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Title	Detecting a downward shift in a proportion using a geometric CUSUM chart
Authors	Bourke, Patrick D.
Publication date	2019-12-19
Original Citation	Bourke, P. D. (2020) 'Detecting a downward shift in a proportion using a geometric CUSUM chart, Quality Engineering, 32:1, pp. 75-90, doi: 10.1080/08982112.2019.1664750
Type of publication	Article (peer-reviewed)
Link to publisher's version	https://www.tandfonline.com/doi/citedby/10.1080/08982112.2019.1664750 - 10.1080/08982112.2019.1664750
Rights	© 2018 Informa UK Limited. This is an Accepted Manuscript of an article published by Taylor & Francis in Quality Engineering on 19 December 2019, available online: http://www.tandfonline.com/10.1080/08982112.2019.1664750
Download date	2026-11-06 12:21:30
Item downloaded from	https://hdl.handle.net/10468/9765



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Detecting a Downward Shift in a Proportion

Using a Geometric CUSUM Chart

by

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Version 5: August 2019

ABSTRACT

In monitoring an ordered stream of discrete items from a repetitive process, the geometric CUSUM chart may be used to detect sudden shifts from an acceptable level for a process-proportion (p) such as fraction nonconforming. Much of the investigative effort for this CUSUM scheme has been concentrated on the detection of upward shifts, and a recent paper has provided guidance to quality engineers in choosing the parameters (k, h) of such a scheme. In this article, the corresponding task of aiding the choice of parameters for detecting a downward shift is addressed. It is shown, using extensive numerical investigations, that the use of a value for the parameter k based on the Sequential Probability Ratio analysis is not optimal when one is using steady-state evaluation of the detection performance of the CUSUM scheme. Tables are presented listing recommended values of parameters for detection of five sizes of downward shift, for each of 27 in-control levels for p in the range 0.20 to 0.001. Interpolation and extrapolation to find parameter values for other in-control levels of p are also considered, and a range of examples presented. There is an equivalence between a geometric CUSUM scheme and a Bernoulli CUSUM scheme, so that the results of this investigation may also be used in choosing parameter values for a Bernoulli CUSUM chart.

Key-words: Statistical Process Control; Process Monitoring; Cumulative Sum Scheme; Bernoulli CUSUM; Exponential CUSUM.

1. Introduction

The methodology of statistical process control is often viewed as consisting of two phases: phase 1 and phase 2. In phase 1, the focus is on the identification and elimination of Special Causes of variation, with the aim of getting the process into a state of statistical control. For a process in statistical control, there is still variation in the observed performance of the process, due to a range of less influential Common Causes which are too expensive or impractical to control or eliminate. This article is concerned with the phase 2 methodology of process monitoring, so that we already have information on the established in-control performance of the process.

We consider the monitoring of a stream of discrete items arising from a repetitive process, and we assume that it is feasible to observe items in the order of production, with the observed items being assigned to either of two categories. In industrial quality control, these categories are labelled as conforming and nonconforming, or saleable and non-saleable, and the process measure of interest is (usually) the proportion of items in the less desirable category. It is often the case in manufacturing that there is a focus on the "yield" from the process, yield being that proportion of the initial number of manufactured items that is saleable. In environmental applications, the process measure is quite often taken as the proportion in the more desirable category (e.g. the proportion of air-samples at a central location of a large city that satisfy safety standards).

Our interest here is the detection of downward shifts in a process-proportion. For quality engineers, an example of this is where we hope to demonstrate that adjustments made to the process have resulted in a reduction in the proportion of nonconforming items from the process, this being equivalent to an increase in the yield from the process.

There are several ways of organizing the observational data for monitoring purposes. One could group the items into successive "samples" of some convenient and meaningful size n, and determine the number (R, say) of nonconforming items in each sample. A time-order plot of the $\{R_i\}$ data is of course the familiar np-chart. One could also set up a CUSUM chart, using the $\{R_i\}$ data, and this is the Binomial CUSUM chart.

