To appear in:
Communications in Mathematical Physics

QUANTUM STOCHASTIC COCYCLES AND COMPLETELY BOUNDED SEMIGROUPS ON OPERATOR SPACES II

J. MARTIN LINDSAY AND STEPHEN J. WILLS

Dedicated to the memory of Ola Bratteli

Abstract. Quantum stochastic cocycles provide a basic model for time-homogeneous Markovian evolutions in a quantum setting, and a direct counterpart in continuous time to quantum random walks, in both the Schrödinger and Heisenberg pictures. This paper is a sequel to one in which correspondences were established between classes of quantum stochastic cocycle on an operator space or C^*-algebra, and classes of Schur-action 'global' semigroup on associated matrix spaces over the operator space. In this paper we investigate the stochastic generation of cocycles via the generation of their corresponding global semigroups, with the primary purpose of strengthening the scope of applicability of semigroup theory to the analysis and construction of quantum stochastic cocycles. An explicit description is given of the affine relationship between the stochastic generator of a completely bounded cocycle and the generator of any one of its associated global semigroups. Using this, the structure of the stochastic generator of a completely positive quasicontractive quantum stochastic cocycle on a C^*-algebra whose expectation semigroup is norm continuous is derived, giving a comprehensive stochastic generalisation of the Christensen–Evans extension of the GKS&L theorem of Gorini, Kossakowski and Sudarshan, and Lindblad. The transformation also provides a new existence theorem for cocycles with unbounded structure map as stochastic generator. The latter is applied to a model of interacting particles known as the quantum exclusion Markov process, in particular on integer lattices in dimensions one and two.

Contents

2000 *Mathematics Subject Classification.* Primary 81S25, 46L53; Secondary 46N50, 47D06.

Key words and phrases. Quantum dynamical semigroup, quantum Markov semigroup, completely positive, quasicontractive, generator, operator space, operator system, matrix space, Schur-action, Markovian cocycle, stochastic semigroup, quantum exclusion process.

Introduction

This paper is a continuation of our operator space analysis of quantum stochastic cocycles begun in [LW_5]. In that paper we established correspondences between classes of completely bounded quantum stochastic cocycle on an operator space and associated 'global' semigroups of Schur-action maps on corresponding matrix spaces. This was done for a variety of types of operator space, including operator systems, C^*-algebras and column operator spaces, and corresponding cocycle types, such as completely contractive, completely positive and contractive operator cocycles. In the current paper we analyse the relationship between the stochastic generation of such cocycles via quantum stochastic differential equations, and the generation of corresponding global semigroups, and identify the affine transformation from one to the other. This is used to obtain a new existence theorem for quantum stochastic differential equations with unbounded coefficients. We also use it to characterise, directly and in full generality, the stochastic generators of completely positive quasicontractive quantum stochastic cocycles on a C^*-algebra whose expectation semigroups are norm continuous. In view of the fact that norm-continuous semigroups are automatically quasicontractive, this result amounts to a full quantum stochastic extension of the well-known characterisation of the generators of norm-continuous, completely positive semigroups on a C^*-algebra ([ChE]). The new existence theorem for quantum stochastic cocycles is shown to be applicable to 'structure maps' on a C^*-algebra whose domain is matricially square-root closed, by means of semigroup theory. The structure relations are necessary conditions for any generated quantum stochastic cocycle to be a quantum stochastic flow, that is to be unital and *-homomorphic. In turn, we construct such structure maps to obtain quantum Markov exclusion processes on integer lattices governed by quantum stochastic differential equations by applying a theorem of Bratteli and Kishimoto.

In brief, the main results of the paper are as follows. Theorem 4.2 gives the affine transformation from stochastic generator to generator of associated global semigroup, for elementary quantum stochastic cocycles with completely bounded stochastic generator (*elementary*, formerly *cb-Markov regular*, means that each of its associated semigroups is cb-norm continuous). Theorem 5.3 is a new existence theorem for the generation of quantum stochastic cocycles. Theorem 6.4 characterises the stochastic generator of completely positive quasicontractive elementary quantum stochastic cocycles. Theorem 7.4 applies the above-mentioned existence theorem to structure maps, and Theorem 8.1 demonstrates its applicability to quantum exclusion processes.

The prequel of this paper contains a long motivational introduction, and extensive bibliography. A detailed outline of the contents of the current paper follows.

In Section 1 we develop some theory of κ-decomposable completely bounded maps between h-matrix spaces over operator spaces (whose definition is recalled there), for a Hilbert space h with orthonormal basis κ. Under the identification of the matrix space with an actual space of matrices with operator entries, determined by the basis, these κ-decomposable maps correspond to Schur-action maps. Also, each κ-decomposable completely bounded map restricts to a map between spatial tensor products of the space of compact operators on h with the corresponding operator space. We show that this restriction is completely isometrically reversible (Proposition 1.4). The theory is then applied to obtain a characterisation of the generators of norm-continuous completely positive semigroups of κ-decomposable operators on the h-matrix space over a unital C^*-algebra (Theorem 1.7). It is worth recalling that such matrix spaces are operator systems, but typically *not* C^*-algebras.

In Section 2 we investigate in detail the correspondence between continuity properties, in the time parameter t, of completely bounded quantum stochastic cocycles and that of their global semigroups. In Section 3 we relate quantum stochastic cocycles to a basic source, namely quantum stochastic differential equations. In Section 4, by applying quantum stochastic calculus to the appropriate 'diagonal Weyl process' introduced in [LW_5], we compute the affine transformation from the stochastic generator of a completely bounded quantum stochastic cocycle to the generator of the corresponding global semigroup, and give sufficient conditions for its injectivity (Theorem 4.2). Restricting to the global semigroups with respect to an orthonormal basis of the noise dimension space, the transformation is necessarily injective; we give the form of the (partially defined) inverse (Theorem 4.3). In Section 6, bijectivity is demonstrated for specific classes of cocycle.

Section 5 concentrates on completely positive quantum stochastic cocycles on operator systems and C^*-algebras. We first establish a characterisation of contractivity for completely positive quantum stochastic cocycles in terms of the generators of its associated semigroups (Proposition 5.2). This constitutes an infinitesimal counterpart to one of the central results of [LW_5]. Using it we obtain an existence theorem for completely positive quantum stochastic cocycles governed by a quantum stochastic differential equation with unbounded coefficient (Theorem 5.3). In Section 6, Proposition 5.2 and the affine correspondence established in Section 4 are employed to obtain a direct proof of the full quantum stochastic extension of the characterisation theorem for generators of norm-continuous quantum dynamical semigroups on a C^*-algebra A ([ChE], [Lin], [GKS]), namely a characterisation of the stochastic generators of completely positive quasicontractive elementary quantum stochastic cocycles on A with arbitrary noise dimension space, in both the unital and nonunital cases (Theorem 6.4), completing our earlier results [LW_2] and [LiP]. As an application, all *-homomorphic elementary quantum stochastic cocycles are shown to be stochastically generated (Theorem 6.6).

In Section 7 we prove an existence theorem for completely positive unital quantum stochastic cocycles on a unital C^*-algebra (Theorem 7.4). Our hypotheses include necessary conditions on the coefficient map of the governing quantum stochastic differential equation for the solution to be a quantum stochastic flow, by which we mean a quantum stochastic cocycle which is unital and *-homomorphic. As a consequence, the existence theorem dovetails very nicely with recent work on such flows ([DGS]) in which our conclusion coincides with their standing hypothesis. A corollary of the theorem is that completely bounded structure maps generate quantum stochastic cocycles that are necessarily completely positive and contractive.

In the final section we demonstrate the applicability of the existence theorem by constructing quantum exclusion processes as dilations of the quantum Markov semigroups introduced in [Reb]; the methods used in [BW_1] and [BW_2] differ from, but are nicely complementary to, those employed here.

Notation. The symbols $\underline{\otimes}$, $\otimes$ and $\overline{\otimes}$ are used to denote respectively the algebraic, spatial and ultraweak tensor products. The C^*-algebra of compact operators on a Hilbert space h is denoted $K(\mathsf{h})$. For vectors $\xi \in \mathsf{h}$ and $\xi' \in \mathsf{h}'$, using Dirac-inspired notation of bras and kets, we set

$$E_\xi := I_{\mathsf{H}} \otimes |\xi\rangle \in B(\mathsf{H}; \mathsf{H} \otimes \mathsf{h}),\ p(\xi) := I_{\mathsf{H}} \otimes |\xi\rangle\langle\xi| \in B(\mathsf{H} \otimes \mathsf{h}) \text{ and } E^{\xi'} := I_{\mathsf{H}'} \otimes \langle\xi| \in B(\mathsf{H}' \otimes \mathsf{h}'; \mathsf{H}'),$$

where the Hilbert spaces H and H' should always be clear from context. Thus $E^\xi = (E_\xi)^*$, $p(\xi) = E_\xi E^\xi$, and, for $T \in B(\mathsf{H} \otimes \mathsf{h}; \mathsf{H}' \otimes \mathsf{h}')$, $E^{\xi'} T E_\xi = \big(\mathrm{id}_{B(\mathsf{H};\mathsf{H}')} \overline{\otimes}\, \omega_{\xi',\xi}\big)(T)$ for the vector functional $\omega_{\xi',\xi} : B(\mathsf{h};\mathsf{h}') \to \mathbb{C}$, $A \mapsto \langle \xi', A\xi\rangle$. The indicator function of a set A is denoted $\mathbf{1}_A$. This notation is extended to vector-valued functions $f : \mathbb{R}_+ \to V$ and subintervals I of $\mathbb{R}_+$ as follows: f_I denotes the function from $\mathbb{R}_+$ to V which agrees with f on I and is zero outside I. In particular we apply this to vectors in V, by viewing them as constant functions from $\mathbb{R}_+$ to V. Thus, for $v \in V$ and $t \in \mathbb{R}_+$, $v_{[0,t[}$ is the function equal to v on $[0,t[$ and zero on $[t,\infty[$. We use the symbol $\subset\subset$ to denote subset of finite cardinality.

Terminology. Quantum stochastic analysis is the analysis of noncommutative processes adapted to the intrinsic operator filtration of a symmetric Fock space over $L^2(\mathbb{R}_+;\mathsf{k})$ for a (noise, or multiplicity) Hilbert space k ([L]). In particular, quantum stochastic differential equations are with respect to Fock space creation, preservation and annihilation processes in the sense of Hudson and Parthasarathy ([HuP], [Par]). This class of filtrations is universal for tensor independence in various senses (see *e.g.* [SSS]). Thus *quantum stochastic cocycle* is a shorthand for Markovian cocycle, in the sense of Accardi ([Acc], [AFL]), specific to the context of Fock filtrations. We stress that 'classical' processes are not excluded. Indeed these arise naturally when the cocycle is governed by a quantum stochastic differential equation whose coefficient map takes a particular form; this is nicely illustrated in [KuM].

1. Matrix spaces and κ-decomposability

In this section we recall some of the results concerning matrix spaces developed in [LW$_5$] and further extend these. Throughout we work with *concrete* operator spaces, that is, norm-closed subspaces of $B(\mathsf{H};\mathsf{H}')$ for some Hilbert spaces H and H'. The exceptions are Propositions 1.4 and 1.5 which concern operator spaces of completely bounded maps. Excellent references for operator spaces, operator systems and completely bounded maps are [EfR], [Pau] and [Pis]; see also [BlL] for operator algebras from this perspective. For more on matrix spaces, see [L].

For the rest of the paper V and W denote generic (concrete) operator spaces.

Spaces of matrices. For an index set $\mathcal{I}$,

$$\mathrm{M}_{\mathcal{I}}(\mathsf{V})_{\mathrm{b}} := \Big\{ A \in \mathrm{M}_{\mathcal{I}}(\mathsf{V}) : \sup_{\mathcal{J}\subset\subset\mathcal{I}} \|A^{[\mathcal{J}]}\| < \infty \Big\},$$

where $\mathrm{M}_{\mathcal{I}}(\mathsf{V})$ denotes the vector space of matrices $A = [a^i_j]_{i,j\in\mathcal{I}}$ with entries in V, and, for $\mathcal{J} \subset\subset \mathcal{I}$, $A^{[\mathcal{J}]} \in \mathrm{M}_{\mathcal{J}}(\mathsf{V})$ denotes the $\mathcal{J}\times\mathcal{J}$-truncation of A to a finite matrix. This has a natural operator space structure ([EfR], Chapter 10). When V is an operator algebra the *Schur product* on $\mathrm{M}_{\mathcal{I}}(\mathsf{V})$

$$[a^i_j] \cdot [b^i_j] := [a^i_j b^i_j]$$

restricts to $\mathrm{M}_{\mathcal{I}}(\mathsf{V})_{\mathrm{b}}$. Let $\mathrm{D}_{\mathcal{I}}(\mathsf{V})$ and $\mathrm{D}_{\mathcal{I}}(\mathsf{V})_{\mathrm{b}}$ denote the corresponding subspaces of diagonal matrices, so that

$$\mathrm{D}_{\mathcal{I}}(\mathsf{V})_{\mathrm{b}} := \Big\{ [a^i_j] \in \mathrm{M}_{\mathcal{I}}(\mathsf{V}) : a^i_j = 0 \text{ if } i \neq j \text{ and } \sup_{i\in\mathcal{I}} \|a^i_i\| < \infty \Big\}.$$

Matrix spaces. If $B(\mathsf{H};\mathsf{H}')$ is the ambient full operator space of V then, for any Hilbert spaces h and h', we call the operator space

$$\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h};\mathsf{h}') := \big\{ T \in B(\mathsf{H}\otimes\mathsf{h};\mathsf{H}'\otimes\mathsf{h}') : E^x T E_y \in \mathsf{V} \text{ for all } x \in \mathsf{h}', y \in \mathsf{h} \big\}$$

the h-h'-*matrix space* over V (or h-*matrix space* when $\mathsf{h}' = \mathsf{h}$). Note that it suffices to verify the membership condition for x and y running through total subsets of h' and h such as orthonormal bases. Thus $\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ contains the operator space $\mathsf{V} \otimes K(\mathsf{h})$, and is an operator system when V is. Note, however, that it is typically *not* a C^*-algebra when V is, unless h is finite dimensional. To any completely bounded operator $\varphi : \mathsf{V} \to \mathsf{W}$, and Hilbert space h, the map $\varphi \underline{\otimes} \operatorname{id}_{B(\mathsf{h})}$ extends uniquely to a bounded operator $\varphi \otimes_{\mathrm{M}} \operatorname{id}_{B(\mathsf{h})} : \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}) \to \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})$, satisfying

$$E^x(\varphi \otimes_{\mathrm{M}} \operatorname{id}_{B(\mathsf{h})}(T))E_y = \varphi(E^x T E_y), \quad T \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}),\ x, y \in \mathsf{h}.$$

This is consistent with two well-known extensions, namely $\varphi \otimes \operatorname{id}_{K(\mathsf{h})}$ and, when φ is an ultraweakly continuous completely bounded map between dual operator spaces, also $\varphi \overline{\otimes} \operatorname{id}_{B(\mathsf{h})}$ ([EfR]). These h-*matrix liftings* are completely bounded and satisfy $\|\varphi \otimes_{\mathrm{M}} \operatorname{id}_{B(\mathsf{h})}\|_{\mathrm{cb}} = \|\varphi\|_{\mathrm{cb}}$. If W is explicitly of the form $\mathsf{U} \otimes_{\mathrm{M}} B(\mathsf{K}; \mathsf{K}')$, for some operator space U and Hilbert spaces K and K', we write

$$\varphi^{\mathsf{h}} \ \text{ for } \ \Pi \circ (\varphi \otimes_{\mathrm{M}} \operatorname{id}_{B(\mathsf{h})}) : \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}) \to \mathsf{U} \otimes_{\mathrm{M}} B(\mathsf{h} \otimes \mathsf{K}; \mathsf{h} \otimes \mathsf{K}'), \tag{1.1}$$

Π being the tensor flip $\mathsf{U} \otimes_{\mathrm{M}} B(\mathsf{K} \otimes \mathsf{h}; \mathsf{K}' \otimes \mathsf{h}) \to \mathsf{U} \otimes_{\mathrm{M}} B(\mathsf{h} \otimes \mathsf{K}; \mathsf{h} \otimes \mathsf{K}')$. If $B(\mathsf{H}; \mathsf{H}')$ is the ambient full operator space of V then, for any ultraweakly continuous completely bounded map $\chi : B(\mathsf{K}; \mathsf{K}') \to B(\mathsf{h}; \mathsf{h}')$, the map $\operatorname{id}_{B(\mathsf{H};\mathsf{H}')} \overline{\otimes} \chi$ restricts to a map

$$\operatorname{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \chi : \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{K}; \mathsf{K}') \to \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}; \mathsf{h}').$$

Matrix spaces are thus an abstraction of the above operator spaces of matrices. Specifically, any choice of orthonormal basis $\kappa = (e_i)_{i \in \mathcal{I}}$ for h determines the completely isometric isomorphism (*i.e.* linear isomorphism each of whose matrix liftings is an isometry)

$$\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}) \cong \mathrm{M}_{\mathcal{I}}(\mathsf{V})_{\mathrm{b}}, \quad T \longleftrightarrow {}^{\kappa}T = [T^i_j]_{i,j \in \mathcal{I}}, \text{ where } T^i_j := E^{e_i} T E_{e_j} \in \mathsf{V}. \tag{1.2}$$

Let $\mathsf{V} \otimes_{\mathrm{M}} \mathrm{D}_{\kappa}(\mathsf{h})$ denote the subspace of the matrix space tensor product that corresponds to $\mathrm{D}_{\mathcal{I}}(\mathsf{V})_{\mathrm{b}}$ under the above identification. In contrast to $\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$, this operator space is manifestly κ-dependent. Note that

$$\mathsf{V} \otimes_{\mathrm{M}} \mathrm{D}_{\kappa}(\mathsf{h}) = \big\{ T \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}) : Tp(e_j) = p(e_j)T \text{ for all } j \in \mathcal{I} \big\}.$$

We have frequent recourse to ampliations of the following kind, for an operator space V and Hilbert space h:

$$\iota^{\mathsf{V}}_{\mathsf{h}} : \mathsf{V} \to \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}), \quad a \mapsto a \otimes I_{\mathsf{h}}. \tag{1.3}$$

Schur isometries and products. For a Hilbert space h with orthonormal basis $\kappa = (e_i)_{i \in \mathcal{I}}$, let $S_\kappa : \mathsf{h} \to \mathsf{h} \otimes \mathsf{h}$ denote the *Schur isometry* defined by continuous linear extension of the map $e_i \mapsto e_i \otimes e_i$, and let $\Sigma_\kappa : B(\mathsf{h}) \to B(\mathsf{h} \otimes \mathsf{h})$ be the corresponding *Schur homomorphism*, $T \mapsto S_\kappa T S_\kappa^*$. Thus Σ_κ is an injective normal *-homomorphism; its natural left inverse is the normal completely positive unital map $\Upsilon_\kappa : R \mapsto S_\kappa^* R S_\kappa$. Note that

$$\Sigma_\kappa(K(\mathsf{h})) \subset K(\mathsf{h} \otimes \mathsf{h}) \ \text{ and } \ \Upsilon_\kappa(K(\mathsf{h} \otimes \mathsf{h})) = K(\mathsf{h}).$$

Lemma 1.1. *The following hold*:

$$\big(\operatorname{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa\big)\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})\big) \subset \big(\mathsf{V} \otimes K(\mathsf{h})\big) \otimes_{\mathrm{M}} B(\mathsf{h}), \ \textit{and} \tag{1.4}$$

$$\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}) = \big(\operatorname{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Upsilon_\kappa\big)\big((\mathsf{V} \otimes K(\mathsf{h})) \otimes_{\mathrm{M}} B(\mathsf{h})\big). \tag{1.5}$$

Proof. Let $T \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ and $i, j, l \in \mathcal{I}$, and let $B(\mathsf{K}; \mathsf{K}')$ be the ambient full operator space of V. The identity

$$(I_{\mathsf{h}} \otimes \langle e_l|) S_\kappa = |e_l\rangle\langle e_l|$$

gives

$$E^{e_i}\big(\operatorname{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa\big)(T) E_{e_j} = E^{e_i}(I_{\mathsf{K}'} \otimes S_\kappa) T (I_{\mathsf{K}} \otimes S_\kappa^*) E_{e_j} = p(e_i) T p(e_j). \tag{1.6}$$

However
$$p(e_i)Tp(e_j) = E^{e_i}TE_{e_j} \otimes |e_i\rangle\langle e_j| \in \mathsf{V} \otimes K(\mathsf{h}),$$
and so inclusion (1.4) holds. Since $\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Upsilon_\kappa$ is a left inverse for $\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa$, (1.5) follows from (1.4). □

Remarks. Note that under the identification (1.2), equation (1.6) says that $\big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa\big)(T)$ is a matrix of matrices in which the (i,j)-block has only one nonzero component, namely T^i_j in the (i,j)-place of that block.

The map $\mathrm{id}_{B(\mathsf{h})} \overline{\otimes} \Upsilon_\kappa$ takes a matrix of matrices and transforms it into a matrix by selecting appropriately from each block. This is an instructive viewpoint for appreciating the next few results.

The κ-*Schur product* on $B(\mathsf{K} \otimes \mathsf{h})$ is defined by
$$R \bullet_\kappa T := \big(\mathrm{id}_{B(\mathsf{K})} \overline{\otimes} \Upsilon_\kappa\big)\big((R \otimes I_{\mathsf{h}})(I_{\mathsf{K}} \otimes \Pi)(T \otimes I_{\mathsf{h}})(I_{\mathsf{K}} \otimes \Pi)\big)$$
where Π denotes the unitary tensor flip on $\mathsf{h} \otimes \mathsf{h}$. The terminology is justified since, under the identification (1.2),
$$(R \bullet_\kappa T)^i_j = R^i_j T^i_j.$$

The following generalisation of a well-known property of Schur products of complex matrices is easily verified.

Lemma 1.2. *Let $R \in B(\mathsf{K} \otimes \mathsf{h})$ and $T \in I_{\mathsf{K}} \otimes B(\mathsf{h})$. If R and T are both nonnegative then so is $R \bullet_\kappa T$.*

κ-decomposability. Let h be a Hilbert space with orthonormal basis $\kappa = (e_i)_{i \in \mathcal{I}}$. A map $\phi \in B(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h}))$ is called κ-*decomposable* if it satisfies any of the following equivalent conditions: for each $T \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ and $i, j \in \mathcal{I}$,

(i) $E^{e_i}\phi(T)E_{e_j} = E^{e_i}\phi(p(e_i)Tp(e_j))E_{e_j}$; or

(ii) $p(e_i)\phi(T)p(e_j) = p(e_i)\phi(p(e_i)Tp(e_j))p(e_j)$; or

(iii) $p(e_i)\phi(T)p(e_j) = \phi(p(e_i)Tp(e_j))$.

The same definition applies to maps $\psi \in B(\mathsf{V} \otimes K(\mathsf{h}); \mathsf{W} \otimes K(\mathsf{h}))$. We use the respective notations
$$B_{\kappa\text{-dec}}\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})\big) \quad \text{and} \quad B_{\kappa\text{-dec}}\big(\mathsf{V} \otimes K(\mathsf{h}); \mathsf{W} \otimes K(\mathsf{h})\big)$$
and similarly for spaces of completely bounded κ-decomposable maps. Note that these subspaces are norm closed.

Thus a κ-decomposable map $\phi : \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}) \to \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})$ has *Schur action.* That is, under the identifications (1.2) determined by κ, for each $i, j \in \mathcal{I}$,
$$\begin{aligned} &\phi(T)^i_j = \phi^i_j(T^i_j), \quad \text{for the map} \\ &\phi^i_j : a \mapsto E^{e_i}\phi\big(E_{e_i} a E^{e_j}\big)E_{e_j}. \end{aligned} \tag{1.7}$$
The resulting Schur-action map $[\phi^i_j]_\bullet : \mathrm{M}_{\mathcal{I}}(\mathsf{V})_{\mathrm{b}} \to \mathrm{M}_{\mathcal{I}}(\mathsf{W})_{\mathrm{b}}$ is denoted ${}^\kappa\phi$. Note that if $\varphi \in CB(\mathsf{V}; \mathsf{W})$ then $\varphi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})}$ is κ-decomposable, with $(\varphi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})})^i_j = \varphi$ for every i and j.

Note also that if $\phi \in B_{\kappa\text{-dec}}\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})\big)$ then $\phi\big(a \otimes |e_i\rangle\langle e_j|\big) = \phi^i_j(a) \otimes |e_i\rangle\langle e_j|$, for $a \in \mathsf{V}$ and $i, j \in \mathcal{I}$. It follows that
$$\phi\big(\mathsf{V} \otimes K(\mathsf{h})\big) \subset \mathsf{W} \otimes K(\mathsf{h}).$$

Restriction therefore induces a map

$$\Phi : CB_{\kappa\text{-dec}}\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})\big) \to CB_{\kappa\text{-dec}}\big(\mathsf{V} \otimes K(\mathsf{h}); \mathsf{W} \otimes K(\mathsf{h})\big). \tag{1.8}$$

In Proposition 1.4 below, we see that there is a satisfactory extension map inverting Φ.

Lemma 1.3. *Let* $\phi \in CB_{\kappa\text{-dec}}(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h}))$*. Then*

$$\big(\phi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})}\big) \circ \big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa\big) = \big(\mathrm{id}_{\mathsf{W}} \otimes_{\mathrm{M}} \Sigma_\kappa\big) \circ \phi. \tag{1.9}$$

Proof. For $T \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ and $i, j \in \mathcal{I}$, applying (1.6) gives both

$$E^{e_i}\big(\phi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})}\big) \circ \big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa\big)(T)E_{e_j} = \phi\big(E^{e_i}(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa)(T)E_{e_j}\big) = \phi(p(e_i)Tp(e_j))$$

and

$$E^{e_i}\big((\mathrm{id}_{\mathsf{W}} \otimes_{\mathrm{M}} \Sigma_\kappa) \circ \phi\big)(T)E_{e_j} = p(e_i)\phi(T)p(e_j).$$

Thus (1.9) follows by κ-decomposability of ϕ. □

By (1.4), the map $\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa$ co-restricts to a map $\big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa\big)' \in CB\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); (\mathsf{V} \otimes K(\mathsf{h})) \otimes_{\mathrm{M}} B(\mathsf{h})\big)$.

Proposition 1.4. *The map* Φ *defined in* (1.8) *is a completely isometric isomorphism whose inverse* Ψ *is the extension map given by*

$$\Psi : \psi \mapsto \big(\mathrm{id}_{\mathsf{W}} \otimes_{\mathrm{M}} \Upsilon_\kappa\big) \circ \big(\psi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})}\big) \circ \big(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa\big)'. \tag{1.10}$$

Proof. Let Ψ be the map (1.10). Then, for $\psi \in CB_{\kappa\text{-dec}}\big(\mathsf{V} \otimes K(\mathsf{h}); \mathsf{W} \otimes K(\mathsf{h})\big)$, $T \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ and $i, j \in \mathcal{I}$,

$$\begin{aligned} E^{e_i}\Psi(\psi)(T)E_{e_j} &= E^{e_i \otimes e_i}(\psi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})})\big((\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa)(T)\big)E_{e_j \otimes e_j} \\ &= E^{e_i}\psi\big(E^{e_i}(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Sigma_\kappa)(T)E_{e_j}\big)E_{e_j} \\ &= E^{e_i}\psi\big(p(e_i)Tp(e_j)\big)E_{e_j} \end{aligned}$$

by (1.6). Thus $\Psi(\psi)$ is κ-decomposable. To see that Ψ inverts Φ, pick $\phi \in CB_{\kappa\text{-dec}}\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})\big)$ and set $\psi = \Phi(\phi)$ then $\psi(p(e_i)Tp(e_j)) = \phi(p(e_i)Tp(e_j))$ and so

$$E^{e_i}\Psi(\psi)(T)E_{e_j} = E^{e_i}\phi(T)E_{e_j}$$

by κ-decomposability of ϕ and ψ. Thus $\Psi(\psi) = \phi$, hence Ψ is a left inverse for Φ. On the other hand, let $\psi \in CB_{\kappa\text{-dec}}\big(\mathsf{V} \otimes K(\mathsf{h}); \mathsf{W} \otimes K(\mathsf{h})\big)$ and $T \in \mathsf{V} \otimes K(\mathsf{h})$ then $\psi(T) \in \mathsf{W} \otimes K(\mathsf{h})$ and, by κ-decomposability of ψ,

$$E^{e_i}\psi(T)E_{e_j} = E^{e_i}\psi(p(e_i)Tp(e_j))E_{e_j} = E^{e_i}\Psi(\psi)(T)E_{e_j}.$$

It follows that $\Phi(\Psi(\psi)) = \psi$. Thus Ψ is a right inverse too.

