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JOURNAL OF MATHEMATICAL PHYSICS **59**, 103101 (2018)

Resonant interactions of rotational water waves in the equatorial *f*-plane approximation

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We investigate here the resonance phenomenon in periodic unidirectional water waves in flows of constant vorticity governed by the equatorial f-plane approximation. The relevance of such water waves displaying a one dimensional wave vector is also underlined in the paper—in the context of equatorial capillary-gravity water waves—and serves as the basis for the resonance analysis which is carried out by means of dispersion relations for equatorial water waves that were quite recently derived [see the work of Constantin, Differ. Integr. Equations **26**(3-4), 237–252 (2013) and Martin, Nonlinear Anal.: Theory, Methods Appl. **96**, 1–17 (2014)]. We show that, while gravity water waves do not exhibit three-wave resonance, the four-wave resonance occurs irrespective of the vorticity. *Published by AIP Publishing.*
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I. INTRODUCTION

Simulation of water wave profiles is an essential component in several engineering investigations and applications.[27,48,49,54] Time-histories derived from simulations are used for various purposes such as vibration control of offshore platforms and floating wind turbine structures[5,26] or control of wave energy converters for maximizing power output.[43] These applications necessitate that the simulated waveforms satisfy some physical characteristics which are observed from field measurements. Hence, an empirical approach is usually followed in engineering applications by generating the waveforms from the average spectra represented in the frequency domain, e.g., JONSWAP (Joint North Sea Wave Project), satisfying some stochastic criteria.

The use of an empirical approach circumvents the difficulty associated with the generation of wave profiles based on a rigorous theoretical description requiring the satisfaction of the equations of fluid mechanics (e.g., momentum and continuity). Even when a mechanistic based approach is followed for describing wave profiles, mostly a linear and in some cases a higher order nonlinear wave theory is used. However, water waves, in general (described by Euler equations), and, those of large amplitudes, in particular, are known to be highly nonlinear with a nonlinear free surface boundary condition. In addition, the presence of currents makes the flow rotational too. These nonlinear, rotational effects are usually unaccounted for in the applications mentioned earlier. Furthermore, since the wave spectra contain waves of several frequencies (due to their panchromatic nature), the possibility of the presence of nonlinear resonant interactions emerges. The intention of this paper is to investigate these nonlinear interactions in rotational water waves with a view to inform the research on engineering applications so that these effects can be incorporated for relevant applications as necessary.

We are concerned here with two major aspects of the water wave propagation over flows exhibiting constant vorticity and subjected to the Coriolis force. One of them regards the dimensionality of such flows. More precisely, we show that capillary-gravity wave trains governed by the f-plane

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approximation near the equator propagate at the free surface of a water flow of constant non-zero vorticity over a flat bed only if the flow is two dimensional. Our qualitative result enlarges the plethora of the new findings concerning wave-current interactions in the f-plane approximation, direction of research initiated by Constantin[13] (see also Refs. 25, 31, 32, and 34).

The wave trains in flows with constant non-zero vorticity are possible only for two-dimensional flows is a result proven first rigorously by Constantin and Kartashova[8] for capillary waves and by Constantin[11] for gravity flows. The intermediate situation of wave trains for flows driven by gravity and surface tension was clarified in Ref. 40. Studies showing the two-dimensionality of certain water flows with constant non-zero vorticity were also performed by Stuhlmeier[51] and Wahlén.[53]

The scenario of two-dimensional flows is of high significance, since it allows for in-depth studies of wave-current interactions, a phenomenon that cannot be overlooked, given the ubiquity of currents in oceans and seas. Indeed, thorough analytical investigations of rotational two-dimensional flows have been initiated during the last two decades by Constantin and co-authors and concern the existence of solutions to the full nonlinear water wave problem,[6,10,22] the regularity of solutions,[12] the particularities of the flow beneath the surface wave pertaining to particle trajectories[7] or to pressure,[1,2,4,9,19,30,33,52] and, more recently, the incorporation of geophysical effects,[13–17,20,21] the aspect that we will also consider here. The latter aspect is of fundamental importance when analyzing ocean flows, since the effect of the Earth's rotation manifests differently in the two Earth hemispheres.[23,24] The impact of geophysical effects on the development of tsunamis was recently analysed in Ref. 29.

Moreover, high precision numerical simulations for the travelling waves in flows with constant non-zero vorticity over a flat bed presented in Refs. 18 and 42 raise awareness on the significant differences that occur between these flow patterns and the irrotational water-wave propagation; see also the discussions in Refs. 8, 11, and 41.

