialRing(GF(5))
b=x^3+x^2+1
c=x^2+x+1
b.quo_rem(c)
```

yields $(x, 4x + 1)$. The set of all integers modulo a prime p is sometimes called $\mathrm{GF}(p)$.

We can define our own function to calculate Bézout coefficients, which does exactly what the `xgcd()` function does on polynomials of one variable:

```
def bezpoly(b,c):
    p,q,r=1,0,b
    s,t,u=0,1,c
    while u!=0:
        if r.degree()>u.degree():
            p,q,r,s,t,u=s,t,u,p,q,r
        Q=u//r
        s,t,u=s-Q*p,t-Q*q,u-Q*r
    return (r,p,q)
```

Then the code

```
R.<t> = PolynomialRing(QQ)
f=(2*t+1)*(t+1)*(t-1)
g=(2*t+1)*(t-1)*t^2
bezpoly(f,g)
```

returns $\left(2t^2 - t - 1, -t + 1, 1\right)$.

Interpolation

> *Il faut bien s'arrêter quelque part, et pour que la science soit possible, il faut s'arrêter quand on a trouvé la simplicité.*
>
> We have to stop somewhere, and for science to be possible, we must stop when we have found simplicity.
>
> — Henri Poincaré
> La Science et L'Hypothèse

Theorem 9.6. *Over any field, if we specify distinct values $x_0, \ldots, x_n$ and arbitrary values $y_0, \ldots, y_n$, there is a unique polynomial $p(x)$ of degree n so that $p(x_i) = y_i$.*

The polynomial $p(x)$ *interpolates* the values y_i at the points x_i.

Proof. We associate to the numbers $x_0, \ldots, x_n$ the polynomials

$$p_j(x) = \frac{(x-x_0)\ldots(x-x_{j-1})(x-x_{j+1})\ldots(x-x_n)}{(x_j-x_0)\ldots(x_j-x_{j-1})(x_j-x_{j+1})\ldots(x_j-x_n)}.$$

for $j = 0, 1, 2, \ldots, n$. The reader can check that $p_j(x_j) = 1$ while $p_j(x_i) = 0$ for $i \neq j$. Then

$$p(x) = y_0p_0(x) + y_1p_1(x) + \cdots + y_np_n(x)$$

is our required polynomial. To check that it is unique, if there are two such polynomials, say $p(x)$ and $q(x)$, both of degree at most n, both equal to y_i at $x = x_i$, their difference $p(x) - q(x)$ vanishes at $x = x_i$, so is divisible by $(x-x_0)(x-x_1)\ldots(x-x_n)$, which has degree $n+1$. But their difference has degree at most n, so is zero. □

9.16 What can go wrong with interpolation if the numbers are drawn from the remainders modulo 6?

9.17* A similar problem to secret coding: sharing the power to make decisions. Suppose 100 politicians are given the power to start a war, but at least 51 of them must agree to send the signal to go to war. More generally, a group of N people arrange that any $k+1$ of them can agree to transmit a signal (for example, to go to war), but no k of them can. How can you use modular arithmetic to create a secret signal that any $k+1$ people can, working together, come up with, but no k of them can?

Chapter 10

Real polynomials, complex polynomials

I cherish the greatest respect towards everybody's religious obligations, never mind how comical.

— Herman Melville
Moby Dick

Hell is an idea first born on an undigested apple-dumpling.

— Herman Melville
Moby Dick

In this chapter (and only this chapter), we assume that the reader is familiar with real numbers, with continuity, and with the intermediate value theorem: every continuous function $y = f(x)$ defined on an interval $a \leq x \leq b$ takes on all values between $f(a)$ and $f(b)$. We also assume that the reader knows that polynomials are continuous. See [18] for the complete story of continuity and the intermediate value theorem.

Polynomials over the real numbers

If $f(x)$ is a polynomial with positive coefficients, like

$$f(x) = -4 + 7x + 17x^2 + 674x^4,$$

then making x very large makes $f(x)$ very large positive, clearly, since x^4 eventually gets much larger than all other terms. The equation $f(x) = 0$ must have a solution $x > 0$, because $f(x)$ starts at $f(0) = -4$ (negative) and then gets much larger than anything we like eventually for large enough x, so gets positive, so must be zero somewhere in between.

10.1 Prove that the equation

$$394x^3 - 92843x^2 + 209374x - 2830$$

has a real number solution x. More generally, prove that any odd-order polynomial equation in a real variable has a real solution.

Counting positive roots

A biologist, a physicist and a mathematician drink coffee and watch a house across the street from them. They see two people walk into the house. An hour later, three people walk out.
Physicist: *We measured incorrectly.*
Biologist: *No, they reproduced.*
Mathematician: *No, if one more person enters, the house will be empty.*

Theorem 10.1 (Descartes)**.** *The number of positive roots, counted with multiplicities, of a real polynomial $b(x)$ is either equal to the number of changes of sign of its coefficients (when we write the polynomial $b(x)$ with terms in order of degree), or less by a multiple of 2.*

The polynomial
$$b(x) = 12x^7 + x^5 - x^3 - x^2 + x + 1$$
has coefficients with signs $++--++$. We ignore the zero coefficients, for example in front of x^6 and x^4. So as we travel along the signs, they change twice, once from $+$ to neighboring $-$, and once from $-$ to neighboring $+$. So this $b(x)$ has at most 2 roots $x > 0$, and the number of roots with $x > 0$ is even. So there are either 2 positive roots or none.

The polynomial $c(x) = x^5 - 3x^4 + 3x^3 + 9x^2 - 27x + 3$ has 4 sign changes, so 4 or 2 or 0 roots x with $x > 0$. If we look at the polynomial $c(-x)$, expand out to get $c(-x) = -x^5 - 3x^4 - 3x^3 + 9x^2 + 27x + 3$ has 1 sign change, so 1 root with $x > 0$. Therefore $c(x)$ has 1 root with $x < 0$.

Proof. Take a polynomial $b(x)$. If x divides $b(x)$, dividing out factors of x from $b(x)$ doesn't change positive roots or signs of coefficients, so we can assume that there are no factors of x. In other words, assume that the constant coefficient of $b(x)$ is not zero:
$$b(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$$
with $b_0 \neq 0$ and $b_n \neq 0$. For large positive numbers x, $b(x)$ has the same sign as b_n:
$$\frac{b(x)}{x^n} = \frac{b_0}{x^n} + \frac{b_1}{x^{n-1}} + \cdots + \frac{b_{n-1}}{x} + b_n \to b_n.$$

Suppose that $b(x)$ has no positive roots. By the intermediate value theorem, $b(x)$ stays that same sign for all $x > 0$. For $x = 0$, this sign is $b(x) = b_0$, while for x large it is the sign of b_n. So if $b(x)$ has no positive roots, then b_0 and b_n have the same sign, so there are an even number of sign changes, to start and end at the same sign. In particular, the theorem is true for any polynomial $b(x)$ with no positive roots.

Suppose that $b(x)$ has a positive root, say at $x = a$. Factor out:
$$b(x) = (x - a)c(x)$$

for some polynomial $c(x)$. We claim that $b(x)$ has one more sign change than $c(x)$. Write out the coefficients of $c(x)$, say as

$$c(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}.$$

Expand out the equation

$$b(x) = (x - a)c(x)$$

to find the relations between the coefficients:

$$b_j = c_{j-1} - a c_j.$$

Starting at the top, $b_n = c_{n-1}$, the top coefficients match in sign. As we go down the coefficients by degree, pick out the first sign change in the coefficients of $c(x)$, say $c_{j-1} < 0$ while $c_j > 0$. Then $b_j = c_{j-1} - a c_j < 0$, so the sign of the b_j coefficient is the same as that of c_{j-1}. If the $b(x)$ coefficients have not already changed sign before we got to b_j, then they certainly must change sign by the time we get to b_j. The same argument works for the next sign change and so on: at least as many sign changes. Since the signs of the highest coefficients of $b(x)$ and $c(x)$ agree, while the signs of the lowest coefficients disagree, the total number of sign changes of $b(x)$ must be greater than those of $c(x)$ by an odd number: one more sign change or three more or five more, etc.

By induction, the number of positive roots of $c(x)$ is equal to the number of sign changes of $c(x)$ or less by a multiple of two. Since $b(x)$ has one more positive root, and one more sign change or three more or five more, etc., the result holds by induction on degree. □

Sage

The number of sign changes in a polynomial is easy to spot by eye, but we can also write sage code. Each polynomial `p` has associated list of coefficients `p.list()`. This list has the coefficients in order, but with zero entries in the list at zero coefficients. For example, if `p=t^5-2*t+7` then `p.list()` yields $[7, -2, 0, 0, 0, 1]$. We will travel along the coefficients one by one, with a for loop. The code (which we explain below):

```
def descartes(p):
    sign = 0
    sign_changes = 0
    for c in p.list():
        if c != 0:
            if sign == 0:
                if c < 0:
                    sign = -1
                else:
                    if c > 0:
                        sign = 1
            else:
                if c*sign < 0:
                    sign_changes = sign_changes + 1
                    sign = -sign
    return sign_changes
```

We store the sign of the last nonzero coefficient in a variable `sign`, which is set to zero initially to mean that we haven't yet encountered a nonzero coefficient. If we encounter our first nonzero coefficient, we just set `sign` to its value. But if we find any subsequent nonzero coefficient, we compare whether it has the same sign as `sign`, and if it has an opposite sign, i.e. if `L[i]*sign<0`, then we count a new sign change. Now calling the function:

```
t = var('t')
descartes(t^(79)-4*t^(17)+t^(10)+t^6+t-1)
```

yields 3.

Counting roots of a real polynomial in an interval

Suppose that $p(x)$ is a polynomial with real coefficients. Let $p_0(x)$ be just another name for $p(x)$. Let $p_1(x) := p'(x)$. From then on, compute quotients and remainders: $p_2(x)$ is the negative of the remainder of $p_0(x)$ divided by $p_1(x)$, and so on: $p_{j+1}(x)$ is the negative of the remainder of $p_{j-1}(x)$ divided by $p_j(x)$, until you get to have no remainder, say $p_m(x)$ divides into $p_{m-1}(x)$. The *Sturm sequence* of $p(x)$ is the sequence $p_0(x), p_1(x), \ldots, p_m(x)$.

The Sturm sequence of $p(x) = x^6 - 12x + 10$ is:

$$\begin{aligned}
p_0(x)&=x^6 - 12x + 10, \\
p_1(x)&=6x^5 - 12, \\
p_2(x)&=10x - 10, && x^6 - 12x + 10 = \tfrac{x}{6}(6x^5 - 12) - (10x - 10) \\
p_3(x)&=6. && 6x^5 - 12 = \tfrac{3}{5}(x^4 + x^3 + x^2 + x + 1)(10x - 10) - 6
\end{aligned}$$

Note that $p_3(x)$ divides into $p_2(x)$, because $p_3(x)$ is a nonzero constant.

Take a polynomial $p(x)$, and its Sturm sequence $p_0(x), p_1(x), \ldots, p_m(x)$. For any real number x, let $s(x)$ be the number of sign changes (ignoring zeroes) in the sequence of numbers

$$p_0(x), p_1(x), \ldots, p_m(x).$$

The *expected number of distinct roots* of $p(x)$ in the interval $a < x \leq b$ is $s(a) - s(b)$.

Let's find the expected number of distinct roots of $p(x) = x^6 - 12x + 10$ over $0 < x \leq 1$. Note that $p(0) = 10$ and $p(1) = -1$ are not roots, so not multiple roots.

$$\begin{aligned}
p_0(0)&= 10, & p_0(1)&= -1, \\
p_1(0)&= -12, & p_1(1)&= -6, \\
p_2(0)&= -10, & p_2(1)&= 0, \\
p_3(0)&= 6, & p_3(1)&= 6,
\end{aligned}$$

so $s(0) = 2$ sign changes, and $s(1) = 1$ sign changes, with expected number of roots $2 - 1 = 1$, i.e. we expect that $p(x)$ has one of its roots in the interval

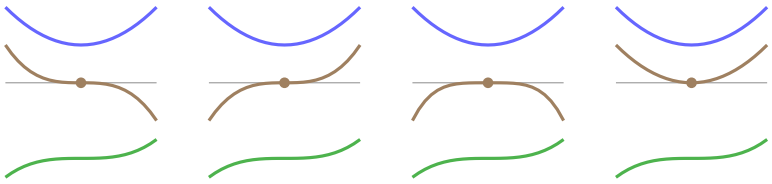
$0 < x \leq 1$.

Theorem 10.2 (Sturm)**.** *If a polynomial $p(x)$ with real coefficients has no multiple roots at $x = a$ or at $x = b$ then the expected number of distinct roots $p(x)$ in the interval $a < x \leq b$ is the number of distinct roots of $p(x)$ in that interval.*

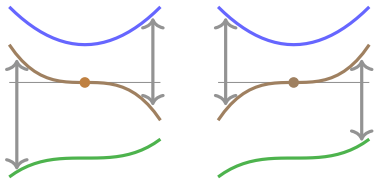
Proof. If two successive polynomials in the Sturm sequence share a root, say if $p_{i-1}(x)$ and $p_i(x)$ have a common root at $x = x_0$, then take quotient and remainder: $p_{i-1}(x) = q(x)p_i(x) - p_{i+1}(x)$, and plug in to see that $p_{i+1}(x_0) = 0$ too. The same idea works backwards: if $p_{i+1}(x)$ and $p_i(x)$ have a common root at $x = x_0$, then take quotient and remainder: $p_{i-1}(x) = q(x)p_i(x) - p_{i+1}(x)$, and plug in to see that $p_{i-1}(x_0) = 0$ too. Hence the common roots of any two successive polynomials in the Sturm sequence lie at precisely the multiple roots of $p(x)$.

Suppose that $x = x_0$ is a root of $p_m(x)$. By definition, $p_m(x)$ is the greatest common divisor (up to $\pm$ sign) of $p(x)$ and $p'(x)$, so divides into all of the $p_0(x), p_1(x), \ldots, p_m(x)$. So its zeroes are already zeroes of all of those, and so $x = x_0$ is a root of all of $p_0(x), p_1(x), \ldots, p_m(x)$. Since $p_0(x) = p(x)$ and $p_1(x) = p'(x)$, $x = x_0$ is a zero of both $p(x)$ and $p'(x)$, so at least a double root of $p(x)$.

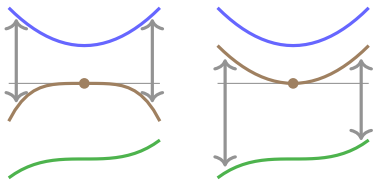
Take a root $x = x_0$ of one of the polynomials $p_i(x)$ in the Sturm sequence. Suppose that $p_i(x)$ is not $p_0(x)$ or $p_m(x)$, at the two ends of the sequence, but just one of the polynomials in middle of the sequence. Suppose that $x = x_0$ is not a multiple root of $p(x)$. From the above reasoning, $x = x_0$ is not a root of either $p_{i-1}(x)$ or of $p_{i+1}(x)$. Take quotient and remainder: $p_{i-1}(x) = q(x)p_i(x) - p_{i+1}(x)$ and plug in to see that $p_{i-1}(x_0) = -p_{i+1}(x_0)$. So if $x = x_0$ is not a root of all of the polynomials in the chain, but a root of one of them, p_i, then his neighbours p_{i-1}, p_{i+1} disagree in their sign at this point $x = x_0$, and so also disagree near $x = x_0$.



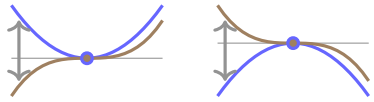
Suppose in addition that $p_i(x)$ has an odd number of roots at $x = x_0$. As x travels from a little less than $x = x_0$ to a little more than $x = x_0$, the sign of $p_i(x)$ changes across this gap, but that of $p_{i-1}(x)$ and $p_{i+1}(x)$ stay the same, since they don't have a root at $x = x_0$. So they together contribute the same total number of sign changes for $x > x_0$ as they did for $x < x_0$.



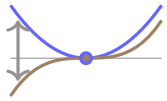
Suppose instead in addition that $p_i(x)$ has an even number of roots at $x = x_0$. As x travels from a little less than $x = x_0$ to a little more, the sign of all of $p_{i-1}(x), p_i(x), p_{i+1}(x)$ stay the same across this gap. So they together contribute the same total number of sign changes for $x > x_0$ as they did for $x < x_0$.



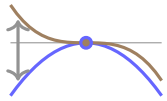
Lets go back to the start of the Sturm sequence. If $p(x)$ has an odd number of roots at some point $x = x_0$, then $p'(x)$ has an even number, and so $p(x)$ changes sign as x goes from $x < x_0$ to $x > x_0$, while $p'(x)$ doesn't change sign.



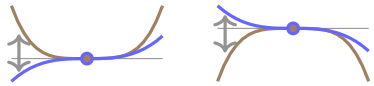
If $p(x)$ increases across this gap, then $p'(x) > 0$ on either side, so the number of sign changes contributed by $p_0(x), p_1(x)$ decreases:



Similarly if $p(x)$ decreases:



Similarly if $p(x)$ has an even number of roots:



Hence as x pass through any root of $p(x)$, the terms $p_0(x), p_1(x)$ contribute different numbers of sign changes in the Sturm sequence by at least one. □

10.2 Explain why, at each step in the Sturm sequence, we can allow ourselves to divide or multiply each polynomial $p_i(x)$ by any positive constant number. (That can help to simplify the computation.)

10.3 Apply the Sturm theorem to the polynomial $p(x) = 4x^4 + 2x^2 - 1$ to find the number of roots between $x = 0$ and $x = 1$.

10.4 Apply the Sturm theorem to the polynomial $p(x) = x^4 + 4x^3 - 1$ to find the number of roots between $x = 0$ and $x = 1$.

10.5 Apply the Sturm theorem to the polynomial $p(x) = x^6 + 4x^3 - 2$ to find the number of roots between $x = 0$ and $x = 1$.

10.6 Apply the Sturm theorem to the polynomial $p(x) = 2x^4 + 4x^3 + 2x^2 - 1$ to find the number of roots between $x = -2$ and $x = 1$.

10.7 Apply the Sturm sequence to the polynomial

$$p(x) = x^5 + 5x^4 - 1$$

to find the number of roots between $x = 0$ and $x = 1$.

10.8 What do the signs in a Sturm sequence say when you plug in a quadratic function $p(x) = ax^2 + bx + c$, say with $a \neq 0$?

To count out zeroes:

```
def count_sign_changes(L):
    n=0
    # Collect up only the nonzero elements of the list L
    M=[ x for x in L if x!= 0 ]
    for i in range(0,len(M)-1):
        p=M[i]*M[i+1]
        if p<0:
            n=n+1
    return n
def expected_number_of_zeroes(p,a,b):
    if p == 0:
        return oo
        # Returns infinity
    # Create a list called L containing the Sturm sequence.
    L = [p,diff(p(x),x)]
    n = 2
    q,r=L[n-2].quo_rem(L[n-1])
    while r!=0:
        # Every nonzero remainder r gets -r stuck in the Sturm sequence.
        L.append(-r)
        n=n+1
        q,r=L[n-2].quo_rem(L[n-1])
    # Make a list A of values of the Sturm sequence polynomials at x=a.
    A=[ t.subs(x=a) for t in L ]
    B=[ t.subs(x=b) for t in L ]
    return count_sign_changes(A)-count_sign_changes(B)
```

Then the code:

```
R.<x>=PolynomialRing(QQ)
b=(x-1)*x*(x+1)^3
expected_number_of_zeroes(b,0,1)
```

yields 1.

10.9 For a polynomial $p(x) = a_n x^n + \cdots + a_0$ with real coefficients, find a constant $c > 0$ so that every real root x of $p(x)$ lies in the interval $-c \leq x \leq c$. (By repeatedly cutting this interval in half, and applying Sturm's theorem, we can rapidly zoom in on the zeroes of $p(x)$, using a computer.)

Factoring complex polynomials

> *For every complex problem there is an answer that is clear, simple, and wrong.*
>
> — H. L. Mencken

The complex numbers are in a very strong sense free of the deficiencies of the rational and real numbers:

Theorem 10.3 (The fundamental theorem of algebra)**.** *Every nonconstant polynomial function*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

with complex number coefficients $a_0, a_1, \ldots, a_n$ has a complex root, i.e. a complex number $z = z_0$ so that $p(z_0) = 0$.

The proof of the theorem uses analysis, but we want to focus on algebra; see [18] p. 513 chapter 2 theorem 2 for a complete proof. For every complex polynomial problem, there is an answer that is perhaps unclear, complex and right.

10.10 Draw the roots of $z^5 - 1$.

Theorem 10.4. *Every complex coefficient polynomial function*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

of any degree n splits into a product of linear factors

$$p(z) = a_n (z - r_1)(z - r_2) \ldots (z - r_n).$$

In particular, a complex coefficient polynomial is irreducible just when it is linear.

Proof. Apply the fundamental theorem of algebra to find a root r_1, and then divide $p(z)$ by $z - r_1$ and apply induction. □

Factoring real polynomials

Theorem 10.5. *Every real coefficient polynomial function*

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

of any degree n splits into a product of real linear and real quadratic factors

$$p(x) = a_n (x - r_1)(x - r_2) \ldots (x - r_k)\, q_1(x) q_2(x) \ldots q_\ell(x),$$

where $q_j(x) = x^2 + b_j x + c_j$ is real quadratic with no real roots. In particular, a real coefficient polynomial is irreducible just when it is linear or quadratic $ax^2 + bx + c$, with $a \neq 0$ and with $b^2 - 4ac < 0$ (i.e. no real roots).

Proof. If the polynomial $p(x)$ has a real root, divide off the associated linear factor, and apply induction on degree of $p(x)$. So we can assume $p(x)$ has no real roots. Take a complex root, say $z_1 = x_1 + y_1 i$. Because all coefficients of $p(x)$ are real, $p(x) = \overline{p(x)}$, for any real number x, and more generally for a complex number z:

$$\overline{p(z)} = p(\bar{z})\,.$$

In particular, if $z_1 = x_1 + y_1 i$ is a root, then $\bar{z}_1 = x_1 - y_1 i$ is also a root. So $p(z)$ is divisible by

$$(z - z_1)\,(z - \bar{z}_1) = z^2 - 2x_1 z + |z_1|^2\,,$$

a quadratic function with real coefficients. Divide this factor off; applying complex conjugate, find that the quotient also has real coefficients. Apply induction. □

Partial fractions

Lemma 10.6. *Take any rational function*

$$\frac{p(x)}{q(x)}$$

over any field. Suppose that its denominator factors, say into coprime factors $q(x) = u(x)v(x)$*. Then it splits into a sum:*

$$\frac{p(x)}{q(x)} = \frac{t(x)p(x)}{u(x)} + \frac{s(x)p(x)}{v(x)}$$

where $s(x), t(x)$ *are the Bézout coefficients of* $u(x), v(x)$*.*

Proof. Write out $s(x)u(x) + t(x)v(x) = 1$ and then multiply both sides by $p(x)$ and divide both sides by $q(x)$. □

A *partial fraction* is a rational function $\frac{b(x)}{c(x)^n}$ where $b(x)$ has smaller degree than $c(x)$, and $c(x)$ is irreducible. A *partial fraction decomposition* of a rational function $\frac{p(x)}{q(x)}$ is an expression as a sum of a polynomial function together with finitely many partial fractions, all with distinct denominators, so that the denominators of the partial fractions divide $q(x)$.

Theorem 10.7. *Every rational function over any field has a partial fraction decomposition, unique up to reordering the partial fractions.*

Proof. Take a rational function $b(x)/c(x)$. Repeat the process described in lemma 10.6 to get to a sum of terms of the form $b(x)/c(x)^n$ where $c(x)$ is irreducible. If $b(x)$ does not have smaller degree than $c(x)$, then divide $c(x)$ into $b(x)$ to get quotient and remainder:

$$\begin{aligned}\frac{b(x)}{c(x)^n} &= \frac{q(x)c(x) + r(x)}{c(x)^n}, \\ &= \frac{q(x)}{c(x)^{n-1}} + \frac{r(x)}{c(x)^n},\end{aligned}$$

and apply induction, replacing $b(x)$ by $q(x)$. We let the reader prove uniqueness. □

10.11* Prove that partial fraction decomposition is unique.

$$\frac{4}{x^2+2x-3} = \frac{-1}{x+3} + \frac{1}{x-1}.$$

Over the field of real numbers, the rational function

$$\frac{p(x)}{q(x)} = \frac{x^9 - 2x^6 + 2x^5 - 7x^4 + 13x^3 - 11x^2 + 12x - 4}{x^7 - 3x^6 + 5x^5 - 7x^4 + 7x^3 - 5x^2 + 3x - 1}$$

has partial fraction decomposition

$$\frac{p(x)}{q(x)} = x^2 + 3x + 4 + \frac{1}{x-1} + \frac{1}{(x-1)^3} + \frac{x+1}{x^2+1} + \frac{1}{(x^2+1)^2}.$$

Note that we cannot get rid of the square in the last denominator, because we have to allow powers. Also note that if we allow complex numbers here instead of real numbers, then we can factor $x^2 + 1 = (x - i)(x + i)$, so there is a different partial fraction decomposition over the complex numbers, with lower degrees in denominators.

In sage:

```
x=var('x')
f = 4/(x^2+2*x-3)
f.partial_fraction(x)
```

yields $-\frac{1}{x+3} + \frac{1}{x-1}$.

Integrals

Corollary 10.8. *Every rational function $f(x)$ with real coefficients has indefinite integral $\int f(x)\,dx$ a polynomial in functions of the form*

$$x, \frac{1}{x-c}, \log(x-c), \log\left(x^2+bx+c\right), \arctan(x-c),$$

for finitely many constant numbers b, c.

Proof. The partial fraction decomposition of $f(x)$ is a sum of a polynomial and some partial fractions. The integral of a polynomial is a polynomial, and the integral of a sum is a sum, so it is enough to prove the result assuming that $f(x)$ is a single partial fraction $\frac{p(x)}{q(x)^n}$. Write out $p(x)$ as a sum of terms, and thereby break up the integral, so it is enough to assume that $p(x) = x^k$ for some integer $k \geq 0$. Since $q(x)$ is irreducible, it is linear or quadratic with no real roots.

Suppose that $q(x)$ is linear and $n = 1$. Since the degree of $p(x)$ is smaller than that of $q(x)$, $p(x)$ is a constant, and we can rescale to get $p(x) = 1$ and $q(x)$ monic, and translate in x by a constant to get $q(x) = x$:

$$\int f(x)\,dx = \int \frac{dx}{x} = \log x + C.$$

Suppose that $q(x)$ is quadratic and $n = 1$. Rescale to get $q(x)$ and $p(x)$ monic. Since the degree of $p(x)$ is smaller than that of $q(x)$, $p(x)$ is a constant or linear. This $q(x)$ has two complex roots, say z_1 and $\bar{z_1}$. Translate x by a real number constant to get z_1 to have vanishing real part: $z_1 = y_1 i$. Rescale variables to get $y_1 = 1$, so that $q(x) = x^2 + 1$. If $p(x)$ is linear, say $p(x) = x + c$, then

$$\frac{p(x)}{q(x)} = \frac{x}{q(x)} + c\frac{1}{q(x)},$$

so we can assume that $p(x) = x$ or $p(x) = 1$. So then

$$f(x) = \frac{x}{x^2+1}, \quad \int f(x)\,dx = \frac{\log\left(x^2+1\right)}{2} + C,$$

or

$$f(x) = \frac{1}{x^2+1}, \quad \int f(x)\,dx = \arctan(x) + C.$$

If $q(x) = (x-c)^n$ with $n \geq 2$, translate to get $q(x) = x^n$ so that we can reduce the problem to

$$\int \frac{dx}{x^n} = \frac{x^{1-n}}{1-n} + C.$$

If $q(x)$ is an irreducible quadratic, brought to a power $n \geq 2$, translation gives us $q(x) = 1 + x^2$ as above. If $p(x) = 1$, then we use

$$\int \frac{dx}{(1+x^2)^{n+1}} = \frac{x}{2n\,(1+x^2)^n} + \frac{2n-1}{2n}\int \frac{dx}{(1+x^2)^n},$$

by induction. If $p(x) = x^{2k+1}$ an even power, then use substitution $u = x^2$, to reduce the problem down by induction. In general, differentiate the expression

$$\frac{x^k}{(1+x^2)^n}$$

and integrate both sides to find

$$\frac{x^k}{(1+x^2)^n} + C = -2n\int \frac{x^{k+1}\,dx}{(1+x^2)^{n+1}} + k\int \frac{x^{k-1}dx}{(1+x^2)^n}.$$

This enables us, by induction, to lower the order of the numerator, at the cost of raising the order of the denominator, to reduce the integration to a previous case. □

Sage can integrate:

```
integral((1/x)+(x-1)/x^3,x)
```

yields $-\frac{2\,x-1}{2\,x^2} + \log(x)$.

Chapter 11

Factoring polynomials

> Silva: *You've got many refinements. I don't think you need to worry about your failure at long division. I mean, after all, you got through short division, and short division is all that a lady ought to cope with.*
>
> — Tennessee Williams
> Baby Doll and Tiger Tail, Act I, Scene 2

Factoring with integer coefficients

11.1 Suppose that $p(x)$ is a nonconstant polynomial with integer coefficients. Prove that there are infinitely many positive integers x at which $p(x)$ is not prime.

Proposition 11.1. *Working with coefficients remainders modulo a prime, if* $0 = b(x)c(x)$ *for two polynomials* $b(x), c(x)$*, then either* $b(x) = 0$ *or* $c(x) = 0$*.*

Proof. The highest term of the product is the product of the highest terms, so has coefficient the product of the coefficients of the highest terms. In problem 5.14 on page 35, we saw that if a product of remainders modulo a prime vanishes (modulo that prime), then one of the factors is zero. So the coefficient of the highest term of either $b(x)$ or of $c(x)$ is zero. But then by definition that isn't the highest term. □

Lemma 11.2 (Gauss's lemma)**.** *Any polynomial* $p(x)$ *with integer coefficients factors over rational numbers just when it factors over integers. In other words, suppose that we can factor it as* $p(x) = b(x)c(x)$ *into polynomials* $b(x)$*,* $c(x)$ *with rational coefficients. Then we can rescale each of* $b(x)$ *and* $c(x)$*, by multiplying by nonzero rational numbers, to arrange that* $p(x) = b(x)c(x)$ *still but now* $b(x)$ *and* $c(x)$ *have integer coefficients.*

Proof. The idea is to just clear denominators. First, take a least common denominator d for all of the coefficients of $b(x)$ and $c(x)$, and scale both sides with it, so that now we have an equation $d\,p(x) = b(x)c(x)$ with new polynomials $b(x), c(x)$, but with integer coefficients. Expand d into its prime factorization, say

$$d = p_1 p_2 \dots p_n,$$

where some of these primes might appear several times in the factorization. Quotient out the coefficients of both sides of $d\,p(x) = b(x)c(x)$ by the prime p_1 to see $0 = \bar{b}(x)\bar{c}(x)$. So then one of $\bar{b}(x)$ or $\bar{c}(x)$ vanishes (in other words, all of its coefficients vanish). Say, for example, $\bar{b}(x) = 0$. But then p_1 divides into all coefficients of $b(x)$, so we can divide both sides of $d\,p(x) = b(x)c(x)$ by p_1. Repeat until you have divided away all of the prime factors in d. □

A polynomial with integer coefficients is *irreducible* if it does not split into a product $b(x)c(x)$ except for having $b(x) = \pm 1$. (Note that this notion of irreducibility is different from being irreducible over a field.)

$$\begin{aligned} 2x^3 - 4x + 2 &= (2)(x-1)(x^2+x-1), \\ &= (-2)(-x+1)(x^2+x-1), \\ &= (-2)(x-1)(-x^2-x+1) \end{aligned}$$

are three different factorizations into irreducibles.

Corollary 11.3. *Suppose that $p(x)$ is a polynomial with coprime integer coefficients. Then it is irreducible as an integer coefficient polynomial just when it is irreducible as an rational coefficient polynomial.*

Proof. By Gauss' Lemma above, if $p(x)$ factors into rational polynomials, then it factors into integer polynomials. Conversely, since $p(x)$ has coprime coefficients, if $p(x)$ factors into integer polynomials $p(x) = b(x)c(x)$ then neither $b(x)$ nor $c(x)$ can be constant polynomials. □

We saw on page 60 that $x^3 - 3x - 1$ has no rational roots. If it is reducible over the field of rational numbers, then it has a rational root; see problem 9.9 on page 59. So therefore it is irreducible over the field of rational numbers. Its coefficients are coprime, so it is irreducible over the integers.

Theorem 11.4. *Every nonzero integer coefficient polynomial has a factorization into irreducible integer coefficient polynomials, unique up to the order in which we write down the factors and up to perhaps multiplying factors by -1.*

Proof. Let d be the greatest common divisor of the coefficients of $p(x)$, so that $p(x) = d\,P(x)$, where the coefficients of $P(x)$ are coprime. Since d factors uniquely into primes, it suffices to prove that $P(x)$ can be factored uniquely into irreducibles. Thus we may assume that coefficients of $p(x)$ are coprime.

Factor $p(x)$ into irreducibles over the field of rational numbers. By Gauss' Lemma, such a factorization yields a factorization of $p(x)$ into integer coefficient factors, each a constant rational number multiple of the rational coefficient factors. We want to check that the factors remain irreducible. Since the coefficients of $p(x)$ are coprime, the coefficients in each of these factors are also coprime: a common divisor would pull out of the whole factorization. By corollary 11.3, each factor is irreducible over the integers.

Suppose that we have two factorizations of $p(x)$ into irreducible integer coefficient polynomials. Recall that the factorization over rationals is unique up to reordering factors and scaling factors by nonzero constant rationals. Therefore our two factorizations are each obtained from the other by such tricks. So if we take out one of the factors $P(x)$ from one factorization, and the corresponding factor $Q(x)$ from the other, then $Q(x) = \frac{a}{b}P(x)$ where a, b are integers, so $bQ(x) = aP(x)$. The coefficients of $P(x)$ are coprime integers, and so are those of $Q(x)$. Taking greatest common divisor, we find $b = \pm a$. So $P(x) = \pm Q(x)$. □

Sage can test to see if a polynomial is irreducible:

```
R.<x> = PolynomialRing(QQ)
b=x^3+x+2
b.is_irreducible()
```

yields False. Indeed `factor(b)` yields $(x+1)\cdot(x^2-x+2)$. To work over the finite field with 2 elements:

```
R.<x> = PolynomialRing(GF(2))
b=x^3+x+2
b.is_irreducible()
```

yields False, while `factor(b)` yields $x\cdot(x+1)^2$.

Eisenstein's criterion: checking that there are no more factors

Proposition 11.5 (Eisenstein's criterion)**.** *Take a polynomial*

$$q(x) = a_n x^n + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$$

with integer coefficients $a_0, a_1, \ldots, a_n$. Suppose that there is a prime p so that

a. *p does not divide the highest degree coefficient a_n and*

b. *p divides all of the other coefficients and*

c. *p^2 does not divide the lowest coefficient a_0.*

Then $q(x)$ is irreducible over the field of rational numbers.

Proof. By corollary 11.3 on the preceding page, if $q(x)$ factors over the field of rational numbers, then it factors over the integers, say

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = (b_r x^r + \cdots + b_0)\,(c_s x^s + \cdots + c_0)$$

with integer coefficients

$$b_0, b_1, \ldots, b_r, c_0, c_1, \ldots, c_s.$$

Since p, but not p^2, divides the lowest coefficient $a_0 = b_0 c_0$, p divides exactly one of b_0, c_0, say b_0. Now from the equation

$$a_1 = b_0 c_1 + b_1 c_0,$$

and the fact that p divides all but the highest degree coefficient a_n, we see that p divides b_1. From the equation

$$a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0,$$

we see that p divides b_2. By induction, we find that p divides all coefficients

$$b_0, b_1, \ldots, b_r.$$

This contradicts the condition that p does not divide a_n. □

The polynomial $x^9 + 14x + 7$ is irreducible by Eisenstein's criterion.

For any prime p and positive integer d, the polynomial $x^d - p$ satisfies Eisenstein's criterion. Therefore there are no rational number square roots, cube roots, and so on, of any prime number p.

The polynomial $x^2 + 4x + 4 = (x + 2)^2$ is not irreducible, so we *cannot* always apply Eisenstein's criterion with nonprime integers p, for instance with $p = 4$.

11.2 Give some examples of polynomials to which Eisenstein's criterion applies.

11.3 Suppose that $b(x)$ is a polynomial with integer coefficients of degree d. Prove that $c(x) := x^d b(1/x)$ is also a polynomial with integer coefficients of degree d. Prove that $b(x)$ is irreducible just when $c(x)$ is. Apply to

$$b(x) = 1 + 4x^{27} + 8x^{28} + 2x^{30}.$$

11.4 Explain why the lowest terms in y of a product $b(x, y)c(x, y)$ are obtained by expanding the product of the lowest y terms in each of $b(x, y)$ and $c(x, y)$. Use this to prove that

$$p(x, y) = 5 + 25x + 8x^2 + 7xy + 19y^3$$

is irreducible.

11.5 Explain why the highest terms in y of a product $b(x, y)c(x, y)$ are obtained by expanding the product of the highest y terms in each of $b(x, y)$ and $c(x, y)$. Use this to prove that

$$q(x, y) = 2 + 9xy + 40x^3y^2 + 5y^3 + 25xy^3 + 8x^2y^3$$

is irreducible.

11.6 Suppose that p is a prime number. Prove that the polynomial $c(x) := x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible. Hint: problem 2.10 on page 16.

Eisenstein's criterion in sage

To check Eisenstein's criterion, tell sage to work with integer coefficient polynomials:

```
R.<x> = PolynomialRing(ZZ)
```

Write a function to find a list of prime factors of any integer:

```
def prime_factors(x):
    x=abs(x)
    list=[]
    p=2
    while p<=x:
        if x%p==0:
```

```
            list=list+[p]
            while x%p==0:
               x=x//p
         p=p+1
      return list
```

Note that [] represents an empty list, and we add lists by concatenating them. So `prime_factors(-6)` yields `[2,3]`, the prime factors in order. Finally, we make a function `Eisenstein(b)` to apply to integer coefficient polynomials, which returns the lowest prime p for which Eisenstein's criterion applies to our polynomial $b(x)$, or returns 0 if no such prime exists.

```
def Eisenstein(b):
   c=b.coefficients()
   highest=c.pop()
   possible_primes=prime_factors(gcd(c))
   for p in possible_primes:
      if (highest%p!=0) and (c[0]%(p^2)!=0):
         return p
   return 0
```

For example

