Generating Voting Rules from Random Relations

Extended Abstract

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ABSTRACT

We consider a way of generating voting rules based on a random relation, the winners being alternatives that have the highest probability of being supported. We consider different notions of support, such as whether an alternative dominates the other alternatives, or whether an alternative is undominated, and we consider structural assumptions on the form of the random relation, such as being acyclic, asymmetric, connex or transitive. We give sufficient conditions on the supporting function for the associated voting rule to satisfy various properties such as Pareto and monotonicity. The random generation scheme involves a parameter p between zero and one. Further voting rules are obtained by tending p to zero, and by tending p to one, and these limiting rules satisfy a homogeneity property, and, in certain cases, Condorcet consistency. We define a language of supporting functions based on eight natural properties, and categorise the different rules that can be generated for the limiting p cases.

KEYWORDS

Voting rules; random relations; limiting probabilities

ACM Reference Format:

Nic Wilson. 2019. Generating Voting Rules from Random Relations. In *Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), Montreal, Canada, May 13–17, 2019,* IFAAMAS, 3 pages.

1 INTRODUCTION

We develop in this paper a framework for aggregating multi-agent preferences, including many interesting instances (i.e., different aggregation methods), based on a novel probabilistic model. Our approach involves using a weighted relation v to pick a random binary preference relation between alternatives. The numerical support for an alternative x is the chance that the randomly picked relation R (logically) supports x, i.e., $\Pr(R \in \mathrm{Sp}_x)$, where Sp_x is defined to be the set of relations that support x. The output is the set of winners, i.e., the set of alternatives with maximal numerical support. Many different notions of logical support are possible, leading to different definitions of Sp_x.

Randomly generating relation R

Let A be a finite set of alternatives. Define Δ to be the set of pairs (x, y) with $x \neq y$. A subset R of Δ is thus an irreflexive binary relation on A. We define $\mathcal{V}$ to be the set of all functions v from Δ to the non-negative reals. An element v of $\mathcal{V}$ will be intended to represent some degree of preference for alternative x over alternative y. For instance, in a voting scenario, it could represent the number of voters preferring x to y. We generate a random irreflexive binary relation R, based on parameter $p \in (0, 1)$, as follows. For each $(x, y) \in \Delta$ we (independently) omit (x, y) from R with chance $(1 - p)^{v(x,y)}$, so the probability that R contains (x, y) equals $1 - (1 - p)^{v(x,y)}$. Based on this, the chance $Pr_p^v(\{R\})$ that the randomly chosen relation is equal to a particular R ($\subseteq \Delta$) is defined as follows:

$$Pr_p^v(\{R\}) = \prod_{(x,y)\in R} (1 - (1 - p)^{v(x,y)}) \times \prod_{(x,y)\in \Delta\setminus R} (1 - p)^{v(x,y)}.$$

Example 1.1. Consider the set of alternatives $A = \{a, b, c, d\}$ and $v \in \mathcal{V}$ represented by the following table, with e.g., $v(a, b) = 5$; v may, for example, arise from a profile with eight voters: two voters with preference order $a > b > c > d$, and three voters with each of $a > b > d > c$ and $c > b > d > a$.

$v(x, y)$	a	b	c	d
a	–	5	5	5
b	3	–	5	8
c	3	3	–	5
d	3	0	3	–

Let $R = \mathrm{O}_b = \{(b, a), (b, c), (b, d)\}$. Let $q = 1 - p$ and let $r = \sum_{(x,y)\in\Delta\setminus \mathrm{O}_b} v(x, y) = 32$. Then $Pr_p^v(\{R\})$ equals $(1 - q^{v(b,a)})(1 - q^{v(b,c)})(1 - q^{v(b,d)})q^r$, i.e., $(1 - q^3)(1 - q^5)(1 - q^8)q^{32}$. □

Defining winners, given $p \in (0, 1)$

A supporting function Sp associates a set of relations Sp_x ($\subseteq 2^\Delta$) with each alternative x in A. Given $v \in \mathcal{V}$ and a value $p \in (0, 1)$, we consider, for each alternative x, the probability $\Pr_p^v(\mathrm{Sp}_x)$ of Sp_x, i.e., $\sum_{R\in \mathrm{Sp}_x} Pr_p^v(\{R\})$. This generates a social choice rule in the obvious way: we define $W_p^{\mathrm{Sp}}(v)$ to be the set of alternatives x that maximise $\Pr_p^v(\mathrm{Sp}_x)$, so that $x \in W_p^{\mathrm{Sp}}(v)$ if and only if for all alternatives y, $\Pr_p^v(\mathrm{Sp}_x) \geq \Pr_p^v(\mathrm{Sp}_y)$.