The other facet that undergoes our investigation (and which becomes easier to be dealt with due to the first aspect concerning dimensionality) is the resonant interaction between two or more waves that combine to build a new one. This process, of resonant interaction, is one of the most fundamental nonlinear phenomena that can happen between waves. It has a profound influence on the waves evolution through the significant energy transfer among the dominant wave trains and provides insights into the effects of weak turbulence, **as emphasized in the studies of Pushkarev and Zakharov[45,46] and in the paper by Pushkarev, Resio and Zakharov[47], for the context of irrotational flows.** A pioneering investigation on three-wave resonances in rotational water waves was performed by Constantin and Kartashova,[8] whose upshot was that rotational capillary waves do exhibit three wave resonances. The latter does not occur in irrotational capillary waves, cf. Ref. 35. The previous work was extended to rotational capillary-gravity water waves in Ref. 40. The resonance problem for geophysical water waves in the equatorial f-plane approximation is considered in Sec. IV of this paper following the discussion in Sec. III which shows the two-dimensionality of capillary-gravity wave trains of constant vorticity in the equatorial f-plane approximation. We show that, while gravity water waves do not exhibit three-wave resonance, the four-wave resonance occurs for arbitrary values of the (constant) vorticity. The latter type of resonance has received constant attention over the last decades.[38,50] For a recent confirmation (of experimental nature) of the existence of four wave resonances among gravity waves, we refer the reader to Ref. 3. A comprehensive account on water wave resonances can be found in Refs. 36 and 37.

II. THE PHYSICS OF THE PROBLEM

This section is concerned with presenting the governing equations for equatorial capillary-gravity water waves arising as the free surface of a rotational water flow of constant vorticity, of finite depth d which is governed by the equatorial f-plane approximation, cf. Ref. 44. We choose a rotating framework with the origin at a point on the Earth's surface, with the x axis pointing horizontally due East, the y-axis horizontally due North, while the z-axis is oriented upward. We will consider here regular wave trains of water waves propagating steadily in the direction of the horizontal x axis, periodic (of period L) in the variable x, and exhibiting no variation in the y direction. The latter assumption does not mean that we consider two dimensional flows; this feature will emerge as a consequence of the analysis performed hereafter.

Consequently, the fluid domain is bounded below by the impermeable flat bed $z = -d$ and above by the free surface $z = \eta(x - ct)$, where the function η gives the wave profile and $c > 0$ is the wave speed. We will also assume that η has mean zero over one period, that is,

$$\int_0^L \eta(s)\, ds = 0. \tag{1}$$

Without loss of generality, we may also assume that the wave crest is located at $x = 0$, which corroborated with (1) yields $\eta(0) > 0$.

The governing equations in the f-plane approximation near the equator are the equation of momentum conservation

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + 2\omega w &= -\tfrac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z &= -\tfrac{1}{\rho}P_y \\ w_t + uw_x + vw_y + ww_z - 2\omega u &= -\tfrac{1}{\rho}P_z - g \end{aligned} \tag{2}$$

and the equation of mass conservation

$$u_x + v_y + w_z = 0. \tag{3}$$

Here (u, v, w) is the velocity field, t is the time variable, P represents the pressure in the fluid, ρ denotes the density, g is the constant acceleration of gravity, and $\omega = 73 \times 10^{-6}$ rad/s is the constant rotational speed of the Earth round the polar axis.

To single out the water wave problem from a vast range of the hydrodynamical ones, we impose the kinematic boundary conditions

$$w = (u - c)\eta_x \quad \text{on} \quad z = \eta(x - ct) \tag{4}$$

and

$$w = 0 \quad \text{on} \quad z = -d, \tag{5}$$

together with the surface boundary condition

$$P = P_{atm} - \sigma \frac{\eta_{xx}}{(1 + \eta_x^2)^{3/2}} \quad \text{on} \quad z = \eta(x - ct). \tag{6}$$

An essential feature of most water flows—especially of those in the equatorial zone of the ocean—is the presence of vorticity $\Omega(x, y, z, t)$, defined as the curl of the velocity field, that is,

$$\Omega = (w_y - v_z, u_z - w_x, v_x - u_y). \tag{7}$$

Taking the curl in (2), we obtain the following evolution equation for the vorticity:

$$\Omega_t + (\mathbf{u} \cdot \nabla)\Omega - 2\omega\big(u_y, v_y, w_y\big) = (\Omega \cdot \nabla)\mathbf{u}. \tag{8}$$

Remark II.1. Throughout the paper, we shall work under the assumption that $\Omega_2 + 2\omega \neq 0$. *This is a reasonable assumption, since* $\omega \approx 0.73 \times 10^{-4}$ *rad/s, while a typical value for* Ω_2 *in the equatorial Pacific is* $25 \times 10^{-3}\ s^{-1}$, *cf. Ref.* 13.