```
Eisenstein(2*x^8+27*x^4+3*x^2+6)
```

yields 3. The expression `c=b.coefficients()` yields a list of coefficients of the polynomial, in order from lowest to highest degree. The expression `c.pop()` returns the last element in the list, and at the same time deletes that element from the list. If `Eisenstein(p)` is not [], then `p` is irreducible by Eisenstein's criterion.

Factorization over rational functions and over polynomials

11.7 Over the

a. complex numbers,

b. real numbers,

c. rational numbers or

d. integers,

e. integer remainders modulo 2,

f. integer remainders modulo 3,

is

$$b(x,y) = x^2 + y^2$$

reducible as a polynomial in two variables x, y?

Proposition 11.6 (Gauss' lemma)**.** *Suppose that $p(x,y)$ is a polynomial in two variables over a field, and that $p(x,y)$ factors as $p(x,y) = b(x,y)c(x,y)$, where $b(x,y)$ and $c(x,y)$ are polynomial in x with coefficients rational functions of y. Then $p(x,y)$ also factors in polynomials in x, y. To be precise, after perhaps multiplying $b(x,y)$ by a rational function of y, and $c(x,y)$ by its reciprocal, we can arrange that $b(x,y)$ and $c(x,y)$ are polynomials in both x and y.*

Proof. The coefficients in x on the right hand side of the equation $p(x,y) = b(x,y)c(x,y)$ are rational in y, hence are quotients of polynomials in y. Multiplying through by a common denominator we obtain an equation

$$d(y)p(x,y) = b(x,y)c(x,y)$$

with new $b(x,y), c(x,y)$ which are polynomials in x, y, and $d(y)$ is a nonzero polynomial. If $d(y)$ is constant, divide it into $c(x,y)$, and the result is proven.

So we can assume that $d(y)$ is nonconstant and write $d(y)$ as a product of irreducible polynomials

$$d(y) = d_1(y)d_2(y)\dots d_n(y).$$

Expand out $b(x,y)$ and $c(x,y)$ into powers of x:

$$b(x,y) = \sum b_j(y)x^j,$$
$$c(x,y) = \sum c_j(y)x^j.$$

Then $d_1(y)$ divides into all of the terms in

$$b(x,y)c(x,y) = \sum_n \sum_{j,k} b_j(y)c_k(y)x^{j+k}.$$

In particular, $d_1(y)$ divides into the lowest term $b_0(y)c_0(y)$, so into one of the factors, say into $b_0(y)$. Suppose that $d_1(y)$ divides into all of the terms $b_0(y), b_1(y), \dots, b_{j-1}(y)$, and also all of the terms $c_0(y), c_1(y), \dots, c_{k-1}(y)$. The x^{j+k} term has coefficient

$$\underline{b_0(y)}c_{j+k}(y) + \underline{b_1(y)}c_{j+k-1}(y) + \dots + \underline{b_{j-1}(y)}c_{k+1}(y)$$
$$+ b_j(y)c_k(y) + b_{j+1}(y)\underline{c_{k-1}(y)} + \dots + b_{j+k}(y)\underline{c_0(y)}.$$

So $d_1(y)$ divides into the underlined expressions, and must divide into one of $b_j(y)$ or $c_k(y)$. So we can increase the value of j, or the value of k, or both. By induction on both j and k, $d_1(y)$ divides into all terms in $b(x,y)$ (and hence divides into $b(x,y)$), or divides into all terms in $c(x,y)$ (and hence divides into $c(x,y)$). We cancel out a copy of $d_1(y)$ from both sides of the equation

$$d(y)p(x,y) = b(x,y)c(x,y)$$

and proceed by induction on the degree of $d(y)$. □

Proposition 11.7 (Eisenstein's criterion)**.** *Take a polynomial $q(x,y)$ over a field. Expand in powers of x:*

$$q(x,y) = a_n(y)x^n + a_{n-1}(y)x^{n-1} + \dots + a_1(y)x + a_0(y),$$

so that each $a_0(y), a_1(y), \dots, a_n(y)$ is a polynomial. Suppose that there is an irreducible polynomial $p(y)$ so that

a. *$p(y)$ does not divide the highest degree term $a_n(y)$ and*
b. *$p(y)$ divides all of the other terms $a_0(y), \dots, a_{n-1}(y)$ and*
c. *$p(y)^2$ does not divide the lowest coefficient $a_0(y)$.*

Then $q(x,y)$ is irreducible.

The proof is identical to the previous proof of Eisenstein's criterion.

$x^{13} + xy + y$ is irreducible, as it has a factor of y in each term in x, except the highest term.

11.8 Is

$$q(x) = 10000x^{10000} + 999x^{999} + 333x^{333} + 111$$

irreducible over the rational numbers?

11.9 Is

$$a(x, y) = y^9x^9 + (y^2 + 1)x^8 + (y^2 + 1)x^7 + y + 1$$

irreducible as a polynomial in two variables x, y with coefficients remainders modulo 2?

11.10 Is

$$b(x, y) = (y + 1)x^{10000} + (y^2 + y + 1)^{999}x^{999} + (y^2 + y + 1)^{333}x^{333} + y^2 + y + 1$$

irreducible as a polynomial in two variables x, y with coefficients

a. remainders modulo 2?

b. integers?

11.11 State and prove a Chinese remainder theorem for polynomials of one variable over any field. Apply it to find the smallest degree polynomial $p(x)$ so that, over the integer remainders modulo 2,

$$\begin{aligned} p(x) &= x + 1 \pmod{x^2 + x + 1}, \\ p(x) &= x \pmod{x^3 + x + 1}. \end{aligned}$$

Homogeneous polynomials

The *degree* of a term in a multivariable polynomial is the sum of the degrees in each of the variables; the *degree* of the polynomial is the highest degree of any nonzero term. A polynomial is *homogeneous* if all of its terms have the same degree.

11.12 Prove that a polynomial $b(x)$ of degree m in some variables $x_1, x_2, \ldots, x_n$ over an infinite field is homogeneous just when $b(tx) = t^m b(x)$ for every t. Give a counterexample over a finite field.

Lemma 11.8. *Over any field, every factor of a homogeneous polynomial is homogeneous.*

Proof. Write the polynomial factorized, as $b(x)c(x)$. The highest degree term in $b(x)$ multiplies by the highest degree term in $c(x)$ to give the highest degree term in $b(x)c(x)$, and the same for the lowest degree terms. Since all terms in $b(x)c(x)$ have the same degree, so do all terms in $b(x)$ and in $c(x)$. □

Chapter 12

Fields

A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies.

— Stefan Banach

Names for our favourite sets

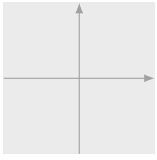
The following are the standard names for various collections of numbers (or remainders):

$\mathbb{Z}$	the set of all integers,
$\mathbb{Q}$	the set of all rational numbers,
$\mathbb{R}$	the set of all real numbers,
$\mathbb{C}$	the set of all complex numbers,
$m\mathbb{Z}$	the set of all integer multiples of some integer m,
$\mathbb{Z}/m\mathbb{Z}$	the set of all remainders modulo a positive integer m.

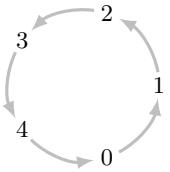
We can draw $\mathbb{R}$ as the number line,

and draw $\mathbb{Z}$ as equispaced dots on the number line,

and $\mathbb{C}$ as the plane:



We could also draw the elements of $\mathbb{Z}/m\mathbb{Z}$ much like we would draw a clock:



12.1 Explain how to add real numbers, how to add complex numbers, and how to add remainders modulo an integer, using these pictures. Explain how to multiply complex numbers using these pictures.

Rings

All of the above sets are examples of rings. A *ring* is a set of objects S together with two operations called *addition* and *multiplication*, denoted like usual addition and multiplication, so that if b and c belong to S, then $b+c$ and bc also belong to S, and so that:

Addition laws:

a. The associative law: For any elements a, b, c in S: $(a+b)+c = a+(b+c)$.

b. The identity law: There is an element 0 in S so that for any element a in S, $a+0=a$.

c. The existence of negatives: for any element a in S, there is an element b in S (denote by the symbol $-a$) so that $a+b=0$.

d. The commutative law: For any elements b, c in S, $b+c=c+b$.

Multiplication laws:

a. The associative law: For any elements a, b, c in S: $(ab)c = a(bc)$.

The distributive law:

a. For any elements a, b, c in S: $a(b+c) = ab+ac$ (left distributive) and $(b+c)a = ba+ca$ (right distributive).

Clearly $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, m\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ (for any positive integer m) are rings.

If X is a set, and S is a ring, then we can add and multiply any two functions mapping X to S by adding and multiplying the values of the functions as usual. The set T of all functions mapping X to S is a ring.

If S is a ring, and x is an abstract variable (really just a pure symbol that we write down) then a *polynomial* in x with values in S is a formal expression we

write down of the form

$$a_0 + a_1 x + \cdots + a_n x^n$$

with coefficients $a_0, a_1, \ldots, a_n$ drawn from S. A polynomial is equal to 0 just when all of its coefficients are 0. Denote the set of all polynomials in x with coefficients from S as $S[x]$. Similarly, if we have two abstract variables x and y, we treat them as commuting: $xy = yx$, and then we define $S[x, y]$ to be the set of all polynomials in x and y with coefficients drawn from S.

$$\frac{1}{3} - 2x + \frac{4}{7}y + 8x^2 y + y^3$$

belongs to $\mathbb{Q}[x, y]$.

$$2x + 1 + 4x^9 y^7 + z$$

belongs to $\mathbb{Z}/15\mathbb{Z}[x, y, z]$.

If S is a ring, a *rational function* with coefficients drawn from S is a formal ratio

$$\frac{b(x)}{c(x)}$$

of polynomials $b(x), c(x)$ in $S[x]$ with $c(x)$ not the zero polynomial. We declare such a formal expression to be equal to

$$\frac{a(x)b(x)}{a(x)c(x)}$$

for any nonzero polynomial $a(x)$. Let $S(x)$ be the collection of all rational functions with coefficients in S in a variable x. *Danger:* even though we call $\frac{b(x)}{c(x)}$ a *rational function*, it is not, strictly speaking, actually a function, but only a formal expression. For example, if $S = \mathbb{Z}/2\mathbb{Z}$, then the rational function

$$\frac{1}{x(x+1)}$$

is not defined for any value of x in $\mathbb{Z}/2\mathbb{Z}$. So we have to think of x not as a variable that varies over some numbers, but as an abstract symbol. Similarly, a rational function is really not a function but an abstract combination of symbols. *Danger:* In some rings, we could have some nonzero $c(x)$ and $a(x)$ but still have $a(x)c(x)$ equal the zero polynomial; this won't happen in our examples below, so we won't worry about it.

If x, y are two abstract variables, we similarly define $S(x, y)$ to be the ring of all rational functions in x, y, i.e. formal ratios of polynomials in x, y with coefficients drawn from S.

If S is a ring, $S\left[x, x^{-1}\right]$ means the ring whose elements are $b(x) + c\left(x^{-1}\right)$, polynomials in an abstract variable x and an abstract variable called x^{-1}, but so that when we multiply those two variables together, we get 1. All elements of $S\left[x, x^{-1}\right]$ are rational functions, but not all rational functions have this form, for example

$$\frac{1}{x+1}$$

is not in $S\left[x, x^{-1}\right]$.

If R and S are rings, let $R \oplus S$ be the set of all pairs (r, s) where r is in R and s is in S. Define addition and multiplication by

$$(r_1, s_1) + (r_2, s_2) \coloneqq (r_1 + r_2, s_1 + s_2)\,,$$
$$(r_1, s_1)\,(r_2, s_2) \coloneqq (r_1 r_2, s_1 s_2)\,.$$

Similarly we define a finite (or even an infinite) sum of rings, as the set of finite (or infinite) sequences of elements, one from each ring, with obvious addition and multiplication, element at a time.

If S is a ring and n is a positive integer, then $n \times n$ matrices with entries in S form a ring, sometimes denoted $S^{n\times n}$, using usual matrix addition and multiplication.

12.2 Let S be the set of all polynomials with real coefficients in one variable. To add elements of S, use usual polynomial addition. Take the "multiplication"operation to be composition of polynomials, *not* usual polynomial multiplication. Prove that S satisfies all of the laws above, except distributivity, and satisfies right but not left distributivity.

If a ring R is a subset of a ring S, and the addition and multiplication operations of R are just the ones of S applied to elements of R, then R is a *subring* of S.

Clearly $\mathbb{Z}$ is a subring of $\mathbb{Q}$, which is subring of $\mathbb{R}$, which is a subring of $\mathbb{C}$.

Special types of rings

A ring is *a ring with identity* if it satisfies the *identity law*: There is an element 1 in S so that for any element a in S, $a1 = a$.

A ring is *a division ring* if it satisfies the zero divisors law: for any elements a, b of S, if $ab = 0$ then $a = 0$ or $b = 0$.

A ring is *commutative* if it satisfies the *commutative law*: for any elements a, b of S, $ab = ba$. Note that we require *every* ring to have commutative addition; a *commutative* ring is one with commutative multiplication.

The associative law for addition, applied twice, shows that $(a + b) + (c + d) = a + (b + (c + d))$, and so on, so that we can add up any finite sequence, in any order,

and get the same result, which we write in this case as $a + b + c + d$. A similar story holds for multiplication.

In any ring, we write $2a$ to mean $a + a$, and so on, and $-2a$ to mean $-a - a$ and so on.

12.3 Let S be the set of all 2×2 matrices with integer coefficients, using the usual laws of matrix addition and multiplication as the addition and multiplication in S. Prove that S is a ring with identity, but *not* commutative.

12.4 Let $2\mathbb{Z}$ be the set of all even integers using the usual laws of integer addition and multiplication as the addition and multiplication in $2\mathbb{Z}$. Prove that $2\mathbb{Z}$ is a commutative ring but without identity.

12.5 Let $2S$ be the set of all 2×2 matrices with even integer coefficients, using the usual laws of matrix addition and multiplication as the addition and multiplication in $2S$. Prove that $2S$ is a noncommutative ring without identity.

12.6 Suppose that S is a ring with identity in which $0 = 1$. Prove that $S = \{ 0 \}$.

From here on, we will only be interested in commutative rings with identity.

Fields

A *unit* in a ring is an element that has a multiplicative inverse, i.e. a reciprocal.

In $\mathbb{Z}/7\mathbb{Z}$ every nonzero element is a unit, because 7 is a prime number.

12.7 What are the units in $\mathbb{Z}/6\mathbb{Z}$?

In $\mathbb{Z}$ the units are ± 1.

If S is a ring, we denote by $S^\times$ the set of units of S.

$(\mathbb{Q}[x])^\times = \mathbb{Q}^\times$: no polynomial of positive degree has a reciprocal polynomial.

Every nonzero rational function is a unit in $\mathbb{Q}(x)$: $(\mathbb{Q}(x))^\times = \mathbb{Q}(x) - \{ 0 \}$.

12.8 Prove that, for any ring S with identity, if b and c are units then bc is a unit.

12.9 Prove that every ring lies in a ring with identity.

12.10 Suppose that R is a ring and that $b \in R$ is both a zero divisor and a unit. Prove that R has exactly one element, the element zero.

A *field* is a commutative ring with identity, in which every nonzero element is a unit, i.e. has a multiplicative inverse, and in which $1 \neq 0$.

Lemma 12.1. *For any positive integer m, $\mathbb{Z}/m\mathbb{Z}$ is a field if and only if m is prime.*

Proof. If m factors, say as $m = bc$, with $b, c \geq 2$, then $b \neq 0$ modulo m, but $bc = 0$ modulo m, so both b and c are zero divisors, and therefore are not units. If m does not factor, then m is prime, and we know that every nonzero element of $\mathbb{Z}/m\mathbb{Z}$ is a unit; we even have a recipe to find its multiplicative inverse. □

The set $\mathbb{Q}$ of rational numbers is a field, as is the set $\mathbb{R}$ of real numbers and the set $\mathbb{C}$ of complex numbers.

Any field k lies inside the field $k(x)$ of rational functions with coefficients in k. In particular, every field is contained inside an infinite field.

12.11 Note that in the field $k = \mathbb{Z}/13\mathbb{Z}$, we have the peculiar equation $13 = 0$. The *characteristic* of a ring with identity R is the smallest positive integer $c > 0$ so that c is zero in R (or in other words, if $e \in R$ is the identity element, then the element $c \cdot e = 0 \in R$). If there is no such positive number c, we say that R has characteristic zero. Prove that the characteristic of a division ring R is prime, say p, and that there is a unique map $\mathbb{Z}/p\mathbb{Z} \to R$ which takes 0 to 0, 1 to 1, and sums to sums and products to products.

12.12 A *zero divisor* in a ring R is an element $r \neq 0$ so that $rs = 0$ for some element $s \neq 0$ of R. Suppose that k is a field. Find the (i) zero divisors and (ii) the units in

a. k

b. $k[x]$

c. $k\left[x, x^{-1}\right]$

d. $k[x] \oplus k[y]$.

Let k be the set of all real numbers of the form $a + b2^{1/3} + c2^{2/3}$ for any a, b, c rational. Clearly all rational numbers (in particular 0 and 1) have this form. It is easy to check that if add any two such numbers, or subtract, we get another such. If we multiply two such numbers:

$$(a + b2^{1/3} + c2^{2/3})(p + q2^{1/3} + r2^{2/3}) = \alpha + \beta 2^{1/3} + \gamma 2^{2/3}$$

where

$$\begin{aligned} \alpha &= ap + 2br + 2cq, \\ \beta &= aq + bp + 2cr, \\ \gamma &= ar + cp + bq. \end{aligned}$$

We need to check division; this turns out to be difficult. Associate to each such number $a + b2^{1/3} + c2^{2/3}$ the vector (a, b, c) in $\mathbb{Q}^3$. The product $(a + b2^{1/3} + c2^{2/3})(p + q2^{1/3} + r2^{2/3})$ has associated vector

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

The determinant of the matrix

$$A = \begin{pmatrix} a & 2c & 2b \\ b & a & 2c \\ c & b & a \end{pmatrix}$$

is $a^3 + 2b^3 + 4c^3 - 6abc$. We need to prove that this determinant is nonzero, i.e. that there is an inverse matrix. Indeed the reader can check that

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \bar{a} & 2\bar{c} & 2\bar{b} \\ \bar{b} & \bar{a} & 2\bar{c} \\ \bar{c} & \bar{b} & \bar{a} \end{pmatrix}$$

where $\bar{a} = a^2 - 2bc$, $\bar{b} = 2c^2 - ab$, $\bar{c} = b^2 - ac$. Note that this tells us precisely that

$$\frac{1}{a + b2^{1/3} + c2^{2/3}} = \frac{\bar{a} + \bar{b}2^{1/3} + \bar{c}2^{2/3}}{\det A}.$$

So in order to see that k is a field, and check the above expression for division in k, we only need to check $\det A \neq 0$. In other words, let's check that if a, b, c are rational numbers, not all zero, then $a^3 + 2b^3 + 4c^3 \neq 6abc$. Note that we can rescale all of a, b, c by the same constant, and both sides of the inequality $a^3 + 2b^3 + 4c^3 \neq 6abc$ scale by the same factor. So we can scale a, b, c so that they become coprime integers. If $a^3 + 2b^3 + 4c^3 = 6abc$, clearly a^3 is even, so a is even, say $a = 2\alpha$. Then $a^3 + 2b^3 + 4c^3 = 6abc$ becomes $8\alpha^3 + 2b^3 + 4c^3 = 12\alpha bc$. Divide out 2 to get $4\alpha^3 + b^3 + 2c^3 = 6\alpha bc$. Clearly b is even, say $b = 2\beta$; plug in: $4\alpha^3 + 8\beta^3 + 2c^3 = 12\alpha\beta c$. Divide out 2 to get $2\alpha^3 + 4\beta^3 + c^3 = 6\alpha\beta c$. Finally we see that c is even, contradicting the fact that a, b, c are coprime integers. Hence we conclude that k is a field.

Linear algebra

Most of linear algebra works identically over any field: matrices, Gaussian elimination, invertibility, solvability of linear systems of equations, determinants, and the theory of eigenvalues and eigenvectors.

12.13 Let k be the Boolean numbers $k := \mathbb{Z}/2\mathbb{Z}$, and A the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

thought of as having entries from k. Is A invertible? If so, find A^{-1}.

12.14 If A is a matrix whose entries are rational functions of a variable t over a field k, prove that the rank of A is constant in t, except for finitely many values of t.

12.15 If A is a matrix whose entries are integers, let A_p be the same matrix, but with coefficients taken modulo a prime p. Prove that A is invertible over the rational numbers (i.e. has an inverse matrix with rational number entries) just when, for all but finitely many primes p, the matrix A_p is invertible over $\mathbb{Z}/p\mathbb{Z}$. Prove that A is not

invertible over the rational numbers just when, for every prime number p, A_p is not invertible over $\mathbb{Z}/p\mathbb{Z}$. Prove that A is invertible over the integers just when, for every prime number p, the matrix A_p is invertible over $\mathbb{Z}/p\mathbb{Z}$.

Factoring in many variables

Theorem 12.2. *Every nonzero polynomial $p(x_1, \ldots, x_n)$ with coefficients in a field admits a factorization*

$$p(x_1, \ldots, x_n) = p_1(x_1, \ldots, x_n) \ldots p_\ell(x_1, \ldots, x_n)$$

into irreducible factors, unique up to reordering and scaling by nonzero constants.

Proof. We prove the result for two variables x, y, but the same proof works for any number of variables. The existence of a factorization is clear by induction on degree. Take two factorizations:

$$p(x, y) = p_1(x, y) \ldots p_\ell(x, y) = q_1(x, y) \ldots q_m(x, y).$$

If the highest order term in one of these $p_j(x, y)$ or $q_j(x, y)$ is in the x variable, then replace x, y by $x + \lambda y, y$, for some constant λ. But for a poor choice of λ, this might knock out the highest order term in y in another $p_j(x, y)$ or $q_j(x, y)$.

To avoid this, we replace our field by the field of rational functions in an abstract variable λ. We claim no cancellations now take place. Consider why. Fix your eye on one particular factor $p_1(x, y)$. The highest degree term in y in the output $p_1(x+\lambda y, y)$ arises from the terms of the same degree in the input $p_1(x, y)$, but in x and y together. Each $x^k y^{n-k}$ in the input contributes a $\lambda^k y^n$, one for each k, to the y^n term in the output. No two such contributions have the same power of λ, so they can't cancel one another. So we get a nonzero y^n term in every factor.

We can reverse the process by replacing x, y by $x - \lambda y, y$, so factorisations are preserved and reflected. If we can prove unique factorisation in this larger field, then any factorisation in the original field will still be unique in the original field. In particular, we can suppose that none of the factors is constant in y, and that the highest order term in y is constant in x.

Thinking of p as a polynomial in y, with coefficients rational in x, by induction the factorisation in such functions is unique. So after perhaps reordering, each $p_j(x, y)$ is equal to some $q_j(x, y)$ up to some rational function $b_j(x)/c_j(x)$, with no common factor in numerator and denominator. But then $c_j(x)p_j(x, y) = b_j(x)q_j(x, y)$, and we expand out in powers of y to get $c_j(x)$ dividing into every coefficient in $q_j(x, y)$, so into the highest coefficient, which has no x in it. □

12.16 Prove that every rational function over a field has an expression

$$\frac{b(x_1, \ldots, x_n)}{c(x_1, \ldots, x_n)}$$

as a ratio of polynomials with no common nonconstant factor, unique up to rescaling both numerator and denominator by the same nonzero constant.

Chapter 13

Field extensions

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

— Bertrand Russell

Quotienting a polynomial

We return to polynomials in a single variable x. We write that $b(x) = c(x)$ modulo a polynomial $p(x)$ to mean that $b(x) - c(x)$ is a multiple of $p(x)$. Just as for integers, we add modulo $p(x)$, subtract modulo $p(x)$, and multiply modulo $p(x)$, all of which makes sense since multiples of $p(x)$ add, subtract, and multiply to give multiples of $p(x)$. When we work modulo $p(x)$, write $b(x)^{-1}$ to mean a remainder so that $b(x)b(x)^{-1} = 1$ modulo $p(x)$. To find $b(x)^{-1}$, we use Bézout coefficients.

Let $p(x) := x^3 + 2x + 1$ over the field of rational numbers. Modulo $p(x)$, clearly $x^3 = -2x - 1$. It is common to use a Greek letter, like α, for the remainder of x modulo $p(x)$, instead of calling it x. So remainders modulo $p(x)$ are just expressions like $\alpha, 7 - 4\alpha, \alpha^2/3$, but when we compute, we reduce modulo $\alpha^3 + 2\alpha + 1$, i.e. we change any α^3 to $\alpha^3 = -2\alpha - 1$. So, for example,

$$\begin{aligned}(\alpha^2 + 1)(\alpha^2 + \alpha + 1) &= \alpha^4 + \alpha^3 + 2\alpha^2 + \alpha + 1,\\ &= \alpha^3\alpha + \alpha^3 + 2\alpha^2 + \alpha + 1,\\ &= (-2\alpha - 1)\alpha + (-2\alpha - 1) + 2\alpha^2 + \alpha + 1,\\ &= -2\alpha^2 - \alpha - 2\alpha - 1 + 2\alpha^2 + \alpha + 1,\\ &= -2\alpha.\end{aligned}$$

If $p(x) = x^2$, and again letting α be the remainder of x modulo $p(x)$, then quotienting out $p(x)$ yields remainders of the form $b + c\alpha$, since $\alpha^2 = 0$. The remainder α is very much like the "very small quantities" that physicists talk about, so small that the square is negligibly small and can be dropped from calculations.

If $p(x) = x(x-1)$, and again letting α be the remainder of x modulo $p(x)$, then quotienting out $p(x)$ yields remainders of the form $b + c\alpha$, but with $\alpha(\alpha - 1) = 0$, so $\alpha^2 = \alpha$. Think of α as a number which can't decide whether it wants to be zero or one, and is somehow behaving like both zero and one at the same time.

Let $p(x) := x^2 + x + 1$ with coefficients over the field of remainders modulo 2. Write the remainder of x modulo $p(x)$ as α. To find α^{-1}, write out $0 = p(\alpha)$:

$$0 = \alpha^2 + \alpha + 1,$$

and subtract the constant term to the other side:

$$1 = \alpha^2 + \alpha,$$

and then factor out an α from the nonconstant terms:

$$1 = \alpha(\alpha + 1)$$

or in other words $\alpha^{-1} = \alpha + 1$.

13.1 With coefficients being integers modulo 3, find x^{-1} modulo $x^9 + 2x^2 + 1$.

13.2 Guess: what do you think the Chinese remainder theorem might be for remainders modulo polynomials, instead of working with remainders modulo integers?

We can find reciprocals more generally, not just of α but of any remainder modulo any polynomial, using Bézout coefficients. Let $p(x) := x^3 + x + 1$ with coefficients over the field of remainders modulo 2. Write the remainder of x modulo $p(x)$ as α. To find $(\alpha^2 + \alpha + 1)^{-1}$, find Bézout coefficients of $p(x)$ and $x^2 + x + 1$:

$$\begin{pmatrix} 1 & 0 & x^2+x+1 \\ 0 & 1 & x^3+x+1 \end{pmatrix} \text{ add } x(\text{row } 2) \text{ to row } 1,$$

$$\begin{pmatrix} 1 & 0 & x^2+x+1 \\ x & 1 & x^2+1 \end{pmatrix} \text{ add row } 2 \text{ to row } 1,$$

$$\begin{pmatrix} 1+x & 1 & x \\ x & 1 & x^2+1 \end{pmatrix} \text{ add } x(\text{row } 1) \text{ to row } 2,$$

$$\begin{pmatrix} 1+x & 1 & x \\ x^2 & x+1 & 1 \end{pmatrix}.$$

Hence

$$(x^2)(x^2 + x + 1) + (x + 1)(x^3 + x + 1) = 1.$$

Modulo out $p(x)$ to get

$$(\alpha^2)(\alpha^2 + \alpha + 1) = 1.$$

Hence

$$\frac{1}{\alpha^2+\alpha+1} = \alpha^2.$$

13.3 Work out the complete multiplication table for the remainders of polynomials in x when we quotient out by $x^2 + x$ over the field of remainders modulo 3. Which elements have reciprocals?

Field extensions

If K is a field and $k \subset K$ is a subring containing some nonzero element, and containing the reciprocals of all of its nonzero elements, k is a *subfield* of K and K is an *extension* of k.

The complex numbers are a field extension of the real numbers.

The real numbers are a field extension of the rational numbers.

Consider how we could write down field extensions. If $k \subset K$ is a subfield, a *spanning set* of K over k is a collection of elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ of K so that every element of K has the form

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

for some coefficients

$$a_1, a_2, \ldots, a_n$$

from k. A *basis* is a minimal spanning set. The *degree* of K over k is the smallest number of elements in a basis.

We can write every element of $\mathbb{C}$ as $x \cdot 1 + y \cdot i$, for some real coefficients x, y. So $\mathbb{C}$ is a degree 2 extension of $\mathbb{R}$ with $1, i$ as a basis.

Clearly $1, \sqrt{2}$ is a basis for $\mathbb{Q}\left(\sqrt{2}\right)$ over $\mathbb{Q}$.

The real numbers $\mathbb{R}$ as an extension of $\mathbb{Q}$ has no basis; it is an infinite degree extension. This is a difficult fact to prove.

If $k \subset K \subset L$ are finite degree extensions then clearly $k \subset L$ is a finite degree extension, with degree the product of the degrees of $k \subset K$ and $K \subset L$. Take a basis α_i for $k \subset K$ and a basis β_j for $K \subset L$; then $\alpha_i\beta_j$ is a basis for $k \subset L$.

Any finite "tower" of extensions, each of finite degree, has degree the product of the degrees, so is a finite degree extension.

Adding a root

Let's return to the example on page 94. Look for roots of $p(x) := x^2 + x + 1$ over the finite field $k := \mathbb{Z}/2\mathbb{Z}$. Try $x = 0$:

$$p(0) = 0^2 + 0 + 1 = 1,$$

no good. Try $x = 1$:

$$p(1) = 1^2 + 1 + 1 = 1 + 1 + 1 = 1,$$

since $1 + 1 = 0$. No good. So $p(x)$ has no roots in k.

Let K be the ring of remainders modulo $p(x)$. As before, we write the remainder of x modulo $p(x)$ as α. Modulo $p(x) = x^2 + x + 1$, of course $x^2 = x + 1$, i.e. $\alpha^2 = \alpha + 1$. So any polynomial of degree 2 or more in α reduces to one of degree 1, by replacing α^2 by $\alpha + 1$. So every element of K is a polynomial of degree 0 or 1 in α, with coefficients 0 or 1:

$$\{0, 1, \alpha, \alpha + 1\} = K.$$

Using only the fact that $1 + 1 = 0$, compute out the addition table

$+$	0	1	α	$\alpha+1$
0	0	1	α	$\alpha+1$
1	1	0	$\alpha+1$	α
α	α	$\alpha+1$	0	1
$\alpha+1$	$\alpha+1$	α	1	0

Using only the fact that $\alpha^2 = \alpha + 1$, compute out the multiplication table:

$\cdot$	0	1	α	$\alpha+1$
0	0	0	0	0
1	0	1	α	$\alpha+1$
α	0	α	$\alpha+1$	1
$\alpha+1$	0	$\alpha+1$	1	α

Looking for reciprocals, we find that

$$\begin{aligned} \frac{1}{0} &\text{ does not exist,} \\ \frac{1}{1} &= 1, \\ \frac{1}{\alpha} &= \alpha + 1, \\ \frac{1}{\alpha+1} &= \alpha. \end{aligned}$$

Every nonzero element has a reciprocal: K is a field. So $K = \{0, 1, \alpha, \alpha + 1\}$ is an extension of k. The polynomial $p(x)$ splits over this field: $p(x) = (x + \alpha)(x + 1 + \alpha)$. (Now we see why we like to call our remainder by a Greek letter: we get to use x again for an abstract variable.)

Theorem 13.1. *Suppose that $p(x)$ is a polynomial over a field k, Let α be the remainder of x modulo $p(x)$, and write $k[\alpha]$ for the ring of remainders modulo $p(x)$. Thinking of $p(x)$ as a polynomial, $p(x)$ has α as a root in $k[\alpha]$. The ring $k[\alpha]$ is a field just when $p(x)$ is irreducible over k. If this happens, then $k(\alpha) = k[\alpha]$ and $k(\alpha)$ is an extension field of k and $p(x)$ has a linear factor over the field $k(\alpha)$.*

Proof. By definition, $p(\alpha)$ means $p(x)$ but with x replaced by α, i.e. by the remainder of x modulo $p(x)$, i.e. by x, but quotienting out any copies of $p(x)$ from the resulting expression $p(x)$, i.e. $p(\alpha)$ is just 0. Every element of $k[\alpha]$ is a polynomial $b(\alpha)$ in α, but of degree less than the degree of $p(x)$, since we can quotient out any higher order terms by $p(x)$. So $k[\alpha]$ is a field just when any nonzero $b(\alpha)$ in $k[\alpha]$ has a reciprocal in $k[\alpha]$.

Suppose that $p(x)$ is irreducible. Take Bézout coefficients for $b(x), p(x)$: $s(x)b(x) + t(x)p(x) = d(x)$, so $d(x)$ is the greatest common factor of $b(x), p(x)$. Then $b(x), p(x)$ have greatest common divisor 1, i.e. $d(x) = 1$ (after perhaps a constant rescaling), so mod out $p(x)$ to get $s(\alpha)b(\alpha) = 1$, i.e. $b(\alpha)$ has a reciprocal. So every nonzero polynomial in α has a reciprocal, i.e. $k(\alpha) = k[\alpha]$.

Suppose that $p(x)$ is reducible, say $p(x) = b(x)c(x)$, into lower degree factors, so $0 = b(\alpha)c(\alpha)$. We want to prove that $b(\alpha)$ has no reciprocal. If $b(\alpha)$ has a reciprocal, multiply both sides by it to get $0 = c(\alpha)$, i.e. $c(x)$ is a multiple of $p(x)$, but has degree less than that of $p(x)$, so $c(x) = 0$, so $p(x) = b(x)c(x) = 0$, a contradiction. □

We say that the field $k[\alpha]$ arises from k by "adding a root" α to the polynomial $p(x)$. Careful: the trick only works if $p(x)$ is irreducible. If $p(x)$ is reducible, split it into irreducible factors, and you can add roots as you like for each factor.

This theorem explains why, in our last example, adding a root to $p(x) = x^2 + x + 1$ over $k = \mathbb{Z}/2\mathbb{Z}$ gave a field $K = \{\, 0, 1, \alpha, \alpha + 1 \,\}$, not just a ring.

An element $u \in K$ is *algebraic* for an extension $k \subset K$ if it is a root of a nonconstant polynomial in $k[x]$. If such a polynomial factors, u is a root of one of the factors. If u is a root of two polynomials, it is a root of the greatest common divisor. Hence there is a smallest degree nonconstant polynomial $p(x) \in k[x]$ so that $p(u) = 0$. This $p(x)$ is uniquely determined if we rescale to make it monic, and then called the *minimal polynomial* of u for the extension $k \subset K$.

Corollary 13.2. *An element $u \in K$ is algebraic for an extension $k \subset K$ just when $k[u] = k(u) \subset K$ is a finite degree field extension of k lying inside K. If this happens, that field extension is isomorphic to $k[\alpha]$ where, as usual, α is the remainder of x modulo $p(x)$, and where $p(x) \in k[x]$ is the minimal polynomial of u.*

Proof. For any $u \in K$, if $k[u] \subset K$ is a finite dimensional vector space then the powers $1, u, u^2, \ldots$ acheive some linear relation after finitely many steps. Such a relation, say

$$0 = a_0 + a_1 u + \cdots + a_n u^n,$$

is a polynomial in u, so u is algebraic; we can assume that $a_n = 1$. Reverse steps: having $k[u]$ of finite dimension is equivalent to u being algebraic.

The minimal polynomial of u is then

$$p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n.$$

Clearly $a_0 \neq 0$ since we could otherwise factor out some power of x. So then we can solve for u^{-1}: divide both sides of the relation by $a_0 u$:

$$0 = u^{-1} + \frac{a_1}{a_0} + \cdots + u^{n-1},$$

so we solve for u^{-1} as a polynomial in u. But the same argument works for any element of our finite dimensional vector space: $k[u]$ is a field inside K. Map $x \in k[x] \mapsto u \in k[u] = k(u)$ onto, a linear map over k. The kernel is precisely the set of polynomials divisible by $p(x)$, so the map descends to an isomorphism from the quotient vector space $k[\alpha] \to k[u]$, preserving addition and multiplication. □

Corollary 13.3. *Every finite degree field extension consists entirely of algebraic elements.*

Corollary 13.4. *A field extension by finitely many algebraic elements is algebraic.*

Proof. It has finite degree. □

Theorem 13.5. *The only finite degree extensions of $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$.*

Proof. The extension must be algebraic, so either is $\mathbb{R}$ or contains an extension by adding a root of an irreducible. The irreducibles are linear or quadratic, and the linear have roots in $\mathbb{R}$, so the extension is by adding a root of an irreducible quadratic polynomial $x^2 + bx + c$, so some

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

with negative discriminant. Translating α by a real number, the field extension is the same field we get by adding just a square root to the discriminant $b^2 - 4c$. Scaling, it is the same field we get by adding a $\sqrt{-1}$, i.e. isomorphic to $\mathbb{C}$. We can now factor every quadratic, by the same formula, so now our field is algebraically closed, and so has no extension. □

Splitting fields

A polynomial $p(x)$ with coefficients in a field k *splits* if it is a product of linear factors. If $k \subset K$ is a subfield, a polynomial $p(x)$ in $k[x]$ *splits over* K if it splits into a product of linear factors when we allow the factors to have coefficients from K.

The real-coefficient polynomial $x^2 + 1$ splits over $\mathbb{C}$:

$$x^2 + 1 = (x - i)(x + i).$$

An extension K of a field k is a *splitting field* of a polynomial $p(x)$ over k if $p(x)$ splits over K and every element of K is expressible as a rational function (with coefficients from k) of the roots of $p(x)$.

13.4 Denote by $\mathbb{Q}\left(\sqrt{2}\right)$ the field extension of $\mathbb{Q}$ generated by $\sqrt{2}$, i.e. the smallest subfield of $\mathbb{C}$ containing $\sqrt{2}$. Prove that $\mathbb{Q}\left(\sqrt{2}\right)$ consists precisely of the numbers $b + c\sqrt{2}$ for b, c rational. Prove that $\mathbb{Q}\left(\sqrt{2}\right)$ is a splitting field for $x^2 - 2$ over $\mathbb{Q}$.

Theorem 13.6. *Every polynomial $p(x)$ over any field k has a splitting field K. If $p(x)$ has degree n then K has degree at most $n!$ over k.*

Proof. Split $p(x)$ into irreducible factors, and add a root of one of them. When you add a root, the elements of the extension field are polynomials of degree less than n, so the degree of extension is at most n, equal to n just when $p(x)$ is itself irreducible. In the extension, $p(x)$ splits off a linear factor $p(x) = (x - \alpha)q(x)$, so the next time you add a root, you are adding it to some $q(x)$ of lower degree. By induction, the total degree is at worst $n!$. □

The splitting field is more-or-less unique, but this is difficult to make precise; see theorem 21.7 on page 185.

Over $k = \mathbb{R}$ the polynomial $p(x) = x^2 + 1$ has splitting field $\mathbb{C}$:

$$x^2 + 1 = (x - i)\,(x + i)\,.$$

Every polynomial over $\mathbb{C}$ splits into linear factors, so for any subfield $k \subset \mathbb{C}$ the splitting field K of any polynomial over k lies inside $\mathbb{C}$, a field of complex numbers.

13.5 Consider the polynomial

$$p(x) = x^3 + x^2 + 1$$

over the field $k = \mathbb{Z}/2\mathbb{Z}$. Suppose that that splitting field K of $p(x)$ contains a root α of $p(x)$. Prove that α^2 and $1 + \alpha + \alpha^2$ are the two other roots. Compute the addition table and the multiplication table of the 8 elements

$$0, 1, \alpha, 1 + \alpha, \alpha^2, 1 + \alpha^2, \alpha + \alpha^2, 1 + \alpha + \alpha^2.$$

Use this to prove that

$$K = \left\{0, 1, \alpha, 1 + \alpha, \alpha^2, 1 + \alpha^2, \alpha + \alpha^2, 1 + \alpha + \alpha^2\right\}$$

so K is a finite field with 8 elements.

Algebraic closure

A field k is *algebraically closed* if every polynomial over k splits into linear factors in k. An extension K of a field k is a *splitting field* of a collection of polynomials over k if every one of those polynomials splits over K and K is generated by the roots of all of these polynomials put together. If that collection consists of *all* of the polynomials defined over k, then we say that K is an *algebraic closure* of k. We won't prove:

Theorem 13.7. *Every field k has an algebraic closure, denoted $\bar{k}$, unique up to an isomorphism which is the identity map on k. The field $\bar{k}$ is algebraically closed.*

See [6] p. 544, proposition 30 for a proof.

Clearly $\bar{\mathbb{R}} = \mathbb{C}$.

Danger: $\bar{\mathbb{Q}} \neq \mathbb{C}$. The number $\pi = 3.14\ldots$ does not satisfy any polynomial with rational coefficients, so π belongs to $\mathbb{C}$ but doesn't belong to $\bar{\mathbb{Q}}$. However, $\bar{\mathbb{Q}}$ is a subfield of $\mathbb{C}$. Sit $\mathbb{Q}$ inside $\mathbb{C}$ as usual. Take the subfield of $\mathbb{C}$ generated by all roots of all polynomials with rational coefficients, all roots of polynomials with all of those as coefficients, and so on, and take the union K of all such fields. By the theorem, K is isomorphic to $\bar{\mathbb{Q}}$, by an isomorphism fixing every rational number.

If k is any finite field, say with elements

$$k = \{\, \alpha_1, \alpha_2, \ldots, \alpha_n \,\},$$

then let

$$p(x) := 1 + (x - \alpha_1)\,(x - \alpha_2)\ldots(x - \alpha_n)\,.$$

Clearly $p(\alpha) = 1$ for any $\alpha \in k$. Therefore k is not algebraically closed: every algebraically closed field is infinite.

13.6 Prove that, for any finite field, and any integer, there is a finite extension of that field which has more elements than that integer.

Chapter 14

Formal power series

If the doors of perception were cleansed every thing would appear to man as it is, Infinite. For man has closed himself up, till he sees all things thro' narrow chinks of his cavern.

— William Blake
The Marriage of Heaven and Hell

Introduction

Recall that many elementary functions are expressed in *power series* (also called *Taylor series*):

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

some of which are not expanded around the origin:

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n},$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

Each of these converges only in some interval near the value of x around which we are expanding. Since we are interested in algebra, we ignore convergence. A *formal power series* in variables x, y, z is a formal sum

$$\sum_{p,q,r=0}^{\infty} c_{pqr} x^p y^q z^r$$

where the coefficients c_{pqr} belong to some ring R, and we don't require the sum to converge. (For a general ring R, there is really no concept of convergence anyway).

The formal power series

$$0! + 1!x + 2!x^2 + 3!x^3 + \dots$$

over the real numbers converges nowhere but at $x = 0$.

Similarly, if we have variables $x_1, \dots, x_n$, take integers, positive or zero, $\alpha = (\alpha_1, \dots, \alpha_n)$ and write x^α to mean $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ A *formal power series* in $x_1, \dots, x_n$ over a ring R is a formal sum

$$\sum_\alpha c_\alpha x^\alpha,$$

with coefficients c_α in R. The set of all formal power series in some variables $x_1, \dots, x_n$ over a ring R is denoted $R[[x_1, \dots, x_n]]$, or $R[[x]]$ for short if $x = (x_1, \dots, x_n)$ is understood.

Arithmetic

Add, subtract, multiply and differentiate by the usual rules for polynomials, applied term by term to the whole power series, treating all coefficients as commuting with all variables. In other words, add by

$$\left(\sum_\alpha b_\alpha x^\alpha\right) + \left(\sum_\alpha c_\alpha x^\alpha\right) := \sum_\alpha (b_\alpha + c_\alpha)\, x^\alpha,$$

subtract by

$$\left(\sum_\alpha b_\alpha x^\alpha\right) - \left(\sum_\alpha c_\alpha x^\alpha\right) := \sum_\alpha (b_\alpha - c_\alpha)\, x^\alpha,$$

scale by

$$c\left(\sum_\alpha b_\alpha x^\alpha\right) := \sum_\alpha (cb_\alpha)\, x^\alpha,$$

and

$$\left(\sum_\alpha b_\alpha x^\alpha\right) c := \sum_\alpha (b_\alpha c)\, x^\alpha,$$

(so that, in effect, we are declaring that all variables x_i commute with all coefficients from R), and multiply by

$$\left(\sum_\alpha b_\alpha x^\alpha\right)\left(\sum_\alpha c_\alpha x^\alpha\right) := \sum_\alpha \left(\sum_{\beta+\gamma=\alpha} b_\beta c_\gamma\right) x^\alpha.$$

If R is commutative, or a ring with identity, so is $R[[x_1, \dots, x_n]]$.

If R is a ring with identity, a *monomial* is an expression x^α. When we multiply a formal power series by a monomial, we add α to the power of every term, with the same coefficients appearing only shifted in powers. Therefore, if a formal power series $f(x)$ can be factored by a monomial x^α, the factorization is unique, and we write it as $f(x)/x^\alpha$.

Define

$$\frac{\partial}{\partial x_i}\sum_\alpha b_\alpha x^\alpha = \sum_{\alpha_i>0} b_\alpha \alpha_i \frac{x^\alpha}{x_i}.$$

If we let

$$b(x) = \sum_\alpha b_\alpha x^\alpha,$$

write $b(0)$ to mean b_0, but otherwise we can not make sense of setting x to a constant inside $b(x)$, since we have no notion of convergence. In the same vein, we can write $b(0, x_2, \ldots, x_n)$ to mean the sum of all terms in the expansion of $b(x)$ which have zero power of x_1.

It is convenient to write $x^\alpha|$ to mean the operator which associates to each formal power series its x^α coefficient, so

$$x^3\Big|\,(1 - 7x + 4x^2 - 12x^3 + 2x^4) = -12$$

while

$$x^3\Big|\,(1 + 2 + 3x + 4x^2) = 0.$$

14.1 Prove that

$$1 = (1 - x)(1 + x + x^2 + \ldots)$$

over any ring with identity.

For each $\alpha = (\alpha_1, \ldots, \alpha_n)$, it is traditional to write $|\alpha|$ to denote $\alpha_1 + \cdots + \alpha_n$, and to say that x^α has *order* $|\alpha|$.

The role played by highest order terms in polynomials is often played by lowest in formal power series. The *lowest order part* of a formal power series is the sum of all nonzero terms of lowest order.

In two variables x, y, the lowest order part of $x^3 + 4x^2y + x^4 + y^5 + \ldots$ is $x^3 + 4x^2y$.

Infinite sums and products

We can't make sense of infinite sums in an arbitrary ring, since we don't have a notion of convergence. We allow ourselves to define an infinite sum of formal power series

$$f_1(x) + f_2(x) + \ldots$$

only if the lowest orders of the $f_j(x)$ go to infinity with j; only finitely many $f_j(x)$ add to the term at each order.

14.2 Let

$$f_1(x) = 1 + x + x^2 + \ldots,$$

and let $f_k(x) = x^{k-1} f_1(x)$. Find $f_1(x) + f_2(x) + \ldots$.

Similarly, an infinite product

$$f_1(x) f_2(x) \ldots$$

is defined as long as the lowest order of each $f_j(x)$ goes to ∞ with j.

If

$$\begin{aligned} f_1(x) &= 2 + 2x + 2x^2 + 2x^3 + 2x^4 + \dots, \\ f_2(x) &= 1 + 3x^2 + 3x^3 + 3x^4 + \dots, \\ f_3(x) &= 1 + 4x^3 + 4x^4 + \dots, \\ &\vdots \end{aligned}$$

then

$$f_1(x)f_2(x)f_3(x)\cdots = 2 + 2x + 8x^2 + \dots$$

14.3 Over the rational numbers, find the terms up to order 3 in

$$\prod_{n=1}^{\infty} \exp(x^n).$$

Division

Theorem 14.1. *A formal power series in a single variable, over a commutative ring with identity, has a reciprocal formal power series if and only if its constant term is a unit.*

Proof. Divide off the constant term, so we can assume it is 1, say $f(x) = 1 - g(x)$ with $g(0) = 0$. Take the identity

$$1 = (1 - x)(1 + x + x^2 + \dots)$$

and replace x by the formal power series $g(x)$:

$$1 = f(x)(1 + g(x) + g(x)^2 + \dots).$$

□

Corollary 14.2. *If $f(x), g(x)$ are formal power series in a single variable, over a commutative ring with identity, and $g(0)$ is a unit, then there is a unique formal power series $h(x)$ so that $f(x) = h(x)g(x)$, and $h(x)$ is therefore denoted $f(x)/g(x)$.*

14.4 Over a field, take any two formal power series $b(x), c(x)$ in a single variable x. Lets try to divide $b(x)$ by $c(x)$. Prove that there is a unique polynomial $r(x)$ of degree less than the lowest order of $c(x)$ and a unique formal power series $q(x)$ so that $b(x) = q(x)c(x) + r(x)$.

Composition

When we compose polynomial functions $y = f(x)$ and $z = g(y)$, we obtain a polynomial function $z = g(f(x)) = (g \circ f)(x)$. Note that $g(f(x))$ has constant term $g(f(0))$. So we cannot hope to extend composition to formal power series, since $g(y)$ might not make sense with x is replaced by $f(0)$. So we will need to assume that $g(y)$ is

a polynomial or that $f(0) = 0$. On the other hand, if $f(0) = 0$, then $g(f(0)) = g(0)$ is defined. For polynomials, the degree of the composition is at most the product of the degrees, since the highest possible term in $g(f(x))$ comes from taking that in $f(x)$ and plugging into that of $g(y)$. By the same reasoning, the term in $g(f(x))$ of a given degree n, perhaps not the highest degree, arises only from terms of degree p in $f(x)$ plugged into terms of degree q in $g(y)$ with $pq = n$. So we can define the composition $g(f(x))$ as long as $f(0) = 0$ or $g(y)$ is a polynomial.

Over any field k of characteristic zero, the *exponential series* is

$$\exp(x) = 1 + x + \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We cannot expand out $\exp(\exp(x))$ since $\exp(0) = 1 \neq 0$. We *can* expand out $\exp(-1 + \exp(x))$:

$$\begin{aligned}
\exp(-1 + \exp(x)) &= 1 + (-1 + \exp(x)) + (-1 + \exp(x))^2 + \dots, \\
&= 1 + \left(x + \frac{x^2}{2} + \dots\right) + \frac{\left(x + \frac{x^2}{2} + \dots\right)^2}{2} + \dots, \\
&= 1 + x + \frac{x^2}{2} + \frac{x^2}{2} + \dots, \\
&= 1 + x + x^2 + \dots
\end{aligned}$$

14.5 Find the next term in $\exp(-1 + \exp(x))$.

14.6 Is it true that

$$\prod_{k=1}^{\infty} \exp(x^k) = \exp\left(\sum_{k=1}^{\infty} x^k\right)?$$

Formal maps

A *formal transformation* or *formal map* $y = f(x)$, for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ variables, over a ring R, is a choice of formal power series $y_i = f_i(x)$ for each y_i variable and with $y_i(0) = 0$. For a matrix A with nonnegative integer entries, we let

$$x^A := \left(\prod_j (x_j)^{A_{1j}}, \dots, \prod_j (x_j)^{A_{mj}}\right).$$

The reader can check that $(x^A)^B = x^{(BA)}$: exponents multiply under composition, but not quite how we might guess. Each formal transformation is

$$y = f(x) = \sum_A b_A x^A,$$

where b_A are matrices with entries in R. If there are only finitely many nonzero terms, the map is a *polynomial map*.

Define the *composition* of two formal maps $z = g(y)$ and $y = f(x)$, with $f(0) = 0$ or with $g(y)$ polynomial, to have terms of order n given by taking the terms of order k in $f(x)$ plugged into terms of degree $n - k$ in $g(y)$, to produce a polynomial map, and then taking its terms of order n.

Consider the expression

$$(a_0 + a_1 x + a_2 x^2 + \dots)^n$$

over a commutative ring. Expanding out,

$$a_0^n + n a_0^{n-1} a_1 x + \dots$$

the coefficient of x^ℓ is

$$\sum_{0<k\le\ell} \sum_{i_1+\dots+i_k=\ell} a_{i_1} \dots a_{i_k}.$$

The only way to get an a_ℓ in there is in the term

$$n a_0^{n-1} a_\ell,$$

and no coefficient a_k for any $k > \ell$ can appear in that term. So if we write it as

$$(a_0 + a_1 x + a_2 x^2 + \dots)^n = b_0 + b_1 x + \dots$$

then

$$\begin{aligned} b_0 &= a_0^n, \\ b_1 &= n a_0^{n-1} a_1, \\ b_2 &= n a_0^{n-1} a_2 + \binom{n}{2} a_0^{n-2} a_1^2, \\ &\vdots \\ b_\ell &= n a_0^{n-1} a_\ell + \dots, \end{aligned}$$

We try to solve these equations for $a_0, a_1, \dots$ in terms of $b_0, b_1, \dots$. Suppose that $b_0 \neq 0$. We only need to solve $a_0^n = b_0$ for a_0. After that, each equation for b_ℓ is linear in the unknown a_ℓ, so has a unique solution as long as $n a_0^{n-1}$ has a reciprocal.

In summary:

Theorem 14.3. *Any formal power series in a single variable has an n^{th} root formal power series, as long as*

- *it is over a commutative ring with identity in which n is a unit and*
- *its lowest degree term has degree a multiple of n and*
- *its lowest degree term has coefficient an n^{th} power of a unit.*

There is then exactly one n^{th} root of the formal power series for each n^{th} root of that coefficient.

Proof. We look for an n^{th} root of some formal power series

$$y = b_0 x^{kn} + b_1 x^{kn+1} + \cdots = x^{kn}(b_0 + b_1 x + \dots),$$

for some multiple kn. We can take n^{th} root x^k of x^{kn}, so we need only take n^{th} root of $b_0 + b_1 x + \dots$, as above. All subsequent a_ℓ are uniquely determined after the choice of a_0, i.e. n^{th} root of b_0. If we change the choice of a_0, say to some A_0, which is also an n^{th} root of b_0, then

$$\frac{A_0^{n-1}}{a_0^n}$$

is a reciprocal of A_0, so A_0 is also a unit. □

Theorem 14.4. *Suppose that $p(x, y)$ is a formal power series over a commutative ring, with $p(0, 0) = 0$. Suppose that the coefficient of the linear term in y is a unit in the commutative ring. Then there is a unique formal power series $y = y(x)$, over the same commutative ring, with $y(0) = 0$ and with $p(x, y(x)) = 0$ as a composition of formal power series. The terms of this formal power series $y(x)$ are given by polynomials, with integer coefficients, in the terms of the formal power series $p(x, y)$, after divided $p(x, y)$ by that unit.*

Proof. We can rescale $p(x, y)$ to arrange that the coefficient of y is -1, so that we can write the formal equation $0 = p(x, y)$ as $y = q(x, y)$, where $q(0, 0) = 0$ and where $q(x, y)$ has vanishing y term. Expand out $q(x, y) = \sum b_{pq} x^p y^q$ to make the equation $y = q(x, y)$ into

$$\begin{aligned}
y =& (b_{10}\,x + b_{20}x^2 + b_{30}\,x^3 + \dots) \\
&+ (b_{11}\,x + b_{21}x^2 + b_{31}\,x^3 + \dots)y \\
&+ (b_{02} + b_{12}\,x + b_{22}x^2 + b_{32}\,x^3 + \dots)y^2 \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \\
&+ (b_{0n} + b_{1n}x + b_{2n}x^2 + b_{3n}x^3 + \dots)y^n \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}$$

Write out $y = c_1 x + c_2 x^2 + \dots$, and plug in to get

$$\begin{aligned}
& c_1\ x + c_2 x^2 + c_3\ x^3 \dots \\
=& (b_{10}\,x + b_{20}x^2 + b_{30}\,x^3 + \dots) \\
&+ (b_{11}\,x + b_{21}x^2 + b_{31}\,x^3 + \dots)(c_1 x + c_2 x^2 + c_3 x^3 + \dots) \\
&+ (b_{02} + b_{12}\,x + b_{22}x^2 + b_{32}\,x^3 + \dots)(c_1 x + c_2 x^2 + c_3 x^3 + \dots)^2 \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \\
&+ (b_{0n} + b_{1n}x + b_{2n}x^2 + b_{3n}x^3 + \dots)(c_1 x + c_2 x^2 + c_3 x^3 + \dots)^n \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots
\end{aligned}$$

Expand out term by term to find equations

$$\begin{aligned} c_1 &= b_{10}, \\ c_2 &= b_{20} + b_{11}c_1 + b_{02}c_1^2 \\ c_3 &= b_{30} + b_{11}c_2 + b_{21}c_1 + b_{12}c_1^2 + 2b_{02}c_1c_2 + b_{03}c_1^3, \\ &\vdots \end{aligned}$$

It is clear that we can calculate out order by order, but let us prove it. Consider the general induction argument to arrive at the general term $c_\ell = \ldots$. We can write the above equations as

$$y = xQ_0(x) + xQ_1(x)y + \sum_{n=2}^{\infty} q_n(x)y^n.$$

We know all of the coefficients of $Q_1(x), Q_2(x), q_2(x), q_3(x), \ldots$. We want y to be divisible by x; write $y = xz$ and plug in to get

$$z = Q_0(x) + xQ_1(x)z + \sum_{n=2}^{\infty} q_n(x)x^{n-1}z^n.$$

Take a formal power series $z = c_1 + c_2x + \ldots$ and plug in. Each term c_k appearing on the right hand side appears in $xQ_1(x)z$, multiplied by at least one x, so forces itself into the equation for c_{k+1} on the left hand side. But c_k also it appears in $q_n(x)x^{n-1}z^n$. In that term, n factors c_k multiply together, say of degrees $k_1, k_2, \ldots, k_n$, and give a term of order at least $n - 1 + k_1 + \cdots + k_n$, with $n \geq 2$, so a term of higher order that any of $c_{k_1}, \ldots, c_{k_n}$. Hence each term c_k on the right hand side only occurs in equations that give rise to terms c_ℓ on the left hand side, with $k < \ell$. Hence every equation, term by term, solves for some c_ℓ in terms of lower order c_k. So there is a unique equation determining each term, and it determines it as a polynomial, with integer coefficients, in the lower order terms. □

Corollary 14.5. *At any smooth point of a plane algebraic curve, given by one non-trivial polynomial equation $0 = p(x, y)$ in two variables, we can arrange by a linear change of variables that the curve passes through the origin of coordinates, and is not tangent to the vertical coordinate axis there. Then the equation of the curve $0 = p(x, y)$ has a unique formal power series solution $y = y(x)$ at that point.*

14.7 Prove that the polynomial $p(x, y) = y^2 - x^2(1+x)$ is irreducible as a polynomial over any commutative ring with identity. (Hint: a stronger result is perhaps easier: $y^2 - f(x)$ is irreducible just when the polynomial $f(x)$ is a square.) Over the real numbers, its zeroes form an algebraic plane curve:

Explain how to factor $p(x, y)$ into a product of two formal power series, each linear in x (so it is *not* irreducible as a formal power series), over any commutative ring with identity. Roughly speaking, near the origin, the algebraic plane curve consists of the "graphs" of two "functions", but the "functions" might just be formal power series.

Consider a formal power series $x = a_1 y + \ldots$; can we solve for $y = y(x)$, as a formal power series in x? According to our theorem, if a_1 is a unit, then we can. But suppose that $0 = a_1 = a_2 = \cdots = a_{n-1}$ but $a_n \neq 0$? The simplest example is $x = y^2$. No formal power series can solve this: if the constant term of $y(x)$ is nonzero, so is that of $y(x)^2$. If not zero, then $y(x)^2$ has degree 2 or more in x in its lowest term. So we have to introduce a formal root, i.e. work modulo the ideal generated by $z = x^2$, for some new variable z. This is the only obstruction:

Theorem 14.6. *Take a formal power series $x = a_1 y + \ldots$ whose lowest order nonzero coefficient, say a_n, is a unit which has an n^{th} root. Then the formal power series has a formal inverse function, i.e. we can solve it for $y = y(x^{1/n})$, term by term in a formal power series, to ensure identity composition in either direction, after we introduce an n^{th} root $x^{1/n}$. There is precisely one such inverse for each n^{th} root of a_n.*

Proof. Let c_0 be an n^{th} root of a_n, the first nonzero term of $x = x(y)$. Write $x^{1/n}$ for a formal n^{th} root of x, i.e. the remainder of z modulo $z^n - x$. Apply theorem 14.3 on page 106 to take an n^{th} root formal power series

$$1 + b_1 y + b_2 y^2 + \ldots$$

of the formal power series

$$1 + \frac{a_{n+1} y}{a_n} + \frac{a_{n+2} y^2}{a_n} + \ldots$$

Apply theorem 14.4 on page 107 to the equation

$$0 = -x^{1/n} + c_0 y(1 + b_1 y + b_2 y^2 + \ldots),$$

to give a formal power series

$$y = c_1 x^{1/n} + c_2 x^{2/n} + \ldots.$$

So this solves the n^{th} power of

$$x^{1/n} = c_0 y(1 + b_1 y + b_2 y^2 + \ldots),$$

which expands out to

$$x = a_n y^n + a_{n+1} y^{n+1} + \ldots$$

□

Theorem 14.7 (Euclidean algorithm)**.** *Over a field, take a formal power series $f(x, y)$ in n variables $x = (x_1, \ldots, x_n)$ and one variable y*

$$f(x, y) = \sum a_j(x) y^j.$$

Suppose that $a_0(x), \ldots, a_{n-1}(x)$ all have zero constant terms, while $a_n(x)$ has nonzero constant term. Given any formal power series $g(x, y)$ there is a unique quotient and remainder

$$g(x, y) = q(x, y) f(x, y) + r(x, y)$$

with formal power series quotient $q(x, y)$ and remainder polynomial in y (with degree at most $n - 1$) with coefficients formal power series in x.

Proof. Given any formal power series $b(x,y)$, let $\overleftarrow{b}(x,y)$, the *head*, be the terms of degree at most $n-1$ in y. All terms not in the head are divisible by y^n, so divide out that y^n and define the *tail* $\vec{b}(x,y)$, by the equation

$$b(x,y) = \overleftarrow{b}(x,y) + y^n \vec{b}(x,y).$$

For example,

$$\overrightarrow{cx^a y^b} = \begin{cases} cx^a y^{b-n} & \text{if } b \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

If we can find $q = q(x,y)$ with $\vec{g} = \overrightarrow{qf}$, then $\overrightarrow{g - qf} = 0$, so $g - qf$ is some remainder. Splitting up f into head and tail, we need precisely to solve

$$\begin{aligned} \vec{g} &= \overrightarrow{q\overleftarrow{f}} + \overrightarrow{qy^n \vec{f}}, \\ &= \overrightarrow{q\overleftarrow{f}} + q\vec{f}. \end{aligned}$$

By hypothesis, we can divide by $\vec{f}$. Let $z := q\vec{f}$ and

$$\varphi := \frac{\overleftarrow{f}}{\vec{f}}.$$

So we need to solve

$$\vec{g} = \overrightarrow{z\varphi} + z.$$

Think of this as operating on z,

$$z \mapsto z + \overrightarrow{z\varphi}.$$

and we have to get $\vec{g}$ as output. By hypothesis, $\overleftarrow{f}$ has zero constant term in x, so φ does too. So the second term of our operation

$$z \mapsto \overrightarrow{z\varphi},$$

maps every formal power series z into one with a higher degree of lowest term in x. To solve

$$\vec{g} = z + \overrightarrow{z\varphi},$$

the lowest degree terms on the left hand side are the lowest terms of z. Proceed inductively: the terms we have already solved for appear arising in the $\overrightarrow{z\varphi}$ bit, and then we pull those across to the left hand side, and solve for one more layer of terms in z on the right hand side. □

A *Weierstrass polynomial* in variables $x = (x_1, \ldots, x_n)$ and a variable y is a formal power series

$$y^n + b_{n-1}(x)y^{n-1} + \cdots + b_0(x)$$

where each $b_i(x)$ has zero constant term. Warning: a Weierstrass polynomial is only required to be polynomial in y, not in x.

Theorem 14.8 (Weierstrass preparation)**.** *Over a field, take a formal power series $f(x,y)$ in n variables $x = (x_1, \ldots, x_n)$ and one variable y*

$$f(x,y) = \sum a_j(x) y^j.$$

Suppose that $a_0(x), \ldots, a_{n-1}(x)$ all have zero constant terms, while $a_n(x)$ has nonzero constant term. Then

$$f(x, y) = p(x, y)u(x, y),$$

where $u(0, 0) \neq 0$, $u(x, y)$ is a formal power series, and $p(x, y)$ is a Weierstrass polynomial.

Intuitively and roughly, this says that solving an equation in formal power series for one of the variables in terms of the others is the same as solving a polynomial equation in one variable, but with coefficients formal power series in the other variables.

Proof. By theorem 14.7 on page 109, we can find a quotient and remainder for y^n, say $y^n = qf + r$. Note that the terms of degree less than n in y have zero constant term in x, both on the right hand side and in f, so also in r. Writing out $q = q_0 + q_1 y + \ldots$, and $f = \cdots + a_n y^n + \ldots$, with a_n a unit, multiply out to find that the y^n term at $x = 0$ is $1 = q_0(0)a_n(0)$. So $q_0(x)$ has a reciprocal formal power series $u(x, y)$ with $u(0, 0) \neq 0$. So $f = u(y^n - r)$: there is such an expression for f. Such an expression is unique: split any two such into powers of y. □

Laurent series

A *formal Laurent series* is defined just as a formal power series, but allowing in addition finitely many terms of negative degree:

$$f(x) = \sum_{\alpha} b_\alpha x^\alpha,$$

for $x = (x_1, \ldots, x_n)$ and with the sum over $\alpha = (\alpha_1, \ldots, \alpha_n)$, for any integers α_i, but with only finitely many terms having any of the $\alpha_i < 0$.

$$\frac{1}{x} + 1 + x + x^2 + \ldots$$

is a formal Laurent series in one variable, while

$$\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots$$

is *not*, nor is

$$x^{1/2} + x,$$

The series

$$\frac{1}{xy} + \frac{1}{x} + \frac{1}{y} + 1 + x^3 + x^5 + x^7 + \ldots$$

is a formal Laurent series in two variables, but

$$x^2 y^{-1} + x^3 y^{-2} + x^4 y^{-3} + \ldots,$$

is not.

We denote the set of all formal Laurent series in variables $x = (x_1, \ldots, x_n)$ over a ring R by $R((x))$.

14.8 Prove that the formal Laurent series, in a single variable, over a field, form a field.

Puiseaux series

A *Puiseaux series* is a formal sum

$$y(x) = c_1 x^{p_1/q} + c_2 x^{p_2/q} + \dots$$

for an increasing sequence of integers $p_1, p_2, \dots$, finitely many of which can be negative, and a single positive integer q. In other words, it is a formal Laurent series in $x^{1/q}$.

$$\frac{1}{x} + \frac{1}{x^2} + \dots = x^{-1} + x^{-2} + \dots$$

is *not* a Puiseaux series, nor is

$$x^{.1} + x^{.11} + x^{.111} + \dots = x^{1/10} + x^{11/100} + x^{111/1000} + \dots$$

14.9 Prove that the set of all Puiseaux series over a field k in a single variable x forms a field.

Algebraic closures

Theorem 14.9. *If k is an algebraically closed field of characteristic zero, then the algebraic closure of the field of formal Laurent series in a single variable over k is the field of Puiseaux series in that variable over k.*

Given an algebraic curve $0 = p(x, y)$, we want to write out a Puiseaux series $y = y(x)$ to "solve" the equation $0 = p(x, y)$, and we want to consider both the existence and the uniqueness of this series. We already see that there is no such formal power series in integer powers of x. We will prove (after we define the term *order* of a point of a plane algebraic curve):

Theorem 14.10 (Newton–Puiseaux)**.** *Take a plane algebraic curve over an algebraically closed field k. For simplicity, we suppose that the origin lies on the curve, and that the curve is irreducible. There is a Puiseaux series $y(x)$ so that $0 = p(x, y(x))$, in the ring of Puiseaux series. The number of distinct such series agrees with the order of the origin as a point of the algebraic curve.*

> *Whoever you are, go out into the evening,*
> *leaving your room, of which you know every bit;*
> *your house is the last before the infinite,*
> *whoever you are.*
>
> — Rainer Maria Rilke

Chapter 15

Resultants and discriminants

We found that Cayley and other mathematicians of the period understood many of the concepts which today are commonly thought of as modern and quite recent. Thus, in an 1848 note on the resultant, Cayley in fact laid out the foundations of modern homological algebra. We were happy to enter into spiritual contact with this great mathematician.

— I. M. Gelfand, M. .M. Kapranov, A. M. Zelevinsky
Discriminants, Resultants and Multidimensional Determinants

The resultant

For any polynomial

$$b(x) = b_0 + b_1 x + \cdots + b_m x^m,$$

denote its vector of coefficients as

$$\vec{b} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

Clearly the vector of coefficients of $xb(x)$ is the same, with an extra row added to the top, with a zero in that row. When we multiply two polynomials $b(x)$ and

$$c(x) = c_0 + c_1 x + \cdots + c_n x^n$$

the vector of coefficients of $b(x)c(x)$ is

$$\begin{pmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & \ddots & \ddots & \\ b_m & \ddots & \ddots & b_0 \\ & b_m & \ddots & b_1 \\ & & \ddots & \vdots \\ & & & b_m \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix},$$

(with zeroes represented as blank spaces) and this $(m+n+1)\times(n+1)$ matrix we denote by

$$[b] = \begin{pmatrix} b_0 & & & \\ b_1 & b_0 & & \\ \vdots & \ddots & \ddots & \\ b_m & \ddots & \ddots & b_0 \\ & b_m & \ddots & b_1 \\ & & \ddots & \vdots \\ & & & b_m \end{pmatrix}.$$

So the vectors of coefficient are related by $\overrightarrow{bc} = [b]\vec{c}$.

The *resultant* of two polynomials

$$b(x) = b_0 + b_1 x + \cdots + b_m x^m \text{ and}$$
$$c(x) = c_0 + c_1 x + \cdots + c_n x^n,$$

denoted $\operatorname{res}_{b,c}$, is the determinant of the matrix $\begin{pmatrix}[b] & [c]\end{pmatrix}$, given by stacking the matrices $[b]$ and $[c]$ of suitable sizes beside one another, to get a square matrix of size $(m+n)\times(m+n)$. To remember the sizes: the number of columns in $[b]$ is the degree of c, and vice versa.

If

$$b(x) = 4 + 3x + 7x^2 + x^3,$$
$$c(x) = 5 + 2x,$$

then $\operatorname{res}_{b,c}$ is the determinant of the 4×4 matrix

$$\begin{pmatrix} 4 & 5 & 0 & 0 \\ 3 & 2 & 5 & 0 \\ 7 & 0 & 2 & 5 \\ 1 & 0 & 0 & 2 \end{pmatrix},$$

which is $\operatorname{res}_{b,c} = 197$.

If we swap b and c, we swap columns, so clearly $\operatorname{res}_{c,b} = (-1)^{mn}\operatorname{res}_{b,c}$.

Take $b(x) = x^2 + 4x + 7$ and $c(x) = x^3$ and compute

$$\operatorname{res}_{b,c} = \det \begin{pmatrix} 7 & 0 & 0 & 0 & 0 \\ 4 & 7 & 0 & 0 & 0 \\ 1 & 4 & 7 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} = 7^3.$$

The easiest examples of arbitrary degrees arise when we take any $b(x)$ but try $c(x) = x^n$:

$$\operatorname{res}_{b,c} = \det \begin{pmatrix} b_0 & & & & & & \\ \vdots & b_0 & & & & & \\ \vdots & \ddots & \ddots & & & & \\ b_m & \ddots & \ddots & b_0 & & & \\ & b_m & \ddots & \vdots & 1 & & \\ & & \ddots & \vdots & & \ddots & \\ & & & b_m & & & 1 \end{pmatrix} = b_0^n.$$

Sage

Computing resultants in sage requires us to tell sage what sort of numbers arise as the coefficients of our polynomials.

```
P.<x> = PolynomialRing(QQ)
a=x^3+x+7
b=x^2+2
a.resultant(b)
```

yields 51. The expression `P.<x> = PolynomialRing(QQ)` tells sage that we are working with polynomials with rational coefficients (`QQ` means rational) in a variable `x`.

We could write our own resultant function, just to see how it might work:

```
def resultant(b,c):
    B=b.list()
    C=c.list()
    m=len(B)-1
    n=len(C)-1
    A=matrix(m+n,m+n)
    for j in range(0,n):
        for i in range(0,m+n):
            if (0<=i-j) and (i-j<=m):
                A[i,j]=B[i-j]
            else:
                A[i,j]=0
    for j in range(n,m+n):
        for i in range(0,m+n):
            if (0<=i-j+n) and (i-j+n<=n):
                A[i,j]=C[i-j+n]
            else:
                A[i,j]=0
    return det(A)
```

Try it out:

```
t=var('t')
p = t^2+2*t+1
q = t+1
resultant(p,q)
```

yields 0.

Common factors and resultants

Lemma 15.1. *Two polynomials $b(x)$ and $c(x)$ of degrees m and n, over any field, have a nonconstant common factor just when*

$$0 = u(x)b(x) + v(x)c(x)$$

for two polynomials $u(x)$ and $v(x)$ of degrees $n-1$ and $m-1$ at most, not both zero.

Proof. Two polynomials $b(x)$ and $c(x)$ of degrees m and n have a nonconstant common factor just when we can write the polynomials factored, say as $b(x) = v(x)d(x)$ and $c(x) = -u(x)d(x)$. But then

$$0 = u(x)b(x) + v(x)c(x).$$

On the other hand, suppose that

$$0 = u(x)b(x) + v(x)c(x)$$

for two polynomials $u(x)$ and $v(x)$ of degrees $n-1$ and $m-1$ at most. Suppose that $b(x)$ and $c(x)$ have no common factor, so their greatest common divisor is 1. Write out Bézout coefficients

$$1 = s(x)b(x) + t(x)c(x)$$

and compute

$$\begin{aligned} v &= vsb + vtc, \\ &= vsb + tvc, \\ &= vsb - tub, \\ &= b\left(vs - tu\right). \end{aligned}$$

But the degree of v is smaller than that of b, so $v = 0$. Swapping roles of b, c and of u, v and of s, t, we get $u = 0$. □

Proposition 15.2. *Two polynomials, over any field, have a nonconstant common factor just when their resultant vanishes.*

Proof. Take two polynomials

$$\begin{aligned} u(x) &= u_0 + u_1 x + \cdots + u_{n-1}x^{n-1}, \\ v(x) &= v_0 + v_1 x + \cdots + v_{m-1}x^{m-1}. \end{aligned}$$

The vector of coefficients of $u(x)b(x) + v(x)c(x)$ is

$$\overrightarrow{bu + cv} = \begin{pmatrix} [b] & [c] \end{pmatrix} \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}$$

By linear algebra, the determinant vanishes just when the matrix has a nonzero null vector, i.e. a nonzero vector in its kernel. So the resultant vanishes just when there is a choice of coefficients

$$u_0, u_1, \ldots, v_0, v_1, \ldots,$$

not all zero, so that $u(x)b(x) + v(x)c(x) = 0$. In other words, there are polynomials $u(x)$ and $v(x)$ of required degrees so that $u(x)b(x) + v(x)c(x) = 0$. The theorem now follows from lemma 15.1 on the facing page. □

15.1 Find the resultants, and use them to decide if there are nonconstant common factors.

a. $x^2 - 5x + 6, x^3 - 3x^2 + x - 3$

b. $x^2 + 1, x - 1$

The resultant is brutal to compute, even for polynomials of fairly small degree. Its importance, like that of the determinant of a matrix, arises from its theoretical power. At this point, the reader should be annoyed: we already know how to find common divisors, even the greatest common divisor, by a fast calculation, so we surely don't need a slow resultant calculation to see if there is a common divisor of positive degree, since we can already actually find that divisor, and more quickly. The reader is right, but surprisingly we will find many uses for the resultant.

15.2 Over the field of remainders modulo 2, calculate the resultant of $b(x) = x^3 + x^2 + x + 1, c(x) = x^3 + x$. *Warning*: the 6×6 determinant you get is actually easier than it looks at first, so look for tricks to find it without any hard calculation.

15.3 Calculate a resultant to prove that, for any prime integer $p \geq 2$, when we work over the field of remainders modulo p, the polynomials $x^2 + 1$ and $x^2 + 3x$ have a common factor just exactly when $p = 2$ or $p = 5$.

15.4 Suppose that $b(x)$ and $c(x)$ are polynomials in one variable x, with coefficients in the field of integer remainders modulo 2. Suppose that $b(x)$ is a sum of 1126 nonzero terms, while $c(x)$ is a sum of 8080 nonzero terms. Note: we don't say anything about the degree of either one. Prove that they have a common factor of positive degree.

Lemma 15.3. *Take two polynomials $b(x), c(x)$ in a variable x of degrees m, n. The resultant $r = \mathrm{res}_{b,c}$ is expressible as $r = u(x)b(x) + v(x)c(x)$ where $u(x)$ and $v(x)$ are polynomials of degrees $n-1, m-1$. One can make a particular choice of $u(x), v(x)$ so that the coefficients of $u(x)$ and of $v(x)$ are expressible as polynomial expressions in the coefficients of $b(x)$ and $c(x)$, and those polynomial expressions have only 1 or -1 as coefficient in each term.*

Proof. Look at our matrix:

$$r = \det \begin{pmatrix} b_0 & & & & c_0 & & & \\ b_1 & b_0 & & & c_1 & c_0 & & \\ \vdots & b_1 & \ddots & & \vdots & c_1 & \ddots & \\ \vdots & \ddots & \ddots & b_0 & \vdots & \ddots & \ddots & c_0 \\ b_m & \ddots & \ddots & b_1 & c_n & \ddots & \ddots & c_1 \\ & b_m & \ddots & \vdots & & c_n & \ddots & \vdots \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & b_m & & & & c_n \end{pmatrix}$$

Add to the first row the second multiplied by x and then the third multiplied by x^2 and so on, which doesn't change the determinant:

$$r = \det \begin{pmatrix} b(x) & xb(x) & \dots & x^{n-1}b(x) & c(x) & xc(x) & \dots & x^{m-1}c(x) \\ b_1 & b_0 & & & c_1 & c_0 & & \\ \vdots & b_1 & \ddots & & \vdots & c_1 & \ddots & \\ \vdots & \ddots & \ddots & b_0 & \vdots & \ddots & \ddots & c_0 \\ b_m & \ddots & \ddots & b_1 & c_n & \ddots & \ddots & c_1 \\ & b_m & \ddots & \vdots & & c_n & \ddots & \vdots \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & b_m & & & & c_n \end{pmatrix}.$$

Expand the determinant across the top row to see that

$$r = u(x)b(x) + v(x)c(x)$$

where

$$u(x) = \det \begin{pmatrix} 1 & x & \dots & x^{n-1} & & & & \\ b_1 & b_0 & & & c_1 & c_0 & & \\ \vdots & b_1 & \ddots & & \vdots & c_1 & \ddots & \\ \vdots & \ddots & \ddots & b_0 & \vdots & \ddots & \ddots & c_0 \\ b_m & \ddots & \ddots & b_1 & c_n & \ddots & \ddots & c_1 \\ & b_m & \ddots & \vdots & & c_n & \ddots & \vdots \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & b_m & & & & c_n \end{pmatrix}$$

and

$$v(x) = \det \begin{pmatrix} & & & & 1 & x & \dots & x^{m-1} \\ b_1 & b_0 & & & c_1 & c_0 & & \\ \vdots & b_1 & \ddots & & \vdots & c_1 & \ddots & \\ \vdots & \ddots & \ddots & b_0 & \vdots & \ddots & \ddots & c_0 \\ b_m & \ddots & \ddots & b_1 & c_n & \ddots & \ddots & c_1 \\ & b_m & \ddots & \vdots & & c_n & \ddots & \vdots \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & b_m & & & & c_n \end{pmatrix}.$$

Note that we can expand out any determinant using only multiplication and addition of entries and some $\pm$ signs. □

If we scale up $b(x)$ by a factor λ, the matrix gets a λ in each column in the left n columns, where n is the degree of $c(x)$. The resultant of $b(x), c(x)$ scales by λ^n. So practically we only need to compute resultants of monic polynomials.

Proposition 15.4. *Given two monic polynomials over any field, each factored into linear factors,*

$$\begin{aligned} b(x) &= (x-\beta_1)(x-\beta_2)\dots(x-\beta_m), \\ c(x) &= (x-\gamma_1)(x-\gamma_2)\dots(x-\gamma_n), \end{aligned}$$

then the resultant is

$$\begin{aligned} \operatorname{res}_{b,c} &= (\gamma_1-\beta_1)(\gamma_1-\beta_2)\dots(\gamma_n-\beta_m), \\ &= b(\gamma_1)\,b(\gamma_2)\dots b(\gamma_n). \\ &= (-1)^{mn} c(\beta_1)\,c(\beta_2)\dots c(\beta_m), \end{aligned}$$

In particular, the resultant vanishes just when $b(x)$ and $c(x)$ have a common root.

Proof. If we expand out the expression for $b(x)$ into a sum of monomials, with coefficients being expressed in terms of these β_i, and similarly for $c(x)$, then the resultant is a huge polynomial expression in terms of the various β_i and γ_j. This polynomial vanishes whenever $\beta_i = \gamma_j$. Thinking of β_i and γ_j as abstract variables, the expression $\gamma_j - \beta_i$ is a linear function. Assume for the moment that our field is infinite. Then the resultant is divisible by $\gamma_j - \beta_i$, for any i and j, by lemma 20.2 on page 174. Therefore the resultant is divisible by the product

$$(\gamma_1-\beta_1)(\gamma_1-\beta_2)\dots(\gamma_n-\beta_m).$$

Imagine expanding out the resultant as a determinant, with $b(x), c(x)$ monic. Each term in the determinant contains exactly n of the various b_j, and each b_j has $m-j$ β's. So the highest term in the β's is the one from the diagonal. The diagonal has $b_0, b_0, \dots, b_0, 1, 1, \dots, 1$, giving a term b_0^n in the determinant. Expand in β's as $b_0 = (-1)^m \beta_1 \dots \beta_m$ to give a term $(-1)^{mn}\beta_1^n \dots \beta_m^n$ in the resultant. But the product gives exactly the same highest term in β's. By swapping columns, we see that our product and resultant also give exactly the same highest term in γ's, and so

the resultant equals the product. For a finite field, embed into an infinite field, as in chapter 12. □

If $b(x) := x^7 + 2x + 1$ and $c(x) := x - 1$, then $c(x)$ has root $x = 1$, and both are monic, so

$$\operatorname{res}_{b,c} = b(1) = 1^7 + 2 + 1 = 4.$$

15.5 Suppose that $b(x)$ is monic. Prove that

$$\operatorname{res}_{b,c} = (-1)^{mn} c(\beta_1)\, c(\beta_2) \dots c(\beta_m)\,.$$

Lemma 15.5. *For any polynomials $b(x), c(x), d(x)$,*

$$\operatorname{res}_{bd,c} = \operatorname{res}_{b,c} \operatorname{res}_{d,c}\,.$$

Proof. Split all of the polynomials into linear factors (over a splitting field):

$$\begin{aligned}\operatorname{res}_{bd,c} &= b(\gamma_1)\, d(\gamma_1) \dots b(\gamma_n)\, d(\gamma_n)\,,\\ &= b(\gamma_1)\, b(\gamma_2) \dots b(\gamma_n)\, d(\gamma_1)\, d(\gamma_2) \dots d(\gamma_n)\,,\\ &= \operatorname{res}_{b,c} \operatorname{res}_{d,c}\,.\end{aligned}$$

□

15.6 Suppose that $c(x)$ is a monic polynomial. Prove that

$$\operatorname{res}_{b+dc,c} = \operatorname{res}_{b,c}$$

for any polynomials $b(x)$ and $d(x)$. A surprise: note that the degree of $b + dc$ could be either that of b or that of dc, or something smaller than either, as any number of terms in $b + dc$ might cancel one another. But surprisingly, we don't need to know the degree of $b + dc$.

15.7 A fast trick to find resultants: use the result of the previous problem to compute $\operatorname{res}_{b,c}$ where

$$\begin{aligned}b(x) &= x^8 + 4x^2 + 2x + 1,\\ c(x) &= x^6 + 4.\end{aligned}$$

We can make sage code for our fast trick:

```
R.<x> = PolynomialRing(QQ)
def fast_resultant(b,c):
    m=b.degree()
    n=c.degree()
    if m<n:
        return (-1)^(m*n)*fast_resultant(c,b)
    if n==0:
        if m==0:
            return 1
        return c(0)^m
    lb=b.leading_coefficient()
```

```
    lc=c.leading_coefficient()
    B=b/lb
    C=c/lc
    r=B % C
    if (r==0):
        return 0
    return (-1)^(m*n)*lb^n*lc^m*fast_resultant(C,r)
```

so that `fast_resultant(x^3+x+7,x^2+2)` yields 51. We can run a speed test,

```
b=sum([ZZ.random_element(0,100)*x^j for j in [0..100]])
c=sum([ZZ.random_element(0,100)*x^j for j in [0..100]])
print("The built in resultant function from sage:")
%timeit b.resultant(c)
print("Our slow resultant function:")
%timeit resultant(b,c)
print("Our fast resultant function:")
%timeit fast_resultant(b,c)
```

Sage's built in routine easily wins, but we can see that the fast method is faster than the slow one:

```
The built in resultant function from sage:
4.48 ms ± 25.3 µs per loop (mean ± std. dev. of 7 runs, 100 loops each)
Our slow resultant function:
72.8 ms ± 171 µs per loop (mean ± std. dev. of 7 runs, 10 loops each)
Our fast resultant function:
53.2 ms ± 146 µs per loop (mean ± std. dev. of 7 runs, 10 loops each)
```

The discriminant

For any polynomial $p(x)$, with any sort of coefficients, say

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

we define the *derivative*

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^n,$$

just imitating calculus. There is no interpretation of this as a "rate of change", since the coefficients could be remainders modulo some integer, or something much stranger.

The *discriminant* Δ_p of a polynomial $p(x)$ is the resultant with the derivative: $\Delta_p := \mathrm{res}_{p,p'}$. By definition, the discriminant vanishes just when the polynomial has a common factor with its derivative. For example, over the complex numbers, the discriminant vanishes just when the polynomial has a "multiple root", i.e. a repeated linear factor:

$$p(z) = (z - z_0)^2 \ldots.$$

15.8 Find the discriminants of (a) $ax^2 + b + c$, (b) $x^3 + x^2$, (c) $x^3 + x^2 + 1$, (d) $x^3 + 2x - 1$ and explain what they tell you about these polynomials.

The polynomial $f(x) = 1 + 2x^2 + x^4$ over the real numbers has no real roots. Its derivative is $f'(x) = 4x + 4x^3$, so its discriminant is

$$\Delta_f = \det \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 & 4 & 0 \\ 1 & 0 & 2 & 0 & 4 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \end{pmatrix} = 0$$

(after a long calculation). So $f(x)$ has a common factor with $f'(x)$. We can see the common factor more clearly if we factor:

$$f(x) = 1 + 2x^2 + x^4 = \left(1 + x^2\right)^2,$$
$$f'(x) = 4x + 4x^3 = 4x\left(1 + x^2\right).$$

So $f(x), f'(x)$ share a common factor of $1 + x^2$. They don't share a root over the real numbers, but over the complex numbers

$$f(x) = (x - i)^2 (x + i)^2,$$
$$f'(x) = 4x(x - i)(x + i),$$

so they share roots at $x = \pm i$, the double roots of $f(x)$.

The polynomial $p(x) = x^3 + 2x^2 + 2x + 2$ is irreducible over the rational numbers by Eisenstein's criterion, so has no rational roots. It has odd degree so at least one real root. There is no positive real root since all terms are positive. The derivative is $p'(x) = 3x^2 + 4x + 2$. By the quadratic formula, the zeroes of $p'(x)$ are at

$$x = -\frac{2}{3} \pm \frac{2\sqrt{2}}{3} i,$$

not real. Since $p'(x)$ has only positive coefficients, $p'(x) > 0$ for $x > 0$. But $p'(x)$ has no real zeroes, so $p'(x) > 0$ for all real values of x. Therefore $p(x)$ is increasing, so has precisely one real root, irrational and negative. The two remaining complex roots are nonreal conjugates. The discriminant of $p(x)$ is

$$\Delta = \det \begin{pmatrix} 2 & 0 & 2 & 0 & 0 \\ 2 & 2 & 4 & 2 & 0 \\ 2 & 2 & 3 & 4 & 2 \\ 1 & 2 & 0 & 3 & 4 \\ 0 & 1 & 0 & 0 & 3 \end{pmatrix} = -44.$$

Since $\Delta \neq 0$, we see that there are no double roots over the complex numbers, which is already clear.

15.9 Suppose that a is a real number. Find the discriminant of $q(x) = x^3 + ax + 1$. For which real numbers a does $q(x)$ have a multiple root? What is the multiplicity of

that root?

15.10 Prove that the sign of the discriminant Δ of a monic cubic polynomial tells us that the number of roots is:

$\Delta > 0$	3 distinct roots
$\Delta = 0$	a triple root or 1 real double root and 1 real single root
$\Delta < 0$	1 real and 2 complex roots.

15.11 Prove that, for any monic polynomial $p(x)$ with roots $x_1, x_2, \ldots, x_n$,

$$\Delta_{p(x)} = (-1)^{n(n-1)/2} \prod_{i<j} (x_i - x_j)^2.$$

Sage

Sage computes discriminants:

```
R.<t> = PolynomialRing(QQ)
(t^2-t+7).discriminant()
```

yields -27.

Equations of parameterized curves

Take the curve in the plane parameterized by

$$(x(t), y(t)) = \left(t^2 - 1, t\left(t^2 - 1\right)\right):$$

We want to find an equation for this curve, as a polynomial in x and y.

Proposition 15.6. *Draw a curve*

$$x(t) = \frac{p(t)}{q(t)}, y(t) = \frac{u(t)}{v(t)},$$

where $p(t), q(t), u(t), v(t)$ are polynomials with coefficients in a field k. The resultant in the variable t of the polynomials

$$p(t) - xq(t), u(t) - yv(t)$$

vanishes along the points (x, y) in the plane which lie along this curve; we can allow the values of x and y to be in the algebraic closure $\bar{k}$.

Proof. The resultant vanishes just at those points (x, y) where the two polynomials have a common factor. If there is a common factor of two polynomials, that factor has a root over $\bar{k}$, so there is a common root t of both polynomials, in other words a value of t where $x = p(t)/q(t)$ and $y = u(t)/v(t)$. On the other hand, a common root gives a common linear factor. So the resultant vanishes just along the image of the curve in the "plane". Careful: $k = \mathbb{R}$ then this "plane" has points given as pairs (x, y) of complex numbers, so it is not the complex plane, but two copies of the complex plane. □

For the curve in the plane parameterized by

$$(x(t), y(t)) = \left(t^2 - 1, t\left(t^2 - 1\right)\right),$$

we have to take the resultant of

$$t^2 - 1 - x, t\left(t^2 - 1\right) - y,$$

as a polynomial in t; treat x and y as constants:

$$\det \begin{pmatrix} -(1+x) & & & -y & \\ 0 & -(1+x) & 0 & -1 & -y \\ 1 & 0 & -(1+x) & 0 & -1 \\ & -1 & 0 & 1 & 0 \\ & & -1 & & 1 \end{pmatrix} = y^2 - x^2 - x^3$$

So the curve lies among the set of points (x, y) so that $y^2 = x^2 + x^3$. We will see that the curve consists precisely of those points.

For real variables, the curve $x = t^2, y = t^4$ has both $x \geq 0$ and $y \geq 0$ since $t^2 \geq 0$ and $t^4 \geq 0$. So the curve is the right half of the parabola $y = x^2$ on which $x \geq 0$. The resultant comes out as

$$\det \begin{pmatrix} -x & 0 & 0 & 0 & -y & 0 \\ 0 & -x & 0 & 0 & 0 & -y \\ 1 & 0 & -x & 0 & 0 & 0 \\ 0 & 1 & 0 & -x & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} = \left(x^2 - y\right)^2.$$

So the resultant vanishes along $y = x^2$, the whole parabola. So we don't quite get what we wanted: we get the algebraic equation $y = x^2$ that that curve satisfies, but we miss out on the inequality $x \geq 0$.

15.12 Find a nonconstant polynomial equation $f(x, y) = 0$ satisfied by the points of the curve

$$x(t) = t^2, y(t) = t^3 - t.$$

15.13 Find a nonconstant polynomial equation $f(x, y) = 0$ satisfied by the points of the curve

$$x(t) = t^2 + t, y(t) = \frac{1}{t}.$$

15.14 Find a nonconstant polynomial equation $f(x, y) = 0$ satisfied by the points of the curve

$$x(t) = t^2 + 1, y(t) = t + \frac{1}{t}.$$

15.15 With coefficients over the field of remainders modulo 2, find a polynomial equation $0 = p(x, y)$, with $p(x, y)$ not constant, satisfied by the parameterized plane curve $(x(t), y(t)) = (t^3 + t, 1/t)$.

Sage

Sage handles 2 variable polynomials:

```
P.<x,y> = PolynomialRing(QQ)
a = x + y
b = x^3 - y^3
a.resultant(b)
```

yields $-2y^3$. If we want to use the resultant in the other variable, we specify which variable to get rid of: `a.resultant(b, y)` yields $2x^3$. Sage will take a parameterisation of a curve and give us the equation of the curve. Our example above becomes:

```
R.<x,y,t> = PolynomialRing(QQ)
(t^2-1-x).resultant(t*(t^2-1)-y,t)
```

yields $-x^3 - x^2 + y^2$.

Sage computes resultants of polynomials over any field.

Let K be the splitting field of $x^2 + x + 1$ over $k = \mathbb{Z}/2\mathbb{Z}$, with generator a. We compute the resultant and greatest common divisor of

$$p(t) = (at + 1)^2(at - 1)^2, q(t) = (at + 1)(at + a^2)^2.$$

via

```
R.<x> = PolynomialRing(GF(2))
K.<a> = (x^2 + x + 1).splitting_field()
S.<t> = PolynomialRing(K)
p = S((a*t+1)^2*(a*t-1)^2)
q = S((a*t+1)*(a*t+a^2)^2)
print(p.gcd(q))
print(p.resultant(q))
```

yielding $t + a + 1$ and 0.

Chapter 16

Permuting roots

This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind.

— Hermann Weyl
On Évariste Galois's letter, written the night before Galois died in a pistol duel. Galois's letter is about permutations of roots of polynomials.

Vieta's formulas

For now, we work over any field. Which polynomials have which roots? How are the coefficients related to the roots?

To get a quadratic polynomial $p(x)$ to have roots at 3 and 7, we need it to have $x-3$ and $x-7$ as factors. So it has to be

$$\begin{aligned} p(x) &= a(x-3)(x-7), \\ &= a\left(x^2 - 7x - 3x + (-3)(-7)\right), \\ &= a\left(x^2 - (3+7)x + 3\cdot 7\right). \end{aligned}$$

A polynomial is *monic* if its leading coefficient is 1. We can make any polynomial monic by dividing off the leading term:

$$3x^2 - 4x + 1 = 3\left(x^2 - \frac{4}{3}x + \frac{1}{3}\right).$$

The only monic quadratic polynomial $p(x)$ with roots $x=3$ and $x=7$ is

$$p(x) = x^2 - (3+7)x + 3\cdot 7.$$

The only quadratic monic polynomial $p(x)$ with roots $x=r$ and $x=s$, by the same steps, must be

$$p(x) = x^2 - (r+s)x + rs.$$

By the same steps, the only cubic monic polynomial $q(x)$ with roots at $x=r$, $x=s$ and $x=t$ is

$$q(x) = (x-r)(x-s)(x-t).$$

If we multiply it all out, we get

$$q(x) = x^3 - (r+s+t)x^2 + (rs+rt+st)x - rst.$$

Ignoring the minus signs, the coefficients are

$$\begin{aligned} &1 \\ &r+s+t \\ &rs+rt+st \\ &rst \end{aligned}$$

There is some pattern:

a. There is a minus sign, turning on and off like a light switch, each time we write down a term.

b. Each coefficient is a sum of products of roots, taken in all possible ways with a fixed number of roots.

Proposition 16.1 (Vieta's formula (Vieté))**.** *If a monic polynomial $p(x)$ splits into a product of linear factors, say*

$$p(x) = (x - r_1)(x - r_2)\dots(x - r_n)$$

then the numbers $r_1, r_2, \dots, r_n$ are the roots of $p(x)$, and the coefficients of $p(x)$, say

$$p(x) = x^n - a_1 x^{n-1} + \cdots \pm a_{n-1}x \pm a_n,$$

(with signs switching each term in front of the a_j coefficients) are computed from the roots by

$$\begin{aligned} a_1 &= r_1 + r_2 + \cdots + r_n, \\ a_2 &= \sum_{i<j} r_i r_j, \\ &\vdots \\ a_i &= \sum_{j_1<j_2<\cdots<j_i} r_{j_1} r_{j_2} \dots r_{j_i}, \\ &\vdots \\ a_{n-1} &= r_1 r_2 \dots r_{n-1} + r_1 r_2 \dots r_{n-2} r_n + \cdots + r_2 r_3 \dots r_n, \\ a_n &= r_1 r_2 \dots r_n. \end{aligned}$$

The *elementary symmetric polynomials* are the polynomials $e_1, \dots, e_n$ of variables $t_1, t_2, \dots, t_n$, given by

$$\begin{aligned}
e_1(t_1, t_2, \dots, t_n) &= t_1 + t_2 + \cdots + t_n, \\
e_2(t_1, t_2, \dots, t_n) &= t_1t_2 + t_1t_3 + \cdots + t_{n-1}t_n, \\
&= \sum_{i<j} t_it_j, \\
&\vdots \\
e_i(t_1, t_2, \dots, t_n) &= \sum_{j_1<j_2<\cdots<j_i} t_{j_1}t_{j_2}\dots t_{j_i}, \\
&\vdots \\
e_{n-1}(t_1, t_2, \dots, t_n) &= t_1t_2\dots t_{n-1} + t_1t_2\dots t_{n-2}t_n + \cdots + t_2t_3\dots t_n, \\
e_n(t_1, t_2, \dots, t_n) &= t_1t_2\dots t_n.
\end{aligned}$$

So we can restate our proposition as:

Proposition 16.2 (Vieta's formula)**.** *If a monic polynomial $p(x)$ splits into a product of linear factors, say*

$$p(x) = (x - r_1)(x - r_2)\dots(x - r_n)$$

then the numbers $r_1, r_2, \dots, r_n$ are the roots of $p(x)$, and the coefficients of $p(x)$:

$$p(x) = x^n - e_1x^{n-1} + \dots (-1)^{n-1}e_{n-1}x + (-1)^n e_n,$$

are the elementary symmetric polynomials

$$e_i = e_i(r_1, r_2, \dots, r_n)$$

of the roots $r_1, r_2, \dots, r_n$.

Proof. We can see this immediately for linear polynomials: $p(x) = x - r$, and we checked it above for quadratic and cubic ones. It is convenient to rewrite the elementary symmetric polynomials as

$$e_i(r_1, r_2, \dots, r_n) = \sum r_1^{b_1}r_2^{b_2}\dots r_n^{b_n}$$

where the sum is over all choices of numbers $b_1, b_2, \dots, b_n$ so that

a. each of these b_j is either 0 or 1 and

b. so that altogether

$$b_1 + b_2 + b_3 + \cdots + b_n = i,$$

or in other words, "turn on" i of the roots, and turn off the rest, and multiply out the ones that are turned on, and then sum over all choices of which to turn on.

If we expand out the product

$$p(x) = (x - r_1)(x - r_2)\dots(x - r_n)$$

we do so by pick whether to multiply with x or with $-r_1$ from the first factor, and then whether to multiply with x or with $-r_2$ from the second, and so on, and we add

over all of these choices. Each choice write as $b_1 = 0$ if we pick to multiply in the x, but $b_1 = 1$ if we pick to multiply in the $-r_1$, and so on:

$$\begin{aligned} p(x) &= \sum_{b_1,\dots,b_n} (-r_1)^{b_1} (-r_2)^{b_2} \dots (-r_n)^{b_n} x^{(1-b_1)+(1-b_2)+\dots+(1-b_n)}, \\ &= \sum_j \sum_{b_1+\dots+b_n=j} (-r_1)^{b_1} (-r_2)^{b_2} \dots (-r_n)^{b_n} x^{n-j}. \end{aligned}$$

□

Symmetric polynomials

A polynomial $f(t_1, t_2, \dots, t_n)$ is *symmetric* if it is unchanged by permuting the variables $t_1, t_2, \dots, t_n$.

$t_1 + t_2 + \dots + t_n$ is symmetric.

For any numbers or variables $t = (t_1, t_2, \dots, t_n)$ let

$$P_t(x) := (x - t_1)(x - t_2) \dots (x - t_n).$$

Clearly the roots of $P_t(x)$ are precisely the entries of the vector t. Let $e(t) := (e_1(t), e_2(t), \dots, e_n(t))$, so that e maps vectors to vectors (with entries in our field).

Recall that a field is *algebraically closed* if every polynomial with coefficients in that field splits into a product of linear factors with coefficients in that field. Recall the fundamental theorem of algebra (theorem 10.3 on page 72): the field of complex numbers $\mathbb{C}$ is algebraically closed.

Lemma 16.3. *For each vector c with entries in a field, there is a vector r, with entries in a finite degree extension field, so that $e(r) = c$.*

Proof. Let $r_1, r_2, \dots, r_n$ be the roots of the polynomial

$$P(x) = x^n - c_1 x^{n-1} + c_2 x^{n-2} + \dots + (-1)^n c_n$$

in its splitting field. By proposition 16.2 on the previous page, $P_r(x)$ has coefficients precisely the same as those of $P(x)$, i.e. $P_r(x) = P(x)$. □

Lemma 16.4. *Given some variables over a field, if there are more elements of the field than variables, then there is no nontrivial polynomial equation satisfied by the elementary symmetric polynomials.*

Consequently, any such polynomial relation disappears after some field extension.

Proof. Any relation $p(e(t)) = 0$ between the elementary symmetric polynomials must be satisfied at all points c, so is trivial for any function p. For a polynomial p, which we don't think of as a function, over a large enough field, the expression $p(e(t))$ vanishing forces $p(c)$ to vanish for each c. Varying any one component of c, $p(c)$ is a polynomial function of that component with too many roots, so is zero in that component for all other components fixed, so the zero polynomial by induction. □

Lemma 16.5. *Over any field, the entries of two vectors u and v are permutations of one another just when $e(u) = e(v)$, i.e. just when the elementary symmetric polynomials take the same values at u and at v.*

Proof. The values of the entries of $e(u), e(v)$ are the coefficients of $P_u(x), P_v(x)$ whose roots are the entries of u, v. Clearly the following are equivalent:

- The entries of u, v agree up to order.
- The monic polynomials with those roots agree.
- $P_u(x) = P_v(x)$.
- $P_u(x), P_v(x)$ have the same coefficients.
- $e(u) = e(v)$.

□

Lemma 16.6. *Over any field, a list of rational functions $f(t) = (f_1(t), \ldots, f_n(t))$, with $t = (t_1, \ldots, t_n)$, are precisely the functions $t = (t_1, \ldots, t_n)$ up to reordering just when they satisfy*

$$e(f(t)) = e(t).$$

Proof. If our field is k, apply lemma 16.5 over $k(t_1, \ldots, t_n)$. □

We want to see that every symmetric polynomial $f(t_1, t_2, \ldots, t_n)$ has the form $f(t) = h(e(t))$, for a unique polynomial h, and conversely if h is any polynomial at all, then $f(t) = h(e(t))$ determines a symmetric polynomial. We want a recipe to write down each symmetric polynomial in terms of the elemenary symmetric polynomials, to find this mysterious h.

Take the polynomial

$$f = 3x^2yz + 3xy^2z + 3xyz^2 + 5xy + 5xz + 5yz.$$

Clearly the terms with the 5's look like an elementary symmetric polynomial:

$$f = 3x^2yz + 3xy^2z + 3xyz^2 + 5e_2.$$

(We write e_2 to mean the polynomial $e_2(x, y, z) = xy + xz + yz$, the 2nd elementary symmetric polynomial.) But what about the terms with the 3's? Factor them all together as much as we can:

$$f = 3xyz(x + y + z) + 5e_2.$$

Then it is clear:

$$f = 3e_3e_1 + 5e_2.$$

If $a = (a_1, a_2, \ldots, a_n)$, write t^a to mean $t_1^{a_1}t_2^{a_2}\ldots t_n^{a_n}$. Order terms by "alphabetical" order, also called *weight*: order monomials by the order in t_1; if two monomials have the same order in t_1, break the tie by looking at the order in t_2, and so on.

The term

$$t_1^5t_2^3$$

has higher weight than any of

$$t_1^3 t_2^5, t_1^5 t_2^2, 1, \text{ and } t_2^{1000}.$$

Showing the highest order term, and writing dots to indicate lower order terms,

$$\begin{aligned} e_1(t) &= t_1 + \dots, \\ e_2(t) &= t_1 t_2 + \dots, \\ &\vdots \\ e_j(t) &= t_1 t_2 \dots t_j + \dots \\ &\vdots \\ e_n(t) &= t_1 t_2 \dots t_n. \end{aligned}$$

If a symmetric polynomial contains a term, then it also contains every term obtained by permuting the variables:

$$t_1^3 t_2^5 + t_1^5 t_2^3.$$

It is somehow easier to read x, y, z than t_1, t_2, t_3, so take a symmetric polynomial

$$f(x, y, z) = 6x^{16} y^9 z^7 + \dots$$

where we only write out the highest weight term. In such notation,

$$\begin{aligned} e_1(x, y, z) &= x + y + z = x + \dots, \\ e_2(x, y, z) &= xy + xz + yz = xz + \dots, \\ e_3(x, y, z) &= xyz. \end{aligned}$$

Imagine a term in f: if the highest power in that term is in y or z, permute the variables to get that highest power into x, giving a higher weight term in f. So the highest weight term contains the variables x, y, z, with highest power in x and successively lower powers in each of y, z. By the same reasoning, inside the highest weight term, each of x, y, z appears to at least to the power of the z variable: in this case, at least to a power of 7. Factor out 7 powers of each variable; we underline that factored out part so you can see it:

$$f = 6x^9 y^2 \underline{(xyz)^7} + \dots$$

In the remaining factors, there are now no z variables, and each variable appears at least to a power of 2, so we factor out 2 of each:

$$f = 6x^7 \underline{(xy)^2} \, (xyz)^7 + \dots$$

So finally it is clear that f has the same highest weight as $6e_1^7 e_2^2 e_3^7$. Hence

$$f = 6e_1^7 e_2^2 e_3^7 + \dots$$

up to terms of lower weight.

Theorem 16.7. *Every symmetric polynomial over any commutative ring has exactly one expression as a polynomial in the elementary symmetric polynomials.*

Proof. Write the highest weight term as some coefficient times

$$t^a = t_1^{a_1} t_2^{a_2} \dots t_n^{a_n}.$$

If a_1 is smaller than a_2, swap t_1 with t_2 to get a higher weight term, a contradiction, so $a_1 \geq a_2$. In the same way, $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$. Let

$$\begin{aligned} d_1 &\coloneqq a_1 - a_2, \\ d_2 &\coloneqq a_2 - a_3, \\ &\vdots \\ d_{n-1} &\coloneqq a_{n-1} - a_n, \\ d_n &\coloneqq a_n. \end{aligned}$$

Let's search for the highest weight term in e^d:

$$\begin{aligned} e^d &= e_1^{d_1} \dots e_n^{d_n}, \\ &= (t_1 + \dots)^{d_1} (t_1 t_2 + \dots)^{d_2} \dots (t_1 \dots t_n)^{d_n}, \\ &= t_1^{d_1} (t_1 t_2)^{d_2} \dots (t_1 \dots t_n)^{d_n} + \dots \\ &= t_1^{a_1 - a_2} (t_1 t_2)^{a_2 - a_3} \dots (t_1 \dots t_{n-1})^{a_{n-1} - a_n} (t_1 \dots t_n)^{a_n} + \dots, \\ &= t_1^{a_1} t_2^{a_2} \dots t_n^{a_n} + \dots \\ &= t^a + \dots \end{aligned}$$

So whatever coefficient of t^a we find in our given polynomial, subtract that multiple of e^d from the polynomial, and we reduce the weight. By induction, we can somehow write the lower weight terms as polynomials in the elementary symmetric polynomials. □

The sum of squares of two variables is symmetric:

$$x^2 + y^2 = (x + y)^2 - 2xy.$$

To compute out these expressions: $f(x, y) = x^2 + y^2$ has highest term x^2. The polynomials $e_1(x, y) = x + y$ and $e_2(x, y) = xy$ have highest terms x and xy. So we subtract off $e_1(x, y)^2$ from $f(x, y)$, and find $f(x, y) - e_1(x, y)^2 = -2xy = -2e_2(x, y)$.

16.1 Express each of the following polynomials as polynomials in the elementary symmetric polynomials:

a. $x^3y^3z + x^3yz^3 + xy^3z^3$

b. $4xyz^3 + 4xzy^3 + 4yzx^3$

c. $x^4y^4z^4$

d. $xyz + x^2y^2z^2$

Symmetric functions

A *symmetric function* is a rational function of several variables, invariant under permutation of those variables.

16.2 Prove that every symmetric function over any field is a ratio of polynomials in the elementary symmetric polynomials.

Sage

Sage can compute with elementary symmetric polynomials. It writes them in a strange notation. The polynomials we have denoted by e_3 sage denotes by `e[3]`. More strangely, sage denotes $e_1e_2^3e_5^2$ as `e[5,5,2,2,2,1]` which is the same in sage as `e[2]^3*e[1]*e[5]^2`. You can't write `e[1,2,2,2,5,5]`; you have to write it as `e[5,5,2,2,2,1]`: the indices have to decrease. Set up the required rings:

```
P.<w,x,y,z>=PolynomialRing(QQ)
S=SymmetricFunctions(QQ)
e=S.e()
```

This creates a ring $P = \mathbb{Q}[w, x, y, z]$, and then lets S be the ring of symmetric polynomials. The last (mysterious) line sets up the object e to be the elementary symmetric polynomials of the ring S. We can then define a polynomial in our variables:

```
f = w^2+x^2+y^2+z^2+w*x+w*y+w*z+x*y+x*z+y*z
e.from_polynomial(f)
```

which prints out $e_{1,1} - e_2$, the expression of f in terms of elementary symmetric polynomials. To expand out a symmetric polynomial into w, x, y, z variables:

```
q = e[2,1]+e[3]
q.expand(4,alphabet=['w','x','y','z'])
```

prints out

$$w^2x+wx^2+w^2y+4wxy+x^2y+wy^2+xy^2+w^2z+4wxz+x^2z+4wyz+4xyz+y^2z+wz^2+xz^2+yz^2$$

Sums of powers

Define $p_j(t) = t_1^j + t_2^j + \cdots + t_n^j$, the sums of powers.

Lemma 16.8 (Isaac Newton)**.** *The sums of powers are related to the elementary symmetric polynomials by*

$$\begin{aligned}
0 &= e_1 - p_1, \\
0 &= 2\,e_2 - p_1\,e_1 + p_2, \\
&\vdots \\
0 &= k\,e_k - p_1\,e_{k-1} + p_2\,e_{k-2} - \cdots + (-1)^{k-1}p_{k-1}\,e_1 + (-1)^k p_k,
\end{aligned}$$

Using these equations, we can write the sums of powers in terms of the elementary symmetric polynomials over any commutative ring. On the other hand, we can only write the elementary symmetric polynomials in terms of the sums of powers over any field of characteristic zero or larger than n.

Proof. Let's write $t^{(\ell)}$ for t with the ℓ^{th} entry removed, so if t is a vector with n entries, then $t^{(\ell)}$ is a vector with $n-1$ entries.

$$p_j e_{k-j} = \sum_\ell t_\ell^j \sum_{i_1<i_2<\cdots<i_{k-j}} t_{i_1} t_{i_2} \dots t_{i_{k-j}}$$

Either we can't pull a t_ℓ factor out of the second sum, or we can:

$$\begin{aligned} &= \sum_\ell t_\ell^j \sum_{i_1<i_2<\cdots<i_{k-j}}^{i_1,i_2,\dots\neq\ell} t_{i_1} t_{i_2} \dots t_{i_{k-j}} + \sum_\ell t_\ell^{j+1} \sum_{i_1<i_2<\cdots<i_{k-j-1}}^{i_1,i_2,\dots\neq\ell} t_{i_1} t_{i_2} \dots t_{i_{k-j-1}} \\ &= \sum_\ell t_\ell^j e_{k-j}\left(t^{(\ell)}\right) + \sum_\ell t_\ell^{j+1} e_{k-j-1}\left(t^{(\ell)}\right). \end{aligned}$$

Putting in successive terms of our sum,

$$\begin{aligned} p_j e_{k-j} - p_{j+1} e_{k-j-1} &= \sum_\ell t_\ell^j e_{k-j}\left(t^{(\ell)}\right) + \sum_\ell t_\ell^{j+1} e_{k-j-1}\left(t^{(\ell)}\right) \\ &\quad - \sum_\ell t_\ell^{j+1} e_{k-j-1}\left(t^{(\ell)}\right) - \sum_\ell t_\ell^{j+2} e_{k-j-2}\left(t^{(\ell)}\right) \\ &= \sum_\ell t_\ell^j e_{k-j}\left(t^{(\ell)}\right) - \sum_\ell t_\ell^{j+2} e_{k-j-2}\left(t^{(\ell)}\right). \end{aligned}$$

Hence the sum collapses to

$$\begin{aligned} p_1 e_k - p_2 e_{k-1} + \cdots + (-1)^{k-1} p_{k-1} e_1 &= \sum_\ell t_\ell e_{k-1}\left(t^{(\ell)}\right) + (-1)^{k-1} \sum_\ell t_\ell^k \cdot e_0\left(t^{(\ell)}\right) \\ &= k\, e_k + (-1)^{k-1} p_k. \end{aligned}$$

□

Proposition 16.9. *Every symmetric polynomial over a field of characteristic zero is a polynomial in the sums of powers.*

Proof. We can solve recursively for the sums of powers in terms of the elementary symmetric polynomials and conversely. □

The invariants of a square matrix

Over any field, a rational function $f(A)$ in the entries of a square matrix A, with coefficients in the field, is *invariant* if $f\left(FAF^{-1}\right) = f(A)$ for any invertible matrix F with coefficients in the field. An invariant is independent of change of basis: if $T\colon V \to V$ is a linear map on an n-dimensional vector space, we can define the value $f(T)$ of any invariant f of $n \times n$ matrices, by letting $f(T) := f(A)$ where A is the matrix associated to T in some basis of V.

The determinant is an invariant of square matrices.

For any $n \times n$ matrix A, write

$$\det(A - \lambda I) = \chi_A(\lambda) = e_n(A) - e_{n-1}(A)\lambda + e_{n-2}(A)\lambda^2 + \cdots + (-1)^n\lambda^n.$$

The expressions $e_1(A), e_2(A), \ldots, e_n(A)$ are invariants, while $\chi_A(\lambda)$ is the *characteristic polynomial* of A.

If we write the trace of a matrix A as $\operatorname{tr} A$, then the expressions

$$p_k(A) = \operatorname{tr}\left(A^k\right)$$

are invariant polynomials in the entries of A.

16.3 If A is diagonal, say

$$A = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix},$$

then prove that $e_j(A) = e_j\,(t_1, t_2, \ldots, t_n)$, the elementary symmetric polynomials of the eigenvalues.

16.4 Generalize the previous exercise to A diagonalizable.

16.5 Prove that the entries of A^{-1} are rational functions of the entries of the square matrix A, over any field.

Let Δ_A be the discriminant of the characteristic polynomial of A: $\Delta_A := \Delta_{\chi_A}$. The map $A \mapsto \Delta_A$ is also a polynomial invariant of A, as the coefficients of the characteristic polynomial are.

16.6 Take a square matrix A. Suppose that the characteristic polynomial of A splits into linear factors. Prove that $\Delta_A = 0$ just when A has an eigenvalue of multiplicity two or more.

Theorem 16.10. *Every invariant polynomial (or invariant rational function) of square matrices, over any field with more elements than the number of columns of the matrices, has exactly one expression as a polynomial (rational function) in the elementary symmetric polynomials of the eigenvalues.*

We can replace the elementary symmetric polynomials of the eigenvalues by the sums of powers of the eigenvalues, over any field of characteristic zero or larger than the number of columns.

Proof. Take an invariant polynomial $f(A)$. (The same argument will work for an invariant rational function.) Every invariant polynomial $f(A)$ determines an invariant polynomial $f(t)$ by setting

$$A = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix}.$$

Taking F any permutation matrix, invariance tells us that $f\left(FAF^{-1}\right) = f(A)$. But $f\left(FAF^{-1}\right)$ is given by applying the associated permutation to the entries of t. Therefore $f(t)$ is a symmetric polynomial. Therefore $f(t) = h(e(t))$, for some polynomial h; so $f(A) = h(e(A))$ for diagonal matrices. Replace f by $f(A) - h(e(A))$ to arrange that $f(A) = 0$ on all diagonal matrices A. By invariance, $f(A) = 0$ on all diagonalizable matrices.

The equation $f(FAF^{-1}) = f(A)$ holds for all F over our field. Imagine that F has abstract variables as entries. The difference $f(FAF^{-1}) - f(A)$ is a rational function in the entries of F. Its numerator vanishes for any choice of values of those abstract variables, and so vanishes. Hence $f(FAF^{-1}) = f(A)$ for F with abstract variable entries. In particular, if our field lies in a larger field, then f remains invariant over the larger field, because we can plug into the entries of F the values in the larger field.

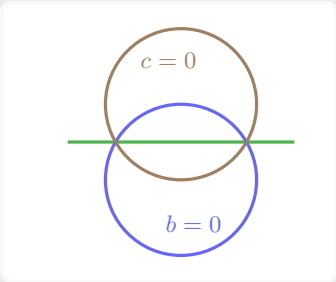
Recall that a matrix has an eigenvector for each eigenvalue, so if there are n distinct eigenvalues, then A is diagonalizable. This occurs just when the characteristic polynomial $\chi_A(\lambda)$ splits into distinct linear factors, and so $\Delta_A \neq 0$. Conversely, if $\Delta_A \neq 0$ and $\chi_A(\lambda)$ splits into linear factors, then they are distinct, and so $f(A) = 0$. By theorem 13.6 on page 99, we can sit our field into a larger field in which the characteristic polynomial $\chi_A(\lambda)$ splits into linear factors, and therefore if $\Delta_A \neq 0$ then $f(A) = 0$.

Pick any matrix A_0 whose eigenvalues are all distinct. In particular, $\Delta_{A_0} \neq 0$. Take any matrix A_1. For an abstract variable t, let $A_t := (1-t)A_0 + tA_1$. Since Δ_{A_t} is a polynomial, not vanishing at $t = 0$, it is a nonzero polynomial in t. So $\Delta_{A_t} \neq 0$ except for finitely many t. Hence $f(A_t) = 0$ except for finitely many t. But $f(A)$ is a polynomial, so vanishes for all t. □

The function $f(A) = e_j\left(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\right)$, where the matrix A, with complex number entries, has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, is a continuous invariant function of a real matrix A, and is *not* a polynomial in $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Resultants and permutations

If $b(x, y) = x^2 + y^2 - 1$ and $c(x, y) = x^2 + (y-1)^2 - 1$, how can we find the points (x, y) at which both $b(x, y) = 0$ and $c(x, y) = 0$?



Think of $b(x, y)$ and $c(x, y)$ as polynomials in x, with coefficients rational functions of y. Then the resultant $r(y)$ of $b(x, y), c(x, y)$ is a rational function of y. Compute it: $r(y) = (2y-1)^2$ vanishes at $y = 1/2$. So there is a common factor there:

$$b(x, 1/2) = c(x, 1/2) = \left(x - \frac{\sqrt{3}}{2}\right)\left(x + \frac{\sqrt{3}}{2}\right).$$

The common factor came from two common roots.

16.7 Working over $k = \mathbb{R}$, let $b(x, y) := xy^2 + x^{10}y + 1$ and $c(x, y) := -xy^2 + 2x^{13}y + 1$. Let B be the plane algebraic curve with equation $0 = b(x, y)$ and C the one with equation $0 = c(x, y)$. Use a resultant to find the points (x, y) of the plane where the curves meet.

In chapter 15, we saw that for any two polynomials split into linear factors

$$b(x) = (x - \beta_1)(x - \beta_2)\dots(x - \beta_m),$$
$$c(x) = (x - \gamma_1)(x - \gamma_2)\dots(x - \gamma_n),$$

the resultant is

$$\mathrm{res}_{b,c} = (\gamma_1 - \beta_1)(\gamma_1 - \beta_2)\dots(\gamma_n - \beta_m).$$

So the resultant is homogeneous of degree mn in the variables β_i, γ_j. The resultant is invariant under any permutation of the roots of $b(x)$, and also under any permutation of the roots of $c(x)$. Indeed the resultant is by definition expressed in terms of the coefficients of $b(x)$ and $c(x)$, not the roots. The coefficients are homogeneous polynomials in the roots, elementary symmetric polynomials. Expanding out the coefficients

$$\begin{aligned} b(x) &= x^m + b_{m-1}x^{m-1} + \dots + b_0, \\ &= x^m - e_1(\beta)x^{m-1} + e_2(\beta)x^{m-2} + \dots \pm e_m(\beta), \\ c(x) &= x^n + c_{n-1}x^{n-1} + \dots + c_0, \\ &= x^n - e_1(\gamma)x^{n-1} + e_2(\gamma)x^{n-2} + \dots \pm e_n(\gamma), \end{aligned}$$

we see that b_j has degree $m - j$ in the roots.

Suppose now that we just take any polynomial $b(x)$ of degree m with coefficients being abstract variables b_j. We will now invent a different concept of *weight*. Assign

each coefficient b_j the *weight* $m - j$ and then define the *weight* of a monomial in the b_j to be the sum of weights of its factors.

Lemma 16.11. *With this notion of weight, every term in the resultant has weight mn.*

Proof. By theorem 13.6 on page 99, we can replace our field by some larger field, to arrange that $b(x)$ and $c(x)$ split into linear factors. We assign each root a weight of one; this gives the correct weight to each coefficient b_j, c_j. The weight of each term in the resultant is just its degree in the roots, so mn as above. □

Proposition 16.12. *Given any two polynomials $b(x, y)$ of total degree m and $c(x, y)$ of total degree n over a field, in two variables x, y, either there are at most mn values of y for which there is some point (x, y) which satisfies both $b(x, y) = 0$ and $c(x, y) = 0$ (and the resultant in x vanishes at those values), or $b(x, y)$ and $c(x, y)$ have a common factor as polynomials in x, y. If the polynomials are homogeneous then either they have a homogeneous common factor, or their resultant is homogeneous nonzero of degree mn.*

Proof. Think of the polynomials as polynomials in x with coefficients rational in y:

$$b(x, y) = \sum_{j+k \leq m} b_{jk} x^k y^j = \sum_j b_j(y) x^j,$$

we see that $b_j(y)$ has degree at most $m - j$, exactly the weight of the coefficient $b_j(y)$ as it enters into the resultant. Therefore the resultant is a polynomial of degree at most mn in y. The resultant vanishes at those values $y = a$ for which there is a common factor between $b(x, a)$ and $c(x, a)$, and in particular if there is a common root it vanishes. So there are at most mn such points or the resultant is everywhere zero. If the resultant vanishes everywhere, then $b(x, y)$ and $c(x, y)$ have a common factor with coefficients rational functions of y. By Gauss's lemma (proposition 11.6 on page 81) they also have a common factor in polynomials in x, y.

For homogeneous polynomials, each term $b_j(y)$ has degree exactly $m - j$, so the resultant either vanishes everywhere or has degree exactly mn. □

Chapter 17

Morphisms

We dance round in a ring and suppose, but the secret sits in the middle and knows.

— Robert Frost
The Secret Sits, A Witness Tree, 1942

Definition

A *morphism* of rings $f\colon R \to S$ is a map f taking elements of a ring R to elements of a ring S, so that $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all elements b, c of R. (Morphisms are also often called *homomorphisms*.)

The map $f\colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $f(b) = \bar{b}$, remainder modulo m, is a morphism.

17.1 Prove that the map $f\colon \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$, $f(b) = 4b$ is a morphism. Draw pictures to explain it.

17.2 Find all morphisms $f\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$, where p and q are prime numbers and $p \neq q$.

Sage

We define $R = \mathbb{Q}[x, y]$ and $T = \mathbb{Q}[z]$ and define a morphism $\phi\colon R \to S$ by $\phi(x) = z^3$ and $\phi(y) = 0$:

```
R.<x,y> = PolynomialRing(QQ)
T.<z> = PolynomialRing(QQ)
phi = R.hom([z^3, 0],T)
phi(x^2+y^2)
```

yielding $\phi(x^2 + y^2) = z^6$.

Kernel and image of a morphism

If $f\colon R \to S$ is a morphism of rings, the *kernel* of f is the set

$$\ker f := \{\, r \in R \mid f(r) = 0 \,\}.$$

The *image* of f is the set

$$f(R) := \{\, s \in S \mid s = f(r) \text{ for some } r \in R \,\}.$$

The morphism $x \in \mathbb{R} \mapsto x + 0i \in \mathbb{C}$ has kernel $\{\, 0 \,\}$ and image the set of all numbers $x + 0i$. (We often write 0 to mean $\{\, 0 \,\}$ in algebra.)

The morphism $a \in \mathbb{Z} \mapsto \bar{a} \in \mathbb{Z}/m\mathbb{Z}$ has kernel $m\mathbb{Z}$.

17.3 Prove that the kernel of a morphism $f\colon R \to S$ of rings is a subring of R, while the image is a subring of S.

17.4 Prove that the morphism $p(x) \in \mathbb{R}[x] \mapsto p(i) \in \mathbb{C}$ has kernel consisting of all polynomials of the form

$$\left(1 + x^2\right) p(x),$$

for any polynomial $p(x)$.

17.5 Suppose that R is a ring and that $f, g\colon \mathbb{Q} \to R$ are morphisms and that $f(1) = g(1)$. Prove that $f = g$.

A morphism $f\colon R \to S$ with kernel 0 is *injective*, also called *one-to-one*, while a morphism $f\colon R \to S$ with image $f(R) = S$ is *surjective* or *onto*. A surjective and injective morphism is an *isomorphism*. If there is an isomorphism between rings R and S, they are *isomorphic*, denoted $R \cong S$. If two rings are isomorphic, then they are identical for all practical purposes. An isomorphism $f\colon R \to R$ from a ring to itself is an *automorphism*.

Take some positive integers $m_1, m_2, \ldots, m_n$. Let

$$m := m_1 m_2 \ldots m_n.$$

For each $b \in \mathbb{Z}/m\mathbb{Z}$, write its remainder modulo m_1 as $\bar{b}_1$, and so on. Write $\bar{b}$ to mean

$$\bar{b} := \left(\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_n\right).$$

Let

$$S := (\mathbb{Z}/m_1\mathbb{Z}) \oplus (\mathbb{Z}/m_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/m_n\mathbb{Z}).$$

Map $f\colon \mathbb{Z}/m\mathbb{Z} \to S$ by $f(b) = \bar{b}$. It is clear that f is a ring morphism. The Chinese remainder theorem states that f is a ring *isomorphism* precisely when any two of the integers

$$m_1, m_2, \ldots, m_n$$

are coprime. For example, if $m = 3030 = 2 \cdot 3 \cdot 5 \cdot 101$, then

$$\mathbb{Z}/3030\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z}) \oplus (\mathbb{Z}/101\mathbb{Z}).$$

17.6 Prove that $\mathbb{Z}/4\mathbb{Z}$ is *not* isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$.

17.7 Prove that any two rings that each contain exactly one element are isomorphic, by a unique isomorphism.

Kernels

The main trick to understand complicated rings is to find morphisms to them from more elementary rings, and from them to more elementary rings.

17.8 Prove that every subring of any given ring is the image of some morphism of rings.

It is natural to ask which subrings of a ring are kernels of morphisms. An *ideal* of a commutative ring R is a subring I of R so that if i is in I and r is anything in R, in I or not, then ri is in I: an ideal is *sticky* under multiplication.

The even integers form an ideal $2\mathbb{Z}$ inside the integers.

The integers do *not* form an ideal inside the rational numbers. They are not sticky, because $i = 2$ is an integer and $r = \frac{1}{3}$ is rational, but $ri = \frac{2}{3}$ is *not* an integer.

The real polynomials in a variable x vanishing to order 3 or more at the origin form an ideal $x^3\mathbb{R}[x]$ inside $\mathbb{R}[x]$. Note why they are sticky: if $i = x^3p(x)$ and $r = q(x)$ then $ri = x^3p(x)q(x)$ has a factor of x^3 still, so also vanishes to order 3 or more. This is a good way to think about ideals: they remind us of the ideal of functions vanishing to some order somewhere, inside some ring of functions.

Take a commutative ring R with identity. Let $P := R[x]$ be the ring of polynomials in a variable x. The polynomials of even degree form a subring $E \subset P$. But E is not an ideal, since x^2 belongs to E, and x to P, but $x^2x = x^3$ is *not* in E.

17.9 Prove that the only ideals in $\mathbb{Q}$ are 0 and $\mathbb{Q}$.

17.10 Prove that a nonempty subset I of a commutative ring R is an ideal just when, if i, j are in I then $i - j$ is in I and if i is in I and r is in R then ri is in I.

Lemma 17.1. *The kernel of any morphism of rings is an ideal.*

Proof. Take a ring morphism $f\colon R \to S$ and let $I := \ker f$. Pick any i in I and any r in R. We need to prove that ir is in I, i.e. that f kills ir, i.e. that $0 = f(ir)$. But $f(ir) = f(i)f(r) = 0f(r) = 0$. □

We will soon see that kernels of morphisms are precisely the same as ideals.

17.11 Prove that the ideals of $\mathbb{Z}$ are precisely $0, \mathbb{Z}$ and the subsets $m\mathbb{Z}$ for any integer $m \geq 2$. (Of course, we could just say that the ideals are the sets $m\mathbb{Z}$ for any integer $m \geq 0$.)

Lemma 17.2. *Given a commutative ring R and any set A of elements of R, there is a unique smallest ideal containing A, denoted (A). To be precise, (A) is the set of all finite sums*

$$r_1 a_1 + r_2 a_2 + \cdots + r_n a_n$$

for any elements

$$r_1, r_2, \ldots, r_n$$

of R and any elements

$$a_1, a_2, \ldots, a_n$$

of A.

Proof. Let (A) be that set of finite sums. Clearly A lies in (A), and (A) is an ideal. On the other hand, if A lies in some ideal, that ideal must contain all elements of A, and be sticky, so contain all products ra for r in R and a in A, and so contains all finite sums of such, and so contains I. □

The ideal $I := (A)$ generated by $A := \{ 12, 18 \}$ inside $R := \mathbb{Z}$ is precisely $6\mathbb{Z}$, because we use Bézout coefficients r_1, r_2 to write 6 as $6 = r_1 \cdot 12 + r_2 \cdot 18$, and so 6 lies in $I = (12, 18)$. But then $I = (12, 18) = (6)$.

More generally, the ideal $I := (m_1, m_2, \ldots, m_n)$ generated by some integers $m_1, m_2, \ldots, m_n$ is precisely $I = (m) = m\mathbb{Z}$ generated by their greatest common divisor. So we can think of ideals in rings as a replacement for the concept of greatest common divisor.

The same idea works for polynomials, instead of integers. Working, for example, over the rational numbers, the ideal $I := (p_1(x), p_2(x), \ldots, p_n(x))$ inside the ring $R = \mathbb{Q}[x]$ is just $I = p(x)\mathbb{Q}[x]$ where

$$p(x) = \gcd \{p_1(x), p_2(x), \ldots, p_n(x)\} .$$

In $R := \mathbb{Q}[x, y]$, the story is more complicated. The ideal $I := \left(x, y^2\right)$ is not expressible as $p(x, y)R$, because I cannot be expressed using only one generator $p(x, y)$. If there were such a generator $p(x, y)$, then x and y^2 would have to be divisible by $p(x, y)$, forcing $p(x, y)$ to be constant, so not generating I.

Sage

In sage we construct ideals, for example inside the ring $R = \mathbb{Q}[x, y]$, as:

```
R.<x,y>=PolynomialRing(QQ)
R.ideal( [x^3,y^3+x^3] )
```

We can then test whether two ideals are equal:

```
R.ideal( [x^3,y^3+x^3] )==R.ideal( [x^3,y^3] )
```

yields True.

Fractions

Theorem 17.3. *A commutative ring R lies inside a field k as a subring just when the product of any two nonzero elements of R is nonzero. After perhaps replacing k by the subfield generated by R, the field k is then uniquely determined up to isomorphism, and called the* field of fractions *of R. Every nonzero ring morphism $R \to K$ to a field K extends uniquely to an injective field morphism $k \to K$.*

Proof. If a commutative ring R lies in a field k, then clearly any two nonzero elements have nonzero product, as they both have reciprocals. Take two pairs (a, b) and (c, d) with a, b, c, d in R, and $0 \neq b$ and $d \neq 0$. Declare them equivalent if $ad = bc$. Write the equivalence class of (a, b) as $\frac{a}{b}$. Define

$$\begin{aligned}(a,b) + (c,d) &= (ad + bc, bd),\\ (a,b) - (c,d) &= (ad - bc, bd),\\ (a,b) \cdot (c,d) &= (ac, bd).\end{aligned}$$

It is easy to check that these operations applied to equivalent inputs yield equivalent outputs, giving unambiguous definitions on ratios:

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd},\\ \frac{a}{b} - \frac{c}{d} &= \frac{ad - bc}{bd},\\ \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd},\end{aligned}$$

Tedious checking shows that these yield a field k. The crucial step is that the denominators can't vanish. The reciprocal in k is easy to find:

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

if $a \neq 0$ and $b \neq 0$. Map R to k by $a \mapsto \frac{a}{1}$. Check that this map is injective. Given a ring morphism $f\colon R \to K$ to a field, extend by

$$f\left(\frac{a}{b}\right) = \frac{f(a)}{f(b)}.$$

Again tedious checking ensures that this is defined. Every field morphism is injective. □

The field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$.

The field of fractions of the ring

$$\mathbb{Z}\left[\frac{1}{2}\right]$$

of half integers is also $\mathbb{Q}$.

The field of fractions of a field k is k.

The field of fractions of $\mathbb{Z}[x]$ is $\mathbb{Q}(x)$.

The field of fractions of $\mathbb{Q}[x]$ is also $\mathbb{Q}(x)$.

17.12 Prove that these examples above are correct.

We won't prove:

Theorem 17.4. *If k is a field, then $k((x))$ is the field of fractions of $k[[x]]$ for any variables $x = (x_1, \ldots, x_n)$.*

Sage

Sage constructs fields of fractions out of any ring with no zero divisors. For example, let $F := \mathbb{Z}/7\mathbb{Z}$ and $R := F[x, y]$ and let k be the fraction field of R, and then $S = k[t]$ and then compute the resultant of two polynomials $p(t), q(t) \in S$:

```
R.<x,y> = PolynomialRing(GF(7))
k=R.fraction_field()
S.<t> = PolynomialRing(k)
p=S(x*t+t^2/t)
q=S(y*t+1)
p.resultant(q)
```

Linear algebra over commutative rings

An *integral domain* is a commutative ring with identity $1 \neq 0$ so that any two nonzero elements have nonzero product. As above, every integral domain lies inside its field of fractions.

17.13 Suppose that A is a square matrix with entries in an integral domain D. Prove that $\det A = 0$ just when there is a nonzero vector x with entries in D so that $Ax = 0$.

Recall that the determinant of a matrix, over any field (but think of real numbers just to be more comfortable), can be expanded across the first row:

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n},$$

where A_{ij} is the i,j-minor, i.e. the determinant of the matrix we get when we delete row i and column j from A. But we can expand across the second row instead:

$$\det A = a_{21}A_{21} + a_{22}A_{22} + \cdots + a_{2n}A_{2n}.$$

What if we plug in the a_{1j} from the first equation in place of the a_{2j} from the second? We get

$$a_{11}A_{21} + a_{12}A_{22} + \cdots + a_{1n}A_{2n}.$$

This is the determinant of the matrix we get when we take A and replace row 2 by a copy of row 1. That matrix has two equal rows, so determinant 0. We conclude:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \dots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} = \begin{pmatrix} \det A & 0 & \dots & 0 \\ 0 & \det A & \dots & 0 \\ \vdots & & \dots & \vdots \\ 0 & 0 & \dots & \det A \end{pmatrix}$$

Note that the second matrix is A transposed, so lets transpose the whole thing. The *adjugate* of A, denoted $\operatorname{adj} A$, is the matrix with entries $\operatorname{adj} A_{ij} = a_{ji}$.

Theorem 17.5. *For any square matrix A with entries in any commutative ring, $A \operatorname{adj} A = (\operatorname{adj} A)A = (\det A)I$.*

Proof. Above we see the result holds for real matrices: expanding across the rows gave $A \operatorname{adj} A = (\det A)I$. Expanding down columns gives $(\operatorname{adj} A)A = (\det A)I$. Hence the result holds for any matrix A whose entries are real-valued functions: just take the values of the functions at each point. So the result holds for the ring $R = \mathbb{Z}(t_1, \dots, t_n)$ for any variables $t_1, \dots, t_n$, thinking of rational functions as functions, not just as formal expressions, which we can safely do over real numbers or integers.

Consider a matrix T whose each entry is an abstract variable like

$$T = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & \dots & \vdots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix}$$

Since T has entries in $\mathbb{Z}(t_{11}, \dots, t_{nn})$, the result is true for T. The computation of the adjugate of this T only involves multiplying entries, with ± 1 coefficients, so a polynomial with integer coefficients in the various t_{ij} variables.

Take any commutative ring R. If we can prove the result for a commutative ring containing R, that proves it for R. We can put R into a commutative ring with identity, by adding one element 1 and making it an identity. So it suffices to prove the result for any commutative ring R with identity. Take any matrix A with entries A_{ij} in R. We can replace R by the subring of R generated by these entries A_{ij}. Map

$$p(t_{11}, \dots, t_{nn}) \in \mathbb{Z}[t_{11}, \dots, t_{nn}] \mapsto p(A_{11}, \dots, A_{nn}) \in R.$$

This map is a surjective ring morphism. It identifies all of the expressions we compute out in the adjugate, so takes the entries of $T, \operatorname{adj} T, (\det T)I$ to those of $A, \operatorname{adj} A, (\det A)I$. □

Given a polynomial in one variable, like $b(t) = t^2 - t + 1$, and a square matrix A, we let $b(A)$ mean $b(A) = A^2 - A + I$. We even allow $b(t)$ to have matrix coefficients, so if

$$B(t) = B_0 + tB_1 + \cdots + t^p B_p$$

is a polynomial with matrix coefficients, let us *define* $B(A)$ to mean

$$B(t) = B_0 + B_1 A + \cdots + B_p A^p,$$

putting all copies of A on the right hand side.

Take two polynomials $B(t), C(t)$ whose coefficients are square matrices, all of the same size. Expand out

$$B(t) = B_0 + tB_1 + \cdots + t^p B_p,$$
$$C(t) = C_0 + tC_1 + \cdots + t^q C_q.$$

The product $P(t) = B(t)C(t)$ expands out to

$$P(t) = P_0 + tP_1 + \cdots + t^{p+q} P_{p+q},$$

where

$$P_i = \sum_{i=j+k} B_j C_k.$$

So

$$P(A) = P_0 + P_1 A + \cdots + P_{p+q} A^{p+q}.$$

If a square matrix A commutes with all of the coefficient matrices $C_0, C_1, \ldots, C_q$, then in each P_i, A passes through all C_k, so

$$\begin{aligned} P_i A^i &= \sum_{i=j+k} B_j C_k A^i, \\ &= \sum_{i=j+k} B_j C_k A^{j+k}, \\ &= \sum_{i=j+k} B_j A^j C_k A^k. \end{aligned}$$

Putting this back together,

$$P(A) = B(A)C(A)$$

as long as A commutes with all coefficient matrices of $C(t)$.

Theorem 17.6 (Cayley–Hamilton)**.** *For any square matrix A over any commutative ring, A satisfies the characteristic polynomial of A, i.e. if $\chi_A(t) = \det(A - tI)$ is the characteristic polynomial of A, then $\chi_A(A) = 0$.*

Proof. Danger: there is an obvious apparent proof which is *wrong*: just plug $t = A$ into $p(t) = \det(A - tI)$ to get $\det(A - AI) = 0$. This is wrong. For example, apply the same argument to $q(t) = \det(A) - \det(tI)$. What went wrong: we defined "plugging in a matrix into a polynomial" to mean that we expand out the polynomial into a sum of powers of t, and then plug into each power on the right hand side of the matrix coefficient. Expanding out $q(t)$ into powers of t, this wrong proof doesn't make sense anymore:

$$q(t) = \det(A) - t^n$$

so

$$q(A) = \det(A)I - A^n$$

does not vanish for

$$A = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

in any commutative ring for any element $a \neq 0$ for any integer $n \geq 2$.

Suppose that A is $n \times n$. Take an abstract variable t and let $B(t) = \mathrm{adj}(A - tI)$ and $C(t) = A - tI$. So

$$P(t) = B(t)C(t)$$

where

$$P(t) = p(t)I,$$

with $p(t) = \det(A - tI)$ the characteristic polynomial. Clearly A commutes with the coefficient matrices of $C(t)$, so

$$p(A)I = P(A) = B(A)C(A) = B(A)(A - A) = 0.$$

□

17.14 Prove that, for any square matrices A, B with entries in any commutative ring, $\det(AB) = (\det A)(\det B)$.

17.15 Prove that a square matrix A, with entries in a commutative ring with identity, is invertible if and only if its determinant is a unit in that ring.

17.16 Give an example of a matrix A and a vector x, both with entries in the same commutative ring with identity, so that $x \neq 0$, $Ax = 0$ but $\det A \neq 0$.

The *minimal polynomial* of a square matrix A with entries in a field is the smallest degree single variable polynomial $m(t)$, with leading term 1, so that $m(A) = 0$.

Corollary 17.7. *Every square matrix over any field has a unique minimal polynomial, which divides its characteristic polynomial.*

Proof. If there are two polynomials $b(t)$ and $c(t)$ with $b(A) = 0$ and $c(A) = 0$, find their greatest common divisor using the extended Euclidean algorithm

$$g(t) = u(t)b(t) + v(t)c(t),$$

and then clearly

$$g(A) = u(A)b(A) + v(A)c(A) = u(A)0 + v(A)0 = 0.$$

So there is a unique lowest degree polynomial vanishing on A, which divides all others. Rescale it to have leading coefficient 1. □

Algebras

Almost every ring we encounter allows us to multiply by elements from some field.

Matrices over a field k can be multiplied by elements of k.

Polynomials over a field k can be multiplied by elements of k.

Rational functions over a field k can be multiplied by elements of k.

An *algebra* A over a field k is a ring with an operation $s \in k, a \in A \mapsto sa \in A$, called *scalar multiplication* so that

a. $s(a+b) = (sa) + (sb)$,

b. $(r+s)a = (ra) + (sa)$,

c. $(rs)a = r(sa)$,

d. $s(ab) = (sa)b = a(sb)$,

e. $1a = a$,

for $r, s \in k$, $a, b \in A$.

The field k is an algebra over itself.

If $k \subset K$ is a subfield, then K is an algebra over k.

If A is an algebra over k, then the $n \times n$ matrices with entries in A are an algebra over k.

If A is an algebra over k, then the polynomials $A[x]$ with coefficients in A are an algebra over k.

Chapter 18

Galois theory

We have made every effort to understand Galois's proof. His reasoning is not sufficiently clear, sufficiently developed, for us to judge its correctness and we can give no idea of it in this report.

— Siméon-Denis Poisson

Automorphisms

How can we save work in linear algebra? The 2×2 matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

has eigenvalues $\lambda = 2 \pm \sqrt{2}$, as the reader can check. When we compute the eigenvectors, we find that the $\lambda = 2 - \sqrt{2}$ eigenspace is spanned by an eigenvector

$$\begin{pmatrix} 1 \\ \sqrt{2} + 1 \end{pmatrix}.$$

We can guess that the other eigenvalue has corresponding eigenvector given by changing $\sqrt{2} \mapsto -\sqrt{2}$, so

$$\begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix}.$$

Why is this correct? Note that A has only rational coefficients. Write elements of $K = \mathbb{Q}(\sqrt{2})$ as $a + b\sqrt{2}$ with $a, b \in k = \mathbb{Q}$. Then clearly $\sqrt{2} \mapsto -\sqrt{2}$ preserves addition and multiplication, so preserves all of the arithmetic operations used to find eigenvalues and eigenvectors. It also preserves all entries of A, since they are rational: $a + 0\sqrt{2}$. The eigenvalues lie in K. The operations used to find eigenvectors, pure linear algebra, never take us out of K. So $\sqrt{2} \mapsto -\sqrt{2}$, applied to all entries in the first eigenvector with the first eigenvalue give us a second eigenvector with the second eigenvalue.

By the same argument, the eigenvalues and eigenvectors of any real matrix need not be real, but are interchanged by complex conjugation.

An *automorphism* of a ring R is a morphism $f\colon R \to R$ which has an inverse $f^{-1}\colon R \to R$, which is also a morphism.

18.1 Prove that the set of all automorphisms of any ring R is a group, the *automorphism group* of R.

The set of rational numbers of the form $b + c\sqrt{2}$, for $b, c \in \mathbb{Q}$, forms a field, usually denoted $\mathbb{Q}(\sqrt{2})$. Indeed, when we add such numbers

$$(7 + 3\sqrt{2}) \; + \; (4 - \sqrt{2}) = (7 + 4) + (3 - 1)\sqrt{2}.$$

Similarly if we subtract. If we multiply,

$$(7 + 3\sqrt{2})(4 - \sqrt{2}) = (7 \cdot 4 + 3 \cdot 1 \cdot 2) + (7 \cdot (-1) + 3 \cdot 4)\sqrt{2}.$$

Finally, to compute reciprocals,

$$\frac{1}{7 + 6\sqrt{2}} = \frac{1}{7 + 6\sqrt{2}} \frac{7 - 6\sqrt{2}}{7 - 6\sqrt{2}} = \frac{7 - 6\sqrt{2}}{7^2 - 2 \cdot 6^2}$$

which the reader can simplify. The map

$$f(b + c\sqrt{2}) = b - c\sqrt{2}$$

is an automorphism, i.e. a morphism of the field to itself. Note that $f^{-1} = f$.

18.2 Prove that every morphism $f\colon R \to S$ of rings preserves subtraction.

Suppose that R is a ring. Suppose that, for any elements b, c of R with $c \neq 0$, there is a unique element d of R so that $dc = b$. Write this element as $d = b/c$ and say that R has *right inverses*.

Lemma 18.1. *Every morphism $f\colon R \to S$ of rings with right inverses preserves right inverses.*

Proof. If $d = b/c$ then $dc = b$ so $f(dc) = f(b) = f(d)f(c)$ so $f(d) = f(b)/f(c)$. □

Lemma 18.2. *Every automorphism of a ring preserves* 0*. Every automorphism of a ring with identity preserves the identity element.*

Proof. Every element b of R satisfies $b = 0 + b = b + 0$. Any automorphism $h\colon R \to R$ satisfies $h(b) = h(0 + b) = h(0) + h(b) = h(b + 0) = h(b) + h(0)$ for all $b \in R$. Since h has an inverse, h is one-to-one and onto, so every element c of R has the form $c = h(b)$ for some $b \in R$. So $c = h(0) + c = c + h(0)$ for all c in R. In particular, taking $c = 0$, $0 = h(0) + 0 = h(0)$.

Every element b of R satisfies $b = 1b = b1$. Any automorphism $h\colon R \to R$ satisfies $h(b) = h(1b) = h(1)h(b) = h(b1) = h(b)h(1)$ for all $b \in R$. Since h has an inverse, h is one-to-one and onto, so every element c of R has the form $c = h(b)$ for some $b \in R$. So $c = h(1)c = ch(1)$ for all c in R. In particular, taking $c = 1$, $1 = h(1)1 = h(1)$. □

Lemma 18.3. *The only automorphism of any of the fields $\mathbb{Q}, \mathbb{R}$ or $\mathbb{Z}/p\mathbb{Z}$ is the identity map $b \mapsto b$.*

Proof. Suppose that $k = \mathbb{Q}$ or $k = \mathbb{R}$ or $k = \mathbb{Z}/p\mathbb{Z}$. If $f\colon k \to k$ is an automorphism, then $f(1) = 1$. Therefore $f(1+1) = f(1)+f(1) = 1+1$. Similarly $f(1+1+1) = 1+1+1$, and so on. Therefore if $k = \mathbb{Z}/p\mathbb{Z}$ we have exhausted all elements of k and $f(b) = b$ for every element b of k. In the same way, if $k = \mathbb{Q}$ or $k = \mathbb{R}$, then $k(n) = n$ for all positive integers n. Similarly, $f(0) = 0$. But then $f(-n) = f(0-n) = f(0) - f(n) = 0 - f(n) = -f(n)$. So f is the identity map on the integers. Apply f to a ratio b/c of integers, and since f preserves right inverses, $f(b/c) = f(b)/f(c) = b/c$. Hence if $k = \mathbb{Q}$, then f is the identity map.

We can assume that $k = \mathbb{R}$. Say that a number x in $\mathbb{R}$ is *positive* if $x \neq 0$ and $x = y^2$ for some number y. Clearly positivity is preserved under any automorphism. Write $x < y$ to mean that $y - x$ is positive, and $x \leq y$ to mean that $y - x$ is positive or zero. Automorphisms preserve the ordering of real numbers. Take any real number x and approximate x from below by rational numbers $r_1, r_2, \ldots, \to x$ and from above by rational numbers $s_1, s_2, \cdots \to x$. Then $r_i < x < s_i$ so applying f: $r_i < f(x) < s_i$. It follows that $f(x) = x$, as real numbers as completely determined by their digits, i.e. by approximation by rationals. □

Lemma 18.4. *There are two automorphisms of $\mathbb{Q}(\sqrt{2})$: the identity map $\iota(b+c\sqrt{2}) = b + c\sqrt{2}$ and the map*

$$f(b + c\sqrt{2}) = b - c\sqrt{2}.$$

Proof. The same argument as above shows that any automorphism h of $\mathbb{Q}(\sqrt{2})$ fixes all rational numbers $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$. So $h(b + c\sqrt{2}) = h(b) + h(c)h(\sqrt{2}) = b + ch(\sqrt{2})$: it suffices to determine $h(\sqrt{2})$. Note that $\sqrt{2}^2 = 2$, so apply h to get $h(\sqrt{2})^2 = 2$, i.e. $h(\sqrt{2}) = \pm\sqrt{2}$. □

Galois groups

The *Galois group* of a field extension $k \subset K$, denoted $\operatorname{Aut} K/k$ is the group of automorphisms of K which are the identity on k.

We found above that $\operatorname{Aut}\mathbb{Q}(\sqrt{2})/\mathbb{Q} = \{\pm 1\}$ where we let -1 denote the transformation $b + c\sqrt{2} = b - c\sqrt{2}$.

If $b(x)$ is a polynomial with coefficients in k, and g is a automorphism of K fixing all elements of k, then $g(b(x)) = b(x)$, i.e. g fixes all coefficients. Therefore if α is a root of $b(x)$ lying in K, then so is $g\alpha$: $0 = b(\alpha) = \sum b_j \alpha^j$ so

$$0 = g(b(\alpha)) = \sum b_j g(\alpha)^j.$$

Hence the Galois group $\operatorname{Aut} K/k$ of a field extension K/k acts on the roots in K of all polynomials over k. If a polynomial splits into factors over k, then clearly the Galois group $\operatorname{Aut} K/k$ permutes the K-roots of each factor individually.

The Galois group of $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ permutes $\sqrt{2}, -\sqrt{2}$, the roots of $x^2 - 2$. Note that the elements of $\mathbb{Q}(\sqrt{2})$ have the form $a + b\sqrt{2}$, for a, b rational. Hence the elements of $\mathbb{Q}(\sqrt{2}$ fixed by the Galois group are precisely the elements of $\mathbb{Q}$.

A *Galois extension* is an extension K/k so that the elements of K fixed by $\operatorname{Aut} K/k$ are precisely the elements of k.

In chapter 12 we checked that the field $K := \mathbb{Q}(\sqrt[3]{2})$ consists of the numbers $a + b2^{1/3} + c2^{2/3}$ for a, b, c rational. Note that K is a subfield of the field of real numbers. There is only one real number α that satisfies $\alpha^3 = 2$: the real number $\alpha = 2^{1/3}$. Therefore there is only one element α of K that satisfies $\alpha^3 = 2$: the element $\alpha = 2^{1/3}$. So every automorphism of K sends $2^{1/3}$ to itself. The Galois group of $K/\mathbb{Q}$ is $\operatorname{Aut} K/\mathbb{Q} = \{ 1 \}$. The extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is *not* a Galois extension; the Galois group is too small, fixing everything.

Let K be the splitting field of $p(x) = x^3 - 2$ over $k = \mathbb{Q}$. There are 3 complex cube roots of 2: the real number $2^{1/3}$, and its two rotations by a third of a revolution around the origin: call them α and $\bar{\alpha}$. As we will see, by theorem 21.8 on page 186, because $p(x)$ is irreducible over k, any two of these roots are swapped by some element of the Galois group of K/k. We can draw one of these permutations: complex conjugation preserves $2^{1/3}$, and preserves the polynomial, and swaps the angles of the other roots, preserving their lengths, so swaps the other two roots. Any permutation of the three roots arises as a unique element of the Galois group: once we swap roots to get any root we like to sit in the spot $2^{1/3}$, we can then permute the other two by complex conjugation if they are not already where we want them. So the Galois group of K/k is the symmetric group on 3 letters. In particular, K/k is a Galois extension.

18.3 Prove that, over any field, the rational functions in finitely many variables form a Galois extension of the symmetric functions in those variables, with Galois group the group of permutations of the variables.

Adding roots

Start with a field k, and take two elements $b, c \in k$. The roots of the equation $0 = x^2 + bx + c$ lie in some extension K. If $b^2 - c$ is a square in k, then $K = k$. Otherwise, the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

shows that the roots of our quadratic equation lie in the field $K = k(\sqrt{b^2 - 4c})$, as long as k has characteristic not equal to 2. In fields of characteristic 2, the quadratic formula doesn't hold, and the splitting field is more complicated.

Theorem 18.5. *Suppose that K is a splitting field of an irreducible polynomial $p(x)$ over a field k. If $\alpha, \beta \in K$ are roots of $p(x)$ then the Galois group of K over k has an element which takes α to β. In particular, K is a Galois extension of k.*

We will prove this result in chapter 21.

Return to our previous example: in chapter 12 we checked that the field $K := \mathbb{Q}(\sqrt[3]{2})$ consists of the numbers $a + b2^{1/3} + c2^{2/3}$ for a, b, c rational. Above, we saw that K has trivial Galois group over $k = \mathbb{Q}$. The polynomial $p(x) = x^3 - 2$ has a root in K, but not all of its roots. Our theorem says that the splitting field L of $p(x)$ over k is Galois. Let $\alpha = \sqrt[3]{2}$, so $K = k(\alpha)$. Factor $p(x) = (x - \alpha)(x^2 + \alpha x + \alpha^2)$. But $x^2 + \alpha x + \alpha^2$ has no roots in K, since its roots are not real (as we will see), while K lies inside $\mathbb{R}$. We need to extend to L to find roots for $x^2 + \alpha x + \alpha^2$, i.e. the two remaining roots of $p(x)$. Since $x^2 + \alpha x + \alpha^2$ is quadratic, L is a quadratic extension of K, i.e. $L = K(\beta)$ is given by adding

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\alpha^2}}{2} = -\frac{\alpha}{2}\left(1 \pm \sqrt{-3}\right),$$

i.e. $L = K(\sqrt{-3}) = k(\sqrt[3]{2}, \sqrt{-3})$. Automorphisms of L fixing k are determined by their action on $\sqrt[3]{2}$ and $\sqrt{-3}$. Complex conjugation is one such automorphism. Let ω be the cube root of 1 lying in the upper half plane. The three roots of $p(x)$ are $\sqrt[3]{2}, \omega\sqrt[3]{2}$ and $\bar{\omega}\sqrt[3]{2}$. Complex conjugation swaps the last two roots. Some other automorphism in the Galois group swaps the first two roots, by the theorem. That map then must leave the other root alone, as it must remain a root. So we can identify two automorphisms in the Galois group. Composing these two, we see that every permutation of the roots $\sqrt[3]{2}, \omega\sqrt[3]{2}$ and $\bar{\omega}\sqrt[3]{2}$ is obtained by a unique automorphism in the Galois group of L over k, i.e. $\operatorname{Aut} L/k$ is the symmetric group on three letters.

Pick a prime p. Over $k = \mathbb{Q}$, factor $x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1)$. The second factor is irreducible by problem 11.6 on page 80. The Galois group then takes any p-th root of 1 to any other (except for 1 itself, which is fixed). Once we take the root $\omega = e^{2\pi i/p}$ to some other root, ω^n, this determines the automorphism, as all other roots are powers of ω. The Galois group is thus given by automorphisms $\phi_n(\omega) = \omega^n$, which clearly commute, forming the cyclic group of order p.

18.4 Suppose that $p(x)$ is a polynomial with coefficients in a field k, and with roots in some extension K of k, say $\alpha_1, \alpha_2, \ldots, \alpha_n$. Prove that, for any symmetric polynomial q with coefficients in k, $q(\alpha_1, \ldots, \alpha_n)$ belongs to k.

Theorem 18.6. *Suppose that K is the splitting field of an irreducible monic polynomial $p(x)$ over a field k. Let d be the discriminant of $p(x)$, i.e. the resultant of $p(x), p'(x)$. Then $(-1)^{n(n-1)/2}d$ is a square in K, say α^2, so has a square root α in K. The field extension $k(\alpha) \subset K$ has Galois group $\operatorname{Aut} K/k(\alpha) \subset \operatorname{Aut} K/k$ consisting precisely of the elements of $\operatorname{Aut} K/k$ which act as even permutations of the roots.*

Proof. Problem 15.11 on page 123 shows that

$$d = (-1)^{n(n-1)/2} \prod_{i<j} (x_i - x_j)^2,$$

so we can take

$$\alpha = \prod_{i<j}(x_i - x_j),$$

or any multiple by any square of -1, or the same expression after any permutation of roots. This α changes sign when we permute roots, according to the sign of the permutation.

Every automorphism of K fixing all elements of $k(\alpha)$ fixes all elements of $k \subset k(\alpha)$, so $\operatorname{Aut} K/k(\alpha) \subset \operatorname{Aut} K/k$. Given an automorphism g in $\operatorname{Aut} K/k$, it permutes the roots of $p(x)$, so acts on α as the sign of that permutation, so fixes α just when it is even as a permutation. But it fixes k in any case, so fixes $k(\alpha)$ just when it fixes α. □

Return to the splitting field K of $x^3 - 2$ over $k = \mathbb{Q}$. We saw that $\operatorname{Aut} K/k$ is the symmetric group on three letters. We compute the discriminant $\Delta = \Delta_{x^3-2} = 108$. So $d = (-1)^{3(2)/2}\Delta = -108$, which has no square root in $k = \mathbb{Q}$. We adjoin a square root to -108, or equivalently to -3 since $-108 = 2^2 3^2(-3)$, so $\alpha = \sqrt{-3}$, and $\operatorname{Aut} K/k(\alpha)$ is the alternating group on three letters.

Radical extensions

Why is there a quadratic equation but not an equation like it to solve higher order polynomial equations in one variable? A *simple extension* of a field k is an extension K, denoted $k(\alpha)$, with an element α of K, so that every element of K is a rational function $b(\alpha)/c(\alpha)$ with coefficients in k. A *simple radical extension* of a field k is an extension $k(\alpha)$ for which α satisfies an equation $\alpha^n = c$ for some element c of k; we usually write such an extension as $k(\sqrt[n]{c})$. A *radical extension* is the result of repeated simple radical extensions $k(\alpha_1)(\alpha_2)\dots(\alpha_n)$.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2},$$

for a quadratic equation $0 = x^2 + bx + c$ (over any field k not of characteristic 2) puts the two roots into the simple radical extension field $K = k(\sqrt{b^2 - 4c})$. In terms of elementary symmetric polynomials, we can take variables t_1, t_2 to represent roots, and then our polynomial equation is $0 = x^2 - e_1 x + e_2$. Our quadratic formula tells us that

$$k(t_1, t_2) = k(e_1, e_2)(\sqrt{e_1^2 - 4e_2}).$$

Any cubic equation of the form

$$0 = t^3 + pt + q$$

(called a *depressed cubic*) has the solutions

$$C - \frac{p}{3C} \quad \text{with} \quad C = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

over any field k not of characteristic 2 or 3, for any choice of square and cube roots in a suitable radical extension.

We can "depress" a general cubic

$$0 = x^3 - e_1 x^2 + e_2 x - e_3$$

as

$$x = t + \frac{e_1}{3}$$

so that t satisfies the depressed cubic with

$$p = \frac{3e_2 - e_1^2}{3},$$
$$q = \frac{-2e_1^2 + 9e_1 e_2 - 27e_3}{27}.$$

This gives a complicated expression for the roots of the general cubic in terms of the coefficients, but one that lies in a radical extension

$$k(t_1, t_2, t_3) = k(e_1, e_2, e_3, \alpha, \beta_1, \ldots, \beta_6)$$

where α is a square root for

$$\frac{q^2}{4} + \frac{p^3}{27}$$

and the β_i are the 6 choices of values of C, for the possible choices of cube roots of

$$-\frac{q}{2} \pm \alpha.$$

In the same way, any formula to solve a polynomial equation, over a given field k, and of a given order n, by taking coefficients, rational functions thereof, successive square roots, cube roots, ..., and rational functions thereof, lies in some radical extension of $k(e_1, \ldots, e_n)$. Since it solves for the roots, that radical extension contains $k(t_1, \ldots, t_n)$. If it only solves for one root, we can make a radical extension in which we solve for all others by symmetry of the coefficients under permutations.

The extension $\mathbb{Q}(x, y)/\mathbb{Q}$ (adding two abstract variables) admits an automorphism $x \mapsto y$, $y \mapsto x$. But the further extension $\mathbb{Q}(\sqrt{x}, y)$ doesn't admit this automorphism, since it has no $\sqrt{y}$. This is easy to fix: just add a $\sqrt{y}$.

In the remainder of this section, we assume some familiarity with group theory [2, 4, 6, 12].

Lemma 18.7. *Suppose that K is an extension of a field k, and that* $\operatorname{Aut} K/k$ *is a finite group. For every radical extension E of K, there is a radical extension F of E so that every automorphism* $\operatorname{Aut} K/k$ *extends to an automorphism on F.*

Proof. If we have some element α of K and some radical $\sqrt[n]{\alpha}$, we just need to add some element $\sqrt[n]{g\alpha}$ for every g in $\operatorname{Aut} K/k$, and repeat as needed. □

Consider the equation $x^n = 1$ over some field k. Its roots in any splitting field are called the n^{th} roots of 1 or n^{th} roots of unity.

Lemma 18.8. *If ω is a p^{th} root of 1 over a field k, and $\omega \neq 1$, then the p^{th} roots of 1 are precisely*

$$1, \omega, \omega^2, \dots, \omega^{p-1}.$$

Proof. Take a splitting field K for $x^p - 1$. If α, β are two roots in K of this equation, then clearly $\alpha\beta$ is too. Clearly 0 is not a root, so the roots have reciprocals, which are also roots. So the roots form an abelian group A with precisely p elements, one of which is 1 in k. There might be others in k as well. If we take any element of that group, it generates a subgroup. By Lagrange's theorem, the order of a subgroup divides the order of the group, so this subgroup has order 1 or p, i.e. the subgroup is $\{1\}$ or A. □

If we allow repeated radical extensions, we might as well extend by a prime root each time, since $\sqrt[6]{\alpha}$ can be added by first adding $\beta = \sqrt[2]{\alpha}$ and then adding $\sqrt[3]{\beta}$.

A simple radical extension $k(\alpha)/k$ is *tame* if α is a p^{th} root of some element of k, p prime, and either

- α is a p^{th} root of 1 or
- every p^{th} root of 1 in $k(\alpha)$ already lies in k.

Lemma 18.9. *Every radical extension arises by a sequence of tame simple radical extensions.*

Proof. Replace every simple radical extension by a prime root each time as above. If introducing some α, not a p^{th} root of 1, also introduces some p^{th} root of 1, we can just first introduce that p^{th} root of 1 and then introduce α. □

Lemma 18.10. *The Galois group of any tame simple radical extension is a finite abelian group.*

Proof. Take a radical extension $K = k(\alpha)$ of a field k, say with α a root of $x^p = c$, each element the Galois group of K/k is determined by how it acts on α. But it has to preserve the equation $x^p = c$, so moves α around to some element β with $\beta^p = c$. So then β/α is a p^{th} root of 1. Suppose that every p^{th} root of 1 in $k(\alpha)$ already lies in k, so $\beta/\alpha \in k$. We associate to each element g of the Galois group our root of 1: let

$$\varphi\colon g \in \operatorname{Aut} K/k \mapsto \frac{g\alpha}{\alpha} \in \{\gamma \in k^\times \mid \gamma^p = 1\} \subseteq k^\times.$$

We would like to know whether this map is a group morphism. If it is, it has image in the p^{th} roots of 1, an abelian group, and kernel trivial, so our Galois group is abelian, and is either

- just the identity transformation or
- the group of all p^{th} roots of 1, a group with $p-1$ elements.

Note that

$$\frac{\varphi(gh)}{\varphi(g)\varphi(h)} = \frac{\alpha}{h\alpha} g \frac{h\alpha}{\alpha}.$$

But

$$\frac{h\alpha}{\alpha} \in k$$

is invariant under any g in $\operatorname{Aut} K/k$. So φ is a group morphism.

Next suppose instead that α is p^{th} root of 1. We can assume that α is a primitive p^{th} root of 1. Every element g of $\operatorname{Aut} K/k$ takes α to some other p^{th} root of 1, and these are of course just powers α^i. So another, say h with $h\alpha = \alpha^j$, has $gh\alpha = hg\alpha = \alpha^{ij}$. Since these automorphisms are determined by how they act on α, they commute. □

Corollary 18.11. *The Galois group of any radical extension is a finite solvable group.*

Proof. Any radical extension consists of repeated tame simple radical extensions, with abelian Galois groups, hence repeated extensions by abelian groups, so solvable. The problem is to relate the Galois groups. Imagine two steps of our extension: we extend $k \subset k(\alpha) \subset k(\alpha, \beta) \subset \ldots$. We claim that $k(\alpha)$ is invariant under $\operatorname{Aut} k(\alpha, \beta)/k$.

If α is a p^{th} root of 1, then every $g \in \operatorname{Aut} k(\alpha, \beta)/k$ takes α to some other p^{th} root of 1, but these are all just powers α^i, so already lie in $k(\alpha)$. Hence $k(\alpha)$ is invariant under $\operatorname{Aut} k(\alpha, \beta)/k$.

If α is not a p^{th} root of 1, then every p^{th} root of 1 in $k(\alpha)$ already lies in k. Take any $g \in \operatorname{Aut} k(\alpha, \beta)/k$. Then $g\alpha/\alpha$ is a p^{th} root of 1. But we have no reason why is belongs to $k(\alpha)$.

We can assume that when we carry out our repeated tame extensions, we add all of the roots of unity first, and then all of the other roots later. So we can suppose that $g\alpha/\alpha$ is in k, our earlier field. Therefore $g\alpha = (g\alpha/\alpha)\alpha$ is in $k(\alpha)$. Hence $k(\alpha)$ is invariant under $\operatorname{Aut} k(\alpha, \beta)/k$.

So $\operatorname{Aut} k(\alpha, \beta)/k$ acts on $k, k(\alpha), k(\alpha, \beta)$, each contained in the next, acting on $k(\alpha)$ by a morphism of groups

$$\operatorname{Aut} k(\alpha, \beta)/k \to \operatorname{Aut} k(\alpha)/k$$

with kernel precisely $\operatorname{Aut} k(\alpha, \beta)/k(\alpha)$, an exact sequence of groups

$$1 \to \operatorname{Aut} k(\alpha, \beta)/k(\alpha) \to \operatorname{Aut} k(\alpha, \beta)/k \to \operatorname{Aut} k(\alpha)/k \to 1.$$

The sequence starts and ends abelian, so the middle is solvable. Repeating this argument, at each step we extend a solvable by another abelian. □

We want to prove that there is no analogue of the quadratic equation for high degree polynomials.

Theorem 18.12. *The general equation of degree* 5 *or more is not solvable in radicals. In other words, the radical extensions of $k(e_1, \ldots, e_n)$, for any field k, do not contain any extension field isomorphic to $k(t_1, \ldots, t_n)$, where $e_1, \ldots, e_n$ are the elementary symmetric polynomials in variables $t_1, \ldots, t_n$.*

Proof. Write $e = (e_1, \ldots, e_n)$ and $t = (t_1, \ldots, t_n)$. Radical extensions have solvable Galois group. We can make the extensions so that they are Galois extensions. Suppose we have a radical extension $K/k(e)$ which contains $k(t)$. By definition K is also a radical extension of $k(t)$. By lemma 18.7 on page 158, we can extend further if needed so that the Galois group of $k(t)/k(e)$ extends to lie in the Galois group of $K/k(e)$.

The extension $k(t)/k(e)$ is Galois with Galois group the symmetric group (the group of permutations of variables $t_1, \ldots, t_n$). We leave the reader to pursue sufficient group theory to prove that the symmetric group of permutations of 5 or more variables is not a solvable group (because, for example, it is simple and not abelian). So the solvable Galois group of $K/k(e)$ contains the nonsolvable symmetric group. The reader can, once again, master sufficient group theory to prove that every subgroup of any solvable group is solvable. □

Chapter 19

Plane algebraic curves

Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.

— Felix Klein

Example

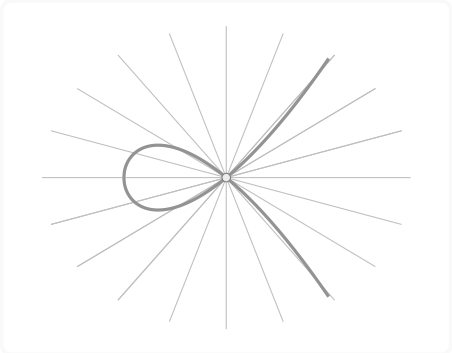
Let's find the solutions of the algebraic equation $y^2 = x^2 + x^3$. The solutions form a curve in the plane, intuitively because the equation is one constraint on two variables. There is one obvious solution: $(x, y) = (0, 0)$. A minor miracle: almost every line through $(0, 0)$ passes through another point of the curve, which we can compute. To see this, write any line through $(0, 0)$, say with slope t, as $y = tx$. Plug this in to the equation $y^2 = x^2 + x^3$ to see if we can find any more solutions:

$$(tx)^2 = x^2 + x^3,$$

which we simplify to get either $x = 0$ or, dividing by x^2,

$$t^2 = 1 + x.$$

Solving for x, we find $x = t^2 - 1$. Plug in to the equation of the line $y = tx$ to get $y = t\left(t^2 - 1\right)$. Every solution, except $(x, y) = (0, 0)$, has to lie on some line through $(0, 0)$, and that line will be either $y = tx$ or a vertical line $x = 0$. But on the line $x = 0$, the only solution is $y = 0$, the origin. Moreover, if we let $t = \pm 1$, we get the solution $(x, y) = (0, 0)$ too. So we get all of the solutions.



Each line strikes the curve at two points (one of which is the origin); except for the vertical line.

Definition

A *plane algebraic curve* is the set of zeroes of a nonconstant polynomial in two variables; the *degree* of the curve is the degree of the polynomial.

degree	appellation
1	line
2	conic
3	cubic
4	quartic
5	quintic
6	sextic
7	septic
8	octic

19.1 Use the same trick for the circle $x^2 + y^2 = 1$ as follows. Show first (using the formula for the slope of a line) that for any point (x, y) of the circle, the line through $(0, 1)$ and (x, y) passes through the horizontal axis at the point $(t, 0)$ where

$$t = \frac{x}{1 - y}.$$

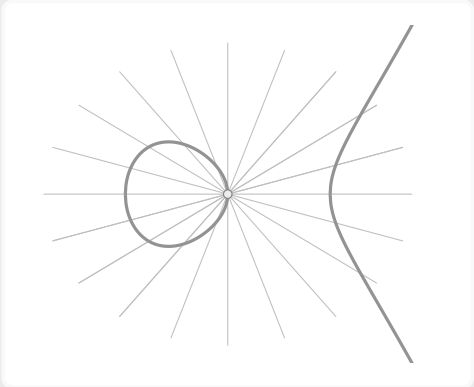
It helps to draw a picture. Solve for x: $x = t(1 - y)$, and plug into the equation of the circle to find that

$$(x, y) = \frac{1}{t^2 + 1}\left(2t, t^2 - 1\right).$$

Explain now why every rational point of the circle corresponds to a rational value of t and vice versa. Explain why the same conclusion holds over any field k.

Let's try a different cubic curve. Take the cubic curve

$$y^2 = (x + 1)x(x - 1).$$



We see right away that some lines strike the curve in more than two points, one at the origin, and two others. Calculate as above that those two others are

$$x = \frac{t^2 \pm \sqrt{t^4 + 4}}{2}, \ y = tx.$$

Fermat's Last Theorem says that there are no integer solutions to $a^n + b^n = c^n$ with nonzero integers a, b, c and $n \geq 3$. Dividing both sides by c, if there were a solution, we would solve

$$x^n + y^n = 1$$

with nonzero rational numbers x, y. Conversely, any nonzero rational numbers x, y solving this equation, after clearing denominators, give a solution to $a^n + b^n = c^n$. So Fermat's Last Theorem is equivalent to the statement that the plane algebraic curve $x^n + y^n = 1$ has no rational number solutions except $(x, y) = (0, 1), (0, -1), (1, 0)$ and $(-1, 0)$. In particular, we are interested in plane algebraic curves over the rational numbers, not just the real numbers.

Looking for integer solutions of integer polynomial equations, can then look at those equations modulo a prime, and get equations over a finite field. So it is natural to consider algebraic curves over any field.

Consider the polynomial equation $x^2 + y^2 = -7$. There are no real values of x, y that satisfy this, so the algebraic curve is empty.

To remedy this problem, we allow the variables x and y to take on complex values. Sticking in a value for one of the variables, or perhaps for the other, with a suitable choice of value, we will find that we have a nonconstant polynomial in one variable, so there is a solution (x, y) by the fundamental theorem of algebra. Hence algebraic curves have complex points, but perhaps no real points.

Take a field k and an algebraic closure $k \subset \bar{k}$. A *plane algebraic curve* X over k given by a nonconstant polynomial equation $f(x, y) = 0$ with coefficients in k is the set X of its $\bar{k}$-points, i.e. points (x, y) with x, y in the algebraic closure $\bar{k}$ satisfying $f(x, y) = 0$. There are infinitely many $\bar{k}$-points, because we can force either x or y to any constant value, and for infinitely many such constants $f(c, y)$ or $f(x, c)$ is not constant so has a root in $\bar{k}$.

The equation of an algebraic curve is not uniquely determined. The curve $x^2 - y = 0$ is the same curve as $\left(x^2 - y\right)^9 = 0$. The curve $xy = 0$ splits into $x = 0$ or $y = 0$, two curves. If the equation of a curve has nonconstant factors, we say that the curve is *reducible* and the equations formed from the factors, or the curves they cut out, are *components* of the curve. We can assume that the polynomial $f(x, y)$ in the equation $f(x, y) = 0$ of a curve is not a square, cube, etc. of any polynomial, nor is any constant multiple of $f(x, y)$.

Lemma 19.1. *An irreducible plane algebraic curve determines its equation uniquely up to scaling by a nonzero constant.*

Proof. Work over a field k with algebraic closure $\bar{k}$. Take two irreducible equations $0 = b(x, y)$ and $0 = c(x, y)$ with the same curve, i.e. the same points over $\bar{k}$. For any constant $x = a$ over $\bar{k}$, the equations $0 = b(a, y)$ and $0 = c(a, y)$ have the same solutions. Unless the x variable does not appear in either equation, these are infinitely many different points in common on the curve. But if x does not appear, swap x with y and repeat. By proposition 16.12 on page 139, $b(x, y)$ and $c(x, y)$ have a common factor. □

A *regular function* on a plane algebraic curve is the restriction of a polynomial function.

On the plane algebraic curve $(y^2 = x^2 + x^3)$, the functions y^2 and $x^2 + x^3$ take on the same values at every point, and therefore are the same regular function.

If C is a plane algebraic curve over field k, let $k[C]$ be the set of regular functions on C with coefficients in k; clearly $k[C]$ is an algebra over k.

A *regular morphism* of plane algebraic curves $f\colon C \to D$ is a map which can be expressed somehow as $f(x, y) = (s(x, y), t(x, y))$ for two polynomials $s(x, y), t(x, y)$, so that f, applied to any point of C, yields a point of D.

The map $f(x, y) = (s, t) = \left(x^2 - 2, x(x^2 - 1)\right)$ maps $C = (y = 0)$ to $D = (t^2 = s^2 + s^3)$, as we saw previously.

A regular morphism with an inverse regular morphism is *biregular*, and the associated curves are *biregular*. Clearly biregular plane algebraic curves have isomorphic algebras of regular functions.

The map $f(x, y) = (s, t) = \left(x, x^2\right)$ maps the line $C = (x = y)$ to the parabola $D = (t = s^2)$, with inverse $g(s, t) = (x, y) = (s, s)$. These maps are not inverses of one another in the plane, but they invert one another along the curves C and D in the plane.

19.2 Prove that every isomorphism of algebras of regular functions of irreducible plane algebraic curves arises uniquely from a biregular morphism.

Plane conics

We will see that, over any field of characteristic not 2, any conic is biregular to at most one of

Equation	Geometry	Regular function algebra
$y = 0$	line	$k[x]$
$xy = 0$	pair of intersecting lines	$k[x] \oplus k[y]$
$xy = 1$	hyperbola	$k\left[x, x^{-1}\right]$
$y = x^2$	parabola	$k[x]$
$y = \pm 1$	pair of disjoint lines	$k[x] \oplus k[y]$
$y = \pm\alpha$	pair of disjoint lines over $k(\alpha), \alpha^2 \in k, \alpha \notin k$	$k(\alpha)[x]$

Take a conic $0 = f(x, y)$ and expand out

$$f(x, y) = ax^2 + bxy + cy^2 + rx + sy + t.$$

If $0 = a = b = c$ then this is a line, not a conic, so we can assume that at least one of a, b, c is not zero. First, suppose that $0 = a = c$, so

$$f(x, y) = bxy + rx + sy + t.$$

Rescale to arrange that $b = 1$. Factor as

$$f(x, y) = (x + s)(y + r) + t - rs.$$

Change variables by translation, replacing $x + s$ by x and $y + r$ by y and let $u := rs - t$ to get

$$f(x, y) = xy - u.$$

If $u = 0$, then our conic is a pair of lines intersecting at a single point. If $u \neq 0$, rescale x by $1/u$ to arrange that our conic is $xy = 1$.

Suppose that one or more of a or b is not zero, and swap variables if needed to arrange that $a \neq 0$ and rescale the equation $0 = f(x, y)$ to get $a = 1$.

$$f(x, y) = x^2 + bxy + cy^2 + rx + sy + t.$$

Suppose we work over a field not of characteristic 2, i.e. where $2 \neq 0$, so we can divide by 2:

$$f(x, y) = \left(x + \frac{by}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) y^2 + rx + sy + t.$$

Replace $x + by/2$ by a new variable called x and replace $c - b^2/4$ by a new constant called c:

$$f(x, y) = x^2 + cy^2 + rx + sy + t.$$

Complete the square in x

$$f(x, y) = \left(x + \frac{r}{2}\right)^2 + cy^2 + rx + sy + t - \frac{r^2}{4}.$$

Rename $x + r/2$ to x and $t - r^2/4$ to t:

$$f(x, y) = x^2 + cy^2 + sy + t.$$

If $c = 0$ then

$$f(x, y) = x^2 + sy + t.$$

If $s \neq 0$ then replace $sy + t$ by a new variable, which we call $-y$, to get

$$f(x, y) = x^2 - y.$$

Note that then our curve is a parabola $y = x^2$. On the parabola, every polynomial $g(x, y)$ is just $g\big(x, x^2\big)$, a polynomial in x. If $0 = s = t$ then our conic is actually a line $x = 0$; linearly transform variable to get the conic to be $y = 0$. If $0 = s$ and $t \neq 0$ then our conic is $x^2 = -t$. Swap x with y to get the conic $y^2 = -t$. If $-t$ has a square root in k, say a, then our equation is $y = \pm a$, and rescale the y variable to get $a = 1$, so a pair of disjoint lines $y = \pm 1$. If $-t$ has no square root in k, then an empty set of points with coordinates in k, but a pair of disjoint lines with coordinates in $k(\alpha)$, $\alpha = \sqrt{-t}$: $y = \pm\alpha$. This α is uniquely determined up to scaling by an element of $k^\times$. For example, over the rational numbers, the equations $x^2 = -t$ for t any prime or $x^2 = t$ for t any prime give conics with no rational points. Different primes give different conics, and the reader can work out why they are not biregular over the rational numbers. One the other hand, over the real numbers, we only have a single conic $x^2 = -1$ with no real points.

We still have the case $a \neq 0$ and $c \neq 0$, which is more complicated, so there are still some conics whose regular function algebra is not classified on our list. For example, the circle is not on our list. We can easily see that, after completing the square in y, and some finite field extension to add some square roots as needed, we can get to a hyperbola, but that doesn't quite give the description of the algebra over the original field.

Rational functions

A *rational function* on a irreducible plane algebraic curve $C = (c(x, y) = 0)$ over a field k is a rational function of x, y (which we think of as actually a function on the $\bar{k}$-points of the curve), whose denominator does not vanish everywhere on the curve.

On the curve $C = (y = x^2)$, the rational function $f = x/(x - y)$ might seem to be undefined near the origin, but we can also write it as

$$\begin{aligned} f &= \frac{x}{x - y}, \\ &= \frac{x}{x - x^2}, \\ &= \frac{1}{1 - x}. \end{aligned}$$

The circle $C = (x^2 + y^2 = 0)$ is irreducible over $k = \mathbb{R}$, so has well defined field of rational functions $\mathbb{R}(C)$. Note that C is reducible over $\mathbb{C}$: $x^2 + y^2 = (x + iy)(x - iy)$, so we can't make a field $\mathbb{C}(C)$, of complex coefficient rational

functions, since $x+iy, x-iy$ are two rational functions which multiply to zero as complex coefficient rational functions on C. Elements of $\mathbb{R}(C)$ are ratios

$$\frac{b(x)+c(x)y}{d(x)+e(x)y},$$

for any $b(x), c(x), d(x), e(x)$ real coefficient polynomials, but with $y^2 = -x^2$, i.e. we can formally write y as ix, allowing complex coefficients into any terms of positive degree in x.

19.3 Find the field of rational functions on every conic from our list above, over any field not of characteristic 2.

19.4 Prove that the rational functions on an irreducible curve are the field of fractions of the ring of regular functions.

A rational function f is *regular* near a point of X if f can be written as $f = b/c$ for some polynomials b, c with c nonzero at that point.

A *rational morphism* of plane algebraic curves $f\colon C \to D$ is a map, which can be expressed somehow as $f(x,y) = (s(x,y), t(x,y))$ for two rational functions $s(x,y), t(x,y)$, each of which is regular at some point of C, so that f, applied to any point of C at which both $s(x,y)$ and $t(x,y)$ are regular, yields a point of D. A rational morphism with a rational inverse is *birational*, and the associated curves are *birational*. A rational morphism is *regular* near a point of C if $s(x,y)$ and $t(x,y)$ are regular near that point, i.e. can be expressed near that point as ratios of polynomials not vanishing near that point.

19.5 Prove that any plane algebraic curve is irreducible just when its algebra of regular functions is an integral domain, so that its field of fractions is defined.

Lemma 19.2. *Irreducible plane algebraic curves, over any field k, are birational just when their fields of rational functions are isomorphic extensions of k.*

Proof. Suppose that the curves are birational. Clearly the rational functions compose with the birational map, and with its inverse, identifying the fields. Conversely, suppose that the fields are isomorphic, say by an isomorphism

$$\phi\colon k(C) \to k(D).$$

Write the coordinates of the plane in which C lives as x, y, and those of the plane in which D lives as u, v. Let $s(x,y) := \phi^{-1}u$ and $t(x,y) := \phi^{-1}v$. □

A curve birational to a line in the plane (with equation $y = 0$, for example) is a *rational curve*. A plane algebraic curve C is rational just when $k(C) \cong k(x)$.

19.6 Imitate our picture of drawing lines through a point of $y^2 = x^3 + x^2$, but replacing that cubic curve by an irreducible conic. Prove that conics are rational.

19.7 Prove that every plane algebraic curve has infinitely many points with coordinates in any algebraic closure of the underlying field. Hints: Pick a point not on the curve. (You need to prove that such a point exists, using the fact that every algebraically closed field is infinite.) Change variables by translation to arrange that

this point is the origin. Any line through the origin strikes the curve at some point, except perhaps for finitely many exceptional lines. (Why?) Two lines through the origin intersect only at the origin. So any two lines through the origin strike the curve at different points. There are infinitely many lines through the origin. (Why?)

Integrals

Suppose that $g(x, y) = 0$ is the equation of an irreducible plane algebraic curve over the field of real numbers, for example $y^2 = x^2 + x^3$. Suppose that we can locally write the curve as the graph of a function $y = y(x)$, for example $y = \sqrt{x^2 + x^3}$. Pick a rational function $f(x, y)$ on the curve, and consider the integral $\int f\,dx$, by which we mean

$$\int f(x, y)\,dx = \int f(x, y(x))\,dx.$$

If the curve is rational, then we can parameterize it by $x = x(t), y = y(t)$, as rational functions of a new variable t. So then the integral becomes

$$\int f\,dx = \int f(x(t), y(t))\,\frac{dx}{dt}\,dt$$

and this is a composition of rational functions, so a rational function of t. Using partial fractions, we can integrate this explicitly. For example, on the curve $y^2 = x^2 + x^3$, if we take the rational function

$$f := \frac{x^3}{y},$$

we can write our integral as

$$\int f\,dx = \int \frac{x^3\,dx}{\sqrt{x^2 + x^3}}.$$

We then solve this integral by parameterising as on page 161,

$$x = x(t) = t^2 - 1, y = y(t) = t\left(t^2 - 1\right)$$

to get

$$\begin{aligned}\int \frac{x^3\,dx}{\sqrt{x^2 + x^3}} &= \int \frac{x^3\,dx}{y},\\ &= \int \frac{\left(t^2 - 1\right)^3 2t\,dt}{t\left(t^2 - 1\right)},\\ &= 2\int \left(t^2 - 1\right)^2\,dt.\end{aligned}$$

19.8 Simplify this integral completely into a function of x.

19.9 Recall that the circle is rational, as it can be parameterized by

$$x(t) = \frac{1 - t^2}{1 + t^2}, y(t) = \frac{2t}{1 + t^2}.$$

But the circle can also be parameterized using trigonometry:

$$x(\theta) = \cos\theta, y(t) = \sin\theta.$$

Use these two facts to explain how to solve all integrals of the form

$$\int f(\cos\theta, \sin\theta)\, d\theta,$$

where f is any rational function.

Ideals

Given a plane algebraic curve C and a point $p_0 \in C$ (say, to be precise, that $p_0 \in C(k)$ is a point defined over k), we are naturally interested in the regular functions on C vanishing at p_0. If p_0 has coordinates $p_0 = (x_0, y_0)$, then the polynomial functions $x - x_0, y - y_0$ vanish simultaneously only at p_0. For simplicity, translate the plane so that p_0 lies at the origin. A regular function on C vanishes at p_0 just when it is a sum $xg(x,y) + yh(x,y)$, since every polynomial function vanishing at the origin is expressible in a finite Taylor series expansion. In other words, the regular functions on C vanishing at the origin are precisely the ideal $I = (x, y)$.

Similarly, the regular functions on C vanishing at $(x, y) = p_0 = (x_0, y_0)$ are the polynomials of the form $(x - x_0)\, g(x,y) + (y - y_0)\, h(x,y)$, so constitute the ideal $I = (x - x_0, y - y_0)$. This I is the kernel of the morphism $p(x,y) \mapsto p(x_0, y_0)$, mapping $k[C] \to k$.

This is the motivation for the name "ideal": an ideal is like a idealized notion of a point.

The regular functions on the real number line are the polynomials $p(x)$, and those which vanish at a point $x = x_0$ are those of the form $(x - x_0)\, p(x)$. Similarly, if we take two points $x_0 \neq x_1$, then the regular functions on the line vanishing at both points are the polynomials of the form $(x - x_0)\,(x - x_1)\, p(x)$, the polynomials divisible $(x - x_0)\,(x - x_1)$.

If we imagine running the two points into one another, so that they collide at $x = 0$, then these polynomials approach polynomials divisible by x^2. We could think of the equation $x^2 = 0$, or of the ideal $I = \left(x^2\right)$, as representing a "double point".

Working over $k := \mathbb{Q}$ on the curve $C := (x^2 = y)$, the ideal $I := (x - 2)$ does not correspond to any point of C defined over k. Again, we can think of I as an "ideal point".

19.10 Prove that a polynomial $p(x, y)$ vanishes as a regular function on some irreducible plane algebraic curve $C = (c(x, y) = 0)$ just when $c(x, y)$ divides $p(x, y)$.

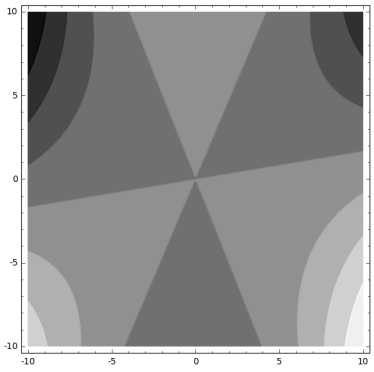
The surface $S = (x^2 + y^2 = z^2)$ over $\mathbb{R}$ is a cone. The subset of S cut out by the equation $z = 1$ is a circle on that cone, a curve on the surface. The subset of S cut out by the equation $z^2 + 1 = 0$ is empty, but has complex points. Even over the real numbers, we have an ideal $I = (z^2 + 1) \subset \mathbb{R}[S]$, which we can think of as a kind of "ideal curve". So intuitively, ideals represent "ideal geometric objects", which perhaps we can't really draw.

Sage

Sage can usually plot the real points of a plane algebraic plane curve. For any equation like $y^3 + x^3 - 6x^2y = 0$ the code