Two basic supporting functions, Opt and U

We say that x *is dominating in relation* R if R contains the set $\mathrm{O}_x = \{(x, y) : y \neq x\}$, so that x is preferred to every other alternative with respect to preference relation R. We say that the supporting function Sp satisfies the property Opt if Sp_x only contains relations R in which x is dominating. In this case, R supports x only if x is dominating in relation R. We say that Sp satisfies property U if $R \in \mathrm{Sp}_x$ implies x is undominated in R, i.e., $R \cap \mathrm{D}_x = \emptyset$, where $\mathrm{D}_x = \{(y, x) : y \neq x\}$, so no alternative dominates x.

Proc. of the 18th International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2019), N. Agmon, M. E. Taylor, E. Elkind, M. Veloso (eds.), May 13–17, 2019, Montreal, Canada.

Example 1.1 continued: Suppose we define Sp_x to be U_x for $x \in A$; with this definition, relation R supports b if and only if b is not dominated in R, i.e., there exists no pair of the form (x, b) in R. In other words, $R \in \mathrm{Sp}_b$ if and only if $R \subseteq \Delta \setminus \mathrm{D}_b$, where $\mathrm{D}_b = \{(a, b), (c, b), (d, b)\}$. $Pr_p^v(\mathrm{U}_b) = \sum_{R \subseteq \Delta \setminus \mathrm{D}_b} Pr_p^v(\{R\})$, which can be shown to be equal to $\prod_{\psi \in \mathrm{D}_b} q^{v(\psi)} = q^s$ where $s = \sum_{\psi \in \mathrm{D}_b} v(\psi) = 8$. Similarly, $Pr_p^v(\mathrm{U}_a) = q^9$, $Pr_p^v(\mathrm{U}_c) = q^{13}$, and $Pr_p^v(\mathrm{U}_d) = q^{18}$. This shows that whatever value is chosen for $p \in (0, 1)$, b is the unique winner, since it has the highest probability of being supported: $Pr_p^v(\mathrm{U}_b) > Pr_p^v(\mathrm{U}_x)$ for $x \neq b$. In fact, Proposition 1.2 below implies that $\mathrm{Sp}_x = \mathrm{U}_x$ generates the Borda voting rule for any value of p.

Suppose we instead define Sp_x to be $\mathrm{U}_x \cap \mathrm{Opt}_x$ for all $x \in A$. Now Sp_x contains all relations R in which (i) x is undominated and (ii) x dominates the other alternatives. It can be shown that $Pr_p^v(\mathrm{Sp}_b) = (1 - q^3)(1 - q^5)(1 - q^8)q^8$ and $Pr_p^v(\mathrm{Sp}_a) = (1 - q^5)^3 q^9$. With e.g., $p = 0.5$, this makes b the unique winner; in fact, b is the unique winner unless p is very small (less than around 0.0693), when a becomes the winner. □

We consider sufficient conditions for desirable properties on the voting rule. In particular, if Sp satisfies both Opt and U then we show that the voting rule satisfies natural monotonicity and Pareto properties. This therefore gives a method for generating a large family of voting (and other aggregation) rules that have some good properties.

Further supporting functions

As well as properties Opt and U we consider a weaker form TOpt of property Opt, (relating to whether x is dominating in the transitive closure of R) and OOpt, which means that, for $R \in \mathrm{Sp}_x$, R only contains elements of the form (x, z), i.e., $R \subseteq \mathrm{O}_x$. We also consider structural properties that restrict the form of the relation: asymmetry, acyclicity, connex, and transitivity properties. We consider a simple language $\mathfrak{L}$ of logical support, based on these eight properties, with a supporting function being generated by a subset of the eight properties.