In (Calvin 1983) and in (Bourke 1991) it was proposed that one observe the number (X, say) of conforming items between successive nonconforming items, and there are several types of chart that use the ordered $\{X_i\}$ data. This approach requires that it be feasible to retain information on the order of production of items. The observations (X_i) here are termed conforming run-lengths (CRLs), and three types of chart were proposed in (Bourke 1991): the RL_1 chart where the monitoring statistic has a geometric distribution, the RL_2 chart where the monitoring statistic has a negative binomial distribution, and the geometric CUSUM chart. Some authors have introduced the terminology Cumulative Count of Conformances (CCC), a term which first appeared in 1992, but this is essentially the same as conforming run-length (CRL).

It is stated in (Szarka and Woodall 2011, page 738) that "*Although the use of runs rules can improve the performance of the CCC approach, we recommend the use of CUSUM charts*." The use of a geometric CUSUM for detection of both upward and downward shifts from an in-control value of a proportion (denoted by p_a) was considered in (Bourke 1991), and limited tables of recommended parameter values were provided for detecting an upward shift, as well as some tables for detecting a downward shift.

There is another way of viewing the observational data. Following inspection on item *i*, the notation $B_i = 1$ may be used to denote the outcome that item *i* is nonconforming, while $B_i = 0$ denotes that this item is conforming. If we assume that the underlying proportion of nonconforming items remains constant at level *p*, and that there is independence among the items, then we may view the process as a Bernoulli process. In monitoring such a process, we hope to detect a sudden shift from an acceptable level (p_a) for the proportion. The use of a Bernoulli CUSUM scheme to monitor the process using the data-stream $\{B_i\}$ was proposed in (Reynolds and Stoumbos 1999) and independently in (Bourke 2001a), for detection of upward shifts.

There is a discussion of methods for using data from Bernoulli-type processes in (Szarka and Woodall 2011), and it is recommended that either a geometric CUSUM chart or a Bernoulli CUSUM chart be used for monitoring a proportion. The methodology for designing a Bernoulli CUSUM chart to detect a downward shift in a proportion has been developed in (Reynolds 2013), but there is still a considerable burden on quality engineers in implementing this methodology to find appropriate values for the parameters of a Bernoulli CUSUM chart. The importance of developments that help to bridge the gap between theory and practice in process monitoring has been stressed in (Woodall 2017). The task of aiding quality engineers in choosing values for the two parameters of a geometric CUSUM chart for detecting an *upward* shift in a proportion is addressed in (Bourke 2018), and extensive tables are provided, together with methods for interpolation and extrapolation for values of the in-control proportion not included in the tables. The present paper is a companion paper to (Bourke 2018), in that this paper seeks to provide guidance to quality engineers in the design of a geometric CUSUM chart for detection of a *downward* shift in a proportion.

The article is organized as follows. In the next Section, there is a brief introduction to the geometric CUSUM chart and its parameters *k* and *h*. Following this introduction, measures of performance for the geometric CUSUM are considered, and we set out the recommended design procedure for the chart. The choice of values for the parameters *k* and *h* is considered next, and we describe an extensive computational investigation to aid this choice. In the Section "Tables of *k, h* values for a Geometric CUSUM Chart", we provide five tables containing recommended parameter values for CUSUM schemes to detect five specific shift-sizes, for p_a values ranging from 0.20 to 0.001. (p_a is the acceptable level for the proportion being monitored.) These five tables form the central part of this article. The problem of detecting a range of shift sizes with a single CUSUM scheme is considered next, and some recommendations are made. Using the afore-mentioned five tables, it is noted that (for a specific shift-size and false-alarm rate) there is

stability in the quantities $h \times p_a$ and h/k across a range of values for p_a, and especially so for values of p_a below 0.05. Some interpolation results are given in the next Section for finding CUSUM schemes for values of p_a within the range (0.1 to 0.001) and these interpolations are based on the stability of the quantity h/k. A numerical example from manufacturing to illustrate the task of finding parameters for a geometric CUSUM chart is then given. Extrapolations are considered next, this time using the stability of the quantity $h \times p_a$, and examples are given of CUSUM schemes for $p_a = 0.0005$ and 0.0002. The final Section provides a summary of the paper, and indications of areas of further research.