```
x,y=var('x,y')
contour_plot(y^3+x^3-6*x^2*y==0, (x,-10,10), (y,-10,10))
```

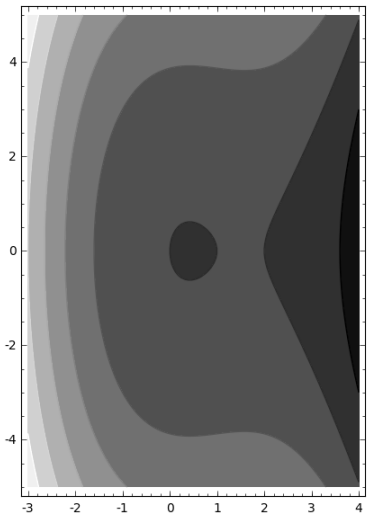
yields a picture of the level sets of the function $y^3 + x^3 - 6x^2y$:



from which we can see that our curve is a union of three lines. Similarly

```
x,y=var('x,y')
contour_plot(y^2-x*(x-1)*(x-2)==0, (x,-3,4), (y,-5,5))
```

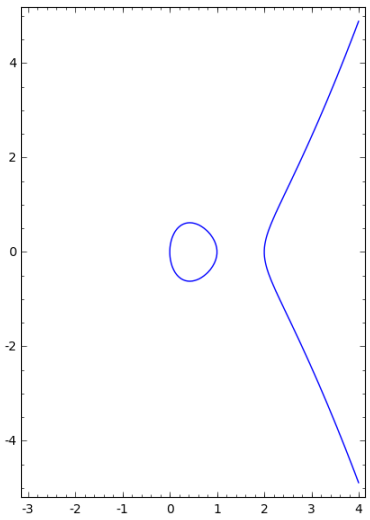
yields level sets like



and we can also plot the plane algebraic curve, without the other level sets, using

```
f(x,y) = y^2-x*(x-1)*(x-2)
implicit_plot(f, (-3, 4), (-5, 5))
```

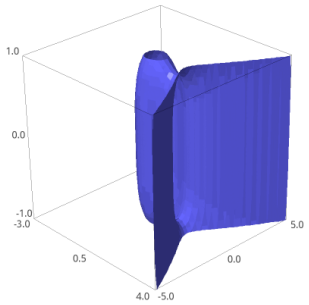
yielding



Similarly, we can draw algebraic surfaces in space:

```
f(x,y,z) = y^2-x*(x-1)*(x-2)+z^4-1
implicit_plot3d(f, (-3, 4), (-5, 5), (-1,1))
```

yielding


1.0
0.0
-1.0
-3.0
0.5
4.0
-5.0
0.0
5.0

Chapter 20

Where plane curves intersect

Among their codified and systematic sciences is geometry, which is totally useless. The sum of the three angles in a triangles is two right angles—what benefit does it have? These theorems that are close to their hearts—what purpose do they serve?

— Ahmad al-Fārūqī al-Sirhindī
Maktubat

Vanishing polynomials

Over the finite field with two elements, $t(t+1)$ vanishes for any value of t in that field.

Over any finite field, the polynomial

$$q(t) = \prod_c (t - c)$$

(where the product is over all constants c in the field) vanishes for any value of the variable t in that field.

Lemma 20.1. *Take a nonzero polynomial in several variables, over a field k, that vanishes for all values of those variables. Then the field k is finite and the polynomial is expressible as*

$$p(x) = \sum_{i=1}^{n} p_i(x) q(x_i)$$

where $x = (x_1, x_2, \ldots, x_n)$ and each $p_i(x)$ is a polynomial and

$$q(t) = \prod_c (t - c)$$

is our polynomial that vanishes for all values of t in k. In particular, $p(x)$ has degree at least equal to the number of elements in the field in at least one of the variables.

Proof. In one variable, we can factor each root as in corollary 9.2 on page 59. Suppose we have two variables x, y and a polynomial $p(x, y)$. Set y to zero, and find that

by induction, the resulting polynomial $p(x,0)$ is divisible as required by $q(x)$, say $p(x,0) = q(x)p_1(x)$. So $p(x,y) = q(x)p_1(x) + yh(x,y)$, say. It is good enough to prove the result for $yh(x,y)$ and add to $q(x)p_1(x)$. So we can assume that $p(x,y) = yh(x,y)$. Moreover, since $p(x,y)$ vanishes for all x,y values in our field, $h(x,y)$ vanishes for all x,y values in our field as long as $y \neq 0$.

Define a polynomial $\delta(t)$ by

$$\delta(t) = \prod_{c \neq 0} \left(1 - \frac{t}{c}\right),$$

where the product is over all nonzero constant elements c in our field. Check that $\delta(0) = 1$ while $\delta(b) = 0$ for any nonzero element b of our field. Therefore

$$h(x,y) - \delta(y)h(x,0)$$

vanishes for all values of x,y in our field. By induction on degree, we can write

$$h(x,y) - \delta(y)h(x,0) = h_1(x,y)q(x) + h_2(x,y)q(y).$$

Plugging back into $p(x,y)$ gives the result. □

Linear factors

A *linear function* is a polynomial of the form

$$f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

If not all of the coefficients $a_1, a_2, \ldots, a_n$ are zero, the set of points $x = (x_1, x_2, \ldots, x_n)$ at which $f(x) = 0$ is called a *linear hyperplane*.

Lemma 20.2. *Suppose that in variables $x = (x_1, x_2, \ldots, x_n)$*

a. *$p(x)$ is a polynomial and*

b. *$f(x)$ is a nonzero linear function and*

c. *$p(x) = 0$ at every point x where $f(x) = 0$ and*

d. *the field we are working over contains more elements than the degree of $p(x)$ in any variable.*

Then $f(x)$ divides $p(x)$, i.e. there is a polynomial $q(x)$ so that $p(x) = q(x)f(x)$.

Proof. By a linear change of variables, we can arrange that $f(x) = x_1$. Expand $p(x)$ in powers of x_1. If there is a "constant term", i.e. a monomial in $x_2, x_3, \ldots, x_n$, then setting $x_1 = 0$ we will still have some nonzero polynomial in the variables $x_2, x_3, \ldots, x_n$. But this polynomial vanishes for all values of these variables, so is zero by lemma 20.1 on the preceding page. □

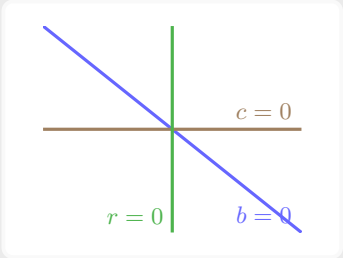
Resultants in many variables

Lemma 20.3. *Take two polynomials $b(x), c(x)$ in a variable x, with coefficients in a commutative ring S, of degrees m, n. Then the resultant $r = \mathrm{res}_{b,c}$ is expressible as $r = u(x)b(x) + v(x)c(x)$ where $u(x)$ and $v(x)$ are polynomials, with coefficients in the same commutative ring S, of degrees at most $n-1, m-1$.*

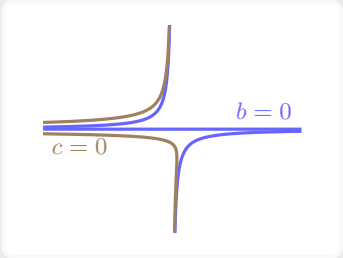
The proof is identical to the proof of lemma 15.3 on page 117.

Take two polynomials $b(x, y)$ and $c(x, y)$ and let $r(x)$ be the resultant of $b(x, y), c(x, y)$ in the y variable, so thinking of b and c as polynomials in y, with coefficients being polynomials in x. So $r(x)$ is a polynomial in x.

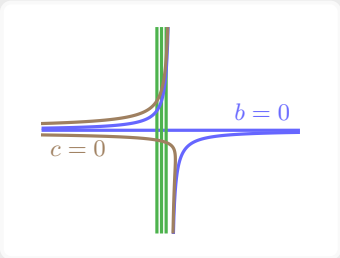
If $b(x, y) = y + x$, $c(x, y) = y$, then $r(x) = x$ vanishes just at the value $x = 0$ where there is a common factor: y.



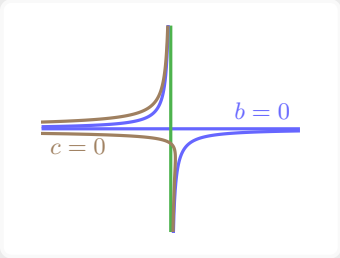
Let $b(x, y) := xy^2 + y$, $c(x, y) := xy^2 + y + 1$.



Look at the picture at $x = 0$ and near $x = 0$: because the polynomials drop degrees, the number of roots of $b(0, y)$ on the vertical line $x = 0$ is smaller than the number of roots of $b(a, y)$ on the vertical line $x = a$ for constants $x = a$ near 0. We can see roots "flying away" to infinity on those lines.

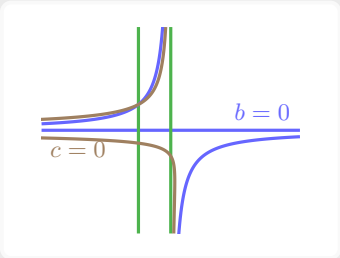


Finding the determinant of the associated 4×4 matrix tells us that $r(x) = x^2$, which vanishes just at the value $x = 0$ where $b(0, y) = y$ and $c(0, y) = y + 1$ drop in degrees, *not* due to a common factor.



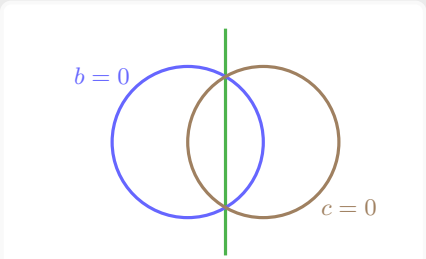
The resultant of y and $y + 1$ is *not* $r(0)$ since the degrees drop, so we would compute resultant of y and $y + 1$ using a 2×2 matrix, and find resultant -1.

If $b(x, y) = xy^2 + y$ and $c(x, y) = 2xy^2 + y + 1$ then $r(x) = x(x + 1)$ vanishes at $x = 0$ and $x = -1$. At $x = 0$, $b(0, y) = y, c(0, y) = y + 1$ have no common factor, but drop degrees. At $x = -1$, $b(-1, y) = -y(y - 1)$, $c(-1, y) = -2(y - 1)(y - 1/2)$ have a common factor, but they don't drop degrees.



If $b(x, y) = x^2 + y^2 - 1$ and $c(x, y) = (x - 1)^2 + y^2 - 1$ then $r(x) = (2x - 1)^2$ vanishes at $x = 1/2$, where there are *two* different intersection points, a double common factor:

$$b(1/2, y) = c(1/2, y) = \left(y - \frac{\sqrt{3}}{2}\right)\left(y + \frac{\sqrt{3}}{2}\right).$$



Lemma 20.4. *Suppose that k is a field and*

$$x = (x_1, x_2, \ldots, x_n)$$

are variables and y is a variable. Take $b(x, y)$ and $c(x, y)$ two nonconstant polynomials in $k[x, y]$. Then in some finite degree extension of k there are constants

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

so that in the expressions $b(x + \lambda y, y), c(x + \lambda y, y)$, among the monomials of highest degree in x, y, one of those monomials is a constant multiple of a power of y.

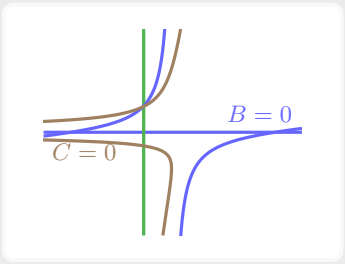
Return to our earlier example of $b(x, y) = xy^2 + y$ and $c(x, y) = 2xy^2 + y + 1$ and let

$$\begin{aligned} B(x, y) &:= b(x + \lambda y, y) = \lambda y^3 + xy^2 + y, \\ C(x, y) &:= c(x + \lambda y, y) = 2\lambda y^3 + 2xy^2 + y + 1. \end{aligned}$$

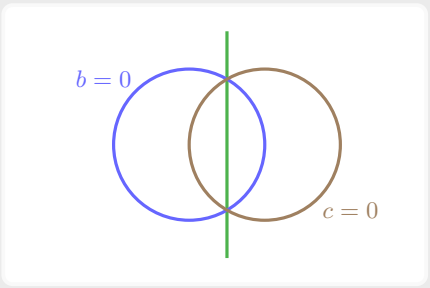
For any nonzero constant $\lambda \neq 0$, the resultant of $B(x, y), C(x, y)$ is

$$r(x) = \det \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ x & 1 & 0 & 2x & 1 & 1 \\ l & x & 1 & 2\lambda & 2\,x & 1 \\ 0 & l & x & 0 & 2\lambda & 2\,x \\ 0 & 0 & l & 0 & 0 & 2\lambda \end{pmatrix} = -\lambda^2 (\lambda + 1 + x).$$

So the resultant now vanishes just when $x = -(\lambda + 1)$, which is precisely when $B(x, y), C(x, y)$ have a common factor of $y - 1$. With a small value of $\lambda \neq 0$, the picture changes very slightly: we change x, y to $x + \lambda y, y$, a linear transformation which leaves the x-axis alone, but tilts the y-axis to the left. Crucially, we tilt the asymptotic line at which the two curves approached one another (where $b(x, y)$ and $c(x, y)$ dropped degrees), but with essentially no effect on the intersection point:



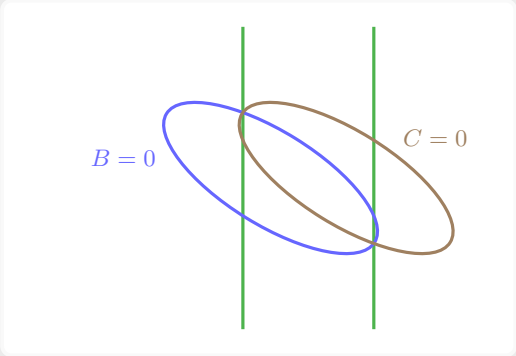
If $b(x, y) = x^2 + y^2 - 1$ and $c(x, y) = (x-1)^2 + y^2 - 1$, then $r(x) = (2x-1)^2$ vanishes at $x = 1/2$, where there are *two* different intersection points.



If instead we pick any nonzero constant λ and let $B(x, y) := (x + \lambda y)^2 + y^2 - 1$ and $C(x, y) := (x + \lambda y - 1)^2 + y^2 - 1$ then the resultant

$$r(x) = 4\left(\lambda^2 + 1\right)\left(x - \frac{1 + \lambda\sqrt{3}}{2}\right)\left(x - \frac{1 - \lambda\sqrt{3}}{2}\right)$$

vanishes at two distinct values of x corresponding to the two distinct roots. In the picture, the two roots now lie on different vertical lines (different values of x).



Proof. It is enough to prove the result for the highest total degree terms of b, so we can assume that b is homogeneous of degree d. The expression $b(x, 1)$ is a nonzero

polynomial, so doesn't vanish everywhere, at least after some finite field extension, by lemma 20.1 on page 173. Pick λ to be a value of x for which $b(\lambda, 1) \neq 0$. Then

$$b(x + y\lambda, y) = y^d b(\lambda, 1) + \dots$$

where $\dots$ are lower order in y. Similarly for two polynomials $b(x, y), c(x, y)$, or for any finite set of polynomials. □

Corollary 20.5. *Take one variable y and several variables $x = (x_1, x_2, \dots, x_n)$ and two polynomials $b(x, y)$ and $c(x, y)$ over a field. Let $r(x)$ be the resultant of $b(x, y), c(x, y)$ in the y variable. For any constant a in our field, if $b(a, y)$ and $c(a, y)$ have a common root in some algebraic extension of our field then $r(a) = 0$.*

If the coefficient of highest order in y of both $b(x, y)$ and $c(x, y)$ is constant in x (for example, perhaps after the normalization described in lemma 20.4 on page 177) then $r(a) = 0$ at some value $x = a$ in our field just exactly when $b(a, y)$ and $c(a, y)$ have a common factor. Moreover $r(x)$ is then the zero polynomial just when $b(x, y)$ and $c(x, y)$ have a common factor which is a polynomial in x, y of positive degree in y.

Proof. If, at some value $x = a$, both the degrees of $b(x, y)$ and $c(x, y)$ don't drop, then the resultant in y is expressed by the same expression whether we set $x = a$ to a constant value or leave x as an abstract variable, compute resultant, and then set $x = a$.

Work over the ring of polynomials in y, with coefficients rational in x. The resultant in y being zero as a function of x forces a common factor in that ring, i.e.

$$b(x, y) = d(x, y)B(x, y),$$
$$c(x, y) = d(x, y)C(x, y),$$

where $d(x, y), B(x, y)$ and $C(x, y)$ are rational in x and polynomial in y and $d(x, y)$ has positive degree in y. In particular, $c(x, y)$ factorises over that ring. By the Gauss lemma (proposition 11.6 on page 81), $c(x, y)$ factorises over the polynomials in x, y. But $c(x, y)$ is irreducible, so one factor is constant, and it isn't $d(x, y)$, so it must be $C(x, y)$, so we rescale by a nonzero constant to get $d(x, y) = c(x, y)$, i.e. $c(x, y)$ divides $b(x, y)$. □

20.1 Prove that every nonconstant rational morphism of plane algebraic curves maps only finitely many points to a given point.

Corollary 20.6. *Given a finite collection of polynomial functions over a field k, for $x = (x_1, x_2, \dots, x_n)$, either*

- *in some finite extension of k, there is at least one point on which all polynomials in the collection vanish or*
- *in some finite extension of k, every polynomial lies in the ideal generated by this collection.*

Proof. Suppose our collection consists of just one polynomial, perhaps in several variables. If its values at all points are the same, even after taking any finite extension, then subtract that constant and apply corollary 20.1 on page 173 to find it becomes zero, so was a constant, say $c \neq 0$. Write any polynomial $p(x)$ as $p(x) = c(p(x)/c)$ to see it is a multiple of that constant c, so lies in the ideal. If it its values are not all the same, perhaps after taking some finite extension, take two points where it takes on

different values. Draw the line between them. It is not constant on that line. Restrict it to that line and we reduce the problem down to one variable parameterizing that line. Extend the field to add a root.

Suppose we just have two polynomials $p_1(x)$ and $p_2(x)$ in our collection. After perhaps a finite extension, and a linear change of variables, we compute a resultant to find the values of one fewer of the variables on which there are simultaneous zeroes.

Suppose instead that there are finitely many polynomials in our collection. We repeatedly replace pairs by such resultants, to eventually reduce the number of variables, and apply induction on the variables.

In the end, when we have only one variable: all resultants are constants, and if they are not all zero, then there are no simultaneous solutions.

Suppose that there is a nonzero resultant in among them. Rescale to get its value to be 1. That nonzero resultant is expressed as a linear combination as in lemma 20.3 on page 175. By induction we construct polynomials $q_j(x)$ so that

$$1 = q_1(x)p_1(x) + q_2p_2(x) + \cdots + q_s(x)p_s(x).$$

So 1 lies in the ideal generated by these $p_1(x), p_2(x), \ldots, p_s(x)$. But then any polynomial $f(x)$ has the form $f(x) = f(x) \cdot 1$, so also lies in that ideal. □

20.2 In corollary 20.6, prove that either one case or the other occurs, but not both.

Chapter 21

Quotient rings

. . . , whereas in Ireland, two and two are just as likely to make five, or three, or are still more likely to make nothing at all.

— E. Œ. Somerville & Martin Ross
The Irish R.M.

Recall once again that all of our rings are assumed to be commutative rings with identity. Suppose that R is a ring and that I is an ideal. For any element $r \in R$, the *translate* of I by r, denoted $r + I$, is the set of all elements $r + i$ for any $i \in I$.

If $R = \mathbb{Z}$ and $I = 12\mathbb{Z}$ is the multiples of 12, then $7 + I$ is the set of integers which are 7 larger than a multiple of 12, i.e. the set of all numbers

$$\ldots, 7 - 12, 7, 7 + 12, 7 + 2 \cdot 12, 7 + 3 \cdot 12, \ldots$$

which simplifies to

$$\ldots, -5, 7, 19, 31, 53, \ldots$$

But this is the same translate as $-5 + I$, since $-15 + 12 = 7$ so $-5 + I$ is the set of all numbers

$$\ldots, -5 - 12, -5, -5 + 12, -5 + 2 \cdot 12, -5 + 3 \cdot 12, \ldots$$

which is just

$$\ldots, -15, -5, 7, 19, 31, 53, \ldots$$

the same sequence.

Working again inside $\mathbb{Z}$, $2 + 2\mathbb{Z} = 2\mathbb{Z}$ is the set of even integers.

If $R = \mathbb{Q}[x]$ and $I = (x)$, then $\frac{1}{2} + I$ is the set of all polynomials of the form

$$\frac{1}{2} + xp(x)$$

for any polynomial $p(x)$, in other words the set of polynomials whose constant term is $\frac{1}{2}$.

If i is in I then $i + I = I$.

We add two translates by

$$(r_1 + I) + (r_2 + I) := (r_1 + r_2) + I,$$

and multiply by

$$(r1 + I)\,(r_2 + I) = (r_1 r_2) + I.$$

Lemma 21.1. *For any ideal I in any ring R, two translates $a + I$ and $b + I$ are equal just when a and b differ by an element of I.*

Proof. We suppose that $a + I = b + I$. Clearly a belongs to $a + I$, because 0 belongs to I. Therefore a belongs to $a + I = b + I$, i.e. $a = b + i$ for some i from I. □

Lemma 21.2. *The addition and multiplication operations on translates are well defined, i.e. if we find two different ways to write a translate as $r + I$, the results of adding and multiplying translates don't depend on which choice we make of how to write them.*

Proof. You write your translates as $a_1 + I$ and $a_2 + I$, and I write mine as $b_1 + I$ and $sb_2 + I$, but we suppose that they are the same subsets of the same ring: $a_1 + I = b_1 + I$ and $a_2 + I = b_2 + I$. By lemma 21.1, $a_1 = b_1 + i_i$ and $a_2 = b_2 + i_2$ for some elements i_1, i_2 of I. So then

$$\begin{aligned} a_1 + a_2 + I &= b_1 + i_1 + b_2 + i_2 + I, \\ &= b_1 + b_2 + (i_1 + i_2) + I, \\ &= b_1 + b_2 + I. \end{aligned}$$

The same for multiplication. □

If I is an ideal in a commutative ring R, the *quotient ring* R/I is the set of all translates of I, with addition and multiplication of translates as above. The reader can easily prove:

Lemma 21.3. *For any ideal I in any commutative ring R, R/I is a commutative ring. If R has an identity element then R/I has an identity element.*

If $R := \mathbb{Z}$ and $I := m\mathbb{Z}$ for some integer m then $R/I = \mathbb{Z}/m\mathbb{Z}$ is the usual ring of remainders modulo m.

If $p(x, y)$ is an irreducible nonconstant polynomial over a field k and $I = (p(x, y)) = p(x, y)k[x, y]$ is the associated ideal, then $k[x, y]/I = k[C]$ is the ring of regular functions on the plane algebraic curve $C = (p(x, y) = 0)$.

Lemma 21.4. *An ideal I in a ring R with identity is all of R just when 1 is in I.*

Proof. If 1 lies in I, then any r in R is $r = r1$ so lies in I, so $I = R$. □

An ideal I in a ring R is *prime* if I is not all of R and, for any two elements a, b of R, if ab is in I then either a or b lies in I.

The ideal $p\mathbb{Z}$ in $\mathbb{Z}$ generated by any prime number is prime.

The ring R is *not* prime in R, because the definition explicitly excludes it.

Inside $R := \mathbb{Z}[x]$, the ideal (x) is prime, because in order to have a factor of x sitting in a product, it must lie in one of the factors.

Inside $R := \mathbb{Z}[x]$, the ideal $(2x)$ is *not* prime, because $a = 2$ and $b = x$ multiply to a multiple of $2x$, but neither factor is a multiple of $2x$.

A polynomial $p(x)$ in several variables, over any field, is irreducible just when $(p(x))$ is a prime ideal.

The regular functions on an algebraic curve $p(x, y) = 0$ over a field k constitute the ring R/I where $R = k[x, y]$ and $I = (p(x, y))$. The curve is irreducible just when the ideal is prime.

More generally, intuitively, if we think of any ring R as being something like the polynomial functions in some variables, then an ideal is like the equations of some geometric object (so, roughly, ideals are like geometric objects), prime ideals are like irreducible geometric objects, and quotient rings are like the polynomial functions on those geometric objects.

Take a field k. In the quotient ring $R = k[x, y, z]/(z^2 - xy)$, the ideal (z) contains $z^2 = xy$, but does not contain x or y, so is not prime.

Lemma 21.5. *An ideal I in a commutative ring with identity R is prime just when R/I has no zero divisors, i.e. no nonzero elements α, β can have $\alpha\beta = 0$.*

Proof. Write α and β as translates $\alpha = a + I$ and $\beta = b + I$. Then $\alpha\beta = 0$ just when $ab + I = I$, i.e. just when ab lies in I. On the other hand, $\alpha = 0$ just when $a + I = I$, just when a is in I. □

A ideal I in a ring R is *maximal* if I is not all of R but any ideal J which contains I is either equal to I or equal to R (nothing fits in between).

The ideal of regular functions vanishing at a point of an algebraic curve is maximal. Intuitively, we picture any ring as the ring of "regular functions" on some "geometric object". The maximal ideals we picture as the "points" of that object, while the prime ideals are the "geometric subobjects", like curves

in a surface, and so on.

The maximal ideals in the integers are precisely the ideal generated by prime numbers, so the same as the prime ideals. In this sense, the integers provide a poor example, as we don't see the difference between prime and maximal.

Lemma 21.6. *If I is an ideal in a commutative ring R with identity, then I is a maximal ideal just when R/I is a field.*

Proof. Suppose that I is a maximal ideal. Take any element $\alpha \neq 0$ of R/I and write it as a translate $\alpha = a + I$. Since $\alpha \neq 0$, a is not in I. We know that $(a) + I = R$, since I is maximal. But then 1 lies in $(a) + I$, so $1 = ab + i$ for some b in R and i in I. Expand out to see that

$$\frac{1}{\alpha} = b + I.$$

So nonzero elements of R/I have reciprocals.

Suppose instead that R/I is a field. Take an element a in R not in I. Then $a + I$ has a reciprocal, say $b + I$. But then $ab + I = 1 + I$, so that $ab = 1 + i$ for some i in I. So $(a) + I$ contains $(ab) + I = (1) = R$. □

If p is a prime number, then $I := p\mathbb{Z}$ is maximal in $R := \mathbb{Z}$, because any other ideal J containing I and containing some other integer, say n, not a multiple of p, must contain the greatest common divisor of n and p, which is 1 since p is prime. We recover the fact that $\mathbb{Z}/p\mathbb{Z}$ is a field.

The ideals in $R := \mathbb{Z}/4\mathbb{Z}$ are $(0), (1) = \mathbb{Z}/4\mathbb{Z}$ and $(2) \cong \mathbb{Z}/2\mathbb{Z}$, and the last of these is maximal.

Take a field k and let $R := k[x]$. Every ideal I in R has the form $I = (p(x))$, because each ideal is generated by the greatest common divisor of its elements. We have seen that R/I is prime just when $p(x)$ is irreducible. To further have R/I a field, we need $I = (p(x))$ to be maximal. Take some polynomial $q(x)$ not belonging to I, so not divisible by $p(x)$. Since $p(x)$ is irreducible, $p(x)$ has no factors, so $\gcd\{p(x), q(x)\} = 1$ in $k[x]$, so that the ideal generated by $p(x), q(x)$ is all of R. Therefore $I = (p(x))$ is maximal, and the quotient $R/I = k[x]/(p(x))$ is a field.

If k is an algebraically closed field and $R := k[x]$ then every irreducible polynomial is linear, so every maximal ideal in R is $I = (x - c)$ for some constant c.

21.1 For any positive integer $m \geq 2$, what are the prime ideals in $\mathbb{Z}/m\mathbb{Z}$, and what are the maximal ideals?

21.2 The *radical* $\sqrt{I}$ of an ideal I in a commutative ring R is the ideal generated by all elements r of R so that r^n is in I for some integer $n \geq 1$. Prove that $\sqrt{I}$ is an ideal of R. A *radical ideal* is an ideal I for which $\sqrt{I} = I$. Prove that every prime ideal is radical.

Sage

We take the ring $R = \mathbb{Q}[x, y]$ and then quotient out by the ideal $I = (x^2y, xy^3)$ to produce the ring $S = R/I$. The generators x, y in R give rise to elements $x + I, y + I$ in S, which are renamed to a, b for notational convenience.

```
R.<x,y> = PolynomialRing(QQ)
I = R.ideal([x^2*y,x*y^3])
S.<a,b> = R.quotient_ring(I)
(a+b)^4
```

which yields `a^4 + b^4`.

We can define $R = \mathbb{Q}[x, y]$, $I = (y^2)$, $S = R/I$ where we label $x + I, y + I$ as a, b, $T = \mathbb{Q}[z]$ and define a morphism $\phi\colon S \to T$ by $\phi(a) = z^3, \phi(b) = 0$:

```
R.<x,y> = PolynomialRing(QQ)
I = R.ideal([y^2])
S.<a,b> = R.quotient_ring(I)
T.<z> = PolynomialRing(QQ)
phi = S.hom([z^3, 0],T)
phi(a^2+b^2)
```

yielding z^6.

Automorphisms of splitting fields

Theorem 21.7. *Suppose that k is a field and $p(x)$ is a nonconstant polynomial over k, say of degree n. Any two splitting fields for $p(x)$ over k are isomorphic by an isomorphism which is the identity on k.*

Proof. We already know that splitting fields exist, by adding roots to irreducible factors of $p(x)$. Each time we add a root, the dimension of the extension field over k is the degree of the irreducible factor. We then split off at least one linear factor in the extension field, and repeat. So at most $n!$ degree in total.

Given any splitting field K for an irreducible polynomial $p(x)$, pick any root $\alpha \in K$ of $p(x)$ and map

$$f(x) \in k[x] \mapsto f(\alpha) \in K,$$

to see that $K \cong k[x]/I$. Hence K is uniquely determined up to isomorphism.

Suppose instead that $p(x)$ is reducible, and that K, L are splitting fields of $p(x)$. Pick an irreducible factor $q(x)$ of $p(x)$. Then the subfields of K, L generated by adding a root of $q(x)$ to k are isomorphic, since $q(x)$ is irreducible, so we can assume that these subfields are the same field. We need only split $p(x)$ over that field, or quotient out all copies of $q(x)$ from $p(x)$ and split the result. Apply induction on the degree of $p(x)$. □

Adding a root to $x^2 + 1$ over $\mathbb{R}$ yields the splitting field $\mathbb{C}$.

Theorem 21.8. *Suppose that K is a splitting field of an irreducible polynomial $p(x)$ over a field k. If $\alpha, \beta \in K$ are roots of $p(x)$ then the Galois group of K over k has an element which takes α to β. In particular, K is a Galois extension of k.*

Proof. As in the proof of theorem 21.7 on the preceding page, there is a field isomorphism $k[x]/I \to K$ taking $x \mapsto \alpha$, and another field isomorphism $k[x]/I \to K$ taking $x \mapsto \beta$. □

Chapter 22

Field extensions and algebraic curves

Transcendental numbers

A number $\alpha \in \mathbb{C}$ is *algebraic* if it is the solution of a polynomial equation $p(\alpha) = 0$ where $p(x)$ is a nonzero polynomial with rational coefficients. A number which is not algebraic is called *transcendental*. More generally, given a field extension K of a field k, an element $\alpha \in K$ is *algebraic* over k if it is a root of a nonzero polynomial $p(x)$ with coefficients in k.

Theorem 22.1. *For any field extension K of a field k, an element $\alpha \in K$ is transcendental if and only if the field*

$$k(\alpha) = \left\{ \frac{p(\alpha)}{q(\alpha)} \;\middle|\; \frac{p(x)}{q(x)} \in k(x) \text{ and } q(\alpha) \neq 0 \right\}$$

is isomorphic to $k(x)$.

Proof. If α is transcendental, then the map

$$f\colon \frac{p(x)}{q(x)} \in k(x) \mapsto \frac{p(\alpha)}{q(\alpha)} \in k(\alpha)$$

is clearly well defined, onto, and preserves all arithmetic operations. To show that f is 1-1 is the same as showing that f has trivial kernel. Suppose that $p(x)/q(x)$ lies in the kernel of f. Then $p(\alpha) = 0$, so $p(x) = 0$, so $p(x)/q(x) = 0$. So the kernel is trivial, and so f is a bijection preserving all arithmetic operations, so f is an isomorphism of fields.

On the other hand, take a element α in K and suppose that there is some isomorphism of fields

$$g\colon k(x) \to k(\alpha).$$

Let $\beta := g(x)$. Because g is a field isomorphism, all arithmetic operations carried out on x must then be matched up with arithmetic operations carried out on β, so

$$g\left(\frac{p(x)}{q(x)}\right) = \frac{p(\beta)}{q(\beta)}.$$

Because g is an isomorphism, some element must map to α, say

$$g\left(\frac{p_0(x)}{q_0(x)}\right) = \alpha.$$

So

$$\frac{p_0(\beta)}{q_0(\beta)} = \alpha.$$

So $k(\beta) = k(\alpha)$. Any algebraic relation on α clearly gives one on β and vice versa. Therefore α is algebraic if and only if β is. Suppose that β is algebraic. Then $q(\beta) = 0$ for some polynomial $q(x)$, and then g is not defined on $1/q(x)$, a contradiction. □

Finite fields

We are going to make a complete list of all finite fields. Pick a prime number p and work over the field $k = \mathbb{Z}/p\mathbb{Z}$. For any integer $n > 0$, the polynomial $x^{p^n} - x$ has two obvious linear factors: x and $x - 1$, and then factors as

$$x^{p^n} - x = x(x-1)(x^{-2+p^n} + x^{-3+p^n} + \cdots + x + 1).$$

Lemma 22.2. *For any prime p, over any extension field K of the field $k = \mathbb{Z}/p\mathbb{Z}$, the polynomial*

$$x^{p^n} - x$$

has no multiple roots.

Proof. Let $c(x) := x^{p^n} - x$. As above, write $x(x-1)b(x) = c(x)$. Note that $b(0) = 1 \neq 0$ and $b(1) = p^n - 1 = -1 \neq 0$. If, in some field K, we can factor out a multiple root from $b(x)$, that multiple root is not at $x = 0$ or $x = 1$, so we can factor out the same multiple root from $c(x) = x(x-1)b(x)$. Take the derivative of $c(x)$:

$$c'(x) = \frac{d}{dx}\left(x^{p^n} - x\right) = p^n x^{-1+p^n} - 1 = -1,$$

since $p = 0$ in k. But then $c'(x)$ has no roots in any field extension K of k, so $c(x)$ has no multiple roots over K, so $b(x)$ has no multiple roots over K. □

Lemma 22.3. *For any finite field K of characteristic p, the* Frobenius morphism *$f(x) = x^p$ is an automorphism of fields $f\colon K \to K$, i.e. is a bijection preserving addition, subtraction, multiplication and division, taking $0 \mapsto 0$ and $1 \mapsto 1$.*

Proof. Clearly $0^p = 0$ and $1^p = 1$. For any $b, c \in K$, clearly $(bc)^p) = b^p c^p$. The binomial theorem gives

$$(b+c)^p = b^p + pb^{p-1}c + \frac{p(p-1)}{2}b^{p-2}c^2 + \cdots + \frac{p!}{j!(p-j)!}b^{p-j}c^j + \cdots + c^p.$$

Every term except b and c has a factor of p in it, and $p = 0$ in K, so

$$(b+c)^p = b^p + c^p.$$

The same for $(b-c)^p = b^p - c^p$. If $f(b) = 0$, then $b^p = 0$ so $b \cdot b^{p-1} = 0$ so $b = 0$ or $b^{p-1} = 0$, and by induction, we find that $b = 0$. If $f(b) = f(c)$ then $f(b-c) = 0$ so $b - c = 0$ so $b = c$, i.e. f is 1-1. But then, since K is finite, f is a bijection. □

Theorem 22.4. *For any prime p, the splitting field K of the polynomial*

$$c(x) = x^{p^n} - x$$

over the field $k = \mathbb{Z}/p\mathbb{Z}$ has p^n elements, every one of which is a root of $c(x)$. Every finite field K is obtained uniquely as the splitting field of $c(x)$ for some prime number p and integer $n \geq 1$, and so is uniquely determined up to isomorphism by its number of elements p^n.

Proof. There are p^n roots of $c(x) = x^{p^n} - x$ over its splitting field K, since $b(x)$ splits into linear factors, and, by lemma 22.2 on the preceding page each linear factor gives a distinct root. Given any roots α, β of $c(x)$,

$$\alpha^{p^n} = \alpha,\ \beta^{p^n} = \beta.$$

Apply the Frobenius morphism:

$$(\alpha + \beta)^{p^n} = \alpha^{p^n} + \beta^{p^n} = \alpha + \beta,$$

so $c(\alpha + \beta) = 0$. Similarly

$$(\alpha\beta)^{p^n} = \alpha^{p^n}\beta^{p^n} = \alpha\beta,$$

so $c(\alpha\beta) = 0$. Hence the roots of $c(x)$ form a field, over which $c(x)$ splits, and so $b(x)$ splits. So every element of K is a root of $c(x)$, and so K is a splitting field of $c(x)$ with p^n elements. By the existence and uniqueness of splitting fields in theorem 21.7 on page 185, K is the unique splitting field of $c(x)$.

Take any finite field L, say of characteristic p. We map $0, 1, 2, \ldots, p-1$ in $k = \mathbb{Z}/p\mathbb{Z}$ to $0, 1, 2, \ldots, p-1$ in L, a morphism of fields. Hence L is a vector space over k, of finite dimension, say n, and so has p^n elements. The Frobenius morphism is an invertible k-linear map of L, generating a subgroup of the group of invertible k-linear maps $L \to L$. This subgroup is finite, since there are only finitely many elements of L, so finitely many permutations of those elements. This subgroup is generated by a single element, so is cyclic. By the classification of cyclic groups, this subgroup is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$, for some integer $\ell > 0$, so every element of L satisfies some equation $x^{p^\ell} = x$. □

For any finite field k, say with p^n elements, we can associate to any $\alpha, \beta \in k$ the quadratic equation $(x-\alpha)(x-\beta) = 0$ which has roots α, β. On the other hand, given a quadratic equation $x^2 + bx + c = 0$, we can ask whether it has roots. Those with roots are given as $x^2 + bx + c = (x - \alpha)(x - \beta)$ for some α, β, unique up to swapping the order in which we write them down. As there are p^n elements, there are

$$\frac{p^n(p^n + 1)}{2}$$

pairs α, β of possible roots, up to swapping order. But there are p^{2n} quadratic equations $x^2 + bx + c$: choose any b, c from k. So

$$p^{2n} - \frac{p^n(p^n + 1)}{2} = \frac{p^n(p^n - 1)}{2}$$

quadratic equations over k have no solution in k.

In particular, let $k_1 \subset k_2 \subset k_3 \ldots$ be finite fields or orders $2, 4, 8$ and so on. Each k_{n+1} is the splitting field of the polynomial $z^2 + z + \alpha$ over k_n where α is any element of k_n not belonging to k_{n-1}. In particular, we can see which quadratic equations have solutions in which fields of characteristic 2. Note that we can't use the quadratic formula in fields of characteristic 2, because the quadratic formula divides by 2.

22.1 Prove by induction that if k is a finite field of characteristic 2, then every element α of k is a square, i.e. $\alpha = \beta^2$ for some β.

Sage

Lets get sage to construct the splitting field K of $p(x) := x^2 + x + 1$ over the field $k = \mathbb{Z}/2\mathbb{Z}$. Sage constructs splitting fields by first building a ring, which we will call R, of polynomials over a field. In our case, we build $R = k[x]$. Then we let K be the field generated by an element a, which corresponds to the variable x in the polynomial ring R.

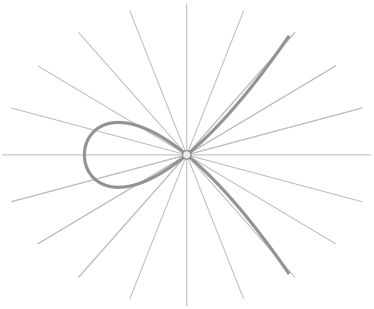
```
R.<x> = PolynomialRing(GF(2))
K.<a> = (x^2 + x + 1).splitting_field()
```

We can then make a list of some elements of K, call it S, and compute out addition and multiplication tables:

```
S=[0,1,a,a+1]
print("Addition table:")
for i in range(0,4):
    for j in range(0,4):
        print("({})+({})={}".format(S[i],S[j],S[i]+S[j]))
print("Multiplication table:")
for i in range(0,4):
    for j in range(0,4):
        print("({})*({})={}".format(S[i],S[j],S[i]*S[j]))
```

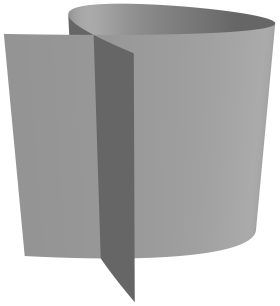
Drawings and field extensions

We can sometimes draw pictures of field extensions. Take an irreducible algebraic curve $C = (p(x, y) = 0)$ in the plane, and look at its field of rational functions $k(C)$, our favourite example of a field extension of a field k.

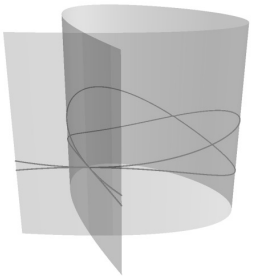


Suppose that K is a field extension of $k(C)$, say $K = k(C)(\alpha)$ given by extending by an element α. We want to draw a curve D so that $K = k(D)$.

If α is transcendental, then $k(C)(\alpha) \cong k(C)(z)$ for some abstract variable z. But $k(C)$ is already a field of rational functions of two variables x, y, quotiented out to enforce the relation $p(x, y) = 0$. So K is just the rational functions on the "cylinder" $C \times k$ inside 3-dimensional space, with variables x, y, z; our cylinder has the equation $p(x, y) = 0$, independent of z.

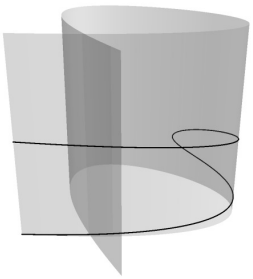


Suppose instead that α is algebraic, say satisfying a polynomial equation $f(z) = 0$ with $f(z)$ a polynomial with coefficients from $k(C)$. So each coefficient of $f(z)$ is itself a rational function on C. Each such function is expressible as a rational function of x, y. Clearing denominators, we get a polynomial equation $f(x, y, z) = 0$ with coefficients in the underlying field k. The solutions form a surface in 3-dimensional space. But the resulting surface depends on how we pick out rational functions of x, y to represent our coefficients of $f(z)$ from $k(C)$. So in fact we have to restrict our x, y variables to lie in C, i.e. we consider the *curve* D cut out by the equations $p(x, y) = f(x, y, z) = 0$.



The map $(x, y, z) \in D \mapsto (x, y) \in C$ is regular.

We wonder now whether there is a $k(C)$-root to $f(z) = 0$, say β. This β is in $k(C)$, so a rational function on C, so that $f(\beta) = 0$. So this is a rational map $z = \beta(x, y)$ from C to D, lifting the curve C up from the plane into space, tracing out D on the cylinder. In particular, there is no such map if D is irreducible and $D \to C$ is many to one:



So in such a case, $k(C) \subset k(D)$ is a nontrivial field extension. We get a picture giving some intuition for nontrivial algebraic field extensions: typically we expect to see an irreducible curve D mapping to a curve C by a many-to-one map. The Galois group of $k(D)/k(C)$ swaps strands of the curve D up and down in the picture, so each point of D remains over the same point of C as before the swap.

The story continues similarly in higher dimensions: a transcendental extension increases dimension as a "cylinder", while an algebraic extension, say of degree n, preserves dimension, giving a new geometric object, sitting inside such a "cylinder", mapping regularly to the old one, by an n-to-1 map.

Rational curves

Recall that the *degree* of a field extension $k \subset K$, denoted $[K : k]$, is the smallest integer n so that there are n elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ of K so that every element of K is a linear combination $\sum a_j \alpha_j$ with coefficients $a_1, a_2, \ldots, a_n$ in k. The degree of the field extension is infinite if there is no such integer n.

22.2 Suppose that $k \subset K \subset L$ are field extensions. Prove that $[L : k] = [L : K][K : k]$. Note: this works even if one or more degree is infinite: let $n\infty = \infty n = \infty\infty = \infty$ for any positive integer n. You have to prove these infinite degree cases as well.

22.3 Prove that a field extension is an isomorphism if and only if it has degree one.

22.4 Suppose that $p(x)$ is an irreducible polynomial over a field k. Let $K := k[x]/(p(x))$. Prove that $[K : k]$ is the degree of $p(x)$.

22.5 Prove that if $u(t)$ is a nonconstant rational function over a field k, then there are infinitely many values of $u(t)$ as t varies over $\bar{k}$. In particular, $k \subset k(u(t))$ is a transcendental extension.

Recall that a curve $C = (0 = f(x, y))$ is *irreducible* if $f(x, y)$ is irreducible as a polynomial.

Theorem 22.5. *An irreducible plane algebraic curve C over a field k is rational just when there is a nonconstant rational map $k \to C$. In other words, C is rational just when there are rational functions $x(t), y(t)$, not both constant, for which the point $(x(t), y(t))$ lies in C for all t in the algebraic closure of the field.*

Proof. If our curve C is rational then by definition C is birational to a line, so there is such a map. Suppose on the other hand that there is such a map $k \to C$, say written as $t \in k \mapsto (x(t), y(t)) \in C$ with

$$f(x(t), y(t)) = 0$$

for some rational functions $x(t), y(t)$, not both constant. As we vary t through $\bar{k}$, the moving point $(x(t), y(t))$ moves through infinitely many different points of the plane. Take a rational function on C, say $g(x, y) = b(x, y)/c(x, y)$. Then $c(x, y) \neq 0$ at all but finitely points of C. (Note that this requires that our curve C is irreducible, as otherwise we might have $g(x, y)$ vanishing on a component of C.) So at infinitely many values of t, $g(x(t), y(t))$ is defined. We map $g(x, y) \in k(X) \mapsto g(x(t), y(t)) \in k(t)$, a morphism of fields. The image is some subfield $K = k(x(t), y(t)) \subset k(t)$. The result follows from the following theorem applied to K. □

Theorem 22.6 (Lüroth's theorem)**.** *Suppose that $k \subset K \subset k(t)$ is a finitely generated extension of a field k, where t is an abstract variable. Then $K = k(f(t))$ is generated by a single rational function $f(t)$ in $k(t)$. In particular, either $K = k$ (if $f(t)$ is a constant function) or K is isomorphic to $k(t)$ by $t \mapsto f(t)$.*

The proof of this theorem requires a lemma:

Lemma 22.7. *Take a nonconstant rational function $f(x) = b(x)/c(x)$, with $b(x), c(x)$ coprime, over a field k. Then, as a polynomial in a variable y with coefficients in $k(f(x))$, the expression*

$$f(x)c(y) - b(y)$$

in $k(f(x))[y]$ is irreducible.

Proof. This expression is not zero, because it is nonconstant in x. It vanishes at $y = x$. Hence x satisfies this polynomial (in y) expression with coefficients in $k(f(x))$. So $k(x)$ is an algebraic extension of $k(f(x))$. Since $k(x)$ is transcendental over k, $k(f(x))$ is transcendental over k. So $f(x)$ solves no polynomial equation over k. Take an abstract variable t. Map $k(t, y) \to k(f(x), y)$ by $t \mapsto f(x)$. This map is an isomorphism of fields. Similarly $k[t, y] \to k[f(x), y]$ is an isomorphism of rings. In order to prove that $tc(y) - b(y)$ is irreducible in $k(t)[y]$, it suffices to prove that it is irreducible in $k[t, y]$, i.e. clear t from denominators, by Gauss's lemma (proposition 11.6 on page 81). If we can factor $tc(y) - b(y) = g(t, y)h(t, y)$ in $k[t, y]$ then one of $g(t, y)$ or $h(t, y)$ has degree 1 in t, since the product does, and so one of $g(t, y)$ or $h(t, y)$ has degree zero in t, depending only on y, say $g(t, y) = g(y)$. Then $g(y)h(t, y) = tc(y) - b(y)$, so $g(y)$ divides $tc(y) - b(y)$, and since t is an abstract variable, $g(y)$ divides $b(y)$ and $c(y)$, which are coprime, so $g(y)$ is a nonzero constant. □

We now prove Lüroth's theorem:

Proof. For each element $f(t) = b(t)/c(t)$ in K, let n be the larger of the degrees of $b(t), c(t)$. Pick an element $f(t) = b(t)/c(t)$ in K, not in k, for which the value of n is as small as possible. By the previous lemma, the polynomial $B(y) = f(x)c(y) - b(y)$ is irreducible in $k(f(x))[y]$, so the map

$$x, y \in k(f(x))[y]/(B(y)) \mapsto x, x \in k(x)$$

is an isomorphism of fields and $[k(x) : k(f(x))] = n$. The expression $B(y)$ is the polynomial in $K[y]$ of smallest positive y-degree which is satisfied by $y = x$, since a smaller degree one would give a smaller value for n. Indeed the y-degree of $B(y)$ is n. In particular, $B(y)$ is also irreducible in $K[y]$ and the map

$$x, y \in K[y]/(B(y)) \mapsto x, x \in k(x)$$

is an isomorphism. In particular, $[k(x) : K] = n$. But

$$\begin{aligned} n &= [k(x) : k(f(x))], \\ &= [k(x) : K][K : k(f(x))], \\ &= n[K : k(f(x))] \end{aligned}$$

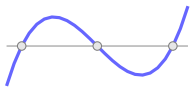
so $[K : k(f(x))] = 1$. □

Chapter 23

The projective plane

The projective line

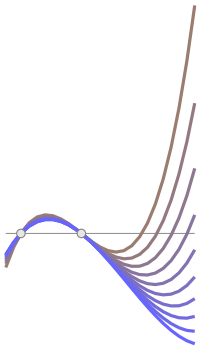
A cubic polynomial in one variable might have three roots:



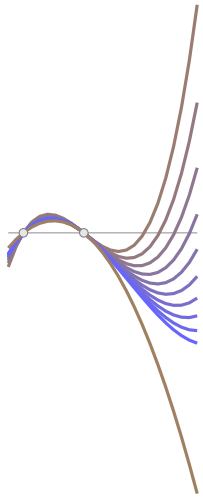
We can put them wherever we like by picking suitable coefficients. Imagine one of them moves far away, so at time t, we want our roots to be at $x = 0, x = 1, x = t$. We could pick the polynomial to be $x(x-1)(x-t)$; but this has no limit as t gets large, since there is a factor of t in our polynomial. Fix this: divide out by t, say take our polynomial to be instead $x(x-1)(x-t)/t$, which does not change the roots. Expand this out:

$$\frac{x(x-1)(x-t)}{t} = \frac{x^3}{t} - \left(1 + \frac{1}{t}\right) x^2 + x.$$

For nonzero t, the coefficients remain bounded as t gets large.



So our root moves far away, while the coefficients remain bounded. What happens as t becomes infinite?



Our polynomial reaches a limit

$$-x^2 + x = x(1 - x).$$

The two fixed roots stay in place, but the polynomial drops degree when the moving root disappears to infinity.

There is something frustrating about the possibility that a moving polynomial can drop degree like this, so that in the limit, roots can disappear to infinity. Consider how we can "hold on" to the roots. Take a cubic polynomial

$$p(x) = ax^3 + bx^2 + cx + d.$$

Associate to it the homogeneous polynomial in two variables

$$P(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

by sticking in a power of y in each term so that the term reaches total degree 3 in x and y together. The original polynomial is $p(x) = P(x, 1)$. So geometrically, $p(x)$ is just $P(x, y)$ but only calculated along the line $y = 1$.

23.1 Homogenize $p(x) = x^4 + x + 1$.

All terms have total degree 3 so, when we rescale the variables, every term scales by 3 factors of the scaling

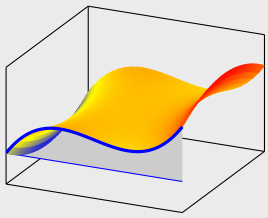
$$P(sx, sy) = s^3 P(x, y)$$

In particular, if $P(x, y)$ vanishes at some point $(x, y) = (x_0, y_0)$ of the plane, it also vanishes at every rescaling of that point. So it also vanishes on the line through (x_0, y_0) and $(0, 0)$. Thus the zeroes of a homogeneous polynomial form a collection of lines through the origin.

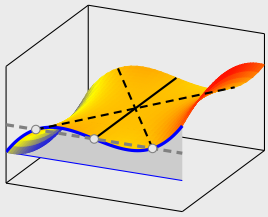
Taking $p(x) = x(x - 1)(x + 1)$, we find

$$P(x, y) = x(x - y)(x + y)$$

by making each factor homogeneous. On the graph of $P(x, y)$ we see the graph of $p(x)$ along the line $y = 1$:



So $P(x, y)$ vanishes on the lines $x = 0$, $x = y$, $x = -y$ in the plane. These lines intersect the line $y = 1$ at $x = 0$, $x = 1$, $x = -1$, the roots of $p(x)$.



23.2 Starting with $p(x) = x^3 + x$, compute $P(x, y)$, find the real roots of $p(x)$ and the lines through the origin on which $P(x, y)$ vanishes. Draw the graph of $P(x, y)$; indicate on it where the graph of $p(x)$ lies.

Passing from the polynomial $p(x)$ to the homogeneous polynomial $P(x, y)$, we lose no information, since $p(x) = P(x, 1)$. The same trick works for polynomials of any degree. But the roots now behave better as we allow them to move: we picture that the lines only turn around, but they can't disappear, because they continue to pass through the origin.

Return to our example of

$$p(x) = \frac{x(x-1)(x-t)}{t}.$$

Just think of t as a constant for now; we will soon let it move. Homogenize:

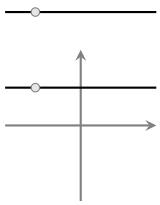
$$P(x, y) = \frac{x(x-y)(x-ty)}{t}.$$

So the roots are the lines $x = 0$, $x = y$, $x = ty$. As we let t get large, the line $x = ty$ can be written as $x/t = y$, so becomes $0 = y$ in the limit.

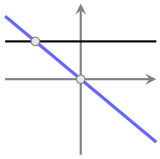
Take any polynomial $p_t(x)$ whose coefficients depend on some parameter t (or maybe several parameters). Imagine we compute out the associated homogeneous polynomial, call it $P_t(x, y)$. Suppose that $p_t(x)$ has some degree d, except maybe for certain "special" values of t. As long as the coefficients of $p_t(x)$ don't all disappear at once, i.e. for the same value of t or limit of t, the same is true of $P_t(x, y)$, since they

are the same coefficients. All terms in the homogeneous polynomial $P_t(x, y)$ have degree exactly d. Since they never all vanish, $P_t(x, y)$ doesn't drop degree for any value or limit of t.

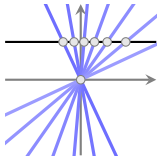
Each point x becomes a unique point $(x, y) = (x, 1)$ on the line $y = 1$.



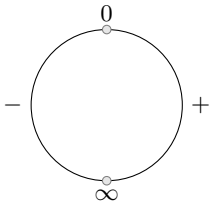
Conversely, each point $(x, y) = (x, 1)$ arises in this way from a point x. But each point $(x, 1)$ also spans a line through the origin.



Every line through the origin arises uniquely in this way except $y = 0$, which we can think of as $x = \infty$. Spin the line around the origin.



As we spin, our line strikes $y = 1$ at some point $(x, 1)$, which we think of as a number x; picture the line spinning clockwise, so $x \to \infty$. When our spinning line becomes horizontal, it becomes $y = 0$, so $x = \infty$. After that, as the angle of the line goes further around clockwise, we now find it strikes a point $(x, 1)$ of our line $y = 1$ but with $x < 0$. So we have "passed through infinity" and found ourselves wrapping around to negative numbers:



The *projective line* of a field k is the set of all lines through the origin in the plane k^2; each is uniquely expressed as $x = ty$ for some element t from k (and so is naturally thought of as an element t of k), except for the line $y = 0$ (which we naturally write as $t = \infty$). Every polynomial $p(x)$, in one variable x, of degree at most some positive

integer d, is naturally identified with a homogeneous polynomial $P(x, y)$ of degree exactly d, which then has roots in the projective line; to make sense of this, we must fix the choice of d, some integer which is at least as large as the actual degree of $p(x)$. If $p(x)$ has degree less than d, say some degree d_0, we can treat that as meaning that $P(x, y)$ has $d - d_0$ factors of y, or say that $p(x)$ has $d - d_0$ "roots at infinity". We are not afraid to work with roots at infinity, since we can always write out $P(x, y)$ to make precise what we mean. Roots won't disappear as we vary the coefficients of $p(x)$, when we compute roots in the projective line; they only move to infinity, or even move across infinity. We are forced to look for our roots on the projective line whenever we have a polynomial whose coefficients vary with some parameter.

23.3 Suppose that we change variables x, y by a linear transformation, say to

$$\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that the line $x = ty$ changes to the line $X = TY$ given by the associated *projective transformation*

$$T = \frac{at + b}{ct + d}.$$

The projective plane

> *The more systematic course in the present introductory memoir . . . would have been to ignore altogether the notions of distance and metrical geometry . . . . Metrical geometry is a part of descriptive geometry, and descriptive geometry is all geometry.*
>
> — Arthur Cayley

The old fashioned term *descriptive geometry* means the geometry of straight lines. Straight lines in the plane remain straight when you rescale the plane, or translate the plane to the left or right, up or down, or when you rotate the plane. They even remain straight when you carry out any linear change of variables, as you know from linear algebra.

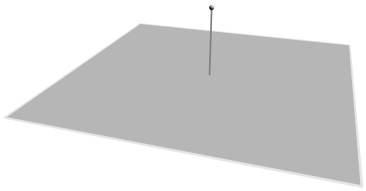
> *To see a World in a Grain of Sand,*
> *And Heaven in a Wild Flower.*
> *Hold Infinity in the palm of your hand,*
> *And Eternity in an hour.*
>
> — William Blake
> Auguries of Innocence

Railway tracks built on a flat plane appear to meet "at infinity".

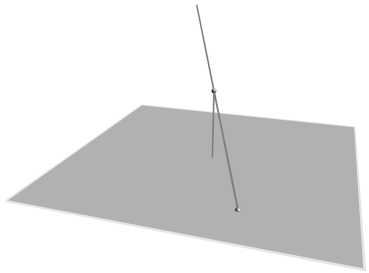
Imagine that we were to add a point to the plane "at infinity" in the direction where these tracks (straight lines) appear to meet. *Danger:* in order to add only one point, imagine that the two rails meet at the *same* point in one direction as they do in the other, making them both into circles. The projective plane is the set of all of the usual points of the plane (which we can think of as the "finite points" of the projective plane) together with some sort of "points at infinity" to represent the directions in the plane.

> *But you must note this: if God exists and if He really did create the world, then, as we all know, He created it according to the geometry of Euclid and the human mind with the conception of only three dimensions in space. Yet there have been and still are geometricians and philosophers, and even some of the most distinguished, who doubt whether the whole universe, or to speak more widely the whole of being, was only created in Euclid's geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity.*
>
> — Fyodor Dostoyevsky
> The Brothers Karamazov

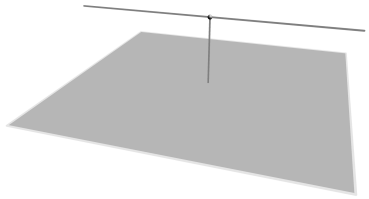
We would like a more precise mathematical definition of points "at infinity". Place yourself at a vantage point above the plane.



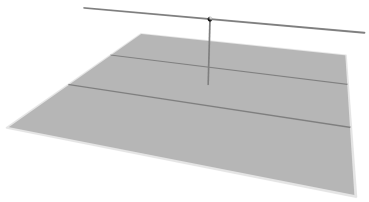
Every "finite" point of the plane lies on a line through your vantage point: the line of sight.



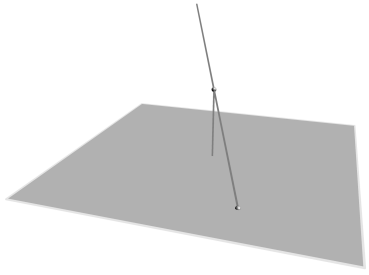
You can also look out to the points at infinity, along lines:



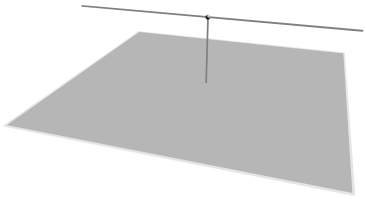
For example, there is such a line parallel to the lines of our train track:



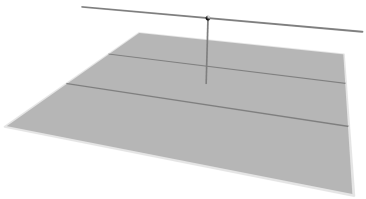
On the other hand, if we take any line through the vantage point, either it hits a "finite" point of the plane:



or it is parallel to the plane

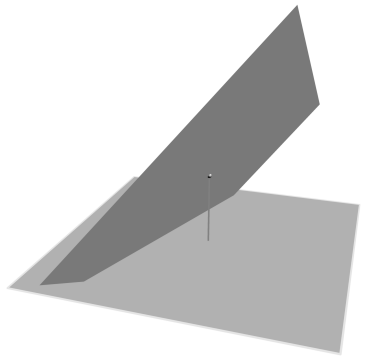


so that it is pointed along the direction of some train tracks (lines) in the plane, to some "point at infinity".



The points of the projective plane, finite points together with infinite points at infinity, are in 1-1 correspondence with lines through the vantage point.

This picture gives us a rigorous definition of points at infinity: the *projective plane* is the set of all lines through a chosen point of 3-dimensional space, called the vantage point. The *points* of the projective plane (using this definition) are the lines through the vantage point. The "finite points" are the lines not parallel to the horizontal plane, while the "infinite points" are the lines which are parallel the horizontal plane. The set of finite points is the *affine plane*, which we think of as just the usual xy-plane. The *lines*, also called *projective lines*, of the projective plane are the planes through the vantage point.



Homogeneous coordinates

We can make this more explicit by writing each point of 3-dimensional space $\mathbb{R}^3$ as a triple (x, y, z). Draw the plane not as the usual xy-plane, but as the plane $z = z_0$, for some nonzero constant z_0. Take the vantage point to be the origin. Any linear change

of the 3 variables x, y, z takes lines (and planes) through the origin to one another: there are more symmetries of the projective plane than we have ever encountered in the affine plane.

Every point of 3-dimensional space not at the origin lies on a unique line through the origin. So every point (x, y, z) with not all of x, y, z zero lines on a unique line through the origin. The points of this line are precisely the rescalings (tx, ty, tz) of that point. So each point of the projective plane can be written as a triple (x, y, z), not all zero, and two such triples represent the same point just when each is a rescaling of the other. Denote such a point as $[x, y, z]$. For example, the point $[3, 1, 2]$ of the projective plane means the line through the origin consisting of the points of the form $(3t, t, 2t)$ for all values of a variable t, including $(3, 1, 2)$, $(3 \cdot 2, 1 \cdot 2, 2 \cdot 2)$, $(-3, -1, -2)$, and so on. We write this as:

$$[3, 1, 2] = [3 \cdot 2, 1 \cdot 2, 2 \cdot 2] = [-3, -1, -2].$$

The coordinates x, y, z are the *homogeneous coordinates* of a point $[x, y, z]$ of the projective plane.

In our pictures, the plane $z = z_0$ was below the vantage point, but it is more traditional to take the plane to be $z = 1$, above the vantage point. The *affine plane* is just the usual xy plane, but identified either with the plane $z = 1$, or with the subset of the projective plane given by points $[x, y, 1]$.

The roles of x, y, z are all the same now, and we can see that any linear change of variables of x, y, z takes the points of the projective plane to one another, and takes the lines of the projective plane to one another. Each linear change of variables is represented by an invertible matrix

$$g = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix}.$$

Conventionally we label our matrix columns using $0, 1, 2$ rather than $1, 2, 3$. Such a matrix takes a point $p = [x, y, z]$ of the plane to the point $q = [X, Y, Z]$ where

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = g \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We denote this relation as $q = [g]p$ and call $[g]$ a *projective automorphism*. Clearly if g and h are 2 invertible 3×3 matrices, then $[gh] = [g][h]$, i.e. we carry out the transformation of lines through the origin by carrying out the multiplications by the matrices.

Lemma 23.1. *If a square matrix g rescales every nonzero vector by a scalar, i.e. $gx = \lambda(x)x$ for some number $\lambda(x)$, for every vector $x \neq 0$, then $\lambda(x)$ is a constant scalar independent of x, and $g = \lambda I$ is a constant multiple of the identity matrix.*

Proof. Suppose that g is $n \times n$. If $n = 1$ then every square matrix is a constant scalar, so suppose that $n \geq 2$. Write our vectors as $x = \sum_j x_j e_j$ in the standard basis of $\mathbb{R}^n$

and calculate

$$\begin{aligned} gx &= \lambda(x)\, x, \\ &= \sum_j \lambda(x)\, x_j e_j, \\ &= \sum_j x_j g e_j, \\ &= \sum_j x_j \lambda(e_j)\, e_j. \end{aligned}$$

So for any x with $x_j \neq 0$, we have $\lambda(x) = \lambda(e_j)$. But then if $x_i \neq 0$, and if $i \neq j$, then

$$\begin{aligned} \lambda(x) &= \lambda(e_i)\,, \\ &= \lambda(e_i + e_j)\,, \\ &= \lambda(e_j)\,. \end{aligned}$$

So λ is a constant. □

Lemma 23.2. *Two projective automorphisms $[g], [h]$ have precisely the same effect on all points $[x, y, z]$ of the projective plane just when the matrices agree up to a constant $h = \lambda g$, some number $\lambda \neq 0$.*

Proof. Let $\ell := h^{-1}g$, so that $[\ell] = [h]^{-1}[g]$ and $[\ell]$ fixes every point of the projective plane just when ℓ rescales every vector in $\mathbb{R}^3$ by a scalar. □

In particular, the points of the projective plane are permuted by the projective automorphisms, as are the projective lines.

The *projective plane* over any field k, denoted $\mathbb{P}^2(k)$, is the set of lines through the origin in k^3. In k^3, the line through two points

$$p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, q = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

is just the set of all points $tp + (1-t)q$ for t in k.

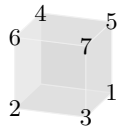
Example: the Fano plane

Let $k := \mathbb{Z}/2\mathbb{Z}$; the *Fano plane* is the projective plane $\mathbb{P}^2(k)$. Each point is $[x, y, z]$ with x, y, z in k defined up to rescaling. But the only nonzero scalar you can use to rescale with is 1, since $k = \{\, 0, 1 \,\}$. Therefore we can just write $[x, y, z]$ as (x, y, z). Taking all possibilities for x, y, z from k, not all zero, we get 7 points

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

corresponding to the numbers $1, 2, \ldots, 7$ written in base 2.

We can also draw this diagram as a cube; for each point, you draw the corresponding point in $\mathbb{R}^3$ with the same coordinates:



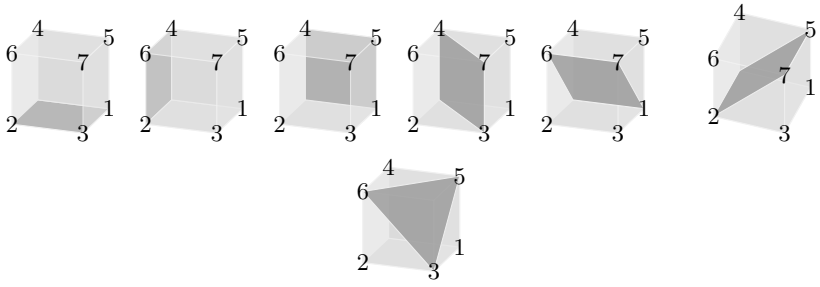
with one vertex not marked; the unmarked vertex is at the origin, so doesn't represent any point of the Fano plane. Every line in the Fano plane has three vertices. Take two numbers from 1 to 7, say $1, 5$, and write out their binary digits under one another:

$$\begin{array}{c} 001 \\ 101 \end{array}.$$

Look at these as vectors in k^3, so add them without carrying digits:

$$\begin{array}{c} 001 \\ 101 \\ \hline 100 \end{array}$$

The sum vector lies on the same line in the Fano plane, giving the lines:



Projective space

Similarly, *projective space* $\mathbb{P}^n(k)$ is the set of lines through the origin of k^{n+1}, and a *projective automorphism* of projective space is the result of applying an invertible square matrix to those lines. Again, two invertible square matrices yield the same projective automorphism just when they agree up to a scalar multiple, by the same proof.

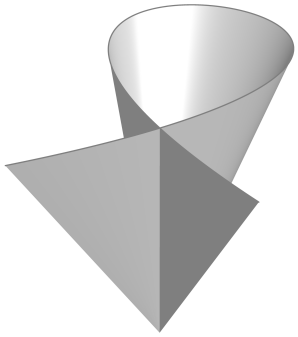
Chapter 24

Algebraic curves in the projective plane

Homogenisation of plane curves

Take an algebraic curve $y^2 = x^2 + x^3$ in the affine plane over a field k.

The *cone* on the curve is the surface in k^3 given by $y^2z = x^2z + x^3$.



The recipe: given an algebraic equation, like $y^2 = x^2 + x^3$:

a. Find the degree n of the highest term; in this example $n = 3$.

b. Invent a new variable z.

c. Multiply each term by a power of z so that the total degree of each term reaches n.

A polynomial is *homogeneous* if all of its terms have the same total degree, i.e. sum of degrees in all variables.

24.1 Homogenise $x^2y - x^2 + y = 1 + 2x - y^2x^3$.

In sage:

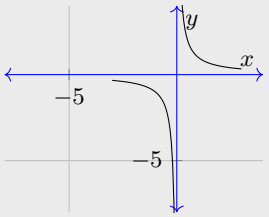
```
R.<x,y,z> = PolynomialRing(QQ)
p = x^3+x*y+1
p.homogenize(z)
```

yields $x^3 + xyz + z^3$.

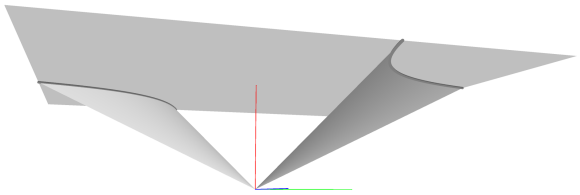
The homogenised polynomial equation $y^2z = xz^2 + x^3$ has all terms of degree 3, so if we rescale x, y, z to tx, ty, tz, we rescale both sides of the equation by t^3, so solutions (x, y, z) rescaled remain solutions (tx, ty, tz). Thefore every solution lies on a line through the origin consisting of other solutions. The set of solutions is a surface, which is a union of lines through the origin. A *projective plane curve* is the set of lines through the origin satisfying a nonzero homogeneous polynomial.

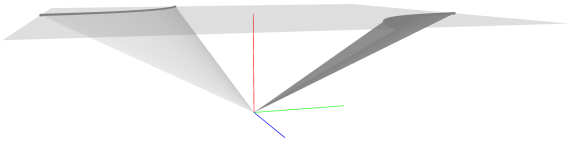
Conics

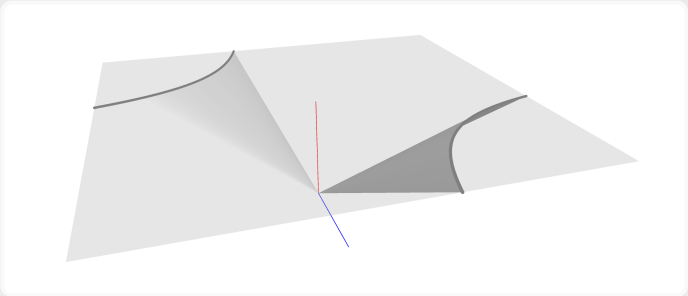
Take a hyperbola $xy = 1$ in the affine plane,



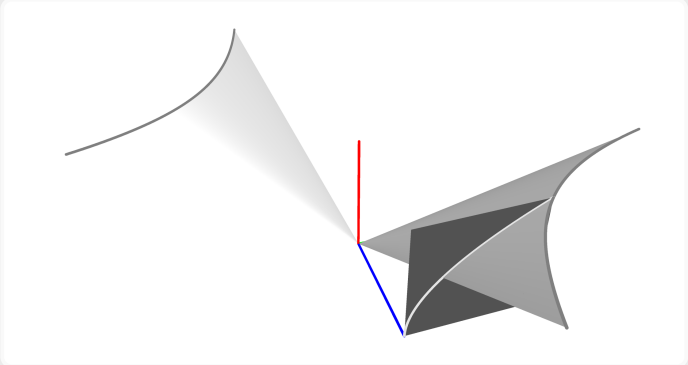
and homogenise to $xy = z^2$, as we see from three different perspectives:





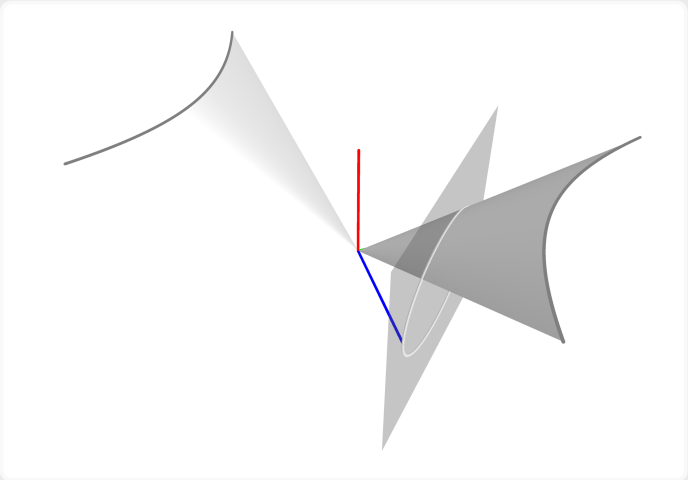


If we slice the surface along a plane like $z = 2$, parallel to the plane we drew but higher up, that plane $z = 2$ intersects our surface in a dilated hyperbola. But if we slice the surface along the plane $x = 1$, we get $xy = z^2$ to become $y = z^2$, a parabola:



Suitable linear changes of variable interchange the x and z axes, so suitable projective automorphisms identify the two affine curves: the hyperbola and the parabola. In particular, the two curves have isomorphic fields of rational functions. They *don't* have isomorphic rings of regular functions.

If instead we slice our surface with the plane $x + y = 1$,



we get $y = 1 - x$ plugged in to $xy = z^2$, which simplifies to

$$\left(x - \frac{1}{2}\right)^2 + z^2 = \frac{1}{4},$$

a circle of radius $\frac{1}{2}$. So again the circle is birational to the parabola and to the hyperbola, and they all have the same rational function fields. In these pictures we can see why the conics are called *conics*: they are all slices of the same cone.

The affine curve $x^2 + y^2 = -1$ has no real points on it, but has complex points. It homogenises to $x^2 + y^2 + z^2 = 0$, a "sphere of zero radius", with the origin as its only real point, but with again many complex points. The complex linear change of variables $X = x - iy, Y = x + iy, Z = iz$ gives us $XY = Z^2$, the same projective curve as before. So these curves are all birational over the complex numbers, and indeed are all identified by automorphisms of the complex projective plane.

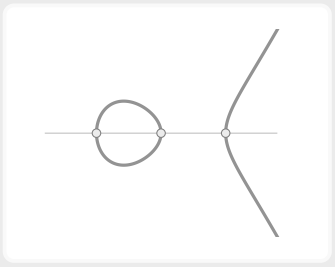
24.2 By the same trick of homogenising and dehomogenising in various variables, draw the same sort of pictures for the circle $x^2+y^2 = 1$ to see what cone it homogenises to, and what curves it makes when you intersect that cone with the planes (a) $x = 1$, (b) $y = 1$ and (c) $z = y + 1$.

Counting intersections of curves

Who are you going to believe, me or your own eyes?

— Chico Marx

A line through a cubic curve

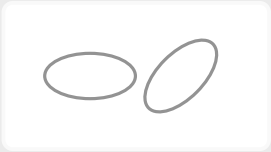


can hit as many as 3 points.

Two conics

can intersect at as many as 4 points, but sometimes only at 3 points:

Some conics don't appear to intersect at all

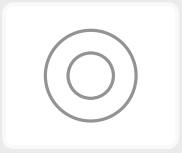


but keep in mind that our points can have coordinates in the algebraic closure, so they actually intersect.

The conics $x^2 + y^2 = 1$ and $(x-4)^2 + y^2 = 1$:

intersect at $(x, y) = (2, \pm i\sqrt{15})$.

Some conics appear not to have even complex intersections, such as $x^2+y^2 = 1$ and $x^2 + y^2 = 2$.



But in the projective plane, we can homogenise to $x^2 + y^2 = z^2$ and $x^2 + y^2 = 2z^2$, and then intersect with the plane $x = 1$, to get $1+y^2 = z^2$ and $1+y^2 = 2z^2$, which intersect at $(y, z) = (\pm i, 0)$.

24.3 Find all points of the curve $x^3 + x^2y + xy^2 + y^3 + x^2 = 1$ which lie "at infinity", i.e. on the line $z = 0$ in homogeneous coordinates, over the field $k = \mathbb{C}$.

24.4 Find all points of the curve $y^2 + y = x^3 + x + 1$ on the projective plane over the field $k = \mathbb{Z}/5\mathbb{Z}$.

Lemma 24.1. *Suppose that k is an infinite field. Then for any finite set of lines in $\mathbb{P}^2(k)$, there are infinitely many points of $\mathbb{P}^2(k)$ not on any of those lines.*

Proof. If we work in the affine plane, the points (x, y) lying on those lines must satisfy one of the associated linear equations, say $y = m_ix + b_i$ or $x = c_i$. Pick a number x_0 not among the constants c_i. Now we have finitely many values $y = m_ix_0 + b_i$ to avoid; pick y to not be one of those. □

Lemma 24.2. *Every nonzero homogeneous polynomial of degree n in two variables over a field k has n roots, counted with multiplicities, in $\mathbb{P}^1\big(\bar{k}\big)$, and so splits into a product of homogeneous linear factors over $\bar{k}$.*

Proof. As part of the proof, we define multiplicities of zeroes of a homogeneous polynomial $p(x, y)$ to be multiplicities of linear factors. So we are saying precisely that $p(x, y)$ factors into n linear factors. If $p(x, y)$ has a linear factor, divide it out and apply induction. Since y is not a linear factor of $p(x, y)$, $p(x, 1)$ has degree n, say $p(x, 1) = a(x - x_1) \dots (x - x_n)$. The homogeneous polynomial

$$r(x, y) := p(x, y) - a(x - x_1y)(x - x_2y) \dots (x - x_ny)$$

vanishes at $y = 1$ for any x, so by homogeneity vanishes for any $y \neq 0$. Fixing any value of x, there are infinitely many values of y at which $0 = r(x, y)$, and so $r(x, y)$ vanishes everywhere. □

The polynomial

$$p(x, y) = x^3 + x^2y - 2xy^2$$

becomes

$$p(x, 1) = x^3 + x^2 - 2x = x(x - 1)(x + 2),$$

so that

$$p(x, y) = x(x - y)(x + 2y).$$

The family of curves with $y^2 = ax$ contains the curve $y^2 = 0$ when $a = 0$, a line. So a family of curves of degree 2 contains a curve of degree 1. It is natural to think of the curve being a "double line" when $a = 0$.

Similarly, we can say that a *multiple component* of a curve means a curve given by an equation which is a positive integer power of an irreducible homogeneous polynomial, say $0 = f(x, y, z)^k$ giving a component with multiplicity k. From now on, we allow projective plane curves to have components with positive integer multiplicities.

Breaking curves into components

Theorem 24.3 (Study)**.** *Every algebraic curve in the projective plane has a unique decomposition into irreducible algebraic curves, its* components*. To be more precise, every nonzero homogeneous polynomial in* 3 *variables $f(x, y, z)$ has a decomposition*

$$f = a_0 f_1^{k_1} f_2^{k_2} \dots f_n^{k_n},$$

into a product of a nonzero constant a_0 and several coprime irreducible homogeneous polynomials $f_j(x, y, z)$ to various positive integer powers, unique up to rescaling the constant and the polynomials by nonzero constants, and writing down the polynomials in perhaps a different order.

Proof. Suppose that g is an irreducible polynomial and that $f = 0$ at every point in $\bar{k}$ where $g = 0$. We want to prove that g divides f. Work in the affine plane $z = 1$. Think of each equation as a polynomial in one variable x, but with coefficients rational functions of y. By corollary 20.5 on page 179, after perhaps a linear change of variables, the resultant in y vanishes just at those x values where there are simultaneous roots. In particular, the number of different values of x where such roots occur is never more than the degree of the resultant, as a polynomial in x. For each value $x = c$ (over $\bar{k}$) for which $g(c, y)$ is not constant, the values of y at which $g(c, y) = 0$ also satisfy $f(c, y) = 0$, so the resultant vanishes. Therefore this resultant is the zero polynomial. If the curve where $f = 0$ is the union of distinct irreducible curves given by equations $g_i = 0$, then each of these g_i divides f, so their product divides f. Apply induction on the degree of f. □

Theorem 24.4. *Every rational morphism of projective plane algebraic curves $f\colon B \to C$ is constant or onto a finite union of components of C, at most one for each component of B.*

Proof. We can assume that B is irreducible, and we have already assumed that our field is algebraically closed by definition of the points of the curve. Our rational morphism is

$$f(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z)).$$

If $B = (b(x, y, z) = 0)$ then the system of equations

$$b(a, b, c) = 0, a = a(x, y, z), b = b(x, y, z), c = c(x, y, z)$$

has (after perhaps a linear change of x, y, z variables) resultant eliminating x, y, z given by some algebraic equation in a, b, c which specifies which points lie in the image. By corollary 20.5 on page 179 this equation is not trivial. Split C into irreducibles $C_1, C_2, \ldots, C_N$, say $C_j = (c_j = 0)$. Then $c_1 c_2 \cdots c_N \circ f = 0$ on B, since f takes B to C. So if $B = (b(x, y, z) = 0)$ then b is divisible by $c_1 c_2 \cdots c_N \circ f$ in any affine plane given by taking $x = 1$ or $y = 1$ or $z = 1$. If $c_1 \circ f$ is not zero on B then divide by it to see that already $c_2 \cdots c_N \circ f = 0$ on B. So f takes B into $C_2 \cup \cdots \cup C_N$. We can assume that $c_j \circ f = 0$ on B for every j, i.e. that $B \subset f^{-1} C_j$ for every j, i.e. that

$$f(B) \subset \bigcap_j C_j :$$

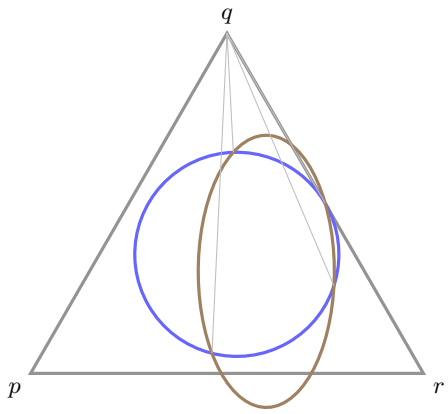
there is only one such C_j, i.e. C is irreducible. Hence the image is precisely C. □

Counting intersections

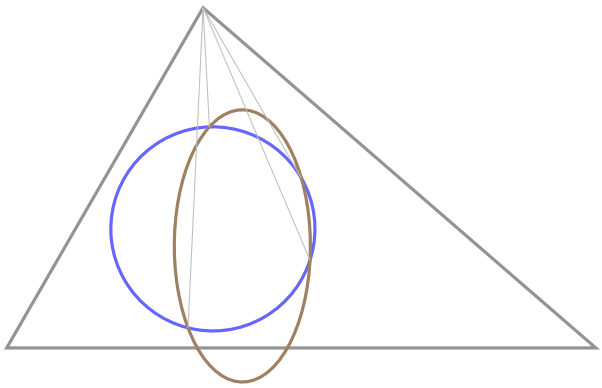
A triangle pqr is *generic* for two algebraic projective plane curves B, C if

a. neither curve passes through q and

b. no line connecting two intersection points of B and C lies through q.



A triangle pqr is *very generic* for curves B and C if it is generic and there is no intersection point of B and C along the line qr. Keep in mind that we allow intersection points, and points of the triangle, to have coordinates in the algebraic closure of our field.



By projective automorphism, we can make any triangle become the one whose vertices are

$$[0,0,1],[1,0,0],[0,1,0].$$

This projective automorphism is unique up to rescaling the variables. In the affine plane, these vertices are the points $(0,0),(\infty,0),(0,\infty)$. Suppose that our two curves have equations $B = (b(x,y) = 0)$ and $C = (c(x,y) = 0)$. The triangle is generic just when

a. $(0,\infty) = [0:1:0]$ is not on B or C, and

b. no vertical line intersects two intersection points of B and C.

In a more algebraic description:

a. $b(x,y)$ and $c(x,y)$ both have nonzero constants in front of their highest terms in y and

b. for any constant $x = x_0$, $b(x_0,y)$ and $c(x_0,y)$ have at most one common root in y.

If the triangle is generic, it is also very generic just when there is no zero of the homogeneous polynomials $b(x,y,z)$, $c(x,y,z)$ on the line at infinity, i.e. the line $z = 0$, or, in other words, the highest order terms of $b(x,y)$ and $c(x,y)$ (which are homogeneous of the same degree as $b(x,y)$ and $c(x,y)$) have no common root on $\mathbb{P}^1$. The resultant $\mathrm{res}_{b,c}(x)$ in y depends, up to constant factor, only on the choice of curves B and C and the choice of triangle. The *intersection multiplicity* of B and C at a point $p = (x,y)$, denoted BC_p, is the multiplicity of x as a zero of $\mathrm{res}_{b,c}(x)$. If p is not a point of both B and C then define $BC_p := 0$. If p lies on a common component of B and C then let $BC_p := \infty$. Let $BC := \sum_p BC_p$ summed over all points p, called the *intersection number*. Danger: so far, the intersection multiplicity and intersection number both depend on the choice of triangle.

Let $b(x,y) := y + x$ and $c(x,y) := y$. The intersection points of the lines $(b(x,y) = 0)$ and $(c(x,y = 0)$ in the affine plane are at $0 = y = y + x$, i.e. at the origin. In the projective plane, there is no further intersection point, as the equations are already homogeneous. So the standard triangle is very generic for these two lines. The resultant of $b(x,y), c(x,y)$ in y is $r(x) = x$. So the two lines $B = (y + x = 0)$ and $C = (y = 0)$ have $BC_{(0,0)} = 1$. Similarly

any two lines have intersection multiplicity 1 at their intersection point.

Let $B := (xy^2 + y = 0)$ and $C := (xy^2 + y + 1 = 0)$. The curves intersect in the affine plane at

$$(x, y) = \left(-2, -\frac{1}{2}\right)$$

(assuming that $2 \neq 0$ in our field). Homogenize to find also the intersection point $(0, \infty) = [0 : 1 : 0]$. So the coordinate triangle is not generic. As previously, if we change x to $x + \lambda$ for any constant $\lambda \neq 0$, our curves change to

$$B = (\lambda y^3 + xy^2 + y = 0), C = (2\lambda y^3 + 2xy^2 + y + 1).$$

They now intersect at

$$(x, y) = (-1 - \lambda, 1) = [-1 - \lambda, 1, 1].$$

(We previously saw this from looking at the resultant $r(x) = -\lambda^2(\lambda + 1 + x)$.) After homogenizing and then setting $y = 1$, we also find another intersection at $[x, y, z] = [-\lambda, 1, 0]$. The standard triangle is generic for these curves, since the two intersection points lie on different lines through $[0, 1, 0]$. But it is not very generic, because our second intersection point lies outside the affine plane, on the line at infinity. We can only calculate the intersection multiplicity at the first point, where the resultant vanishes to degree 1, so an intersection multiplicity of

$$BC_{(-\lambda-1,1)} = 1.$$

The other point is not in the affine plane, so intersection multiplicity is not defined at that point, at least by our definition.

Theorem 24.5 (Bézout)**.** *Over any field, take two projective algebraic plane curves B and C not sharing a common component. Over some finite degree extension of the field there is a very generic triangle for the curves. For any generic triangle, $BC \leq \deg B \deg C$. A generic triangle is very generic just when equality holds.*

Proof. Split B and C into irreducibles and add up intersections by multiplying resultants: without loss of generality we can assume that B and C are irreducible.

By proposition 16.12 on page 139, there are at most $\deg B \deg C$ values of x that occur on intersection points, and by the same reasoning at most that number of y values, so finitely many intersection points.

Since there are finitely many intersection points, there are finitely many lines through them. By lemma 24.1 on page 212 we can pick one vertex of our triangle to avoid lying on any of them (after perhaps replacing by an extension field), while arbitrarily picking the other two vertices to be any two distinct points. So we have a generic triangle, and we can assume that its vertices are $(0, 0)$, $(\infty, 0)$ and $(0, \infty)$. We can then move the line at infinity to avoid the intersection points (again in some extension field), so there is a very generic triangle.

Pick any generic triangle. No two intersection points lie on any line through $(0, \infty)$, i.e. on any vertical line: all of the intersection points have distinct x values. There is no intersection point at $(0, \infty)$, i.e. the highest order terms in y in the equations have no common zero. Therefore even when $b(a, y)$ drops degree at some value of $x = a$,

$c(a, y)$ doesn't, so the resultant is the product of the values of $b(a, y)$ at the roots y of $c(a, y)$, vanishing precisely at the values of x where the curves have an intersection. In the proof of proposition 16.12 on page 139, we found the degree of the resultant to be at most $\deg b \deg c$ and nonzero. There are at most a number of intersection points given by the degree of the resultant as a polynomial. Hence the sum of the intersection multiplicities over the intersections in the affine plane is the sum of the degree of root over all roots of the resultant, so at most the degree of the resultant. Our triangle is very generic just when this sum covers all intersection points, as there are none on the line at infinity.

We still need to see that for our very generic triangle, the degree of the resultant is precisely $\deg B \deg C$. The assumption of being very generic is expressed algebraically as saying that the homogeneous polynomials $b(x, y, z), c(x, y, z)$ have no common roots along $z = 0$. In other words, $b(x, y, 0)$ and $c(x, y, 0)$ have no common roots on $\mathbb{P}^1$. So the resultant of $b(x, y, 0), c(x, y, 0)$ in y is nonzero, and has degree $\deg B \deg C$. But the degree of resultant of $b(x, y, z), c(x, y, z)$ can be no less for variable z as for the fixed value $z = 0$. Therefore the resultant has the required degree. □

Theorem 24.6. *To any two projective algebraic plane curves B and C over a field k and any point p of the projective plane defined over the algebraic closure $\bar{k}$ of k, there is a unique quantity BC_p, the* intersection multiplicity*, so that*

a. *$BC_p = CB_p$ and*

b. *$BC_p = 0$ just when p does not lie on any common point of B and C defined over $\bar{k}$ and*

c. *$BC_p = \infty$ just when p lies on a common component of B and C defined over $\bar{k}$ and*

d. *BC_p is a positive integer otherwise and*

e. *two distinct lines meet with intersection multiplicity 1 at their unique point of intersection and*

f. *if B splits into components (perhaps with multiplicities) B_1 and B_2, then $BC_p = B_1C_p + B_2C_p$ and*

g. *if B and C are defined by homogeneous polynomials $b(x, y, z)$ and $c(x, y, z)$ and E is the curve defined by $bh + c$ where $h(x, y, z)$ has degree $\deg h = \deg b - \deg c$, then $BC_p = BE_p$.*

Moreover, BC_p can be computed as defined above using any generic triangle.

Proof. In this proof, we write bc_p instead of BC_p if B is cut out by the equation $0 = b(x, y, z)$ and C by $0 = c(x, y, z)$. First we prove that the conditions *a.–g.* determine the multiplicity uniquely. Since they are independent of choice of affine chart, this ensures that the multiplicity is also independent. We then check that our definition above satisfies these, so must be determined by these conditions independent of the choice of affine chart made in the definition above.

Any two notions of multiplicity agree on common components of B and C, and on points not belonging to B or C, by the conditions above. So we only need to check points of positive finite multiplicity. We can assume by induction that B and C are both irreducible, and cut out by irreducible homogeneous polynomials b and c. Suppose that we have two different methods of calculating an intersection multiplicity satisfying our various conditions, one of which we can take to be the definition by the

resultant above. By induction suppose that they agree wherever they both assign a value less than the larger of the two values that they assign as bc_p.

Take affine coordinates in which $p = (0, 0)$. If $\deg b(x, 0) = 0$ then $b(x, 0) = 0$ so $b(x, y)$ is divisible by y, not irreducible, so our result holds by induction. The same holds if $\deg c(x, 0) = 0$. So both b and c have positive degree in each of x and y when the other variable is set to zero. Rescale to get both to have unit coefficient in x:

$$b(x, y) = x^\beta + \dots,$$
$$c(x, y) = x^\gamma + \dots.$$

Suppose (after perhaps swapping the names of b and c) that $\beta \leq \gamma$ and let

$$h(x, y) := c(x, y) - x^{\gamma-\beta} b(x, y).$$

By construction, $h(x, 0)$ has degree in x at most $\gamma - 1$. Our last property of intersection multiplicity above demands that $bc_p = bh_p$. Therefore we can replace c by h and repeat, lowering the degree in x by induction. This proves uniqueness of intersection multiplicities satisfying our axioms.

We want to prove existence, i.e. that the recipe we defined above satisfies our axioms above.

a. Resultants change sign when we swap polynomials, so $\operatorname{res}_{b,c} = -\operatorname{res}_{c,b}$.

b. Follows from theorem 24.5 on page 216.

c. Follows from theorem 24.5 on page 216.

d. Follows from theorem 24.5 on page 216.

e. Follows by direct computation of the resultant.

f. Follows from lemma 15.5 on page 120.

g. Follows from proposition 15.4 on page 119, and the existence of splitting fields.

□

We restate Bézout's theorem without reference to generic triangles, since the previous theorem proves that they are irrelevant.

Theorem 24.7 (Bézout)**.** *Over any field k, take two projective algebraic plane curves B and C not sharing a common component. Over the algebraic closure $\bar{k}$ of the field, the intersection multiplicities of the intersection points sum to the intersection number, which is*

$$BC = \deg B \deg C.$$

24.5 Prove that any nonconstant rational function on a curve has poles, perhaps over a finite degree extension field.

Sage

Pieter Belmans wrote the following sage code to find the intersection multiplicity of two algebraic curves at a point. First we find the minimum degree of any term in a polynomial $f(x)$ of one variable.

```
def ldegree(f):
    minimum = infinity
    for (n, m) in f.dict():
        minimum = min(minimum, n)
    return minimum
```

Given any two polynomials b, c in two variables, we want to figure out what the two variables are:

```
def determine_variables(b, c):
    if len(b.variables()) == 2:
        return b.variables()
    if len(c.variables()) == 2:
        return c.variables()
    if len(b.variables()) == len(c.variables()) == 1:
        if b.variables() == c.variables():
            return (b.variable(0), 0)
        else:
            return (c.variable(0), b.variable(0))
    return (0,0)
```

Finally, we use our definition of intersection multiplicity, and induction, to compute intersection multiplicities recursively.

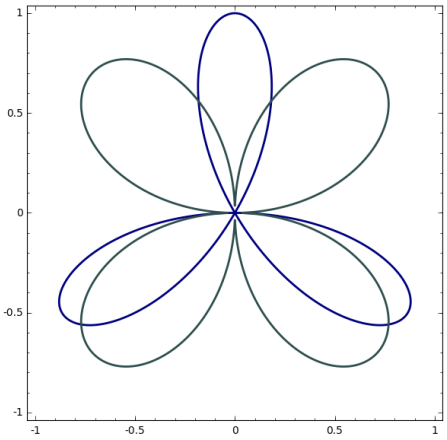
```
def intersection_multiplicity(b, c, point = (0,0)):
    (x,y) = determine_variables(b, c)
    # translate both curves to origin and calculate it there
    b = b.subs({x:x + point[0], y:y + point[1]})
    c = c.subs({x:x + point[0], y:y + point[1]})
    # if $b(0,0)\neq 0$ or $c(0,0)\neq 0$ they don't intersect in the origin
    if b.subs({x:0, y:0}) != 0 or c.subs({x:0, y:0}) != 0:
        return 0
    # if $b$ or $c$ are zero they don't intersect properly
    if b == 0 or c == 0:
        return Infinity
    # we only look at factors of $x$
    f = b.subs({y:0})
    g = c.subs({y:0})
    # $b$ contains a component $y=0$
    if f == 0:
        # $c$ contains a component $y=0$ too, no proper intersection
        if c == 0:
            return infinity
        # remove common $y^n$ in $b$, count degree of $x$ in $c$ and recurse
        else:
            f = b.quo_rem(y)[0]
            return ldegree(g) + intersection_number(f, c)
    # $b$ does not contain a component $y=0$
    else:
        # $c$ *does* contain a component $y=0$
        if g == 0:
```

```
        g = c.quo_rem(y)[0]
        return ldegree(f) + intersection_number(b, g)
    # we recurse by removing factors of $x$
    else:
        p, q = f.lc(), g.lc()
        r, s = f.degree(), g.degree()
        # we drop the highest degree term
        if r <= s:
            return intersection_multiplicity(b, p*c - q*x^(s-r)*b)
        else:
            return intersection_multiplicity(q*b - p*x^(r-s)*c, c)
```

A test run:

```
P.<x,y> = PolynomialRing(QQ)
b = P(x^2+y^2)^2+3*x^2*y-y^3
c = P(x^2+y^2)^3-4*x^2*y^2
intersection_multiplicity(b,c)
```

yields 14, the intersection multiplicity at the origin of the curves:



Chapter 25

Families of plane curves

All happy families are alike; each unhappy family is unhappy in its own way.

— Leo Tolstoy
Anna Karenina

Pencils of curves

Every curve C of degree n has a homogeneous equation $c(x, y, z) = 0$ of degree n, unique up to rescaling, by Study's theorem (theorem 24.3 on page 213). Take two curves B and C of the same degree n, with equations $b(x, y, z) = 0$ and $c(x, y, z) = 0$. The level sets

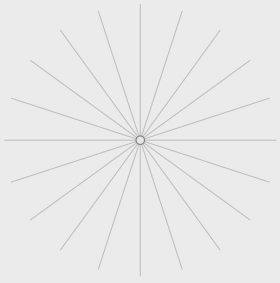
$$\frac{b(x, y, z)}{c(x, y, z)} = r$$

of the ratio form a collection of curves C_r, called the *pencil* of curves containing B and C. The curve B is C_0 while C is C_∞. Every point $[x, y, z]$ in the plane lies on a unique one of these curves, with value r given by taking the value of $b(x, y, z)/c(x, y, z)$ at that point, except for the points in $B \cap C$, for which the ratio is never defined. To be more precise, each curve C_r can also be written as

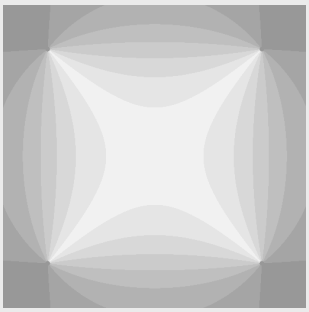
$$b(x, y, z) = r\, c(x, y, z),$$

(except for $C_\infty = C$). The points of $B \cap C$ belong to *all* curves C_r of the pencil. Moreover the degree of C_r is the same as that of B and C.

The lines through a point form a pencil: if $B = (x = 0)$ and $C = (y = 0)$ then $C_r = (x = ry)$.

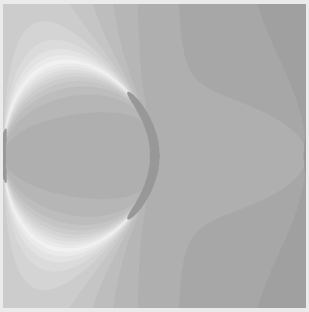


The pencil of conics containing the circle $x^2 + y^2 = 1$ and the reducible conic $x^2 = y^2$ (which is a pair of lines $x = \pm y$):



Look at the 4 points where the circle intersects the two lines; these points lie in every conic in the pencil.

The pencil of cubics containing $y^2 = x + x^3$ and $xy^2 = 0$:



Danger: for some values of r, C_r might actually be of lower order than B or C. For example, if $B = (y^2 = x)$ and $C = (-y^2 = x)$, two conics, the pencil has curves C_r with equation

$$(1 - r)\, y^2 = (1 + r)x,$$

which drops order at $r = 1$, i.e. C_1 is a line. But note that if we homogenize,

$$(1 - r)\, y^2 = (1 + r)zx,$$

the curve C_1 is actually a pair of lines. In a pencil C_r for which C_0 and C_∞ are irreducible, we rarely but occasional find reducible curves, as in this example, and we might picture that the moving curve C_r "bubbles" into reducible components.

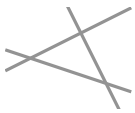
Lemma 25.1. *Suppose that two projective plane curves B and C of order n intersect at exactly n^2 points (the maximum possible by Bézout's theorem). Write n as $n = p+q$. If exactly pn of these points lie on a curve E of degree p (again the maximum possible) then the remaining points, qn of them, lie on a curve F of degree q (again the maximum possible).*

Proof. Write B and C as C_0 and C_∞ in a pencil C_r. Pick a point s different from the intersection points of C_0 and C_∞. We can pick such a point, because we can replace our field by its algebraic closure if needed, or more simply by any infinite field containing. The numbers of intersection points can't increase when we work in a larger field, since Bézout's theorem tells us that they are maximal already. The point s then belongs to a unique element of the pencil, say to C_{r_0}.

In particular, we can pick s to belong to E. But then s sits in C_{r_0} along with the points of $B \cap C$ and s also sits in E along with pn points of $B \cap C$. So E and C_{r_0} contain $1 + pn$ points of $B \cap C$. By Bézout's theorem, the degrees of E and C_{r_0} are p and n, they contain a common irreducible component. Since E is irreducible by assumption, E is that component of C_{r_0}. So $C_{r_0} = E \cup F$, for some curve F of degree at most q, i.e. the pencil bubbles into two components. The remaining intersection points of $B \cap C$ don't lie in E, but all lie in C_{r_0}, so they lie in F. □

Polygons

In a projective plane, we have a concept of line, but not of line segment, or of angle or distance. In a projective plane a *triangle* is a triple of points not all on the same line, together with the lines that connect them. So a triangle looks like

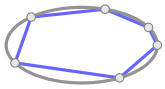


rather than

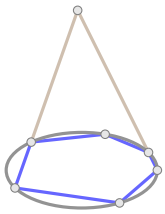
Similarly, a *quadrilateral* is a quadruple of points, called the *vertices*, no three on the same line, with a chosen ordering defined up to cyclically permuting, so there is no first point or second point, but if you pick a point from the quadruple, there is a next point, and a next, and so on until you come back to the point you started with. In the same way, we define a *polygon*. The lines through subsequent vertices are the *edges*. We can't define the notion of a square or rectangle since there is no notion of length or angle available.

Pascal's mystic hexagon

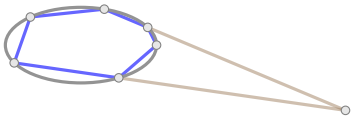
Draw a hexagon with vertices sitting inside a conic.



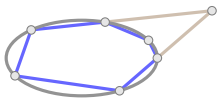
Since there is an even number of sides, each side has an "opposite" side, half way around the hexagon. Pick a side and its opposite, and draw lines joining them.



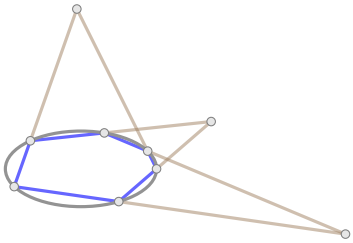
The same for the next pair of opposite sides.



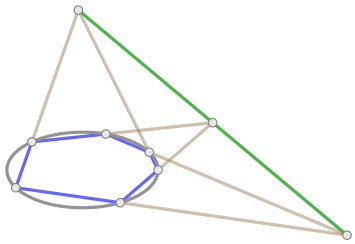
And finally for the third pair of opposite sides.



Draw all of these lines together in one picture.



Mysteriously, all three intersection points lie on a line.



Theorem 25.2. *Given a hexagon in the projective plane over a field, the intersection points of opposite sides lie on a line.*

Proof. Let B be the reducible cubic curve consisting of three of the lines, no two of them in succession (for example, the first, third and fifth lines). Let C be the reducible cubic curve consisting of the remaining three lines of the hexagon. Let E be the quadric. The result follows immediately from lemma 25.1 on page 223. □

The space of curves of given degree

Consider the vector space V_d of homogeneous polynomials of degree d in 3 variables. Each such polynomial has a coefficient of $x^a y^b z^c$ as long as $a+b+c=d$. Imagine that we take $d+2$ sticks sitting in a row. Colour 2 of them white and the rest black. Take a to be the number of black sticks before the first white one, b the number of black sticks between the two white ones, and c the number of black sticks after the second white one. Clearly $a+b+c=d$. So the dimension of V_d as a vector space is

$$\binom{d+2}{2}.$$

But since the equation $p_C = 0$ of a curve C is defined only up to rescaling, the curves of degree d are identified with points of $\mathbb{P}^{d^*}(k)$ where

$$d^* = -1 + \binom{d+2}{2} = \frac{d(d+3)}{2}.$$

d	d^*
1	2
2	5
3	9
4	14
5	20
6	27
7	35
8	44
9	54

The set of lines in $\mathbb{P}^2(k)$ is a 2-dimensional projective space, a projective plane, the *dual projective plane*.

More generally, our projective space $\mathbb{P}^{d^*}(k)$ contains all curves of degree d, reducible or irreducible.

A pencil $C_r = (p = rq)$ of two curves B and C is just a line inside $\mathbb{P}^{d^*}(k)$. More generally, a *linear system* is collection of plane curves whose equations form a projective linear subspace in $\mathbb{P}^{d^*}(k)$.

The condition that a point $q_0 = (x_0, y_0, z_0)$ lies in a curve $C = (c(x, y, z) = 0)$ is a linear equation $c(x_0, y_0, z_0) = 0$ on the coefficients of the polynomial $c(x, y, z)$, so is satisfied by a linear projective hyperplane inside $\mathbb{P}^{d^*}(k)$. For example, the condition that a point lie in a line is one equation, so determines a line in the dual projective plane. Similarly, the condition that a curve has order at least k at a point q_0 of the plane is a collection of $k(k+1)/2$ linearly independent equations, killing the terms in the Taylor series up to order k (if we arrange that q_0 is the origin, for example). A collection of points $p_1, p_2, \ldots, p_n$ in the projective plane is in *general position* for curves of degree d if the linear system of curves passing through them has either (1) the minimal dimension possible $d(d+3)/2 - n$ if this quantity is not negative and (2) is empty otherwise.

Lemma 25.3. *The definition of general position makes sense, in that if we fix n points, then the linear system consisting of all degree d curves through those points has dimension at least*

$$d^* - n = \frac{d(d+3)}{2} - n.$$

After perhaps replacing our field with an infinite extension field, if $d^* - n \geq 0$ then equality is acheived for some collection of n points, while if $d^* - n < 0$ then there is a collection of n points not lying on any curve of degree d. In particular, if $d^* = n$, then (after perhaps an infinite field extension) there is a set of n points for which there is a unique curve of degree d passing through those n points.

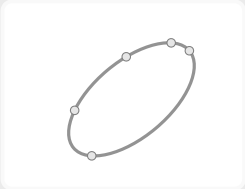
Proof. The proof is by induction. For $d = 1$, there is a pencil of lines through a point ($n = 1$) and a unique line through any 2 points ($n = 2$) and no line through 3 or more points, if we pick our 3 points correctly. Suppose that the result is true for all values of n and d less than some particular choices of n and $d \geq 2$. Note that

$$d^* = \frac{d(d+3)}{2} = d + 1 + \frac{(d-1)(d-1+3)}{2} = d + 1 + (d-1)^*.$$

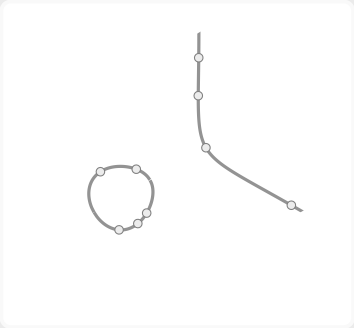
Pick $d+1$ points on a line L, and then pick $(d-1)^*$ points in general position, so that they lie on a unique curve C of degree $d-1$. By Bézout's theorem, the line L is a component of any curve D of degree d through the first $d+1$ points. Therefore such a curve D is just $C \cup L$. Hence D is the unique curve of its degree through the given points. So through some collection of $d(d+3)/2$ points, there is a unique curve of degree d. If we add any more points, choosing them not to lie on the that curve, then there is no curve of degree d through those points. If we take away any points, say down to some number n of point, then the dimension goes up by at most one for each removed point, giving us our result by induction. □

Through any 2 points there is precisely one line:

Through any 5 points in general position, there is precisely one conic.



Through any 9 points in general position, there is precisely one cubic.



Through any 14 points in general position, there is precisely one quartic.

Through any 20 points in general position, there is precisely one quintic.

Lemma 25.4. *Through any* 5 *points in the plane, there is a conic.*

- *No three of the points lie on a line just when the conic is irreducible.*
- *No four of the points lie on a line just when the conic is unique.*
- *Four but not five of the points lie on a line just when the conics through all five form a pencil, with each curve in the pencil being a pair of lines: the line through the four colinear points, and any line through the fifth point.*
- *All five points lie on a line just when the conics form a* 2*-dimensional linear system, with each curve in the system being a pair of lines: the line through the five points, and any other line.*

Proof. The space of conics through four points has dimension at least $d^* - 4 = 2^* - 4 = 5 - 4 = 1$. Indeed there is a pair of lines covering those four points. Add a fifth point, not colinear with any three of the other four. The condition of containing the fifth point is an additional linear equation on the quadratic polynomial of the conic, independent of those for the first four points, as it is not satisfied for that pair of lines. So the linear system of conics through the five points is a smaller dimensional projective space, but still not empty.

If no three of the points are colinear, then repeating the argument above, each point adds an independent condition, not a linear combination of the others, so there is a unique conic through the five points. If that conic is reducible, it is a pair of lines, and then all five points lie on two lines, so at least three points lie on one of the lines. Since some line contains three of the points, by Bézout's theorem, that line is a component of the conic, so the conic must be a pair of lines, one of which is determined.

If no line contains four of the points, then each line is determined by the two points it contains which are not on the other line, and the conic is uniquely determined. □

Prescribing singularities

Lemma 25.5. *The curves of a given degree d which pass through a given point of the plane, having a singularity of order n or more at that point, form a linear system of codimension $n(n+1)/2$ in the degree d curves, if $d \geq n$.*

Proof. There is such a curve, since $d \geq n$, so the space of these curves is not empty. Take the point to be the origin in affine coordinates. Then the polynomial equations of such curves are those with all terms of degree less than n vanishing, so we can count the terms. □

Theorem 25.6. *The curves of a given degree d which pass through given points, having given orders $n_1, n_2, \ldots, n_k$ at given points $p_1, p_2, \ldots, p_k$ form a linear system of codimension no more than*

$$\frac{1}{2}\sum n_j(n_j+1).$$

Proof. We intersect the linear systems of the previous lemma. □

Genera

The *arithmetic genus* of an irreducible plane algebraic curve of degree d is $g = (d-2)(d-1)/2$. Its *geometric genus*, also called its *deficiency*, is

$$\frac{(d-2)(d-1)}{2} - \frac{1}{2}\sum n_j(n_j-1),$$

where $n_1, \ldots, n_k$ are the orders of the singularities.

Theorem 25.7. *The geometric genus of any irreducible plane algebraic curve is not negative.*

Proof. For a line or irreducible conic, there are no singularities and the arithmetic genus is zero. So we can assume the degree of our curve is 3 or more. By lemma 27.2 on page 241,

$$\frac{1}{2}\sum n_j(n_j-1) \leq d(d-1).$$

Consider the spaces of all curves of degree $d-1$ which have singularities at the same points as our given curve, and with multiplicity no less than $n-1$ wherever our

curve has a singular point of order n. By theorem 25.6 on the preceding page, the set of all such curves is a linear system, if not empty, of codimension

$$\frac{1}{2}\sum_j n_j(n_j-1)$$

so, if not empty, has dimension

$$\delta := \frac{(d-1)(d+2)}{2} - \frac{1}{2}\sum_j n_j(n_j-1).$$

Moreover, if $\delta \geq 0$ then the linear system is not empty. The space of all curves of degree $d-1$ has dimension $(d-1)(d+2)/2$, and

$$\frac{1}{2}\sum_j n_j(n_j-1) \leq d(d-1) < \frac{1}{2}(d-1)(d+2).$$

So $\delta > 0$ and our space of curves is not empty: there are curves of degree $d-1$ with orders at least $n_1-1,\ldots,n_k-1$ at the given points.

Denote the singular points by $p_1, p_2, \ldots, p_k$ and the curve as C. After perhaps taking an algebraic extension of the field we work over, we can find δ regular points on our curve, say $q_1, q_2, \ldots, q_\delta$. Take the linear system of curves of degree $d-1$ with singularity of order at least $n_j - 1$ at each point p_j, and passing through $q_1, q_2, \ldots, q_\delta$. Each point q_j imposes at most one linear condition, so there is at least one curve D in this linear system.

Curve D has lower degree than C, and C is irreducible, so D and C meet only at finitely many points. By Bèzout's theorem (theorem 24.7 on page 218), C and D intersect with multiplicities summing to $d(d-1)$. They have the same singular points, where orders are at least n_j and $n_j - 1$, and then they intersect at δ regular points, so

$$d(d-1) \geq \sum_j n_j(n_j-1) + \delta.$$

□

Independence of linear equations on curves

A points $p_1, p_2, \ldots, p_n$ in the plane *impose independent conditions on curves of degree* d if the linear system of curves through those points has dimension $d^* - n$ (or is empty if $d^* - n < 0$). Lemma 25.3 on page 226 says that (after perhaps a field extension) failure to impose independent conditions on curves of degree d occurs just when the linear system of curves through those points has dimension more than $d^* - n$ (or is not empty, if $d^* - n < 0$), and also says that we can choose n points that impose independent conditions. We want to examine the failure of small numbers of points to impose independent conditions.

Imagine we try to prove that points $p_1, p_2, \ldots, p_n$ impose independent conditions on curves of degree d. Suppose we prove that there is a curve of degree d passing through $p_1, p_2, \ldots, p_{n-1}$ but not p_n. In the vector space of degree d polynomials, the condition of passing through p_n as well is an additional linear equation not always satisfied, so cuts out a linear subspace of one lower dimension. If the choice of which

point is p_n is arbitrary, then the same will hold for any of the points p_i in place of p_n, giving n linearly independent equations, i.e. a linear subspace of n lower dimensions, so the points impose independent conditions on curves of degree d.

Theorem 25.8. *Suppose that $p_1, p_2, \ldots, p_n$ are distinct points in the plane, and $n \leq 2d+2$. Then these points fail to impose independent conditions on curves of degree d just when either*

- *$d+2$ of the points are colinear or*
- *$n = 2d+2$ and the points $p_1, p_2, \ldots, p_n$ lie on a conic.*

Proof. Suppose that $d+2$ of the points are colinear, say on a line L, and say they are $p_1, p_2, \ldots, p_{d+2}$. By Bézout's theorem, any curve of degree d containing those points contains L so has equation factoring into a product of the equation of L and an equation of degree $d-1$. The set of curves of degree d containing L is a projective space of dimension $(d-1)^*$, and so imposing that our curve has degree d and contains L is imposing $d^* - (d-1)^* = d+1$ conditions. The remaining points are $p_{d+3}, \ldots, p_n$, so $n-(d+2)$ points, and so together impose at most $n-(d+2)$ conditions, i.e. $p_1, \ldots, p_n$ together impose at most $d+1+n-(d+2) = n-1$ conditions at most, so the points fail to impose independent conditions.

Suppose that $n = 2d+2$ and that the points $p_1, p_2, \ldots, p_n$ lie on a conic. Construct a curve of degree d through the points by constructing a conic through the points, and then constructing any curve of degree $d-2$, i.e. $(d-2)^*$ dimensions of curves at least. If the points were to impose independent conditions, we would have $d^* - n = d^* - (2d+2)$ dimensions of curves. But

$$(d-2)^* = \frac{d^2-d-2}{2} > \frac{d^2-d-4}{2} = d^* - n.$$

Now we suppose that $p_1, p_2, \ldots, p_n$ fail to impose independent conditions, and that $n \leq 2d+2$. Some special cases to get started:

- $n = 0$: no conditions, so failure is impossible, i.e. d^* dimensions of curves.
- $n = 1$: one point cannot fail to impose independent conditions, as there is a curve of any degree $d > 0$ not passing through that point (at least after a field extension).
- $n = 2$: for any two points, take d lines through one point none of which contain the other (at least after a field extension). Their union is a degree d curve.
- $n = 3$: if the points are not colinear, a line through two of the points and not the third, taken d times, is a degree d curve. If the points are colinear, and $d = 1$, they fail to impose independent conditions, but satisfy our theorem. So we can suppose that the points are colinear, and $d \geq 2$. Pick three more points q_1, q_2, q_3 so that the five points p_1, p_2, q_1, q_2, q_3 contain no three colinear points. (We can do this after perhaps a field extension.) The conic through these five points is irreducible, so does not contain the line through p_1, p_2, p_3, by Bézout's theorem. Add some lines to that conic, none of them through p_3, to get a degree d curve through p_1, p_2 missing p_3.
- $d = 1$: degree d curves are lines. We have $n \leq 2d+2 = 4$ points. Our n points fail to impose independent conditions on lines just when either
  - $n \geq 3$ and the points are colinear or

  - $n = 4$: all 4 points lie on a conic, for example on a pair of lines.

- $d = 2$: degree d curves are conics. Suppose that no $d + 2 = 4$ points lie on a line. By hypothesis, $n \leq 2d + 2 = 6$, so we need to consider $n = 4, 5, 6$. The conic through any 5 of the points is unique, by lemma 25.4 on page 227.
  - $n = 5$: the points impose independent conditions because $d^* - n = 0$ and the space of conics through all 5 points is zero dimensional.
  - $n = 4$: add a fifth point not on any line through the four points, and get a unique conic through all five, so the five points impose independent conditions, and so the four points do.
  - $n = 6$: any five of the points lie on a conic, so the sixth point fails to give an independent condition just when all six lie on the same conic.
- $n \leq d + 1$: pick lines $L_1, L_2, \ldots, L_{n-1}$, each line L_i passing through the point p_i and no other of the points. Then $C := L_1 \cup L_2 \cup \cdots \cup L_{n-1}$ has degree $n - 1$. If $n - 1 < d$ then pick some curve D of degree $d - n + 1$, not passing through p_n, and we get a curve $C \cup D$ of degree d through all but one point. To get this to work, we need $n - 1 \leq d$, i.e. $n \leq d + 1$.

By induction, suppose we have proven our result for all smaller values of d, and, for the given d, for all smaller values of n. Suppose that among $p_1, \ldots, p_n$, the points $p_1, \ldots, p_{d+1}$ lie on a line L. If any more of our points lie on that line, the result is proven. Let $d' := d - 1$ and $n' := n - (d + 1)$ and let

$$p'_1, \ldots, p'_{n'}$$

be the points

$$p_{d+2}, \ldots, p_n.$$

If $p'_1, \ldots, p'_{n'}$ impose independent conditions on curves of degree d', then we find a curve C of degree $d' = d - 1$ through all but one of $p_{d+2}, \ldots, p_n$, and then $L \cup C$ has degree d and passes through all but one of $p_1, \ldots, p_n$, so $p_1, \ldots, p_n$ are independent conditions. Hence $p'_1, \ldots, p'_{n'}$ fail to impose independent conditions.

By induction, either

- $d' + 2$ of the points $p'_1, \ldots, p'_{n'}$ are colinear or
- $n' = 2d' + 2$ and the points $p'_1, \ldots, p'_{n'}$ lie in a conic.

The second possibility, $n' = 2d' + 2$, expands out to become $n = 3d + 1$. But $n \leq 2d + 2$, which forces $d \leq 1$, not possible. Hence $d' + 2$ of the points $p'_1, \ldots, p'_{n'}$ are colinear, i.e. $d + 1$ of the points $p_{d+2}, \ldots, p_n$ are colinear, say on a line L'. So the conic $L \cup L'$ contains $2d + 2$ of the points $p_1, \ldots, p_n$. But $n \leq 2d + 2$, so $n = 2d + 2$. So we have a contradiction: no $d + 1$ of our points are colinear.

Suppose that some number ℓ of our points are colinear, say $p_1, p_2, \ldots, p_\ell$. Let $d' := d - 1$ and $n' := n - \ell$ and let

$$p'_1, \ldots, p'_{n'}$$

be the points

$$p_{\ell+1}, \ldots, p_n.$$

If $p'_1, \ldots, p'_{n'}$ impose independent conditions on curves of degree d', then we find a curve C of degree d' through all but one of $p'_1, \ldots, p'_{n'}$. Then $L \cup C$ has degree d and passes through all but one of $p_1, \ldots, p_n$, so $p_1, \ldots, p_n$ are independent conditions, a contradiction. Hence $p'_1, \ldots, p'_{n'}$ fail to impose independent conditions.

By induction, either

- $d'+2$ of the points $p'_1, \ldots, p'_{n'}$ are colinear or
- $n' = 2d'+2$ and the points $p'_1, \ldots, p'_{n'}$ lie in a conic.

The first possibility, $d+1$ of the points are colinear, we have already ruled out. So $n' = 2d'+2$, expands out to become $n = 2d+2+\ell-2$. But $n \leq 2d+2$, which forces $\ell \leq 2$. So no 3 of our points are colinear.

Suppose that $p_3, \ldots, p_n$ impose independent conditions on curves of degree $d-1$. Then there is a curve C of degree $d-1$ through all of $p_4, \ldots, p_n$ and not through p_3. But then $p_1p_2 \cup C$ has degree d and misses p_3, a contradiction. So any $n-2$ of our points fail to impose independent conditions on curves of degree $d-1$. We let $n' = n-2$ and $d' = d-1$, and apply induction to see that $n' \leq 2d'+2$, so either

- $d'+2$ points are colinear, or
- $n' = 2d'+2$ and $p_3, \ldots, p_n$ lie on a conic.

The first case expands out to $d+1$ points are colinear, a contradiction to the results above. The second case expands out to $n = 2d+2$ and $p_3, \ldots, p_n$ lie on a conic. But this is true for any reordering of our points: any set of $n-2$ of our points lie on a conic. If $d = 1$ or $d = 2$, we have already checked our result above. For $d \geq 3$, since we have found that $n = 2d+3 \geq 9$, so $n-2 \geq 7$. If you choose $n-2$ points and I choose $n-2$ points, they must have at least $n-4 \geq 5$ points in common. Any 5 of our points lie on at most one conic, since no three are colinear, by lemma 25.4 on page 227. So all n points lie on a conic. □

Chapter 26

Elliptic curves

It is possible to write endlessly on elliptic curves. (This is not a threat.)

— Serge Lang
Elliptic Curves: Diophantine Analysis

Cubic curves

Theorem 26.1 (Chasles)**.** *Suppose that C_1 and C_2 are plane cubic curves intersecting at 9 points over the algebraic closure of the field. Every plane cubic curve passing through any 8 of those points passes through all 9. The plane cubic curves passing through those 9 points are precisely the pencil through C_1 and C_2.*

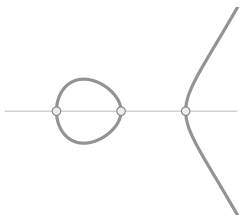
Proof. By theorem 25.8 on page 230, with $n = 8$ and $d = 3$, either

a. 8 of the points impose independent conditions or

b. 5 of the points are colinear or

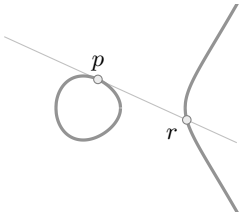
c. all 8 of the points lie on a conic.

If 7 of the points lie on a conic, then by Bézout's theorem that conic lies in both C_1 and C_2, so C_1 and C_2 have infinitely many points in common, a contradiction. If 4 of the points lie on a line, then by Bézout's theorem that line lies in both C_1 and C_2, so C_1 and C_2 have infinitely many points in common, a contradiction. So all 8 points inmpose independent conditions, i.e. the projective space of cubics through all 8 points has dimension $3^* - 8 = 1$. Since $C_1 \neq C_2$, the projective space of cubics through all 9 points has dimension at least 1: the pencil through C_1 and C_2 is 1-dimensional. But the family through all 9 points is a projective subspace of the family through all 8 points. Both projective spaces have the same dimension, hence these two projective spaces are equal. □



For any two distinct points p, q of a cubic curve C, the line through p and q intersects C in three points; let r be the third point. If p and q are defined in some field k, then so is the line through p and q. Suppose that C is also defined over k. On that line, parameterized over k as $y = mx + b$, the cubic equation of C (having coefficients over k) restricts to have at least two roots in k, corresponding to the points p and q. So that cubic polynomial on the line factors into linear factors over k. That third root gives the location of r on that line: the point r is also defined over k.

We can extend this definition to allow $p = q$, by taking r to be the point of C at which the tangent line to ℓ to C at p touches C again.



With a little algebra by hand, and some sage code to solve the mess, we can find the intersection points of tangent lines. Consider the curve

$$C = (y^2 - (x+1)x(x-1))$$

and the point p with $p = (x, y)$ with $x = -35/100$:

```
X=-35/100
Y=sqrt((X+1)*X*(X-1))
# The slope of the tangent line is given by implicit differentation.
M=(3*X^2-1)/(2*Y)
# Write the equation of the curve restricted to the tangent line.
x=var('x')
f=(Y+M*(x-X))^2-(x+1)*x*(x-1)
print("The tangent line strikes the curve at ",solve(f,x))
XX=(201601/196560)
YY=-sqrt((XX+1)*XX*(XX-1))
print("The other intersection point is ",XX.n(),YY.n())
plot(f,x,-2,2)
```

Elliptic curves

A *elliptic curve* over a field k is a smooth plane cubic curve C defined over k with a chosen point o of C, also defined over k, which we think of as the "origin" of C. Given any point p of C, the line op contains three points of C (counting with multiplicity if needed), say $p, o, \bar{p}$. Define the *addition* of points p, q by taking the line pq, which then intersects C at 3 points, say p, q, r, and then defining $p + q$ to be $\bar{r}$, so $p + q, r, o$ are the three points at which a line intersects C. Danger: note that this addition is *not* the same as adding the coordinates of the points, as you would do in linear algebra.

What if $p = q$? The tangent line to C at p intersects C at some point not equal to p, say r, and define $p + p = \bar{r}$.

Theorem 26.2. *Any elliptic curve is an abelian group under the addition operation.*

Proof. If p, q, r are the points at which a line intersects a cubic C, then so are q, p, r, and so $p + q = \bar{r} = q + p$. By definition, $p, o, \bar{p}$ are on a line, so $\bar{\bar{p}} = p$ and so $p + o = \bar{\bar{p}} = p$.

Consider the point $\bar{o}$, i.e. so that the line $\bar{o}o$ is the tangent line to C at o. Let $-p = \overline{p + \bar{o}}$, i.e. $-p, p, \bar{o}$ are on a line. So then $p + (-p) = \bar{\bar{o}} = o$.

Any equation like $r + q = p + q$ forces the points $r, q, \overline{r + q}$ to be $r, q, \overline{p + q}$, and hence to be $p, q, \overline{p + q}$, so $p = r$, we can cancel.

By definition of addition, for any two points p, q of C,

$$p, \overline{p + q}, \overline{p + \overline{p + q}}$$

lie in a line, while

$$p, q, \overline{p + q}$$

do as well, so

$$q = \overline{p + \overline{p + q}},$$

i.e.

$$\bar{q} = p + \overline{p + q},$$

a "funny cancellation".

Take three points p, q, r of C. We want to prove that $p + (q + r) = (p + q) + r$. Suppose that $p = o$ or $q = o$ or $r = o$ or $p = r$. The equation $p + (q + r) = (p + q) + r$ follows from any of these. So we can suppose that none of these is satisfied.

Define lines

$$\begin{aligned} L_1 &= p(q + r), \\ L_2 &= rq, \\ L_3 &= o(p + q), \\ L_1' &= r(p + q), \\ L_2' &= pq, \\ L_3' &= o(r + q). \end{aligned}$$

Note that $p, q + r$ lie on L_1, and so therefore does $\overline{p + (q + r)}$. Similarly,

$$\begin{aligned} p, q + r, \overline{p + (q + r)} &\in L_1, \\ r, q, \overline{q + r} &\in L_2, \\ o, p + q, \overline{p + q} &\in L_3, \\ r, p + q, \overline{(p + q) + r} &\in L_1', \\ p, q, \overline{p + q} &\in L_2', \\ o, q + r, \overline{q + r} &\in L_3'. \end{aligned}$$

Suppose that $L_1 = L_1'$. So the points $p, q + r, \overline{p + (q + r)}$ are some permutation of the points $r, p + q, \overline{(p + q) + r}$. This forces $p = r$ or $p = p + q$ or $q + r = r$ or $q + r = p + q$, and after cancellation these becomes $p = r$ or $q = o$, contradicting our hypotheses.

So $L_1 \neq L_1'$. Take the point s where L_1 and L_1' intersect. If s lies on C, then s lies on $p, q + r, s$, so $\bar{s} = p + (q + r)$, but also s lies on $r, p + q, s$, so $\bar{s} = (p + q) + r$. So s lies on C just when $p + (q + r) = (p + q) + r$.

Consider the 8 points $o, p, q, r, q+r, p+q$, and a ninth point s. The cubic $L_1 \cup L_2 \cup L_3$ contains all 9 points, as does the cubic $L'_1 \cup L'_2 \cup L'_3$. Suppose that these cubics have no component in common. Then by Chasles's theorem (theorem 26.1 on page 233), the plane cubics containing the 8 points contain all 9, so s belongs to C.

Suppose that these cubics share a component, say $L_i = L'_j$. The 3 points of C on L_i and the 3 points on L'_j must be the same 3 points. No other point of the cubic C can lie on that line, by Bézout's theorem. So the remaining 5 points lie on the other lines. Each pair of remaining lines is a conic. By lemma 25.4 on page 227, these conics are equal or share a common line through 4 of the points. But we can't have 4 points of C on the same line, by Bézout's theorem. So these conics are equal, i.e. all three lines L_1, L_2, L_3 are the same as L'_1, L'_2, L'_3 up to reordering.

We know that $L_1 \neq L'_1$, so $L_1 = L'_2$ or $L_1 = L'_3$. Swapping p and r swaps primed and unprimed lines.

If $L_2 = L'_2$, the points $r, q, \overline{q+r}$ are the same as $p, q, \overline{p+q}$, up to reordering. If this forces $r = p$, we have seen that $p + (q + r) = (p + q) + r$ follows. Otherwise, we find $\overline{q+r} = p$ and $r = \overline{p+q}$, and then

$$p + (q + r) = p + \bar{p} = \bar{o},$$

and

$$(p + q) + r = \bar{r} + r = \bar{o}.$$

So we can assume that $L_2 \neq L'_2$.

If $L_3 = L'_3$, the points $o, p + q, \overline{p+q}$ are the same as $o, r + q, \overline{r+q}$. Since we suppose that $p \neq r$, $r + q = \overline{p+q}$. By funny cancellation,

$$p + (q + r) = p + \overline{p+q} = \bar{q},$$

while

$$(p + q) + r = \overline{q+r} + r = r + \overline{r+q} = \bar{q},$$

so again $p + (q + r) = (p + q) + r$.

If $L_2 = L'_3$ then $r = o$ or $q = o$, a contradiction.

Hence L_1, L_2, L_3 must be some permutation of L'_1, L'_2, L'_3, but we can't have $L_1 = L'_1$ or $L_2 = L'_2$ or $L_2 = L'_3$ or $L_3 = L'_3$. So $L_1 = L'_3$. Permutation of p and r then forces $L_3 = L'_1$, so $L_2 = L'_3$, a contradiction. □

We always take an elliptic curve to be a smooth cubic C with a chosen point $o \in C$, giving an addition operation as above. If we pick some other point o' of our cubic curve C to be the "origin" of an elliptic curve, then we get another addition operation, say $+'$. The map

$$p \in C \mapsto p + (o' - o),$$

using the operations defined above is an isomorphism of groups from C with $+$ to C with $+'$. So the choice of point o is not very important.

Chapter 27

The tangent line

Lines crossing a curve

Proposition 27.1. *Pick an irreducible plane algebraic curve C of degree d and a point p of the plane not lying on C. Among all of the lines through p, every one of them strikes C in exactly d distinct points, except for at most $d(d-1)$ lines.*

Proof. First assume that C is irreducible. Arrange that the point p lies at the origin in an affine plane $z = 1$, so that the lines through p are those of the form $y = mx$ or $x = 0$. Also arrange that C is not entirely on the line at infinity, so contains some finite points. From the equation $q(x, y) = 0$ of the curve C, the intersections lie on $q(x, mx) = 0$, a polynomial of degree at most d in x. This polynomial is not the zero polynomial, because the origin does not lie on C. Therefore the number of intersections is d. The multiplicities will all be 1 except at values of m where the discriminant (as a polynomial in x) of $q(x, mx) = 0$ vanishes. The discriminant is an equation of degree at most $d(d-1)$ in m. If the discriminant vanishes as a polynomial in m, it vanishes as a rational function of m, and so $q(x, mx)$ has a zero of higher multiplicity as a polynomial in x with rational coefficients in m: $q(x, mx) = (x - f(m))^2 Q(x, m)$. But replacing m by y/x we find that $q(x, y)$ is reducible as a polynomial in x with coefficients rational in y, and so by the Gauss' lemma $q(x, y)$ is reducible as a polynomial in x, y, a contradiction. So the discriminant is a nonzero polynomial, and if it is constant we can homogenise and dehomogenise to get it to be nonconstant. □

Given a curve in the projective plane, we can take any of the three homogeneous coordinate variables $[x, y, z]$ and rescale to 1, say to $x = 1$, and see the curve as lying on the affine plane. As we have seen, changing the choice of variable, say to $y = 1$ or to $z = 1$, involves rehomogenising the equation and then plugging in $y = 1$ or $z = 1$, giving us a birational curve.

Looking at our curve in the affine plane, by setting $x = 1$, the curve "homogenises" into its cone in k^3, by homogenising the equation. The tangent line in the affine plane $x = 1$ homogenises into a plane through the origin, the tangent plane to the cone. When we set $y = 1$ or $z = 1$, or more generally we look at how our homogenised cone surface passes through some plane not containing the origin, the tangent plane to the cone becomes a tangent line to the resulting curve. Therefore we can speak of a *tangent line* to an algebraic plane curve in the projective plane, by which we mean the projective line which corresponds to the tangent plane to the cone.

To calculate that tangent line, we can either operate directly in some affine plane, say $x = 1$, and just differentiate implicitly as usual, or we can homogenise, and then differentiate the equation of the surface via partial derivatives.

The curve $y^2 = x^2 + x^3$ has tangent line given by differentiating implicitly:

$$2yy' = 2x + 3x^2,$$

so that at the point $(x, y) = (3, 6)$, we have

$$2(6)y' = 2(3) + 3(3)^2,$$

i.e. the slope of the tangent line is

$$y' = \frac{11}{4},$$

giving the tangent line

$$y - 6 = \frac{11}{4}(x - 3),$$

at the point $(x, y) = (3, 6)$.

For the same curve, we could instead homogenise to the cone

$$y^2 z = x^2 z + x^3,$$

and then treat any one of the variables as a function of the other two, say treating $z = z(x, y)$, so take partial derivatives:

$$\begin{aligned} y^2 \frac{\partial z}{\partial x} &= 2xz + x^2 \frac{\partial z}{\partial x} + 3x^2, \\ 2yz + y^2 \frac{\partial z}{\partial y} &= x^2 \frac{\partial z}{\partial y}. \end{aligned}$$

We are interested in the point $(x, y) = (3, 6)$ in the affine plane $z = 1$, so the point $(x, y, z) = (3, 6, 1)$. We plug in these values to our equations:

$$\begin{aligned} 6^2 \frac{\partial z}{\partial x} &= 2(3)(1) + 3^2 \frac{\partial z}{\partial x} + 3(3)^2, \\ 2(6)(1) + 6^2 \frac{\partial z}{\partial y} &= 3^2 \frac{\partial z}{\partial y}. \end{aligned}$$

Solve these to find

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{11}{9}, \\ \frac{\partial z}{\partial y} &= -\frac{4}{9}, \end{aligned}$$

giving an equation of the tangent plane:

$$z - 1 = \frac{11}{9}(x - 3) - \frac{4}{9}(y - 6),$$

which simplifies to

$$z = \frac{11}{9}x - \frac{4}{9}y.$$

This is the same solution, because along the affine plane $z = 1$ it gives

$$1 = \frac{11}{9}x - \frac{4}{9}y,$$

which, solving for y, is

$$y - 6 = \frac{11}{4}(x - 3),$$

exactly the same solution we had above.

At the point $(x, y) = (0, 0)$, the process above fails to yield a tangent line to the curve $y^2 = x^2 + x^3$ since differentiating implicitly:

$$2yy' = 2x + 3x^2,$$

and plugging in the point $(x, y) = (3, 6)$, we have $0 = 0$, no equation at all. In general, our procedure yields a tangent line to an irreducible curve precisely at the points (x, y) where the polynomial $p(x, y)$ in the equation $p(x, y) = 0$ of the curve has at least one nonzero partial derivative, so that we can solve for $y'(x)$ or for $x'(y)$ implicitly. In this example,

there are clearly 2 different lines tangent to the curve. We find these by taking the homogeneous terms of lowest degree in x, y, $y^2 = x^2$, dropping the x^3 because it has higher order, and then factoring into $y = \pm x$. We can justify this over the real numbers (or the complex numbers), by saying that our equation $y^2 = x^2 + x^3$ has solutions $y = \pm\sqrt{x^2 + x^3} \cong \pm x^2$, asymptotically as $x \to 0$.

Singular points

That's what love is. It's a recognition of singularity.

— Robert Graves
Conversations with Robert Graves

Take any algebraic curve C in the projective plane, and any point p_0 of C. Write out an affine plane $z = 1$ in which p_0 is the origin $(x, y) = (0, 0)$, and write out the equation $0 = p(x, y)$ of the curve. We can take $p(x, y)$ to be a product of irreducibles $p_i(x, y)$. Since the origin belongs to C, at least one of the factors $p_i(x, y)$ vanishes there. Expand each $p_i(x, y)$ into sums of constants times monomials. Without loss of generality, we can work over an infinite field. Rescale (x, y) by a nonzero constant, and then let that constant "get small", i.e. look at the powers of that constant, thinking

of higher powers as "smaller". Each monomial term rescales by some power of that factor. Rescale both sides of the equation $0 = p(x, y)$ to get rid of the lowest power. For example if C is $0 = x(y^2 - x^2 - x^3)$, rescaling by a factor λ gives

$$0 = \lambda x \left(\lambda^2 y^2 - \lambda^2 x^2 - \lambda^3 x^3\right).$$

Divide out the lowest power of λ, in this case λ^2, to get

$$0 = x \left(y^2 - x^2 - \lambda^3 x^3\right).$$

The invariant terms give a homogeneous polynomial $x\left(y^2 - x^2\right)$. The zeroes of this polynomial form the tangent lines to the curve. By homogeneity, the number of tangent lines is the degree of the homogeneous polynomial, which is at most the degree of the curve. The *order* of a point p_0 on a curve C is the number of tangent lines, counted with multiplicity. A *regular point* (also called a *smooth point*) of an algebraic curve is a point of order 1; any other point is a *singular point*. A curve without singular points (over the algebraic closure of the field) is *smooth* or *regular*.

At any point of order 1, make a projective change of variables to put the point at the origin, and the line to be the horizontal line $y = 0$. So our curve has equation $p(x, y) = 0$ with $\frac{\partial p}{\partial y} \neq 0$ and with $\frac{\partial p}{\partial x} = 0$ at the origin. After a constant rescaling, our equation is $y = ax^2 + bxy + cy^2 + \ldots$. Note that we cannot rid our equation of these higher order terms in y.

The curve $y^3 = x^4$ over the real numbers is the graph of a differentiable (but not polynomial) function $y = x^{4/3}$. Its tangent lines at the origin arise from keeping the lowest degree homogeneous terms: $y^3 = 0$, i.e. there is a unique tangent line $y = 0$. Nonetheless, over the complex numbers this curve is *not* the graph of a differentiable complex function of a complex variable.

The singular points of an affine algebraic curve C with equation $0 = p(x, y)$ are precisely the points where

$$0 = p(x, y) = \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y},$$

by definition. Similarly, the singular points of a projective algebraic curve are the singular points of one of its representatives in an affine plane.

Take a reducible curve and write it as a union of irreducible component curves. Correspondingly, write its equation as a product of two polynomials. Any two of the irreducible components intersect somewhere (at least in the points defined over the algebraic closure of the field), say at a point p. Take affine coordinates in which p is the origin. Each of the two components has equation given by a polynomial vanishing at the origin, so the original curve is a product of such polynomials. Therefore the equation of the reducible curve vanishes to at least second order. Therefore every reducible curve is singular.

Consequently, every nonsingular curve is irreducible.

The curve $y = x^n$ in the affine plane $z = 1$ has no singularities in that affine plane, if $n \geq 1$ is an integer. In the affine plane $x = 1$ it becomes $z^{n-1}y = 1$ which has singularities just when

$$z^{n-1}y = 1 \text{ and } z^{n-1} = 0 \text{ and } (n-1)z^{n-2}y = 0,$$

which never happen simultaneously. Therefore over any field, the curve $y = x^n$ is regular for every $n = 1, 2, \ldots$. In particular, there are regular curves of all degrees.

27.1 Suppose that every intersection point of two projective algebraic plane curves B and C, over an algebraically closed field, has multiplicity one. Prove that B and C are regular at these points.

27.2 Prove that every regular projective algebraic plane curve is irreducible. Give an example of reducible regular affine algebraic plane curve.

Lemma 27.2. *The number of singular points of an irreducible algebraic curve of degree d is at most $d(d-1)$. Moreover, if the singular points have orders $n_1, n_2, \ldots, n_k$, then*

$$\sum_k n_k(n_k - 1) \leq d(d-1).$$

Proof. The singular points are the solutions of

$$0 = p(x, y) = \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y}.$$

If one or both of these partial derivatives vanishes, then $p(x, y)$ is a function of one variable only, and the result is trivial to prove. If the two functions

$$p(x, y), \frac{\partial p}{\partial x}$$

have a common factor, then since $p(x, y)$ is irreducible, this factor must be $p(x, y)$. But $\frac{\partial p}{\partial x}$ is a polynomial of lower degree in x than $p(x, y)$, a contradiction. So the number of common roots is $p(x, y)$ and $\frac{\partial p}{\partial x}$ is $d(d-1)$ when multiplicities are counted. Take a singular point, and arrange that it is the origin. If order is n, so $p(x, y)$ expands out to have lowest nonzero term homogeneous of degree n, then $\frac{\partial p}{\partial x}$ has lowest nonzero term homogeneous of degree $n - 1$. □

27.3 What are the singularities, and their orders, of $y^p = x^{p+q}$ for $1 \leq p, q$?

Theorem 27.3. *The tangent line to a projective plane algebraic curve at a regular point is the unique line that has intersection multiplicity 2 or more at that point. More generally, the intersection multiplicity of a line ℓ and a curve $C = (0 = f)$ at a point p is the multiplicity of the zero of p as a zero of the restriction of f to ℓ.*

Proof. We can arrange that our coordinates axes form a generic triangle as usual, and that our intersection point is at $(x, y) = (1, 1)$. We can assume that our projective plane algebraic curve is

$$0 = f(x, y) = \sum_j f_j(x) y^j.$$

We can arrange that our line is $y = 1$. The intersection multiplicity is the multiplicity of $x = 1$ as a zero of the resultant

$$\det \begin{pmatrix} -1 & & & & f_0(x) \\ 1 & -1 & & & f_1(x) \\ & 1 & -1 & & f_2(x) \\ & & \ddots & \ddots & \vdots \\ & & & 1 & -1 & f_{n-1}(x) \\ & & & & 1 & f_n(x) \end{pmatrix},$$

and we add each row to the next

$$\begin{aligned} &= \det \begin{pmatrix} -1 & & & & f_0(x) \\ & -1 & & & f_0(x) + f_1(x) \\ & & -1 & & f_0(x) + f_1(x) + f_2(x) \\ & & & \ddots & \vdots \\ & & & -1 & f_0(x) + f_1(x) + \cdots + f_{n-1}(x) \\ & & & & f_0(x) + f_1(x) + \cdots + f_n(x) \end{pmatrix}, \\ &= (-1)^n \left(f_0(x) + f_1(x) + \cdots + f_n(x) \right), \\ &= (-1)^n f(x, 1). \end{aligned}$$

So the intersection multiplicity of the line and the curve is the multiplicity of vanishing of $f(x, 1)$ at $x = 1$. Since $f(1, 1) = 1$, the intersection multiplicity is positive, and is two or more just when

$$\left. \frac{d}{dx} \right|_{x=1} f(x, 1) = 0,$$

i.e. just when $y = 1$ is a tangent line. □

Theorem 27.4. *Denote by C_p the order of a projective algebraic plane curve C at a point p. Then for any point p and projective algebraic plane curves C, D:*

$$C_p D_p \leq C D_p,$$

with equality just when no line defined over any extension field is tangent to both curves at p. In particular, the order C_p is the intersection multiplicity $C\ell_p$ for any line ℓ through p which is not tangent to C at p.

Proof. If C or D is a line, our result is precisely theorem 27.3 on the previous page. If either C or D is reducible, our result follows from induction on the degrees of the components and adding up multiplicities.

After perhaps replacing our field by a finite degree extension, pick a generic triangle for the two curves, and arrange that its vertices are $(0, 0), (\infty, 0), (0, \infty)$ as usual. It is easier to compute tangent lines if we shift the axes by adding constants to x and y to arrange that the vertices are at $(-1, -1), (\infty, -1), (-1, \infty)$. We can then arrange that our point p is $(x, y) = (0, 0)$. Write the equations of our curves as $C = (f = 0)$ and

$D = (g = 0)$. Then CD_p is the order of $x = 0$ as a zero of the resultant $r(x) = \mathrm{res}_{f,g}(x)$ in y. Let $r \coloneqq C_p$, $s \coloneqq D_p$. The lowest order homogeneous polynomial terms have order equal to the order of the point on the curve, so we can write out

$$\begin{aligned} f(x,y) &= f_0(x)x^r + f_1(x)x^{r-1}y + \cdots + f_r(x)y^r \\ &\quad + f_{r+1}(x)y^{r+1} + \cdots + f_m(x)y^m, \\ g(x,y) &= g_0(x)x^s + g_1(x)x^{s-1}y + \cdots + g_s(x)y^s \\ &\quad + g_{s+1}(x)y^{s+1} + \cdots + g_n(x)y^n. \end{aligned}$$

The tangent lines are the zero lines of the homogeneous terms of lowest order in x, y:

$$\begin{aligned} 0 &= f_0(0)x^r + f_1(0)x^{r-1}y + \cdots + f_r(0)y^r, \\ 0 &= g_0(0)x^s + g_1(0)x^{s-1}y + \cdots + g_s(0)y^s. \end{aligned}$$

We can assume that our axes are chosen so that the horizontal and vertical axes are not tangent to either curve at the origin, i.e. to arrange that neither of these equations are satisfied by $(x, y) = (1, 0)$ or by $(x, y) = (0, 1)$, i.e. none of

$$f_0(0), g_0(0), f_r(0), g_s(0)$$

vanishes. Since these tangent line equations are homogeneous, to find tangent lines we can restrict to $y = 1$:

$$\begin{aligned} 0 &= f_0(0)x^r + f_1(0)x^{r-1} + \cdots + f_r(0), \\ 0 &= g_0(0)x^s + g_1(0)x^{s-1} + \cdots + g_s(0). \end{aligned}$$

The resultant of these two polynomials doesn't vanish just when these polynomials have no common factor, i.e. just when the curves C and D have no common tangent line; called this resultant R_1.

The point $(x, y) = (0, 0)$ is by assumption an intersection point of C and D. We can arrange, making our triangle generic enough, again perhaps using a field extension, that the lines $x = 0$ and $y = 0$ do not contain any other intersection points of C and D other than $(x, y) = (0, 0)$. This is precisely saying that our polynomials

$$\begin{aligned} f(x,y) &= f_0(x)x^r + f_1(x)x^{r-1}y + \cdots + f_r(x)y^r \\ &\quad + f_{r+1}(x)y^{r+1} + \cdots + f_m(x)y^m, \\ g(x,y) &= g_0(x)x^s + g_1(x)x^{s-1}y + \cdots + g_s(x)y^s \\ &\quad + g_{s+1}(x)y^{s+1} + \cdots + g_n(x)y^n. \end{aligned}$$

do not both vanish when we substitute $x = 0$, except at $y = 0$, and vice versa. Plugging in $x = 0$:

$$\begin{aligned} f(0,y) &= f_r(0)y^r + f_{r+1}(0)y^{r+1} + \cdots + f_m(0)y^m, \\ g(0,y) &= g_s(0)y^s + g_{s+1}(0)y^{s+1} + \cdots + g_n(0)y^n. \end{aligned}$$

Dividing out common factors of y:

$$\begin{aligned} \frac{f(0,y)}{y^r} &= f_r(0) + f_{r+1}(0)y + \cdots + f_m(0)y^{m-r}, \\ \frac{g(0,y)}{y^s} &= g_s(0) + g_{s+1}(0)y + \cdots + g_n(0)y^{n-s}. \end{aligned}$$

These are both polynomials and don't vanish at $y = 0$, since f and g vanish exactly to orders r and s at $y = 0$. These two polynomials also can't both vanish anywhere else, since there are no common roots of f and g on $x = 0$ except at $y = 0$. Hence the resultant of these two doesn't vanish; call this resultant R_2.

We next claim that $\mathrm{res}_{f,g}(x) = x^{rs}R_1R_2 + \ldots$ has lowest order term given by these two resultants. Before we prove our claim, note that if our claim is true, then CD_p is given by the order of this resultant, so is at least rs, proving the theorem. Moreover this order is rs just when $R_1 \neq 0$ (since we know that $R_2 \neq 0$), i.e. just when there are no common tangent lines.

We are left to justify this claim, i.e. to compute out the lowest order term of

$$\mathrm{res}_{f,g}(x) = \det \begin{pmatrix} f_0(x)x^r & & g_0(x)x^s & \\ f_1(x)x^{r-1} & \ddots & g_1(x)x^{s-1} & \ddots \\ \vdots & \ddots & \vdots & \ddots \\ f_m(x) & \ddots & g_n(x) & \ddots \\ 0 & \ddots & 0 & \ddots \end{pmatrix}.$$

The lowest order term in x is the same as the lowest order term of what happens when we plug $x = 0$ in all of the polynomials:

$$\det \begin{pmatrix} f_0(0)x^r & & g_0(0)x^s & \\ f_1(0)x^{r-1} & \ddots & g_1(0)x^{s-1} & \ddots \\ \vdots & \ddots & \vdots & \ddots \\ f_m(0) & \ddots & g_n(0) & \ddots \\ 0 & \ddots & 0 & \ddots \end{pmatrix}.$$

So it suffices to prove our result for the special case that all $f_j(x)$ and $g_j(x)$ are constants, say:

$$\begin{aligned} f(x,y) &= a_0x^r + a_1x^{r-1}y + \cdots + a_ry^r \\ &\quad + a_{r+1}y^{r+1} + \cdots + a_my^m, \\ g(x,y) &= b_0x^s + b_1x^{s-1}y + \cdots + b_sy^s \\ &\quad + b_{s+1}y^{s+1} + \cdots + b_ny^n. \end{aligned}$$

Then the resultant is

$$\mathrm{res}_{f,g}(x) = \det \begin{pmatrix} a_0x^r & & b_0x^s & \\ a_1x^{r-1} & \ddots & b_1x^{s-1} & \ddots \\ \vdots & \ddots & \vdots & \ddots \\ a_m & \ddots & b_n & \ddots \\ 0 & \ddots & 0 & \ddots \end{pmatrix}.$$

We need to prove that this is $x^{rs}R_1R_2 + \ldots$. Write out

$$R_1 = \det \begin{pmatrix} a_0 & & b_0 & \\ a_1 & \ddots & b_1 & \ddots \\ \vdots & \ddots & \vdots & \ddots \\ a_r & \ddots & b_s & \ddots \\ 0 & \ddots & 0 & \ddots \end{pmatrix}$$

and

$$R_2 = \det \begin{pmatrix} a_r & & b_s & \\ a_{r+1} & \ddots & b_{s+1} & \ddots \\ \vdots & \ddots & \vdots & \ddots \\ a_m & \ddots & b_n & \ddots \\ 0 & \ddots & 0 & \ddots \end{pmatrix}.$$

Let's try a simple case: if $r = 2, s = 1, m = 3, n = 2$, the resultant is

$$\operatorname{res}_{f,g}(x) = \det \begin{pmatrix} a_0x^2 & 0 & b_0x & 0 & 0 \\ a_1x & a_0x^2 & b_1 & b_0x & 0 \\ a_2 & a_1x & b_2 & b_1 & b_0x \\ a_3 & a_2 & 0 & b_2 & b_1 \\ 0 & a_3 & 0 & 0 & b_2 \end{pmatrix}.$$

Sadly, when we expand out this determinant, we get a huge mess:

$$\begin{aligned} \operatorname{res}_{f,g}(x) = x^2 &R_1R_2 \\ &+ x^3(a_3^2b_0^3 - 2\,a_1a_3b_0^2b_2 + 3\,a_0a_3b_0b_1b_2 + a_1^2b_0b_2^2 - 2\,a_0a_2b_0b_2^2 - a_0a_1b_1b_2^2) \\ &+ x^4a_0^2b_2^3 \end{aligned}$$

where

$$R_1 = a_2b_0^2 - a_1b_0b_1 + a_0b_1^2$$

and

$$R_2 = a_2b_2 - a_3b_1.$$

Nevertheless, the lowest term (as expected) is $\operatorname{res}_{f,g}(x) = x^2R_1R_2 + \ldots$.

Returning to the general problem: luckily, we don't really need to compute any of these resultants at all. Suppose that $r \le s$. After rescaling f, x and y we can assume that $a_r = 1$ and that $a_m = 1$. Every term in $g - b_sx^{s-r}f$ contains a factor of y, so write it as $g - b_sx^{s-r}f = y\tilde{g}$. We will see that replacing g by $\tilde{g}$, the three resultants we have to calculate then all compute the same values. By problem 15.6 on page 120:

$$\begin{aligned} \operatorname{res}_{f(x,y),g(x,y)}(x) &= \operatorname{res}_{f(x,y),y\tilde{g}(x,y)}(x), \\ &= \operatorname{res}_{f(x,y),y}(x)\operatorname{res}_{f(x,y),\tilde{g}(x,y)}(x), \\ &= f(x,0)\operatorname{res}_{f(x,y),\tilde{g}(x,y)}(x), \\ &= x^r \operatorname{res}_{f(x,y),\tilde{g}(x,y)}(x) \end{aligned}$$

while, if we let F be the lowest order part of f, G the lowest order part of g, and $\tilde{G}$ the lowest order part of $\tilde{g}$,

$$R_1 = \operatorname{res}_{F(x,1),G(x,1)} = \operatorname{res}_{F(x,1),\tilde{G}(x,1)}$$

and

$$R_2 = \operatorname{res}_{\frac{f(0,y)}{y^r},\frac{y\tilde{g}(0,y)}{y^s}} = \operatorname{res}_{\frac{f(0,y)}{y^r},\frac{\tilde{g}(0,y)}{y^{s-1}}} .$$

Hence our claim is true for f, g just when it is true for $f, \tilde{g}$. Similarly, if $r < s$, swap the roles of f and g and repeat. Apply induction on r and s, driving at least one of them to become smaller at each step, with the other staying the same. The induction stops when one of them is zero, and C and D cease to intersect at p. □

Chapter 28

Inflection points

Traditionally, an *inflection point* meant a point where the graph of a function changes from being convex down to being convex up, or vice versa. Clearly at any inflection point, the derivative goes from decreasing to increasing, or vice versa, so the second derivative vanishes. Notions of convexity depend very strongly on having some idea of positive and negative numbers, so they don't apply in the context of algebraic curves. The closest we can come to this is to *define* an *inflection point* or *flex* of an algebraic plane curve to be a point at which the tangent line meets the curve to order 3 or more.

To find such points, we define the *Hessian* $\det b''(x,y,z)$ of a homogeneous polynomial $b(x,y,z)$ by

$$\det b''(x,y,z) = \det \begin{pmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{xy} & b_{yy} & b_{yz} \\ b_{xz} & b_{yz} & b_{zz} \end{pmatrix}$$

where

$$b_{xy} := \frac{\partial^2 b}{\partial x \, \partial y}$$

and so on. So if $b(x,y,z)$ is homogeneous of degree n, then b_{xy} is homogeneous of degree $n-2$, so $\det b''(x,y,z)$ is homogeneous of degree $3(n-2)$. If $b(x,y,z)$ has degree 2, so that the associated curve is a conic, then $\det b''(x,y,z)$ is a constant, the determinant of the symmetric matrix of coefficients of $b(x,y,z)$. So $\det b''(x,y,z)$ is a nonzero constant, i.e. H is empty, just when $b(x,y,z)$ is a smooth conic. Similarly, $\det b''(x,y,z)$ vanishes everywhere for a line or a singular conic (a pair of lines).

If the Hessian is not constant, the *Hessian curve* of the curve $B = (0 = b(x,y,z))$ is the curve $H = (0 = \det b''(x,y,z))$.

Sage can compute Hessians for us:

```
P.<x,y,z> = PolynomialRing(QQ)
b = x^3-y^2*z+z^3
Hb = matrix([[diff(b,x,x),diff(b,x,y),diff(b,x,z)],
             [diff(b,x,y),diff(b,y,y),diff(b,y,z)],
             [diff(b,x,z),diff(b,y,z),diff(b,z,z)]])
factor(Hb.det())
```

yields $(-24) \cdot x \cdot (y^2 + 3z^2)$.

Working over any field of characteristic 3, the polynomial $b(x,y,z) := x^3 + y^3 - z^3$ has derivatives $0 = b_x = b_y = b_z$, and so has Hessian $\det b''(x,y,z) = 0$.

Lemma 28.1 (Euler)**.** *Every homogeneous polynomial $b(x, y, z)$, say of degree n, over any commutative ring with identity, satisfies*

$$nb(x, y, z) = x\frac{\partial b}{\partial x} + y\frac{\partial b}{\partial y} + z\frac{\partial b}{\partial z}.$$

Proof. Since $b(x, y, z)$ is a sum of homogeneous terms, it is enough to check one such term, which is easy for the reader to check. □

Lemma 28.2. *The Hessian $\det b''(x, y, z)$ vanishes at a smooth point of a plane algebraic curve $B = (0 = b(x, y, z))$ just when the tangent line at that point meets the curve to order 3 or more.*

Proof. Euler's lemma:

$$nb(x, y, z) = x\frac{\partial b}{\partial x} + y\frac{\partial b}{\partial y} + z\frac{\partial b}{\partial z}$$

applies to the homogeneous polynomials $b_x := \frac{\partial b}{\partial x}$ and so on to yield

$$\begin{aligned}(n-1)b_x &= xb_{xx} + yb_{xy} + zb_{xz},\\ (n-1)b_y &= xb_{xy} + yb_{yy} + zb_{yz},\\ (n-1)b_z &= xb_{xz} + yb_{yz} + zb_{zz}.\end{aligned}$$

Multiply row 1 of the Hessian determinant by x, row 2 by y, and add to the third row multiplied by z to get

$$z \det b''(x, y, z) = (n-1) \det \begin{pmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{xy} & b_{yy} & b_{yz} \\ b_x & b_y & b_z \end{pmatrix}.$$

Apply the same trick across the columns instead of the rows:

$$z^2 \det b''(x, y, z) = (n-1)^2 \det \begin{pmatrix} b_{xx} & b_{xy} & b_x \\ b_{xy} & b_{yy} & b_y \\ b_x & b_y & \frac{nb}{n-1} \end{pmatrix}.$$

At singular points of the curve $B = (0 = b(x, y, z))$, the final row vanishes, so the determinant vanishes. At any point where the curve $B = (0 = b(x, y, z))$ is not singular, which we can arrange that point to be $(x, y, z) = (0, 0, 1)$. Arrange that the tangent line of B at that point is the x-axis, so $b_y = 1$ while $b_x = 0$. Euler's lemma gives $b_z = 0$. Plug into equation 28 to get $\det b''(0, 0, 1) = (n-1)^2 b_{xx}(0, 0, 1)$. The multiplicity of b restricted to $x = 0$ is then given by expanding out $b(0, y) = b(0, 0) + b_y(0, 0)y + b_{yy}(0, 0)y^2/2 + \ldots$, clearly at least 3.

By implicit differentiation, at any smooth point, the slope of the tangent line is

$$0 = b_x + b_y y'$$

and so differentiating again gives

$$0 = b_{xx} + 2b_{xy}y' + b_{yy}(y')^2 + b_y y'',$$

as the derivative of y'. So

$$y' = -\frac{b_x}{b_y},$$

while

$$y'' = -\frac{b_{xx} + 2b_{xy}y' + b_{yy}(y')^2}{b_y}.$$

Expanding out,

$$\begin{aligned} y'' &= \frac{-b_{xx}b_y^2 + 2b_{xy}b_xb_y - b_{yy}b_x^2}{b_y^3}, \\ &= \frac{1}{b_y^3}\begin{pmatrix} b_{xx} & b_{xy} & b_x \\ b_{xy} & b_{yy} & b_y \\ b_x & b_y & 0 \end{pmatrix} \end{aligned}$$

So at any point where $b = 0$, this is

$$y'' = \frac{\det b''(x, y, 1)}{(n-1)^2 b_y^3}.$$

If we take a point (x_0, y_0) at which $b_y \neq 0$, let y_0' be y' at that point, and compute out that on the tangent line,

$$\begin{aligned} b(x, y) &= b(x_0 + \Delta x, y_0 + y_0'\Delta x), \\ &= b(x_0, y_0) + b_x(x_0, y_0)\Delta x + \dots, \\ &= -b_y(x_0, y_0)y_0'\Delta x - b_y(x_0, y_0)y_0''\frac{\Delta x}{2} + \dots \end{aligned}$$

So the Hessian vanishes just where the tangent line meets the curve to order 3 or more, and this occurs just where $y'' = 0$. □

Lemma 28.3. *Take a plane algebraic curve $B = (0 = b(x, y, z))$. Suppose that the characteristic of the field is coprime to the exponents appearing in all terms of $b(x, y, z)$. Then $b(x, y, z)$ has vanishing Hessian just when B is a union of lines. In other words, B has Hessian curve H defined just when B is not a union of lines.*

Proof. Continuing the previous proof, if we suppose that the Hessian vanishes everywhere, we see that y'' is everywhere zero, i.e. we see that at every point where $b = 0$,

$$0 = y'' = (y')_x + (y')_y y'.$$

Plug in $y' = -b_x/b_y$ to get

$$\left(\frac{b_x}{b_y}\right)_x = \left(\frac{b_x}{b_y}\right)\left(\frac{b_x}{b_y}\right)_y.$$

This last equation turns a derivative in x into a product, with the first factor having no derivative in x. Note that it only holds at points of the curve, so holds modulo b. Hence, when we differentiate both sides, the result hold modulo b, b_x, and so on. By induction, at a point where $0 = b = b_x$ and where $0 \neq b_y$, if we differentiate both sides repeatedly, we find a higher order x derivative of b vanishing each time. So at any point where $0 = b = b_x$ we also have $0 = b = b_x = b_{xx} = \dots$. Because the exponents arising in the terms of $b(x, y, z)$ are coprime to the characteristic of the field, the vanishing of derivatives $b_x, b_{xx}, \dots$ implies that the coefficients of $x, x^2, \dots$ in $b(x, y, 1)$ are zero. Every term in b contains a factor of y, so $b = 0$ along the line $y = 0$. □

Lemma 28.4. *Take a plane algebraic curve $B = (0 = b(x, y, z))$, which has Hessian curve H. Then H passes through all singular points of B, intersecting B at $3n(n-2)$ points at most, counting with multiplicity. The curves H and B meet at finitely many points. Moreover, H meets B at a smooth point of B with multiplicity r just when the tangent line at that point meets B with multiplicity $r+2$.*

Proof. We can assume that the point in question is the origin in the affine plane, and that the tangent line is $y = 0$. Factoring out all copies of x from the terms with no y in them, the equation of B is now

$$0 = b(x, y) = y\, f(x, y) + x^{r+2} g(x),$$

with $f(0,0) \neq 0$ and $g(0) \neq 0$. We can rescale x, y to arrange $f(0,0) = 1$ and $g(0) = 1$.

The associated homogeneous polynomial is

$$b(x, y, z) = y\, f(x, y, z) + x^{r+2} g(x, z).$$

Note that the affine equations $f(0,0) = 1$ and $g(0) = 1$ become homogeneous equations $f(0,0,z) = z^{n-1}$ and $g(0,z) = z^{n-r-2}$. The Hessian $\det b''(x,y,z)$ is extremely complicated in terms of f, g. The first derivatives are

$$\begin{aligned} b_x &= y f_x + (r+2) x^{r+1} g + x^{r+2} g_x, \\ b_y &= f + y f_y + x^{r+2} g_y, \\ b_z &= y f_z + x^{r+2} g_z, \end{aligned}$$

and the second derivatives are

$$\begin{aligned} b_{xx} &= y f_{xx} + (r+2)(r+1) x^r g + 2(r+2) x^{r+1} g_x + x^{r+2} g_{xx}, \\ b_{xy} &= f_x + y f_{xy} + (r+2) x^{r+1} g_y + x^{r+2} g_{xy}, \\ b_{xz} &= y f_{xz} + (r+2) x^{r+1} g_z + x^{r+2} g_{xz}, \\ b_{yy} &= 2 f_y + y f_{yy} + x^{r+2} g_{yy}, \\ b_{yz} &= f_z + y f_{yz} + x^{r+2} g_{yz}, \\ b_{zz} &= y f_{zz} + x^{r+2} g_{zz}. \end{aligned}$$

Modulo y, x^r, i.e. dropping terms with y or x^r in them, we find

$$\det b''(x,y,z) = \det \begin{pmatrix} 0 & f_x & 0 \\ f_x & 2f_y & f_z \\ 0 & f_z & 0 \end{pmatrix} = 0.$$

So every term in $\det b''(x,y,z)$ contains either y or x^r:

$$\det b''(x,y,z) = yF(x,y,z) + x^r G(x,y,z)$$

for some polynomials F, G. We can absorb any y terms in G into F, so

$$\det b''(x,y,z) = yF(x,y,z) + x^r G(x,z)$$

for some polynomials F, G. Compute $\det b''(x,y,z)$ modulo y, and divide out x^r from the first row, and then set $x = 0$, to find

$$G(0,1) = \frac{\det b''}{x^r}(0,0,1) = -(r+2)(r+1) g\,(0,1)\, f_z\,(0,0,1)^2\,.$$

We can actually get sage to compute this for us:

```
y=var('y')
z=var('z')
r=var('r')
f = function('f')(x,y,z)
g = function('g')(x,z)
b(x,y,z) = y * f(x,y,z) + x^(r+2)*g(x,z)
m=matrix([
      [diff(b(x,y,z),x,x),diff(b(x,y,z),x,y),diff(b(x,y,z),x,z)],
      [diff(b(x,y,z),x,y),diff(b(x,y,z),y,y),diff(b(x,y,z),y,z)],
      [diff(b(x,y,z),x,z),diff(b(x,y,z),y,z),diff(b(x,y,z),z,z)]])
h=m.det()
factor(simplify(expand((h.subs(y=0)/x^r).subs(x=0))))
```

prints out $-(r+2)(r+1)g\left(0,z\right)\mathrm{D}_2\left(f\right)\left(0,0,z\right)^2$. (The sage notation for derivatives is peculiar, and we leave the reader to work out why D_2 means derivative in the third variable.) Since we know that $f(0,0,z)=z^{n-1}$ and $g(0,0,1)=1$, we can compute out

$$G(0,1)=\frac{\det b''}{x^r}(0,0,1)=-(r+2)(r+1)(n-1)^2.$$

Returning to affine coordinates, at the origin $(x,y)=(0,0)$ in the (x,y) affine plane,

$$G(0)=-(r+2)(r+1)(n-1)^2\neq 0.$$

We are looking for $H\cap B$ near the origin $(0,0,1)$, i.e. we are looking for common zeroes, with multiplicity, of

$$\begin{aligned}0&=yf(x,y,z)+x^{r+2}g(x,z),\\0&=yF(x,y,z)+x^rG(x,z).\end{aligned}$$

Dehomogenize by setting $z=1$:

$$\begin{aligned}0&=yf(x,y)+x^{r+2}g(x),\\0&=yF(x,y)+x^rG(x).\end{aligned}$$

If $r=0$, then the second equation doesn't vanish near $(x,y)=(0,0)$, so the origin is not an intersection point, contradicting our assumptions. So $r\geq 1$.

Suppose that B and H share a common component, and suppose that the origin belongs to that component. If $y=0$ on some intersection point, then plug in to get that either $x=0$ or $g(x)=0$. But $g(0)=1$, so $g(x)$ is zero at only finitely many points, none near the origin. The same holds over any algebraic extension of our field. So $(y=0)$ is not a component of $B\cap H$, i.e. if $B\cap H$ contains a curve, it is not inside the line $(y=0)$. Solve for x^r as

$$x^r=\frac{y}{(r+1)(r+2)(1+\dots)},$$

and plug in to get

$$\frac{x^2y}{(r+1)(r+2)(1+\dots)}=-y\frac{1+\dots}{1+\dots}.$$

Cancel the y, since $(y=0)$ is not in $B\cap H$:

$$x^2=-(r+1)(r+2)\frac{1+\dots}{1+\dots},$$

and clear denominators:

$$x^2(1+\dots) = -(r+1)(r+2)(1+\dots).$$

on any common component of B and H. But this is not satisfied at the origin, so the common component does not meet the origin. Recall that we picked any point of $H \cap B$ and made it the origin. Hence B and H share no common component.

We will compute the multiplicity of intersection BH_p at any smooth point p of B. We can assume that we have chosen very generic affine coordinates, so that the line through any two intersection points of B and H does not touch $(0,1)$ (even after an algebraic extension of the field). We can still arrange that p is the origin $(0,0)$ and that the tangent line to B at $(0,0)$ is $y=0$. Take the resultant $R(x)$ of $yf(x,y)+x^{r+2}g(x)$ and $yF(x,y)+x^rG(x)$. By theorem 24.6 on page 217, BH_p is the order of vanishing of $R(x)$ at the origin: if we write

$$f(x,y) = \sum y^j f_j(x),$$
$$F(x,y) = \sum y^j F_j(x),$$

then

$$R(x) = \det \begin{pmatrix} x^{r+2}g(x) & & & & x^rG(x) & & & \\ f_1(x) & x^{r+2}g(x) & & & F_1(x) & x^rG(x) & & \\ \vdots & f_1(x) & \ddots & & \vdots & F_1(x) & \ddots & \\ f_k(x) & \ddots & \ddots & x^{r+2}g(x) & F_\ell(x) & \ddots & \ddots & x^rG(x) \\ & f_k(x) & \ddots & f_1(x) & & F_\ell(x) & \ddots & F_1(x) \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & f_k(x) & & & & F_\ell(x) \end{pmatrix}.$$

Expand across the top row to see that $R(x)$ is a multiple of x^r. We want to prove that $R(x)$ has order exactly x^r, no something higher. The x^r term in $R(x)$ doesn't notice terms of order x^{r+1} or higher, so we can assume that $g(x)$ and $G(x)$ are constants, both equal to 1. Terms of higher order than x^r make no contribution to the order r term in $R(x)$, so we only need to find the resultant of $yf(x,y), yF(x,y)+x^r$. If we write dots to indicate terms of higher order than x^r,

$$\begin{aligned} R(x) &= \operatorname{res}_{yf(x,y),yF(x,y)+x^r} + \dots, \\ &= \operatorname{res}_{y,yF(x,y)+x^r} \operatorname{res}_{f(x,y),yF(x,y)+x^r} + \dots, \\ &= x^r \operatorname{res}_{f(x,y),yF(x,y)+x^r} + \dots. \end{aligned}$$

Suppose that the x^r term vanishes. Then $f(0,y)$ and $yF(0,y)$ have a common root (in some algebraic extension), say $y=y_0$. Then $(0,y_0)$ is a common root of $yf(x,y)+x^{r+2}g(x)$ and $yF(x,y)+x^rG(x)$, i.e. an intersection point $(0,y_0) \in B \cap H$. This intersection point is not at the origin, since $f(0,0)=1$. This intersection point is on the line between $(0,1)$ and $(0,0)$, a contradiction. So the x^r term in $R(x)$ doesn't vanish, i.e. the order of intersection BH_p is r. □

Corollary 28.5. *A smooth plane algebraic curve of degree n with a defined Hessian curve has exactly $3n(n-2)$ inflection points, counting multiplicities, in some algebraic extension field.*

Proof. Follows from by Bézout's theorem (theorem 24.7 on page 218) and the degree of the Hessian being $3(n-2)$ as above. □

Chapter 29

Conics and quadratic forms

I like mathematics because it is not human and has nothing particular to do with this planet or with the whole accidental universe—because, like Spinoza's god, it won't love us in return.

— Bertrand Russell

Quadratic forms

A *quadratic form* is a homogeneous polynomial of degree 2.

Proposition 29.1 (Sylvester's Law of Inertia)**.** *Every quadratic form in n variables over the real numbers is identified, by some linear change of variables, with precisely one of*

$$x_1^2 + x_2^2 + \cdots + x_p^2 - \left(y_1^2 + y_2^2 + \cdots + y_q^2\right)$$

where $p + q \leq n$. The triple of numbers (p, q, r) so that $p + q + r = n$ is called the signature *of the quadratic form.*

Proof. Write the quadratic form as

$$q(x) = a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{nn}x_n^2,$$

or in other words as $q(x) = \langle Ax, x\rangle$ for a symmetric matrix A. Apply an orthogonal change of variables which diagonalizes the matrix A. Then rescale each variable to get the eigenvalues of the matrix to be ± 1 or zero. □

Lemma 29.2. *Every quadratic form in n variables over a field k of characteristic not 2 can be brought to the "diagonal" form*

$$q(x) = \sum a_i x_i^2$$

by a linear change of variables. The a_i are uniquely determined up to multiplying by squares $a_i b_i^2$ of nonzero elements b_i of k.

Proof. Take a vector x for which $q(x) \neq 0$. Make a linear change of variables to get $x = e_1$, so that $q(e_1) \neq 0$. Hence after rescaling

$$q(x) = c_1x_1^2 + \ldots,$$

say

$$q(x) = c_1x_1^2 + c_2x_1x_2 + \cdots + c_nx_1x_n + \ldots$$

where x_1 does not appear in the remaining terms. Replace $x_1, x_2, \ldots, x_n$ with $X_1, x_2, \ldots, x_n$ where

$$X_1 := x_1 - \frac{c_2}{2c_1}x_2 - \cdots - \frac{c_n}{2c_1}x_2$$

and check that now

$$q(x) = c_1 X_1^2 + \ldots$$

where the $\ldots$ don't involve X_1, so we can apply induction. □

For the real numbers, any positive real number has a square root, so we recover the previous lemma.

For $k = \mathbb{C}$, any complex number has a square root, so every quadratic form in n complex variables becomes

$$q(z) = z_1^2 + z_2^2 + \cdots + z_p^2,$$

for a unique $p \leq q$.

For $k = \mathbb{Q}$, any rational number, up to multiplying by a nonzero square of a rational number, is a *square-free* integer, i.e. has the form

$$\alpha = \pm p_1 p_2 \ldots p_s$$

where all of the primes

$$p_1, p_2, \ldots, p_s$$

are distinct. So every quadratic form over k has the form

$$q(x) = \sum \alpha_i x_i^2$$

for square-free integers α_i. In the associated conic curve, we can rescale the coefficients, so we can assume that the square-free integers α_i have no common factor. We can also assume, by changing all signs if needed, that there are at least as many positive α_i as negative.

Since every element of any field obtains a square root in some extension, the quadratic forms over any algebraically closed field have the form

$$q(z) = z_1^2 + z_2^2 + \cdots + z_p^2,$$

for a unique $p \leq q$.

Theorem 29.3. *Over a field k, take a quadratic form q, say in n variables. Then after perhaps repeatedly replacing k by a quadratic extension of k, finitely many times, and a linear change of variables, we can arrange that either*

$$q(x_1, \ldots, x_n) = x_1 x_2 + x_3 x_4 + x_5 x_6 + \cdots + x_{m-1} x_m$$

or

$$q(x_1, \ldots, x_n) = x_1^2 + x_2x_3 + x_4x_5 + \cdots + x_{m-1}x_m,$$

for some integer m.

Proof. A *null vector* of q is a vector v so that $q(v + x) = q(x)$, i.e. translating q by a null vector doesn't change the value of q. The *null space* of q is the set of all null vectors. The null space is a linear subspace of k^n since if v_1 and v_2 belong to the null space, then $q(v_1 + v_2 + x) = q(v_1 + x) = q(x) = q(v_2 + x)$, and if v belongs to the null space, and c is any constant in k, then $q(cv + x) = c^2\, q(v + x/c) = c^2\, q(x/c) = q(x)$.

Make a linear change of variables to get various of the variables to parameterize this subspace, i.e. the subspace is given by setting the first j variables $x_1, x_2, \ldots, x_j$ to zero and the last $n - j$ variables $x_{j+1}, \ldots, x_n$ to any values. So q is a function of those first j variables only. We can reduce our problem to the study of those first j variables, i.e. we can assume that the null space is zero.

If $n = 1$, $q = ax^2$, we only need to ensure that a has a square root in k. If $n = 2$, $q = ax^2 + bxy + cy^2$, we only need to ensure that $ax^2 + bx + c$ has a root in k, and then q factors into linear factors, and we change variables to arrange that $q = xy$. So suppose that $n \geq 3$.

A *plane* is a 2-dimensional linear subspace of k^n. On any plane P, we can arrange that $q = 0$ or $q = x_1x_2$ or $q = x_1^2$, as above, after some linear change of variables. For the moment, suppose that there is a plane P on which we can arrange that $q = x_1x_2$, after perhaps a quadratic extension of k.

Consider the subset Q of k^n consisting of those vectors w so that $q(v + w) = q(v) + q(w)$ for all v in P. Expand out this equation to see that this is a linear equation in w, so Q is a linear subspace. Moreover, we only need to check the equation $q(v_i + w) = q(v_i) + q(w)$ for any two vectors v_i forming a basis of P, so Q is cut out by two linear equations, so has dimension at least $n - 2$. Check that $P \cap Q = \{\, 0 \,\}$, because on P, $q = x_1x_2$. So Q is a linear subspace of k^n of dimension $n - 2$ exactly.

Make a linear change of variables so that P is spanned by the first two basis vectors (i.e. by setting any values to x_1 and x_2 while setting all remaining variables equal to zero) and Q by the remaining $n - 2$ basis vectors: $k^n = P \oplus Q = k^2 \oplus k^{n-2}$. Change the first two variables if needed to arrange that $q = x_1x_2$ on P, as we saw is possible. By induction, we can arrange that on Q, either

$$q(0, 0, x_3, \ldots, x_n) = x_3x_4 + x_5x_6 + \cdots + x_{n-1}x_n$$

or

$$q(0, 0, x_3, \ldots, x_n) = x_3x_4 + x_5x_6 + \cdots + x_{n-2}x_{n-1} + x_n^2.$$

and so

$$q(x_1, \ldots, x_n) = x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{n-1}x_n$$

or

$$q(x_1, \ldots, x_n) = x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{n-2}x_{n-1} + x_n^2.$$

Now suppose that there is no plane P in k^n on which we can arrange that q has the form x_1x_2. Suppose that this remains true even after taking finitely many quadratic extensions. So on any plane P in k^n, either $q = 0$ or we can arrange that $q = x_1^2$.

Let U be the set of all vectors in k^n on which q vanishes. Take two linearly independent vectors on which q vanishes, i.e. two points of U. On the plane they

span, q vanishes on at least two lines, so q vanishes everywhere on that plane. So U is a linear subspace of k^n. Arrange by linear change of variables that U is given by the linear equations $0 = x_{s+1} = \cdots = x_n$. If $s \geq 2$, then the plane parameterized by $(x_1, x_2, 0, \ldots, 0)$ intersects U only at the origin, i.e. $q = 0$ on that plane only at the origin. After a quadratic extension, as we have seen, we can arrange that $q = x_1 x_2$, a contradiction. Hence $s = 0$ or $s = 1$. But $s = 0$ just when $U = k^n$, i.e. just when $q = 0$ everywhere. So we can assume that $s = 1$, i.e. $0 = q(0, x_2, \ldots, x_n)$, i.e. $q = 0$ whenever $x_1 = 0$, i.e. q has a factor of x_1, and so splits into a product of linear factors. □

Plane conics

Look at the curves $x^2 = y^2$ and the curves $x^2 + y^2 = 0$, defined over $k = \mathbb{R}$; each is a singular plane conic. The first is a pair of lines $x = \pm y$ while the second, a circle of radius zero, has the origin as its only real point, but splits into $x = \pm iy$ over $\bar{k} = \mathbb{C}$.

Lemma 29.4. *If a plane conic over a field k has a singular point with coordinates in k then it is either*

a. *a pair of lines defined over k or else*
b. *a pair of lines defined over a quadratic extension $k(\sqrt{\alpha})$ meeting at a single point defined over k, the singular point, and no other point of the curve is defined over k.*

Proof. At a singular point of a plane conic, in an affine plane in which the singular point is the origin, the equation of the conic becomes a quadratic form in x, y. This factors over k or over a quadratic extension of k, i.e. the conic is a pair of lines in a quadratic extension. If any point (except the origin) on one of those lines has coordinates in k, then the line does, and so does the associated linear factor, and then by dividing that linear factor in, we see that the other linear factor is also defined over k. Suppose on the other hand that the linear factors don't exist over k, only over the quadratic extension. Then no other point of the affine plane with coordinates in k belongs to the curve, except the singular point. Write out the equation of the curve as $ax^2 + bxy + cy^2 = 0$, say. Switch to the affine plane $y = 1$ where the curve is $ax^2 + bx + c$. Since the homogeneous equation has no roots in $\mathbb{P}^1(k)$, this quadratic equation has no root in k, so there is no point in this affine plane $y = 1$. Therefore the projective conic has precisely one point with coordinates in k. □

Proposition 29.5. *If a conic in the projective plane over a field k has a regular point with coordinates in k, it is either a pair of distinct lines, and taken to the conic $xy = 0$ by some projective automorphism, or is irreducible and taken to the conic $y = x^2$ by some projective automorphism.*

Proof. Suppose that our conic is not a pair of lines. It has a regular point with coordinates in k, and is not a pair of lines, so it has no singular points, and so is irreducible. Arrange by projective automorphism that the regular point lies at the origin of the affine plane $z = 1$ with tangent line $y = 0$, so the curve has the form

$$y = ax^2 + bxy + cy^2.$$

If $a = 0$, we get a factor of y on both sides, so the equation is reducible. Rescale y to arrange $a = 1$, and write the equation as $x^2 = y\,(1 - bx - cy)$. Change variables by a projective automorphism $X = x, Y = y, Z = z - bx - cy$ to get to $x^2 = y$. □

Corollary 29.6. *If an irreducible conic in the projective plane over a field k has a regular point with coordinates in k then it is birational to a line.*

Proof. Suppose that the conic, call it C, has equation $x^2 = y$, i.e. $x^2 = yz$ homogeneously. Map

$$[s, t] \in \mathbb{P}^1(k) \mapsto \left[st, s^2, t^2\right] \in C \subset \mathbb{P}^2(k)\,.$$

The inverse map is

$$[x, y, z] \in C \mapsto \begin{cases} [x, y], & \text{if } y \neq 0, \\ [z, y], & \text{if } z \neq 0. \end{cases}$$

Clearly this identifies the rational functions.

More geometrically, pick a regular point p_0 on the curve, and arrange that it is the origin in an affine plane. Then take a line ℓ in the plane, and identify each point $p \in \ell$ of that line with the intersection point of the line pp_0 with C. We leave the reader to check that this gives the same map. □

Chapter 30

Projective duality

There does not exist a category of science to which one can give the name applied science. There are sciences and the applications of science, bound together as the fruit of the tree which bears it.

— Louis Pasteur
Revue Scientifique (1871)

The set of lines in the projective plane $P = \mathbb{P}^2(k)$ is called the *dual plane* and denoted $P^* = \mathbb{P}^{2*}(k)$. Each line has a homogeneous linear equation $0 = ax + by + cz$, unique up to rescaling, and so the point $[a, b, c]$ is determined, allowing us to identify P^* with $\mathbb{P}^2(k)$ itself. The points of P^* correspond to lines in P. Through each point of P, there is a pencil of lines; for example if the point is (x_0, y_0) in the affine chart $z = 1$, then the lines are $a(x - x_0) + b(y - y0) = 0$, or expanding out: $ax + by + c = 0$ where $c = -ax_0 - by_0$. Hence in P^* the pencil is the line consisting of those points $[a, b, c]$ so that $ax_0 + y_0 + c = 0$. So each point p of P is identified with a line p^* in P^*, consisting of those lines in P passing through p. Each line λ of P is identified with a point λ^* in P^*. Given two points p, q of P and the line pq through them, the corresponding lines p^* and q^* in P^* intersect at the point corresponding to $(pq)^*$. We can drop all of this notation and write p^* also using the symbol p, and just say that each point p of P is a line in P^*, and vice versa.

Dual curves

Take a projective algebraic curve C in $P = \mathbb{P}^2(k)$ over a field k with algebraic closure $\bar{k}$, with equation $0 = p(x, y)$ in some affine chart.

30.1 Denote partial derivatives as $p_x := \frac{\partial p}{\partial x}$, etc. Show that at any point (x_0, y_0) of C at which $p_x \neq 0$ or $p_y \neq 0$, the equation of the tangent line is

$$0 = p_x(x_0, y_0)(x - x_0) + p_y(x_0, y_0)(y - y_0).$$

Lemma 30.1. *In homogeneous coordinates $[x, y, z]$ on $P = \mathbb{P}^2$, if the equation of a curve C is $0 = p(x, y, z)$ with $p(x, y, z)$ homogeneous, then the tangent line at a point $[x_0, y_0, z_0]$ of C has homogeneous equation*

$$0 = xp_x(x_0, y_0, z_0) + yp_y(x_0, y_0, z_0) + zp_z(x_0, y_0, z_0).$$

Proof. Differentiate to get tangent line

$$0 = p_x(x_0, y_0, z_0)(x - x_0) + p_y(x_0, y_0, z_0)(y - y_0) + p_z(x_0, y_0, z_0)(z - z_0).$$

Take the equation that says that $p(x, y, z)$ is homogeneous of degree d:

$$p_x(\lambda x, \lambda y, \lambda z) = \lambda^d p(x, y, z),$$

and differentiate it in λ to get

$$xp_x(\lambda x, \lambda y, \lambda z) + yp_y(\lambda x, \lambda y, \lambda z) + zp_z(\lambda x, \lambda y, \lambda z) = d\lambda^{d-1}p(x, y, z).$$

Now set $\lambda = 1$ to get

$$xp_x + yp_y + zp_z = dp.$$

Plug into the homogeneous equation for the tangent line to get

$$0 = xp_x(x_0, y_0, z_0) + yp_y(x_0, y_0, z_0) + zp_z(x_0, y_0, z_0) - dp(x_0, y_0, z_0)\,.$$

But at $[x_0, y_0, z_0]$ we know that $p(x_0, y_0, z_0) = 0$. □

Each line $[a, b, c]$ in P^* has homogeneous equation $ax+by+cz = 0$, so the equation of the tangent line at each point $[x_0, y_0, z_0]$ of a curve with homogeneous equation $0 = p(x, y, z)$ is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}.$$

The *dual curve* of C is the smallest algebraic curve C^* in P^* containing the tangent lines of C with coordinates in $\bar{k}$.

Corollary 30.2. *Every curve has a dual curve, i.e. the map*

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

taking the smooth points of C to P^* has image inside an algebraic curve.

Proof. Assume that the field is algebraically closed. Suppose that the image contains finitely many points. The original curve has finitely many tangent lines, so is a finite union of lines. Following Study's theorem (theorem 24.3 on page 213) we can assume that C is irreducible, say $C = (p(x, y, z) = 0)$. Take the four equations $p = 0$, $a = p_x$, $b = p_y$, $c = p_z$, rescaling out any one of the x, y, z variables by homogeneity (or some linear combination of them), and compute resultants for them in one of the x, y, z variables, treating them as having coefficients rational in the other two x, y, z variables and in a, b, c. Repeat until we get rid of x, y, z entirely. Since we have polynomial expressions to begin with in all of our variables, each resultant is also polynomial in those variables. By corollary 20.5 on page 179, perhaps making a linear change of x, y, z variables before taking our first resultant, the resultants we end up are not all zero polynomials. So the image of the duality map lies in an algebraic curve in the dual plane. But the resultant vanishes just precisely on the points in the image of the duality map, since these equations specify precisely those points $[a, b, c]$. □

For any irreducible conic C, given by an equation $0 = \langle Ax, x\rangle$ for a symmetric matrix A, with homogeneous coordinates

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

and dual plane homogeneous coordinates

$$a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

differentiate to get that the partial derivatives of $q(x) = \langle Ax, x \rangle$ are $a = q'(x) = 2Ax$. Solve for x: $x = 2A^{-1}a$. Careful to note that we cannot solve for x over a field of a characteristic 2. Plug in to the equation of the conic: $0 = \left\langle a, A^{-1}a \right\rangle$. So, for any conic over any field not of characteristic 2, the dual conic has the inverse matrix.

Each singular point of C is tangent to all lines through it, so by definition that collection of lines belongs to the dual curve. But that collection of lines forms a line in the dual plane. So the dual curve is a union of irreducible algebraic curves, finitely many of which are lines, precisely one line for each singular point of C.

Proposition 30.3. *Suppose that C is an irreducible plane algebraic curve and the degree of C is denoted d. The dual curve has degree $d(d-1)$, containing no lines if C is not singular, and containing a line for each singular point of C in the closure of the field of definition, with multiplicity of that line given by the multiplicity of the singularity.*

Proof. We can assume the field is algebraically closed. The degree of C^* is the number of intersections of C^* with the generic line L in the plane. Each line in the dual plane is the dual $L = p^*$ of a point p in the original plane, i.e. the set of points in the dual plane corresponding to lines through the point p of the plane. The intersection of L^* with C^* occurs at the points of C^* corresponding to the lines tangent to C passing through p. There are $d(d-1)$ tangent lines through p, counting multiplicity, by proposition 27.1 on page 237. Among these lines are all lines connecting p to any singular point of C, corresponding to lines in C^*. □

Chapter 31

Polynomial equations have solutions

If the followers of the prophets err in their proofs, have no fear; the foundation of their knowledge is the authority of the prophets. They provide proofs only for added strength. Following the authority of the prophets is sufficient for them, unlike the philosophers. The philosophers stray from the authority of the prophets and instead rely on their proofs to establish an argument. Misguided, they misguide.

— Ahmad al-Fārūqī al-Sirhindī
MAKTUBAT

To argue with a man who has renounced the use and authority of reason, and whose philosophy consists in holding humanity in contempt, is like administering medicine to the dead.

— Thomas Paine
THE CRISIS V

Affine space

For any field k, *affine space* k^n is the set of all points

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

with coordinates $c_1, c_2, \ldots, c_n$ in k.

We want to say: if you have infinitely many polynomial equations, in a finite set of variables, you can throw away all but finitely many without changing the solutions. Let us be more precise. Recall that an *ideal* I in $k[x]$ is a collection of polynomials so that we can add and multiply polynomials in I without leaving I, and I is *sticky*: if $f \in I$ and $g \in k[x]$ then $fg \in I$. Any set S of polynomials generates an ideal (S): the ideal is the set of all expressions

$$g_1 f_1 + g_2 f_2 + \cdots + g_s f_s$$

where $f_1, f_2, \ldots, f_s$ is any finite collection of polynomials in S and $g_1, g_2, \ldots, g_s$ are any polynomials at all in $k[x]$. Given any set of polynomials, we can replace that set

by the ideal of polynomials it generates inside $k[x]$ without changing the roots of the polynomials.

Theorem 31.1 (Hilbert basis theorem)**.** *For any field k and any variables $x = (x_1, x_2, \ldots, x_n)$, every ideal I in $k[x]$ is finitely generated. In other words, there are finitely many polynomials*

$$f_1(x), f_2(x), \ldots, f_s(x)$$

in I so that the polynomials in I are precisely those of the form

$$g_1(x)f_1(x) + g_2(x)f_2(x) + \cdots + g_s(x)f_s(x),$$

for any polynomials $g_1(x), g_2(x), \ldots, g_s(x)$ in $k[x]$.

Proof. The result is obvious if $n = 0$ since the only ideals in k are k and 0. It is also obvious if $n = 1$, since any ideal contains the greatest common divisor of its elements, and therefore contains the ideal generated by it, but is clearly contained in it, so every ideal is generated by one polynomial $f_1(x)$. By induction, suppose that the result is true for any number of variables less than or equal to n. Take an ideal $I \subset k[x, y]$ where $x = (x_1, x_2, \ldots, x_n)$ and y is a variable. Write each element $f \in I$ as

$$f(x, y) = f_0(x) + f_1(x)y + \cdots + f_d(x)y^d,$$

so that $f_d(x)$ is the *leading coefficient*. Let J be the set of all leading coefficients of elements of I. Clearly J is an ideal in $k[x]$, so finitely generated by induction. So we can pick polynomials $f_1(x, y), f_2(x, y), \ldots, f_s(x, y)$ in I whose leading coefficients generate the ideal J of all leading coefficients.

To be more precise, let us specify that we pick $f_1(x, y)$ of minimal y degree in I. Subsequently choose each $f_j(x, y)$ of minimal y degree among polynomials in I not already in the ideal generated by $f_1(x, y), \ldots, f_{j-1}(x, y)$. Only finitely many times will we pick a polynomial of a given y degree, because those of given y degree have leading terms forming an ideal inside J, so also finitely generated. Each choice makes a strictly large ideal inside J. After finitely many choices, these polynomials will have leading terms generating J, since every leading term not generated by one of these polynomials arises at some degree.

Let N be the maximum degree in y of any of the $f_j(x, y)$. Take some polynomial $p(x, y)$ from I of degree greater than N in y, say with leading coefficient $\check{p}(x)$. Express that leading coefficient in terms of the leading coefficients $\check{f}_j(x)$ of the polynomials $f_j(x, y)$ as

$$\check{p}(x) = \sum p_j(x)\check{f}_j(x).$$

The y degrees of these $f_j(x, y)$ whose leading terms appear in this sum cannot exceed that of $p(x, y)$, since otherwise we would have picked $p(x, y)$ in place of some $f_j(x, y)$. If we raise the y degree of $f_j(x, y)$ suitably to match the y degree of $p(x, y)$, say by multiplying by y^{k_j}, some positive integer k_j, then

$$\sum p_j(x)y^{k_j} f_j(x, y)$$

has leading coefficient the same as $p(x, y)$, so

$$p(x, y) = \sum y^{k_j} p_j(x)f_j(x, y) + \ldots$$

up to lower order in y. Inductively we can lower the order in y of p, replacing p by the ... . At each step, we modify p by something in the ideal generated by the f_j. So every polynomial in I of degree N or less in y is in the ideal generated by the f_j. Add to our list of $f_j(x,y)$ some additional $f_j(x,y) \in I$ to span all of the polynomials in I of degree up to N in y. By induction we have generated all of the polynomials in I. □

Theorem 31.2 (Weak nullstellensatz)**.** *Given a collection of polynomial equations in variables*

$$x = (x_1, x_2, \ldots, x_n) \, .$$

over a field k, either there is a point $x = c$ defined in a finite algebraic extension field of k where all of these polynomials vanish or else the ideal generated by the polynomials is all of $k[x]$.

Proof. Follows from theorem 31.1 on the preceding page and corollary 20.6 on page 179. □

31.1 Given a point $c \in k^n$, prove that the ideal

$$I_c := \{ \, f(x) \mid f(c) = 0 \, \}$$

of functions vanishing at $x = c$ is a maximal ideal. Prove that if $x = p$ and $x = q$ are distinct points of k^n, then $I_p \neq I_q$.

Corollary 31.3. *Take variables*

$$x = (x_1, x_2, \ldots, x_n) \, .$$

over an algebraically closed field k. Every maximal ideal in $k[x]$ is the set of all polynomial functions vanishing at some point c, for a unique point c.

Proof. Take any ideal $I \subset k[x]$ Take a point c so that all polynomials in I vanish there, using theorem 31.4. Then $I \subset I_c$. □

An *affine variety* $X = X_S$ is the collection of zeroes of a set S of polynomials in variables

$$x = (x_1, x_2, \ldots, x_n) \, .$$

Note that we allow these zeroes to lie in any finite field extension. If $I = (S)$ is the ideal generated by S, then clearly $X_I = X_S$. Given any set X of points, let I_X be the ideal of polynomials vanishing on X.

Theorem 31.4 (Nullstellensatz)**.** *Take variables*

$$x = (x_1, x_2, \ldots, x_n) \, .$$

over an algebraically closed field k. For any ideal J in $k[x]$, $\sqrt{J} = I_X$ where $X = X_J$.

Proof. In other words, we have to prove that, given any polynomials $p_1(x), \ldots, p_\ell(x)$, a polynomial $p(x)$ vanishes everywhere where all of those polynomials do, at points x in any finite field extension of k, just when

$$p(x)^s = a_1(x)p_1(x) + \cdots + a_\ell(x)p_\ell(x),$$

for some integer s and some polynomials $a_1(x), \ldots, a_\ell(x)$.

If not, consider

$$1 - tp(x), p_1(x), \ldots, p_\ell(x),$$

as polynomials in one more variable t. At a common zero (x, t), all $p_1(x), \ldots, p_\ell(x)$ vanish, so $p(x)$ vanishes, so $1 - tp(x)$ doesn't. Hence there is no common zero. By the nullstellensatz above, there are polynomials

$$g_0(t, x), g_1(t, x), \ldots, g_\ell(t, x),$$

so that

$$1 = g_0(t, x)(1 - tp(x)) + g_1(t, x)p_1(x) + \cdots + g_\ell(t, x)p_\ell(x).$$

Substitute $t := 1/p(x)$, so that in the field of fractions, i.e. in the field of rational functions,

$$1 = g_1(t, x)p_1(x) + \cdots + g_\ell(t, x)p_\ell(x),$$

where $t = 1/p(x)$. Expand into a common denominator, say of $p(x)^s$:

$$1 = \frac{a_1(x)p_1(x) + \cdots + a_\ell(x)p_\ell(x)}{p(x)^s}.$$

Clear denominators. □

Closed sets

A *closed set* in affine space k^n (over a field k) is just another name for an affine variety.

The entire set k^n is closed: it is the zero locus of the polynomial function $p(x) = 0$.

Every finite set in k is closed: the set $\{ c_1, c_2, \ldots, c_n \}$ is the zero locus of

$$p(x) = (x - c_1)(x - c_2)\ldots(x - c_n).$$

Any infinite subset of k, except perhaps k itself, is *not* closed. We know that every ideal I in $k[x]$ is generated by a single polynomial (the greatest common divisor of the polynomials in I), say $I = p(x)$, so the associated closed set is the finite set of roots of $p(x)$.

A subset of k^2 is closed just when it is either (a) all of k^2 or (b) a finite union of algebraic curves and points.

An *open set* is the complement of a closed set. An open set containing a point c of k^n is a *neighborhood* of c. The intersection of any collection of closed sets is a closed set: just put together all of the equations of each set into one set of equations. In particular, the intersection of all closed sets containing some collection X of points of k^n is a closed set, the *closure* of X. A set S is dense in a closed set X if X is the closure of S.

If the set X in k^n is the zero locus of some polynomials $f_i(x)$ and the set Y in k^n is the zero locus of some polynomials $g_j(x)$, then $X \cup Y$ in k^n is the zero locus of the polynomials $f_i(x)g_j(x)$. Hence the union of finitely many closed sets is closed.

31.2 Give an example of a field k and a polynomial $b(x, y)$ over k so that the points in the image of the polynomial (i.e. the points of the form $z = b(x, y)$ for any x and y in k) do *not* form an affine subvariety of k.

Projective varieties

Just as an algebraic curve in the projective plane is, by definition, the zero locus of nonzero homogeneous polynomial in three variables, so a *projective variety* X in projective space $\mathbb{P}^n$ is the collection of all lines on which some homogeneous polynomials in $n + 1$ variables vanish. Note that these zeroes can lie in the projective space of the algebraic closure of the field of definition of X, as was our definition for algebraic curves. By the Hilbert basis theorem, we can assume that the collection of polynomials is finite. As before, taking affine coordinates, i.e. setting one of the variables to 1, gives an affine chart on projective space, and X intersects that chart in an affine variety.

If our set of homogeneous polynomials is empty, we find that projective space is a projective variety inside itself.

In the projective plane, with homogeneous coordinates $[x, y, z]$, the homogeneous polynomials x^2, y^2, z^2 don't vanish at any point of projective space, as their common zero locus in k^3 is at the origin, so they don't vanish on any line in k^3.

The points of any projective variety are allowed to arise in some field extension. So a projective variety has no points just when the homogeneous polynomials cutting it out vanish nowhere or only at the origin. By the nullstellensatz, they generate an ideal containing a nonzero constant function, or the radical ideal they generate is the ideal generated by all homogeneous linear functions.

To each symmetric $(n + 1) \times (n + 1)$ matrix A over any field, the associated *quadric hypersurface* $Q = Q_A$ in projective space $\mathbb{P}^n$ is the set of all null lines of A; a line is *null* for A if it is the span of a vector x in the kernel of A.

Among the $p \times q$ matrices A, those which have at most a given rank, say rank k, are precisely those all of whose $(k + 1) \times (k + 1)$ minor determinants vanish. These determinants are homogeneous in the coordinates of the matrix, so a collection of homogeneous polynomials. In the projective space $\mathbb{P}^{pq-1}$, we find a variety X. Each point of X is a nonzero matrix A, defined up to rescaling, and with rank at most k, and every such matrix, up to rescaling,

gives a different point of X.

The intersection of projective varieties is a projective variety, with equations given by those of each of the original varieties put together.

A projective variety is *reducible* if it is the union of two projective subvarieties, neither contained in the other. A projective variety X is *reducible* in an open set U if $X \cap U$ is the union of $X_1 \cap U$ and $X_2 \cap U$, for two projective subvarieties X_1, X_2, neither contained in the other. A projective variety is *irreducible* at a point p_0 if irreducible in every open set containing p_0. A *projective hypersurface* is a projective variety X locally cut out by a single equation, i.e. every point of X lies in an open set U of projective space in which there is a nonconstant regular function f for which $X \cap U$ is the variety given by $f = 0$.

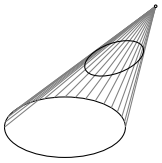
31.3 Suppose that X is a projective hypersurface, and U an open subset of projective space in which $X \cap U$ is the variety given by an equation $f = 0$. Suppose that f is irreducible. Prove that X is irreducible.

31.4 Prove that every analytic hypersurface is, near each of its points, the union of finitely many hypersurfaces irreducible at that point.

We use the same terminology of *closed sets* for projective varieties, and *open sets* for their complements, as we did for affine varieties.

Projection and elimination

Take a curve X, a plane H and a point x_0 not on X or H, all sitting in $\mathbb{P}^3$. We want to throw a shadow of X, from a light source at x_0, onto H.



More generally, we could just take a variety X, a hyperplane H and a point x_0 all in projective space $\mathbb{P}^n$. Arrange by linear change of variables that, in some affine space $x_0 = 1$, we have $H = (x_1 = 1)$ and x_0 is the origin. Take one of the polynomial equations that cuts out X, say $0 = b(x)$. Take a point y on H, so $y_1 = 1$. Clearly y is the shadow of a point x of X just when $x = ty$ for some value of a variable t. So the points y of our shadow are the points for which $0 = b(ty)$ has a solution (t, y), i.e. we eliminate the variable t from the equation $0 = b(ty)$, for every equation $0 = b(x)$ satisfied on X. Hence the linear projection of a variety is a variety, whose equations can be computed using resultants.

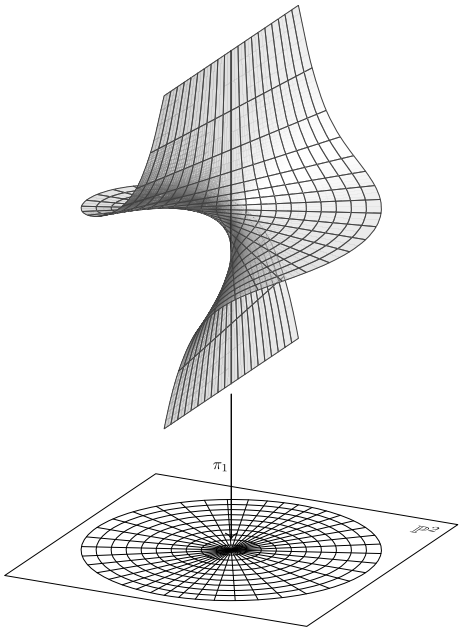
31.5 Find the linear projection of the curve $y = x^2 + 1, z = x^3$ from the origin to the plane $y = -1$. Draw the picture, supposing the variables are real.

Chapter 32

Blow up

Nothing like a little disaster for sorting things out.

— Michelangelo Antonioni
Blow Up



Blowing up curves

Take the plane curve $C = (0 = x^2 + x^3 - y^2)$:

The origin is a singular point. Consider new variables $X = x, Y = y/x$, which

we can write as $x = X, y = XY$. In these variables, the same curve is

$$\begin{aligned} 0 &= X^2 + X^3 - (XY)^2, \\ &= X^2(1 + X - Y^2). \end{aligned}$$

The curve was an irreducible cubic, but is now a product of a line $X = 0$, appearing twice, and a conic $1 + X = Y^2$.

This is surprising: an irreducible curve reducing, and a cubic which is *not* a union of a line a conic, becomes just that. Part of the explanation is that the change of variables is not defined along the line $x = 0$, so it is a badly behaved change of variables. But we can map $(X, Y) \mapsto (x, y) = (X, XY)$ everywhere. When we do, the *exceptional line* $(X = 0)$ (i.e. every point (X, Y) with $X = 0$) gets mapped to the origin. So the map is many to one. Away from the exceptional line, the map is bijective, but only maps to points (x, y) with $x \neq 0$. So the image of the map is the (x, y)-plane, sliced along the line $(x = 0)$, but then with the origin glued back in.

Take any plane algebraic curve $C = (0 = p(x, y))$, with singularity at the origin. Expand out in homogeneous polynomials by degree:

$$p(x, y) = p_d(x, y) + p_{d+1}(x, y) + \cdots + p_n(x, y),$$

Change variables linearly if needed so that the vertical axis $(x = 0)$ is not a tangent line, i.e. $p_d(0, y)$ is not the zero polynomial. Plug in the same $(x, y) = (X, XY)$ variables to get

$$\begin{aligned} p(X, Y) = X^d p_d(1, Y) + X^{d+1} p_{d+1}(1, Y) + \cdots + X^n p_n(1, Y), \\ + X^d (p_d(1, Y) + X p_{d+1}(1, Y) + \cdots + X^{n-d} p_n(1, Y)). \end{aligned}$$

So the curve C splits into two curves: the exceptional line $(X = 0)$, which arises with multiplicity d, and the curve

$$C' = (0 = p_d(1, Y) + X p_{d+1}(1, Y) + \cdots + X^{n-d} p_n(1, Y)),$$

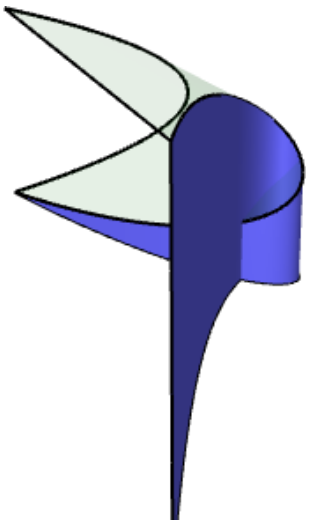
a curve of degree $n - d$, the *proper transform* of C. The curve C when written in these (X, Y) variables as above is the *total transform* of C, i.e. just the proper transform together with the multiplicity d copy of the exceptional line.

Three dimensional pictures

Take a plane algebraic curve $C = (0 = p(x, y))$ through the origin. Draw the "cylinder" X consisting of points (x, y, z) in three dimensions with $0 = p(x, y)$. Intersect the cylinder with the "saddle surface" S consisting of points (x, y, z) with $y = xz$. The intersection curve $\tilde{C} := X \cap S$ consists of points $0 = p(x, y)$ and $y = xz$. These equations together force $0 = p(x, xz)$. So the intersection curve projects to the (x, z)-plane to become two curves: the proper transform and the exceptional line. We always find the line $(x = y = 0)$ lying in the intersection curve; this is the exceptional line in this picture.

Draw the original curve C on a flat plane, i.e. with constant z. In dark blue, draw the "cylinder" X connecting C with that intersection curve $\tilde{C} = X \cap S$, but we don't draw S (just to avoid clutter). The curve along the top is $\tilde{C}$, and on the bottom is C.

The proper transform C' is the projection of $\tilde{C}$ to the (x, z)-plane, i.e. we project C' onto a flat wall to the left of the picture; the flat wall is a plane of constant y.



In sage, you can rotate the picture with your mouse; try