III. THE RELEVANT DIMENSION FOR EQUATORIAL CAPILLARY-GRAVITY WAVE TRAINS

With the considerations of Sec. II in mind, we can formulate the following result on the dimensionality of capillary-gravity wave trains:

Theorem III.1. *Capillary-gravity wave trains propagate at the free surface of a water flow over a flat bed governed by Eqs.* (2)–(6) *exhibiting a constant vorticity only if the flow is two dimensional. More precisely, the velocity field* (u, v, w) *and the pressure* P *are independent of the variable* y, *and* $\Omega_1 = \Omega_3 = 0$.

a. Proof. The assumption of constant vorticity and Eq. (8) provides us with the equation

$$(\Omega \cdot \nabla)\mathbf{u} + 2\omega(u_y, v_y, w_y) = 0. \tag{9}$$

Our intention in the first step is to prove that $\Omega_3 = 0$. We assume for the sake of contradiction that $\Omega_3 \neq 0$. Arguing similarly as in Ref. 11, using (9) and the kinematic boundary condition (5), we obtain that $w = 0$ throughout the fluid domain. Hence, from (7), it follows that the identities

$$u_z = \Omega_2 \quad \text{and} \quad v_z = -\Omega_1 \tag{10}$$

hold within the fluid domain. Integrating in (10) with respect to z, we infer the existence of two functions $\overline{u} = \overline{u}(x, y, t)$ and $\overline{v} = \overline{v}(x, y, t)$ such that

$$\begin{aligned} u(x, y, z, t) &= \overline{u}(x, y, t) + \Omega_2 z, \\ v(x, y, z, t) &= \overline{v}(x, y, t) - \Omega_1 z. \end{aligned} \tag{11}$$

With the help of (3), we see that $\overline{u}, \overline{v}$ satisfy

$$\overline{u}_x + \overline{v}_y = 0,$$

a relation which provides us with a function $\psi(x, y, t)$ satisfying

$$\overline{u} = \psi_y, \quad \overline{v} = -\psi_x. \tag{12}$$

We further deduce that

$$\psi_{xx} + \psi_{yy} = -\Omega_3 \tag{13}$$

and

$$\begin{aligned} \Omega_1 \psi_{xy} + (\Omega_2 + 2\omega)\psi_{yy} + \Omega_2 \Omega_3 = 0, \\ -\Omega_1 \psi_{xx} - (\Omega_2 + 2\omega)\psi_{xy} - \Omega_1 \Omega_3 = 0. \end{aligned} \tag{14}$$

Equations (13) and (14) yield

$$\psi_{xx} = \frac{-\Omega_1^2 \Omega_3 - 2\omega(\Omega_2 + 2\omega)\Omega_3}{\Omega_1^2 + (\Omega_2 + 2\omega)^2}, \tag{15}$$

$$\psi_{xy} = -\frac{\Omega_1 \Omega_2 \Omega_3}{\Omega_1^2 + (\Omega_2 + 2\omega)^2}, \tag{16}$$

$$\psi_{yy} = -\frac{\Omega_2(\Omega_2 + 2\omega)\Omega_3}{\Omega_1^2 + (\Omega_2 + 2\omega)^2}. \tag{17}$$

Therefore, the function ψ has the following structure:

$$\psi(x, y, t) = Ax^2 + Bxy + Cy^2 + a(t)x + b(t)y + \overline{c}(t), \tag{18}$$

where

$$\begin{aligned} A &= \frac{1}{2} \cdot \frac{-\Omega_1^2 \Omega_3 - 2\omega(\Omega_2 + 2\omega)\Omega_3}{\Omega_1^2 + (\Omega_2 + 2\omega)^2}, \\ B &= -\frac{\Omega_1 \Omega_2 \Omega_3}{\Omega_1^2 + (\Omega_2 + 2\omega)^2}, \\ C &= -\frac{1}{2} \cdot \frac{\Omega_2(\Omega_2 + 2\omega)\Omega_3}{\Omega_1^2 + (\Omega_2 + 2\omega)^2}, \end{aligned}$$