The Geometric CUSUM Chart for Detecting a Downward Shift

The data for this CUSUM scheme is the ordered sequence of CRL values, denoted by $\{X_i\}$. The CUSUM (G_i) is updated whenever a nonconforming item (in the ordered sequence of items) is found, as follows:

$$G_i = \max (0, X_i - k + G_{i-1}) \qquad i = 1, 2 \ldots \tag{1}$$

The quantity k is one of the two parameters of the CUSUM scheme, and the choice of a value for k depends on the size of shift we wish to detect. Clearly, whenever the X_i values exceed k, the resulting G_i values will increase, and may eventually reach or exceed a specified level h (which is the other parameter of the CUSUM scheme). When this happens, we take this to be a signal from the CUSUM scheme that the value of the proportion has decreased. The parameter k is termed the *reference value* of the CUSUM, while the second parameter (h) has been termed the *decision interval* value.

For this geometric CUSUM we can see that there are two possible monitoring procedures:

Procedure A:

Each time a nonconforming item is observed, we update the CUSUM value using equation (1), and then we check to see if $G_i \geq h$. If we find that $G_i \geq h$, we have a signal. This procedure has the advantage of being simple to operate.

On reflection, it may be seen that when we find that G_i *exceeds* h with Procedure A, a signal could have been generated by an earlier *conforming* item. Thus, if we rely on the updating of the CUSUM at each nonconforming item and only then check for a signal, this will delay the occurrence of a signal.

Procedure B:

If one is using Procedure A, it is very likely that the CUSUM value (G_n say) for the CRL that has generated a signal, will *exceed* h. But if we had been checking the value of $(X - k + G_{n-1})$ after *each item*, then we would have found that $(X - k + G_{n-1})$ can increase to *exactly* reach the value h. (Here, we are using the notation X to denote an incomplete CRL, i.e. the next item may be conforming or nonconforming). Thus, a signal can be produced by a conforming item, i.e. without waiting for the nonconforming item that marks the end of the current CRL. When this occurs, we have from equation (1)

$$h = X - k + G_{n-1} \quad (2)$$

which requires that the value of X (the current incomplete CRL) has reached the value $(h + k - G_{n-1})$. If a nonconforming item had occurred before X reached the value $(h + k - G_{n-1})$ the CUSUM value would have to be updated using equation (1).

Thus, following the updating of the CUSUM to value G_{n-1}, we should evaluate $(h + k - G_{n-1})$ and proceed to keep a record of the number (X) of conforming items since the most recent nonconforming item. As each subsequent item is observed and found to be conforming (thereby increasing X by 1), we continue to check to see if the current conforming run-length (X) has reached the value $(h + k - G_{n-1})$, in which case we have a signal. If a nonconforming item is found before a signal, we update the CUSUM value (to G_n say) and resume the comparison of X and $(h + k - G_n)$ as each subsequent item is observed (until we get either a signal or the next nonconforming item).

One can think of Procedure B as a curtailed geometric CUSUM in that one need not wait for the next nonconforming item to check for a signal, as in Procedure A. If Procedure B is used, it is possible to detect a downward shift more quickly than with Procedure A, particularly in the case of large shifts. In this article, we use Procedure B. It is arguable that the operation of this curtailed geometric CUSUM is a little simpler than the operation of a Bernoulli CUSUM scheme, which requires updating of the CUSUM value after each item is observed. Procedure B is an item-by-item monitoring scheme, whereas Procedure A is not.