```
x(t)=t^2-1
y(t)=t*(t^2-1)
z(t)=t
u,v=var('u,v')
cylindr=parametric_plot3d(
   (x(u), y(u), v*z(u)+(1-v)*(-2)),
   (u,-2,2),
   (v,0,1),
   boundary_style={"color": "black",
                   "thickness": 2,
                   "plot_points":400},
   frame=False,
   plot_points=[300,300])
other_cylindr=parametric_plot3d(
   (x(u), v*y(u)+(1-v)*(-6), z(u)),
   (u,-2,2),
   (v,0,1),
   color="green",
   opacity=.1,
   boundary_style={"color": "black",
                   "thickness": 2,
                   "plot_points":400},
   frame=False,
   plot_points=[300,300])
show(cylindr+other_cylindr)
```

Blowing up singularities

Let us reconsider the story algebraically. In some algebraic extension, we factor $p_d(x,y)$, say into

$$p_d(x,y) = (y - m_1 x)(y - m_2 x)\dots(y - m_d x).$$

There is no x factor, because we supposed that the vertical axis is not tangent. The equation of C becomes in X, Y:

$$\begin{aligned} 0 &= p_d(X, XY) + \dots, \\ &= (XY - m_1 X)(XY - m_2 X)\dots(XY - m_d X) + \dots, \\ &= X^d (Y - m_1)(Y - m_2)\dots(Y - m_d) + \dots. \end{aligned}$$

So the Y variable keeps track of the slopes of the tangent lines. Indeed since $Y = y/x$, we can think of Y as a slope. The proper transform has equation

$$0 = (Y - m_1)(Y - m_2)\dots(Y - m_d) + X p_{d+1}(1, Y) + \dots + X^{n-d} p_n(1, Y)$$

so at $X = 0$, the roots of the proper transform are the slopes of the tangent lines.

Again take the plane curve $C = (0 = x^2 + x^3 - y^2)$:

The proper transform is $C' = (1 - Y^2 + X)$, so has $X = 0$ term $1 - Y^2 = (1 - Y)(1 + Y)$, the two slopes of the two tangent lines are $Y = \pm 1$.

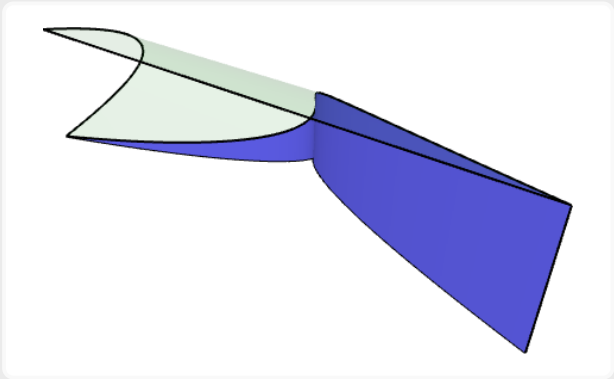
Each root $Y = m_i$ occurs as the intersection of the proper transform C' with the line $X = 0$, i.e. the exceptional line, with multiplicity equal to the number of factors of $y - m_i x$ in the lowest degree terms $p_d(x,y)$. By theorem 27.4 on page 242, that intersection number is no less than the multiplicity of C' at each intersection point $(X, Y) = (0, m_i)$.

The intersection number of each tangent line $\ell = (y = m_i x)$ with C is, from theorem 27.3 on page 241, given by plugging in $y = m_i x$ to the equation of C, i.e. into $p(x,y)$, and counting the degree of $p(x, m_i x)$ as a function of x.

The multiplicity of C at the origin is the sum of multiplicities of tangent lines. So the multiplicities of C' along the exceptional line add up to at most the multiplicity of C at the origin, with equality just when there is a unique tangent line to C (in any extension field), so p_d is a power of a single linear factor.

The cuspidal cube $C = (y^2 = x^3)$

has proper transform $C' = (Y^2 - X)$, which intersects the exceptional line $(X = 0)$ as $Y^2 = 0$, i.e. with multiplicity 2.



Blowing up the plane

The pictures help, especially if you get sage to draw them in three dimensions. But there is still the uncomfortable feeling that (i) we had to arrange that the vertical axis is not a tangent line, which seems artificial and (ii) the story occurs in the affine plane, but the true story should arise in the projective plane.

Rather than thinking of variables $X = x, Y = y/x$, think that Y is the slope of the line from the origin to (x, y). That line is determined by its slope, so Y parameterizes the lines through the origin by their slopes. Pick a point, which we can think of as the origin, and draw all of the lines through it. The *blow up* B_{p_0} at a point p_0 of the projective plane is the set of all pairs (p, ℓ) so that p is a point of the projective plane, ℓ a projective line on the plane passing through p_0, and p is also a point of ℓ.

If we draw the real projective plane, or more precisely affine coordinates, and draw each line ℓ through the origin of affine coordinates, so that we lift it up higher if it has higher slope, we get a picture (due to Charles Staats III) at the beginning of this chapter. The blow up is the surface drawn twisting around above the plane.

Let's become more closely acquanted with the picture. Each point q of the blow up drops straight down to a point p of the plane. If p is not the origin, then the line ℓ is the line through the origin and p, and we can recover q as the unique point of the blow up surface lying directly above p. If p is the origin, then the point q lies on the *exceptional line*, the line drawn directly above the origin. But then the line ℓ is the unique line in the plane so that the line on the blow up which projects down to ℓ passes through q.

Algebraically: take the projective plane P and its dual plane P^*. We can write points of P as $p = [x_0, x_1, x_2]$ and points of P^* as $\ell = [a_0, a_1, a_2]$, corresponding to the line $0 = a_0x_0 + a_1x_1 + a_2x_2$ in P. The line ℓ goes through the origin $p_0 = [0, 0, 1]$

of the affine plane just when $a_2 = 0$. The line ℓ also goes through the point p just when $0 = a_0x_0 + a_1x_1$. So the blow up surface B_{p_0} is the set of all pairs

$$(p, \ell) = ([x_0, x_1, x_2], [a_0, a_1, 0]) \in P \times P^*$$

so that $a_0x_0 + a_1x_1 = 0$. So the blow surface is the intersection of the two three dimensional varieties: the *incidence variety* $(a_0x_0 + a_1x_1 + a_2x_2 = 0) \subset P \times P^*$, consisting of pointed lines, and the variety $P \times p_0^*$.

The *exceptional divisor* E_{p_0} is the subset of B_{p_0} consisting of points (p_0, ℓ) where ℓ is a line through p_0. The set of these lines ℓ form a projective line inside the projective plane P^*.

Take an algebraic curve C in the projective plane P and a point p_0 of P. The points of B_{p_0} away from the exceptional divisor are points (p, ℓ) where $p \neq p_0$ and ℓ is the line p_0p. Each such point is uniquely identified with the point p. The points of B_{p_0} lying over the points of C, and not over the point p_0, are precisely those of the form (p, ℓ) where p is a point of C and $\ell = p_0p$, so identified with the points p of C. The closure of that set of points is the *strict transform* of C.

Chapter 33

Schubert calculus

One service mathematics has rendered the human race. It has put common sense back where it belongs. It has put common sense back where it belongs, on the topmost shelf next to the dusty canister labelled discarded nonsense.

— Eric T. Bell

Exterior powers

Suppose that V is a finite dimensional vector space over a field k. A *multilinear function* is a function

$$f\colon \underbrace{V \times V \times \cdots \times V}_{q \text{ copies of } V} \to k,$$

so that

$$f(v_1, v_2, \ldots, v_q)$$

is linear in each vector variable $v_1, \ldots, v_q$. It is *alternating* if it changes sign when we swap any two of the variables.

33.1 Prove that a function is alternating just when it changes sign when we swap any successive two of the variables.

33.2 Prove that a function is alternating just when it changes sign by the sign of a permutation when we apply that permutation to the variables.

Clearly the set of all alternating multilinear functions of q vector variables is a vector space, denoted $\Lambda^q V^*$. In particular, when $q = 1$, $\Lambda^1 V^* = V^*$ is just the dual space.

Take a linear map $\varphi\colon V \to W$ between vector spaces. Recall that φ determines a transpose $\varphi^*\colon W^* \to V^*$ by $\varphi^*\xi = \xi \circ \varphi$ for each $\xi \in W^*$. In other words, each linear function ξ on W, composed with φ, gives a linear map on V. In the same way, define a composition for multilinear functions: define

$$\varphi^*\colon \Lambda^q W^* \to \Lambda^q V^*,$$

by

$$(\varphi^* f)(v_1, \ldots, v_q) = f(\varphi v_1, \ldots, \varphi v_q).$$

Maybe we don't like that φ^* maps things to do with W to things to do with V. Careful to note where we make use of dual spaces: we define $\Lambda^q V$ to be $[\Lambda^q(V^*)]^*$, the

q^{th} *exterior power* of V. Abstractly, each element of $\Lambda^q V$ is an alternating multilinear function on V^*, so takes q linear functions on V as its arguments, and computes out a number. This is very abtract, but will quickly become more concrete.

Each vector $v_0 \in V$ determines a linear function f on linear functions ξ on V by: $f(\xi) := \xi(v_0)$. So $\xi \in V^*$, and so $f \in V^{**}$. In this way, we map $V \to V^{**}$, $v_0 \mapsto f$.

33·3 Since V has finite dimension, this map is an isomorphism.

Henceforth, we allow ourselves to write this function f as v_0, so that we write

$$v_0(\xi) := \xi(v_0).$$

Now we imitate this construction for multilinear functions. Fix any vectors $v_1, \ldots, v_q$ in V. Define a map on multilinear functions: plug this vectors into the function. So to each multilinear function f, we associate the number

$$f(v_1, \ldots, v_q).$$

We call this function

$$f \mapsto f(v_1, \ldots, v_q)$$

by the name

$$v_1 \wedge \cdots \wedge v_q.$$

So

$$(v_1 \wedge \cdots \wedge v_q)(f) = f(v_1, \ldots, v_q).$$

Try $q = 1$: $\Lambda^1 V = V$, with $v(f) = f(v)$ as above.

Take vectors $v_1, \ldots, v_n$ in V. Suppose that I is a set of p distinct integers from 1 to n, say $I = \{ i_1, i_2, \ldots, i_q \}$ with $1 \le i_1 < i_2 < \cdots < i_p \le n$. Let

$$v_I := v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

Lemma 33.1. *For any basis $e_1, \ldots, e_n$, the set of vectors e_I form a basis of $\Lambda^q V$. In particular,*

$$\dim \Lambda^q V = \binom{\dim V}{q}.$$

This suggests the notation $\binom{V}{q}$ instead of $\Lambda^q V$, which might have been a better choice, but is not standard.

If $V = k^4$, then $\Lambda^1 V = V$ and $\Lambda^2 V$ has basis

$$e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4.$$

As we mentioned above, it is convenient to write $e_{1,2}$ to mean $e_1 \wedge e_2$. If we work with fewer than 10 vectors, skip the comma: define e_{12} to mean $e_1 \wedge e_2$, and so on:

$$e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}.$$

33.4 Suppose that

$$f\colon \underbrace{V \times V \times \cdots \times V}_{q \text{ times}} \to W$$

is an alternating multilinear map of vector spaces. Prove that there is a unique linear map $F\colon \Lambda^q V \to W$ so that

$$F(v_1 \wedge \cdots \wedge v_q) = f(v_1, \ldots, v_q)$$

for any vectors $v_1, \ldots, v_q$ in V.

The exterior algebra

Let

$$\Lambda^* V := \bigoplus_q \Lambda^q V.$$

If $x \in \Lambda^q V$, let $(-1)^x := (-1)^q$.

Lemma 33.2. *There is a unique* wedge product *on $\Lambda^* V$, denoted $\wedge$, so that*

a. *$\Lambda^* V$ is an associative algebra, and*

b. *$x \wedge y = (-1)^x (-1)^y y \wedge x$ and*

c. *for any vectors v_i in V, if I and J are disjoint sequences of increasing indices, with all indices of I less than those of J, then $v_I \wedge v_J = v_{IJ}$.*

Proof. Suppose that there is such an operation. For any vectors v_I, v_J, if all elements of I exceed those of J, then $v_I \wedge v_J = (-1)^{|I|+|J|} v_J \wedge v_I$, so these products are determined by our rules. For any other combinations, we can use this rule and associativity to slide elements v_i, v_j across one another one at a time. Taking these v_i to be a basis, we expand out a basis v_I, and we find that combining these rules inductively gives precisely one way to compute out each wedge product, so the wedge product is defined. □

If $V = k^4$, we compute in $\Lambda^* V$. The four vectors e_1, e_2, e_3, e_4 anticommute with one another, i.e. $e_2 \wedge e_1 = -e_1 \wedge e_2$ and so on. By the same rule, $e_1 \wedge e_1 = 0$ and so on. The only other rule is associativity. It is convenient as above to denote $e_1 \wedge e_2$ as $e_{1,2}$. Since our indices run only from 1 to 4, we can drop commas and denote $e_{1,2}$ by e_{12}, and so on. Remember that $e_{11} = e_1 \wedge e_1 = 0$, and so also $e_{112} = 0$. But then $e_{121} = -e_{112}$ by antisymmetry, and so $e_{121} = -e_{112} = 0$. More generally, repeated indices force zero. For instance,

$$\begin{aligned}
(e_{12} + e_{13}) \wedge (e_4 + e_2 + e_{32}) &= e_{12} \wedge (e_4 + e_2 + e_{32}) \\
&\quad + e_{13} \wedge (e_4 + e_2 + e_{32}), \\
&= e_{124} + e_{122} + e_{1232} \\
&\quad + e_{134} + e_{132} + e_{133}, \\
&= e_{124} + 0 + 0 \\
&\quad + e_{134} - e_{123} + 0, \\
&= e_{124} + e_{134} - e_{123}.
\end{aligned}$$

The Grassmannian

Take integers $0 < p \leq n$ and a vector space V of dimension n. The set of all p-dimensional linear subspaces of V is denoted $\mathrm{Gr}_p V$ or $\mathrm{Gr}_{p,n}$, and called the *Grassmannian* of p-dimensional subspaces of V, a strange but standard name.

Take p linearly independent vectors $v_1, \ldots, v_p$ from V. If we change the vectors $v_1, \ldots, v_p$ to some other vectors spanning the same subspace, say to $v_i' := \sum a_{ij} v_j$, for an invertible matrix $A = (a_{ij})$, then clearly $v_1 \wedge \cdots \wedge v_p$ changes to

$$v_1' \wedge \cdots \wedge v_p' = \det A v_1 \ldots v_p.$$

So $v_1 \wedge \cdots \wedge v_p$ is defined uniquely, up to a nonzero constant multiple, by the linear subspace spanned by $v_1, \ldots, v_p$, say $W \subset V$. To each p-dimensional linear subspace $W \subset V$, we pick a basis $v_1, \ldots, v_p$ of W and associate to W the line which is the span of $v_1 \wedge \cdots \wedge v_p$, so a point of the projective space $P = \mathbb{P}\Lambda^p V \cong \mathbb{P}^d$ where $d := \binom{\dim V}{p}$. We want to see that the Grassmannian is a projective variety.

33·5 An element $x \in \Lambda^p$ has the form $v_1 \wedge \cdots \wedge v_p$ for some vectors $v_1, \ldots, v_p$ just when the linear map

$$w \in V \mapsto w \wedge x$$

has kernel of dimension p or more, in which case the kernel has dimension p or dimension n.

Theorem 33.3. *The Grassmannian $\mathrm{Gr}_p V$ is a projective variety, with each p-dimensional linear suspace $W \in \mathrm{Gr}_p V$ sitting in projective space $\mathbb{P}\Lambda^p V$ as the line spanned by the wedge product $v_1 \wedge \cdots \wedge v_p$ of a basis $v_1, \ldots, v_p$ of W. Each rational function on the Grassmannian is a ratio of homogeneous polynomial functions of the components of elements of $\Lambda^p V$, in any basis of V.*

Proof. From among the various elements x of $\Lambda^p V$, the associated linear map $w \mapsto w \wedge x$ has rank p or less just when, in a basis, its $(p+1) \times (p+1)$ minors all vanish. Hence the set of all such elements x is an affine variety in $\Lambda^p V$ cut out by these homogeneous equations. The associated projective variety is identified with the Grassmannian, by the lemma above: each nonzero x with all such minors vanishing arises as $x = v_1 \wedge \ldots v_p$ for linearly independent $v_1, \ldots, v_p$, uniquely determined up to invertible linear combinations. □

The product $X \times Y$ of two projective varieties, X in $\mathbb{P}^p$ and Y in $\mathbb{P}^q$, is a projective variety, but not obviously so. Indeed, each $X \times \{ y_0 \}$ and each $\{ x_0 \} \times Y$ is a projective subvariety with obvious identification with X or Y. We want the rational functions on $X \times Y$ to be the field generated by those on X and those on Y, with a product of functions regular near some points x_0 on X and y_0 on Y being regular near (x_0, y_0). Each point (x, y) of $X \times Y$ gives a point of $\mathbb{P}^p$, so a line ℓ_x in k^{p+1}, and a point of $\mathbb{P}^q$, so a line ℓ_y in k^{q+1}. Hence a plane $\ell_x \times \ell_y \in k^{p+q+2}$. So we see $X \times Y$ sitting in $\mathrm{Gr}_{2,p+q+2}$ which sits in $\mathbb{P}^d$ where $d = \binom{p+q+2}{2}$. The equations of $X \times Y$ in $\mathbb{P}^d$ are those of $\mathrm{Gr}_{2,p+q+2}$ in $\mathbb{P}^d$, and then those of X inside its projective space and Y in its; it is not practical to follow through the entire process to find polynomial equations of $X \times Y$.

Any product of projective spaces is a projective variety.

The blowup of the projective plane P at a point p_0 is a projective variety, as it is the subvariety of $P \times P^*$ given by equations $0 = a_0 x_0 + a_1 x_1$, as we saw previously.

Flags

A *flag* $F = \{ V_0, \ldots, V_n \}$ in V is a collection of linear subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V,$$

where V_i has dimension i.

Every flag F becomes the flag $V_j = E_{1\ldots j}$ in a suitable basis: just pick e_1 to be any nonzero element of V_1, e_2 any nonzero element of V_2 so that e_1, e_2 are linearly independent, and so on. Such a basis is *adapted* to the flag.

33.6 A invertible linear transformation of V preserves a flag (an *automorphism* of the flag) just when it is strictly upper triangular in some, hence any, adapted basis.

Given a linear subspace $W \subset V$, we record the dimensions d_j of $W \cap V_j$. Note that $d_0 = 0$ and $d_n = p$ for any W.

Take a basis $e_1, \ldots, e_n$ of V adapted to a flag $F = \{ V_i \}$. Let $I \coloneqq \{ i_1, i_2, \ldots, i_p \}$ with

$$1 \leq i_1 < i_2 < \cdots < i_p \leq n.$$

Let E_I be the span of $e_{i_1}, \ldots, e_{i_p}$, called a *coordinate subspace*.

For $V = k^4$, we have coordinate subspaces

$$\begin{aligned} E_1, E_2, E_3, E_4 &\in \mathrm{Gr}_{1,4} = \mathbb{P}^3, \\ E_{12}, E_{13}, E_{14}, E_{23}, E_{24} &\in \mathrm{Gr}_{2,4}, \\ E_{123}, E_{124}, E_{134}, E_{234} &\in \mathrm{Gr}_{2,4} = \mathbb{P}^{3*}. \end{aligned}$$

33.7 Take the vectors $e_{i_1}, \ldots, e_{i_\ell}$, where ℓ is chosen as large possible so that $i_\ell \leq j$. These vectors are a basis of $E_I \cap V_j$. In particular, d_j is this integer ℓ.

Try $I \coloneqq \{ n-p+1, n-p+2, \ldots, n-1, n \}$; then $E_I \cap V_j$ has basis $e_{n-p+1}, \ldots, e_j$ if $n - p + 1 \leq j$, and is zero otherwise. Hence

$$0 = d_0 = d_1 = \cdots = d_{n-p}, d_{n-p+1} = 1, d_{n-p+2} = 2, \ldots, d_{n-1} = p-1, d_n = p.$$

Lemma 33.4. *Take a flag in a finite dimensional vector space V and an adapted basis for the flag. For any linear subspace $W \subseteq V$, there is a unique coordinate subspace E_I with the same intersection dimensions d_j with the flag subspaces. There is a flag automorphism taking E_I to W.*

Proof. If W lies inside V_{n-1}, then the result holds by induction. So we can suppose that W does not lie inside V_{n-1}. Let W' be the span of $e_n, W \cap V_{n-1}$. Clearly W' has the same intersection dimensions d_j as W. Moreover, if a coordinate subspace has those same intersection dimensions, it must contain e_n. By induction, we can replace $W' \cap V_{n-1} = W \cap V_{n-1}$ by a unique coordinate subspace E_I of V_{n-1} without changing its intersection dimensions. Then E_{In} is the unique coordinate subspace of V with the same intersection dimensions d_j as W. Inductively, take an automorphism of V_{n-1} preserving the flag and taking $W \cap V_{n-1}$ to $E_I \cap V_{n-1}$. Extend it to an automorphism of V by taking $w_n \mapsto e_n$ for some $w_n \in W - V_{n-1}$. □

Schubert cells and Schubert varieties

The *Schubert cell* $C_I \subseteq \mathrm{Gr}_p V$ of the flag is the set of all p-dimensional linear subspaces$W \subseteq V$ whose intersection dimensions agree with those of E_I. Since every linear subspace W has such intersection dimensions, every linear subspace belongs to a union Schubert cell, i.e. $\mathrm{Gr}_p V$ is the disjoint union of Schubert cells. The *Schubert variety* $X_I \subseteq \mathrm{Gr}_p V$ of the flag is the set of all p-dimensional linear subspaces$W \subseteq V$ whose intersection dimensions at at least those of E_I. Each Schubert variety X_I is clearly a union of Schubert cells

$$X_I = \bigcup_J C_J,$$

the union being taken over all $J = \{ j_\ell \}$ so that $j_\ell \leq i_\ell$ for all ℓ. Suppose that W has a basis $v_1, \ldots, v_p$. The dimension of $W \cap V_j$ is at least some integer d_j just when the linear map

$$v \mapsto v \wedge v_1 \wedge \cdots \wedge v_p$$

has rank at most $j - d_j$. So the various X_I are cut out by rank conditions, i.e. by minor determinants of various matrices in our basis for V. Hence X_I is a projective variety. We want to understand how to compute intersections $X_I \cap X_J$, which will tell us how conditions on intersection dimensions reinforce one another, an essential but sophisticated phenomenon in linear algebra.

If $V = k^4$, in $\mathrm{Gr}_{2,4}$ we have Schubert varieties

$$X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}.$$

So C_{12} consists of those planes $W \subset V$ with the same intersection dimensions as E_{12}. We find these: $E_{12} \cap V_1 = V_1$ has dimension 1, and $E_{12} \cap V_2 = V_2$ has dimension 2. So C_{12} consists of planes $W \subset V$ so that $W \cap V_1$ is a line and $W \cap V_2$ is a plane, so $W = V_2 = E_{12}$. Hence C_{12} is a point. Compute out intersection dimensions:

	V_1	V_2	V_3
E_{12}	1	2	2
E_{13}	1	1	2
E_{14}	1	1	1
E_{23}	0	1	2
E_{24}	0	1	1
E_{34}	0	0	1

we see that each plane W in C_{ij} has a unique basis of the given form:

$$\begin{array}{ll} C_{12} & e_1, e_2 \\ C_{13} & e_1, a_{32}e_2 + e_3 \\ C_{14} & e_1, a_{42}e_2 + a_{43}e_3 + e_4 \\ C_{23} & a_{21}e_1 + e_2, a_{31}e_1 + e_3 \\ C_{24} & a_{21}e_1 + e_2, a_{41}e_1 + a_{43}e_3 + e_4 \\ C_{34} & a_{31}e_1 + a_{32}e_2 + e_3, a_{41}e_1 + a_{42}e_2 + e_4. \end{array}$$

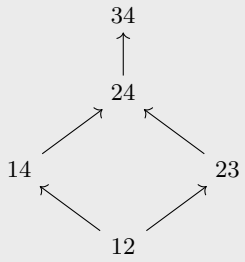
Take a plane W spanned by $e_1, a_{32}e_2 + e_3$. If $a_{32} \neq 0$, and we let $t = 1/a_{32}$, it is also spanned by

$$e_1, e_2 + te_3,$$

and so we picture that, in the "limit as $t \to 0$", we would expect e_1, e_2 to lie in the closure. Indeed, by our definition of X_{13}, X_{13} includes C_{13} and also all C_{ij} for which the intersection dimensions above are no smaller. So to find X_{13}, look above at all E_{13} having intersection dimensions $1, 1, 2$. The only other E_{ij} with intersection dimensions not smaller is E_{12}. So $X_{13} = C_{13} \cup C_{12}$. In the same way,

$$\begin{aligned} X_{12} &= C_{12}, \\ X_{13} &= C_{12} \cup C_{13}, \\ X_{14} &= C_{12} \cup C_{13} \cup C_{14}, \\ X_{23} &= C_{12} \cup C_{13} \cup C_{23}, \\ X_{24} &= C_{12} \cup C_{13} \cup C_{23} \cup C_{24}, \\ X_{34} &= C_{12} \cup C_{12} \cup C_{14} \cup C_{23} \cup C_{24} = \mathrm{Gr}_{2,4}\,. \end{aligned}$$

We track the inclusions of the Schubert varieties by drawing an arrow from one variety to another to represent containment, writing ij to represent X_{ij}:



Since distinct Schubert cells are disjoint, we see immediately how the various Schubert varieties intersect.

33.8 Repeat this for $\mathrm{Gr}_{3,6}$.

Parameterizing Schubert cells

The linear transformation in lemma 33.4 on page 281 is not uniquely determined. For example, rescaling all basis vectors, by a diagonal matrix, preserves all coordinate

subspaces. More generally, flag automorphisms preserving a coordinate subspace E_I are precisely those for which, for each j in I, column j of g has zeroes everywhere except in rows $i \leq j$ with i in I. On the other hand, a flag automorphism is *I-complementary* if its matrix agrees with the identity matrix in all positions g_{ij} with i in I or j not in I. Inductively, we can use our proof of lemma 33.4 on page 281 above to prove that there is a unique I-complementary g with $W = gE_I$.

Corollary 33.5. *Every element of C_I is uniquely expressed as $W = UE_I$ where U is I-complementary. Hence we identify C_I with the set of all I-complementary flag automorphisms, i.e. with the set of all matrices g equal to the identity matrix except at positions g_{ij} for i not in I and j in I, so $C_I = k^d$ for some integer d.*

33.9 Prove that X_I is the closure of C_I.

A linear transformation preserves the subspace E_{I_0} spanned by $e_1, \ldots, e_p$ just when, in any adapted basis, it is a block matrix

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \text{ of sizes } \begin{pmatrix} p \times p & p \times q \\ 0 & q \times q \end{pmatrix},$$

a *Borel matrix*. For I as above, let E_I be the span of the $e_{i_1}, \ldots, e_{i_p}$. If we rescale coordinate axes, i.e. multiple by a diagonal invertible matrix, the subspaces of dimension p that are preserved are precisely the coordinate subspaces. So the group of diagonal invertible matrices has these points E_I of $\mathrm{Gr}_p V$ as its fixed points.

Chapter 34

More projective planes

So full of shapes is fancy
That it alone is high fantastical.

— William Shakespeare
Twelfth Night

The pure mathematician, like the musician, is a free creator of his world of ordered beauty

— Bertrand Russell

We know that man has the faculty of becoming completely absorbed in a subject however trivial it may be, and that there is no subject so trivial that it will not grow to infinite proportions if one's entire attention is devoted to it.

— Leo Tolstoy
War and Peace

Projective planes

A *projective plane* (in a more general sense of the term than we have used previously) is a set P, whose elements are called *points*, together with a set L, whose elements are called *lines*, and a defined notion of incidence, i.e. of a point being "on" a line (or we could say that a line is "on" or "through" a point) so that

a. Any two distinct points both lie on a unique line.

b. Any two distinct lines both lie on a unique point.

c. There are 4 points, with no 3 on the same line.

We usually denote a projective plane as P, leaving its set L of lines understood, and its notion of incidence understood.

Lemma 34.1. *The projective plane $\mathbb{P}^2(k)$ over any field is a projective plane in this more general sense.*

Proof. Given any two distinct points of $\mathbb{P}^2(k)$, they arise from two distinct vectors in k^3, determined up to rescaling. Such vectors lie in a basis of k^3, and span a plane in k^3 through the origin. The lines in that plane lying through the origin consitute a projective line through the two points. Similarly, any two distinct projective lines are the lines through the origin of two distinct planes through the origin. Those planes are each cut out by a linear equation, and the linear equations have linearly independent coefficients, because they are distinct planes. But then the simultaneous solutions of the two equations in three variables form a line through the origin, a projective point. To find 4 points, with no 3 on the same line, consider the 3 points

$$\begin{bmatrix}1\\a\\b\end{bmatrix}, \begin{bmatrix}0\\1\\c\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

for any constants a, b, c from k. For these points to be on the same projective line, the corresponding vectors

$$\begin{pmatrix}1\\a\\b\end{pmatrix}, \begin{pmatrix}0\\1\\c\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

must lie on the same plane, i.e. satisfying the same linear equation with not all coefficients zero. But these vectors are linearly independent. Now take the 4 points

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

and check that any 3 do not lie in a projective line. □

34.1 Prove that every line in a projective plane contains at least three points.

Duality

We defined polygons in projective planes over fields in section 25 on page 223; we employ the same definitions in any projective plane. If P is a projective plane with set of lines L, the *dual projective plane*, denoted P^*, has L as its set of "points" and P as its set of "lines", and the same incidence notion, i.e. we use interchange the words "point" and "line".

Lemma 34.2. *The dual P^* of a projective plane P is a projective plane, and $P^{**} = P$.*

Proof. We have only to check that there is a quadrilateral in P^*. Pick a quadrilateral in P, and we need to see that its 4 edges, in the obvious cyclic ordering, are the vertices of a quadrilateral in P^*. Three of those vertices in P^* lie on a line in P^* just when three of the lines of the original quadrilateral lie through the same point, say p. So we need to prove that no quadrilateral has three of its edges through the same point. Taking the remaining edge as the fourth edge, the first three edges of the quadrilateral pass through the same point, so that point is the first and the second vertex. But then the third vertex lies on a line with the first two. □

For example, if V is a 3-dimensional vector space over a field k, the associated projective plane $\mathbb{P}V$ is the set of lines through the origin of V. A line in $\mathbb{P}V$ is a plane through the origin of V. Every plane through the origin is the set of solutions of a linear equation, i.e. is the set of zeroes of a linear function $f\colon V \to k$, i.e. of an element $0 \neq f \in V^*$ of the dual vector space. Such a function is uniquely determined by its zeroes, up to rescaling. So each line in $\mathbb{P}V$ is identified with a 2-plane through the origin of V and thereby with a linear function $f \in V^*$ defined up to rescaling, and thereby with a line $k \cdot f \subset V^*$ through the origin of V^*, and thereby with a point of $\mathbb{P}V^*$: $\mathbb{P}(V^*) = \mathbb{P}(V)^*$.

Projective spaces

A *projective space* is a pair of sets, called the sets of *points* and *lines*, together with a definition of *incidence* of points on lines, so that

a. Any two distinct points p, q are on exactly one line pq.

b. Every line contains at least three points.

c. For any four points p, q, r, s, with no three on the same line, if pq intersects rs then pr intersects qs.

Points are *collinear* if they lie on a line, and lines *intersect* at a point if they both lie on that point. By *a.* and *b.*, every line of a projective space is determined uniquely by the set of points lying on it. Therefore it is safe to think of lines as sets of points.

34.2 Prove that $\mathbb{P}^n(k)$ is a projective space, for any field k.

Given a point p and a line ℓ, let $p\ell$, the *plane* spanned by p and ℓ, be the set of points q so that $q = p$ or pq intersects ℓ. A *line* of a plane $P = p\ell$ in a projective space is a line of the projective space which intersects two or more points of P.

Lemma 34.3. *Every plane in a projective space is a projective plane.*

Proof. Take a projective space S, and a plane $P = p\ell$ in S. Looking at the definition of projective space, the first requirement ensures the first requirement of a projective plane: any two distinct points line in a unique line. Take two distinct lines m, n. We need to show they intersect at a unique point, and that this point also lies in P.

First, suppose that $m = \ell$ and that p lies on n. We need to prove that n intersects ℓ. By definition of lines in a plane, there are at least two points of P on each line of P. Pick a point $q \neq p$ from P on n. Then $n = pq$ intersects $m = \ell$ at a point of ℓ, by definition of a plane.

Next, suppose that $m = \ell$ but that p does not lie on n. We need to prove that n intersects ℓ. By definition of n being a line on the plane P, n contains two points of P. Pick two points q, q' of P lying on n. If either point lies on ℓ, we are done. Let r, r' be the intersection points of pq, pq' with ℓ, which exist by definition of a plane. Then $pq = qr$ and $pq' = q'r'$ so qr and $q'r'$ intersect at p. Suppose that three of q, q', r, r' lie on the same line. But $n = qq'$ so that line is n. But then r is on n, so n intersects ℓ, and we are done. So we can assume that q, q', r, r' are not collinear. By property *c.*, since qr and $q'r'$ intersect (at p), $qq' = n$ intersects $rr' = \ell$.

Next, suppose that $m \neq \ell$. Repeating the above if $n = \ell$, we can assume that $n \neq \ell$. From the above, m and n both intersect ℓ, say at points a and b. If $a = b$ then m and n intersection and we are done. If $a \neq b$, take points $a' \neq a$ in m and

$b' \neq b$ in n both lying in P. Repeating the above, $a'b'$ intersects ℓ. So either three of a, a', b, b' are collinear, or $aa' = m$ intersects $bb' = n$. Suppose that three of a, a', b, b' are collinear. Then one of b or b' lies in m, or one of a, a' lies in n, so m and n intersect at a point of P. Therefore we can assume that no three of a, a', b, b' are collinear. By property c. on the preceding page, since $a'b'$ and ab intersect, $aa' = m$ intersects $bb' = n$. We have established that any two lines of P intersect in S. We have to prove that this intersection point c lies in the plane P. The line pa is a line of P, and so intersects ℓ, m and n. The line $n = bc$ is also a line of P. If three of p, a, b, c are collinear, then p lies in n or a lies in n, and we are done as above. By property c. on the previous page, since pa and bc intersect, pc intersects $ab = \ell$. By definition of a plane, the point c belongs to P. So we have established that any two distinct lines of P lie on a unique point of P.

Finally, take two distinct points a, b on ℓ and $a' \neq a$ on pa and $b' \neq b$ on pb. By definition, a, b, a', b' all lie on P. Collinearity of any three of these points would, by the same argument, force $a = b$. □

Corollary 34.4. *If a line lies in a plane in a projective space, then every point lying on that line lies on that plane.*

A *projective subspace* of a projective space *of dimension* $-\infty$ means an empty set, *of dimension zero* means a point, and inductively of dimension $n+1$ means the union of all points on all lines through a projective subspace of dimension n and a disjoint point. A line of a projective subspace is a line of the projective space containing two or more points of the subspace. Clearly every projective subspace is itself a projective space, by induction from the proof that every plane in a projective space is a projective plane.

Lemma 34.5. *In any 3-dimensional projective space, every line meets every plane.*

Proof. Pick a plane P. We want to see that every line ℓ meets P. Suppose that $\ell = ab$ for two points a, b. If either of a, b are on P, then ℓ meets P, so suppose otherwise.

Pick a point q_0 not in P. Suppose that $a \neq q_0$ and $b \neq q_0$. Each of a, b lies on a line q_0a', q_0b', for some a', b' on P, since our projective space has dimension 3. If $a = a'$ then a is on P and ℓ so ℓ meets P. Similarly if $b = b'$. Therefore we can suppose that $a \neq a'$ and $b \neq b'$. So aa' and bb' intersect at q_0. Suppose that three of a, a', b, b' lie on a line. Then $aa' = q_0a'$ contains b or b', so equals $ab = \ell$ or $a'b'$ lying on P. In the first case, $ab = \ell$ then contains a', which lies on P, so ℓ meets P. In the second case, $a'b'$ lies in P and contains a which lies on ℓ, so ℓ meets P. So we can assume that no three of a, a', b, b' lie on a line. By c., ab intersects $a'b'$, so ℓ intersects a line in P. □

Lemma 34.6. *Any two distinct planes in a 3-dimensional projective space meet along a unique line.*

Proof. Take planes $P \neq P'$. Take a point p_0 in P but not P'. Pick two distinct lines ℓ, m in P through p_0. Let a be the intersection of ℓ and P', and b the intersection of m and P'. The line ab lies in P' and P. If some common point c of P and P' does not lie in ab, then the plane abc lies in P and P', so $P = P'$. □

The quaternionic projective spaces

The *quaternions* over a commutative ring R are the quadruples $a = (a_0, a_1, a_2, a_3)$ of elements of R, each written as

$$a = a_0 + a_1 i + a_2 j + a_3 k.$$

They form the ring $\mathbb{H}_R$ with addition component by component and multiplication given by the rules

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j,$$

and conjugation defined by

$$\bar{a} = a_0 - a_1 i - a_2 j - a_3 k.$$

The *quaternions* $\mathbb{H}$ (with no underlying commutative ring specified) are the quaternions over $R = \mathbb{R}$.

34·3 Prove that $\mathbb{H}_R$ is associative, by a direct computation.

A *skew field* is an associative ring with identity, not necessarily commutative, in which every element a has a multiplicative inverse: $a^{-1}a = aa^{-1} = 1$.

34·4 Check that

$$a\bar{a} = \bar{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

In particular, if k is a field in which sums of squares are never zero unless all elements are zero (as when $k = \mathbb{R}$), prove that $\mathbb{H}_k$ is a skew field.

34·5 Suppose that k is a skew field. Define $\mathbb{P}^2(k)$ by analogy with the definition for a field. Explain why $\mathbb{P}^2(k)$ is then a projective plane.

The *quaternionic projective plane* $\mathbb{P}^2(\mathbb{H})$ is the projective plane of the *quaternions*.

Take a skew field k. Add vectors x in k^n as usual component by component, but multiply by elements ra in k on the right:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} a = \begin{pmatrix} x_1 a \\ x_2 a \\ \vdots \\ x_n a \end{pmatrix}.$$

Multiply by matrices on the left as usual.

A *vector space* V over a skew field k is a set V equipped with an operation $u + v$ and an operationa va so that

Addition laws:

1. $u + v$ is in V
2. $(u + v) + w = u + (v + w)$
3. $u + v = v + u$

for any vectors u, v, w in V,

Zero laws:

1. There is a vector 0 in V so that $0 + v = v$ for any vector v in V.
2. For each vector v in V, there is a vector w in V, for which $v + w = 0$.

Multiplication laws:

1. va is in V
2. $v \cdot 1 = v$
3. $v(ab) = (va)b$
4. $v(a + b)v = va + vb$
5. $(u + v)a = ua + va$

for any numbers $a, b \in k$, and any vectors u and v in V.

For example, $V = k^n$ is a vector space, with obvious componentwise addition and scaling. A *subspace* of a vector space V is a nonempty subset U of V so that when we apply the same scaling and addition operations to elements of U we always get elements of U. In particular, U is also a vector space.

A *linear relation* between vectors $v_1, v_2, \ldots, v_p \in V$ is an equation

$$0 = v_1 a_1 + v_2 a_2 + \cdots + v_p a_p$$

satisfied by these vectors, in which not all of the coefficients $a_1, a_2, \ldots, a_p$ vanish. Vectors with no linear relation are *linearly independent*. A *linear combination* of vectors $v_1, v_2, \ldots, v_p \in V$ is vector

$$v_1 a_1 + v_2 a_2 + \cdots + v_p a_p$$

for some elements $a_1, a_2, \ldots, a_p$ from k. One weird special case: we consider the vector 0 in V to be a linear combination of an *empty* set of vectors. The *span* of a set of vectors is the set of all of their linear combinations. A *basis* for V is a set of vectors so that every vector in V is a unique linear combination of these vectors. Note that some vector spaces might not have any basis. Also, if $V = \{ 0 \}$ then the empty set is a basis of V.

Theorem 34.7. *Any two bases of a vector space over a skew field have the same number of elements.*

Proof. If $V = \{ 0 \}$, the reader can check this special case and see that the empty set is the only basis. Take two bases $u_1, u_2, \ldots, u_p$ and $v_1, v_2, \ldots, v_q$ for V. If needed, swap bases to arrange that $p \leq q$. Write the vector v_1 as a linear combination

$$v_1 = u_1 a_1 + u_2 a_2 + \cdots + u_p a_p.$$

If all of the elements $a_1, a_2, \ldots, a_p$ vanish then $v_1 = 0$, and then zero can be written as more than one linear combination, a contradiction. By perhaps reordering our basis vectors, we can suppose that $a_1 \neq 0$. Solve for u_1:

$$u_1 = (v_1 + u_2 (-a_2) + \cdots + u_p (-a_p)) \, a_1^{-1}.$$

We can write every basis vector $u_1, u_2, \ldots, u_p$ as a linear combination of $v_1, u_2, \ldots, u_p$, and so the span of $v_1, u_2, \ldots, u_p$ is V. If there is more than one way to write a vector in V as such a linear combination, say

$$v_1 b_1 + u_2 b_2 + \cdots + u_p b_p = v_1 c_1 + u_2 c_2 + \cdots + u_p c_p$$

then the difference between the left and right hand sides is a linear relation

$$0 = v_1 d_1 + u_2 d_2 + \cdots + u_p d_p.$$

Solve for v_1 in terms of u_1 and plug in:

$$0 = u_1 a_1 d_1 + u_2 (d_2 + a_2 d_1) + \cdots + u_p (d_p + a_p d_1).$$

Since these $u_1, u_2, \ldots, u_p$ form a basis,

$$0 = a_1 d_1 = d_2 + a_2 d_1 = \cdots = d_p + a_p d_1.$$

Multiplying the first equation by a_1^{-1} gives $d_1 = 0$, and then that gives $0 = d_2 = \cdots = d_p$, a contradiction. Therefore $v_1, u_2, \ldots, u_p$ is a basis. By induction, repeating this process, we can get more and more of the vectors $v_1, v_2, \ldots, v_q$ to replace the vectors $u_1, u_2, \ldots, u_p$ and still have a basis. Finally, we will find a basis $u_1, u_2, \ldots, u_p$ consisting of a subset of p of the vectors $v_1, v_2, \ldots, v_q$. But then every vector is a unique linear combination of some subset of these vectors, a contradiction unless that subset is all of them, i.e. $p = q$. □

The *dimension* of a vector space is the number of elements in a basis, or ∞ if there is no basis. A *linear map* $f\colon V \to W$ between vector spaces V, W over a skew field k is a map so that $f(v_1 + v_2) = f(v_1) + f(v_2)$ for any v_1, v_2 in V and $f(va) = f(v)a$ for any v in V and a in k.

34.6 Prove that for any linear map $f\colon V \to W$ between vector space $V = k^q$ and $W = k^p$, there is a unique $p \times q$ matrix A with entries from k so that $f(v) = Av$ for any v in V.

The *kernel* of a linear map $f\colon V \to W$ is the set of all vectors v in V so that $f(v) = 0$. The *image* of a linear map $f\colon V \to W$ is the set of all vectors w in W of the form $w = f(v)$ for some v in V.

34.7 Prove that the kernel and image of a linear map $f\colon V \to W$ are subspaces of V and of W and that their dimensions add up to the dimension of V.

34.8 Prove that for any basis $v_1, v_2, \ldots, v_n$ for a vector space V, there is a unique linear map $f\colon V \to W$ with has any given values $f(v_i) = w_i$.

34.9 Imitating the proof of lemma 34.1 on page 285, prove that $\mathbb{P}^2(k)$ is a projective plane for any skew field k.

34.10 Prove that $\mathbb{P}^n(k)$ is a projective space for any skew field k.

Incidence geometries

> *The world is all that is the case.*
> *The world is the totality of facts, not of things.*
>
> — Ludwig Wittgenstein
> Tractatus Logico-Philosophicus

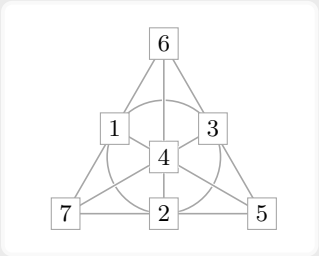
We want to consider more general configurations of points and lines than projective planes or spaces. An *incidence geometry*, also called a *configuration*, is set P, whose elements are called *points*, together with a set L, whose elements are called *lines*, and a defined notion of incidence, i.e. of a point being "on" a line (or we could say that a line is "on" or "through" a point).

As points, take $P := \mathbb{Z}/m\mathbb{Z}$ and as lines takes $L := \mathbb{Z}/m\mathbb{Z}$ and then say that a point p is on a line ℓ if $p = \ell$ or $p = \ell + 1$. Call this a *polygon* or an m-gon. If $m = 3$, call it a *triangle*, and if $m = 4$ a *quadrilateral*.

Any projective plane or projective space is a configuration.

The affine plane is a configuration.

If we just draw any picture on a piece of paper of some dots ("points") and some squiggly curves ("lines", not necessarily straight), with some indication which lines intersect which points, that describes a configuration. For example, we can draw the Fano plane as:



where the "points" are the numbered vertices and the "lines" are the straight lines, including the three straight lines through the center, and the circle.

A *morphism* of configurations

$$f\colon (P_0, L_0) \to (P_1, L_1)$$

is a pair of maps, both called f, $f\colon P_0 \to P_1$ and $f\colon L_0 \to L_1$ so that if a point p lies on a line ℓ, then $f(p)$ lies on $f(\ell)$.

Every projective plane contains a quadrilateral, so has a morphism from the quadrilateral $P = \mathbb{Z}/4\mathbb{Z}$.

Take the Fano plane, using our diagram, and delete the one line corresponding to the circle; call the result the *kite*. Every projective plane $\mathbb{P}^2(k)$ over a field k contains the points $[x, y, z]$ for x, y, z among $0, 1$ and not all zero, and contains the various lines between them, i.e. contains the kite. So there is a morphism,

injective on points and lines, from the kite to every projective plane over any field. In fact, if we start off with any quadrilateral in any projective plane, and we draw from it all lines through any two of its vertices, and then all of the intersections of those lines, and then all lines through any two vertices of that configuration, and so on, we end up with the kite.

If k has a field extension K, then the obvious map $\mathbb{P}^2(k) \to \mathbb{P}^2(K)$ is a morphism.

An *isomorphism* is a morphism whose inverse is also a morphism, so that a point lies on a line just exactly when the corresponding point lies on the corresponding line. An *automorphism* of a configuration is an isomorphism from the configuration to itself. A *monomorphism* is a morphism injective on points and on lines, so that a point lies on a line after we apply the morphism just when it did before.

By definition, every projective plane admits a monomorphism from a quadrilateral.

The map $\mathbb{P}^2(\mathbb{R}) \to \mathbb{P}^2(\mathbb{C})$ is *not* an isomorphism, as it is not onto on either points or lines, so has no inverse, but it is a monomorphism.

The map $f\colon \mathbb{P}^n(k) \to \mathbb{P}^{n+1}(k)$ taking

$$f\left[x_0, x_1, \ldots, x_n\right] = \left[x_0, x_1, \ldots, x_n, 0\right]$$

is a monomorphism.

Invertible matrices with entries in a field k act as projective automorphisms on $\mathbb{P}^2(k)$, and similarly in all dimensions; call these *linear* projective automorphisms or k-projective automorphisms.

Let $k := \mathbb{Q}(\sqrt{2})$ and let $f\colon k \to k$ be the automorphism

$$f\left(a + b\sqrt{2}\right) = a - b\sqrt{2}$$

for any a, b in $\mathbb{Q}$. Then clearly f extends to k^{n+1} for any integer n, and so extends to $\mathbb{P}^n(k)$, giving an automorphism which is *not* linear.

34.11 Any two quadrilaterals in the projective plane $\mathbb{P}^2(k)$ over any field k can be taken one to the other by a linear automorphism. Hint: think about the linear algebra in k^3, get the first three vectors where you want them to go, and think about rescaling them to get the last vector in place.

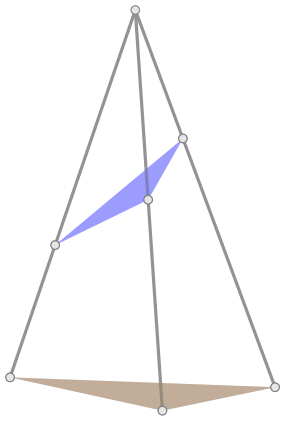
The *dual* of a configuration has as line the points and as points the lines and the same incidence relation. If $f\colon A \to B$ is a morphism of configurations, define the *dual morphism* to be the same maps on points and lines, but interpreted as lines and points respectively.

Desargues' theorem

> *Beauty is the first test: there is no permanent place in the world for ugly mathematics*
>
> — G. H. Hardy

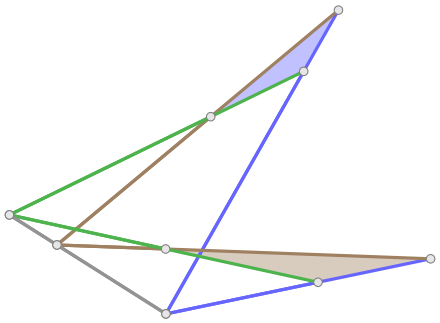
> *What would your good be doing if there were no evil, and what would the earth look like if shadows disappeared from it? After all, shadows are cast by objects and people. There is the shadow of my sword. But there are also shadows of trees and living creatures. Would you like to denude the earth of all the trees and all the living beings in order to satisfy your fantasy of rejoicing in the naked light?*
>
> — Mikhail Bulgakov
> The Master and Margarita

Two triangles with a chosen correspondence between their vertices are *in perspective* from a point if the lines through corresponding vertices pass through that point.



34.12 Prove that, on any projective plane in which every line has at least 4 points, there are two triangles which are *not* in perspective from any point.

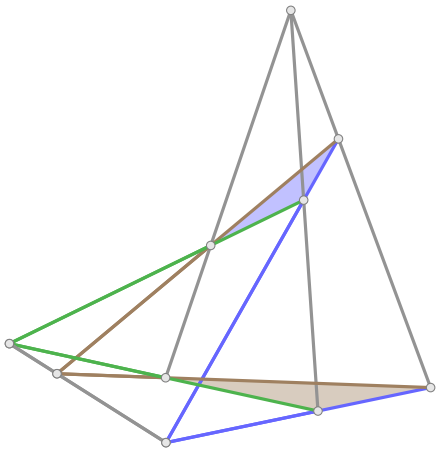
The two triangles are *in perspective* from a line if corresponding edges intersect on that line.



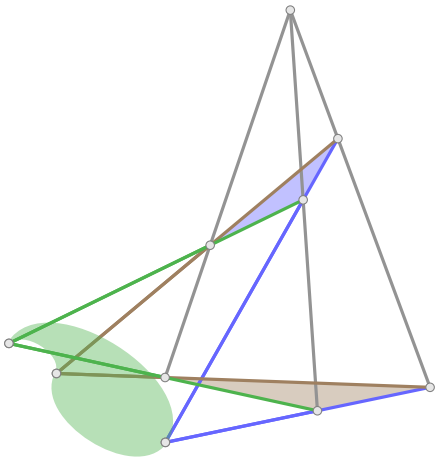
34.13 Prove that a pair of triangles in a projective plane are in perspective from a line just when the dual triangles are in perspective from a point.

34.14 Prove that every projective plane has a pair of triangles in perspective from a point, and a pair of triangles in perspective from a line.

A configuration is *Desargue* if every pair of triangles in perspective from some point is also in perspective from some line. If we draw the picture very carefully, it appears that the real projective plane is Desargues. The incidence geometry



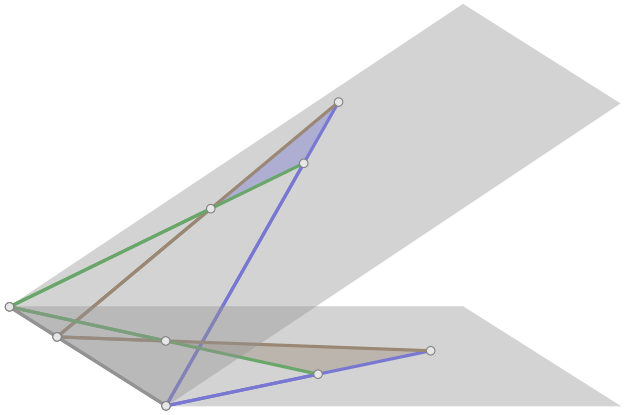
is the *standard Desargues configuration*. The same configuration with three curves added through those 3 points to determine a triangle is the *broken Desargues configuration*.



A configuration is thus Desargues just when every morphism to it from the configuration of a pair of triangles in perspective extends to a morphism from a standard Desargue configuration, or equivalently the configuration admits no monomorphisms from the broken Desargues configuration. Not every projective plane is Desargues, as we will see.

34.15 Prove that a projective plane is Desargues precisely when, for any pair of triangles in perspective, their dual triangles in the dual projective plane are also in perspective.

Lemma 34.8. *Every* 3*-dimensional projective space is Desargues.*



Proof. Suppose that the two triangles lie on different planes. Then these two planes intersect along a single line. The lines we drew along the sides of either triangle lie in one or the other of those two different planes, and can only intersect along that line. Therefore the intersection points lie on that line, i.e. the intersection points are collinear.

But so far we have assumed that the two triangles in perspective lie in different planes. Suppose that they lie in the same plane. We want to "perturb" the triangles out of that plane. Pick a point outside that plane, which exists by axiom *c.*, and call it the *perturbed origin* o'. On the line $o'a$ from the perturbed origin o' to one of the vertices a of the first triangle, pick a point, the *perturbed vertex* a' of the first triangle, not lying in the original plane. Take the vertex A in the second triangle corresponding to a. The points o, a, A are collinear by definition of triangles in perspective. The points o, o', a, A therefore are coplanar. The point a' by construction lines in this plane. So the lines oa' and $o'A$ lie in that plane, and intersect in a point A', the *perturbed vertex* of the second triangle. The triangles with perturbed vertices are now in perspective from the origin o (*not* from the perturbed origin). The perturbed triangles lie in different planes, since otherwise the entire perturbed picture lies in a single plane, which contains the other vertices of our two triangles, so lies in the original plane.

Essentially we just perturbed a vertex of the first triangle, and then suitably perturbed the corresponding one in the second triangle to get a noncoplanar pair of triangles in perspective. The intersection points p', q', r' of the perturbed standard Desargues configuration lie in a line $p'q'r'$ by the previous reasoning. Take the plane containing that line $p'q'r'$ and the perturbed origin o'. That plane intersects the plane P of the original unperturbed triangles along a line ℓ. The lines $o'p', o'q', o'r'$ intersect the plane P at the three intersection points p, q, r of the unperturbed configuration, so these lie in ℓ. □

Theorem 34.9. *Every projective space of dimension* 3 *or more is Desargue.*

Proof. Start with any triangle: it lies on a plane. Add one more point: the picture still lies in a 3-dimensional projective space. But then we can start our sequence of pictures above constructing the two triangles in perspective, and then the entire Desargues configuration, inside that 3-dimensional projective space. □

For example, projective space $\mathbb{P}^n(k)$ of any dimension $n \geq 3$ over any skew field k is Desargues.

Corollary 34.10. *If a projective plane embeds into a projective space of some dimension $n \geq 3$, then the projective plane is Desargues.*

Projective space $\mathbb{P}^n(k)$ of any dimension $n \geq 2$ over any field k is Desargues.

Generating projective planes

A monomorphism $A \to P$ from a configuration to a projective plane P is *universal* if every monomorphism $A \to Q$ to a projective plane Q factors through a morphism $P \to Q$, i.e. $A \to P \to Q$. A configuration is *planar* if any two distinct points of A lie on at most one line. A planar configuration is *spanning* if there are 4 points with no three collinear.

Lemma 34.11. *A configuration is planar if and only if it has a monomorphism to a projective plane. A planar configuration is spanning if and only if it has a universal monomorphism to a projective plane; this projective plane, the plane* generated *by the configuration, is then unique up to a unique isomorphism.*

Proof. Note that any two lines of a planar configuration A intersect at, at most, one point, since otherwise they would contain two distinct points lying on two different lines.

For any configuration B, let B_+ have the same points as B, but add in a new line for each pair of points that don't lie on a line. To be more precise, we can let the lines of B_+ be the lines of B together with the unordered pairs of points $\{p, q\}$ of B that don't lie on a line of B.

For any configuration B, let B^+ have the same lines as B, but add in a new point for each pair of lines that don't lie on a point. To be more precise, we can let the points of B^+ be the points of B together with the unordered pairs of lines $\{m, n\}$ of B that don't lie on a point of B. Inductively, let $A_0 := A$, $A_{2j+1} := A_{2j,+}$ and $A_{2j+2} := A^+_{2j+1}$.

Let P be the configuration whose points are all points of $A_1, A_2, \ldots$, and whose lines are the lines of $A_1, A_2, \ldots$. Clearly any two distinct points of P lie on a unique line, and any two distinct lines of P lie on a unique point. Suppose that A is spanning. Three noncollinear points of A will remain noncollinear in A_+, because we only add new points. They will also remain noncollinear in A^+, because we only add new lines through pairs of points, not through triples. Therefore they remain noncollinear in P, so that P is a projective plane.

If $f\colon A \to Q$ is a monomorphism to a projective plane, define $f\colon A_+ \to Q$ by $f\{p, q\} = f(p)f(q)$, and similarly for $A^+ \to Q$, to define a morphism $f\colon P \to Q$.

Given another universal morphism $f'\colon A \to P'$, both monomorphisms must factor via $A \to P \to P'$ and $A \to P' \to P$. These factoring maps $P \to P'$ and $P' \to P$ are isomorphisms, as one easily checks, by looking step by step at how they act on points and lines. □

A *trivalent configuration* is one in which every point is on at least three lines and every line is on at least three points.

Lemma 34.12. *Take a spanning planar configuration A and its universal morphism to a projective plane $A \to P$. Every monomorphism $T \to P$ from a finite trivalent configuration factors through $T \to A \to P$.*

Proof. At each stage of constructing P, we never put in a new line passing through more than two points. So among the lines of T, the last to appear in the construction of P contains only two points, a contradiction, unless T lies in A. □

Let A have 4 points and no lines, like ::, which is clearly spanning and planar. Take P to be the projective plane generated by A. By the lemma, no line in P contains 3 points. As every line in the Desargues configuration contains 3 points, P does not contain the Desargues configuration. Start with the quadrilateral given by the points of A, thought of as one point and one triangle of P. Since every line in any projective plane contains at least 3 points, we can then build a pair of triangles in perspective in P out of our triangle and point, as in our pictures of the Desargues configuration. Therefore P is not Desargues.

Octonions

In this section, we will allow the use of the word *ring* in a somewhat more general context. A *ring* is a set of objects S together with two operations called *addition* and *multiplication*, and denoted by $b+c$ and bc, so that if b and c belong to S, then $b+c$ and bc also belong to S, and so that:

Addition laws:

a. The associative law: For any elements a, b, c in S: $(a+b)+c = a+(b+c)$.

b. The identity law: There is an element 0 in S so that for any element a in S, $a+0=a$.

c. The existence of negatives: for any element a in S, there is an element b in S (denote by the symbol $-a$) so that $a+b=0$.

d. The commutative law: For any elements a, b in S, $a+b=b+a$.

Multiplication laws:

The Distributive Law:

a. For any elements a, b, c in S: $a(b+c) = ab+ac$.

We do *not* require the associative law for multiplication; if it is satisfied the ring is *associative*.

A ring is *a ring with identity* if it satisfies the *identity law*: There is an element 1 in S so that for any element a in S, $a1=a$. We will only ever consider rings with identity.

A ring is *a division ring* if it satisfies the zero divisors law: for any elements a, b of S, if $ab=0$ then $a=0$ or $b=0$.

A ring is *commutative* if it satisfies the *commutative law*: for any elements a, b of S, $ab=ba$.

The associative law for addition, applied twice, shows that $(a+b)+(c+d) = a+(b+(c+d))$, and so on, so that we can add up any finite sequence, in any order, and get the same result, which we write in this case as $a+b+c+d$. A similar story holds for multiplication *if* R is associative.

Any skew field k has a projective space $\mathbb{P}^n(k)$ for any integer n, defined as the set of tuples

$$x = (x_0, x_1, \ldots, x_n)$$

with $x_0, x_1, \ldots, x_n \in k$, but with two such tuples equivalent just when the second equals the first scaled by a nonzero element:

$$x = (x_0, x_1, \ldots, x_n) \cong xc = (x_0c, x_1c, \ldots, x_nc)\,.$$

This approach doesn't work for nonassociative rings, because it might be that $(xa)b$ can not be written as $x(ab)$ or even as xc for any c. It turns out to be a very tricky problem to define a meaning for the expression $\mathbb{P}^n(R)$ if R is a nonassociative ring, a problem which does not have a general solution.

A *conjugation* on a ring R is a map taking each elements b of R to an element, denoted $\bar{b}$, of R, so that

a. $\overline{b+c} = \bar{b}+\bar{c}$,

b. $\overline{bc} = \bar{c}\bar{b}$,

c. $\bar{1} = 1$ if R has a chosen multiplicative identity element 1,

d. $\bar{\bar{b}} = b$,

for any elements b, c of R. A ring with a conjugation is a *$*$-ring*. If R is a $*$-ring, let R^+ be the set of pairs (a, b) for a, b in R; write such a pair as $a + \iota b$, and write $a + \iota 0$ as a, so that $R \subset R^+$. Endow R^+ with multiplication

$$(a + \iota b)\,(c + \iota d) := ac - d\bar{b} + \iota\,(\bar{a}d + cb)$$

with conjugation

$$\overline{a + \iota b} := \bar{a} - \iota b$$

and with identity 1 and zero 0 from $R \subset R^+$. For example, $\mathbb{R}^+ = \mathbb{C}$, $\mathbb{H} := \mathbb{C}^+$ is the *quaternions* and $\mathbb{O} := \mathbb{H}^+$ is the *octaves*, also called the *octonions*. A *normed ring* is a $*$-ring for which the vanishing of a finite sum

$$0 = a\bar{a} + b\bar{b} + \cdots + c\bar{c}$$

implies $0 = a = b = \cdots = c$.

34.16 If R is a normed ring then prove that R^+ also a normed ring.

34.17 If R is associative, normed and has no zero divisors, then prove that R^+ has no zero divisors.

A $*$-ring is *real* if $a = \bar{a}$ for every a.

34.18 Prove that R^+ is not real for any $*$-ring R.

34.19 Prove that every real $*$-ring is commutative.

34.20 Prove that R^+ is not real, for any $*$-ring R, and is therefore not commutative.

34.21 Prove that R is real just when R^+ is commutative.

34.22 Prove that R is commutative and associative just when R^+ is associative.

The *associator* of a ring R is the expression

$$[a, b, c] = (ab)c - a(bc).$$

Clearly a ring is associative just when its associator vanishes. A ring is *alternative* when its associator alternates, i.e. changes sign when we swap any two of a, b, c.

34.23 Prove that any alternative ring satisfies $b(cb) = (bc)b$ for any elements b, c. We denote these expressions both as bcb, without ambiguity.

34.24 Prove that R is normed and associative just when R^+ is normed and alternative.

34.25 Prove that if R is normed and associative and every nonzero element of the form

$$a\bar{a} + b\bar{b}$$

in R has a left inverse c then every element $a + \iota b$ in R^+ has a left inverse: $c\bar{a} - \iota cb$.

In particular, $\mathbb{O}$ is a normed alternative division ring and every element of $\mathbb{O}$ has a left inverse.

Coordinates

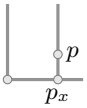
Given a projective plane P, take a quadrilateral:

Call this:

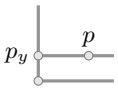
the *line at infinity*. Henceforth, draw it far away. Two lines:

are *parallel* if they meet at the line at infinity. One of the two "finite" points of our quadrilateral we designate the *origin*. We draw lines from it out to the two "far away" points: the *axes*:

Map any point p not on the line at infinity to points on the two axes:



and



The map

$$p \mapsto (p_x, p_y)$$

is invertible: if we denote the "far away" points ∞_x and ∞_y:

$$(p_x, p_y) \mapsto p = (p_x \infty_x)\,(p_y \infty_y)\,,$$

This map takes the finite points p (i.e. not on the line at infinity) to their "coordinates" p_x and p_y on the axes, and takes finite points on the axes to points in the plane. So we write the point p as (p_x, p_y). We still haven't used the fourth point of the quadrilateral:

Draw lines through it parallel to the axes:

Call the points where these lines hit the axes $(1,0)$ and $(0,1)$. Take any point on one axis and draw the line through it parallel to the line $(1,0)(0,1)$:

Identify two points of the two axes if they lie on one of these parallels, but of course we write them as $(x,0)$ or $(0,x)$.

The points of the line at infinity we think of as slopes. Any point of the line at infinity, except the one on the vertical axis, is a "finite slope". Draw lines through the origin.

Each line (except the vertical axis) passes through a point $(1,m)$ and a point on the line at infinity, which we label as m.

A line in the finite plane has *slope* m if it hits the line at infinity in this point labelled m. If it strikes the vertical axis at $(0,b)$, we call b its intercept. Given points x,m,b on the finite horizontal axis, we write $mx+b$ to mean the point y on the finite horizontal axis so that the point with coordinates (x,y) lies on the line with slope m and vertical intercept b. This defines an operation $x,m,b \mapsto mx+b$, which depends on three arguments x,m,b on the finite horizontal line.

A *ternary ring* (sometimes called a *planar ternary ring*) is

a. an operation taking three elements x,m,b of a set R to an element, denoted $mx+b$, of the set R,

b. together with a choice of two elements $0,1$ of R, so that for any x,y,m,n,b,c in X:

1. $m0+b = 0x+b = b$,
2. $m1+0 = m$,
3. $1x+0 = x$,
4. $mx+b = y$ has a unique solution b for any given x,m,y,
5. $mx+b = y$ has a unique solution x for any given b,m,y if $m \neq 0$,
6. $mx+b = nx+c$ has a unique solution x if $m \neq n$,
7. The pair of equations $mx_1+b = y_1$, $mx_2+b = y_2$ has a unique solution m,b if $x_1 \neq x_2$ and $y_1 \neq y_2$.

Warning: a ternary ring is *not* necessarily a ring. Given a ternary ring, we can define $a + b$ to mean $1a + b$, and define ab to mean $ab + 0$, defining an addition and multiplication operation, but these do not necessarily satisfy the axioms of a ring. *Warning:* even worse, with these definitions of addition and multiplication, it might turn out that $(mx) + b$ is not equal to the ternary operation $mx + b$, so we will be safe: we avoid reference to the addition and multiplication operations henceforth.