and a, b, and $\overline{c}$ are the functions depending only on t. Consequently, using the kinematic boundary condition (4), we obtain that the equation

$$[-c + Bx + 2Cy + b(t) + \Omega_2 \eta(x - ct)]\eta'(x - ct) = 0 \tag{19}$$

is valid for all x, y, and t. Owing to the free surface being non flat, we immediately conclude from the above that $C = 0$. The latter and the assumption from Remark II.1 yield that

$$\Omega_2 = 0, \quad B = 0, \quad \text{and} \quad A = -\frac{\Omega_3}{2}.$$

Referring to Eq. (19), we see that the function b equals identically the constant wave speed c. This renders the velocity field in the form

$$\begin{aligned} u(x, y, z, t) &= c, \\ v(x, y, z, t) &= \Omega_3 x - \Omega_1 z - a(t), \\ w(x, y, z, t) &= 0, \end{aligned} \tag{20}$$

for all x, y, z that belong to the fluid domain at time t. The third equation in (2) yields that the pressure P can be written as

$$P(x,y,z,t) = p(x,y,t) + \rho(2\omega c - g)z, \tag{21}$$

for some function $p(x, y, t)$, which, owing to the first and second equations in (2), satisfies

$$p_x = P_x \equiv 0 \quad \text{and} \quad p_y = P_y = \rho(a'(t) - c\Omega_3).$$

The latter equation and (21) yield the quite explicit formula for the pressure

$$P(x,y,z,t) = \rho(a'(t) - c\Omega_3)y + \rho(2\omega c - g)z + \overline{p}(t), \tag{22}$$

for some function $\overline{p}$. By means of the free surface boundary condition (6), we see that the relation

$$P_{atm} + \rho(g - 2\omega c)\eta(x - ct) - \sigma\frac{\eta_{xx}(x)}{(1+\eta_x^2(x))^{3/2}} = \rho(a'(t) - c\Omega_3)y + \overline{p}(t) \tag{23}$$

holds true for all x, y, t. Thus, $a'(t) - c\Omega_3 = 0$, and

$$\rho(g - 2\omega c)\eta(x - ct) - \sigma\frac{\eta_{xx}(x)}{(1+\eta_x^2(x))^{3/2}} = \overline{p}(t) - P_{atm}, \tag{24}$$

for all x and t. Using the periodicity of the function

$$x \to \frac{\eta_x(x)}{\sqrt{1+\eta_x^2(x)}}$$

and taking into account the zero mean property of η from (1), we obtain upon integration from 0 to L in (24) that

$$\rho(g - 2\omega c)\eta(x) = \sigma\frac{\eta_{xx}(x)}{(1+\eta_x^2(x))^{3/2}} \quad \text{for all} \quad x. \tag{25}$$

We use the continuity of η and conclude from $\eta(0) > 0$ that $\eta(x) > 0$ in a neighborhood centered about the wave crest located at $x = 0$. Using $g - 2\omega c > 0$, we deduce from (25) that $\eta_{xx} > 0$ in a neighborhood of $x = 0$. The latter yields that η is convex in a neighborhood of $x = 0$, which contradicts the maximality of η at the crest. The latter contradiction implies that $\Omega_3 = 0$. Hence,

$$v_x - u_y = 0,$$

which implies the existence of a function $\overline{\varphi}(x, y, z, t)$ with

$$\overline{\varphi}_x = u, \qquad \overline{\varphi}_y = v.$$

In addition, using $\Omega_1 = w_y - v_z$ and $\Omega_2 = u_z - w_x$, we infer the existence of a function $\varphi(x, y, z, t)$ such that

$$w = \varphi_z + \Omega_1 y - \Omega_2 x, \quad u = \varphi_x, \quad \text{and} \quad v = \varphi_y. \tag{26}$$

The equation of mass conservation (3) transforms, by means of (26), to

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0, \tag{27}$$

which, differentiated with respect to z, renders w a harmonic function. This discussion is targeted at showing $\Omega_1 = 0$. We will assume, as before, that $\Omega_1 \neq 0$. This immediately implies that the vectors $(\Omega_1, \Omega_2 + 2\omega, 0)$ and $(0, 1, 0)$—the latter giving the direction of the trough line,

$$\mathcal{L} := \{(x_0, y, \eta(x_0, t_0)) : y \in \mathbb{R}\},$$

located at some position x_0 at some moment t_0—are orthogonal and thus linearly independent. Since $\eta_x(x_0) = 0$, we have from the kinematic boundary condition (4) that $w = 0$ along the line $\mathcal{L}$. Using $\Omega_3 = 0$, we can write the third equation in the vorticity equation (9) as