Now Ψ is a composition of complete contractions and Φ is clearly completely contractive. Therefore Φ is a complete isometry and the result follows. □

Remark. Clearly the extension map Ψ preserves complete positivity when V and W are C^*-algebras or operator systems.

We use Proposition 1.4 at the end of this section. A generalisation of the extension procedure in the following result underlies the transformations discussed in Theorems 4.2 and 4.3.

Proposition 1.5. *The prescription* $\varphi \mapsto (\mathrm{id}_{\mathsf{W}} \otimes_{\mathrm{M}} \Upsilon_\kappa) \circ (\varphi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})})$ *defines an injective complete contraction*

$$\Xi : CB\big(\mathsf{V}; \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})\big) \to CB_{\kappa\text{-dec}}\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}); \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})\big). \tag{1.11}$$

Proof. Set $\phi = (\mathrm{id}_{\mathsf{W}} \otimes_{\mathrm{M}} \Upsilon_\kappa) \circ (\varphi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})})$ where $\varphi \in CB\big(\mathsf{V}; \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h})\big)$. Then

$$E^{e_i}\phi(T)E_{e_j} = E^{e_i}\varphi(E^{e_i}TE_{e_j})E_{e_j}, \quad T \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}), i, j \in \mathcal{I}.$$

It follows that ϕ is κ-decomposable, and that

$$E^{e_i}\varphi(a)E_{e_j} = E^{e_i}\phi(a \otimes |e_i\rangle\langle e_j|)E_{e_j}, \quad a \in \mathsf{V}, i, j \in \mathcal{I},$$

which implies that $\varphi = 0$ if $\phi = 0$. Thus the prescription (1.11) defines a linear injection Ξ between the given spaces. Being a composition of complete contractions, Ξ is a complete contraction. □

Remarks. (i) Reverting to the matrix viewpoint, for $\varphi \in CB(\mathsf{V}; \mathsf{W} \otimes_{\mathrm{M}} B(\mathsf{h}))$

$$\Xi(\varphi) \longleftrightarrow \big([a^i_j] \mapsto [\varphi^i_j(a^i_j)]\big) \quad \text{where} \quad \varphi \longleftrightarrow \big[\varphi^i_j := E^{e_i}\varphi(\cdot)E_{e_j}\big]. \tag{1.12}$$

(ii) If h is finite dimensional then Ξ is bijective since it has right inverse $\phi \mapsto \phi \circ \beta$ for the map $\beta : \mathsf{V} \to \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$, $a \mapsto a \otimes |e_{\mathcal{I}}\rangle\langle e_{\mathcal{I}}|$ where $e_{\mathcal{I}} := \sum_{i \in \mathcal{I}} e_i$. However, if h is infinite dimensional then Ξ need not be bijective; for example if $\mathsf{W} = \mathsf{V} \neq \{0\}$ then $\mathrm{id}_{\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})} \notin \operatorname{Ran} \Xi$.

Representations and semigroup generators. We conclude this section by applying the above techniques to obtain an extension of the Christensen–Evans characterisation of the generators of norm-continuous, completely positive semigroups on a C^*-algebra to norm-continuous, κ-decomposable semigroups on a matrix space over a C^*-algebra. The argument is based on the following slight generalisation of the well-known representation theory of $K(\mathsf{k})$.

Lemma 1.6. *Let (Π, H) be a representation of $\mathsf{A} \otimes K(\mathsf{h})$ for a unital C^*-algebra A and Hilbert space h. Then there is a unital representation (π, K) of A and isometry $V \in B(\mathsf{K} \otimes \mathsf{h}; \mathsf{H})$ satisfying*

$$\Pi(A) = V\big(\pi \otimes \mathrm{id}_{K(\mathsf{h})}\big)(A)V^*, \quad A \in \mathsf{A} \otimes K(\mathsf{h}). \tag{1.13}$$

Proof. We may assume that $\mathsf{h} \neq \{0\}$ and so choose a unit vector e in h. Set $\mathsf{K} := P\mathsf{H}$, where $P \in B(\mathsf{H})$ is the orthogonal projection $\Pi(1_{\mathsf{A}} \otimes |e\rangle\langle e|)$, and let J be the inclusion $\mathsf{K} \to \mathsf{H}$. Then $JJ^* = P$ so the prescription $a \mapsto J^*\Pi\big(a \otimes |e\rangle\langle e|\big)J$ defines a unital representation $\pi : \mathsf{A} \to B(\mathsf{K})$. Since

$$\Big\langle \Pi\big(1_{\mathsf{A}} \otimes |x\rangle\langle e|\big)J\xi, \Pi\big(1_{\mathsf{A}} \otimes |x'\rangle\langle e|\big)J\xi' \Big\rangle = \langle x, x'\rangle\langle \xi, \xi'\rangle$$

for $\xi, \xi' \in \mathsf{K}$, $x, x' \in \mathsf{h}$, there is a unique isometry $V \in B(\mathsf{K} \otimes \mathsf{h}; \mathsf{H})$ satisfying

$$V(\xi \otimes x) = \Pi\big(1_{\mathsf{A}} \otimes |x\rangle\langle e|\big)J\xi,$$

for all $\xi \in \mathsf{k}$ and $x \in \mathsf{h}$. To establish (1.13) it suffices, by linearity and continuity, to show that

$$\big\langle \zeta, \Pi(a \otimes |x\rangle\langle x'|)\zeta' \big\rangle = \langle \zeta, V(\pi(a) \otimes |x\rangle\langle x'|)V^*\zeta'\rangle \tag{1.14}$$

for all $a \in \mathsf{A}$, $x, x' \in \mathsf{h}$ and $\zeta, \zeta' \in \mathsf{H}$. But, for such elements, $\Pi(a \otimes |x\rangle\langle x'|)\zeta' = V(\xi' \otimes x)$ where $\xi' = J^*\Pi(a \otimes |e\rangle\langle x'|)\zeta'$, so it is in fact sufficient to take ζ and ζ' of the form $V(\xi \otimes y)$ and $V(\xi' \otimes y')$ where $\xi, \xi' \in \mathsf{K}$ and $y, y' \in \mathsf{h}$. For such choices it is easily verified that equality holds in (1.14) with common value $\langle \xi, \pi(a)\xi'\rangle\langle y, x\rangle\langle x', y'\rangle$. □

Theorem 1.7. *Let A be a unital C^*-algebra acting nondegenerately on $\mathfrak{h}$, let h be a Hilbert space with orthonormal basis κ and let $P = (P_t)_{t \geqslant 0}$ be a norm-continuous semigroup of κ-decomposable bounded operators on the operator system $\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{h})$ with generator ψ. Then P is completely positive if and only if ψ has the form*

$$\psi : A \mapsto T(\pi \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})})(A)T^* + NA + AN^* \tag{1.15}$$

for a unital representation (π, K) *of* A *and operators* $T \in B(\mathsf{K};\mathfrak{h})\overline{\otimes} B(\mathsf{h})$ *and* $N \in B(\mathfrak{h}\otimes\mathsf{h})$. *Moreover,* T *and* N *are necessarily* κ*-diagonal, and may be chosen so that* $N \in \mathsf{A}'' \overline{\otimes} B(\mathsf{h})$.

Proof. Let ψ^0 and P^0 denote the κ-decomposable operator and semigroup on $\mathsf{A}\otimes K(\mathsf{h})$ obtained by restriction of ψ and P. Since completely positive maps as well as maps of the form (1.15) are completely bounded, Proposition 1.4 and its accompanying remark imply that P is a completely positive semigroup if and only if P^0 is, and ψ is given by (1.15), for some unital representation (π,K) of A and (κ-diagonal) operators $T \in B(\mathsf{K};\mathfrak{h})\overline{\otimes} B(\mathsf{h})$ and $N \in B(\mathfrak{h}\otimes\mathsf{h})$, if and only if ψ^0 has the form

$$\psi^0 : A \mapsto T\big(\pi\otimes \mathrm{id}_{K(\mathsf{h})}\big)(A)T^* + NA + AN^* \tag{1.16}$$

for the same π, K, T and N.

Now if ψ^0 has the form (1.16) then it is conditionally completely positive, and hence generates a completely positive semigroup ([EvK], Theorem 4.27).

Conversely, if ψ^0 is the generator of a completely positive semigroup then, by Theorem 3.1 of [ChE], Stinespring's Theorem and Lemma 1.6, it must have the form (1.16), for a unital representation (π,K) of A and operators $T \in B(\mathsf{K};\mathfrak{h})\overline{\otimes} B(\mathsf{h})$ and $N \in \big(\mathsf{A}\otimes K(\mathsf{h})\big)'' = \mathsf{A}''\overline{\otimes} B(\mathsf{h})$. It therefore only remains to show that the κ-decomposability of ψ^0 implies that T and N must be κ-diagonal. We may assume that $\dim \mathsf{h} \geqslant 2$. Let $\kappa = (e_i)_{i\in\mathcal{I}}$ and pick $j\in\mathcal{I}$. Then

$$\psi^0\big(1_{\mathsf{A}}\otimes |e_j\rangle\langle e_j|\big) = TE_{e_j}E^{e_j}T^* + NE_{e_j}E^{e_j} + E_{e_j}E^{e_j}N^*$$

and so, by the κ-decomposability of ψ^0,

$$E^{e_i}TE_{e_j}E^{e_j}T^*E_{e_k} + \delta_{j,k}E^{e_i}NE_{e_j} + \delta_{i,j}E^{e_j}N^*E_{e_k} = 0$$

unless $i = k = j$, where δ is the Kronecker delta. Setting $i = k \neq j$ shows that T is κ-diagonal. Setting $i \neq j = k$ shows that N is κ-diagonal too. □

Remark. Under the identification $\mathsf{A}\otimes_{\mathrm{M}} B(\mathsf{h}) \cong \mathrm{M}_{\mathcal{I}}(\mathsf{A})_{\mathrm{b}}$, induced by the choice of basis κ, and corresponding identifications for the κ-decomposable maps (as in (1.2) and (1.7)), ψ is given by ${}^{\kappa}\psi = [\psi^i_j]_{\bullet}$ where

$$\psi^i_j : a \mapsto t_i\pi(a)t_j^* + n_i a + a n_j^*$$

for $t_i = E^{e_i}TE_{e_i}$ and $n_i = E^{e_i}NE_{e_i}$.

2. Quantum stochastic cocycles and associated semigroups

For the rest of the paper we let $B(\mathfrak{h};\mathfrak{h}')$ be the ambient full operator space of V (or $B(\mathfrak{h})$ in case V is an operator system or C^*-algebra). We also let k be a fixed Hilbert space, referred to as the *noise dimension space.* Set

$$\widehat{\mathsf{k}} = \mathbb{C}\oplus\mathsf{k} \quad \text{and, for each } x\in\mathsf{k}, \quad \widehat{x} := \begin{pmatrix}1\\ x\end{pmatrix}. \tag{2.1}$$

The *quantum Itô projection* is the orthogonal projection $\Delta \in B(\widehat{\mathsf{k}})$ with range $\{0\}\oplus\mathsf{k}$; whenever there is no danger of confusion, its ampliations are denoted by the same symbol. When $\eta = (d_i)_{i\in\mathcal{I}_0}$ is an orthonormal basis for k (with $0\notin\mathcal{I}_0$), we set $\mathcal{I} := \{0\}\cup\mathcal{I}_0$ and define an orthonormal basis $\overline{\eta} = (e_\alpha)_{\alpha\in\mathcal{I}}$ for $\widehat{\mathsf{k}}$ by

$$e_0 := \widehat{d_0} = \begin{pmatrix}1\\0\end{pmatrix} \text{ where } d_0 := 0\in\mathsf{k} \text{ and, for } \ i\in\mathcal{I}_0,\ e_i := \begin{pmatrix}0\\ d_i\end{pmatrix}; \tag{2.2}$$

thus

$$\mathsf{T}(\eta) := \{d_\alpha : \alpha \in \mathcal{I}\} = \{0\} \cup \{d_i : i \in \mathcal{I}_0\} \tag{2.3}$$

is a total subset of k containing 0.

For each subinterval J of $\mathbb{R}_+$ let $\mathcal{F}_J$ denote the symmetric Fock space over $L^2(J;\mathsf{k})$, dropping the subscript when $J = \mathbb{R}_+$. We use *normalised* exponential vectors

$$\varpi(f) := e^{-\frac{1}{2}\|f\|^2}\big((n!)^{-1/2} f^{\otimes n}\big)_{n\geqslant 0}, \quad f \in L^2(\mathbb{R}_+;\mathsf{k});$$

these are linearly independent in $\mathcal{F}$. For any subset T of k containing 0, we set

$$\mathcal{E}_\mathsf{T} := \mathrm{Lin}\{\varpi(f) : f \in \mathbb{S}_\mathsf{T}\},$$

where $\mathbb{S}_\mathsf{T}$ denotes the collection of (right continuous) T-valued step functions in $L^2(\mathbb{R}_+;\mathsf{k})$. The subscript is dropped when $\mathsf{T} = \mathsf{k}$. If T is total in k then $\mathcal{E}_\mathsf{T}$ is total in $\mathcal{F}$ ([Ske]). For a proof of this, together with the basics of quantum stochastic analysis and QS cocycles, and extensive references, we refer to [L]; see also [Fag]. The *Fock–Weyl operators* are the unitary operators on $\mathcal{F}$ defined by continuous linear extension of the following prescription:

$$W(f) : \varpi(g) \mapsto e^{-\mathrm{i}\,\mathrm{Im}\langle f,g\rangle}\varpi(f+g), \qquad f,g \in L^2(\mathbb{R}_+;\mathsf{k}).$$

Quantum stochastic cocycles. By a *quantum stochastic process on* V *with noise dimension space* k we mean a family of linear maps

$$k_t : \mathsf{V} \to L(\mathcal{E}; \mathsf{V} \otimes_{\mathrm{M}} |\mathcal{F}\rangle) \subset L(\mathfrak{h} \underline{\otimes} \mathcal{E}; \mathfrak{h}' \otimes \mathcal{F}), \quad t \in \mathbb{R}_+,$$

which is adapted and pointwise weakly measurable:

$$\begin{aligned} &k_t(a)E_{\varpi(g_{[0,t[})} \in \mathsf{V} \otimes_{\mathrm{M}} |\mathcal{F}_{[0,t[}\rangle \otimes |\varpi(0|_{[t,\infty[})\rangle, \quad \text{and} \\ &s \mapsto \langle \zeta, k_s(a)u\varpi(g)\rangle \text{ is measurable} \end{aligned}$$

for all $a \in \mathsf{V}$, $t \geqslant 0$, $g \in \mathbb{S}$, $u \in \mathfrak{h}$ and $\zeta \in \mathfrak{h}' \otimes \mathcal{F}$. We refer to $\mathfrak{h} \underline{\otimes} \mathcal{E}$ as the *exponential domain* for the process k. A (*completely*) *bounded*, *completely positive* or *completely contractive* QS process on V is a QS process k on V for which each k_t has that property, in which case $k_t(a)$ will also denote the continuous extension of this bounded operator to all of $\mathfrak{h} \otimes \mathcal{F}$. Here we invoke the natural identification (cf. [LW$_5$], Proposition 2.4) and inclusion

$$\begin{aligned} CB(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\mathcal{F})) &= CB(\mathsf{V}; CB(|\mathcal{F}\rangle; \mathsf{V} \otimes_{\mathrm{M}} |\mathcal{F}\rangle)) \\ &\subset L(\mathsf{V}; L(\mathcal{E}; \mathsf{V} \otimes_{\mathrm{M}} |\mathcal{F}\rangle)). \end{aligned}$$

Adaptedness implies that a process k is determined by the family of functions $\{k^{f,g} : f,g \in \mathbb{S}\}$ in $L(\mathsf{V})$ defined by

$$k_t^{f,g} := E^{\varpi(f_{[0,t[})} k_t(\cdot) E_{\varpi(g_{[0,t[})}, \quad t \geqslant 0.$$

A QS process k on V is a (*weak*) *quantum stochastic cocycle on* V if

$$k_0^{f,g} = \mathrm{id}_\mathsf{V} \quad \text{and} \quad k_{r+t}^{f,g} = k_r^{f,g} \circ k_t^{s_r^* f, s_r^* g} \tag{2.4}$$

for all $f,g \in \mathbb{S}$ and $r,t \geqslant 0$ ([LW$_2$], [DLT]). Here $(s_r^*)_{r\geqslant 0}$ denotes the semigroup of left shifts on $L^2(\mathbb{R}_+;\mathsf{k})$. The collection of QS cocycles on V with noise dimension space k is denoted $\mathrm{QSC}(\mathsf{V},\mathsf{k})$. We say that $k \in \mathrm{QSC}(\mathsf{V},\mathsf{k})$ has a *hermitian conjugate QS cocycle* if

$$\mathrm{Dom}\, k_t(a)^* \supset \mathfrak{h}' \underline{\otimes} \mathcal{E} \text{ for all } t \in \mathbb{R}_+, a \in \mathsf{V}, \text{ and } k^\dagger : \Big(a^* \mapsto k_t(a)^*|_{\mathfrak{h}'\underline{\otimes}\mathcal{E}}\Big)_{t\geqslant 0} \in \mathrm{QSC}(\mathsf{V}^\dagger,\mathsf{k})$$

where $\mathsf{V}^\dagger$ is the adjoint operator space $\{a^* : a \in \mathsf{V}\}$. When V is an operator system or C^*-algebra (or, more generally, when V is *hermitian*: $\mathsf{V}^\dagger = \mathsf{V}$), we set

$$\mathrm{QSC}_{\mathrm{h}}(\mathsf{V}, \mathsf{k}) := \{k \in \mathrm{QSC}(\mathsf{V}, \mathsf{k}) : k \text{ is hermitian}\} \tag{2.5}$$

where "k is hermitian" amounts to $k_t(a)^* \supset k_t(a^*)$, for all $t \in \mathbb{R}_+$ and $a \in \mathsf{V}$. When V is an operator system we also set

$$\mathrm{QSC}_{\mathrm{u}}(\mathsf{V}, \mathsf{k}) := \{k \in \mathrm{QSC}(\mathsf{V}, \mathsf{k}) : k \text{ is unital}\} \tag{2.6}$$

where "k is unital" means that $k_t(1) \subset I_{\mathfrak{h}\otimes\mathcal{F}}$ for all $t \in \mathbb{R}_+$.

When k is a completely bounded QS process, condition (2.4) is equivalent to the following, which is more recognisable as a cocycle identity:

$$k_0 = \iota_{\mathcal{F}}^{\mathsf{V}} \quad \text{and} \quad k_{r+t} = \widehat{k}_r \circ \sigma_r \circ k_t, \quad r, t \geqslant 0, \tag{2.7}$$

where $\iota_{\mathcal{F}}^{\mathsf{V}}$ is the ampliation introduced in (1.3) and $\widehat{k}_r$ denotes the natural (matrix-space) tensor extension of k_r to the range of σ_r (see [LW$_5$], Section 5). We refer to such processes as *completely bounded QS cocycles*, and denote the class of these by $\mathrm{QSC}_{\mathrm{cb}}(\mathsf{V}, \mathsf{k})$. When V is a C^*-algebra or operator system, we similarly write

$$\mathrm{QSC}_{\mathrm{cp}}(\mathsf{V}, \mathsf{k}), \quad \mathrm{QSC}_{\mathrm{cpc}}(\mathsf{V}, \mathsf{k}) \ \text{ and } \ \mathrm{QSC}_{\mathrm{cpu}}(\mathsf{V}, \mathsf{k})$$

for the respective subclasses of completely positive, completely positive contractive and completely positive unital QS cocycles.

Remark. In [LW$_5$] all QS cocycles were assumed to be completely bounded (and the measurability condition was not imposed). The reason for dealing with weak QS cocycles here, and elsewhere, is that solutions of the QS differential equation (3.3) with completely bounded coefficients are of this type — but k need not be completely bounded. Indeed, $k_t(a)$ need not even be a bounded operator.

QS cocycles k have *associated semigroups*, defined by

$$\mathcal{P}_t^{x,y} = E^{\varpi(x_{[0,t[})} k_t(\cdot) E_{\varpi(y_{[0,t[})}, \quad x, y \in \mathsf{k},$$

and, for each $f, g \in \mathbb{S}$, the semigroup decomposition of QS cocycles represents $k_t^{f,g}$ in terms of these semigroups ([LW$_5$], Proposition 5.1). When it is bounded a QS cocycle is thereby determined by the family of semigroups $\{\mathcal{P}^{x,y} : x, y \in \mathsf{T}\}$ for any total subset T of k containing 0. A QS cocycle is *elementary* if each of its associated semigroups is completely bounded and cb-norm continuous. *Note.* We have previously used the terminology 'cb-Markov regular'. We write *El*-$\mathrm{QSC}(\mathsf{V}, \mathsf{k})$ for this class of cocycle. For a completely bounded QS cocycle k which is locally bounded in cb-norm, cb-norm continuity for any one of the associated semigroups, such as the *vacuum-expectation semigroup* $\mathcal{P}^{0,0}$, implies that k is elementary. This is a simple consequence of the continuity of the normalised exponential map $\varpi : L^2(\mathbb{R}_+; \mathsf{k}) \to \mathcal{F}$. Similarly for such cocycles either all or none of the associated semigroups are C_0-semigroups. Part (c) of Theorem 2.3 below includes the analogous result for continuity in cb-norm.

A completely bounded QS cocycle k with locally bounded cb-norm is necessarily exponentially bounded in cb-norm: there is $M \geqslant 1$ and $\beta \in \mathbb{R}$ such that

$$\|k_t\|_{\mathrm{cb}} \leqslant M e^{\beta t}, \qquad t \in \mathbb{R}_+ \tag{2.8}$$

([LW$_5$], Proposition 5.4). Following the terminology of semigroup theory, we refer to the quantity $\inf\big\{\beta \in \mathbb{R} : \sup_{t\geqslant 0} e^{-\beta t}\|k_t\|_{\mathrm{cb}} < \infty\big\} \in [-\infty, \infty]$ as the (*exponential*) *growth bound* of k, set

$$\mathrm{QSC}^{\beta}_{\mathrm{qc}}(\mathsf{V},\mathsf{k}) := \big\{k \in \mathrm{QSC}_{\mathrm{cb}}(\mathsf{V},\mathsf{k}) : (2.8) \text{ holds with } M=1\big\}, \text{ and}$$
$$\mathrm{QSC}_{\mathrm{qc}}(\mathsf{V},\mathsf{k}) := \bigcup_{\beta\in\mathbb{R}} \mathrm{QSC}^{\beta}_{\mathrm{qc}}(\mathsf{V},\mathsf{k}),$$

and call elements of the latter class of cocycles *cb-quasicontractive*. Thus, for a C^*-algebra A, El-$\mathrm{QSC}_{\mathrm{cpqc}}(\mathsf{A},\mathsf{k})$ denotes the class of completely positive quasicontractive elementary QS cocycles on A with noise dimension space k.

Remark. Let $k \in \mathrm{QSC}_{\mathrm{cpqc}}(\mathsf{A},\mathsf{k})$ for a C^*-algebra A. If its vacuum expectation semigroup is norm continuous then, by the Christensen–Evans theorem ([ChE]), it has completely bounded generator and so is cb-norm continuous, and therefore (being locally bounded in cb-norm) the QS cocycle k is elementary.

Let $\mathfrak{h}$ be a Hilbert space. A suitably measurable, Fock-adapted family of operators $X = (X_t)_{t\geqslant 0}$ in $B(\mathfrak{h}\otimes\mathcal{F})$ is a *bounded left quantum stochastic operator cocycle* on $\mathfrak{h}$ (with noise dimension space k) if

$$X_0 = I_{\mathfrak{h}\otimes\mathcal{F}} \text{ and } X_{r+t} = X_r\sigma_r(X_t), \quad r,t\in\mathbb{R}_+.$$

For $x,y\in\mathsf{k}$, $P^{x,y} := \big(E^{\varpi(x_{[0,t[})}X_tE_{\varpi(y_{[0,t[})}\big)_{t\geqslant 0}$ defines its (x,y)-*associated semigroup* of operators on $\mathfrak{h}$; X is *elementary* if all of these are norm continuous. Again, if X is locally bounded then norm continuity for any one of its associated semigroups implies that X is elementary. Important examples are the *Weyl cocycles*, $(W(z_{[0,t[}))_{t\geqslant 0}$ $(z\in\mathsf{k})$, much used in quantum stochastic calculus (see [L], pages 247–9). We use the notations El-$\mathrm{QSC}_{\mathrm{b}}(\mathfrak{h},\mathsf{k})$, $\mathrm{QSC}_{\mathrm{qc}}(\mathfrak{h},\mathsf{k})$ etc.

Connections between operator cocycles and mapping cocycles are described in Proposition 5.5 of [LW$_5$]. In this paper we shall use the following one: given bounded left QS operator cocycles X and Y on $\mathfrak{h}$, the prescription

$$k_t(T) := X_t(T\otimes I_{\mathcal{F}})Y_t^* \quad (T\in B(\mathfrak{h}), t\in\mathbb{R}_+)$$

defines a completely bounded cocycle k on $B(\mathfrak{h})$.

Associated Γ-cocycle and global Γ-semigroup. An important idea of Accardi and Kozyrev ([AcK]), expanded on at length in [LW$_5$], is to gather associated semigroups of a QS cocycle into a matrix. For any index set $\mathcal{I}$, let $(\delta^\alpha)_{\alpha\in\mathcal{I}}$ denote the standard orthonormal basis for $l^2(\mathcal{I})$. Then, given a map $\Gamma:\mathcal{I}\to\mathsf{k}$, define unitaries $W_t^\Gamma\in B(l^2(\mathcal{I})\otimes\mathcal{F})$, for $t\in\mathbb{R}_+$, by continuous linear extension of the rule

$$\delta^\alpha\otimes\xi \mapsto \delta^\alpha\otimes W\big(\Gamma(\alpha)_{[0,t[}\big)\xi, \quad \alpha\in\mathcal{I}, \xi\in\mathcal{F}. \tag{2.9}$$

Note that $W_t^\Gamma \in \mathrm{D}_{\mathcal{I}}\big(B(\mathcal{F})\big)_{\mathrm{b}}$ under the identification (1.2). We adapt this notation in two special cases: if $\mathcal{I}=\{0\}\cup\mathcal{I}_0$ and $\eta=(d_i)_{i\in\mathcal{I}_0}$ is a basis for k then we write $W^\eta = W^\Gamma$ for the map $\Gamma:\mathcal{I}\to\mathsf{k}$, $\alpha\mapsto d_\alpha$ (recall our convention (2.2)); if $n\in\mathbb{N}$ and $\mathbf{x}=(x_1,\cdots,x_n)\in\mathsf{k}^n$, set $W^{\mathbf{x}} = W^\Gamma$ for the map $\Gamma:\{1,\cdots,n\}\to\mathsf{k}$, $i\mapsto x_i$.

The following is Proposition 5.9 of [LW$_5$]. Recall the notation (1.1).

Proposition 2.1. *Let $k\in\mathrm{QSC}_{\mathrm{cb}}(\mathsf{V},\mathsf{k})$, with associated semigroups $\{\mathcal{P}^{x,y}: x,y\in\mathsf{k}\}$, and let $B(\mathfrak{h};\mathfrak{h}')$ be the ambient full operator space of V. Then, for any set $\mathcal{I}$ and map $\Gamma:\mathcal{I}\to\mathsf{k}$, setting $\mathsf{h}=l^2(\mathcal{I})$,*

$$\big((I_{\mathfrak{h}'}\otimes W_t^\Gamma)^* k_t^{\mathsf{h}}(\cdot)(I_{\mathfrak{h}}\otimes W_t^\Gamma)\big)_{t\geqslant 0}$$

defines a cocycle $k^\Gamma \in \mathrm{QSC}_{\mathrm{cb}}\big(\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h}), \mathsf{k}\big)$ *that has Schur-action under the identifications* (1.2); *its vacuum-expectation semigroup is given by the Schur-action prescription*

$$\mathcal{P}_t^\Gamma : [a_\beta^\alpha] \mapsto \Big[\mathcal{P}_t^{\Gamma(\alpha),\Gamma(\beta)}(a_\beta^\alpha)\Big]$$

where, for each $x, y \in \mathsf{k}$, $\mathcal{P}^{x,y}$ *is the* (x,y)*-associated semigroup of* k.

Remarks. We use the notation k^η, $\mathcal{P}^{\mathsf{x}}$, etc. in the special cases noted above for Γ. Also, for each set $\mathcal{I}$ and map $\Gamma : \mathcal{I} \to \mathsf{k}$, the semigroup $\mathcal{P}^\Gamma$ clearly leaves $\mathsf{V} \otimes K(\mathsf{h})$ invariant; we denote the resulting restriction by $\mathcal{P}^{\Gamma,K}$.