In starting the CUSUM scheme, one must specify an initial value for G_0. While a zero value seems an obvious choice, there is an advantage to using a positive value such as $h/2$ (termed a head-start level). If the process-proportion is already in a shifted state when the CUSUM is started, this shift will be detected more quickly because of the head start, and the CUSUM is said to have Fast Initial Response (FIR). The use of head starts for CUSUM schemes was proposed in (Lucas and Crozier 1982).

If the CUSUM scheme produces a signal when the proportion is still at the acceptable level p_a, this is a false signal, and we aim to limit the frequency of such signals. But we also want to signal quickly when there has been a sizable downward shift from the acceptable level. A balance should be arrived at between these two interacting aims, taking account of the cost of investigating false signals and the cost of delays in detecting important shifts. In this article, we assume that we know p_a and that we can specify a value for p_r, the maximum size of shifted proportion that should be detected. Consideration is also given to detection of a range of shift sizes.

Measures of Performance for a Monitoring Scheme

We shall use the measure *average number of nonconforming items until a signal* (ANNS) introduced in (Bourke 2018). We will need to evaluate ANNS(p_a) which is used to quantify an acceptable rate of false alarms from the CUSUM scheme. We shall also need to evaluate ANNS(p_r), which quantifies the speed of

detection of a downward shift of interest. A related measure is the average number of observations until a signal (ANOS) which has been recommended in (Reynolds and Stoumbos 1999) and used (inter alia) in (Szarka and Woodall 2011), and (Reynolds 2013). A discussion of the advantage of the ANNS measure is given in (Bourke 2018), and it is noted that ANNS(p) equals p multiplied by ANOS(p). In designing a monitoring scheme, the quality engineer is expected to specify an acceptable minimum level for ANOS(p_a) or alternatively ANNS(p_a). However, values of ANOS(p_a) can be of the order of 100,000 when p_a is as low as 0.001, while corresponding values for ANNS are about 100.

In choosing parameters for any monitoring scheme, the usual approach is to have a specified low frequency for false signals, and fast detection following the occurrence of an important shift. In implementing this approach using the ANNS measure, the aim is to find parameter values so that ANNS(p_r) is as small as possible while ensuring that ANNS(p_a) remains at or closely above some chosen high level.

Evaluation of ANNS(p): Initial-state and Steady-state Evaluations

In (Bourke 2018) there is a discussion of the assumptions (sometimes made implicitly) that have been used in various evaluations of performance measures for a geometric CUSUM chart. In this article, it is assumed that the underlying proportion p remains at a currently-acceptable level (p_a) until a sudden shift occurs to an important lower level (p_r), and that this shift occurs randomly in the stream of items. Thus, up to the shift item, the underlying value of the proportion is at level p_a, and from the shift item onward the underlying value of the proportion remains at level p_r. It is also assumed that prior to the occurrence of a shift, the proportion has remained at the level p_a for a considerable period, and that following any false signals and investigations, the CUSUM is re-set to the head-start level (h/2). Thus, the distribution of CUSUM values prior to a shift will have settled to a steady-state distribution, which will have to be evaluated.

There are two separate ANNS measures that are used in this article:

Measure 1: ANNS(p_a), which is an initial-state evaluation with Fast Initial Response.

ANNS(p_a) = Average number of nonconforming items until a signal, assuming that $p = p_a$.

For this measure, it is assumed that the initial CUSUM value has been set to h/2.

Measure 2: ANNS(p_r), which is a steady-state evaluation following a random downward shift.

ANNS(p_r) = Average number of nonconforming items following a shift until a signal, assuming that p starts out at level p_a, and that (after a lengthy period) there is a random shift to level p_r, which then persists.

These measures correspond to the two ANOS measures used in (Reynolds 2013), which are labelled ICANOS (in-control ANOS) and SSANOS (steady-state ANOS). However, for ICANOS the initial CUSUM value is set at zero in (Reynolds 2013) rather than a head-start value, as in this article.