$$\Omega_1 w_x + (\Omega_2 + 2\omega)w_y = 0, \tag{28}$$

which is a restatement of the fact that w is constant along the vector $(\Omega_1, \Omega_2 + 2\omega, 0)$. The previous considerations now show that $w = 0$ at all points of the plane $z = \eta(x_0, t)$ which is parallel to the flat bed and completely contained in the fluid domain. By means of the Phragmen-Lindelöf maximum

principle, cf. Ref. 28—since w vanishes on $z = -d$ and on $z = \eta(x_0, t)$—we conclude that $w = 0$ in the region of the fluid domain bounded below by the bed and above by the plane $z = \eta(x_0, t)$. Moreover, since w is harmonic, it is also real-analytic at any instant t, cf. Ref. 28. Thus, since $w = 0$ on an open set, it must vanish on the whole fluid domain. A first consequence of the vanishing of w is that we have again a splitting of the velocity components u and v of the kind from (11). Moreover, a similar argument as in the beginning of the proof yields the existence of a function $\psi(x, y, t)$ satisfying

$$\psi_{xx} + \psi_{yy} = 0, \tag{29}$$

and

$$\begin{aligned} \Omega_1\psi_{xy} + (\Omega_2 + 2\omega)\psi_{yy} = 0, \\ -\Omega_1\psi_{xx} - (\Omega_2 + 2\omega)\psi_{xy} = 0, \end{aligned} \tag{30}$$

relations that give

$$\psi_{xx} = \psi_{xy} = \psi_{yy} = 0$$

within the fluid domain. Hence, there are functions $A(t)$, $B(t)$, and $C(t)$ such that

$$\psi(x, y, t) = A(t)x + B(t)y + C(t) \quad \text{for all} \quad x, y, t. \tag{31}$$

The kinematic condition (4) delivers for all x and t the equation

$$[B(t) + \Omega_2\eta(x - ct) - c]\eta'(x - ct) = 0,$$

from which we readily infer that $B(t) \equiv 0$ and $\Omega_2 = 0$. Using the latter two facts and (11), we deduce that

$$u \equiv c \quad \text{and} \quad v = -A(t) - \Omega_1 z. \tag{32}$$

The previous considerations lead to equations similar to those in (24) and (25) that drive us to a contradiction. This contradiction implies now that $\Omega_1 = 0$. We are now ready to prove that the y component of the velocity field is in fact constant. To this end, notice first that the vorticity equation (9) appears like

$$(\Omega_2 + 2\omega)u_y = (\Omega_2 + 2\omega)v_y = (\Omega_2 + 2\omega)w_y = 0, \tag{33}$$

which, in conjunction with Remark II.1, implies that

$$u_y = v_y = w_y = 0.$$

Since

$$\Omega_1 = w_y - v_z = 0, \quad \Omega_3 = v_x - u_y = 0,$$

we also infer [using (33)] that

$$v_x = v_z = 0.$$

We are going to prove that v is also independent of t. To this end, we assume for a moment that v is a function of t. We consider the second equation in (2) and derive that the pressure function satisfies

$$P(x, y, z, t) = -\rho v_t(t)y + f_1(x, z) + f_2(t).$$

From the dynamic boundary condition (6), we infer that the equality

$$-\rho v_t(t)y + f_1(x, \eta(x, t)) + f_2(t) = P_{atm} - \sigma\frac{\eta_{xx}}{(1 + \eta_x^2(x))^{3/2}}$$

holds true for all x, y, and t. Consequently, $v_t(t) = 0$ for all t. From the second equation in (2), we conclude that $P_y = 0$. □

IV. RESONANCES FOR EQUATORIAL GRAVITY WAVES IN THE *f*-PLANE APPROXIMATION

We are concerned in this section with the three-wave resonance problem for rotational flows and incorporating geophysical effects. While the geophysical effects were not considered before, the vorticity was taken into account in the resonance problem also very recently in Refs. 8 and 40 for capillary and capillary-gravity water waves, respectively. Another upshot of the previous studies was

that the number of positive vorticities that trigger a resonance is countable and the size of such a positive constant vorticity is not too small.