Continuity. We now collect together the basic relationships between continuity for QS cocycles and continuity for associated semigroups. We also highlight some implications of strengthened continuity hypotheses which render them inappropriate. Some hybrid locally convex topologies on spaces of the form $B(\mathsf{H} \otimes \mathsf{h}; \mathsf{H}' \otimes \mathsf{h})$ are needed. These are the h-*ultraweak topology* defined by the seminorms

$$p^\omega : T \mapsto \|(\mathrm{id}_{B(\mathsf{H};\mathsf{H}')} \overline{\otimes}\, \omega)(T)\|, \quad \omega \in B(\mathsf{h})_*,$$

and the h-*weak topology* defined by the seminorms

$$p_{x,y} : T \mapsto \|E^x T E_y\|, \quad x, y \in \mathsf{h}.$$

Thus $p_{x,y} = p^{\omega_{x,y}}$ for the vector functional

$$\omega_{x,y} : R \mapsto \langle x, Ry\rangle, \quad x, y \in \mathsf{h}. \tag{2.10}$$

Note that, for any operator space V in $B(\mathsf{H}; \mathsf{H}')$, the matrix space $\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ is the closure of the algebraic tensor product $\mathsf{V} \underline{\otimes} B(\mathsf{h})$ with respect to either of these topologies, and the two topologies coincide on norm-bounded subsets of $B(\mathsf{H} \otimes \mathsf{h}; \mathsf{H}' \otimes \mathsf{h})$. See Sections 2 and 3 of [LW$_5$] for more details. Moreover, $\mathsf{V} \underline{\otimes} K(\mathsf{h})$ is dense in $\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ with respect to the weak and strong operator topologies on $B(\mathsf{H} \otimes \mathsf{h}; \mathsf{H}' \otimes \mathsf{h})$.

For a set $\mathcal{I}$ and *bounded* map $\Gamma : \mathcal{I} \to \mathsf{k}$, set

$$w_t^\Gamma := W_t^\Gamma E_{\varpi(0)} \in B(l^2(\mathcal{I}); l^2(\mathcal{I}) \otimes \mathcal{F}), \quad t \in \mathbb{R}_+.$$

Lemma 2.2. *For all* $t \in \mathbb{R}_+$, $\|w_t^\Gamma - w_0^\Gamma\| = \sqrt{2(1 - e^{-tM})}$, *where* $M = \sup_{\alpha \in \mathcal{I}} \|\Gamma(\alpha)\|^2/2$.

Proof. Let $t \in \mathbb{R}_+$. Then $w_t^\Gamma - w_0^\Gamma \in \mathrm{D}_{\mathcal{I}}(|\mathcal{F}\rangle)_{\mathrm{b}}$ and, for all $\alpha \in \mathcal{I}$,

$$\|(w_t^\Gamma - w_0^\Gamma)\delta^\alpha\|^2 = \|\delta^\alpha \otimes \big(\varpi(\Gamma(\alpha)_{[0,t[}) - \varpi(0)\big)\|^2 = 2(1 - e^{-t\|\Gamma(\alpha)\|^2/2}).$$

The result follows. □

Theorem 2.3. *Let* $k \in \mathrm{QSC}_{\mathrm{cb}}(\mathsf{V}, \mathsf{k})$, *with locally bounded cb-norm. Then the following sets of equivalences hold*:

(a) (i) k *is pointwise ultraweakly continuous.*
(ii) $\mathcal{P}^\Gamma$ *is pointwise ultraweakly continuous for every* k*-valued map* Γ.
(iii) $\mathcal{P}^{x,y}$ *is pointwise ultraweakly continuous for all* $x, y \in \mathsf{k}$.
(iv) $\mathcal{P}^{x,y}$ *is pointwise ultraweakly continuous for all* $x, y \in \mathsf{T}$, *for some total subset* T *of* k *containing* 0.

Suppose that V *is ultraweakly closed and each map* $\mathcal{P}_t^{x,y}$ *is ultraweakly continuous. Then there is a further equivalence*:
(v) $\mathcal{P}^{x,y}$ *is pointwise ultraweakly continuous at* 0 *for some* $x, y \in \mathsf{k}$.

(b) (i) *k is pointwise $\mathcal{F}$-ultraweakly continuous.*
(ii) *$\mathcal{P}^{\Gamma,K}$ is a C_0-semigroup for every* k*-valued map* Γ*.*
(iii) *$\mathcal{P}^{x,y}$ is a C_0-semigroup for all $x, y \in \mathsf{k}$.*
(iv) *$\mathcal{P}^{x,y}$ is a C_0-semigroup for some $x, y \in \mathsf{k}$.*

(c) (i) *The map $\mathbb{R}_+ \to CB(\mathsf{V})$, $s \mapsto (\mathrm{id}_\mathsf{V} \otimes_\mathrm{M} \omega) \circ k_s$ is continuous, for all $\omega \in B(\mathcal{F})_*$.*
(ii) *For every* bounded k*-valued map $\Gamma : \mathcal{I} \to \mathsf{k}$, $\mathcal{P}^\Gamma$ is cb-norm continuous.*
(iii) *k is elementary.*
(iv) *$\mathcal{P}^{x,y}$ is cb-norm continuous for some $x, y \in \mathsf{k}$.*

Proof. The implication (ii)⇒(iii) follows in (a), (b) and (c) since if $\mathbf{x} \in \mathsf{k} \times \mathsf{k}$ then

$$\left[\mathcal{P}^{x_i,x_j}\right] = \mathcal{P}^{\mathbf{x}} = \mathcal{P}^\Gamma,$$

for the map $\Gamma : \{1, 2\} \to \mathsf{k}$, $i \mapsto x_i$; the implication (iii)⇒(iv) is obvious in (a), (b) and (c)

(a) (i)⇒(ii): Assume that (i) holds, fix a map $\Gamma : \mathcal{I} \to \mathsf{k}$ defined on some set $\mathcal{I}$, and set $\mathsf{h} := l^2(\mathcal{I})$, with standard orthonormal basis $(\delta^\alpha)_{\alpha\in\mathcal{I}}$. The key identity is

$$\langle u' \otimes \delta^\alpha, \mathcal{P}^\Gamma_t(A)u \otimes \delta^\beta\rangle = \langle u' \otimes \varpi(\Gamma(\alpha)_{[0,t[}), k_t(E^{e_\alpha} A E_{e_\beta})u \otimes \varpi(\Gamma(\beta)_{[0,t[})\rangle,$$

valid for all $A \in \mathsf{V} \otimes_\mathrm{M} B(\mathsf{h})$, $t \in \mathbb{R}_+$, $u' \in \mathfrak{h}'$, $u \in \mathfrak{h}$ and $\alpha, \beta \in \mathcal{I}$. The result follows by local norm-boundedness of $\mathcal{P}^\Gamma$, norm-totality of the set $\{\omega_{u'\otimes\delta^\alpha,u\otimes\delta^\beta} : u' \in \mathfrak{h}', u \in \mathfrak{h}, \alpha, \beta \in \mathcal{I}\}$ in $B(\mathfrak{h} \otimes \mathsf{h}; \mathfrak{h}' \otimes \mathsf{h})_*$, adaptedness of k and continuity of the map $f \mapsto \varpi(f)$.

(iv)⇒(v): Obvious.

(iv)⇒(i): This follows from the local norm-boundedness of k, the semigroup decomposition of QS cocycles, and the norm-totality in the predual space $B(\mathfrak{h} \otimes \mathcal{F}; \mathfrak{h}' \otimes \mathcal{F})_*$ of the set $\{\omega_{u'\otimes\varepsilon',u\otimes\varepsilon} : u' \in \mathfrak{h}', u \in \mathfrak{h}, \varepsilon', \varepsilon \in \mathcal{E}_\mathsf{T}\}$. See Proposition 4.2 of [LW$_2$] for full details.

(v)⇒(iii): Now suppose that V is ultraweakly closed, and let V_* denote the space of ultraweakly continuous linear functionals on V. Assume that each map $\mathcal{P}^{x,y}_t$ is ultraweakly continuous and the associated semigroup $\mathcal{P}^{x',y'}$ is pointwise ultraweakly continuous at 0. For each $x, y \in \mathsf{k}$, let $Q^{x,y}$ denote the semigroup on V_* induced by $\mathcal{P}^{x,y}$. Pointwise ultraweak continuity for $\mathcal{P}^{x,y}$ amounts to pointwise weak continuity for $Q^{x,y}$. Therefore, by semigroup theory ([Dav], in particular Theorem 6.2.6), (iii) holds if each $\mathcal{P}^{x,y}$ is pointwise ultraweakly continuous at 0. Thus let $x, y \in \mathsf{k}$, $a \in \mathsf{V}$ and $\omega \in \mathsf{V}_*$. Then, for $t \in \mathbb{R}_+$,

$$\omega(\mathcal{P}^{x,y}_t a) - \omega(\mathcal{P}^{x',y'}_t a) = \langle\varpi(x_{[0,t[}), g_t(a)\varpi(y_{[0,t[})\rangle - \langle\varpi(x'_{[0,t[}), g_t(a)\varpi(y'_{[0,t[})\rangle$$

where $g_t := (\omega \overline{\otimes}\, \mathrm{id}_{B(\mathcal{F})}) \circ k_t \in CB(\mathsf{V}; B(\mathcal{F}))$. Therefore the required pointwise continuity at 0 of $\mathcal{P}^{x,y}$ follows from the local boundedness of the family $\{g_t(a) : t \in \mathbb{R}_+\}$ and the continuity of the normalised exponential map $\varpi : L^2(\mathbb{R}_+; \mathsf{k}) \to \mathcal{F}$.

(b) (i)⇒(ii): Let $\mathcal{I}$ be a set, let $\Gamma : \mathcal{I} \to \mathsf{k}$ be a map and let $(\delta^\alpha)_{\alpha\in\mathcal{I}}$ be the standard orthonormal basis of $l^2(\mathcal{I})$. For all $a \in \mathsf{V}$, $\alpha, \beta \in \mathcal{I}$ and $t \in [0, 1[$,

$$\mathcal{P}^{\Gamma,K}_t(a \otimes |\delta^\alpha\rangle\langle\delta^\beta|) = \langle\varpi(\Gamma(\alpha)_{[t,1[}), \varpi(\Gamma(\beta)_{[t,1[})\rangle^{-1} (\mathrm{id}_\mathsf{V} \otimes_\mathrm{M} \omega_{\xi,\zeta})\big(k_t(a)\big) \otimes |\delta^\alpha\rangle\langle\delta^\beta|$$

for $\xi = \varpi(\Gamma(\alpha)_{[0,1[})$ and $\zeta = \varpi(\Gamma(\beta)_{[0,1[})$. Since $\mathcal{P}^{\Gamma,K}$ is locally norm bounded, it follows by totality of $\{a \otimes |\delta^\alpha\rangle\langle\delta^\beta| : a \in \mathsf{V}, \alpha, \beta \in \mathcal{I}\}$ that $\mathcal{P}^{\Gamma,K}$ is strongly continuous at 0, and this extends to all of $\mathbb{R}_+$ by standard semigroup theory.

(iii)⇒(i): This follows from the semigroup decomposition of QS cocycles and the norm-totality of the family $\{\omega_{\varpi(f),\varpi(g)} : f, g \in \mathbb{S}\}$ in $B(\mathcal{F})_*$.

(iv)⇒(iii): This has been remarked upon earlier.

(c) (i)⇒(ii): Fix a set $\mathcal{I}$ and map $\Gamma : \mathcal{I} \to \mathsf{k}$. First note that for $t \in \mathbb{R}_+$

$$\begin{aligned} \mathcal{P}_t^\Gamma &= (I_{\mathfrak{h}'} \otimes w_t^\Gamma)^* k_t^{\mathsf{h}}(\cdot)(I_{\mathfrak{h}} \otimes w_t^\Gamma), \text{ and} \\ \mathcal{P}_t^{0,0} \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})} &= (I_{\mathfrak{h}'} \otimes w_0^\Gamma)^* k_t^{\mathsf{h}}(\cdot)(I_{\mathfrak{h}} \otimes w_0^\Gamma), \end{aligned} \tag{2.11}$$

where $\mathsf{h} = l^2(\mathcal{I})$. Thus $\|\mathcal{P}_t^\Gamma - \mathcal{P}_t^{0,0} \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})}\|_{\mathrm{cb}} \to 0$ as $t \to 0$ by Lemma 2.2. The result follows by standard semigroup techniques, since

$$\|\mathcal{P}_t^{0,0} - \mathrm{id}_{\mathsf{V}}\|_{\mathrm{cb}} = \|(\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \omega_{\varpi(0),\varpi(0)}) \circ (k_t - k_0)\|_{\mathrm{cb}}.$$

(iii)⇒(i): This follows as in part (b).

(iv)⇒(iii): Note that for any set $\mathcal{I}$, if $\mathsf{h} = l^2(\mathcal{I})$ then

$$\mathcal{P}_t^{x,y} \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})} = (I_{\mathfrak{h}'} \otimes_{\mathrm{M}} w_t^{\Gamma_x})^* k_t^{\mathsf{h}}(\cdot)(I_{\mathfrak{h}} \otimes_{\mathrm{M}} w_t^{\Gamma_y})$$

for the constant maps $\Gamma_x, \Gamma_y : \mathcal{I} \to \mathsf{k}$, $\alpha \mapsto x$, respectively $\alpha \mapsto y$. Thus $\lim_{t \to 0} \|\mathcal{P}_t^{x,y} - \mathcal{P}_t^{0,0}\|_{\mathrm{cb}} = 0$ by similar arguments to those used above. □

Remarks. Proposition 2.10 below shows the inappropriateness of strong continuity on all of $\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ as an assumption for $\mathcal{P}^\Gamma$ in part (b).

The hypothesis for (v) of part (a) rectifies an omission in Proposition 5.4 of [LW$_2$].

Intermediate between (b) and (c) there is the further equivalence:

(i) The map $\mathbb{R}_+ \to B(\mathsf{V})$, $s \mapsto (\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \omega) \circ k_s$, is continuous, for all $\omega \in B(\mathcal{F})_*$.

(ii) $\mathcal{P}^{x,y}$ is norm continuous for all $x, y \in \mathsf{k}$.

(iii) $\mathcal{P}^{x,y}$ is norm continuous for some $x, y \in \mathsf{k}$.

The analogous form of Theorem 2.3 for QS operator cocycles is given next; it may be deduced via the natural correspondence between bounded QS operator cocycles on $\mathfrak{h}$ and completely bounded QS mapping cocycles on $|\mathfrak{h}\rangle$, the column-operator space of $\mathfrak{h}$ ([LW$_5$], Proposition 5.5), or proved directly as in [W].

Corollary 2.4. *Let $X \in \mathrm{QSC}_{\mathrm{b}}(\mathfrak{h}, \mathsf{k})$ with locally bounded norm. Then the following sets of equivalences hold*:

(a) (i) *X is strongly continuous.*

(ii) *$P^{x,y}$ is a C_0-semigroup for all $x, y \in \mathsf{k}$.*

(iii) *$P^{x,y}$ is a C_0-semigroup for some $x, y \in \mathsf{k}$.*

(b) (i) *The map $\mathbb{R}_+ \to B(\mathfrak{h})$, $t \mapsto (\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes} \omega)(X_t)$ is continuous, for all $\omega \in B(\mathcal{F})_*$.*

(ii) *X is elementary.*

(iii) *$P^{x,y}$ is norm continuous for some $x, y \in \mathsf{k}$.*

We denote the former class of operator cocycles by C_0-QSC$(\mathfrak{h}, \mathsf{k})$, and the latter class by *El*-QSC$(\mathfrak{h}, \mathsf{k})$.

Remarks. For $X \in \mathrm{QSC}_{\mathrm{b}}(\mathfrak{h}, \mathsf{k})$ with locally bounded norm, it is easily seen that there are constants $M \geqslant 1$ and $\beta \in \mathbb{R}$ such that $\|X_t\| \leqslant Me^{\beta t}$ for all $t \in \mathbb{R}$. Any bounded operator QS cocycle that is weak operator continuous at 0 is necessarily locally norm bounded ([W], Proposition 2.1).

We conclude this subsection with a result concerning continuity in the arguments of k_t, $\mathcal{P}_t^{x,y}$ and $\mathcal{P}_t^\Gamma$ for fixed $t \in \mathbb{R}_+$, when V is ultraweakly closed.

Proposition 2.5. *Let $k \in \mathrm{QSC}_{\mathrm{cb}}(\mathsf{V}, \mathsf{k})$, let T any total subset of k that contains 0, and suppose that V is ultraweakly closed. Then, for $t \in \mathbb{R}_+$, the following are equivalent*:

(i) *k_t is ultraweakly continuous from V to $\mathsf{V}\overline{\otimes} B(\mathcal{F})$.*
(ii) *$\mathcal{P}^\Gamma_t$ is ultraweakly continuous on $\mathsf{V}\overline{\otimes} B\big(l^2(\mathcal{I})\big)$ for each set $\mathcal{I}$ and map $\Gamma : \mathcal{I} \to \mathsf{k}$.*
(iii) *$\mathcal{P}^{x,y}_t$ is ultraweakly continuous on V for each $x, y \in \mathsf{T}$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that (i) holds. Since ultraweakly continuous completely bounded maps ampliate to ultraweakly continuous maps, $\mathcal{P}^\Gamma_t$ is a composition of ultraweakly continuous maps, so (ii) follows.

(ii) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (i): This is covered by Proposition 4.2 of [LW$_2$]. □

Strengthening continuity. Theorem 2.3 identifies the relevant continuity conditions on a QS cocycle k that correspond to standard continuity conditions on its associated semigroups. Thus if V is a C^*-algebra, for example, then pointwise $\mathcal{F}$-ultraweak continuity is the appropriate condition on k, whereas if V is a von Neumann algebra then pointwise ultraweak continuity is the appropriate condition on k. Our next two results show some of the hazards of moving outside these conditions.

Proposition 2.6. *Let $k \in \mathrm{QSC}_{\mathrm{cp}}(\mathsf{M}, \mathsf{k})$ for a von Neumann algebra M, and suppose that k is pointwise $\mathcal{F}$-ultraweakly continuous. Then k is elementary.*

Proof. Let $\{\mathcal{P}^{x,y} : x, y \in \mathsf{k}\}$ be the associated semigroups of k. By Theorem 2.3 the semigroup $\mathcal{P}^{\mathbf{x}}$ is strongly continuous on $\mathrm{M}_n(\mathsf{M})$, for each $n \geqslant 1$ and $\mathbf{x} \in \mathsf{k}^n$. Since $\mathcal{P}^{\mathbf{x}}$ is completely positive (by Proposition 2.1) this implies that $\mathcal{P}^{\mathbf{x}}$ is norm continuous ([Ell]). Since norm-continuous completely positive semigroups have completely bounded generators ([ChE]), it follows that $\mathcal{P}^{\mathbf{x}}$ is cb-norm continuous, and thus each of its component semigroups is too. □

Remark. From this proposition it follows that a strongly continuous completely positive QS contraction cocycle on $B(\mathfrak{h})$ necessarily has cb-norm-continuous associated semigroups. Furthermore, the analysis of [LW$_2$] (or Sections 4–6 below) shows that such a cocycle is governed by the QS differential equation (3.3) with a completely bounded stochastic generator. This appears to have been overlooked in [AcK].

Whilst Proposition 2.6 shows that insisting on strong continuity for k can result in k being necessarily an elementary QS cocycle, the following result and example shows that the converse is false: there are elementary QS cocycles that are not strongly continuous.

Proposition 2.7. *Let k be a strongly continuous *-homomorphic QS cocycle on a unital C^*-algebra A. Then k is unital.*

Proof. If k is nonunital then the projection-valued process $(1_{\mathsf{A}} \otimes I_{\mathcal{F}} - k_t(1_{\mathsf{A}}))_{t\geqslant 0}$ is nonzero for all $t > 0$, by Proposition 5.8 of [LW$_5$]. Thus $\lim_{t\to 0^+} \|k_t(1_{\mathsf{A}}) - 1_{\mathsf{A}} \otimes I_{\mathcal{F}}\| = 1$ and so k is not strongly continuous. □

Since bounded derivations δ on a unital C^*-algebra A satisfy $\delta(1_{\mathsf{A}}) = 0$, norm continuity for a homomorphic semigroup on A implies unitality. This is not so for cocycles on A.

Example 2.8. Let π be a nonunital endomorphism of a unital C^*-algebra A and set $k = k^\phi$ for the completely bounded map

$$\phi : \mathsf{A} \to \mathsf{A} \otimes B(\mathbb{C}^2), \quad a \mapsto \begin{bmatrix} 0 & 0 \\ 0 & \pi(a) - a \end{bmatrix}.$$

Here k^ϕ denotes the solution of the QS differential equation (3.3) determined by ϕ; it is a *-homomorphic elementary QS cocycle on A with noise dimension space $\mathbb{C}$. However $\phi(1_{\mathsf{A}}) \neq 0$, so k is not unital ([LiP], Theorem 5.1) thus k cannot be strongly continuous (by Proposition 2.7). In this case the generator of the global semigroup $\mathcal{P}^\eta$, with respect to the basis $\eta = \{1\}$ for $\mathbb{C}$, is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\frac{1}{2}b \\ -\frac{1}{2}c & \pi(d) - d \end{bmatrix},$$

as follows from Theorem 4.3 below.

For an example of this, with A a separable unital algebra, take A to be the Toeplitz algebra, that is $C^*(T)$ where T is the right-shift operator on l^2, and π to be the map $a \mapsto TaT^*$.

The following lemma leads to a further automatic continuity result that illustrates the problems that can arise if one seeks to enlarge the domain on which the global semigroup acts in Theorem 2.3 (b).

LEMMA 2.9. *Let $(\varphi_t)_{t>0}$ a family of completely bounded maps on V and let K be an infinite dimensional Hilbert space. Then the following are equivalent*:

(i) $\varphi_t \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{K})} \to 0$ *strongly as* $t \to 0$.
(ii) $\|\varphi_t\|_{\mathrm{cb}} \to 0$ *as* $t \to 0$.

Proof. Clearly (ii) implies (i). Suppose therefore that $\|\varphi_t\|_{\mathrm{cb}} \not\to 0$ as $t \to 0$. Then there is $\varepsilon > 0$ and a decreasing sequence $(t_n)_{n\geqslant 0}$ in $]0,\infty[$ with $\lim_{n\to\infty} t_n = 0$ such that $\|\varphi_{t_n}\|_{\mathrm{cb}} \geqslant 2\varepsilon$ for each n. Choose a sequence $(\mathsf{K}_n)_{n\geqslant 0}$ of mutually orthogonal finite-dimensional subspaces of K and a sequence $(A_n)_{n\geqslant 0}$ of operators in $\mathsf{V} \otimes B(\mathsf{K}_n)$ such that $\|A_n\| = 1$ and $\|(\varphi_{t_n} \otimes \mathrm{id}_{B(\mathsf{K}_n)})(A_n)\| \geqslant \varepsilon$. The block-diagonal operator $A = 0' \oplus \bigoplus A_n \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{K})$, where $0'$ denotes the zero element of $\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{K} \ominus \bigoplus \mathsf{K}_n)$, satisfies $\|\varphi_{t_n} \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{K})}(A)\| \geqslant \varepsilon$ for all n. Thus (i) implies (ii). □

PROPOSITION 2.10. *Let $k \in \mathrm{QSC}_{\mathrm{cb}}(\mathsf{V},\mathsf{k})$ with locally bounded cb-norm, and let Γ be a function from some set $\mathcal{I}$ to k. Suppose that $\mathcal{I}$ is infinite, Γ is bounded, and $\mathcal{P}^\Gamma$ is strongly continuous on $\mathsf{V} \otimes_{\mathrm{M}} B(l^2(\mathcal{I}))$. Then k is elementary.*

Proof. In view of the identities (2.11), $\mathcal{P}^{0,0}_t \otimes_{\mathrm{M}} \mathrm{id}_{B(\mathsf{h})} \to \mathrm{id}_{\mathsf{V}\otimes B(\mathsf{h})}$ strongly as $t \to 0$, where $\mathsf{h} = l^2(\mathcal{I})$. Thus Lemma 2.9 implies that $\mathcal{P}^{0,0}$ is cb-norm continuous. The result therefore follows from Theorem 2.3. □

Finally we conclude with a result that is useful for establishing quantum stochastic integrability of certain processes obtained from a QS cocycle.

PROPOSITION 2.11. *Let $k \in \mathrm{QSC}_{\mathrm{cb}}(\mathsf{V},\mathsf{k})$ with locally bounded cb-norm, and suppose that k is pointwise continuous at $t = 0$ with respect to the weak operator topology. Then the following hold*:

(a) *k is pointwise right-continuous on all of $\mathbb{R}_+$ in this topology.*
(b) *If further V is a von Neumann algebra and k is completely positive and contractive then k is pointwise right-continuous on $\mathbb{R}_+$ with respect to the strong operator topology.*

Proof. From the cocycle identity (2.7) it is sufficient to show that for any $a \in \mathsf{V}$ and $t \in \mathbb{R}_+$

$$\lim_{r\downarrow 0} \langle u \otimes \varpi(f), \big[(\widehat{k}_r \circ \sigma_r)(A) - A\big] v \otimes \varpi(g)\rangle = 0$$

where $A = k_t(a)$. However, the definitions and identifications in (2.7) yield the identity

$$\langle u \otimes \varpi(f), (\widehat{k}_r \circ \sigma_r)(A) v \otimes \varpi(g)\rangle = \langle u \otimes \varpi(f_{[0,r[}), k_r(E^{\varpi(s_r^* f)} A E_{\varpi(s_r^* g)}) v \otimes \varpi(g_{[0,r[})\rangle$$

where $(s_r^*)_{r\geqslant 0}$ denotes the C_0-semigroup of left-shifts on $L^2(\mathbb{R}_+;\mathsf{k})$. This fact, coupled with the assumptions on k, gives the desired limit as $r\downarrow 0$.

Suppose now that V is a von Neumann algebra and k completely positive and contractive. Then the operator Schwarz inequality gives, for $A=k_t(a)$ and $\xi\in\mathfrak{h}\otimes\mathcal{F}$,

$$\|\big[(\widehat{k}_r\circ\sigma_r)(A)-A\big]\xi\|^2\leqslant\langle\xi,(\widehat{k}_r\circ\sigma_r)(A^*A)\xi\rangle-2\operatorname{Re}\langle A\xi,(\widehat{k}_r\circ\sigma_r)(A)\xi\rangle+\|A\xi\|^2,$$

and the RHS tends to zero as $r\downarrow 0$ by the first part. The second part follows. $\square$

3. QS cocycles and QS differential equations

As noted earlier, solutions of the QS differential equation (3.3) provide a basic source of QS cocycles. We adopt the notations (1.1), (1.3), (2.1), (2.10) and the definition

$$\chi:\mathsf{k}\times\mathsf{k}\to\mathbb{C},\quad (x,y)\mapsto\tfrac{1}{2}\|x\|^2+\tfrac{1}{2}\|y\|^2-\langle x,y\rangle,\tag{3.1}$$

so that

$$\langle\varpi(f),\varpi(g)\rangle=\exp\Big(-\int\mathrm{d}s\,\chi(f(s),g(s))\Big),\quad f,g\in L^2(\mathbb{R}_+;\mathsf{k}).$$

An operator process $\Lambda_F=\big(\Lambda_F(t)\big)_{t\geqslant 0}$ is defined, for each operator $F\in B(\mathfrak{h}\otimes\widehat{\mathsf{k}};\mathfrak{h}'\otimes\widehat{\mathsf{k}})$, through the identity

$$\Big\langle u\otimes\varpi(f),\Lambda_F(t)v\otimes\varpi(g)\Big\rangle=\int_0^t\mathrm{d}s\,\big\langle u\otimes\widehat{f}(s),F(v\otimes\widehat{g}(s)\big\rangle\langle\varpi(f),\varpi(g)\rangle$$

for all $v\in\mathfrak{h}$, $u\in\mathfrak{h}'$, $f,g\in\mathbb{S}$ and $t\in\mathbb{R}_+$. We also use the following notations for an operator ϕ from V to $\mathsf{V}\otimes_{\mathrm{M}}B(\widehat{\mathsf{k}})$ with domain V_0:

$$\phi_{x,y}:=\big(\mathrm{id}_{\mathsf{V}}\otimes_{\mathrm{M}}\omega_{\widehat{x},\widehat{y}}\big)\circ\phi-\chi(x,y)\,\mathrm{id}_{\mathsf{V}},\quad x,y\in\mathsf{k},\tag{3.2}$$

and Λ_ϕ for the process-valued map $\Lambda_{\phi(\cdot)}$. For subsets T and T' of k which are total and contain 0, a process k on V is said to *satisfy* (3.3) *$\mathcal{E}_{\mathsf{T}'}$-weakly for the domain $\mathfrak{h}\underline{\otimes}\mathcal{E}_{\mathsf{T}}$*, if

$$\langle u\otimes\varpi(f),\big(k_t(a)-a\otimes I_{\mathcal{F}}\big)v\otimes\varpi(g)\rangle=\int_0^t\mathrm{d}s\,\langle u\otimes\varpi(f),k_s\big(E^{\widehat{f}(s)}\phi(a)E_{\widehat{g}(s)}\big)v\otimes\varpi(g)\rangle$$

for all $a\in\mathsf{V}_0,t\in\mathbb{R}_+,u\in\mathfrak{h}',v\in\mathfrak{h},f\in\mathbb{S}_{\mathsf{T}'}$ and $g\in\mathbb{S}_{\mathsf{T}}$. Such a solution is *weakly regular* if, for all $f\in\mathbb{S}_{\mathsf{T}'}$ and $g\in\mathbb{S}_{\mathsf{T}}$, $E^{\varpi(f)}k_t(\cdot)E_{\varpi(g)}$ is bounded with bounds that are locally uniform in t.