Theorem 34.13. *For any quadrilateral in any projective plane, the operation $mx + b$ defined above yields a ternary ring. Every ternary ring arises this way from a unique projective plane with quadrilateral.*

Proof. Start with a projective plane, and follow our steps above to define $mx + b$. To be precise, $y = mx + b$ holds just when (x, y) lies on the line with slope m and intercept $(0, b)$. We need to check that this operation satisfies the identities of a ternary ring. The identity $m0 + b = b$ says that the point $(0, b)$ lies on the line of slope m through $(0, b)$, for example. All of the other statements have equally banal proofs, which the reader can fill in.

Start with a ternary ring R. Let's construct a projective plane. The *finite points* of the projective plane are the points of the set $R \times R$. The *finite slopes* of the projective plane are the points of a copy of the set R. The *infinite slope point* of the projective plane is some point ∞_y, just a symbol we invent which is not an element of R. The *line at infinity* is the set $R \cup \{ \infty_y \}$. The *points* of our projective plane are the elements of the set

$$P := R \times R \cup R \cup \{ \infty_y \}.$$

For any m, b from R, the *line* with slope m and intercept b is the set

$$\{ (x, y) \mid y = mx + b \} \cup \{ m \}.$$

A *line* of our projective plane is either (1) the line at infinity or (2) a line with some slope m and intercept b. Again, boring steps show that the axioms of a projective plane are satisfied, with $(0, 0), (1, 1), 0, \infty_y$ as quadrilateral. □

For example, every ring with identity (not necessarily associative) in which every element has a left inverse becomes a ternary ring by taking $mx + b$ to mean using multiplication and addition in the ring. In particular, the octaves have a projective plane $\mathbb{P}^2(\mathbb{O})$, the *octave projective plane*.

朱漫屠於支益，
千金之家，
三年技成，
而所用其巧。

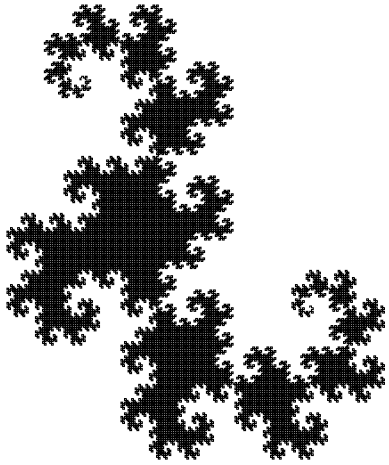
There once lived a man
who learned how to slay dragons
and gave all he possessed
to mastering the art.

After three years
he was fully prepared but,
alas, he found no opportunity
to practise his skills.

— 庄周 (Zhuang Zhou)
子 (Zhuangzi), 列御寇 (Lie Yu-kou)

As a result he began
to teach how to slay dragons.

— René Thom



Hints

1.1.

a. distributive

b. associative multiplication

c. commutative multiplication

1.2. By the commutative law of multiplication, $(a+b)c = c(a+b)$. By the distributive law, $c(a+b) = ca + cb$. By two applications of the commutative law, $ca + cb = ac + cb = ac + bc$.

1.3. For any integer a: $a + 0 = a$. Pick a to be $a = 0$.

1.4.

$$
\begin{aligned}
0 &= 0 + 0 && \text{by problem 1.3,}\\
0 \cdot 0 &= 0 \cdot (0 + 0) && \text{by multiplying by 0,}\\
0 \cdot 0 &= 0 \cdot 0 + 0 \cdot 0 && \text{by the distributive law,}\\
\text{Let } b &= -(0 \cdot 0), &&\\
0 \cdot 0 + b &= (0 \cdot 0 + 0 \cdot 0) + b && \text{by adding } b \text{ to both sides,}\\
0 \cdot 0 + b &= 0 \cdot 0 + (0 \cdot 0 + b) && \text{by the associative law for addition,}\\
0 &= 0 \cdot 0 + 0 && \text{by the definition of } b,\\
0 &= 0 \cdot 0 && \text{by the definition of 0.}
\end{aligned}
$$

1.5.

$$
\begin{aligned}
0 &= 0 + 0 && \text{by problem 1.3,}\\
a \cdot 0 &= a \cdot (0 + 0) && \text{by multiplying by } a,\\
a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{by the distributive law,}\\
\text{Let } b &= -(a \cdot 0), &&\\
a \cdot 0 + b &= (a \cdot 0 + a \cdot 0) + b && \text{by adding } b \text{ to both sides,}\\
a \cdot 0 + b &= a \cdot 0 + (a \cdot 0 + b) && \text{by the associative law for addition,}\\
0 &= a \cdot 0 + 0 && \text{by the definition of } b,\\
0 &= a \cdot 0 && \text{by the definition of 0.}
\end{aligned}
$$

1.6. There is at least one such b, by the existence of negatives. Suppose that $a+b=0$ and that $a+c=0$.

$$\begin{aligned}
(a+b)+c &= 0+c, && \\
&= c+0 && \text{by commutativity of addition,}\\
&= c && \text{by the definition of 0,}\\
&= (a+b)+c && \text{by returning to the start again,}\\
&= a+(b+c) && \text{by associativity of addition,}\\
&= a+(c+b) && \text{by commutativity of addition,}\\
&= (a+c)+b && \text{by associativity of addition,}\\
&= 0+b && \text{by the definition of } c,\\
&= b+0 && \text{by commutativity of addition,}\\
&= b && \text{by the definition of 0.}
\end{aligned}$$

Hence $b=c$.

1.7.

$$\begin{aligned}
a+(-1)a &= a+a(-1) && \text{by commutativity of multiplication,}\\
&= a(1+(-1)) && \text{by the distributive law on the right hand side,}\\
&= a(0) && \text{by the definition of } -,\\
&= 0 && \text{by the definition of 0.}
\end{aligned}$$

So $(-1)a$ fits the definition of $-a$. By problem 1.6 on page 3, $(-1)a=-a$.

1.8. By problem 1.7 on page 3, $(-1)(-1)=-(-1)$ is the unique integer which, added to -1, gives zero, and we know that this integer is 1.

1.12. We need to see that $(-b)+(-c)$ has the defining property of $-(b+c)$: adding to $b+c$ to give zero.

$$\begin{aligned}
(b+c)+((-b)+(-c)) &= b+c+(-b)+(-c) && \text{by as above: parentheses not needed,}\\
&= b+(-b)+c+(-c) && \text{by commutativity of addition,}\\
&= 0+0 && \text{by existence of negatives,}\\
&= 0 && \text{by the definition of 0.}
\end{aligned}$$

1.13. If $b=c+1=d+1$ for two integers c,d, then $c+1=d+1$ so $(c+1)+(-1)=(d+1)+(-1)$, where the existence of negatives ensures that there is a negative -1 of 1. So $c+(1+(-1))=d+(1+(-1))$ by the associative law for addition. So $c+0=d+0$, by the existence of negatives law. So $c=d$, by the identity law for addition.

1.19. Suppose that there are positive integers b,c with $b+c$ not positive. By the law of well ordering, there is a least possible choice of b, and, for that given b, a least possible choice of c. If $b=1$ then $b+c=1+c=c+1$ is positive by the succession law. If $b\neq 1$, then $b=d+1$ for some positive d by the succession law, and then $b+c=(d+1)+c=(d+c)+1$. This is not positive, so by the succession law, $d+c$ is not positive. But then d is smaller than b, so b is not the least possible choice.

1.19. Suppose that there are positive integers b, c with bc not positive. By the law of well ordering, there is a least possible choice of b, and, for that given b, a least possible choice of c. If $b = 1$ then $bc = c$ is positive. If $b \neq 1$, then $b = d + 1$ for some positive d by the succession law, and then $bc = (d + 1)c = dc + c$. This is not positive, so by problem 1.19 on page 4, dc is not positive. But then d is smaller than b, so b is not the least possible choice.

1.21. If $b, c > 0$ or if $b, c < 0$, we know that $bc > 0$. If $b > 0$ and $c < 0$, say $c = -d$ with $d > 0$, then $bc = b(-d$. If we add $bd + b(-d)) = b(d + (-d)) = b0 = 0$. So $b(-d) = -(bd)$ is negative. Similarly, if $b < 0$ and $c > 0$, $bc = cb$ is negative.

1.23.

$$\begin{aligned}
(b + c) - (a + c) &= b + c + (-(a + c)) && \text{by } d - e = d + (-e) \text{ as above,} \\
&= b + c + (-a) + (-c) && \text{by } -(d + e) = (-d) + (-e) \text{ as above,} \\
&= b + c + (-c) + (-a) && \text{by commutativity of addition,} \\
&= b + 0 + (-a) && \text{by existence of negatives,} \\
&= b + (-a) && \text{by definition of zero,} \\
&= b - a && \text{by } d + (-e) = d - e \text{ as above.}
\end{aligned}$$

So $a + c < b + c$ just when $a < b$.

1.24. By distributivity of subtraction, proven in problem 1.16 on page 4, $bc - ac = (b - c)a$. Apply the solution of problem 1.21 on page 4.

1.27. Dividing a nonzero integer a by zero would mean find an integer c so that $a = 0\,c$. But we have seen that $0\,c = 0$, so $a = 0$, a contradiction. Dividing zero by zero would mean find an integer c so that $0 = 0\,c$. But any integer c satisfies this equation, so there is no way to pick out one value c to be $0/0$.

1.31. If there were two, both nonnegative, they would divide one another, so by proposition 1.3 on page 5 each is no larger than the other, hence equal.

1.32.

a. 17

b. 1

c. 1

d. 12

e. 4

f. 49

g. 11111.

1.33. By repeated addition, you can build any positive integer multiple of any integer from your collection. By subtracting an integer from your collection from itself, you can build zero. By repeated subtraction, you can build any negative integer multiple of any integer from your collection. So by adding up, you can build any linear combination $s_1m_1 + s_2m_2 + \cdots + s_nm_n$ of integer multiples $s_1, s_2, \ldots, s_n$ of any elements $m_1, m_2, \ldots, m_n$ from the collection, with any integers $s_1, s_2, \ldots, s_n$. If all integers in our collection are divisible by some integer d, then clearly so are any integer multiples or finite sums of integer multiples. So every common divisor divides all such sums. Hence the common divisors are unchanged if we replace the collection

by the collection of all such sums. So we can suppose that any such sum is already in our collection. Our collection contains some positive integer, because if m_1 is in our collection, then taking s_1 to be 1 if $m_1 > 0$ and -1 if $m_1 < 0$, $s_1 m_1$ is positive. By well ordering, there is a least positive element c in our collection. Take quotient and remainder of any element m in our collection by c: $m = qc + r$. The remainder $r = m - qc$ is also in the collection, not negative, but smaller than c, and therefore is zero. So c is a common divisor of every element in the collection. But c is in the collection, and so every common divisor divides c. Hence c is the greatest common divisor.

1.34. If the collection contains something other than 0, apply problem 1.33 on page 8.

Suppose the collection is empty. Any integer d is a common divisor, since there is nothing to divide into. But then to have d be a greatest common divisor, we also need d to be divisible by all other common divisors. Those other common divisors can be any integers. So a greatest common divisor d is precisely an integer which is divisible by all others, and hence must be zero: the greatest common divisor of the empty set is zero.

Suppose the collection consists of just one integer: 0. Then every integer n divides 0, since $0 = 0 \cdot n$. So every integer is a common divisor of the collection. A greatest common divisor must be precisely an integer divisible by all others. In particular 0 is a common divisor since $0 = 1 \cdot 0$, and is divisible by all integers, so is a greatest common divisor. But any nonzero number d, to be a greatest common divisor, has $d \geq 0$ so positive, and divisible by all integers, so at least as large as any integer by proposition 1.3 on page 5, so infinitely large, impossible. Hence $d = 0$ is the unique greatest common divisor.

2.9. When we expand out $(1 + x)^n$, each term in the expansion arises from choosing either 1 or x from each factor $1 + x$, and multiplying out the choices to produce the term. So terms with x^k arise when we choose x from k of the factors $1 + x$, and 1 from the other $n - k$. So in $(1 + x)^{n+1}$, the x^k terms arise from choosing x from k of the first n factors $1 + x$, and choosing 1 from the last one, or from choosing x from $k - 1$ of the first n factors, and also choosing x from the last one.

2.10. For $n = 1, 2, 3$, we can check by hand:

n	$p_n(x)$	$p_n(1 + x)$
1	1	1
2	$1 + x$	$2 + x = \binom{2}{1}x^0 + \binom{2}{2}x^1$
3	$1 + x + x^2$	$3 + 3x + x^2 = \binom{3}{1}x^0 + \binom{3}{2}x^1 + \binom{3}{3}x^2$

From then on, we will need to use induction:

$$p_{n+1}(x) = p_n(x) + x^n,$$

so

$$\begin{aligned}
p_{n+1}(1+x) &= p_n(1+x) + (1+x)^n \\
&= \sum_{k=0}^{n-1} \binom{n}{k+1} x^k + \sum_{k=0}^{n} \binom{n}{k} x^k, \\
&= \sum_{k=0}^{n-1} \left(\binom{n}{k+1} + \binom{n}{k} \right) x^k + x^n, \\
&= \sum_{k=0}^{n-1} \binom{n+1}{k+1} x^k + x^n, \\
&= \sum_{k=0}^{n} \binom{n+1}{k+1} x^k.
\end{aligned}$$

3.1.

a.

$$\begin{pmatrix} 1 & 0 & 2468 \\ 0 & 1 & 180 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 180 \\ 1 & -13 & 128 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -13 & 128 \\ -1 & 14 & 52 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 14 & 52 \\ 3 & -41 & 24 \end{pmatrix}$$
$$\begin{pmatrix} 3 & -41 & 24 \\ -7 & 96 & 4 \end{pmatrix}$$
$$\begin{pmatrix} -7 & 96 & 4 \\ 45 & -617 & 0 \end{pmatrix}$$
$$(-7)(2468) + (96)(180) = 4$$

b.

$$\begin{pmatrix} 1 & 0 & 79 \\ 0 & 1 & -22 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & -22 \\ 1 & 4 & -9 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 4 & -9 \\ -2 & -7 & -4 \end{pmatrix}$$
$$\begin{pmatrix} -2 & -7 & -4 \\ 5 & 18 & -1 \end{pmatrix}$$
$$\begin{pmatrix} 5 & 18 & -1 \\ -22 & -79 & 0 \end{pmatrix}$$
$$(5)(79) + (18)(-22) = -1$$

c.

$$\begin{pmatrix} 1 & 0 & 45 \\ 0 & 1 & 16 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 16 \\ 1 & -2 & 13 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 & 13 \\ -1 & 3 & 3 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 3 & 3 \\ 5 & -14 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 5 & -14 & 1 \\ -16 & 45 & 0 \end{pmatrix}$$
$$(5)(45) + (-14)(16) = 1$$

d.

$$\begin{pmatrix} 1 & 0 & -1000 \\ 0 & 1 & 2002 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & -1000 \\ 3 & 1 & -998 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 1 & -998 \\ -2 & -1 & -2 \end{pmatrix}$$
$$\begin{pmatrix} -2 & -1 & -2 \\ 1001 & 500 & 0 \end{pmatrix}$$
$$(-2)(-1000) + (-1)(2002) = -2$$

4.2. $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113$

4.4.

$$\begin{aligned}
\gcd\{4864, 3458\} &= 2\gcd\{2432, 1729\}, \\
&= 2\gcd\{1216, 1729\}, \\
&= 2\gcd\{608, 1729\}, \\
&= 2\gcd\{304, 1729\}, \\
&= 2\gcd\{152, 1729\}, \\
&= 2\gcd\{76, 1729\}, \\
&= 2\gcd\{38, 1729\}, \\
&= 2\gcd\{19, 1729\}, \\
&= 2\gcd\{19, 1710\}, \\
&= 2\gcd\{19, 855\}, \\
&= 2\gcd\{19, 836\}, \\
&= 2\gcd\{19, 418\}, \\
&= 2\gcd\{19, 209\}, \\
&= 2\gcd\{19, 190\}, \\
&= 2\gcd\{19, 95\}, \\
&= 2\gcd\{19, 76\}, \\
&= 2\gcd\{19, 38\}, \\
&= 2\gcd\{19, 19\}, \\
&= 2\gcd\{19, 0\}, \\
&= 38
\end{aligned}$$

4.5. Let d be the greatest common divisor of some collection of integers. So each of the integers in the collection is a multiple of d. If some integer d' divides into d, i.e. d is a multiple of d', then each of the integers in the collection is a multiple of d, so a multiple of d'. On the other hand, suppose that d' is an integer which divides into all integers in our collection. Let $g := \gcd\{d, d'\}$, so $d' = D'g$ and $d = Dg$. Clearly D, D' are coprime. Note: d' divides d just when D' divides D. Moreover, all integers in our collection are divisible by g. So we make a new collection of these integers, dividing each integer in the old collection by g. Replacing the old collection by the new, we can assume that d, d' are coprime. So d' divides all integers in our collection, but is coprime to d. Each integer in the collection has the form dm, so d' m, say $dm = dd'm'$. So dd' is a positive common divisor, larger than d if $|d'| > 1$, a contradiction, so $d' = \pm 1$ divides into d.

4.6. Let $d := \gcd\{b, c\}$, so $b = Bd$ and $c = Cd$. If B, C have a positive common divisor k, say $B = kB'$ and $C = kC'$, then $b = B'kd$ and $c = C'kd$ so kd divides b and c, but d is the greatest common divisor so $k = 1$. Therefore $\gcd\{B, C\} = 1$. Multiple by m: $mb = B(md)$ and $mc = C(md)$, so md divides mb and mc. By problem 4.5 on page 27, md divides D, say $D = md\ell$. Then $mb = md\ell b'$ and $mc = md\ell c'$ so $b = d\ell b' = Bd$ so $\ell b' = B$ and similarly $\ell c' = C$. But B and C are coprime, so $\ell = 1$.

5.2. A 1-digit positive integer has the form $b = b_0$ with $0 \leq b_0 \leq 9$ its first digit. A 2-digit positive integer has the form $b = b_0 + 10\,b_1$ with $0 \leq b_0, b_1 \leq 9$. In general, an n-digit positive integer has the form

$$b = b_0 + 10\,b_1 + 10^2\,b_2 + \cdots + 10^{n-1}b_{n-1}.$$

Modulo 9, $10 \equiv 1 \pmod 9$, so modulo 9

$$\begin{aligned} b &\equiv b_0 + 10\,b_1 + 10^2\,b_2 + \cdots + 10^{n-1}b_{n-1} \pmod 9, \\ &\equiv b_0 + b_1 + b_2 + \cdots + b_{n-1} \pmod 9. \end{aligned}$$

5.10. Mod 3, $b(x) = x^4 + x + 2$. We plug in $x = 0, 1, 2$ and take remainder modulo 3 to find that these are not roots:

x	$x^4 + x + 2 \pmod 3$
0	$0^4 + 0 + 2 = 2$
1	$1^4 + 1 + 2 = 1$
2	$2^4 + 2 + 2 = 2$

Any root in integers would reduce modulo 3 to a root in remainders modulo 3.

5.13.

a.

$$\begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & 59 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 13 \\ -4 & 1 & 7 \end{pmatrix}$$
$$\begin{pmatrix} -4 & 1 & 7 \\ 5 & -1 & 6 \end{pmatrix}$$
$$\begin{pmatrix} 5 & -1 & 6 \\ -9 & 2 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -9 & 2 & 1 \\ 59 & -13 & 0 \end{pmatrix}$$
$$(-9)(13) + (2)(59) = 1$$

Answer:$13^{-1} = 50$ modulo 59

b.

$$\begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 11 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 10 \\ -1 & 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 1 & 1 \\ 11 & -10 & 0 \end{pmatrix}$$
$$(-1)(10) + (1)(11) = 1$$

Answer:$10^{-1} = 10$ modulo 11

c.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 193 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 2 \\ -96 & 1 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -96 & 1 & 1 \\ 193 & -2 & 0 \end{pmatrix}$$
$$(-96)(2) + (1)(193) = 1$$

Answer:$2^{-1} = 97$ modulo 193

d.

$$\begin{pmatrix} 1 & 0 & 6003722857 \\ 0 & 1 & 77695236973 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 6003722857 \\ -12 & 1 & 5650562689 \end{pmatrix}$$
$$\begin{pmatrix} -12 & 1 & 5650562689 \\ 13 & -1 & 353160168 \end{pmatrix}$$
$$\begin{pmatrix} 13 & -1 & 353160168 \\ -220 & 17 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -220 & 17 & 1 \\ 77695236973 & -6003722857 & 0 \end{pmatrix}$$
$$(-220)(6003722857) + (17)(77695236973) = 1$$

Answer:$6003722857^{-1} = 77695236753$ modulo 77695236973

6.3. We want to find remainders $(1, 2, 3)$ modulo $(3, 7, 19)$. We let

$$u_1 = 7 \cdot 19 = 133,$$
$$u_2 = 3 \cdot 19 = 57,$$
$$u_3 = 3 \cdot 7 = 21.$$

Modulo $3, 7, 19$ these are

$$u_1 = 1 \mod 3,$$
$$u_2 = 1 \mod 7,$$
$$u_3 = 2 \mod 19.$$

The multiplicative inverses of these, modulo $3, 7, 19$, are obvious by reducing and inspection:

$$v_1 = 1,$$
$$v_2 = 1,$$
$$v_3 = 10.$$

By the Chinese remainder theorem, the answer, up to multiples of $3 \cdot 7 \cdot 19 = 399$, is

$$\begin{aligned} r_1u_1v_1 + r_2u_2v_2 + r_3u_3v_3 &= 1 \cdot 133 \cdot 1 + 2 \cdot 57 \cdot 1 + 3 \cdot 21 \cdot 10, \\ &= 133 + 114 + 630, \\ &= 877, \\ &= 79 + 2 \cdot 399. \end{aligned}$$

So there are at least 79 soldiers in Han Xin's army.

6.4. We have $r_1, r_2 = 2, 8$, $m_1, m_2 = 22, 39$, so $u_1 = 39$, $u_2 = 22$. Modulo 22:

$$v_1 = 39^{-1} = (22 + 17)^{-1} = 17^{-1}.$$

Bézout:

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 17 \\ 0 & 1 & 22 \end{pmatrix} &\quad \text{add } -\text{row 1 to row 2,} \\ \begin{pmatrix} 1 & 0 & 17 \\ -1 & 1 & 5 \end{pmatrix} &\quad \text{add } -3\cdot\text{row 2 to row 1,} \\ \begin{pmatrix} 4 & -3 & 2 \\ -1 & 1 & 5 \end{pmatrix} &\quad \text{add } -2\cdot\text{row 1 to row 2,} \\ \begin{pmatrix} 4 & -3 & 2 \\ -9 & 7 & 1 \end{pmatrix} & \end{aligned}$$

So $(-9)(17) + (7)(22) = 1$, so mod 22, $v_1 = -9 = 22 - 9 = 13$. Modulo 39, $v_2 = 22^{-1}$,

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 22 \\ 0 & 1 & 39 \end{pmatrix} &\quad \text{add } -\text{row 1 to row 2,} \\ \begin{pmatrix} 1 & 0 & 22 \\ -1 & 1 & 17 \end{pmatrix} &\quad \text{add } -\text{row 2 to row 1,} \\ \begin{pmatrix} 2 & -1 & 5 \\ -1 & 1 & 17 \end{pmatrix} &\quad \text{add } -3\cdot\text{row 1 to row 2,} \\ \begin{pmatrix} 2 & -1 & 5 \\ -7 & 4 & 2 \end{pmatrix} &\quad \text{add } -2\cdot\text{row 2 to row 1,} \\ \begin{pmatrix} 16 & -9 & 1 \\ -7 & 4 & 2 \end{pmatrix} & \end{aligned}$$

So $v_2 = 16$. Modulo $39 \cdot 22 = 858$,

$$\begin{aligned} x &= r_1u_1v_1 + r_2u_2v_2, \\ &= (2)(39)(13) + (8)(22)(16), \\ &= 1014 + 2816, \\ &= 3830, \\ &= (4)(858) + 398, \\ &= 398. \end{aligned}$$

6.5. Moduli $m_1, m_2, m_3 = 4, 5, 17$, so $u_1, u_2, u_3 = 85, 68, 20$. Reduce modulo those to get to $1, 3, 3$; we only need reciprocals of these remainders. Easily see that we can take $v_1, v_2, v_3 = 1, 2, 6$ as reciprocals of those remainders. We have $r_1, r_2, r_3 = 3, 4, 0$. Modulo $4 \cdot 5 \cdot 17 = 340$,

$$\begin{aligned} x &= r_1 u_1 v_1 + r_2 u_2 v_2, \\ &= 799, \\ &= 119 \end{aligned}$$

modulo 340.

6.9. Factor $120 = 2^3\, 3^1\, 5^1$. So Euler's totient is $\phi(120) = (2^3 - 2^2)(3^1 - 3^0)(5^1 - 5^0) = 32$. Since 127 is prime, it is coprime to 120. The theorem of Euler's totient function says that $127^{\phi(120)} = 1$ modulo 120. So $127^{32} = 1$ modulo 120. In other words, every time you multiply together 32 copies of 127, modulo 120, it is as if you multiplied together no copies. Check that $162 = 5 \cdot 32 + 2$. So modulo 120, $127^{162} = 127^{5 \cdot 32 + 2} = 127^2$. This is not quite the answer. But we know that $127 = 7$ modulo 120. So modulo 120, $127^{162} = 7^2 = 49$.

8.1. $224/82 = 112/41$, $324/-72 = -9/2$, $-1000/8800 = -5/44$.

8.2. We can start by defining, for any pair of integers (b, c) with $c \neq 0$, and any pair of integers (B, C) with $C \neq 0$, a rational number

$$\frac{\beta}{\gamma} = (Cb + cB, cC).$$

This is well defined, since $cC \neq 0$. If we replace (b, c) by (ab, ac) and replace (B, C) by (AB, AC), then we get rational number

$$\frac{ACab + acAB}{acAC} = \frac{(Aa)(Cb + cB)}{(Aa)(Cc)} = \frac{Cb + cB}{Cc}$$

unchanged. So the resulting β/γ is independent of the choices of pairs, depending only on the rational numbers b/c and B/C. The other proofs are very similar.

8.3. By hand, showing your work, simplify

$$\frac{2}{3} - \frac{1}{2} = \frac{1}{6}, \frac{324}{49} \cdot \frac{392}{81} = 32, \frac{4}{5} + \frac{7}{4} = \frac{51}{20}.$$

8.5. If $\sqrt{3} = x + y\sqrt{2}$, square both sides to get

$$3 = x^2 + 2xy\sqrt{2} + 2y^2.$$

If $y \neq 0$, we can solve for $\sqrt{2}$ as a rational expression in x, y, so a rational number, a contradiction. Hence $y = 0$ and $\sqrt{3} = x$ is rational, a contradiction.

8.6. By definition, every positive rational number is a ratio b/c with b and c positive. We can divide out their greatest common divisor to arrange that b, c are coprime. Take the prime factorization of b, and that of c. As they are coprime, no prime occurs in both factorizations.

9.2. Take the polynomial $b(x) = (1/2)x(x + 1) = (1/2)x^2 + (1/2)x$. If x is an odd integer, then $x + 1$ is even, and vice versa, so $b(x)$ is an integer for any integer x.

9.4. First we apply the Euclidean algorithm: division with remainder

$$x^3 - x^2 - x - 2 = 1(x^3 + x^2 + x) + (-2x^2 - 2x - 2),$$
$$x^3 + x^2 + x = -\frac{x}{2}(-2x^2 - 2x - 2) + 0.$$

Next, find the Bézout coefficients.

$$\begin{pmatrix} 1 & 0 & x^3 + x^2 + x \\ 0 & 1 & x^3 - x^2 - x - 2 \end{pmatrix} \text{ add } -\text{row 1 to row 2},$$
$$\begin{pmatrix} 1 & 0 & x^3 + x^2 + x \\ -1 & 1 & -2x^2 - 2x - 2 \end{pmatrix} \text{ add } -(x/2)\text{row 2 to row 1},$$
$$\begin{pmatrix} 1 + \frac{x}{2} & -\frac{x}{2} & 0 \\ -1 & 1 & -2x^2 - 2x - 2 \end{pmatrix}$$
$$(-1)b(x) + (1)c(x) = -2x^2 - 2x - 2.$$

9.6. Keep in mind that, modulo 3, $-1 = 2, -2 = 1$, $1^{-1} = 1$, $2^{-1} = 2$, so

$$\begin{pmatrix} 1 & 0 & x^3 + x \\ 0 & 1 & x^4 + 1 \end{pmatrix} \text{ add } -x(\text{row 1}) \text{ to row 2},$$
$$\begin{pmatrix} 1 & 0 & x^3 + x \\ -x & 1 & -x^2 + 1 \end{pmatrix} \text{ add } x(\text{row 2}) \text{ to row 1},$$
$$\begin{pmatrix} 1 - x^2 & x & 2x \\ -x & 1 & -x^2 + 1 \end{pmatrix} \text{ add } 2x(\text{row 1}) \text{ to row 2},$$
$$\begin{pmatrix} 1 - x^2 & x & 2x \\ x^3 + x & 1 + 2x^2 & 1 \end{pmatrix}$$

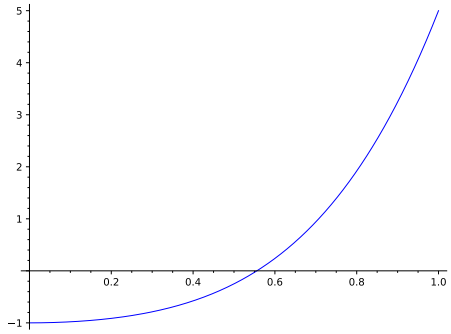
So the Bézout coefficients are $x^3 + x, 1 + 2x^2$ and the gcd is 1. We can check: expand out

$$(x^3 + x)(x^3 + x) + (1 + 2x^2)(x^4 + 1) = (x^6 + 2x^4 + x^2) + (2x^6 + x^4 + 2x^2 + 1) = 1.$$

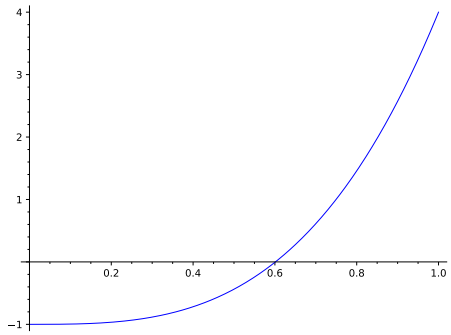
9.13. roots: 0, 1, 3, 4; factorizations: $(x + 2)(x + 3) = x(x + 5) = (5x)(5x + 1)$.

9.17. Pick a large prime number p. Pick a random polynomial $b(t)$ of degree k with coefficients in $\mathbb{Z}/p\mathbb{Z}$. Tell person 1 the value of $b(1)$, person 2 the value of $b(2)$ and so forth. Tell them all the value of p. Send the signal when any group of them tells you $b(0)$. Any $k + 1$ people can use linear algebra (which requires all of the arithmetic operations, including reciprocals, so p has to be prime) to compute all coefficients of $b(t)$ (by interpolation), and hence $b(0)$. Any k or fewer people have no information about $b(0)$, since whatever polynomial $b(t)$ we choose, we can always alter it arbitrarily at any k nonzero values of t, without altering its value at $t = 0$ (proof: interpolation). The signal to start the war is the value of $b(0)$, entered into some computer; if p is enormous, there is essentially no chance of guessing some randomly chosen $b(0)$.

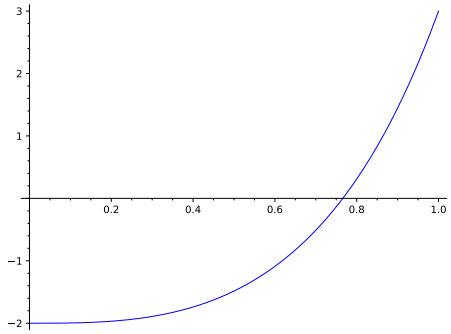
10.3. To get you started: $p_0(x) = 4x^4 + 2x^2 - 1$, $p'(x) = 16x^3 + 4x = 4(4x^3 + x)$, so we can rescale by $1/4$ to simplify to $p_1(x) = 4x^3 + x$, $p_2(x) = 1 - x^2$, $p_3(x) = -5x$ which you can rescale by $1/5$ to get $p_3(x) = -x$, $p_4(x) = -1$. You should find that there is one root in that interval. You can check by letting $x^2 = t$, write as a quadratic equation in t, solve with the quadratic formula, and also look at the graph:



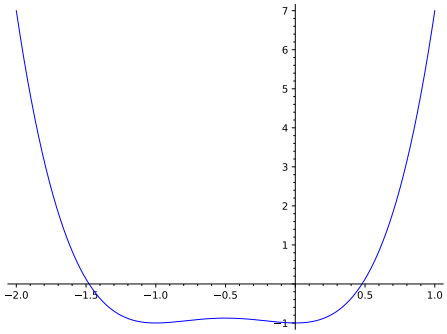
10.4. To get you started: $p_0(x) = x^4 + 4x^3 - 1$, $p'(x) = 4x^3 + 12x^2 = 4(x^3 + 3x^2)$, so we can rescale by $1/4$ to simplify to $p_1(x) = x^3 + 3x^2$, $p_2(x) = 3x^2 + 1$, $p_3(x) = \frac{4}{3}x + 4$, $p_4(x) = -28$. There is one root in that interval:



10.5. The sequence is $x^6 + 4x^3 - 2, 6x^5 + 12x^2, -2x^3 + 2, -18x^2, -2$. These have values at $x = 0$ of $-2, 0, 2, -2$, and at $x = 1$ they have $3, 18, -18, -2$. There is one root in that interval:



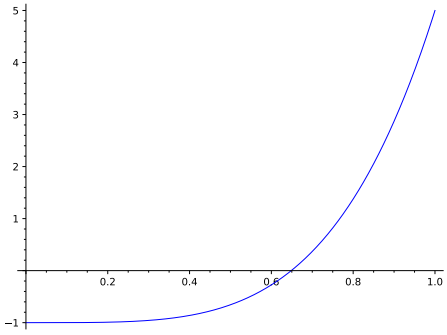
10.6. The sequence is $2x^4 + 4x^3 + 2x^2 - 1, 8x^3 + 12x^2 + 4x, (1/2)x^2 + (1/2)x + 1, 16x + 8, -7/8$. These have values at $x = -2$ of $7, -24, 2, -24, -7/8$, and at $x = 1$ they have $7, 24, 2, 24, -7/8$. There are two roots in that interval:



10.7. Clearly $p'(x) = 5x^4 + 20x^3$. Compute:

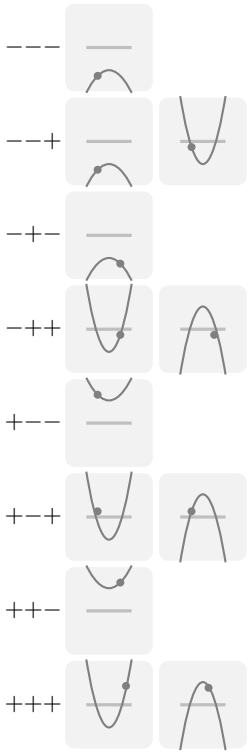
	$x = 0$	$x = 1$
$p_0(x) = x^5 + 5x^4 - 1$	-1	5
$p_1(x) = 5x^4 + 20x^3$	0	25
$p_2(x) = 4x^3 + 1$	1	5
$p_3(x) = \frac{5}{4}x + 5$	5	$\frac{25}{4}$
$p_4(x) = 255$	255	255
	1	0

so $1-0 = 1$ zeroes. (We can cheat to guess the right answer: since $p'(x) = 5x^4+20x^3 = x^3(5x + 20) \geq 0$ for $0 \leq x \leq 1$, and > 0 for $0 < x \leq 1$, we see that $p(x)$ is strictly increasing, so can have at most one root. But $p(0) = -1 < 0$ and $p(1) = 5 > 0$, so one root.)



10.8. Each pattern of signs in the Sturm sequence, followed by a picture of parabolas with that pattern, indicating whether the point where we calculate the Sturm sequence gives a positive or negative value of the quadratic function, whether the point is to the left or right of the vertex of the parabola, and whether the discriminant is positive

or negative:



10.9. If $p(x) = 0$ then

$$a_n x^n + \cdots + a_0 = 0.$$

Subtract off all but the first term,

$$a_n x^n = -a_{n-1}x^{n-1} - \cdots - a_0.$$

Divide by a_n:

$$x^n = -\frac{a_{n-1}x^{n-1}}{a_n} - \cdots - \frac{a_0}{a_n}.$$

Divide by x^{n-1}:

$$x = -\frac{a_{n-1}}{a_n} - \frac{a_{n-2}}{a_n x} - \cdots - \frac{a_0}{a_n x^{n-1}}.$$

So

$$\begin{aligned} |x| &= \left| -\frac{a_{n-1}}{a_n} - \frac{a_{n-2}}{a_n x} - \cdots - \frac{a_0}{a_n x^{n-1}} \right|, \\ &\leq \left| \frac{a_{n-1}}{a_n} \right| + \left| \frac{a_{n-2}}{a_n x} \right| + \cdots + \left| \frac{a_0}{a_n x^{n-1}} \right|. \end{aligned}$$

So if $|x| > 1$,

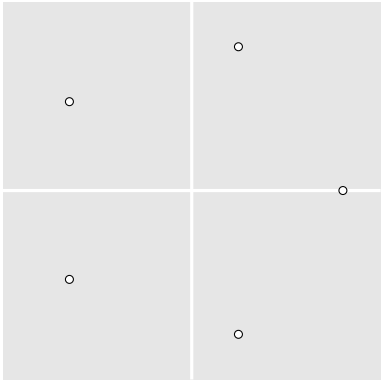
$$|x| \leq \left| \frac{a_{n-1}}{a_n} \right| + \left| \frac{a_{n-2}}{a_n} \right| + \cdots + \left| \frac{a_0}{a_n} \right|.$$

Hence every root lies in either $|x| \leq 1$ or this interval, i.e. we can take c to be either $c = 1$ or

$$c = \frac{|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|}{|a_n|},$$

whichever is larger.

10.10.



10.11. Suppose you find a counterexample: two partial fraction decompositions for the same rational function. Their difference is a partial fraction decomposition for the zero rational function $0/c(x)$ for some $c(x)$. Suppose we have such.

Clearly if our partial fraction decomposition is pure polynomial, with no partial fractions, it can't equal the zero rational function without being entirely zero: clear denominators

Now suppose that we have a partial fraction decomposition for the zero rational function $0/c(x)$. Split $c(x)$ into a product of coprime irreducibles

$$c(x) = c_1(x)^{n_1} \dots c_\ell(x)^{n_\ell}.$$

Suppose that our decomposition is

$$\frac{0}{c(x)} = B(x) + \sum_j \sum_{k=1}^{n_j} \frac{b_{jk}(x)}{c_j(x)^k},$$

Multiply by $c(x)$ to clear denominators:

$$0 = c(x)B(x) + \sum_j \sum_{k=1}^{n_j} b_{jk}(x) c_j(x)^{n_j - k} \prod_{i \neq j} c_i(x)^{n_i}$$

All terms in both sides are divisible by $c_1(x)$, except perhaps a single one:

$$b_{1n_1}(x) \prod_{i \neq 1} c_i(x)^{n_i}.$$

So $c_1(x)^{n_1}$ must also divide this. But all of the c_i are coprime to $c_1(x)$, so $c_1(x)$ divides $b_{1n_1}(x)$, but has larger degree, so $b_{1n_1}(x) = 0$. We can reduce the value of n_1 in our partial fraction decomposition, to find a simpler counterexample with fewer terms.

Apply induction on the number of terms, until the partial fraction decomposition becomes a polynomial, which must agree with the zero polynomial, so must have all coefficients zero.

11.1. We can flip the sign to get $p(x)$ to have a positive leading coefficient. After x is made larger than any of the (finitely many) roots of $p(x)$, we know that $p(x) > 0$. Pick any integer N larger than any of the roots of $p(x)$. In particular, $p(N) > 0$. For any integer ℓ, if we set $x = N + \ell p(N)$, then expanding out $x^k = N^k + \ldots$, the $\ldots$ terms all contain a factor of $p(N)$, so $p(x) = p(N) + \ldots$, where again the $\ldots$ terms all contain a factor of $p(N)$. So $p(x) = 0$ modulo $p(N)$ for all of the infinitely many integers $x = N + \ell p(N)$.

11.6. By problem 2.10 on page 16,

$$\begin{aligned} c(x+1) &= \binom{p}{1} + \binom{p}{2}x + \binom{p}{3}x^2 + \cdots + \binom{p}{p-1}x^{p-2} + \binom{p}{p}x^{p-1}, \\ &= p + \frac{p!}{2!(p-2)!}x + \frac{p!}{3!(p-3)!}x^2 + \cdots + \frac{p!}{(p-1)!1!}x^{p-2} + x^{p-1}. \end{aligned}$$

Clearly there is a factor of p in each term, and no p^2 factor in the constant term.

11.7. Over the complex numbers,

$$b(x,y) = (x+iy)(x-iy).$$

Over all of the other fields, if reducible, it splits into linear factors

$$b(x,y) = (\alpha x + \beta y)(\gamma x + \delta y) = \alpha\gamma x^2 + (\alpha\delta + \beta\gamma)xy + \beta\delta y.$$

None of the coefficients can vanish, since $b(x,y)$ doesn't have a factor of x or y. For real coefficients, $b(x,y)$ vanishes along a line $\alpha x + \beta y = 0$. This is not possible because squares x^2, y^2 can't be negative, so if $x^2 + y^2 = 0$ then $x = y = 0$. Alternate proof: plug in $y = 1$ to get $x^2 + 1 = (\alpha x + \beta)(\gamma x + \delta$, vanishing at $x = -\beta/\alpha$, a square root of -1. It follows that $b(x,y)$ is irreducible over the rationals, because rationals are real. The coefficients of $b(x,y)$ are coprime, so $b(x,y)$ is irreducible over the integers.

11.8. Eisenstein with $p = 3$, $111 = (3)(37)$, and 37 is not divisible by 3, so $q(x)$ is irreducible.

11.9. Take $p(y) = y + 1$, clearly irreducible since linear and not constant. Note that $p(y)^2 = y^2 + 1$. So Eisenstein applies: $a(x,y)$ is irreducible.

11.8. This $b(x,y)$ is irreducible, by Eisenstein's criterion, with with $p(y) = y^2 + y + 1$. We need to know that $p(y)$ is irreducible, and to see this: any reducible quadratic, having degree 2, splits into irreducibles of degrees 1 each, so linear. The only linear polynomials are the two linears $y, y+1$. Their 3 possible products are $y^2, y(y+1), (y+1)^2 = y^2 + 1$, not equal to $y^2 + y + 1$. So irreducible over remainders modulo 2. If it reduces over integers, quotient by 2 to get a reduction in remainders modulo 2, or to get zero. But it remains nonzero and irreducible.

12.9. Starting with any ring R, let S be the set of pairs (n, r) so that n is an integer and $r \in R$. Add by

$$(n_0, r_0) + (n_1, r_1) = (n_0 + n_1, r_0 + r_1).$$

Multiply by

$$(n_0, r_0)(n_1, r_1) = (n_0 n_1, n_1 r_0 + n_0 r_1 + r_0 r_1).$$

12.16. Take two such, cross multiply, and use unique factorisation of polynomials.

13.1.

$$0 = \alpha^9 + 2\alpha^2 + 1,$$
$$-1 = \alpha^9 + 2\alpha^2 = \alpha(\alpha^8 + 2\alpha)$$
$$\frac{-1}{\alpha} = \alpha^8 + 2\alpha,$$

and multiply by -1 to get

$$\frac{1}{\alpha} = -\alpha^8 - 2\alpha = 2\alpha^7 + \alpha$$

since $-1 = 2$.

13.3.

	0	1	2	α	2α	$1+\alpha$	$2+\alpha$	$1+2\alpha$	$2+2\alpha$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	α	2α	$1+\alpha$	$2+\alpha$	$1+2\alpha$	$2+2\alpha$
2	0	2	1	2α	α	$2+2\alpha$	$1+2\alpha$	$2+\alpha$	$1+\alpha$
α	0	α	2α	2α	α	0	α	2α	0
2α	0	2α	α	α	0	2α	2α	α	0
$1+\alpha$	0	$1+\alpha$	$2+2\alpha$	0	2α	$1+\alpha$	$2+2\alpha$	$1+\alpha$	$2+2\alpha$
$2+\alpha$	0	$2+\alpha$	$1+2\alpha$	α	2α	$2+2\alpha$	1	2	$1+\alpha$
$1+2\alpha$	0	$1+2\alpha$	$2+\alpha$	2α	α	$1+\alpha$	2	1	$2+2\alpha$
$2+2\alpha$	0	$2+2\alpha$	$1+\alpha$	0	0	$2+2\alpha$	$1+\alpha$	$2+2\alpha$	$1+\alpha$

14.1. Let's be very careful. The formal definition of the product is

$$\begin{aligned} &x^\ell \Big| (1-x)(1+x+x^2+\dots) \\ &= \sum_{k=0}^{\ell} \left(x^k \Big| (1-x)\right)\left(x^{\ell-k} \Big| (1+x+x^2+\dots)\right). \end{aligned}$$

Clearly

$$x^k \Big| (1-x) = \begin{cases} 1, & \text{if } k=0, \\ -1, & \text{if } k=1, \\ 0, & \text{otherwise,} \end{cases}$$

while

$$x^k \Big| (1+x+x^2+\dots) = 1.$$

So

$$\begin{aligned} x^\ell \Big| (1-x)(1+x+x^2+\dots) &= \sum_{k=0}^{\ell} \begin{cases} 1, & \text{if } k=0, \\ -1, & \text{if } k=1, \\ 0, & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1, & \text{if } \ell = 0, \\ 0, & \text{if } \ell \geq 1. \end{cases} \end{aligned}$$

14.2.

$$\begin{aligned} f_1(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 + \dots, \\ f_2(x) &= x + x^2 + x^3 + x^4 + x^5 + \dots, \\ f_3(x) &= x^2 + x^3 + x^4 + x^5 + \dots, \\ f_4(x) &= x^3 + x^4 + x^5 + \dots, \\ &\vdots \end{aligned}$$

so

$$f_1(x) + f_2(x) + f_3(x) + \cdots = 1 + 2x + 3x^2 + 4x^3 + \dots.$$

14.4. If $b(x)$ has lower lowest order than $c(x)$, we can subtract off some polynomial $r(x)$ from $b(x)$ so that $b(x) - r(x)$ has lowest orderequal to of $c(x)$. Otherwise, we can set $r(x) = 0$. So now $b(x) - r(x)$ has lowest order no less than $c(x)$. We can then multiple $c(x)$ by the power x^k of x that will make $c(x)$ have equal lowest order with $b(x) - r(x)$. So for a suitable constant q_0, $x^{k_0} q_0 c(x)$ has the same lowest order term as $b(x) - r(x)$. So $b(x) - r(x) - q_0 x^{k_0} q_0 c(x)$ has higher lowest order than $b(x) - r(x)$. Repeating, we find that we can write

$$0 = b(x) - r(x) - (q_0 x^{k_0} + q_1 x^{k_1} + q_2 x^{k_2} + \dots)c(x).$$

So $b(x) = r(x) + q(x)c(x)$, with $r(x)$ a polynomial of degree less than the lowest order $c(x)$. If there are two such expressions $r_0(x) + q_0(x)c(x) = r_1(x) + q_1(x)c(x)$, their difference has

$$r_0(x) - r_1(x) = (q_1(x) - q_0(x))c(x).$$

Every term agrees. But every term on the left hand side has degree less than the lowest order of $c(x)$, while every term on the right hand side has degree at least that.

14.7. If $y^2 - f(x)$ is reducible, each factor is linear in y, or else one is constant in y and the other quadratic. In the second case: the factor constant in x is multiplied by a quadratic in x, coefficients polynomial in y, giving $x^2 + \dots$, constant in y, so the factor constant in x and y. In the first case: each linear factor has, up to rescaling, a unit coefficient in front of x

$$x^2 - f(y) = (x + b(y))(x + c(y)) = x^2 + x(b(y) + c(y)) + b(y)c(y),$$

so that $c(y) = -b(y)$, and then $x^2 - f(y) = x^2 - b(y)^2 = (x + b(y))(x - b(y))$. Clearly $y^2(1 + y)$ is not a square, because it has odd degree. But over formal power series,

$$x^2 - y^2(1 + y) = (x - y\sqrt{1 + y})(x + y\sqrt{1 + y})$$

where $\sqrt{1 + y}$ means that we apply our previous results to construct the unique square root of $1 + y$ for which we take $\sqrt{1}$ to be 1.

14.8. Each nonzero formal Laurent series is uniquely expressed as

$$f(x) = x^k F(x),$$

where k is an integer and $F(x)$ is a formal power series with $F(0) \neq 0$. Clearly $F(x)$ has a unique reciprocal formal power series $1/F(x)$. We would like to have

$$\frac{1}{f(x)} = x^{-k} \frac{1}{F(x)}.$$

Call this formal Laurent series $h(x)$. Clearly $h(x)f(x) = 1$.

15.2. Either note the common factor of $x+1$: $b(x) = (x+1)(x^2+x+1)$, $c(x) = (x+1)^2 x$, or compute the determinant of

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

by expanding across the first row:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

and note that two rows are the same so the determinant (and the resultant) is zero.

15.3. The resultant is

$$\operatorname{res}_{x^2+1,x^2+3x} = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \end{pmatrix} = 10.$$

But 10 only has 2 and 5 as prime factors, so this resultant vanishes just exactly when $p = 2$ or $p = 5$. If $p = 2$ the factors are $x^2+1 = (x+1)^2$ and $x^2+3x = x^2+x = x(x+1)$, so a common factor of $x + 1$. If $p = 5$ the factors are $x^2 + 1 = (x - 2)(x - 3)$ and $x^2 + 3x = x(x - 2)$, so a common factor of $x - 2$.

15.4. The resultant is the determinant of a matrix whose every column has an even number of nonzero entries, so sum equal to zero. So the sum of the rows is zero, a linear relation among rows, hence zero determinant.

15.7.

$$b(x) = x^2\, c(x) + 2x + 1,$$

so

$$\begin{aligned} \operatorname{res}_{b(x),c(x)} &= \operatorname{res}_{2x+1,c(x)}, \\ &= \operatorname{res}_{2x+1,x^6+4}, \\ &= 2^6 \operatorname{res}_{x+\frac{1}{2},x^6+4}, \\ &= 2^6 \left(x^6+4\right)\Big|_{x=-\frac{1}{2}}, \\ &= 2^6 \left(\frac{1}{2^6}+4\right), \\ &= 1 + 2^6 \cdot 4, \\ &= 257. \end{aligned}$$

15.8.

a. The derivative of ax^2+bx+c is $2ax+b$. The discriminant of ax^2+bx+c is

$$\det\begin{pmatrix} c & b & 0 \\ b & 2a & b \\ a & 0 & 2a \end{pmatrix} = -a(b^2-4ac)$$

which is surprisingly *not* the usual expression. So a quadratic polynomial has a common factor with its derivative just when $b^2=4ac$.

b. For $p(x)=x^3+x^2$, $p'(x)=3x^2+2x$, so

$$\Delta_p = \det\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 3 & 2 & 0 \\ 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 & 3 \end{pmatrix} = 0$$

due to the row of zeroes, or to the double root at $x=0$.

c. For $p(x)=x^3+x^2+1$, $p'(x)=3x^2+2x$, so

$$\Delta_p = \det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 3 & 2 & 0 \\ 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 & 3 \end{pmatrix} = 31$$

so there is no common factor between $p(x)$ and $p'(x)$, and so no double roots among the complex numbers.

d. For $p(x)=x^3+2x-1$, $p'(x)=3x^2+2$ so

$$\Delta_p = \det\begin{pmatrix} -1 & 0 & 2 & 0 & 0 \\ 2 & -1 & 0 & 2 & 0 \\ 0 & 2 & 3 & 0 & 2 \\ 3 & 0 & 0 & 3 & 0 \\ 0 & 3 & 0 & 0 & 3 \end{pmatrix} = 27$$

so there is no common factor between $p(x)$ and $p'(x)$, and so no double roots among the complex numbers.

15.9. The discriminant is the resultant of $p(x), p'(x)$, so the determinant of

$$\begin{pmatrix} 1 & 0 & a & 0 & 0 \\ a & 1 & 0 & a & 0 \\ 0 & a & 3 & 0 & a \\ 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 0 & 3 \end{pmatrix}$$

which is $27+4a^3$, so vanishes just when $a=-3/\sqrt[3]{4}$. For this value of a, the polynomial actually factors:

$$x^3 - \frac{3}{\sqrt[3]{4}}x + 1 = (x-2^{-1/3})^2(x+2^{2/3}),$$

so a double root at $x=2^{-1/3}$ and a single root at $x=-2^{2/3}$.

15.11. Write out

$$p(x) = (x - x_1)(x - x_2)\dots(x - x_n).$$

Differentiate:

$$\begin{aligned} p'(x) = {} & (x - x_2)(x - x_3)\dots(x - x_{n-1})(x - x_n) \\ & + (x - x_1)(x - x_3)\dots(x - x_{n-1})(x - x_n) \\ & \vdots \\ & + (x - x_1)(x - x_2)(x - x_3)\dots(x - x_n) \\ & + (x - x_1)(x - x_2)(x - x_3)\dots(x - x_{n-1}) \end{aligned}$$

Every term but the first has a factor of $x - x_1$, so if we plug in $x = x_1$:

$$p'(x_1) = (x_1 - x_2)(x_1 - x_3)\dots(x_1 - x_{n-1})(x_1 - x_n) = \prod_{j \neq 1}(x_1 - x_j).$$

By the same reasoning, replacing x_1 by any other root x_i:

$$p'(x_i) = \prod_{j \neq i}(x_i - x_j).$$

So the discriminant is

$$\begin{aligned} \Delta_{p(x)} &= \operatorname{res}_{p(x),p'(x)}, \\ &= (-1)^{n(n-1)} \operatorname{res}_{p'(x),p(x)}, \end{aligned}$$

Note that either n is even or $n - 1$ is even, so $(-1)^{n(n-1)} = 1$:

$$\begin{aligned} \Delta_{p(x)} &= \operatorname{res}_{p'(x),p(x)}, \\ &= \prod_i p'(x_i), \\ &= \prod_i \prod_{j \neq i}(x_i - x_j), \\ &= \prod_{i \neq j}(x_i - x_j). \end{aligned}$$

So now, for example, $x_1 - x_2$ occurs here, as does $x_2 - x_1$, so putting those together into one factor $-(x_1 - x_2)$:

$$\Delta_{p(x)} = \prod_{i<j}(-1)(x_i - x_j)^2.$$

There are $n(n-1)/2$ choices of $i < j$, so finally

$$\Delta_{p(x)} = (-1)^{n(n-1)/2} \prod_{i<j}(x_i - x_j)^2.$$

15.12. Write these equations as polynomials in increasing powers of t, with coefficients rational in x, y:

$$\begin{aligned} 0 &= -x + 0t + t^2, \\ 0 &= -y - t + 0t^2 + t^3. \end{aligned}$$

The resultant is

$$\det \begin{pmatrix} -x & 0 & 0 & -y & 0 \\ 0 & -x & 0 & -1 & -y \\ 1 & 0 & -x & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

You can simplify this determinant by adding x(row 3) to row 1, and similar tricks, to compute it out: $y^2-(x-1)x(x+1)$. So the equation of the curve is $y^2 = (x-1)x(x+1)$.

15.14. $x^2 + (1-x)y^2$

15.15. The equations, in powers of t, are

$$\begin{aligned} 0 &= x + t + t^3, \\ 0 &= 1 + yt. \end{aligned}$$

The resultant is

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 1 & y & 1 & 0 \\ 0 & 0 & y & 1 \\ 1 & 0 & 0 & y \end{vmatrix} = xy^3 + y^2 + 1.$$

So $p(x, y) = xy^3 + y^2 + 1$.

16.1.

a.

$$\begin{aligned} x^3y^3z + x^3yz^3 + xy^3z^3 &= (xyz)(x^2y^2 + x^2z^2 + y^2z^2), \\ &= e_3(x^2y^2 + \dots), \\ &= e_3(e_2^2 + \dots). \end{aligned}$$

But $e_2 = xy + xz + yz$, so

$$e_2^2 = x^2y^2 + x^2z^2 + y^2z^2 + 2x^2yz + 2xy^2z + 2xyz^2,$$

so

$$\begin{aligned} x^3y^3z + x^3yz^3 + xy^3z^3 &= e_3(e_2^2 - [2x^2yz + 2xy^2z + 2xyz^2]), \\ &= e_3(e_2^2 - 2e_3[x + y + z]), \\ &= e_3(e_2^2 - 2e_3e_1) \end{aligned}$$

b. $4xyz^3 + 4xzy^3 + 4yzx^3 = 4e_3(e_1^2 - 2e_2)$

c. $x^4y^4z^4 = e_3^4$

d. $xyz + x^2y^2z^2 = e_3\,(1 + e_3)$

16.2. We will prove that any symmetric function has an expression $b(e_1,\dots,e_n)/c(e_1,\dots,e_n)$ as a ratio of polynomials with no common nonconstant factor, expressed in the elementary symmetric polynomials, unique up to rescaling numerator and denominator by the same nonzero constant. If there were two such, cross multiply and apply unique factorisation of polynomials (theorem 12.2 on page 92). To see that there is one such, write out the function as $p(t_1,\dots,t_n)/q(t_1,\dots,t_n)$. Apply a permutation of the variables, and the solution of problem 12.16 on page 92, to see that each permutation alters the numerator by a nonzero constant multiple, and the denominator by the same multiple. Swapping two variables gives some multiple, and repeating that swap gives the same multiple squared, but returns to the original order of the variables. So any swap either changes sign or leaves both numerator and denominator alone. Replace our expression for our rational function by

$$\frac{p(t_1,\dots,t_n)q(t_1,\dots,t_n)}{q(t_1,\dots,t_n)^2}.$$

So there is an expression in symmetric polynomials. Factor out any common factors. Apply theorem 16.7 on page 133.

16.3.

$$\begin{aligned}\chi_A\lambda &= (t_1-\lambda)(t_2-\lambda)\dots(t_n-\lambda)\\ &= (-1)^n P_t(\lambda)\\ &= e_n(t) - e_{n-1}(t)\lambda + e_{n-2}(t)\lambda^2 + \dots + (-1)^n\lambda^n.\end{aligned}$$

16.5. By theorem 17.5 on page 147, $A^{-1} = (\det A)^{-1}\operatorname{adj} A$, so

$$A^{-1}_{ij} = \frac{(-1)^{i+j}\det(A \text{ with row } j \text{ and column } i \text{ deleted})}{\det A},$$

rational functions of the entries of A.

16.7. The resultant in y involves determinant of smaller matrix than that in x, because the degrees are smaller:

$$\begin{aligned}r(y) &= \det\begin{pmatrix}1 & 0 & 1 & 0\\ x^{10} & 1 & 2x^{13} & 1\\ x & x^{10} & -x & 2x^{13}\\ 0 & x & 0X-x & \end{pmatrix},\\ &= 4x^{27} - 4x^{24} + x^{21},\\ &= x^{21}(2x^3-1)^2,\end{aligned}$$

so

$$x = 0 \text{ or } x = \frac{1}{\sqrt[3]{2}}.$$

If we let $x = 0$ we have no solutions. (This "false solution" is caused by dropping of degree of both $b(x,y)$ and $c(x,y)$, which invalidates the method we are using, since the resultant of lower order polynomials has a different expression.) If $x = 1/\sqrt[3]{2}$, look at $0 = b + c$ to see

$$y = -\frac{2}{x^{10}+2x^{13}} = \frac{1}{\sqrt[3]{2}^2}.$$

17.3. If K is the kernel, an element r of R belongs to K just when $f(r) = 0$. So if r_0, r_1 belong to K, then $r_0 + r_1$ has $f(r_0 + r_1) = f(r_0) + f(r_1) = 0 + 0 = 0$, so $r_0 + r_1$ belongs to K; similarly for $r_0 - r_1$ and for $r_0 r_1$. Zero belongs to K because $f(0) = 0$. So K is a subring.

17.9. If $I \subset \mathbb{Q}$ is an ideal, containing some nonzero element $b \in \mathbb{Q}$, then I also contains $(1/b)b = 1$, and so, for any rational number c, I contains $c \cdot 1 = c$, i.e. $I = \mathbb{Q}$.

17.13. Pass to the field of fractions, where the result is standard linear algebra, and then clear denominators in the entries of x.

17.15. If $\det A$ is a unit, then clearly $(\det A)^{-1} \operatorname{adj} A$ is an inverse. If there is an inverse, say B, then $AB = BA = I$, and we take determinant to find $(\det A)(\det B) = 1$, so $\det A$ is a unit.

18.2. If $z = x - y$ then $f(x) = f(y + z) = f(y) + f(z)$ so $f(z) = f(x) - f(y)$.

18.3. For any field k, take variables $t = (t_1, \ldots, t_n)$ and let $e_1 = e_1(t), \ldots, e_n = e_n(t)$ be the elementary symmetric polynomials and $e = (e_1, \ldots, e_n)$. The field of symmetric functions is by definition the fixed field of $k(t)$ by permutations of the variables. By problem 16.2 on page 134, the field of symmetric functions is precisely the field $k(e)$ generated by $e_1, \ldots, e_n$. We want to see that $k(t)/k(e)$ is a Galois extension, with automorphism group the group of permutations of $t_1, \ldots, t_n$. The symmetric functions $k(e)$ have the rational functions $k(t)$ as a splitting field for the polynomial

$$x^n - e_1 x^{n-1} + e_2 x^{n-2} - \cdots + (-1)^n e_n.$$

By lemma 16.6 on page 131, the automorphisms of $k(t_1, \ldots, t_n)$ which fix $k(e_1, \ldots, e_n)$ are precisely the permutations of $t_1, \ldots, t_n$.

18.4. The elementary symmetric polynomials of the roots are the coefficients of $p(x)$. All symmetric functions are polynomials in the elementary symmetric polynomials. A different proof: all of $\alpha_1, \ldots, \alpha_n$ belong to the splitting field of $p(x)$ inside K. So we can assume without loss of generality that K is the splitting field of $p(x)$. The Galois group of K/k acts as permutations of $\alpha_1, \ldots, \alpha_n$, fixing k, so fixes $q(\alpha_1, \ldots, \alpha_n)$. But K is a Galois extension of k, so the fixed elements under the Galois group are precisely the elements of k.

19.2. Suppose that C has equation $c(x, y) = 0$ and D has equation $d(s, t) = 0$. Take an algebra isomorphism $\Phi\colon k[C] \to k[D]$. This map Φ is uniquely determined by $\Phi(x), \Phi(y)$, because after that every polynomial $f(x, y)$ is mapped to $\Phi(f(x, y)) = f(\Phi(x), \Phi(y))$, as the algebra morphism preserves the addition and multiplication operations by which we construct polynomials. Pick polynomials $x(s, t), y(s, t)$ so that, in $k[D]$, $x(s, t) = \Phi(x)$ and $y(s, t) = \Phi(y)$. Let

$$\psi\colon (s, t) \in \bar{k}^2 \mapsto (x, y) = (x(s, t), y(s, t)) \in \bar{k}^2.$$

Two polynomials agree along the $\bar{k}$-points of the curve D just when their difference vanishes at those points, so just when their difference is divisible by $d(s, t)$, by irreducibility of $d(s, t)$. So these $x(s, t), y(s, t)$ are determined only up to multiples of the irreducible $d(s, t)$.

Since Φ is an isomorphism, it has an inverse isomorphism Φ^{-1}, and so we can similarly find polynomials $s(x, y), t(x, y)$, defined up to adding multiples of $c(x, y)$, so that $s(x, y) = \Phi^{-1} s, t(x, y) = \Phi^{-1} t$ in $k[C]$. Define a map

$$\varphi\colon (x, y) \in \bar{k}^2 \mapsto (s, t) = (s(x, y), t(x, y)) \in \bar{k}^2.$$

Extend Φ to a map $k[x,y] \to k[s,t]$ by

$$\Phi(x) = x(s,t), \Phi(y) = y(s,t)$$

and similarly extend Φ^{-1} to a map $\Psi\colon k[s,t] \to k[x,y]$ by

$$\Psi(s) = s(x,y), \Psi(t) = t(x,y).$$

On any polynomial $f(x,y)$,

$$\begin{aligned} f(\psi(s,t)) &= f(x(s,t), y(s,t)), \\ &= f(\Phi(x), \Phi(y)), \\ &= \Phi(f(x,y)). \end{aligned}$$

Similarly, for any polynomial $f(s,t)$,

$$f(\varphi(x,y)) = \Psi(f(s,t)).$$

But $\Psi \circ \Phi$ reduces to the identity on $k[C] \to k[C]$, so on polynomials,

$$\begin{aligned} \Psi \circ \Phi(x) &= x + a(x,y)c(x,y), \\ \Psi \circ \Phi(y) &= y + b(x,y)c(x,y) \end{aligned}$$

for some polynomials $a(x,y), b(x,y)$. Similarly

$$\begin{aligned} \Phi \circ \Psi(s) &= s + A(s,t)d(s,t), \\ \Phi \circ \Psi(t) &= t + B(s,t)d(s,t) \end{aligned}$$

for some polynomials $A(s,t), B(s,t)$. Composing,

$$\begin{aligned} \psi(\varphi(x,y)) &= (x(\varphi(x,y)), y(\varphi(x,y))), \\ &= (\Psi(x(s,t)), \Psi(y(s,t))), \\ &= (\Psi(\Phi(x)), \Psi(\Phi(y))), \\ &= (x + a(x,y)c(x,y), y + b(x,y)c(x,y)), \end{aligned}$$

Similarly the other way around

$$\varphi(\psi(s,t)) = (s + A(s,t)d(s,t), t + B(s,t)d(s,t)).$$

So on the $\bar{k}$-points of the curves, these are inverses of one another.

19.8.

$$\begin{aligned}
2\int \left(t^2-1\right)^2\,dt &= 2\int \left(t^4-2t^2+1\right)\,dt,\\
&= 2\left(\frac{t^5}{5}-\frac{2}{3}t^3+t\right),\\
&= 2\left(\frac{1}{5}\left(\frac{y}{x}\right)^5-\frac{2}{3}\left(\frac{y}{x}\right)^3+\frac{y}{x}\right),\\
&= 2\left(\frac{1}{5}\frac{y^5}{x^5}-\frac{2}{3}\frac{y^3}{x^3}+\frac{y}{x}\right),\\
&= \frac{2y}{15x^5}\left(3\left(y^2\right)^2-10x^2y^2+15x^4\right),\\
&= \frac{2y}{15x^5}\left(3\left(x^3+x^2\right)^2-10x^2(x^3+x^2)+15x^4\right),\\
&= \frac{2y}{15x}\left(3x^2-4x+8\right),\\
&= \frac{2\sqrt{x^2+x^3}}{15x}\left(3x^2-4x+8\right).
\end{aligned}$$

19.9. The substition

$$\begin{aligned}
\cos\theta &= (1-t^2)/(1+t^2),\\
\sin\theta &= 2t/(1+t^2),
\end{aligned}$$

rewrites the integral in terms of t. (In fact, this substitution is just the famous substition $t=\tan(\theta/2)$ which you can find in the calculus textbooks.)

$$\begin{aligned}
\int f(\cos\theta,\sin\theta)d\theta &= \int \frac{f(\cos\theta,\sin\theta)}{-\sin\theta}(-\sin\theta)d\theta,\\
&= \int \frac{f(\cos\theta,\sin\theta)}{-\sin\theta}d\cos\theta,\\
&= \int \frac{f(x,y)}{-y}dx,\\
&= \int f\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right)\frac{1+t^2}{-2t}d(\frac{1-t^2}{1+t^2}),\\
&= \int f\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right)\frac{1+t^2}{-2t}\frac{(-4t)}{(1+t^2)^2}dt,\\
&= \int f\left(\frac{1-t^2}{1+t^2},\frac{2t}{1+t^2}\right)\frac{2}{1+t^2}dt.
\end{aligned}$$

Take a partial fraction decomposition as in the proof of corollary 10.8 on page 74.

19.10. If $c(x,y)$ divides $p(x,y)$, clearly $p(x,y)=0$ on every $\bar{k}$-point where $c(x,y)=0$. Suppose that $p(x,y)=0$ on those points. For any constant value of x or y in $\bar{k}$, there are points where $c(x,y)=0$ and so the resultant of $p(x,y),c(x,y)$ in the other variable

vanishes. So the resultant vanishes for infinitely many values, so vanishes everywhere. So $p(x,y), c(x,y)$ have a common factor in $k[x,y]$. Recall that $c(x,y)$ is irreducible.

22.1. Let $k_1 \subset k_2 \subset k_3 \ldots$ be finite fields of orders $2, 4, 8$ and so on. Our result is clear if $k = k_1 = \mathbb{Z}/2\mathbb{Z}$. By induction, suppose that our result is true for $k_1, k_2, \ldots, k_{n-1}$. All of the elements α of k_n that are not elements of k_{n-1} satisfy $\alpha^2 + \alpha + c = 0$ for some c in k_{n-1}, as we saw above. By induction, $c = b^2$ for some element b of k_{n-1}. So $\alpha^2 + \alpha + b^2 = 0$, i.e. $\alpha^2 + b^2 = \alpha$. Expand out $(\alpha + b)^2$ to find $(\alpha + b)^2 = \alpha$.

22.3. Take a basis consisting of one element $\alpha \in K$. Then $1 \in K$ is somehow a multiple $1 = a\alpha$ for some $a \in k$ so $\alpha = 1/a \in k$. Any element β of K is $\beta = c\alpha$ for some $c \in k$ so $\beta \in k$.

22.4. Let n be the degree of $p(x)$. We write each element $b(x)$ in $k[x]$ as a polynomial, but if the polynomial has degree n or more, rewrite it as $b(x) = q(x)p(x) + r(x)$, quotient and remainder. Then every element of K is written as the remainder term, i.e. as a polynomial of degree at most $n-1$. Hence the elements $1, x, \ldots, x^{n-1}$ in $k[x]$ map to a spanning set inside K. If not a basis, then there must be some linear relation between them, i.e. a lower degree polynomial in x vanishing in K, i.e. vanishing modulo $p(x)$. But then taking quotient and remainder, we see that this is not possible. Hence K has a basis over k consisting of the images of $1, x, x^2, \ldots, x^{n-1}$ in $k[x]$.

23.1. $P(x,y) = x^4 + xy^3 + y^4$

24.3. Homogenize to $x^3 + x^2y + xy^2 + y^3 + x^2z = z^3$, and then set $z = 0$ to get the homogeneous equation $x^3 + x^2y + xy^2 + y^3 = 0$. Factor: $(x+y)(x^2+y^2) = 0$. So $x + y = 0$, i.e. $[x, -x, 0] = [1, -1, 0]$, or $x^2 + y^2 = 0$, which factors over any field k which contains an element i so that $i^2 = -1$, as $(x - iy)(x + iy) = 0$, i.e. $y = \pm ix$, so $[x, ix, 0] = [1, i, 0]$ and $[x, -ix, 0] = [1, -i, 0]$. Hence the points of our curve that lie on the line at infinity are $[1, -1, 0], [1, i, 0], [1, -i, 0]$.

24.4. Start with the affine plane. Remember (from problem 9.8 on page 59) that you can use the quadratic formula:

$$y = \frac{-1 \pm \sqrt{1 - 4(-x^3 - x - 1)}}{2}.$$

Simplify, using $2^{-1} = 3$ and $4 = -1$, to:

$$y = 3(-1 \pm \sqrt{-x^3 - x})$$

Note that $0^2 = 0, 1^2 = 1, 2^2 = 4, 3^2 = 9 = 4, 4^2 = 16 = 1$, so only $0, 1, 4$ have square roots.

- $x = 0$: $-x^3 - x = 0$, $y = -3 = 2$, $[x, y, z] = [0, 2, 1]$.
- $x = 1$: $-x^3 - x = 3$, no square root of 3, no solution.
- $x = 2$: $-x^3 - x = -8 - 2 = 0$, $y = -3 = 2$, $[x, y, z] = [2, 2, 1]$.
- $x = 3$: $-x^3 - x = -27 - 3 = 0$, $y = -3 = 2$, $[x, y, z] = [3, 2, 1]$.
- $x = 4$: $-x^3 - x = -68 = 2$, no square root of 2 no solution.

We still need the points "at infinity". Homogenize: $y^2z + yz^2 = x^3 + xz^2 + z^3$, and set $z = 0$: $0 = x^3$, so $x = 0$, i.e. $[x, y, z] = [0, y, 0]$, and rescale y to $[0, 1, 0]$. Final answer: $[0, 2, 1], [2, 2, 1], [3, 2, 1], [0, 1, 0]$.

24.5. Every nonconstant rational function in the projective plane has poles along a curve, and curves intersect (at points over the algebraic closure of the field, at least), by Bézout's theorem.

31.2. $p(x, y) = (1 - xy)^2 + x^2$ over the field $k = \mathbb{R}$.

31.3. By the nullstellensatz, X is irreducible in U, since splitting X into a union of distinct subvarieties X_1, X_2, each would satisfy an equation not satisfied on the other, say $f_1 = 0$ and $f_2 = 0$. But then $f = 0$ lies inside $f_1 f_2 = 0$, so f divides f_1 or f_2.

31.5. The projection consists of the points $(x, -1, z)$ for which $t(x, -1, z)$ satisfies our equations for some t. Plug in to get $t^2x^2 + t + 1 = 0$ and $t(t^2x^3 - z) = 0$. Since $y = x^2 + 1 > 0$ on our curve, we need $t < 0$ to get $y = -t$ to lie on the curve. So $z = t^2x^3$. Check that $x(x + z)^2 = z$ is our resulting cubic curve.

33.5. Suppose that x has this form. Either there is a linear relation among $v_1, \ldots, v_p$, so $x = 0$, so this rank is zero, or there is no such relation. We can then take these v_i as the first p vectors in a basis, and expand w in that basis, to see that the kernel of this linear map consists just in w being a linear combination of $v_1, \ldots, v_p$, so kernel of dimension p.

Suppose that this map $w \mapsto w \wedge x$ has such kernel, say of dimension $s \geq p$. Take a basis $e_1, \ldots, e_n$ of V with $e_1, \ldots, e_s$ a basis for that kernel. Expand x in the basis e_I, say as $x = \sum x_I e_I$ with coefficients x_I. Let $I_0 \coloneqq \{\, 1, 2, \ldots, p \,\}$. Then

$$\begin{aligned} 0 &= e_i \wedge x, \\ &= \sum_{i \notin I} x_I e_i \wedge e_I. \end{aligned}$$

Since these e_I are a basis, and $e_i \wedge e_I = \pm e_J$ for $J = \{\, i \,\} \cup I$, $x_I = 0$ unless $i \in I$. Since this holds for all these $i = 1, 2, \ldots, p$, we find that I contains $1, 2, \ldots, p$, and has p indices, so $I = I_0$.

33.9. Since X_I is closed, and contains C_I, X_I contains the closure of C_I. Any homogeneous polynomial vanishing on C_I vanishes on all

$$v_1 \wedge \cdots \wedge v_p$$

for $v_1, \ldots, v_p$ a basis of any W in C_I. We can choose this basis to be

$$\begin{aligned} v_1 &= \sum_{j < i_1} a_{j1} e_j + e_{i_1}, \\ v_2 &= \sum_{j \neq i_1, j < i_2} a_{j2} e_j + e_{i_2}, \\ &\vdots \\ v_\ell &= \sum_{j \notin I}^{j < i_\ell} a_{j\ell} e_j + e_{i_\ell}, \\ &\vdots \\ v_p &= \sum_{j \notin I}^{j < i_p} a_{jp} e_j + e_{i_p}. \end{aligned}$$

Scaling these by constants a_{ji_j}, our homogeneous polynomials vanish on

$$v_1 \wedge \cdots \wedge v_p$$

for

$$\begin{aligned} v_1 &= \sum_{j \le i_1} a_{j1} e_j, \\ v_2 &= \sum_{j \ne i_1, j \le i_2} a_{j2} e_j, \\ &\vdots \\ v_\ell &= \sum_{j \notin I}^{j \le i_\ell} a_{j\ell} e_j, \\ &\vdots \\ v_p &= \sum_{j \notin I}^{j \le i_p} a_{jp} e_j. \end{aligned}$$

But now we can vary these a_{ji_j} to get vanishing on all W lying in all C_J with intersection dimensions at most those of C_I.

34.14. Any projective plane already contains a quadrilateral; pick out any one of those 4 points to make a triangle, and then pick out another to make another triangle. They are in perspective from either of those two points. Taking duals gives perspective from a line.

34.17. To have a nonzero element $a + \iota b$ being a zero divisor on the left, we would need to have

$$\begin{aligned} 0 &= (a + \iota b)(c + \iota d), \\ &= ac - \bar{d}\bar{b} + \iota \left(da + b\bar{c} \right). \end{aligned}$$

Therefore $ac = \bar{d}\bar{b}$ and $da = -b\bar{c}$. Multiplying suitably, $dac = d\bar{d}\bar{b}$ and $dac = -c\bar{c}b$. So $0 = \left(d\bar{d} + c\bar{c} \right) b$ and thus $b = 0$ or else $c = d = 0$. But if $b = 0$ then $ac = 0$ and $da = 0$, so $a = 0$ or else $c = d = 0$.

To have a nonzero element $a + \iota b$ being a zero divisor on the right, we would need to have

$$\begin{aligned} 0 &= (c + \iota d)(a + \iota b), \\ &= ca - \bar{b}\bar{d} + \iota \left(bc + d\bar{a} \right). \end{aligned}$$

Therefore $ca = \bar{b}\bar{d}$ and $bc = -d\bar{a}$. Multiplying suitably, $bca = b\bar{b}\bar{d}$ and $bca = -da\bar{a}$. So therefore $d = 0$ or else $a = b = 0$. But if $d = 0$ then $ca = 0$ and $bc = 0$, so $c = 0$.

Bibliography

If those books are in agreement with the Quran, we have no need of them; and if these are opposed to the Quran, destroy them.
— Omar

We are tired of works of truth.
— Qumran Scrolls 4Q418 69 ii 13-14

[1] Lars V. Ahlfors, *Complex analysis*, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR 510197

[2] M. A. Armstrong, *Groups and symmetry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1988. MR 965514 158

[3] Vladimir I. Arnold, *Ordinary differential equations*, Universitext, Springer-Verlag, Berlin, 2006, Translated from the Russian by Roger Cooke, Second printing of the 1992 edition. MR 2242407

[4] Oleg Bogopolski, *Introduction to group theory*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2008, Translated, revised and expanded from the 2002 Russian original. MR 2396717 158

[5] C. Herbert Clemens, *A scrapbook of complex curve theory*, second ed., Graduate Studies in Mathematics, vol. 55, American Mathematical Society, Providence, RI, 2003. MR 1946768 (2003m:14001)

[6] David S. Dummit and Richard M. Foote, *Abstract algebra*, third ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004. MR 2286236 (2007h:00003) 100, 158

[7] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina, *Introduction to representation theory*, Student Mathematical Library, vol. 59, American Mathematical Society, Providence, RI, 2011, With historical interludes by Slava Gerovitch. MR 2808160

[8] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete mathematics*, second ed., Addison-Wesley Publishing Company, Reading, MA, 1994, A foundation for computer science. MR 1397498 (97d:68003)

[9] Kenneth Hoffman and Ray Kunze, *Linear algebra*, Second edition, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1971. MR 0276251

[10] Thomas W. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York-Berlin, 1980, Reprint of the 1974 original. MR 600654 (82a:00006)

[11] E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1944. MR 0010757

[12] Walter Ledermann, *Introduction to the theory of finite groups*, Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1953, 2d ed. MR 0054593 158

[13] Saunders Mac Lane and Garrett Birkhoff, *Algebra*, third ed., Chelsea Publishing Co., New York, 1988. MR 941522

[14] Tristan Needham, *Visual complex analysis*, The Clarendon Press, Oxford University Press, New York, 1997. MR 1446490

[15] Gian-Carlo Rota, *Combinatorial theory and invariant theory*, Bowdoin College, Maine, U.S.A., 1971, Notes by L. Guibas.

[16] Igor R. Shafarevich, *Basic notions of algebra*, Encyclopaedia of Mathematical Sciences, vol. 11, Springer-Verlag, Berlin, 2005, Translated from the 1986 Russian original by Miles Reid, Reprint of the 1997 English translation [MR1634541], Algebra, I. MR 2164721

[17] Igor R. Shafarevich and Alexey O. Remizov, *Linear algebra and geometry*, Springer, Heidelberg, 2013, Translated from the 2009 Russian original by David Kramer and Lena Nekludova. MR 2963561

[18] Michael Spivak, *Calculus*, Calculus, Cambridge University Press, 2006. 52, 65, 72

[19] The Sage Developers, *Sagemath, the Sage Mathematics Software System* (*Version 7.5.1*), 2018, http://www.sagemath.org.

[20] E. B. Vinberg, *A course in algebra*, Graduate Studies in Mathematics, vol. 56, American Mathematical Society, Providence, RI, 2003, Translated from the 2001 Russian original by Alexander Retakh. MR 1974508 (2004h:00001)

[21] Judy L. Walker, *Codes and curves*, Student Mathematical Library, vol. 7, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 2000, IAS/Park City Mathematical Subseries. MR 1768485 (2001f:14046)

[22] André Weil, *Number theory for beginners*, Springer-Verlag, New York-Heidelberg, 1979, With the collaboration of Maxwell Rosenlicht. MR 532370 (80e:10004)

List of notation

Index