We start by recalling from Ref. 39 the dispersion relation for capillary-gravity water waves in the f-plane approximation which propagate at the free surface of a water flow of mean depth $h > 0$, of constant non-vanishing vorticity γ, and of constant density $\rho = 1$ and exhibiting a current of strength u_0 at the bed. It states that the frequency corresponding to the wave number k is

$$\mathfrak{f}(k) := (u_0 + \gamma h)k - \left(\omega + \frac{\gamma}{2}\right)\tanh(kh) + \frac{1}{2}\sqrt{(2\omega+\gamma)^2\tanh^2(kh) + 4[k(g - 2\omega(u_0+\gamma h)) + k^3\sigma]\tanh(kh)}, \tag{34}$$

provided we bear in mind that the preferred East-West direction of propagation of the surface wave in the equatorial regime. In the above formula, g is the gravitational acceleration, $\sigma > 0$ is the coefficient of surface tension, and $k := |\mathbf{k}|$ is the length of the wave vector.

A. Three-wave resonances

Considering gravity water waves, a vanishing underlying bottom current, and realistic ocean depths h [for which $\tanh(kh) \cong 1$], the formula for the frequency we will effectively work with is

$$\mathfrak{f}(k) := \gamma h k - \left(\omega + \frac{\gamma}{2}\right) + \frac{1}{2}\sqrt{(2\omega+\gamma)^2 + 4k(g - 2\omega h\gamma)}. \tag{35}$$

The three-wave resonance problem amounts to showing that the system

$$\begin{aligned} \mathfrak{f}(\mathbf{k_1}) + \mathfrak{f}(\mathbf{k}_2) &= \mathfrak{f}(\mathbf{k}_3), \\ \mathbf{k}_1 + \mathbf{k}_2 &= \mathbf{k}_3 \end{aligned} \tag{36}$$

admits solutions of the dispersion relation (35). Relying on the main result of Sec. IV, we will assume here that the wave vectors we work with are one dimensional, which amounts to setting $\mathbf{k} = k$. Thus, the system (36) becomes equivalent to

$$\begin{aligned} \sqrt{(2\omega+\gamma)^2 + 4k_1(g-2\omega h\gamma)} + \sqrt{(2\omega+\gamma)^2 + 4k_2(g-2\omega h\gamma)} \\ = 2\omega + \gamma + \sqrt{(2\omega+\gamma)^2 + 4k_3(g-2\omega h\gamma)}, \\ k_1 + k_2 = k_3. \end{aligned} \tag{37}$$

By squaring the first relation in (37) and taking into account the second one, we obtain

$$\left[(2\omega+\gamma)^2 + 4k_1(g-2\omega h\gamma)\right]\cdot\left[(2\omega+\gamma)^2 + 4k_2(g-2\omega h\gamma)\right] = (2\omega+\gamma)^2\left[(2\omega+\gamma)^2 + 4k_3(g-2\omega h\gamma)\right],$$

a relation that is equivalent to

$$16k_1k_2(g-2\omega h\gamma)^2 = 0,$$

which does not hold, given the size of the physical quantities g, ω, h, γ. Thus, the three-wave resonances of equatorial gravity water waves in the f-plane approximation are not possible.

B. Four-wave resonances

The four-wave resonance problem amounts to find positive integers k_1, k_2, k_3, k_4 satisfying

$$\begin{aligned} \mathfrak{f}(k_1) + \mathfrak{f}(k_2) &= \mathfrak{f}(k_3) + \mathfrak{f}(k_4), \\ k_1 + k_2 &= k_3 + k_4, \end{aligned} \tag{38}$$

which is equivalent to

$$\begin{aligned} \sqrt{(2\omega+\gamma)^2 + 4k_1(g-2\omega h\gamma)} + \sqrt{(2\omega+\gamma)^2 + 4k_2(g-2\omega h\gamma)} \\ = \sqrt{(2\omega+\gamma)^2 + 4k_3(g-2\omega h\gamma)} + \sqrt{(2\omega+\gamma)^2 + 4k_4(g-2\omega h\gamma)}, \\ k_1 + k_2 = k_3 + k_4. \end{aligned} \tag{39}$$

Squaring the first relation in (39) and using the second one, we find out that the above system is equivalent to the system

$$k_1k_2 = k_3k_4 \quad \text{and} \quad k_1 + k_2 = k_3 + k_4,$$

which has (regardless of the vorticity γ) the tuples (k_1, k_2, k_1, k_2) and (k_1, k_2, k_2, k_1) as the only possible solutions, with k_1, k_2 positive integers.

C. Remark

It is observed that a four wave resonance is feasible for gravity water waves irrespective of vorticity. Hence, this can impact on the panchromatic wave profiles (with multiple frequencies), and this nonlinearity should be accounted for in simulations from the empirical spectra.

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