Our first result contains a basic criterion for uniqueness of solutions of QS differential equations, extending those of [LiP] and [LW$_1$]. Recall that a *pregenerator* of a C_0-semigroup is a closable operator whose closure is a generator.

Theorem 3.1. *Let ϕ be an operator from V to $\mathsf{V}\otimes_{\mathrm{M}}B(\widehat{\mathsf{k}})$ with domain V_0, and let T and T' be total subsets of k containing 0.*

(a) *Let $k\in\mathrm{QSC}(\mathsf{V},\mathsf{k})$ and suppose that, for all $x\in\mathsf{T}'$ and $y\in\mathsf{T}$, its (x,y)-associated semigroup $\mathcal{P}^{x,y}$ is strongly continuous and has pregenerator $\phi_{x,y}$. Then k is an $\mathcal{E}_{\mathsf{T}'}$-weak solution of the QS differential equation*

$$\mathrm{d}k_t=k_t\cdot\mathrm{d}\Lambda_\phi(t),\quad k_0=\iota_{\mathcal{F}}^{\mathsf{V}}\tag{3.3}$$

on V_0 for the domain $\mathfrak{h}\underline{\otimes}\mathcal{E}_{\mathsf{T}}$.

(b) *Conversely, suppose that for all* $x \in \mathsf{T}'$ *and* $y \in \mathsf{T}$, $\phi_{x,y}$ *is a pregenerator of a* C_0*-semigroup* $\mathcal{P}^{x,y}$ *on* V. *Then* (3.3) *has at most one weakly regular* $\mathcal{E}_{\mathsf{T}'}$*-weak solution on* V_0 *for the domain* $\mathfrak{h}\underline{\otimes}\,\mathcal{E}_{\mathsf{T}}$. *Moreover, any such solution is a (weak) QS cocycle on* V *whose* (x,y)*-associated semigroup is* $\mathcal{P}^{x,y}$, *for each* $x \in \mathsf{T}'$ *and* $y \in \mathsf{T}$.

(c) *Let* k *be an* $\mathcal{E}_{\mathsf{T}'}$*-weak solution of* (3.3) *on* V_0 *for the domain* $\mathfrak{h}\underline{\otimes}\,\mathcal{E}_{\mathsf{T}}$. *Then, for any element* a *of* V_0, *the following are equivalent*:
(i) $k_t(a) = \iota^{\mathsf{V}}_{\mathcal{F}}(a)$ *for all* $t \in \mathbb{R}_+$;
(ii) $\phi(a) = 0$.
In particular, if V *is an operator system and* $1 \in \mathsf{V}_0$ *then* k *is unital if and only if* $\phi(1) = 0$.

Proof. (a) This is a straightforward consequence of the semigroup decomposition of QS cocycles.

(b) Let $f \in \mathbb{S}_{\mathsf{T}'}$ and $g \in \mathbb{S}_{\mathsf{T}}$, let $[t_0, t_1[, [t_1, t_2[, \cdots, [t_n, t_{n+1}[$ be common intervals of constancy of f and g, with $t_0 = 0$ and $t_{n+1} = \infty$, and set $\upsilon := (\phi_{f(t),g(t)})_{t\geqslant 0}$. Since υ is pregenerator-valued, it follows from semigroup theory that the integral equation

$$Q_t a = a + \int_0^t \mathrm{d}s\, Q_s \upsilon_s(a), \qquad a \in \mathsf{V}_0, t \in \mathbb{R}_+ \tag{3.4}$$

has unique strongly continuous $B(\mathsf{V})$-valued solution Q, namely the 'piecewise semigroup evolution' generated by $\overline{\upsilon} := (\overline{\phi}_{f(t),g(t)})_{t\geqslant 0}$, where $\overline{\phi}_{x,y}$ denotes the closure of $\phi_{x,y}$; it is given by

$$Q_t = \mathcal{P}^{f(t_0),g(t_0)}_{t_1-t_0} \circ \cdots \circ \mathcal{P}^{f(t_{i-1}),g(t_{i-1})}_{t_i-t_{i-1}} \circ \mathcal{P}^{f(t_i),g(t_i)}_{t-t_i} \quad \text{for } t \in [t_i, t_{i+1}[\text{ and } i \in \{0, \cdots n\}. \tag{3.5}$$

The result follows since, for any weakly regular $\mathcal{E}_{\mathsf{T}'}$-weak solution k of (3.3) on V_0 for the domain $\mathfrak{h}\underline{\otimes}\,\mathcal{E}_{\mathsf{T}}$, the $B(\mathsf{V})$-valued family $\big(E^{\varpi(f_{[0,t[})}k_t(\cdot)E_{\varpi(g_{[0,t[})}\big)_{t\geqslant 0}$ is strongly continuous and satisfies (3.4), and identity (3.5) then affirms the (weak) cocycle property of k (through semigroup decomposition).

(c) If (i) holds then (ii) follows since, for all $u \in \mathfrak{h}', v \in \mathfrak{h}, x \in \mathsf{T}', y \in \mathsf{T}$ and $T > 0$,

$$e^{-\chi(x,y)T}\langle u, E^{\widehat{x}}\phi(a)E_{\widehat{y}}v\rangle = \lim_{t\to 0} t^{-1}\langle u \otimes \varpi(x_{[0,T[}), (k_t(a) - \iota^{\mathsf{V}}_{\mathcal{F}}(a))v \otimes \varpi(y_{[0,T[})\rangle = 0.$$

The converse is even more straightforward to verify. □

REMARKS. (i) Another version of the uniqueness part of this theorem is proved in [BW$_2$].
(ii) The theorem has a corresponding version appropriate when V is a von Neumann algebra.

Specialising to completely bounded operators ϕ, we next summarise some quantum stochastic lore (see [L] and references therein). Note that, for such ϕ, each $\phi_{x,y}$ is completely bounded and therefore generates a cb-norm-continuous semigroup so Theorem 3.1 applies.

THEOREM 3.2. *Let* $\phi \in CB(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$. *The QS differential equation* (3.3) *has a unique weakly regular* $\mathcal{E}$*-weak solution on the exponential domain* $\mathfrak{h}\underline{\otimes}\,\mathcal{E}$, *denoted* k^ϕ. *Moreover* k^ϕ *is a strong solution and an elementary (weak) QS cocycle on* V *whose* (x,y)*-associated semigroup has generator* $\phi_{x,y}$, *for all* $x, y \in \mathsf{k}$.

If the QS cocycle k^ϕ *is completely bounded then, for any Hilbert space* h, *the cb-process* $\big((k^\phi_t)^{\mathsf{h}}\big)_{t\geqslant 0}$ *on* $\mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$ *equals the elementary QS cocycle* k^Φ, *where* $\Phi = \phi^{\mathsf{h}}$. *Moreover it is* κ*-decomposable for any orthonormal basis* κ *of* h.

Remark. The resulting map

$$\Phi_{\mathsf{V},\mathsf{k}} : CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})) \to El\text{-QSC}(\mathsf{V},\mathsf{k}), \quad \phi \mapsto k^{\phi}. \tag{3.6}$$

is injective, and is referred to as the *QS generation map* (*on* V*, with respect to the noise dimension space* k); ϕ is called the *stochastic generator* of the QS cocycle k^ϕ. For $\phi \in CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$, $\phi^\dagger := \big(a^* \mapsto \phi(a)^*\big) \in CB(\mathsf{V}^\dagger;\mathsf{V}^\dagger \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$, where $\mathsf{V}^\dagger$ is the adjoint operator space $\{a^* : a \in \mathsf{V}\}$, and k^ϕ has hermitian conjugate QS cocycle $k^{\phi^\dagger}$. When $\mathsf{V}^\dagger = \mathsf{V}$, we set

$$\mathfrak{real}(\mathsf{V},\mathsf{k}) := \{\phi \in CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})) : \phi^\dagger = \phi\}, \tag{3.7}$$

thus (see (2.5)) $\Phi_{\mathsf{V},\mathsf{k}}(\mathfrak{real}(\mathsf{V},\mathsf{k})) \subset \mathrm{QSC}_{\mathrm{h}}(\mathsf{V},\mathsf{k})$. Also, when V is an operator system, we set

$$\mathfrak{u}(\mathsf{V},\mathsf{k}) := \{\phi \in CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})) : \phi(1) = 0\}, \tag{3.8}$$

thus (see (2.6)), by part (c) of Theorem 3.1, $\Phi_{\mathsf{V},\mathsf{k}}(\mathfrak{u}(\mathsf{V},\mathsf{k})) \subset \mathrm{QSC}_{\mathrm{u}}(\mathsf{V},\mathsf{k})$.

For a C^*-algebra, the QS generation map is shown to restrict to various bijections of interest in Section 6.

We next give a useful invariance principle, whose straightforward proof follows from the definition of $\mathcal{P}^\Gamma$ (Proposition 2.1), the semigroup decomposition of QS cocycles, and the fact that, for all $x, y \in \mathsf{k}$, $\phi_{x,y}$ is the generator of the (x,y)-associated semigroup of the QS cocycle k^ϕ.

Proposition 3.3. *Let $k \in \mathrm{QSC}_{\mathrm{cb}}(\mathsf{V},\mathsf{k})$ and, for a set $\mathcal{I}$, let $\Gamma : \mathcal{I} \to \mathsf{k}$ be a map whose range is total and contains* 0*. Set $\mathsf{h} = l^2(\mathcal{I})$. Then, for a closed subspace W of V, the following are equivalent*:

(i) $k_t(\mathsf{W}) \subset \mathsf{W}\otimes_{\mathrm{M}} B(\mathcal{F})$ *for all* $t \geqslant 0$.
(ii) $\mathcal{P}^\Gamma_t\big(\mathsf{W}\otimes_{\mathrm{M}} B(\mathsf{h})\big) \subset \mathsf{W}\otimes_{\mathrm{M}} B(\mathsf{h})$ *for all* $t \geqslant 0$.

If these hold then k defines a QS cocycle on W by restriction.

Further equivalences arise if either $k = k^\phi$ for some map $\phi \in CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$, or $\mathcal{P}^\Gamma$ is norm continuous with generator φ. Namely,

(iii) $\phi(\mathsf{W}) \subset \mathsf{W}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$,

respectively,

(iv) $\varphi\big(\mathsf{W}\otimes_{\mathrm{M}} B(\mathsf{h})\big) \subset \mathsf{W}\otimes_{\mathrm{M}} B(\mathsf{h})$.

Remark. Suppose that the map Γ is bounded, $k = k^\phi$ where $\phi \in CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$, and k is completely bounded and locally bounded in cb-norm. Then, by Theorem 4.2 below, $\mathcal{P}^\Gamma$ is cb-norm continuous.

4. Transformations of CB generators

In this section we obtain the affine transformation from (completely bounded) stochastic generators of quantum stochastic cocycles to the generators of the corresponding global semigroups, and establish its injectivity in cases of interest. We also examine the question of inverting the transform in case the global semigroup is determined by an orthonormal basis of the noise dimension space. A basic tool for the section is diagonal Weyl processes W^Γ associated with a k-valued map Γ on an index set (defined in (2.9)).

LEMMA 4.1. *Let $\mathcal{I}$ be a set and $\Gamma : \mathcal{I} \to \mathsf{k}$ a map. Set $\mathsf{h} = l^2(\mathcal{I})$ and let $(\delta^\alpha)_{\alpha\in\mathcal{I}}$ be its standard orthonormal basis. Then the diagonal Weyl process W^Γ is both a left and a right unitary cocycle. It is elementary if and only if Γ is bounded, in which case W^Γ is a strong solution of the QS differential equation*

$$\mathrm{d}X_t = X_t\,\mathrm{d}\Lambda_F(t), \quad X_0 = I_{\mathsf{h}\otimes\mathcal{F}}, \tag{4.1}$$

for the coefficient operator $F = F_\Gamma \in B(\mathsf{h}\otimes\widehat{\mathsf{k}})$ given by

$$F_\Gamma = \begin{bmatrix} -\frac{1}{2}L_\Gamma^* L_\Gamma & -L_\Gamma^* \\ L_\Gamma & 0 \end{bmatrix},$$

in which $L_\Gamma \in B(\mathsf{h};\mathsf{h}\otimes\mathsf{k})$ is the bounded operator determined by the prescription $\delta^\alpha \mapsto \delta^\alpha\otimes\Gamma(\alpha)$. In this case, $j := \big(T \mapsto (W_t^\Gamma)^*(T\otimes I_\mathcal{F})W_t^\Gamma\big)_{t\geqslant 0}$ defines an elementary QS flow j on $B(\mathsf{h})$ with stochastic generator

$$\Theta_\Gamma(T) := F_\Gamma^*(T\otimes I_{\widehat{\mathsf{k}}}) + (T\otimes I_{\widehat{\mathsf{k}}})F_\Gamma + F_\Gamma^*(T\otimes\Delta)F_\Gamma.$$

NOTE. By a *QS flow* we mean a unital *-homomorphic QS cocycle.

Proof. The time-shift identity and Weyl relations,

$$\sigma_t\big(W(f)\big) = W(s_t f) \quad\text{and}\quad W(x_{[0,r[})W(x_{[r,r+t[}) = W(x_{[0,r+t[}) = W(x_{[r,r+t[})W(x_{[0,r}),$$

for $f\in L^2(\mathbb{R}_+;\mathsf{k})$, $x\in\mathsf{k}$ and $r,t\in\mathbb{R}_+$, imply that the unitary process W^Γ is a left and right QS operator cocycle. It follows from Lemma 2.2 and Corollary 2.4 that W^Γ is elementary if and only if Γ is bounded. The unique solution X^F of (4.1) is a left QS operator cocycle. To show that $X^F = W^\Gamma$ first note that for any $\alpha\in\mathcal{I}$ and $x\in\mathsf{k}$

$$F_\Gamma(\delta^\alpha\otimes\widehat{x}) = \delta^\alpha\otimes\begin{pmatrix} -\frac{1}{2}\|\Gamma(\alpha)\|^2 - \langle\Gamma(\alpha),x\rangle \\ \Gamma(\alpha)\end{pmatrix}. \tag{4.2}$$

From this one readily obtains

$$\langle\delta^\alpha, (E^{\widehat{x}}F_\Gamma E_{\widehat{y}} - \chi(x,y))\delta^\beta\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle\delta^\alpha\otimes\varpi(x_{[0,t[}), W_t^\Gamma(\delta^\beta\otimes\varpi(y_{[0,t[})\rangle$$

which is enough to show that the associated semigroups of these two cocycles coincide.

That $j = k^\phi$ for $\phi = \Theta_\Gamma$ is a standard result (see [L] or [LW$_1$], Theorem 7.4). □

Recall the notation k^Γ for the associated Γ-cocycle introduced in Proposition 2.1.

THEOREM 4.2. *Let $\phi\in CB(\mathsf{V};\mathsf{V}\otimes_\mathrm{M} B(\widehat{\mathsf{k}}))$ and, for a set $\mathcal{I}$, let Γ be a bounded map from $\mathcal{I}$ to k. Set $\mathsf{h} = l^2(\mathcal{I})$, with standard orthonormal basis $(\delta^\alpha)_{\alpha\in\mathcal{I}}$. In terms of the operators given in* Lemma 4.1, *define maps $\Phi_\Gamma\in CB(\mathsf{V}\otimes_\mathrm{M} B(\mathsf{h});\mathsf{V}\otimes_\mathrm{M} B(\mathsf{h}\otimes\widehat{\mathsf{k}}))$ and $\varphi_\Gamma\in CB(\mathsf{V}\otimes_\mathrm{M} B(\mathsf{h}))$, by*

$$\Phi_\Gamma := (\mathrm{id}_\mathsf{V}\otimes_\mathrm{M}\upsilon_\Gamma)\circ\phi^\mathsf{h} + \mathrm{id}_\mathsf{V}\otimes_\mathrm{M}\Theta_\Gamma, \text{ and}$$
$$\varphi_\Gamma := (\mathrm{id}_{\mathsf{V}\otimes_\mathrm{M} B(\mathsf{h})}\otimes_\mathrm{M}\omega_{\widehat{0},\widehat{0}})\circ\Phi_\Gamma.$$

where $\upsilon_\Gamma(A) := \widetilde{F}_\Gamma^* A\widetilde{F}_\Gamma$ for the operator $\widetilde{F}_\Gamma := I_{\mathsf{h}\otimes\widehat{\mathsf{k}}} + (I_\mathsf{h}\otimes\Delta)F_\Gamma$. Under the identifications (1.2) *and* (1.7), *the following hold*:

(a) $(e^{t\varphi_\Gamma})_{t\geqslant 0}$ *is the Schur-action semigroup on $\mathsf{V}\otimes_\mathrm{M} B(\mathsf{h})$ comprised of associated semigroups of the (weak) QS cocycle k^ϕ*:

$$e^{t\varphi_\Gamma} = [\mathcal{P}_t^{\Gamma(\alpha),\Gamma(\beta)}]_{\alpha,\beta\in\mathcal{I}}\;\boldsymbol{\cdot} \qquad \text{for } t\in\mathbb{R}_+.$$

(b) *Suppose that the QS cocycle k^ϕ is completely bounded with locally bounded cb-norm and let $\mathcal{P}^\Gamma$ denote its global Γ-semigroup. Then, setting $k^{\phi,\Gamma} = (k^\phi)^\Gamma$,*

$$k^{\phi,\Gamma} = k^{\Phi_\Gamma} \quad \textit{and} \quad \mathcal{P}^\Gamma = (e^{t\varphi_\Gamma})_{t\geqslant 0}.$$

(c) *Suppose that the range of Γ is total and contains 0. Then the affine-linear map $\phi \mapsto \varphi_\Gamma$ is injective.*

Proof. Let $B(\mathfrak{h};\mathfrak{h}')$ be the ambient full operator space of V. For any $\alpha \in \mathcal{I}$ and $x \in \mathsf{k}$, using (4.2),

$$\widetilde{F}_\Gamma(\delta^\alpha \otimes \widehat{x}) = \delta^\alpha \otimes \begin{pmatrix} 1 \\ x + \Gamma(\alpha) \end{pmatrix}, \tag{4.3}$$

and therefore

$$\langle u' \otimes \delta^\alpha, \varphi_\Gamma(A) u \otimes \delta^\beta \rangle = \langle u', \phi_{\Gamma(\alpha),\Gamma(\beta)}(E^{e_\alpha} A E_{e_\beta}) u \rangle, \tag{4.4}$$

for all $A \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$, $u \in \mathfrak{h}$, $u' \in \mathfrak{h}'$ and $\alpha, \beta \in \mathcal{I}$, where $\phi_{x,y}$ denotes the semigroup generator defined in (3.2). Consequently (a) follows; in particular (4.4) shows that φ_Γ has Schur-action.

To prove (b) note that, by Theorem 3.2, the weak QS cocycle k^{Φ_Γ} is elementary. By assumption on k^ϕ and Proposition 2.1, $k^{\phi,\Gamma}$ is a well-defined completely bounded cocycle whose vacuum-expectation semigroup is $\mathcal{P}^\Gamma$. In particular $\mathcal{P}^\Gamma$ is cb-norm continuous by part (a), and so $k^{\phi,\Gamma}$ is also elementary by Theorem 2.3. So to verify that $k^{\phi,\Gamma} = k^{\Phi_\Gamma}$ it suffices to check that their cb-norm-continuous associated semigroups are the same, which can be established by showing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle u' \otimes \delta^\alpha \otimes \varpi(x_{[0,t[}), (k_t^{\Phi_\Gamma} - k_t^{\phi,\Gamma})(A) u \otimes \delta^\beta \otimes \varpi(y_{[0,t[}) \rangle = 0$$

for all $u \in \mathfrak{h}$, $u' \in \mathfrak{h}'$, $\alpha, \beta \in \mathcal{I}$, $x, y \in \mathsf{k}$ and $A \in \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$. This follows from a straightforward if somewhat lengthy computation using (4.2), (4.3) and (3.2).

For (c) let $\phi, \phi' \in CB(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{h})$, and suppose that $\varphi_\Gamma = \varphi'_\Gamma$. Then, by (3.2) and (4.4), $E^{\widehat{\Gamma(\alpha)}}(\phi - \phi')(a) E_{\widehat{\Gamma(\beta)}} = 0$ for all $\alpha, \beta \in \mathcal{I}$ and $a \in \mathsf{V}$. The totality of $\{\widehat{\Gamma(\alpha)} : \alpha \in \mathcal{I}\}$ in $\widehat{\mathsf{k}}$ gives $\phi = \phi'$. $\square$

Remark. The two results above have been proved using the 'matrix of cocycles' idea from Section 5.1 of [LW$_5$]. For example, the matricial QS cocycle $k^{\phi,\Gamma}$ takes the form

$$\Big(\big[(I_{\mathfrak{h}'} \otimes W(\Gamma(\alpha)_{[0,t[}))^* k_t^\phi(\cdot)(I_{\mathfrak{h}} \otimes W(\Gamma(\beta)_{[0,t[})) \big]_{\alpha,\beta\in\mathcal{I}} \Big)_{t\geqslant 0}.$$

For the rest of the section we specialise to the case where Γ is determined by the choice of some orthonormal basis $\eta = (d_i)_{i\in\mathcal{I}_0}$ of k. Thus $l^2(\mathcal{I}_0)$ is identified with k, $l^2(\mathcal{I})$ is identified with $\widehat{\mathsf{k}}$, and, as usual, the convention (2.2) is in operation. Operators such as F_Γ from above will be rebranded as F_η, etc. For example, the operator $L_\eta : \widehat{\mathsf{k}} \to \widehat{\mathsf{k}} \otimes \mathsf{k}$ from Lemma 4.1 is now a decapitated version of the Schur isometry $S_{\overline{\eta}}$ from Section 1.

The spaces $CB(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ and $CB_{\overline{\eta}\text{-dec}}(\mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ can both be viewed as subspaces of $\mathrm{M}_{\mathcal{I}}\big(CB(\mathsf{V})\big)$ through the identifications (1.12) and (1.7) respectively. For the given choice of basis η the map $\phi \mapsto \varphi_\eta$ extends to a map on $\mathrm{M}_{\mathcal{I}}\big(CB(\mathsf{V})\big)$ which is then manifestly bijective.

Indeed, in terms of their matrix components, the transformation and its inverse are as follows:

$$\begin{aligned} \varphi^0_0 &= \phi^0_0, \\ \varphi^i_0 &= \phi^0_0 + \phi^i_0 - \tfrac{1}{2}\,\mathrm{id}_{\mathsf{V}}, \\ \varphi^0_j &= \phi^0_0 + \phi^0_j - \tfrac{1}{2}\,\mathrm{id}_{\mathsf{V}}, \\ \varphi^i_j &= \phi^0_0 + \phi^i_0 + \phi^0_j + \phi^i_j + (\delta^i_j - 1)\,\mathrm{id}_{\mathsf{V}}, \end{aligned} \quad \text{and} \quad \begin{aligned} \phi^0_0 &= \varphi^0_0, \\ \phi^i_0 &= \varphi^i_0 - \varphi^0_0 + \tfrac{1}{2}\,\mathrm{id}_{\mathsf{V}}, \\ \phi^0_j &= \varphi^0_j - \varphi^0_0 + \tfrac{1}{2}\,\mathrm{id}_{\mathsf{V}}, \\ \phi^i_j &= \varphi^i_j - \varphi^i_0 - \varphi^0_j + \varphi^0_0 - \delta^i_j\,\mathrm{id}_{\mathsf{V}} \end{aligned} \tag{4.5}$$

for $i, j \in \mathcal{I}_0$. The difficult task is to determine if, on restriction, we have a surjection $CB(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})) \to CB_{\overline{\eta}\text{-dec}}(\mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$. In Section 6 we determine the image of various subspaces of interest, but for now we must make do with a partially defined inverse. This is given in terms of truncations of ϕ and uses operators that are defined in terms of finite subsets $\mathcal{J}_0$ of $\mathcal{I}_0$. For such subsets, in the notation (2.2), we set $\mathcal{J} = \{0\} \cup \mathcal{J}_0$. The requisite operators are:

$$C_{\mathcal{J}_0} = \sum_{i \in \mathcal{J}_0} |d_i\rangle \in |\mathsf{k}\rangle, \qquad C_{\mathcal{J}} = \sum_{\alpha \in \mathcal{J}} |e_\alpha\rangle \in |\widehat{\mathsf{k}}\rangle, \qquad Q_{\mathcal{J}_0} = \sum_{i \in \mathcal{J}_0} |d_i\rangle\langle d_i| \in B(\mathsf{k}),$$

and the following operators in $B(\widehat{\mathsf{k}})$:

$$\begin{aligned} &\Box_{\mathcal{J}} := C_{\mathcal{J}} C_{\mathcal{J}}^*, \qquad \Delta_{\mathcal{J}} := \begin{bmatrix} 0 & 0 \\ 0 & Q_{\mathcal{J}_0} \end{bmatrix}, \qquad Q_{\mathcal{J}} := \begin{bmatrix} 1 & 0 \\ 0 & Q_{\mathcal{J}_0} \end{bmatrix} = \sum_{\alpha \in \mathcal{J}} |e_\alpha\rangle\langle e_\alpha|, \\ &A_{\mathcal{J}} := \begin{bmatrix} 0 & -\frac{1}{2} C_{\mathcal{J}_0}^* \\ -\frac{1}{2} C_{\mathcal{J}_0} & Q_{\mathcal{J}_0} - C_{\mathcal{J}_0} C_{\mathcal{J}_0}^* \end{bmatrix} \quad \text{and} \quad B_{\mathcal{J}} := \begin{bmatrix} 1 & -C_{\mathcal{J}_0}^* \\ 0 & I_{\mathsf{k}} \end{bmatrix} = I_{\widehat{\mathsf{k}}} - \sum_{i \in \mathcal{J}_0} |e_0\rangle\langle e_i|. \end{aligned} \tag{4.6}$$

For each $\mathcal{J}_0 \subset\subset \mathcal{I}_0$ and $T \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ we define

$$T_{[\mathcal{J}]} := Q_{\mathcal{J}} T Q_{\mathcal{J}},$$

so that $T_{[\mathcal{J}]}$ is a truncation of T when viewed as a matrix according to (1.2). Note that $Q_{\mathcal{J}_0}$, $Q_{\mathcal{J}}$ and $\Delta_{\mathcal{J}}$ are truncations of the bounded operators I_{k}, $I_{\widehat{\mathsf{k}}}$ and Δ, whereas the other operators defined above are truncations of sesquilinear forms, *i.e.* the norms are unbounded as $\mathcal{J}_0$ grows. Recall the left inverse $\Upsilon_{\overline{\eta}}$ of the Schur homomorphism $\Sigma_{\overline{\eta}}$ from Section 1.

Theorem 4.3. *Let* $\phi \in CB\big(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})\big)$ *and let* $\varphi \in CB_{\overline{\eta}\text{-dec}}\big(\mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})\big)$. *Then the following are equivalent*:

(i) $\varphi = (\mathrm{id}_{\mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})} \otimes_{\mathrm{M}} \omega_{\widehat{0},\widehat{0}}) \circ \Phi_\eta$, *as defined in* Theorem 4.2.