A procedure for the evaluation of ANNS(p_a) was developed in (Bourke 1991), but the structure of the transition matrix for detecting a downward shift was not given. This transition matrix is given in Appendix 1, together with an expression for ANNS(p_a). Methodology for the evaluation of ANNS(p_r) was developed in (Bourke 2001b) for the case where the initial CUSUM value is re-set to zero following each false alarm prior to the shift. A simple adaptation of this methodology is needed if we wish to re-set the initial CUSUM value to a head-start value, following any false alarm. A summary of the approach is given in Appendix 2. Simulation is another option for the evaluation of either ANNS(p_a) or ANNS(p_r) and has been used extensively in preparing this article to check the values given by the transition-matrix approach.

It was stated earlier that there are two procedures for operating the geometric CUSUM scheme, and that in this article we assume that procedure B (the curtailed CUSUM) is used. Let us denote the ANOS (and ANNS) for procedure B by $ANOS_B$ (and $ANNS_B$) with similar notation for procedure A. It is possible to develop a transition-matrix representation for Procedure B, but the size of the matrix is much larger than that for Procedure A (because Procedure B is an item-by-item monitoring scheme). Using extensive simulation studies, it was found that $ANOS_B(p_a)$ could be approximated closely using $[ANOS_A(p_a) - 1/p_a]$ and similarly that $ANOS_B(p_r)$ could be well-approximated using $[ANOS_A(p_r) - 1/p_r]$. The transition-matrix evaluations of ANNS in Appendices 1 and 2 are carried out for procedure A, and then we can evaluate $ANNS_B$ from $ANNS_A - 1$.

Finding Values for the Parameters *k, h*

The value for k should be somewhere between $E[X| p = p_a]$ and $E[X| p = p_r]$. Using the sequential probability ratio (SPR) analysis for detecting a shift from p_a to p_r, the following formula for finding a suitable value for k was given in (Bourke 1991).

$$k = \{\ln(p_r / p_a)\}/\{\ln[(1- p_a)/(1- p_r)]\} \qquad (3)$$

We denote a value of k arising from (3) by k_{SPR}. The optimality result for an SPR-based choice of k (Moustakides 1986) is not applicable if one is using a steady-state evaluation for ANNS(p_r), and a similar observation has been made in Sparks (2018).

Values for k from equation (3) can be sizable (e.g. 137.6, for $p_a = 0.01$ and $p_r = 0.005$) and bearing in mind that the CRL data are counts, it seems convenient to use an integer value for k. But when p_a is relatively large at 0.2, with $p_r = 0.15$, equation (3) gives the rounded value 4.75 for k, so that k could be approximated by the integer 5 or the rational number 48/10, or even 475/100. In (Lucas 1985), the use of non-integer rational values for k and h was considered for monitoring Poisson counts, with corresponding consideration for binomial counts in (Gan 1993) and (Bourke 2001a). For the geometric CUSUM schemes presented here, both integer and rational values for k and h are considered. In the next Section, Tables are provided with recommended values for (k, h) to achieve at least specified levels for ANNS(p_a). These specified levels can be achieved more closely when non-integer rational values are used for the parameters (rather than integer values). This is more relevant for larger values of p_a, such as 0.025 and above. As an

illustration, suppose that $p_a = 0.10$, and we wish to find k, h for detection of a possible downward shift to p_r, = 0.10/1.5, with the requirement that ANNS(p_a) be at least 50. If we consider only integer-valued k and h, we would find the scheme $k = 9$, $h = 64$, for which ANNS(p_a) = 50.72, and ANNS(p_r) = 8.32. However, if we consider k, h values that are integer multiples of 0.1, we would find the scheme $k = 9.5$, $h = 54.1$, for which ANNS(p_a) = 50.05, and ANNS(p_r) = 8.22.