When the winners do not depend on p

The following result shows special cases in which the winner is independent of the value $p \in (0, 1)$, in particular the case when $\mathrm{Sp}_x = \mathrm{U}_x$ and the case when $\mathrm{Sp}_x = \mathrm{OOpt}_x$.

Proposition 1.2. *Suppose that for each $x \in A$, Sp_x is of the form $\{R : R \subseteq S_x\}$ for some $S_x \subseteq \Delta$. Then, $Pr_p^v(\mathrm{Sp}_x) = (1 - p)^{v^+(\Delta)} \times (1 - p)^{-v^+(S_x)}$, and $x \in W_p^{\mathrm{Sp}}(v)$ if and only if $x \in argmax_z v^+(S_z)$. In particular, we have $x \in W_p^{\mathrm{U}}(v)$ if and only if $x \in argmin_z v^+(\mathrm{D}_z)$, and $x \in W_p^{\mathrm{OOpt}}(v)$ if and only if $x \in argmax_z v^+(\mathrm{O}_z)$.*

Tending $p \to 1$ or $p \to 0$

A way to ensure a homogeneity property (in which a linear rescaling of the input v makes no difference) is to consider the result of tending p to either 1 or 0. We show that the set of winners is still always non-empty and that we obtain somewhat simpler structures determining the voting rules.

We completely characterise the voting rules for the language $\mathfrak{L}$, for the $p \to 0$ case, and for the $p \to 1$ case in which v is non-zero, leading to seven different voting rules in each case. We show, in particular, that the $p \to 1$ cases lead to a number of well-known voting rules: Borda, the Kemeny rule, Tideman's rule, and maximin.

2 RELATED WORK

The input $v \in \mathcal{V}$ of a $\mathcal{V}$-rule can be viewed as a weighted directed graph on alternatives, with non-negative real weights; this suggests the potential of relationships with weighted tournament solutions, C2 functions in the Fishburn's classification [6, 7]. In particular, in certain cases, there is a correspondence between the $(p \to 1)$-winners and the winners according to a voting rule generated by median orders [1, 6, 8].

There are also some links between between $(p \to 1)$ cases and voting rules generated from Maximum Likelihood Estimators (MLE), and consensus-based voting rules [2, 3, 5], since the maximin rule can be generated, as well as Borda. In a general sense, our approach with limiting p is reminiscent of the construction of the rules $\mathrm{MLE}^{\infty}_{intr}$, MLE^{1}_{intr}, $\mathrm{MLE}^{\infty}_{tr}$ and MLE^{1}_{tr} in [5], and they can give similar rules to the $p \to 1$ rules generated with our framework. However, even when they do, the tie-breaking can be very different from the winners in our framework, as illustrated on pages 190 and 191 of [5], which suggests that there is not a simple correspondence between the different frameworks.

3 DISCUSSION

We have defined and explored a framework for generating voting rules (and $\mathcal{V}$-rules that allow a general form of input), based on winners being alternatives that maximise the probability of being supported. We have given some simple sufficient conditions for certain properties of the voting rule. We defined a simple language of supporting functions, and categorised the rules generated for the two limiting cases with $p \to 0/1$.

Our method allows one to generate large (and continuous) families of voting rules that satisfy some good properties. In particular, if we choose some neutral Sp based on (arbitrarily complicated) sets of relations and add the conditions Opt and U then we obtain neutral $\mathcal{V}$-rules, and thus also voting rules (using an arbitrary strictly monotonic non-zero function of the positive reals), that satisfy Pareto and monotonicity properties. Homogeneity of the voting rule can be enforced by an additional normalisation step. If we additionally consider $p \to 1$ and restrict to asymmetric relations then the voting rule will satisfy the Condorcet property.

For the limiting $p \to 1$ case, it is striking that several well-known voting rules can be generated by choosing different natural choices of the supporting function, including Borda, maximin, the Kemeny rule and Tideman's rule. Therefore, as well as generating new voting rules, the approach gives a new perspective on standard rules, and it would be interesting to pursue a view of the framework as a rationalisation of certain classes of voting rules [4, 5].

ACKNOWLEDGMENTS

This material is based upon works supported by the Science Foundation Ireland under Grant No. 12/RC/2289 which is co-funded under the European Regional Development Fund.

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