(ii) $\varphi = (\mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \widetilde{\Upsilon}_\eta) \circ \phi^{\widehat{\mathsf{k}}} + \mathrm{id}_{\mathsf{V}} \otimes_{\mathrm{M}} \Upsilon'_\eta$, *for the maps*

$$\begin{aligned} &\widetilde{\Upsilon}_\eta : B(\widehat{\mathsf{k}} \otimes \widehat{\mathsf{k}}) \to B(\widehat{\mathsf{k}}), \quad T \mapsto \widetilde{S}_\eta^* T \widetilde{S}_\eta \quad \textit{where} \quad \widetilde{S}_\eta := S_{\overline{\eta}} + \Delta \otimes |\widehat{0}\rangle, \quad \textit{and} \\ &\Upsilon'_\eta : B(\widehat{\mathsf{k}}) \to B(\widehat{\mathsf{k}}), \quad T \mapsto \Upsilon_{\overline{\eta}}(T \otimes \Delta) - \tfrac{1}{2}(T\Delta + \Delta T). \end{aligned}$$

(iii) *For each* $a \in \mathsf{V}$ *and* $\mathcal{J}_0 \subset\subset \mathcal{I}_0$, *the truncation* $\phi_{[\mathcal{J}]} : a \mapsto \phi(a)_{[\mathcal{J}]}$ *is given by*

$$\phi_{[\mathcal{J}]} : a \mapsto (I_{\mathfrak{h}'} \otimes B_{\mathcal{J}})^* \big(\varphi(a \otimes \Box_{\mathcal{J}}) - a \otimes A_{\mathcal{J}}\big)(I_{\mathfrak{h}} \otimes B_{\mathcal{J}}). \tag{4.7}$$

Proof. Set $\varphi_\eta := (\mathrm{id}_{\mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})} \otimes_{\mathrm{M}} \omega_{\widehat{0},\widehat{0}}) \circ \Phi_\eta$. The identities

$$F_\eta E_{\widehat{0}} = (I_{\widehat{\mathsf{k}}} \otimes \Delta) S_{\overline{\eta}} - \tfrac{1}{2} \Delta \otimes |\widehat{0}\rangle \quad \text{and} \quad \widetilde{F}_\eta E_{\widehat{0}} = \widetilde{S}_\eta$$

are verified using the relations (4.2) and (4.3); the equivalence of (i) and (ii) then follows.

The relations

$$\Upsilon_{\overline{\eta}}(\square_{\mathcal{J}} \otimes \Delta) = \Delta_{\mathcal{J}} \quad \text{and} \quad A_{\mathcal{J}} = \Delta_{\mathcal{J}} - \tfrac{1}{2}(\Delta \square_{\mathcal{J}} + \square_{\mathcal{J}} \Delta)$$

show that

$$\Upsilon'_{\eta}(\square_{\mathcal{J}}) = A_{\mathcal{J}}. \tag{4.8}$$

Moreover, since $(C^*_{\mathcal{J}} \otimes \langle e_\beta|)\widetilde{S}_\eta B_{\mathcal{J}} = \langle e_\beta| Q_{\mathcal{J}}$ it follows that

$$B^*_{\mathcal{J}} \widetilde{S}^*_\eta(\square_{\mathcal{J}} \otimes T)\widetilde{S}_\eta B_{\mathcal{J}} = T_{[\mathcal{J}]}, \tag{4.9}$$

first for $T = |e_\alpha\rangle\langle e_\beta|$ and thus for any $T \in B(\widehat{\mathsf{k}})$ by linearity and ultraweak continuity.

Suppose that (ii) holds. Then (iii) follows from (4.8), (4.9) and the totality, with respect to the weak operator topology, of the set of simple tensors in $\mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$.

Finally, suppose that φ is such that (iii) holds. We already know that (iii) holds for φ_η and so

$$\begin{aligned}(I_{\mathfrak{h}'} \otimes B_{\mathcal{J}})^* \varphi(a \otimes \square_{\mathcal{J}})(I_{\mathfrak{h}} \otimes B_{\mathcal{J}}) &= \phi_{[\mathcal{J}]}(a) + a \otimes B^*_{\mathcal{J}} A_{\mathcal{J}} B_{\mathcal{J}} \\ &= (I_{\mathfrak{h}'} \otimes B_{\mathcal{J}})^* \varphi_\eta(a \otimes \square_{\mathcal{J}})(I_{\mathfrak{h}} \otimes B_{\mathcal{J}}).\end{aligned}$$

Invertibility of $B_{\mathcal{J}}$ shows that $\varphi(a \otimes \square_{\mathcal{J}}) = \varphi_\eta(a \otimes \square_{\mathcal{J}})$ for all $a \in \mathsf{V}$ and $\mathcal{J}_0 \subset\subset \mathcal{I}_0$. Since ϕ and ϕ_η are both $\overline{\eta}$-decomposable, this is sufficient to deduce that $\varphi = \varphi_\eta$, so (i) holds. $\square$

5. Generation of completely positive QS cocycles

In this section we consider the stochastic generation of completely positive QS cocycles. Let E be an operator system acting nondegenerately on $\mathfrak{h}$ (so that $1_{\mathsf{E}} = I_{\mathfrak{h}}$), and fix an orthonormal basis $\eta = (d_i)_{i \in \mathcal{I}_0}$ for k and, as usual, set $\mathcal{I} := \{0\} \cup \mathcal{I}_0$ and $\overline{\eta} = (e_\alpha)_{\alpha \in \mathcal{I}}$ with the convention (2.2) in operation, as always. As well as the truncated operators $\Delta_{\mathcal{J}}$, $\square_{\mathcal{J}}$, $A_{\mathcal{J}}$ etc. from the previous section, we define, for $\mathcal{J}_0 \subset\subset \mathcal{I}_0$ and $t \geqslant 0$, the operator:

$$\square_{\mathcal{J},t} := \begin{bmatrix} 1 \\ e^{-t/2} C_{\mathcal{J}_0} \end{bmatrix} \begin{bmatrix} 1 \\ e^{-t/2} C_{\mathcal{J}_0} \end{bmatrix}^* + (1 - e^{-t})\Delta_{\mathcal{J}} \in B(\widehat{\mathsf{k}}).$$

Thus $\square_{\mathcal{J},0} = \square_{\mathcal{J}}$. We focus on the global semigroup $\mathcal{P}^\eta$ on $\mathsf{E} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$ determined by a QS cocycle on E and the given choice of basis. The following summarises Proposition 6.2 and Theorem 6.4 of [LW$_5$], written now in terms of this single semigroup by means of a re-indexing as used in the proof of Theorem 6.5 of that paper. (There we worked instead with the infinite family of semigroups $\{\mathcal{P}^{\mathbf{x}} : \mathbf{x} \in \mathsf{T}^n, n \in \mathbb{N}\}$ for a total subset T of k containing 0.) Recall (2.2), the notation $\overline{\eta}$ for the orthonormal basis $(\widehat{d_\alpha})_{\alpha \in \mathcal{I}}$ of $\widehat{\mathsf{k}}$.

Theorem 5.1 ([LW$_5$]). *Let $\mathcal{P}$ be a semigroup on $\mathsf{E} \otimes K(\widehat{\mathsf{k}})$, or on $\mathsf{E} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$, comprised of completely bounded $\overline{\eta}$-decomposable maps.*

- (a) *Suppose that $\mathcal{P} = \mathcal{P}^\eta$, the global η-semigroup associated to a cocycle $k \in \mathrm{QSC_{cb}}(\mathsf{E}, \mathsf{k})$. Then the following hold:*
  - (i) *k is completely positive if and only if $\mathcal{P}$ is.*
  - (ii) *Suppose that k is completely positive. Then k is contractive if and only if $\mathcal{P}$ satisfies*

$$\mathcal{P}_t(1_{\mathsf{E}} \otimes \square_{\mathcal{J}}) \leqslant 1_{\mathsf{E}} \otimes \square_{\mathcal{J},t} \text{ for all } \mathcal{J}_0 \subset\subset \mathcal{I}_0 \text{ and } t \geqslant 0, \tag{5.1}$$

and k is unital if and only if (5.1) holds with equality.

(b) *Conversely, suppose that $\mathcal{P}$ is completely positive and satisfies* (5.1). *Then there is a unique cocycle $k \in \mathrm{QSC}_{\mathrm{cpc}}(\mathsf{E}, \mathsf{k})$ whose global η-semigroup is $\mathcal{P}$.*

Remarks. In part (a) (i) E may alternatively be assumed to be a (not necessarily unital) C^*-algebra. Parts (a) (ii) and (b) can also be reformulated to cover the case of nonunital C^*-algebras by using operator intervals. See Theorem 6.7 of [LW$_5$] for details.

Note that no continuity in t is assumed in Theorem 5.1. The flexibility as to whether we look at semigroups on $\mathsf{E} \otimes K(\widehat{\mathsf{k}})$ or on $\mathsf{E} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$ follows from Proposition 1.4, however when considering continuity questions one should heed cautionary results such as Proposition 2.10.

Let $\mathcal{J}_0 \subset \mathcal{L}_0 \subset\subset \mathcal{I}_0$. If the inequality (5.1) holds for $\mathcal{L} = \{0\} \cup \mathcal{L}_0$ then, by $\overline{\eta}$-decomposability, it holds for $\mathcal{J}$ too. Thus if k is finite dimensional then (5.1) may be replaced by the inequalities

$$\mathcal{P}_t(1_{\mathsf{E}} \otimes \square_{\mathcal{I}}) \leqslant 1_{\mathsf{E}} \otimes \square_{\mathcal{I},t}, \quad t \geqslant 0.$$

We next derive an infinitesimal adjunct to part (a) of Theorem 5.1.

Proposition 5.2. *Let $\mathcal{P}$ be a C_0-semigroup on $\mathsf{E} \otimes K(\widehat{\mathsf{k}})$ consisting of completely positive, $\overline{\eta}$-decomposable maps, and let φ be its generator. Let $\mathcal{J}_0 \subset\subset \mathcal{I}_0$ and assume that $1_{\mathsf{E}} \otimes \square_{\mathcal{J}} \in \mathrm{Dom}\,\varphi$. Then the following are equivalent*:

(i) $\mathcal{P}_t(1_{\mathsf{E}} \otimes \square_{\mathcal{J}}) \leqslant 1_{\mathsf{E}} \otimes \square_{\mathcal{J},t}$ *for all* $t \in \mathbb{R}_+$.
(ii) $\varphi(1_{\mathsf{E}} \otimes \square_{\mathcal{J}}) \leqslant 1_{\mathsf{E}} \otimes A_{\mathcal{J}}$ *where $A_{\mathcal{J}}$ is the operator defined in* (4.6).

Furthermore, (i) *holds with equality if and only if* (ii) *does.*

Proof. Let us abbreviate $1_{\mathsf{E}} \otimes \square_{\mathcal{J}}, 1_{\mathsf{E}} \otimes \Delta_{\mathcal{J}}$ and $1_{\mathsf{E}} \otimes \Delta$, to $\square_{\mathcal{J}}, \Delta_{\mathcal{J}}$ and Δ respectively. In view of the $\overline{\eta}$-decomposability of $\mathcal{P}$,

$$\widetilde{\mathcal{P}}_t := R_t \mathcal{P}_t(\cdot) R_t \text{ where } R_t = \Delta^{\perp} + e^{t/2}\Delta,$$

defines a completely positive, η-decomposable C_0-semigroup $\widetilde{\mathcal{P}}$ with generator $\widetilde{\varphi}$ given by

$$\mathrm{Dom}\,\widetilde{\varphi} = \mathrm{Dom}\,\varphi, \quad \widetilde{\varphi}(T) = \varphi(T) + \tfrac{1}{2}(\Delta T + T\Delta),$$

and the proposition is established once it is proved that

$$\widetilde{\mathcal{P}}_t(\square_{\mathcal{J}}) \leqslant \square_{\mathcal{J}} + (e^t - 1)\Delta_{\mathcal{J}} \text{ for all } t \geqslant 0 \ \Leftrightarrow\ \widetilde{\varphi}(\square_{\mathcal{J}}) \leqslant \Delta_{\mathcal{J}}.$$

Equality holds on the left when $t = 0$, so the left to right implication is obtained by differentiation. Suppose therefore that $\widetilde{\varphi}$ satisfies the right-hand inequality. Since the semigroup $\widetilde{\mathcal{P}}$ is positive and η-decomposable

$$\begin{aligned}\widetilde{\mathcal{P}}_t(\square_{\mathcal{J}}) &= \square_{\mathcal{J}} + \int_0^t \mathrm{d}t_1\, (\widetilde{\mathcal{P}}_{t_1} \circ \widetilde{\varphi})(\square_{\mathcal{J}}) \\ &\leqslant \square_{\mathcal{J}} + \int_0^t \mathrm{d}t_1\, \widetilde{\mathcal{P}}_{t_1}(\Delta_{\mathcal{J}}) = \square_{\mathcal{J}} + \int_0^t \mathrm{d}t_1\, \Delta_{\mathcal{J}} \cdot \widetilde{\mathcal{P}}_{t_1}(\square_{\mathcal{J}})\end{aligned}$$

where $\cdot$ denotes the η-Schur product. Since $\Delta_{\mathcal{J}} \in 1_{\mathsf{E}} \otimes B(\widehat{\mathsf{k}})_+$, iteration is sanctioned by Lemma 1.2, giving

$$\begin{aligned}\widetilde{\mathcal{P}}_t(\square_{\mathcal{J}}) &\leqslant \square_{\mathcal{J}} + \int_0^t \mathrm{d}t_1\, \Delta_{\mathcal{J}} \cdot \Big(\square_{\mathcal{J}} + \int_0^{t_1} \mathrm{d}t_2\ \Delta_{\mathcal{J}} \cdot \widetilde{\mathcal{P}}_{t_2}(\square_{\mathcal{J}})\Big) \\ &= \square_{\mathcal{J}} + t\Delta_{\mathcal{J}} + \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\ \Delta_{\mathcal{J}} \cdot \widetilde{\mathcal{P}}_{t_2}(\square_{\mathcal{J}}).\end{aligned}$$

Further iteration gives

$$\widetilde{\mathcal{P}}_t(\square_{\mathcal{J}}) \leqslant \square_{\mathcal{J}} + \sum_{k=1}^{n} \frac{t^k}{k!} \Delta_{\mathcal{J}} + \varepsilon_n(t),$$

where

$$\varepsilon_n(t) = \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_n} \mathrm{d}t_{n+1}\, \widetilde{\mathcal{P}}_{t_{n+1}}(\Delta_{\mathcal{J}}).$$

Since $\varepsilon_n(t) \to 0$ as $n \to \infty$, the result follows, noting that the case of equality in (i) and (ii) follows by the same argument. $\square$

Remark. One cannot work with $\mathcal{P}$ and φ directly since $A_{\mathcal{J}}$ is not positive.

Let ϕ be an operator from E to $\mathsf{E} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$ with domain E_0. If k is a completely positive process on E satisfying (3.3) $\mathcal{E}_{\mathsf{T}(\eta)}$-*weakly* for the domain $\mathfrak{h} \underline{\otimes} \mathcal{E}_{\mathsf{T}(\eta)}$ and $1_{\mathsf{E}} \in \mathsf{E}_0$ then k is contractive (respectively unital) if and only if $\phi(1_{\mathsf{E}}) \leqslant 0$ (respectively $\phi(1_{\mathsf{E}}) = 0$); the proof is given in Proposition 5.1 of [LW$_1$]. Moreover, in favourable circumstances (*e.g.* [LW$_1$], Theorem 3.1) such a weak solution k of (3.3) enjoys a semigroup decomposition that, in conjunction with Proposition 5.1 of [LW$_5$], shows that k is a cocycle. In the case that ϕ is unbounded, the missing ingredient at this stage is some hypothesis that ensures that certain affine combinations of components of ϕ are pregenerators of C_0-semigroups.

We now turn this line of argument around and instead prove existence of a solution of (3.3) by going via global semigroups and their associated cocycles. For this we make use of the necessary conditions on ϕ for a CP process weakly satisfying (3.3) to be contractive.

To do this, first note the following identity for $\mathcal{J}_0 \subset\subset \mathcal{I}_0$:

$$\sum_{\alpha,\beta \in \mathcal{J}} \chi(d_\alpha, d_\beta) |e_\alpha\rangle\langle e_\beta| = -A_{\mathcal{J}}, \tag{5.2}$$

where the map χ is defined in (3.1), $\mathcal{J} = \{0\} \cup \mathcal{J}_0$ as usual, and $A_{\mathcal{J}}$ is the operator defined in (4.6). Also, we further specialise the notation $\mathcal{P}^{\Gamma}$ for associated semigroups and cocycles as follows: for $\mathcal{J}_0 \subset\subset \mathcal{I}_0$, $\mathcal{P}^{[\mathcal{J}]}$ denotes the semigroup associated to the map $\Gamma : \mathcal{J} \to \mathsf{k}$, $\alpha \mapsto d_\alpha$. Note that $\mathcal{P}^{[\mathcal{J}]}$ has Schur-action when we make the identification $\mathsf{E} \otimes_{\mathrm{M}} B(l^2(\mathcal{J})) \cong \mathrm{M}_{\mathcal{J}}(\mathsf{E})$. Recall the notations $\mathsf{T}(\eta)$ and $\overline{\eta}$ introduced in (2.3)

Theorem 5.3. *Let ϕ be an operator from E to $\mathsf{E} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$ with domain E_0 satisfying the following conditions*:

(1) $1_{\mathsf{E}} \in \mathsf{E}_0$ *and* $\phi(1_{\mathsf{E}}) \leqslant 0$.

(2) *For all $\alpha, \beta \in \mathcal{I}$, the operator ϕ_{d_α, d_β} on E with domain E_0 (defined in (3.2)) is a pregenerator of a C_0-semigroup $\mathcal{P}^{(\alpha,\beta)}$ on E.*

(3) *For all $\mathcal{J}_0 \subset\subset \mathcal{I}_0$, the semigroup $\mathcal{P}^{[\mathcal{J}]}$ (defined above) is completely positive.*

Then there is a unique process $k \in \mathrm{QSC_b}(\mathsf{E}, \mathsf{k})$ which is locally norm bounded and such that for all $\alpha, \beta \in \mathcal{I}$, $\mathcal{P}^{(\alpha,\beta)}$ is its (d_α, d_β)-associated semigroup. Moreover,

(a) *k is completely positive and contractive, and pointwise $\mathcal{F}$-ultraweakly continuous. Furthermore it is unital if and only if $\phi(1_{\mathsf{E}}) = 0$.*

(b) *k is the unique weakly regular $\mathcal{E}_{\mathsf{T}(\eta)}$-weak solution of the QS differential equation (3.3) on E_0 for the domain $\mathfrak{h} \underline{\otimes} \mathcal{E}_{\mathsf{T}(\eta)}$.*

(c) *k strongly satisfies (3.3) on E_0 for the domain $\mathfrak{h} \underline{\otimes} \mathcal{E}_{\mathsf{T}(\eta)}$, under the following further condition*:

  (4) *$\mathfrak{h}$ and k are separable.*

Proof. The uniqueness part is immediate since every bounded QS cocycle is uniquely determined by its (x, y)-associated semigroups for x and y running through any total subset of k containing 0. We therefore address existence.

First note that, for all $\alpha \in \mathcal{I}$, $\phi_{d_\alpha,d_\alpha} = E^{\widehat{d_\alpha}}\phi(\cdot)E_{\widehat{d_\alpha}}$ since $\chi(d_\alpha, d_\alpha) = 0$, and so $\phi_{d_\alpha,d_\alpha}(1_{\mathsf{E}}) \leqslant 0$ by (1). Moreover, since $\mathcal{P}^{[\mathcal{J}]}$ has Schur-action it follows that each $\mathcal{P}^{(\alpha,\alpha)}$ is positive, and hence is contractive since

$$0 \leqslant \mathcal{P}_t^{(\alpha,\alpha)}(1_{\mathsf{E}}) = 1_{\mathsf{E}} + \int_0^t \mathrm{d}s\, \mathcal{P}_s^{(\alpha,\alpha)}\big(\phi_{d_\alpha,d_\alpha}(1_{\mathsf{E}})\big) \leqslant 1_{\mathsf{E}}.$$

Thus $\mathcal{P}_t^{[\mathcal{J}]}(1_{\mathrm{M}_{\mathcal{J}}(\mathsf{E})}) \leqslant 1_{\mathrm{M}_{\mathcal{J}}(\mathsf{E})}$ for each $\mathcal{J}_0 \subset\subset \mathcal{I}_0$, showing that the completely positive semigroup $\mathcal{P}^{[\mathcal{J}]}$ is contractive.

Linear extension of the prescription

$$\mathcal{P}_t^0 : a \otimes |e_\alpha\rangle\langle e_\beta| \mapsto \mathcal{P}_t^{(\alpha,\beta)}(a) \otimes |e_\alpha\rangle\langle e_\beta|$$

defines an operator $\mathcal{P}_t^0$ on the operator system $\mathsf{E} \otimes K(\widehat{\mathsf{k}})$ with dense domain $\mathsf{E}_0 \underline{\otimes} B_{00}(\overline{\eta})$, where $B_{00}(\overline{\eta}) := \mathrm{Lin}\big\{|e_\alpha\rangle\langle e_\beta| : \alpha, \beta \in \mathcal{I}\big\}$, which inherits complete positivity and contractivity from the family $(\mathcal{P}_t^{[\mathcal{J}]})_{\mathcal{J}_0 \subset\subset \mathcal{I}_0}$. Let $\mathcal{P}_t$ denote its continuous extension to all of $\mathsf{E} \otimes K(\widehat{\mathsf{k}})$. Then $\mathcal{P} := (\mathcal{P}_t)_{t \geqslant 0}$ is a completely positive and contractive $\overline{\eta}$-decomposable C_0-semigroup on $\mathsf{E} \otimes K(\widehat{\mathsf{k}})$. Its generator ψ satisfies

$$\begin{aligned} &\mathrm{Dom}\,\psi \supset \mathsf{E}_0 \underline{\otimes} B_{00}(\overline{\eta}), \quad \text{and} \\ &\psi(a \otimes |e_\alpha\rangle\langle e_\beta|) = \phi_{d_\alpha,d_\beta}(a) \otimes |e_\alpha\rangle\langle e_\beta|, \qquad a \in \mathsf{E}_0,\ \alpha, \beta \in \mathcal{I}. \end{aligned}$$

In particular, using identity (5.2), for all $\mathcal{J}_0 \subset\subset \mathcal{I}_0$

$$\begin{aligned} \psi(1_{\mathsf{E}} \otimes \square_{\mathcal{J}}) &= \sum_{\alpha,\beta \in \mathcal{J}} \phi_{d_\alpha,d_\beta}(1_{\mathsf{E}}) \otimes |e_\alpha\rangle\langle e_\beta| \\ &= \sum_{\alpha,\beta \in \mathcal{J}} \big(E^{\widehat{d_\alpha}}\phi(1_{\mathsf{E}})E_{\widehat{d_\beta}} - \chi(d_\alpha, d_\beta)1_{\mathsf{E}}\big) \otimes |e_\alpha\rangle\langle e_\beta| \\ &= D_{\mathcal{J}}^*\, \phi(1_{\mathsf{E}}) D_{\mathcal{J}} + 1_{\mathsf{E}} \otimes A_{\mathcal{J}} \leqslant 1_{\mathsf{E}} \otimes A_{\mathcal{J}}, \end{aligned}$$

where $D_{\mathcal{J}} = I_{\mathfrak{h}} \otimes \sum_{\alpha \in \mathcal{J}} |\widehat{d_\alpha}\rangle\langle e_\alpha|$. Therefore, by Proposition 5.2,

$$\mathcal{P}_t(1_{\mathsf{E}} \otimes \square_{\mathcal{J}}) \leqslant 1_{\mathsf{E}} \otimes \square_{\mathcal{J},t} \qquad \mathcal{J}_0 \subset\subset \mathcal{I}_0,\ t \in \mathbb{R}_+,$$

and so, by Theorem 5.1, there is a unique cocycle $k \in \mathrm{QSC}_{\mathrm{cpc}}(\mathsf{E}, \mathsf{k})$ whose global η-semigroup on $\mathsf{E} \otimes K(\widehat{\mathsf{k}})$ is $\mathcal{P}$. In particular the (d_α, d_β)-associated semigroup of k is $\mathcal{P}^{(\alpha,\beta)}$.

(a) We have already proved that k is completely positive and contractive. The rest follows from Theorems 2.3 and 3.1.

(b) This follows from Theorem 3.1.

(c) Let K denote the QS cocycle $\big((k_t)^{\widehat{\mathsf{k}}}\big)_{t \geqslant 0}$ on $\mathsf{E} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$. To see that our weak solution is actually a strong solution involves showing that, for each $a \in \mathsf{E}_0$, the process $\big(K_t(\phi(a))\big)_{t \geqslant 0}$ is QS integrable on $(\mathfrak{h} \otimes \widehat{\mathsf{k}}) \underline{\otimes} \mathcal{E}$ which, because k is completely contractive, amounts to proving strong measurability of the vector-valued process $\big(K_t(\phi(a))\xi\big)_{t \geqslant 0}$ for each $\xi \in (\mathfrak{h} \otimes \widehat{\mathsf{k}}) \underline{\otimes} \mathcal{E}$; under assumption (4) this follows from Pettis' Theorem. □

Remarks. (i) In Section 7 we give sufficient conditions on ϕ for assumption (3) to hold.

(ii) The uniqueness here extends that of the QS cocycle generated by a mapping in $CB(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ for an operator space V (see Theorem 3.2).

(iii) An alternative condition to (4), which implies that a weak solution is in fact a strong solution, is that E is a von Neumann algebra so that Proposition 2.11 applies. However, as noted in the remark following Proposition 2.6, strong continuity is an inappropriate assumption in those circumstances.

6. Completely positive elementary QS cocycles

For this section let A be a C^*-algebra acting nondegenerately on $\mathfrak{h}$. The main goal of the section is to reveal the structure of the stochastic generator of a completely positive elementary QS cocycle. To this end, for $R \in B(\mathfrak{h}; \mathfrak{h} \otimes \widehat{\mathsf{k}})$ and $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$, define maps

$$\begin{aligned} &\psi_R \in CB(B(\mathfrak{h}); B(\mathfrak{h} \otimes \widehat{\mathsf{k}})), && T \mapsto RTE^{\widehat{0}} + E_{\widehat{0}}TR^* - T \otimes \Delta, \quad \text{and} \\ &\chi_{\phi,R} \in CB(\mathsf{A}; B(\mathfrak{h} \otimes \widehat{\mathsf{k}})), && \chi_{\phi,R} := \phi - \psi_R. \end{aligned} \tag{6.1}$$

Set

$$\mathfrak{cp}(\mathsf{A}, \mathsf{k}) := \big\{\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})) : \chi_{\phi,R} \text{ is CP for some } R \in B(\mathfrak{h}; \mathfrak{h} \otimes \widehat{\mathsf{k}})\big\}.$$

Note that $\mathfrak{cp}(\mathsf{A}, \mathsf{k}) \subset \mathfrak{real}(\mathsf{A}, \mathsf{k})$.

Proposition 6.1. *Let $\phi \in \mathfrak{cp}(\mathsf{A}, \mathsf{k})$, suppose that A is unital and that the QS cocycle k^ϕ is completely bounded. Then k^ϕ is completely positive.*

Proof. Fix an orthonormal basis η for k. Let φ_η be the generator of the global η-semigroup $\mathcal{P}^\eta$ of the elementary cocycle k^ϕ, as given by Theorem 4.3 whose notations we continue to use. Choose R such that $\chi_{\phi,R}$ is completely positive, then

$$\phi^{\widehat{\mathsf{k}}}(A) = \chi^{\widehat{\mathsf{k}}}_{\phi,R}(A) + \Pi(R \otimes I_{\widehat{\mathsf{k}}})A(I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}} \otimes \langle\widehat{0}|) + (I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}} \otimes |\widehat{0}\rangle)A(R^* \otimes I_{\widehat{\mathsf{k}}})\Pi - A \otimes \Delta,$$

where Π is the unitary tensor flip on $\mathfrak{h} \otimes \widehat{\mathsf{k}} \otimes \widehat{\mathsf{k}}$ exchanging the two copies of $\widehat{\mathsf{k}}$. The identities

$$\begin{aligned} &(\mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \widetilde{\Upsilon}_\eta)(A \otimes \Delta) = (\mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \Upsilon_{\overline{\eta}})(A \otimes \Delta) \quad \text{and} \\ &\big(I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}} \otimes \langle\widehat{0}|\big)(I_{\mathfrak{h}} \otimes \widetilde{S}_\eta) = I_{\mathfrak{h}\otimes\widehat{\mathsf{k}}} \end{aligned}$$

therefore imply that

$$\varphi_\eta(A) = (I_{\mathfrak{h}} \otimes \widetilde{S}_\eta)^* \chi^{\widehat{\mathsf{k}}}_{\phi,R}(A)(I_{\mathfrak{h}} \otimes \widetilde{S}_\eta) + \widetilde{R}A + A\widetilde{R}^*$$

for the operator $\widetilde{R} := (R \otimes I_{\widehat{\mathsf{k}}})\Pi(I_{\mathfrak{h}} \otimes \widetilde{S}_\eta) - \frac{1}{2}I_{\mathfrak{h}} \otimes \Delta$. Applying Stinespring's Theorem to the map $\chi_{\phi,R}$, and then applying Theorem 1.7 to φ_η, it follows that $\mathcal{P}^\eta$ is completely positive. Thus k^ϕ is completely positive by Theorem 5.1. $\square$

Remark. In case $\phi \in \mathfrak{cp}(\mathsf{A}, \mathsf{k})$ but k^ϕ is not completely bounded, the global semigroup $\mathcal{P}^\eta$ still exists and is completely positive. This implies that $\mathcal{P}^{\mathbf{x}}$ is completely positive for all $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} \mathsf{k}^n$, which still implies that the (weak) QS cocycle k^ϕ is completely positive, but now in the following sense: for all $n \in \mathbb{N}$, $A = [a^i_j] \in \mathrm{M}_n(\mathsf{A})_+$ and $\zeta \in (\mathfrak{h} \underline{\otimes} \mathcal{E})^n$,

$$\sum_{i,j=1}^n \langle \zeta^i, k^\phi_t(a^i_j)\zeta^j \rangle \geqslant 0.$$

For A unital, we define the following subsets of $\mathfrak{cp}(\mathsf{A},\mathsf{k})$:

$$\begin{aligned}
\mathfrak{cpqc}_\beta(\mathsf{A},\mathsf{k}) &:= \big\{\phi \in \mathfrak{cp}(\mathsf{A},\mathsf{k}) : \phi(1_{\mathsf{A}}) \leqslant \beta\Delta^\perp\big\},\\
\mathfrak{cpc}(\mathsf{A},\mathsf{k}) &:= \mathfrak{cpqc}_0(\mathsf{A},\mathsf{k}),\\
\mathfrak{cpqc}(\mathsf{A},\mathsf{k}) &:= \bigcup_{\beta\in\mathbb{R}} \mathfrak{cpqc}_\beta(\mathsf{A},\mathsf{k}), \qquad \text{and}\\
\mathfrak{cpu}(\mathsf{A},\mathsf{k}) &:= \mathfrak{cp}(\mathsf{A},\mathsf{k}) \cap \mathfrak{u}(\mathsf{A},\mathsf{k}).
\end{aligned}$$

Remarks. For $\beta \in \mathbb{R}$, the prescriptions

$$k \mapsto \big(e^{-\beta t}k_t\big)_{t\geqslant 0} \quad \text{and} \quad \phi \mapsto \phi - \beta\delta^\perp,$$

in which $\delta^\perp \in CB(\mathsf{A}; \mathsf{A}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ denotes the map $a \mapsto a \otimes \Delta^\perp$, define bijections of $\mathrm{QSC}(\mathsf{A},\mathsf{k})$ and of $CB(\mathsf{A}; \mathsf{A}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ respectively. For a QS cocycle k and real number β, the QS cocycle $(e^{-\beta t}k_t)_{t\geqslant 0}$ is elementary if and only if k is, and the bijections are compatible, in the sense that, for $\phi \in CB(\mathsf{A}; \mathsf{A}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$,

$$\big(e^{-\beta t}k^\phi_t\big)_{t\geqslant 0} = k^{\phi - \beta\delta^\perp}.$$

This is easily verified by appealing to uniqueness of weak solutions to the QS differential equation (3.3), in which $\phi - \beta\delta^\perp$ replaces ϕ, or by making use of the semigroup decomposition of such solutions ([LW$_1$], Theorem 3.1). The remarks so far apply equally to stochastically generated cocycles on *any* operator space V.

The above bijections restrict to bijections

$$\mathrm{QSC}^\beta_{\mathrm{cpqc}}(\mathsf{A},\mathsf{k}) \to \mathrm{QSC}_{\mathrm{cpc}}(\mathsf{A},\mathsf{k}) \quad \text{and} \quad \mathfrak{cpqc}_\beta(\mathsf{A},\mathsf{k}) \to \mathfrak{cpc}(\mathsf{A},\mathsf{k}),$$

noting that, in the notation (6.1), $\psi_R - \beta\delta^\perp = \psi_{R'}$ for $R' = R - \frac{1}{2}\beta E_{\widehat{0}}$.

For the proof of the next result, we require the following easily verified identities associated with an orthonormal basis $\eta = (d_i)_{i\in\mathcal{I}_0}$ for k and the operators defined in (4.6): for $\mathcal{J}_0 \subset\subset \mathcal{I}_0$ and Hilbert spaces K and K', setting $\mathcal{I} := \{0\}\cup\mathcal{I}_0$, $\mathcal{J} := \{0\}\cup\mathcal{J}_0$ as usual and $\overline{\eta} = (e_\alpha)_{\alpha\in\mathcal{I}}$,

$$B^*_{\mathcal{J}}C_{\mathcal{J}} = |\widehat{0}\rangle \tag{6.2}$$

$$B^*_{\mathcal{J}}A_{\mathcal{J}}B_{\mathcal{J}} = \Delta_{\mathcal{J}} - \tfrac{1}{2}\big(|\widehat{0}\rangle C^*_{\mathcal{J}}\Delta B_{\mathcal{J}} + B^*_{\mathcal{J}}\Delta C_{\mathcal{J}}\langle\widehat{0}|\big), \text{ and} \tag{6.3}$$

$$X \otimes \square_{\mathcal{J}} = (I_{\mathsf{K}'}\otimes C_{\mathcal{J}})X(I_{\mathsf{K}}\otimes C_{\mathcal{J}})^*, \text{ for } X \in B(\mathsf{K};\mathsf{K}'). \tag{6.4}$$

For a supplementary Hilbert space H, define the transformation

$$\begin{aligned}
&\alpha_{\mathcal{J},\mathsf{H}} : B(\mathsf{H};\mathfrak{h})\overline{\otimes}\,\mathrm{D}_{\overline{\eta}}(\widehat{\mathsf{k}}) \to B(\mathsf{H};\mathfrak{h})\overline{\otimes}\,|\widehat{\mathsf{k}}\rangle = B(\mathsf{H};\mathfrak{h}\oplus(\mathfrak{h}\otimes\mathsf{k})),\\
&X = \begin{bmatrix} x^0 & 0\\ 0 & X^1\end{bmatrix} \mapsto (I_{\mathfrak{h}}\otimes B_{\mathcal{J}})^*X(I_{\mathfrak{h}}\otimes C_{\mathcal{J}}) = \begin{bmatrix} x^0\\ X^1(I_{\mathsf{H}}\otimes C_{\mathcal{J}_0}) - (I_{\mathfrak{h}}\otimes C_{\mathcal{J}_0})x^0\end{bmatrix},
\end{aligned}$$

which enjoys the following properties:

(1) For any operator space W in $B(\mathsf{H};\mathfrak{h})$,

$$\alpha_{\mathcal{J},\mathsf{H}}\big(\mathsf{W}\otimes_{\mathrm{M}}\mathrm{D}_{\overline{\eta}}(\widehat{\mathsf{k}})\big) \subset \mathsf{W}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle.$$

(2) If $\mathcal{J}_0 \subset \mathcal{L}_0 \subset\subset \mathcal{I}_0$ then

$$Q_{\mathcal{J}}\,\alpha_{\mathcal{L},\mathsf{H}}(\cdot) = \alpha_{\mathcal{J},\mathsf{H}}$$

since $Q_{\mathcal{J}_0}C_{\mathcal{L}_0} = C_{\mathcal{L}_0}$ and $Q_{\mathcal{J}_0}X^1 = X^1Q_{\mathcal{J}_0}$ for all $X^1 \in B(\mathsf{H};\mathfrak{h})\overline{\otimes}\,\mathrm{D}_\eta(\mathsf{k})$.

Recall the QS generation map defined in (3.6).

PROPOSITION 6.2. *Suppose that* A *is unital, and let* $\beta \in \mathbb{R}$. *Then* $\Phi_{\mathsf{A},\mathsf{k}}$ *restricts to bijections*

$$\mathfrak{cpqc}_\beta(\mathsf{A},\mathsf{k}) \to El\text{-QSC}^\beta_{\text{cpqc}}(\mathsf{A},\mathsf{k}), \ \textit{and}$$
$$\mathfrak{cpu}(\mathsf{A},\mathsf{k}) \to El\text{-QSC}_{\text{cpu}}(\mathsf{A},\mathsf{k}).$$

Moreover, for $\phi \in \mathfrak{cpqc}(\mathsf{A},\mathsf{k})$, *there is some* $R \in \mathsf{A}'' \overline{\otimes} |\widehat{\mathsf{k}}\rangle$ *for which* $\chi_{\phi,R}$ *is completely positive.*

Proof. By the above remarks we may assume without loss that $\beta = 0$ and so it is enough to show that $\Phi_{\mathsf{A},\mathsf{k}}$ restricts to a bijection $\mathfrak{cpc}(\mathsf{A},\mathsf{k}) \to El\text{-QSC}_{\text{cpc}}(\mathsf{A},\mathsf{k})$. The second part then follows from the first since for *any* $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$, k^ϕ is unital if and only if $\phi(1_{\mathsf{A}}) = 0$ by Theorems 3.1 and 3.2.

Fix an orthonormal basis η for k, and let $\phi \in \mathfrak{cpc}(\mathsf{A},\mathsf{k})$. Choose $R \in B(\mathfrak{h}; \mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that $\chi_{\phi,R}$ is completely positive. Then φ_η (defined by Theorem 4.2) is the generator of a completely positive semigroup, as shown in the proof of Proposition 6.1. Now $\phi^{\widehat{\mathsf{k}}}(1_{\mathsf{A}} \otimes_{\mathrm{M}} \square_{\mathcal{J}}) \leqslant 0$ since $\phi(1_{\mathsf{A}}) \leqslant 0$, so Theorem 4.3 and the identity (4.8) imply that

$$\varphi_\eta(1_{\mathsf{A}} \otimes \square_{\mathcal{J}}) \leqslant 1_{\mathsf{A}} \otimes \Upsilon'_\eta(\square_{\mathcal{J}}) = 1_{\mathsf{A}} \otimes A_{\mathcal{J}}.$$

Therefore, by Proposition 5.2 and Theorem 5.1, there is a completely positive QS contraction cocycle k whose global η-semigroup $\mathcal{P}^\eta$ is η-decomposable and has generator φ_η, in particular $\mathcal{P}^\eta$ is cb-norm continuous. Moreover, by Theorem 4.2, the component semigroups of $\mathcal{P}^\eta$ are associated semigroups of the QS cocycle k^ϕ, and so, by the semigroup decomposition of QS cocycles, $k = k^\phi$. It follows that $\mathcal{P}^{0,0}$, the vacuum-expectation semigroup of k^ϕ, is cb-norm continuous and so Theorem 2.3 implies that the cocycle k^ϕ is elementary, as required.

Conversely, suppose that $k \in El\text{-QSC}_{\text{cpc}}(\mathsf{A},\mathsf{k})$. Then $\mathcal{P}^{0,0}$, the vacuum-expectation semigroup of k, is cb-norm continuous. Theorem 2.3 therefore implies that the global η-semigroup of k is cb-norm continuous; let $\varphi \in CB_{\overline{\eta}\text{-dec}}(\mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ be its generator. By Theorem 1.7 there is a unital representation (π, K) of A and $\overline{\eta}$-diagonal operators $T \in B(\mathsf{K}; \mathfrak{h}) \overline{\otimes} \mathrm{D}_{\overline{\eta}}(\widehat{\mathsf{k}})$ and $N \in \mathsf{A}'' \overline{\otimes} \mathrm{D}_{\overline{\eta}}(\widehat{\mathsf{k}})$ such that $\varphi(A) = T(\pi \otimes_{\mathrm{M}} \mathrm{id}_{B(\widehat{\mathsf{k}})})(A)T^* + NA + AN^*$.

Our goal now is to show that φ is of the form φ_η, as defined in Theorem 4.2, for a map $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$. Taking our cue from Theorem 4.3 define, for each $\mathcal{J}_0 \subset\subset \mathcal{I}_0$, the map

$$\phi_{\mathcal{J}} : \mathsf{A} \to \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}), \quad a \mapsto (I_{\mathfrak{h}} \otimes B_{\mathcal{J}})^* \big(\varphi(a \otimes \square_{\mathcal{J}}) - a \otimes A_{\mathcal{J}}\big)(I_{\mathfrak{h}} \otimes B_{\mathcal{J}}).$$

Applying identities (6.2)–(6.4) to our formula for φ then gives

$$\phi_{\mathcal{J}}(a) = V_{\mathcal{J}} \pi(a) V_{\mathcal{J}}^* + R_{\mathcal{J}} a E^{\widehat{0}} + E_{\widehat{0}} a R_{\mathcal{J}}^* - a \otimes \Delta_{\mathcal{J}}$$

for the operators

$$V_{\mathcal{J}} = \alpha_{\mathcal{J},\mathsf{K}}(T) \in B(\mathsf{K}; \mathfrak{h} \otimes \widehat{\mathsf{k}}) \quad \text{and} \quad R_{\mathcal{J}} = \alpha_{\mathcal{J},\mathfrak{h}}(N + \tfrac{1}{2} I_{\mathfrak{h}} \otimes \Delta) \in B(\mathfrak{h}) \overline{\otimes} |\widehat{\mathsf{k}}\rangle.$$

Property (1) of the transformations $\alpha_{\mathcal{J},\mathsf{H}}$ implies that $R_{\mathcal{J}} \in \mathsf{A}'' \overline{\otimes} |\widehat{\mathsf{k}}\rangle$ for $\mathcal{J}_0 \subset\subset \mathcal{I}_0$. Property (2) entails the following compatibility: for $\mathcal{J}_0 \subset \mathcal{L}_0 \subset\subset \mathcal{I}_0$,

$$V_{\mathcal{J}} = (I_{\mathfrak{h}} \otimes Q_{\mathcal{J}}) V_{\mathcal{L}}, \quad R_{\mathcal{J}} = (I_{\mathfrak{h}} \otimes Q_{\mathcal{J}}) R_{\mathcal{L}} \quad \text{and so}$$
$$\phi_{\mathcal{J}} = (I_{\mathfrak{h}} \otimes Q_{\mathcal{J}}) \phi_{\mathcal{L}}(\,\cdot\,)(I_{\mathfrak{h}} \otimes Q_{\mathcal{J}}).$$

Up to this point we have used the complete positivity of k and local boundedness of its cb-norms, but not its contractivity in any explicit way. Now, by Theorem 5.1 and Proposition 5.2,

$\phi_{\mathcal{J}}(1_{\mathsf{A}}) \leqslant 0$ for each $\mathcal{J}_0 \subset\subset \mathcal{I}_0$, in other words

$$V_{\mathcal{J}} V_{\mathcal{J}}^* \leqslant -R_{\mathcal{J}} E^{\widehat{0}} - E_{\widehat{0}} R_{\mathcal{J}}^* + I_{\mathfrak{h}} \otimes \Delta_{\mathcal{J}}.$$

Thus, in terms of the block decompositions

$$V_{\mathcal{J}} = \begin{bmatrix} v^0 \\ V^1_{\mathcal{J}_0} \end{bmatrix} \quad \text{and} \quad R_{\mathcal{J}} = \begin{bmatrix} r^0 \\ R^1_{\mathcal{J}_0} \end{bmatrix},$$

we have the inequalities

$$0 \leqslant \begin{bmatrix} v^0 v^{0*} & v^0 {V^1_{\mathcal{J}_0}}^* \\ V^1_{\mathcal{J}_0} v^{0*} & V^1_{\mathcal{J}_0} {V^1_{\mathcal{J}_0}}^* \end{bmatrix} \leqslant \begin{bmatrix} -r^0 - r^{0*} & -{R^1_{\mathcal{J}_0}}^* \\ -R^1_{\mathcal{J}_0} & Q_{\mathcal{J}_0} \end{bmatrix}.$$

Since $\|Q_{\mathcal{J}_0}\| \leqslant 1$, it follows that for each $\mathcal{J}_0 \subset\subset \mathcal{I}_0$ we have $\|V^1_{\mathcal{J}_0}\| \leqslant 1$ and $\|R^1_{\mathcal{J}_0}\| \leqslant \|r^0 + r^{0*}\|^{1/2}$. Compatibility now implies that the nets $(V^1_{\mathcal{J}_0})_{\mathcal{J}_0 \subset\subset \mathcal{I}_0}$ and $(R^1_{\mathcal{J}_0})_{\mathcal{J}_0 \subset\subset \mathcal{I}_0}$ converge in the strong operator topology to column operators $V^1 \in B(\mathfrak{h}; \mathsf{K}) \overline{\otimes} |\mathsf{k}\rangle$ and $R^1 \in \mathsf{A}'' \overline{\otimes} |\mathsf{k}\rangle$ respectively, with $V^1_{\mathcal{J}_0}$ being the $\mathcal{J}_0$-truncation of V^1, and similarly for $R^1_{\mathcal{J}_0}$. Therefore, in terms of the operators

$$V := \begin{bmatrix} v^0 \\ V^1 \end{bmatrix} \quad \text{and} \quad R := \begin{bmatrix} r^0 \\ R^1 \end{bmatrix},$$

$\phi_{\mathcal{J}}$ is the $\mathcal{J}$-truncation of the map $\phi := \chi + \psi_R$, where χ is the completely positive map $a \mapsto V\pi(a)V^*$.

Now ϕ is completely bounded, and is the pointwise limit of the net $(\phi_{\mathcal{J}})_{\mathcal{J}_0 \subset\subset \mathcal{I}_0}$ in the weak operator topology. Since $E^{e_\alpha} \phi(a) E_{e_\beta} = E^{e_\alpha} \phi_{\mathcal{J}}(a) E_{e_\beta}$ as soon as $\mathcal{J}$ contains α and β, ϕ is $\mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$-valued. Moreover

$$R = \text{w.o.-}\lim R_{\mathcal{J}} \in \mathsf{A}'' \overline{\otimes} |\widehat{\mathsf{k}}\rangle.$$

Theorems 4.2 and 4.3 therefore imply that $k = k^\phi$. Finally, since

$$\phi(1_{\mathsf{A}}) = \text{w.o.-}\lim \phi_{\mathcal{J}}(1_{\mathsf{A}}) \leqslant 0,$$

$\phi \in \mathfrak{cpc}(\mathsf{A}, \mathsf{k})$, as required. □

We now turn to the case of *nonunital* C^*-algebras. Let A be such an algebra acting nondegenerately on $\mathfrak{h}$, and denote its unitisation by

$$\mathsf{A}^{\mathrm{u}} = \{a + \lambda I_{\mathfrak{h}} : a \in \mathsf{A}, \lambda \in \mathbb{C}\}.$$

For each $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ and $\beta \in \mathbb{R}$ define an extension by

$$\phi_\beta : \mathsf{A}^{\mathrm{u}} \to \mathsf{A}^{\mathrm{u}} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}), \qquad a + \lambda I_{\mathfrak{h}} \mapsto \phi(a) + \beta\lambda\Delta^\perp,$$

and set

$$\begin{aligned} \mathfrak{cpqc}_\beta(\mathsf{A}, \mathsf{k}) &:= \big\{\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})) : \phi_\beta \in \mathfrak{cpqc}_\beta(\mathsf{A}^{\mathrm{u}}, \mathsf{k})\big\}, \\ \mathfrak{cpc}(\mathsf{A}, \mathsf{k}) &:= \mathfrak{cpqc}_0(\mathsf{A}, \mathsf{k}), \quad \text{and} \\ \mathfrak{cpqc}(\mathsf{A}, \mathsf{k}) &:= \bigcup_{\beta \in \mathbb{R}} \mathfrak{cpqc}_\beta(\mathsf{A}, \mathsf{k}). \end{aligned}$$

Thus $\phi \in \mathfrak{cpqc}_\beta(\mathsf{A}, \mathsf{k})$ if $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ and $\chi_{\phi_\beta, R} \in CB(\mathsf{A}^{\mathrm{u}}; B(\mathfrak{h} \otimes \widehat{\mathsf{k}}))$ is completely positive for some choice of $R \in B(\mathfrak{h}; \mathfrak{h} \otimes \widehat{\mathsf{k}})$.

PROPOSITION 6.3. *Suppose that* A *is nonunital, and let* $k \in \mathrm{QSC}^{\beta}_{\mathrm{cpqc}}(\mathsf{A}, \mathsf{k})$. *Then the prescription*

$$^{\beta}k_t(a + \lambda I_{\mathfrak{h}}) = k_t(a) + e^{\beta t}\lambda I_{\mathfrak{h}\otimes\mathcal{F}}, \quad a \in \mathsf{A}, \lambda \in \mathbb{C}, t \in \mathbb{R},$$

defines a cocycle $^{\beta}k \in \mathrm{QSC}^{\beta}_{\mathrm{cpqc}}(\mathsf{A}^{\mathrm{u}}, \mathsf{k})$. *Moreover, if* k *is elementary then so is* $^{\beta}k$, *and the stochastic generator of* $^{\beta}k$ *(according to* Proposition 6.2*) is of the form* ϕ_β *for a map* $\phi \in \mathfrak{cpqc}_\beta(\mathsf{A}, \mathsf{k})$.

Proof. By the remarks preceding Proposition 6.2, we may suppose without loss that $\beta = 0$. Proposition 2.3, and (the proof of) Theorem 6.7 of [LW$_5$] imply that the given prescription defines a cocycle $^{0}k \in \mathrm{QSC}_{\mathrm{cpu}}(\mathsf{A}^{\mathrm{u}}, \mathsf{k})$. For $x, y \in \mathsf{k}$, the (x, y)-associated semigroup of the QS cocycles ^{0}k and k are related by

$$^{0}\mathcal{P}^{x,y}_t(a + \lambda I_{\mathfrak{h}}) = \mathcal{P}^{x,y}_t(a) + \lambda \exp(-t\chi(x, y))I_{\mathfrak{h}}, \quad a \in \mathsf{A}, \lambda \in \mathbb{C}.$$

Suppose now that k is elementary. Since the map $\mathsf{A}^{\mathrm{u}} \to \mathbb{C}$, $a + \lambda I_{\mathfrak{h}} \mapsto \lambda$ is *-homomorphic and thus contractive, we have $\|a\| \leqslant 2\|a + \lambda I_{\mathfrak{h}}\|$. Thus the semigroup $^{0}\mathcal{P}^{x,y}$ inherits norm continuity from $\mathcal{P}^{x,y}$. Continuity with respect to the cb-norm follows similarly, thus ^{0}k is elementary. Therefore, by Proposition 6.2, $^{0}k = k^{\phi'}$ for some $\phi' \in \mathfrak{cpu}(\mathsf{A}^{\mathrm{u}}, \mathsf{k})$. In view of the invariance principle (Proposition 3.3), ϕ' induces a map $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ by restriction. In particular, $\phi'(a + \lambda I_{\mathfrak{h}}) = \phi(a)$ for all $a \in \mathsf{A}$ and $\lambda \in \mathbb{C}$, so $\phi' = \phi_\beta$ for $\beta = 0$. Thus $\phi \in \mathfrak{cpc}(\mathsf{A}, \mathsf{k})$ and the result follows. □

THEOREM 6.4. *For any* C^**-algebra* A, *the QS generation map* $\Phi_{\mathsf{A},\mathsf{k}}$ *restricts to a bijection*

$$\mathfrak{cpqc}(\mathsf{A}, \mathsf{k}) \to El\text{-}\mathrm{QSC}_{\mathrm{cpqc}}(\mathsf{A}, \mathsf{k}) \tag{6.5}$$

and, for $\phi \in \mathfrak{cpqc}(\mathsf{A}, \mathsf{k})$, *there is some* $R \in \mathsf{A}''\overline{\otimes}\,|\widehat{\mathsf{k}}\rangle$ *for which* $\chi_{\phi,R}$ *is completely positive. In more detail, the following bijections are incorporated in* (6.5):

$$\begin{aligned}
&\mathfrak{cpqc}_\beta(\mathsf{A}, \mathsf{k}) \to El\text{-}\mathrm{QSC}^{\beta}_{\mathrm{cpqc}}(\mathsf{A}, \mathsf{k}) \quad \textit{for each } \beta \in \mathbb{R}, \textit{ in particular,}\\
&\mathfrak{cpc}(\mathsf{A}, \mathsf{k}) \to El\text{-}\mathrm{QSC}_{\mathrm{cpc}}(\mathsf{A}, \mathsf{k}) \quad \textit{and, if } \mathsf{A} \textit{ is unital, also}\\
&\mathfrak{cpu}(\mathsf{A}, \mathsf{k}) \to El\text{-}\mathrm{QSC}_{\mathrm{cpu}}(\mathsf{A}, \mathsf{k}).
\end{aligned}$$

Proof. Courtesy of Propositions 6.2 and 6.3, it only remains to assume that A is nonunital and show that the image of $\mathfrak{cpqc}(\mathsf{A}, \mathsf{k})$ is contained in El-$\mathrm{QSC}_{\mathrm{cpqc}}(\mathsf{A}, \mathsf{k})$. Again by the remark preceding Proposition 6.2, it suffices to verify that the image of $\mathfrak{cpc}(\mathsf{A}, \mathsf{k})$ is contained in El-$\mathrm{QSC}_{\mathrm{cpc}}(\mathsf{A}, \mathsf{k})$. Let $\phi \in \mathfrak{cpc}(\mathsf{A}, \mathsf{k})$, and set $k' := k^{\phi'}$ where $\phi' := \phi_\beta$ for $\beta = 0$. Thus $k' \in El$-$\mathrm{QSC}_{\mathrm{cpu}}(\mathsf{A}^{\mathrm{u}}, \mathsf{k})$, by Proposition 6.2. Since ϕ' extends ϕ, Proposition 3.3 implies that k' restricts to a cocycle $k \in \mathrm{QSC}_{\mathrm{cpc}}(\mathsf{A}, \mathsf{k})$. By uniqueness of weak solutions for the QS differential equation (3.3), it follows that $k = k^{\phi}$, in particular the elementary QS cocycle k^{ϕ} is completely positive and contractive. Since $(\mathsf{A}^{\mathrm{u}})'' = \mathsf{A}''$, the possibility of choosing R to be of the desired form follows from Proposition 6.2. □

REMARKS. Theorem 6.4 completes the quantum stochastic extension of the Christensen–Evans characterisation of the generators of norm-continuous, completely positive semigroups on C^*-algebras ([LiP], [LW$_2$]). This is so because norm-continuous semigroups are automatically quasicontractive.

It is currently not known whether or not completely positive elementary QS cocycles are automatically quasicontractive.

*-homomorphic elementary QS cocycles. We now incorporate the known characterisation of stochastic generators of *weakly multiplicative* elementary QS cocycles into the conclusions of Theorem 6.4. Thus let $\mathrm{QSC}_{\mathrm{mult}}(\mathsf{A},\mathsf{k})$ denote the set of k in $\mathrm{QSC}(\mathsf{A},\mathsf{k})$ that satisfy

$$\operatorname{Dom} k_t(a)^* \supset \mathfrak{h}\underline{\otimes}\,\mathcal{E} \quad \text{and} \quad \big(k_t(a)^*|_{\mathfrak{h}\underline{\otimes}\mathcal{E}}\big)^* k_t(b) = k_t(ab), \qquad t \in \mathbb{R}_+,\ a,b \in \mathsf{A}$$

and, recalling the notations (2.5) and (2.6), set

$$\begin{aligned}&\mathrm{QSC}_{^*\text{-hom}}(\mathsf{A},\mathsf{k}) := \mathrm{QSC}_{\mathrm{mult}}(\mathsf{A},\mathsf{k}) \cap \mathrm{QSC}_{\mathrm{h}}(\mathsf{A},\mathsf{k}) \quad \text{and, when } \mathsf{A} \text{ is unital,}\\ &\mathrm{QSF}(\mathsf{A},\mathsf{k}) := \mathrm{QSC}_{^*\text{-hom}}(\mathsf{A},\mathsf{k}) \cap \mathrm{QSC}_{\mathrm{u}}(\mathsf{A},\mathsf{k}),\end{aligned}$$

referred to as the **-homomorphic quantum stochastic cocycles*, respectively *quantum stochastic flows* (on A with respect to k).

The following known automatic continuity result is relevant.

Lemma 6.5. *Let $\alpha \in L\big(\mathsf{A};\mathcal{O}(\mathcal{D};\mathsf{H})\big)$ where $\mathcal{D}$ is a dense subspace of a Hilbert space H and $\mathcal{O}(\mathcal{D};\mathsf{H})$ denotes the space of operators on H with domain $\mathcal{D}$. Suppose that α is hermitian and weakly multiplicative, equivalently, α satisfies*

$$\alpha(a)^* \supset \alpha(a^*) \quad \textit{and} \quad \alpha(a^*)^*\alpha(b) = \alpha(ab), \qquad a,b \in \mathsf{A}.$$

Then α is bounded operator valued. Moreover the map $\mathsf{A} \to B(\mathsf{H})$, $a \mapsto \overline{\alpha(a)} = \alpha(a)^{**}$, is a *-algebra morphism.

Proof. In case A is not unital, the prescription $a + \lambda 1 \mapsto \alpha(a) + \lambda I_{\mathsf{H}}$ defines a linear extension of α to the unitisation of A and it is readily verified that the extension is also hermitian and weakly multiplicative. We may therefore suppose without loss of generality that A is unital.

For $a \in \mathsf{A}$ and $\zeta \in \mathcal{D}$ with $\|a\| \leqslant 1$, $\|\alpha(1)\zeta\|^2 = \langle \zeta, \alpha(1)\zeta\rangle = \langle \zeta, \alpha(a^*a)\zeta\rangle + \langle \zeta, \alpha(1-a^*a)\zeta\rangle = \|\alpha(a)\zeta\|^2 + \|\alpha((1-a^*a)^{1/2})\zeta\|^2$, so $\|\alpha(a)\zeta\|^2 \leqslant \langle \zeta, \alpha(1)\zeta\rangle$. Taking $a = 1$ this shows that the operator $\alpha(1)$ is bounded with norm at most one; feeding this back then gives $\|\alpha(a)\zeta\|^2 \leqslant \|\zeta\|^2$ for such a and ζ, so $\alpha(a)$ is likewise bounded. It follows that α is bounded operator valued. The second part is straightforward to verify. □

It follows from Lemma 6.5 that

$$\mathrm{QSC}_{^*\text{-hom}}(\mathsf{A},\mathsf{k}) = \big\{k \in \mathrm{QSC}_{\mathrm{b}}(\mathsf{A},\mathsf{k}) : \overline{k_t(\cdot)} \text{ is a } {}^*\text{-algebra morphism for all } t \in \mathbb{R}_+\big\},$$

in particular,

$$\mathrm{QSC}_{^*\text{-hom}}(\mathsf{A},\mathsf{k}) \subset \mathrm{QSC}_{\mathrm{cpc}}(\mathsf{A},\mathsf{k}). \tag{6.6}$$

Now let $\mathfrak{mult}(\mathsf{A},\mathsf{k})$ denote the collection of maps $\phi \in CB(\mathsf{A};\mathsf{A}\otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ which satisfy the following higher-order structure relations:

$$\phi_n(ab) = \sum \phi_{\#\lambda}(a)(\lambda;n)\Delta[\lambda\cap\mu;n]\phi_{\#\lambda}(b)(\mu;n), \qquad n \in \mathbb{N}, a,b \in \mathsf{A}, \tag{6.7}$$

in which the sum is over all pairs of subsets (λ,μ) of $\{1,\cdots,n\}$ whose union is $\{1,\cdots,n\}$. The notation here is as follows: for $k \in \mathbb{N}$, $\phi_k := \phi^{(k)} \circ \cdots \circ \phi^{(1)}$ where $\phi^{(i)} := \phi^{\mathsf{H}}$ for $\mathsf{H} = \widehat{\mathsf{k}}^{\otimes(i-1)}$ (see (1.1)) and, for a subset λ of $\{1,\cdots,n\}$ with cardinality j, $\phi_j(a)(\lambda;n)$ denotes the ampliation of $\phi_j(a)$ to $B(\mathfrak{h}\otimes\widehat{\mathsf{k}}^{\otimes n})$ acting as the identity on each copy of $\widehat{\mathsf{k}}$ labelled by an index from the set $\{1,\cdots,n\}\setminus\lambda$ (see [LW$_3$], for a fuller explanation). Set *-$\mathfrak{hom}(\mathsf{A},\mathsf{k}) := \mathfrak{mult}(\mathsf{A},\mathsf{k}) \cap \mathfrak{real}(\mathsf{A},\mathsf{k})$ and, when A is unital, $\mathfrak{flow}(\mathsf{A},\mathsf{k}) := {}^*$-$\mathfrak{hom}(\mathsf{A},\mathsf{k}) \cap \mathfrak{u}(\mathsf{A},\mathsf{k})$ (recall (3.7) and (3.8)).

THEOREM 6.6. *The QS generation map* $\Phi_{\mathsf{A},\mathsf{k}}$ *restricts to bijections*

$$\begin{aligned} &{}^*\text{-}\mathfrak{hom}(\mathsf{A},\mathsf{k}) \to El\text{-QSC}_{*\text{-hom}}(\mathsf{A},\mathsf{k}) \quad \textit{and, when } \mathsf{A} \textit{ is unital,} \\ &\mathfrak{flow}(\mathsf{A},\mathsf{k}) \to El\text{-QSF}(\mathsf{A},\mathsf{k}). \end{aligned}$$

Proof. Write Φ for $\Phi_{\mathsf{A},\mathsf{k}}$.

(a) Let $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$. By Theorem 3.4 of [LW$_3$], k^ϕ is weakly multiplicative if and only if $\phi \in \mathfrak{mult}(\mathsf{A},\mathsf{k})$. (From the fully coordinate-free perspective adopted here, the proof given there is valid without the stated separability assumption on the noise dimension space.)

(b) By the remark following Theorem 3.2, $\Phi(\mathfrak{real}(\mathsf{A},\mathsf{k})) \subset \mathrm{QSC}_{\mathrm{h}}(\mathsf{A},\mathsf{k})$, and $\Phi(\mathfrak{u}(\mathsf{A},\mathsf{k})) \subset \mathrm{QSC}_{\mathrm{u}}(\mathsf{A},\mathsf{k})$ when A is unital.

(c) If $k \in El\text{-QSC}_{*\text{-hom}}(\mathsf{A},\mathsf{k})$ then, by Theorem 6.4 and the inclusion (6.6), $k \in \operatorname{Ran}\Phi$ and so, by (a) and (b), $k = k^\phi$ where $\phi \in \mathfrak{mult}(\mathsf{A},\mathsf{k}) \cap \mathfrak{real}(\mathsf{A},\mathsf{k}) = {}^*\text{-}\mathfrak{hom}(\mathsf{A},\mathsf{k})$.

(d) If A is unital and $k \in El\text{-QSF}(\mathsf{A},\mathsf{k})$ then, by (c) and (b), $k = k^\phi$ where $\phi \in {}^*\text{-}\mathfrak{hom}(\mathsf{A},\mathsf{k}) \cap \mathfrak{u}(\mathsf{A},\mathsf{k}) = \mathfrak{flow}(\mathsf{A},\mathsf{k})$.

From (a) and (b) it follows that $\Phi({}^*\text{-}\mathfrak{hom}(\mathsf{A},\mathsf{k})) \subset El\text{-QSC}_{*\text{-hom}}(\mathsf{A},\mathsf{k})$ and, when A is unital, $\Phi(\mathfrak{flow}(\mathsf{A},\mathsf{k})) \subset El\text{-QSF}(\mathsf{A},\mathsf{k})$; from (c) and (d) it follows that these inclusions are equalities. The result therefore follows from the injectivity of Φ. □

We end this section by relating the above to the next section. Set

$$\begin{aligned} &{}^*\text{-}\mathfrak{hom}_1(\mathsf{A},\mathsf{k}) := \\ &\quad \big\{\phi \in L(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})) : \phi(a^*a) = \phi(a)^*\iota(a) + \iota(a)^*\phi(a) + \phi(a)^*\Delta\phi(a) \text{ for all } a \in \mathsf{A}\big\}, \end{aligned}$$

in which $\iota := \iota^{\mathsf{A}}_{\widehat{\mathsf{k}}}$, the ampliation introduced in (1.3), and set

$$\mathfrak{struct}(\mathsf{A},\mathsf{k}) := \big\{\phi \in {}^*\text{-}\mathfrak{hom}_1(\mathsf{A},\mathsf{k}) : \ \phi(1) = 0 \text{ if } \mathsf{A} \text{ is unital}\big\}.$$

REMARKS. The following are equivalent conditions on a map $\phi \in L(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ for it to be in ${}^*\text{-}\mathfrak{hom}_1(\mathsf{A},\mathsf{k})$:

(i) $\phi^\dagger = \phi$ and

$$\phi(ab) = \phi(a)\iota(b) + \iota(a)\phi(b) + \phi(a)\Delta\phi(b), \qquad a, b \in \mathsf{A}. \tag{6.8}$$

(ii) ϕ has block matrix form $\left[\begin{smallmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \pi - \iota \end{smallmatrix}\right]$ where $\iota = \iota^{\mathsf{A}}_{\mathsf{k}}$, π is a *-algebra morphism from A to $\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{k})$, δ is a π-derivation from A to $\mathsf{A} \otimes_{\mathrm{M}} |\mathsf{k}\rangle$ (in other words $\delta(ab) = \delta(a)b + \pi(a)\delta(b)$ for all $a, b \in \mathsf{A}$) and $\mathcal{L} \in L(\mathsf{A})$ is hermitian and satisfies

$$\mathcal{L}(ab) - \mathcal{L}(a)b - a\mathcal{L}(b) = \delta^\dagger(a)\delta(b), \qquad a, b \in \mathsf{A}. \tag{6.9}$$

The equivalence with (i) follows from polarisation and the fact that $\mathsf{A}_{\mathrm{h}} = \mathbb{R}\text{-Lin}\,\mathsf{A}_+$.

In particular, we see that if A is unital then

$$\phi(1) = 0 \text{ if and only if } \pi \text{ is unital.}$$

The condition (6.8) is the case $n = 1$ of (6.7), so ${}^*\text{-}\mathfrak{hom}(\mathsf{A},\mathsf{k}) \subset {}^*\text{-}\mathfrak{hom}_1(\mathsf{A},\mathsf{k})$. Conversely, if $\phi \in {}^*\text{-}\mathfrak{hom}_1(\mathsf{A},\mathsf{k})$ then ϕ is necessarily completely bounded (as shown in the next theorem), and further $\phi \in {}^*\text{-}\mathfrak{hom}(\mathsf{A},\mathsf{k})$ when either of the following two conditions obtain: (α) $\phi(\mathsf{A})E_\zeta \subset \mathsf{A} \otimes |\widehat{\mathsf{k}}\rangle$ for all $\zeta \in \widehat{\mathsf{k}}$, or ($\beta$) A is a von Neumann algebra and ϕ is ultraweakly continuous ([LW$_3$], [LW$_4$]). In particular, ${}^*\text{-}\mathfrak{hom}(\mathsf{A},\mathsf{k}) = {}^*\text{-}\mathfrak{hom}_1(\mathsf{A},\mathsf{k})$ if either k or A is finite dimensional.

Recall the definition of ψ_R from (6.1).

THEOREM 6.7. *Let $\phi \in L(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$. Then the following are equivalent*:

(i) $\phi \in {}^*\text{-}\mathfrak{hom}_1(\mathsf{A}, \mathsf{k})$.

(ii) *ϕ has block matrix form $\begin{bmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \pi - \iota \end{bmatrix}$ where, for some $l \in \mathsf{A}'' \overline{\otimes} |\mathsf{k}\rangle$ and $h \in \mathsf{A}''_{\mathrm{h}}$, $\delta = \delta_{\pi,l}$: $a \mapsto la - \pi(a)l$ and $\mathcal{L} = \mathcal{L}_{\pi,l,h} : a \mapsto l^*\pi(a)l - \frac{1}{2}(l^*la + al^*l) + \mathrm{i}(ha - ah)$ in which π is a *-algebra morphism from A to $\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{k})$ and $\iota = \iota^{\mathsf{A}}_{\mathsf{k}}$.*

(iii) *$\phi = L^*\pi(\cdot)L + \psi_R$ where π is a *-algebra morphism from A to $\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{k})$, $L := \begin{bmatrix} l & -I_{\mathfrak{h}\otimes\mathsf{k}} \end{bmatrix} \in \mathsf{A}'' \overline{\otimes} B(\widehat{\mathsf{k}}; \mathsf{k})$ and $R := \begin{bmatrix} \mathrm{i}h - \frac{1}{2}l^*l \\ l \end{bmatrix} \in \mathsf{A}'' \overline{\otimes} |\widehat{\mathsf{k}}\rangle$, for some $l \in \mathsf{A}'' \overline{\otimes} |\mathsf{k}\rangle$ and $h \in \mathsf{A}''_{\mathrm{h}}$.*

Moreover, when these hold, if A is unital then $\phi(1) = -T^*\pi(1)^\perp T$ where $T := \begin{bmatrix} l & I_{\mathfrak{h}\otimes\mathsf{k}} \end{bmatrix}$.

Proof. (i) $\Longrightarrow$ (ii): Let $\phi \in {}^*\text{-}\mathfrak{hom}_1(\mathsf{A}, \mathsf{k})$, and let $\begin{bmatrix} \mathcal{L} & \delta^\dagger \\ \delta & \pi - \iota \end{bmatrix}$ be its block matrix form. Since δ is a π-derivation and $\operatorname{Ran} \delta \subset \mathsf{A}'' \overline{\otimes} |\mathsf{k}\rangle$, Theorem 2.1 of [ChE] implies that $\delta = \delta_{\pi,l}$ for some $l \in \mathsf{A}'' \overline{\otimes} |\mathsf{k}\rangle$. Set $\mathcal{L}_{\pi,l} : a \mapsto l^*\pi(a)l - \frac{1}{2}(l^*la + al^*l)$. Then $\mathcal{L}_{\pi,l}$ is a hermitian map satisfying $\operatorname{Ran} \mathcal{L}_{\pi,l} \subset \mathsf{A}''$ and $\mathcal{L}_{\pi,l}(ab) - \mathcal{L}_{\pi,l}(a)b - a\mathcal{L}_{\pi,l}(b) = \delta^\dagger(a)\delta(b)$ for all $a, b \in \mathsf{A}$. It follows that $\mathcal{L} - \mathcal{L}_{\pi,l}$ is a hermitian derivation with range in A'' and so, by another application of Theorem 2.1 of [ChE], there is $k \in \mathsf{A}''$ such that $\delta = \delta_k : a \mapsto ka - ak$. Since δ_k is hermitian, $(ka^* - a^*k)^* = ka - ak$ for all $a \in \mathsf{A}$, or $k + k^* \in \mathsf{A}' \cap \mathsf{A}''$. It follows that $\delta_k = \delta_{\mathrm{i}h}$ for a hermitian element h of A''.

(ii) $\Longrightarrow$ (i): Let $l \in \mathsf{A}'' \overline{\otimes} |\mathsf{k}\rangle$ and $h \in (\mathsf{A}'')_{\mathrm{h}}$ be such that ϕ has block matrix form $\begin{bmatrix} \mathcal{L}_{\pi,l,h} & \delta^\dagger_{\pi,l} \\ \delta_{\pi,l} & \pi - \iota \end{bmatrix}$. Then $\delta_{\pi,l}$ is a π-derivation and it is easily verified that $\mathcal{L}_{\pi,l,h}$ is a hermitian map satisfying (6.9), so $\phi \in {}^*\text{-}\mathfrak{hom}_1(\mathsf{A}, \mathsf{k})$ by the first remark above.

The equivalence of (ii) and (iii) is readily verified, as is the given consequence of A being unital, and so the proof is complete. $\square$

REMARKS. The above result was proved in [LW$_1$], under the restriction that A is unital and k is separable.

From (iii) we deduce the second of the following inclusions, which should be seen in the light of the inclusion (6.6) and Theorems 6.4 and 6.6.

COROLLARY 6.8. $\mathfrak{struct}(\mathsf{A}, \mathsf{k}) \subset {}^*\text{-}\mathfrak{hom}_1(\mathsf{A}, \mathsf{k}) \subset \mathfrak{cpc}(\mathsf{A}, \mathsf{k})$.

7. From structure map to completely positive QS cocycle

For this section fix a unital C^*-algebra A acting nondegenerately on $\mathfrak{h}$. Our goal is to identify sufficient conditions on an operator ϕ from A to $\mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$ with dense domain A_0 for Theorem 5.3 to apply and yield a completely positive and unital QS cocycle k on A governed by the QS differential equation

$$\mathrm{d}k_t = k_t \cdot \mathrm{d}\Lambda_\phi(t), \quad k_0 = \iota^{\mathsf{A}}_{\mathcal{F}}. \tag{7.1}$$

Our sufficient conditions on ϕ include the necessary algebraic conditions for k to be a *QS flow on* A, that is a unital and *-homomorphic QS cocycle on A (cf. the results of the previous section in which ϕ was defined on *all* of A).

A *structure map for* (A, k) *with domain* A_0 is an operator ϕ from A to $\mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$ with domain A_0 such that

(S1) A_0 is a dense *-subalgebra of A,
(S2) ϕ satisfies $\phi = \phi^\dagger$ and $\phi(ab) = \phi(a)\iota(b) + \iota(a)\phi(b) + \phi(a)\Delta\phi(b)$, for all $a, b \in \mathsf{A}_0$, where $\iota = \iota^{\mathsf{A}}_{\widehat{\mathsf{k}}}$ in the notation (1.3),
(S3) A_0 contains 1_{A} and satisfies $\phi(1_{\mathsf{A}}) = 0$.

Remarks. (i) The notation here is that $\phi^\dagger$ is the map on A given by $a \mapsto \phi(a^*)^*$, thus (S2) includes the condition that ϕ is hermitian.

(ii) If a QS flow on A strongly satisfies (7.1) on a dense *-subalgebra A_0 of A containing 1_{A} then, it follows from the quantum Itô product formula that ϕ is necessarily a structure map.

(iii) If a QS flow j on A is elementary then $j = k^\phi$ for a completely bounded structure map ϕ (with domain A) by Theorem 5.10 of [LW$_2$]. Conversely, necessary and sufficient conditions for a completely bounded map ϕ to stochastically generate a QS flow are given in [LW$_3$]. In favourable cases (notably, if $\phi(\mathsf{A})E_\xi \subset \mathsf{A} \otimes |\widehat{\mathsf{k}}\rangle$ for each $\xi \in \widehat{\mathsf{k}}$, or if A is a von Neumann algebra and ϕ is ultraweakly continuous) these conditions reduce to ϕ simply being a structure map.

Lemma 7.1. *Let $\mu = (\mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \omega) \circ \phi$ where ϕ is a structure map for (A, k) with domain A_0 and $\omega \in B(\widehat{\mathsf{k}})_{*,+}$. Then μ is hermitian, vanishes at 1_{A}, and satisfies*

$$\mu(a^*a) - \mu(a)^*a - a^*\mu(a) \geqslant 0, \qquad a \in \mathsf{A}_0.$$

Proof. Being a composition of hermitian maps, μ is hermitian. Since ϕ vanishes at 1_{A}, μ does too. Let $a \in \mathsf{A}_0$, then

$$\begin{aligned}\mu(a^*a) &= (\mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \omega)\big(\phi(a)^*\iota(a) + \iota(a)^*\phi(a) + \phi(a)^*\Delta\phi(a)\big)\\ &= \mu(a)^*a + a^*\mu(a) + (\mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \omega)\big(\phi(a)^*\Delta\phi(a)\big).\end{aligned}$$

The result follows since $\mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \omega$ is a positive map. □

Remark. It follows that the map μ is conditionally positive: if $a \in \mathsf{A}_0$ and $b \in \mathsf{A}$ are such that $ab = 0$ then $b^*\mu(a^*a)b \geqslant 0$. This is a necessary condition for μ to be the generator of a positive semigroup on A. It is also a sufficient condition if μ is bounded with domain A.

If k is a QS flow then for any map $\Gamma : \mathcal{I} \to \mathsf{k}$ it follows from Proposition 2.1 that k^Γ is a cocycle on (the operator system) $\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{h})$, where $\mathsf{h} = l^2(\mathcal{I})$, and that it is multiplicative when restricted to the *-subalgebra $\mathsf{A} \otimes B(\mathsf{h})$. Thus if $k = k^\phi$ for some $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ and Γ is bounded, then the map Φ_Γ from Theorem 4.3 ought to be a structure map.

We want to establish this directly when starting with an unbounded structure map, and to that end we must extend the use of the notation introduced in (1.1). Let ν be an operator from A to $\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{k}_1; \mathsf{k}_2)$ with domain A_0, and let h be a Hilbert space. Then denote by ν^{h} the operator from $\mathsf{A} \otimes B(\mathsf{h})$ to $(\mathsf{A} \otimes B(\mathsf{h})) \otimes_{\mathrm{M}} B(\mathsf{k}_1; \mathsf{k}_2)$ with domain $\mathsf{A}_0 \underline{\otimes} B(\mathsf{h})$ determined by

$$\nu^{\mathsf{h}}(a \otimes T) := \Pi(\upsilon(a) \otimes T)$$

where Π is the tensor flip from $B(\mathfrak{h} \otimes \mathsf{k}_1 \otimes \mathsf{h}; \mathfrak{h} \otimes \mathsf{k}_2 \otimes \mathsf{h})$ to $B(\mathfrak{h} \otimes \mathsf{h} \otimes \mathsf{h}_1; \mathfrak{h} \otimes \mathsf{h} \otimes \mathsf{k}_2)$. When ϕ is completely bounded, the resulting operator is simply a restriction of that defined in (1.1).

Proposition 7.2. *Let ϕ be a structure map for (A, k) with domain A_0. Then the following hold.*

(a) *Given a Hilbert space H, ϕ^{H} is a structure map for $(\mathsf{A} \otimes B(\mathsf{H}), \mathsf{k})$ with domain $\mathsf{A}_0 \underline{\otimes} B(\mathsf{H})$.*
(b) *Let $\Gamma : \mathcal{I} \to \mathsf{k}$ be a bounded map and set $\mathsf{h} := l^2(\mathcal{I})$. Define Φ_Γ to be the operator from $\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{h})$ to $(\mathsf{A} \otimes_{\mathrm{M}} B(\mathsf{h})) \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}})$ with domain $\mathsf{A}_0 \underline{\otimes} B(\mathsf{h})$ given by*

$$\Phi_\Gamma = (\mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \upsilon_\Gamma) \circ \phi^{\mathsf{h}} + \mathrm{id}_{\mathsf{A}} \otimes_{\mathrm{M}} \Theta_\Gamma$$

for υ_Γ and Θ_Γ as in Theorem 4.2. *Then Φ_Γ is a structure map.*

Proof. (a) That ϕ^{H} is hermitian and satisfies $\phi^{\mathsf{H}}(1_{\mathsf{A}\otimes B(\mathsf{H})}) = 0$ is clear, moreover the identity

$$\phi^{\mathsf{H}}(AB) = \phi^{\mathsf{H}}(A)\iota(B) + \iota(A)\phi^{\mathsf{H}}(B) + \phi^{\mathsf{H}}(A)\Delta\phi^{\mathsf{H}}(B),$$

is readily verified for simple tensors A and B in $\mathsf{A}_0 \underline{\otimes} B(\mathsf{H})$. Part (a) therefore follows by bilinearity.

(b) For ease of reading we suppress all subscripts and ampliations for the rest of the proof. By definition, $\Phi = \Omega + \Theta$ where, for $A \in \mathsf{A}_0 \underline{\otimes} B(\mathsf{h})$,

$$\Omega(A) = \widetilde{F}^*\phi^{\mathsf{h}}(A)\widetilde{F} \quad \text{and} \quad \Theta(A) = F^*\iota(A) + \iota(A)F + F^*(A \otimes \Delta)F, \quad \text{for}$$

$$F := \begin{bmatrix} -\frac{1}{2}L^*L & -L^* \\ L & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{F} := I + \Delta F = \begin{bmatrix} I & 0 \\ L & I \end{bmatrix},$$

in which L is the $\mathfrak{h}$-ampliation of the operator in $B(\mathsf{h}; \mathsf{h} \otimes \mathsf{k})$ defined as in Lemma 4.1, so that

$$\Theta(A) = \begin{bmatrix} L^*(A \otimes I)L - \frac{1}{2}L^*LA - \frac{1}{2}AL^*L & L^*(A \otimes I) - AL^* \\ (A \otimes I)L - LA & 0 \end{bmatrix}.$$

Note that $\widetilde{F}\Delta = \Delta = \Delta\widetilde{F}^*$ and so

$$\iota(A)\widetilde{F} - \widetilde{F}\iota(A) = \begin{bmatrix} 0 & 0 \\ (A \otimes I)L - LA & 0 \end{bmatrix} = \Delta\Theta(A) = \widetilde{F}\Delta\Theta(A).$$

Since ϕ^{h} is a structure map this implies that, for $A \in \mathsf{A}_0 \underline{\otimes} B(\mathsf{h})$,

$$\begin{aligned}
&\Omega(A^*A) - \Omega(A^*)\iota(A) - \iota(A)^*\Omega(A) \\
&\quad = \widetilde{F}^*\phi^{\mathsf{h}}(A)^*\big(\iota(A)\widetilde{F} - \widetilde{F}\iota(A)\big) + \big(\widetilde{F}^*\iota(A)^* - \iota(A)^*\widetilde{F}^*\big)\phi^{\mathsf{h}}(A)\widetilde{F} + \widetilde{F}^*\phi^{\mathsf{h}}(A)^*\Delta\phi^{\mathsf{h}}(A)\widetilde{F} \\
&\quad = \widetilde{F}^*\phi^{\mathsf{h}}(A)^*\widetilde{F}\Delta\Theta(A) + \Theta(A)^*\Delta\widetilde{F}^*\phi^{\mathsf{h}}(A)\widetilde{F} + \widetilde{F}^*\phi^{\mathsf{h}}(A)^*\widetilde{F}\Delta\widetilde{F}^*\phi^{\mathsf{h}}(A)\widetilde{F} \\
&\quad = \Omega(A)^*\Delta\Theta(A) + \Theta(A)^*\Delta\Omega(A) + \Omega(A)^*\Delta\Omega(A).
\end{aligned}$$

Furthermore Θ is a structure map since $F+F^*+F^*\Delta F = 0 = F+F^*+F\Delta F^*$ (*cf.* Lemma 4.1). Thus, since $\Phi = \Omega + \Theta$, this implies that

$$\Phi(A^*A) - \Phi(A^*)\iota(A) - \iota(A)^*\Phi(A) = \Phi(A)^*\Delta\Phi(A)$$

for $A \in \mathsf{A}_0 \underline{\otimes} B(\mathsf{h})$, as required. That Φ is hermitian and $\Phi(1_{\mathsf{A}\otimes B(\mathsf{h})}) = 0$ are easily verified. $\square$

Now fix an orthonormal basis $\eta = (d_i)_{i \in \mathcal{I}_0}$ for k and recall our convention (2.2). Continuing with the notation introduced before Theorem 5.3, we write $\Phi_{[\mathcal{J}]}$ and $\varphi_{[\mathcal{J}]}$ as special cases of Φ_Γ and φ_Γ. Also recall the $\phi_{x,y}$ notation introduced in (3.2).

Proposition 7.3. *Let ϕ be a structure map for (A, k) with domain A_0. Suppose that the following hold*:

(1) *For each $n \in \mathbb{N}$, $\mathrm{M}_n(\mathsf{A}_0)$ is* square-root closed, *that is*

$$\mathrm{M}_n(\mathsf{A}_0)_+ = \big\{A^2 : A \in \mathrm{M}_n(\mathsf{A}_0)_+\big\}, \quad \textit{where } \mathrm{M}_n(\mathsf{A}_0)_+ := \mathrm{M}_n(\mathsf{A})_+ \cap \mathrm{M}_n(\mathsf{A}_0).$$

(2) *For all $\alpha, \beta \in \mathcal{I}$, ϕ_{d_α, d_β} is a pregenerator of a C_0-semigroup $\mathcal{P}^{(\alpha,\beta)}$ on A.*

Then for all $\mathcal{J}_0 \subset\subset \mathcal{I}_0$ the Schur-action semigroup $\mathcal{P}^{[\mathcal{J}]}$ on $\mathrm{M}_{\mathcal{J}}(\mathsf{A})$ is completely positive and unital.

Proof. Let $\mathcal{J}_0 \subset\subset \mathcal{I}_0$. Define a Schur-action operator on $\mathrm{M}_{\mathcal{J}}(\mathsf{A})$ with domain $\mathrm{M}_{\mathcal{J}}(\mathsf{A}_0)$ by

$$\varphi_{[\mathcal{J}]} := \big[\phi_{d_\alpha,d_\beta}\big]_{\alpha,\beta\in\mathcal{J}}.$$

Then $\varphi_{[\mathcal{J}]}$ is a pregenerator of the C_0-semigroup $\mathcal{P}^{[\mathcal{J}]}$, with $\varphi_{[\mathcal{J}]} = (\mathrm{id}_{\mathrm{M}_{\mathcal{J}}(\mathsf{A})} \otimes_{\mathrm{M}} \omega_{\widehat{0},\widehat{0}}) \circ \Phi_{[\mathcal{J}]}$. However, $\Phi_{[\mathcal{J}]}$ is a structure map on $\mathrm{M}_{\mathcal{J}}(\mathsf{A})$ by Proposition 7.2 (b), and so

$$\varphi_{[\mathcal{J}]}(A^*A) - \varphi_{[\mathcal{J}]}(A^*)A - A^*\varphi_{[\mathcal{J}]}(A) \geqslant 0, \qquad A \in \mathrm{M}_{\mathcal{J}}(\mathsf{A}_0)$$

by Lemma 7.1. It follows from Proposition 3.2.22 of [BR$_1$] that $\varphi_{[\mathcal{J}]}$ is dissipative. Thus its closure $\overline{\varphi_{[\mathcal{J}]}}$ is dissipative by Proposition 3.1.15 of [BR$_1$]. However, $\overline{\varphi_{[\mathcal{J}]}}$ is the generator of the C_0-semigroup $\mathcal{P}^{[\mathcal{J}]}$, so $\mathrm{id}_{\mathrm{M}_n(\mathsf{A})} - \alpha\,\overline{\varphi_{[\mathcal{J}]}}$ is surjective for some $\alpha > 0$ by the Hille–Yosida Theorem ([BR$_1$], Theorem 3.1.10). Since its generator is dissipative, $\mathcal{P}^{[\mathcal{J}]}$ is contractive by the Lumer–Phillips Theorem ([BR$_1$], Theorem 3.1.16). Moreover $\mathcal{P}^{[\mathcal{J}]}$ is unital on the C^*-algebra $\mathrm{M}_{[\mathcal{J}]}(\mathsf{A})$ since $\varphi_{[\mathcal{J}]}$ vanishes at the identity. Putting these two properties together shows that $\mathcal{P}^{[\mathcal{J}]}$ is positive ([Pau], Proposition 2.11).

To get complete positivity of $\mathcal{P}^{[\mathcal{J}]}$, note that for any $n \in \mathbb{N}$ the operator $\phi^{\mathbb{C}^n}$ is a structure map for $(\mathrm{M}_n(\mathsf{A}), \mathsf{k})$ by Proposition 7.2 (a), and that conditions (1) and (2) hold for this map. If $\mathcal{Q}^{[\mathcal{J}]}$ denotes the contraction semigroup on $\mathrm{M}_{\mathcal{J}}\big(\mathrm{M}_n(\mathsf{A})\big)$ obtained by running the argument above for this lifted structure map then $\mathcal{Q}^{[\mathcal{J}]} = \Pi\circ(\mathcal{P}^{[\mathcal{J}]}\otimes_{\mathrm{M}}\mathrm{id}_{\mathrm{M}_n(\mathbb{C})})$ where $\Pi : \mathrm{M}_n\big(\mathrm{M}_{\mathcal{J}}(\mathsf{A})\big) \to \mathrm{M}_{\mathcal{J}}\big(\mathrm{M}_n(\mathsf{A})\big)$ is the natural flip isomorphism. The result follows. $\square$

Combined with Theorem 5.3, Proposition 7.3 yields the following stochastic generation theorem for completely positive and unital QS cocycles.

Theorem 7.4. *Let ϕ be a structure map for (A, k) with domain A_0. Suppose that the following hold*:

(1) *For all $n \in \mathbb{N}$, $\mathrm{M}_n(\mathsf{A}_0)$ is square-root closed.*
(2) *There is an orthonormal basis $(d_i)_{i\in\mathcal{I}_0}$ for k such that, for all $\alpha, \beta \in \mathcal{I} := \{0\} \cup \mathcal{I}_0$, ϕ_{d_α,d_β} is a pregenerator of a C_0-semigroup $\mathcal{P}^{(\alpha,\beta)}$ on A.*

Then there is a unique process $k \in \mathrm{QSC_b}(\mathsf{A}, \mathsf{k})$ which is locally norm bounded and such that, for all $\alpha, \beta \in \mathcal{I}$, $\mathcal{P}^{(\alpha,\beta)}$ is its (d_α, d_β)-associated semigroup. Moreover, k is completely positive and unital, and it is the unique weakly regular $\mathcal{E}_{\mathsf{T}(\eta)}$-weak solution of the QS differential equation (7.1) *on A_0 for the domain $\mathfrak{h}\underline{\otimes}\,\mathcal{E}_{\mathsf{T}(\eta)}$; if both $\mathfrak{h}$ and k are separable then it is a strong solution.*

Remarks. Assumption (1) clearly applies when A_0 is the dense *-subalgebra associated with an AF algebra. In the next section we give such an example in which assumption (2) also holds.

Interesting recent work on the construction of QS flows is nicely complementary to ours, in that it takes as *standing hypothesis* the existence of a completely positive and contractive QS cocycle weakly satisfying (7.1) on a dense *-subalgebra containing the identity of the algebra, for a structure map ϕ (see [DGS], Theorem 3.1 and Definitions 2.5 and 2.7). *Warning*: their use of the terminology QS flow is different to ours.

8. The quantum exclusion process

Symmetric quantum exclusion processes have recently been considered by a number of authors. The original paper to tackle the challenge of extending the classical theory of exclusion processes to the quantum domain was [Reb]. This, like subsequent work, has focused on the underlying Markov semigroup. Subsequently conditions were found for the construction of symmetric quantum exclusion *processes* ([BW$_1$]), using the theory of multiple quantum Wiener integrals developed in [LW$_3$]. In this final section we demonstrate how Theorem 7.4 may be employed in the construction of quantum exclusion processes. Our approach is complementary to that of [BW$_1$] and [BW$_2$].

Fixing a nonempty set $\mathcal{R}$, let $\mathsf{A} = CAR(\mathcal{R})$ denote the CAR algebra over $\mathcal{R}$ in its Fock representation ([BR$_2$]). A useful description arises by putting a total order on $\mathcal{R}$. The antisymmetric Fock space over $l^2(\mathcal{R})$ may be naturally identified with the Hilbert space $\mathfrak{h} := l^2(\Gamma_{\mathcal{R}})$ where $\Gamma_{\mathcal{R}} := \{\sigma \subset \mathcal{R} : \#\sigma < \infty\}$, with the Fock space Fermi annihilation and creation operators given by

$$(b_r F)(\sigma) = \mathbf{1}_{r\notin\sigma}\varepsilon(\sigma, r)F(\sigma \cup r) \quad \text{and} \quad (b_r^* F)(\sigma) = \mathbf{1}_{r\in\sigma}\varepsilon(\sigma, r)F(\sigma \setminus r),$$

in which the singleton set $\{r\}$ is abbreviated to r. The notation here is as follows:

$$\varepsilon(\sigma, r) := (-1)^{n(\sigma,r)} \quad \text{where} \quad n(\sigma, r) := \#\{s \in \sigma : s > r\} \quad \text{for} \quad \sigma \in \Gamma_{\mathcal{R}} \text{ and } r \in \mathcal{R}.$$

The anticommutation relations

$$b_r b_s + b_s b_r = 0, \quad b_r b_s^* + b_s^* b_r = \begin{cases} 1_{\mathsf{A}} & \text{if } r = s, \\ 0 & \text{if } r \neq s, \end{cases}$$

imply that $b_r b_r^* b_r = (1_{\mathsf{A}} - b_r^* b_r)b_r = b_r$ $(r \in \mathcal{R})$, so each b_r is a (nonzero) partial isometry. Let A_0 denote *-Alg$\{b_r : r \in \mathcal{R}\}$. Thus A_0 is a dense *-subalgebra of A containing 1_{A}; it is weak operator dense in $B(\mathfrak{h})$. In terms of the standard basis $(e_\sigma)_{\sigma\in\Gamma_{\mathcal{R}}}$ for $\mathfrak{h}$,

$$b_r e_\sigma = \mathbf{1}_{r\in\sigma}\, \varepsilon(\sigma, r)\, e_{\sigma\setminus r} \quad \text{and} \quad b_r^* e_\sigma = \mathbf{1}_{r\notin\sigma}\, \varepsilon(\sigma, r)\, e_{\sigma\cup r}, \quad \sigma \in \Gamma_{\mathcal{R}}, r \in \mathcal{R}.$$

For $\mathcal{S} \subset \mathcal{R}$, set $\mathsf{A}_{\mathcal{S}} := {}^*$- Alg$\{b_s : s \in \mathcal{S}\}$. Then, by the anticommutation relations, each $\mathsf{A}_{\mathcal{S}}$ is a *-subalgebra of A containing 1_{A} with linear basis $\{b_\sigma^* b_\tau : \sigma, \tau \subset\subset \mathcal{S}\}$, where

$$b_\tau := \overrightarrow{\prod_{t\in\tau}} b_t \quad \text{and} \quad b_\sigma^* := (b_\sigma)^* = \overleftarrow{\prod_{s\in\sigma}} b_s^*,$$

with the convention $b_\emptyset = 1_{\mathsf{A}}$. Thus

$$b_\sigma^*\, e_\emptyset = e_\sigma, \text{ for } \sigma \in \Gamma_{\mathcal{R}}, \quad b_\tau e_\emptyset = 0 \text{ for } \tau \in \Gamma_{\mathcal{R}} \setminus \{\emptyset\},$$

in particular $e_\emptyset$ is a cyclic vector for the C^*-algebra A (it is the Fermi Fock vacuum vector), and

$$\mathsf{A}_0 = \bigcup_{\mathcal{S}\subset\subset\mathcal{R}} \mathsf{A}_{\mathcal{S}}.$$

Moreover, for each $\mathcal{S} \subset\subset \mathcal{R}$, $\mathsf{A}_{\mathcal{S}}$ is finite dimensional and thus a closed subspace of A; it is a C^*-algebra isomorphic to $B\big(l^2(\Gamma_{\mathcal{S}})\big)$. Thus A_0 is square-root closed, as is $\mathrm{M}_n(\mathsf{A}_0)$ for each $n \in \mathbb{N}$. In addition, A is separable and an AF algebra if and only if $\mathcal{R}$ is countable.

The elements of $\mathcal{R}$ are used to label sites at which Fermionic particles may exist, with the operator b_r representing the annihilation of a particle at site r, and b_r^* its creation.

Let $\{\alpha_{r,s} : r, s \in \mathcal{R}\}$ be a fixed collection of (complex) *amplitudes*, and set

$$\mathrm{inter}(r) := \{s \in \mathcal{R} : \alpha_{r,s} \neq 0 \text{ or } \alpha_{s,r} \neq 0\}, \text{ and } \mathrm{inter}^+(r) := \mathrm{inter}(r) \cup \{r\}.$$

Thus inter(r) is the collection of sites that interact with site r; $\#\,\mathrm{inter}(r)$ is termed the *valency of the site* r. We make the finite valency assumption

$$\#\,\mathrm{inter}(r) < \infty \text{ for all } r \in \mathcal{R}. \tag{8.1}$$

The transport of a particle from site r to site s with amplitude $\alpha_{r,s}$ is described by the operator $t_{r,s} := \alpha_{r,s} b_s^* b_r$. Also let $\{h_r : r \in \mathcal{R}\}$ be a fixed set of (real) *site energies.* Define bounded operators $\rho_{r,s}$, $\tau_{r,s}$ and δ_r on A by

$$\begin{aligned}
&\rho_{r,s}(a) := [t_{r,s}, a] = \alpha_{r,s}(b_s^* b_r a - a b_s^* b_r),\\
&\tau_{r,s}(a) := -\frac{1}{2}\big(t_{r,s}^* \rho_{r,s}(a) + \rho_{r,s}^\dagger(a) t_{r,s}\big) = -\frac{1}{2}\big(t_{r,s}^*[t_{r,s}, a] + [a, t_{r,s}^*] t_{r,s}\big) \quad \text{and}\\
&\delta_r(a) := \mathrm{i} h_r [b_r^* b_r, a].
\end{aligned}$$

Thus, for each $r, s \in \mathcal{R}$, $\rho_{r,s}$ and δ_r are derivations. In particular $\delta_r(1_{\mathsf{A}}) = \rho_{r,s}(1_{\mathsf{A}}) = 0$ so also $\tau_{r,s}(1_{\mathsf{A}}) = 0$, and the following identity holds: $\tau_{r,s}(ab) - \tau_{r,s}(a)b - a\tau_{r,s}(b) = \rho_{r,s}^\dagger(a)\rho_{r,s}(b)$ for all $a, b \in \mathsf{A}$. Moreover, each $\tau_{r,s}$ is hermitian, as is each δ_r. Noting that, for $\mathcal{S} \subset\subset \mathcal{R}$,

$$[b_s^* b_r, a] = 0 \quad \text{for all } a \in \mathsf{A}_{\mathcal{S}}, r, s \notin \mathcal{S}, \tag{8.2}$$

we see that, for all $\mathcal{S} \subset\subset \mathcal{R}$ and $a \in \mathsf{A}_{\mathcal{S}}$, $\delta_r(a) = 0$ unless $r \in \mathcal{S}$, in which case $\delta_r(a) \in \mathsf{A}_{\mathcal{S}}$, whereas $\rho_{r,s}(a) = 0$ unless either $r \in \mathcal{S}$ and $s \in \mathrm{inter}(r)$, in which case $\rho_{r,s}(a) \in \mathsf{A}_{\mathcal{S}\cup\mathrm{inter}(r)}$, or $s \in \mathcal{S}$ and $r \in \mathrm{inter}(s)$, in which case $\rho_{r,s}(a) \in \mathsf{A}_{\mathcal{S}\cup\mathrm{inter}(s)}$, and similarly for $\tau_{r,s}(a)$. It follows that, under the finite valency assumption (8.1), for all $\mathcal{S} \subset\subset \mathcal{R}$ and $a \in \mathsf{A}_{\mathcal{S}}$, the sets $\{r \in \mathcal{R} : \delta_r(a) \neq 0\}$, $\{q \in \mathcal{R}\times\mathcal{R} : \rho_q(a) \neq 0\}$ and $\{q \in \mathcal{R}\times\mathcal{R} : \tau_q(a) \neq 0\}$ are all finite. Therefore, setting $\mathsf{k} := l^2(\mathcal{R}\times\mathcal{R})$ and letting $\eta = (f_q)_{q\in\mathcal{R}\times\mathcal{R}}$ be its standard orthonormal basis, there is a well-defined operator $\phi := \left[\begin{smallmatrix} \mathcal{L} & \rho^\dagger \\ \rho & 0 \end{smallmatrix}\right]$ from A to $\mathsf{A} \otimes B(\widehat{\mathsf{k}})$ with domain A_0 given by $\mathcal{L} := \delta + \tau$,

$$\delta(a) := \sum_{r\in\mathcal{R}} \delta_r(a), \quad \tau(a) := \sum_{q\in\mathcal{R}\times\mathcal{R}} \tau_q(a) \quad \text{and} \quad \rho(a) := \sum_{q\in\mathcal{R}\times\mathcal{R}} \rho_q(a) \otimes |f_q\rangle.$$

For $\mathcal{S} \subset\subset \mathcal{R}$, set $\mathcal{S}^+ := \bigcup_{r\in\mathcal{S}} \mathrm{inter}^+(r)$ and $\mathsf{k}_{\mathcal{S}} := \mathrm{Lin}\{f_q : q \in \mathcal{S}\times\mathcal{S}\}$, then

$$\delta(\mathsf{A}_{\mathcal{S}}) \subset \mathsf{A}_{\mathcal{S}}, \qquad \tau(\mathsf{A}_{\mathcal{S}}) \subset \mathsf{A}_{\mathcal{S}^+} \quad \text{and} \quad \rho(\mathsf{A}_{\mathcal{S}}) \subset \mathsf{A}_{\mathcal{S}^+} \underline{\otimes} |\mathsf{k}_{\mathcal{S}^+}\rangle,$$

so that ϕ enjoys the approximate invariance property $\phi(\mathsf{A}_{\mathcal{S}}) \subset \mathsf{A}_{\mathcal{S}^+} \underline{\otimes}\, \mathrm{Lin}\big\{|x\rangle\langle y| : x, y \in \widehat{\mathsf{k}_{\mathcal{S}^+}}\big\}$. In particular $\mathrm{Ran}\,\phi \subset \mathsf{A}_0 \underline{\otimes}\, B_{00}(\widehat{\mathsf{k}})$, where B_{00} denotes bounded finite rank operators. Since τ and δ are hermitian, δ is a derivation, ρ an $\iota_{\mathsf{k}}^{\mathsf{A}}$-derivation, and $\tau(ab) - \tau(a)b - a\tau(b) = \rho^\dagger(a)\rho(b)$ for all $a, b \in \mathsf{A}_0$, it follows that ϕ is a structure map for (A, k) with domain A_0.

THEOREM 8.1. *Let $\mathsf{A} = CAR(\mathcal{R})$, in its Fock representation, for a nonempty set $\mathcal{R}$ and let $\phi := \left[\begin{smallmatrix} \mathcal{L} & \rho^\dagger \\ \rho & 0 \end{smallmatrix}\right]$ be the structure map defined as above, in terms of a set of complex amplitudes $\{\alpha_{r,s} : r, s \in \mathcal{R}\}$ satisfying the finite valency condition* (8.1) *and a set of real site energies $\{h_r : r \in \mathcal{R}\}$. Assume that $\mathcal{L}$ is a pregenerator of a C_0-semigroup on A. Then there is a unique process k^ϕ in $\mathrm{QSC}_{\mathrm{cpu}}(\mathsf{A}, \mathsf{k})$ such that $\phi_{x,y} := (\mathrm{id}_{\mathsf{A}} \otimes \omega_{\widehat{x},\widehat{y}}) \circ \phi - \chi(x, y)\, \mathrm{id}_{\mathsf{A}}$ is a pregenerator of its (x, y)-associated semigroup, for all $x, y \in \{0\} \cup \{f_q : q \in \mathcal{R}\times\mathcal{R}\}$. Moreover, k^ϕ is also the unique weakly regular $\mathcal{E}_{\mathsf{T}(\eta)}$-weak solution of the QS differential equation* (7.1) *on A_0 for the domain $\mathfrak{h} \underline{\otimes}\, \mathcal{E}_{\mathsf{T}(\eta)}$, satisfying the equation strongly if the set $\mathcal{R}$ is countable.*

Proof. By identities (4.5),

$$\phi_{0,f_q} = \mathcal{L} + \rho_q^\dagger - \tfrac{1}{2}\,\mathrm{id}_{\mathsf{A}}, \quad \phi_{f_q,0} = (\phi_{0,f_q})^\dagger \quad \text{and} \quad \phi_{f_q,f_{q'}} = \mathcal{L} + \rho_q + \rho_{q'}^\dagger + (\delta_{q,q'} - 1)\,\mathrm{id}_{\mathsf{A}}$$

for all $q, q' \in \mathcal{R}\times\mathcal{R}$. Therefore, for all $x, y \in \{0\} \cup \{f_q : q \in \mathcal{R}\times\mathcal{R}\}$, $\phi_{x,y}$ is a bounded perturbation of $\mathcal{L}$, and so is a pregenerator of a C_0-semigroup on A. As noted already, $\mathrm{M}_n(\mathsf{A}_0)$

is square-root closed, since any $A \in \mathrm{M}_n(\mathsf{A}_0)_+$ belongs to the C^*-algebra $\mathrm{M}_n(\mathsf{A}_{\mathcal{S}})$ for some $\mathcal{S} \subset\subset \mathcal{R}$. Therefore the result follows from Theorem 7.4. $\square$

REMARKS. If an orthonormal basis η' is chosen for k other than the standard one then the resulting QS cocycle k' is unitarily conjugate to k^ϕ via the second quantisation operator $\Gamma(I_{L^2(\mathbb{R}_+)} \otimes V)$, V being the unitary on k which exchanges the bases.

Strong unital *-homomorphic solutions for the QS differential equation (7.1) (where, as is necessary, ϕ is assumed to be a structure map) are sought in [BW$_1$]. Existence is proved assuming uniform boundedness of the valencies and energies, and modulus-symmetry of the amplitudes: $|\alpha_{s,r}| = |\alpha_{r,s}|$ for all $r, s \in \mathcal{R}$, along with certain coupled conditions. These latter constraints, on magnitudes of amplitudes and sizes of valencies, are forged from growth restrictions, on iterates of ϕ applied to the generators b_r, required for the convergence of sums of relevant multiple quantum Wiener integrals. Under strengthened conditions the solution is shown to be a QS cocycle. In the preprint [BW$_2$] these symmetry and valency restrictions are loosened, and existence theorems for weak completely positive contractive solutions of (7.1) are established by means of Feynman–Kac type perturbations of the *-homomorphic processes constructed in [BW$_1$]. The cocycle property for these is then recovered under conditions allowing the application of Theorem 5.3/7.4, as in the current paper.

In Rebolledo's work, and subsequent study ([PMQ]), the coefficients $\alpha_{r,s}$ are assumed to be real; moreover a large part of the analysis is carried out in the W^*-category, focusing on the minimal quantum dynamical semigroup on $B(\mathfrak{h})$. In that context, our results (like those of [BW$_1$] and [BW$_2$]) deliver Feller properties for the resulting semigroups — that is, invariance of the C^*-algebra $\mathsf{A} = CCR(\mathcal{R})$ and strong continuity there.

We finish with a special case of the quantum exclusion process in which the index set $\mathcal{R}$ is an integer lattice, and where the fact that $\mathcal{L}$ is a pregenerator is established using the following result.

THEOREM 8.2 ([BrK], Theorem 4.2.1). *Let L be a dissipative operator on a Banach space $\mathfrak{X}$. Suppose that $(\mathfrak{X}(n))_{n=1}^\infty$ is an increasing sequence of closed subspaces of $\mathfrak{X}$ contained in* $\operatorname{Dom} L$ *such that*

(a) $\mathfrak{X}_0 := \bigcup_{n=1}^\infty \mathfrak{X}(n)$ *is dense in $\mathfrak{X}$, and*
(b) *there are constants $M \geqslant 0$ and $\alpha > 0$ and, for each $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, an operator $L^{n,m}$ from $\mathfrak{X}(n)$ to $\mathfrak{X}$ such that, for all such n and m,*

$$\operatorname{Ran} L^{n,m} \subset \mathfrak{X}(n+m) \quad \text{and} \quad \|L|_{\mathfrak{X}(n)} - L^{n,m}\| \leqslant Mne^{-\alpha m}.$$

Then L is a pregenerator of a (contractive) C_0-semigroup on $\mathfrak{X}$ and $\mathfrak{X}_0$ is a core for $\overline{L}$.

EXAMPLE 8.3. Let $\mathsf{A} = CAR(\mathcal{R})$ acting on the Hilbert space $l^2(\Gamma_{\mathcal{R}})$, where $\mathcal{R} = \mathbb{Z}^d$ which we view as a metric subspace of $l^\infty(\{1, \cdots, d\})$, and let ϕ be the structure map for (A, k) with domain A_0 defined as above, in terms of a given set of amplitudes and energies. We claim that $\mathcal{L}$ is a pregenerator of a C_0-semigroup on A, so that Theorem 8.1 applies, under the following assumptions.

I There is $D \in \mathbb{N}$ such that $\alpha_{r,s} = 0$ whenever the pair (r, s) satisfies $\|r - s\|_\infty > D$.
II There is $K \in \mathbb{R}_+$ such that, for all $(r, s) \in \mathcal{R} \times \mathcal{R} \setminus \{(0, 0)\}$,

$$|\alpha_{r,s}|^2 \leqslant K\big(\max\{\|r\|_\infty, \|s\|_\infty\}\big)^{2-d}.$$

Assumption I strengthens that of (8.1) to there being a uniform limit on the range of interaction and hence a uniform bound on the valencies. In dimensions $d = 1$ and 2, assumption II covers the physically reasonable situation of uniformly bounded amplitudes. Unlike in [BW$_1$], no symmetry condition is imposed on the amplitudes and no bounds on the energies are required.

To see the validity of our claim, first note that $\mathcal{L}$ is conditionally completely positive and so, since its domain is square-root closed and contains 1_{A}, Proposition 3.2.22 of [BR$_1$] implies that $\mathcal{L}$ is dissipative. Let $n \in \mathbb{N}$ and set $\mathsf{A}(n) := \mathsf{A}_{\mathcal{R}_n}$ where $\mathcal{R}_n := \{r \in \mathcal{R} : \|r\|_\infty \leqslant nD\}$; since $\#\mathcal{R}_n = (1+2nD)^d < \infty$, $\mathsf{A}(n)$ is finite dimensional and thus a closed subspace of A. Also note that $\bigcup_{n=1}^\infty \mathsf{A}(n) = \mathsf{A}_0$ and, by assumption I, $\mathcal{R}_n^+ \subset \mathcal{R}_{n+1}$ so $\mathsf{A}_{\mathcal{R}_n^+} \subset \mathsf{A}(n+1)$ for all $n \in \mathbb{N}$. For $m \in \mathbb{Z}_+$ define the map $\mathcal{L}^{n,m} : \mathsf{A}(n) \to \mathsf{A}$ by

$$\mathcal{L}^{n,m} : \mathsf{A}(n) \to \mathsf{A}, \quad a \mapsto \begin{cases} \delta(a) + \sum_{r,s\in\mathcal{R}_n} \tau_{r,s}(a) & \text{if } m = 0, \\ \mathcal{L}(a) & \text{if } m \geqslant 1. \end{cases}$$

Fix $n \in \mathbb{N}$ and $a \in \mathsf{A}(n)$ such that $\|a\| \leqslant 1$. Since $\mathsf{A}(n+1) \subset \mathsf{A}(n+m)$ for all $m \in \mathbb{N}$, the claim follows by appeal to Theorem 8.2 once we have shown that

$$\mathcal{L}^{n,0}(a) \in \mathsf{A}(n) \quad \text{and} \quad \mathcal{L}(a) \in \mathsf{A}(n+1), \tag{8.3}$$

and have found a constant $M = M(d, D, K)$, independent of n and a, such that

$$\|\mathcal{L}(a) - \mathcal{L}^{n,0}(a)\| \leqslant Mn. \tag{8.4}$$

Moreover the relations (8.3) follow from the facts that $\delta(a) \in \mathsf{A}(n)$, $\tau_{r,s}(a) \in \mathsf{A}(n)$ for $r, s \in \mathcal{R}_n$ and, by assumption I, $\tau_{r,s}(a) \in \mathsf{A}(n+1)$ for all $r, s \in \mathcal{R}$.

Assumption I and the commutation relation (8.2) imply that $\tau_{r,s}(a) = 0$ unless $\|r - s\|_\infty \leqslant D$ and $(r, s) \in (\mathcal{R}_n \times \mathcal{R}_{n+1}) \cup (\mathcal{R}_{n+1} \times \mathcal{R}_n)$. Therefore $\mathcal{L}(a) - \mathcal{L}^{n,0}(a) = \sum_{(r,s)\in\mathcal{S}_n} \tau_{r,s}(a)$ where

$$\begin{aligned} \mathcal{S}_n &:= \big\{(r, s) \in [\mathcal{R}_n \times (\mathcal{R}_{n+1} \setminus \mathcal{R}_n)] \cup [(\mathcal{R}_{n+1} \setminus \mathcal{R}_n) \times \mathcal{R}_n] : \|r - s\|_\infty \leqslant D\big\} \\ &\subset \big\{(r, s) : r \in \mathcal{R}_n \setminus \mathcal{R}_{n-1}, s \in (\mathcal{R}_1 + r)\big\} \cup \big\{(r, s) : s \in \mathcal{R}_n \setminus \mathcal{R}_{n-1}, r \in (\mathcal{R}_1 + s)\big\}. \end{aligned}$$

Now $\#(\mathcal{R}_k + t) = (1 + 2kD)^d$, for $k \in \mathbb{N}$ and $t \in \mathcal{R}$, so

$$\begin{aligned} \#\mathcal{S}_n &\leqslant 2\Big(\big(1 + 2nD\big)^d - \big(1 + 2(n-1)D\big)^d\Big)(1 + 2D)^d \\ &\leqslant 4Dd\big(1 + 2nD\big)^{d-1}(1 + 2D)^d \leqslant 4Dd(1 + 2D)^{2d-1} n^{d-1}. \end{aligned}$$

Moreover, for all $(r, s) \in \mathcal{S}_n$, $\|\tau_{r,s}(a)\| \leqslant 2|\alpha_{r,s}|^2$ and so, by assumption II,

$$\|\tau_{r,s}(a)\| \leqslant 2K((n+1)D)^{2-d} = 2KD^{2-d}\big(1 + \tfrac{1}{n}\big)^{2-d} n^{2-d}.$$

Therefore, since $(1+\frac{1}{n})^{2-d} \leqslant 1+\delta_{1,d}$ (Kronecker delta), these estimates combine to yield (8.4), as required, with $M = 8Kd(1 + \delta_{1,d})D^{3-d}(1 + 2D)^{2d-1}$.

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Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, UK
E-mail address: `j.m.lindsay@lancs.ac.uk`

School of Mathematical Sciences, University College Cork, Cork, Ireland
E-mail address: `s.wills@ucc.ie`