When non-integer rational values for k and h are used, the form of the transition matrix for CUSUM values is different and is described in Appendix 1. Transition matrices are also larger: for $h = 412.5$ (which appears in Table 1 for $p_a = 0.025$), the size of the matrix is 4125×4125. If one seeks to use k, h values that are integer multiples of 0.01, the sizes of the transition matrices can be very large: for $h =$ 412.49, the size of the required transition matrix is 41249×41249, with consequential problems for matrix inversion.

Tables are presented in the next Section to assist a quality engineer in choosing the two parameters (k, h) of a geometric CUSUM scheme. In preparing these tables, choices had to be made on the levels to include for the in-control value (p_a) of the proportion that is being monitored, as well as the sizes of downward shift to be detected, and the acceptable levels of ANNS(p_a). It was decided to have the levels of p_a ranging from 0.20 down to 0.001, with 27 levels for p_a. The inclusion of values as high as 0.20 may be surprising. It may be thought that most manufacturing processes operate at quite low levels for p_a (with values less than 0.05), and that values for p_a as high as 0.10 or 0.20 are unrealistic. But manufacturing has a wide range of possible scenarios, from "low-tech" to "very high-tech". When a product is developed, the product has a life-cycle of months, years or even decades. For any new product, the associated process may go through a series of step-changes of improvement, or continuous improvement, or a combination of both. For example, in semiconductor manufacturing, the life-cycle of the product may be just one to two years, with the early period having a high rate of scrapped items, but with a high price per unit (Milor, 2013).

A computational investigation to find the value for k for which ANNS(p_r) is at a minimum (while satisfying the ANNS(p_a) specification) in detecting an *upward* shift is described in (Bourke 2018). It was found that the value arising from equation (3) should be increased by a varying percentage that depended on both the specified level of ANNS(p_a), and the size of shift to be detected. In the present context of detecting a downward shift, a corresponding investigation was conducted. Two methods of conducting searches for suitable (k, h) choices were considered. In the first of these methods, the choice of (k, h) can be made so that the resultant value of ANNS(p_r, k, h) is least among those CUSUM schemes for which the ANNS(p_a) specification (let us say 100) is met. In the second method, the choice of (k, h) is based on interpolated values IANNS(p_r) which are between ANNS(p_r, k, h-1) and ANNS(p_r, k, h), as follows

$$\mathrm{IANNS}(p_r, k, h) = \mathrm{ANNS}(p_r, k, h-1) + \{[100 - \mathrm{ANNS}(p_a, k, h-1)]\frac{\mathrm{ANNS}(p_r, k, h) - \mathrm{ANNS}(p_r, k, h-1)}{\mathrm{ANNS}(p_a, k, h) - \mathrm{ANNS}(p_a, k, h-1)}\} \qquad (4)$$

where ANNS(p_a, k, h) is ≥ the specified level (100), and ANNS(p_a, k, h-1) is less than the specified level of 100. We search for the (k, h) values for which IANNS(p_r) is least among CUSUM schemes for which the ANNS(p_a) specification is met. It was found that when the second method was used, this resulted in somewhat better performance for the interpolations reported later in the article, and it was decided to use this method in developing tables of recommended values for the (k, h) parameters.

The six levels chosen for ANNS(p_a) were: 25, 50, 100, 200, 300 and 500, and for each of these levels, five sizes of downward shift from p_a to p_r were considered, as follows:

$p_r = (p_a)/1.5$ $p_r = (p_a)/2$ $p_r = (p_a)/3$ $p_r = (p_a)/4$ $p_r = (p_a)/5$.

As mentioned above, these searches were conducted for each of 27 levels of p_a, ranging from 0.20 to 0.001, so that, in all, 6×5×27 searches were carried out, yielding 810 CUSUM schemes.

In a typical search, approximately 50 geometric CUSUM schemes were identified for an ordered sequence of k values (for a specified p_a value, a specified shift-size, and a specified ANNS(p_a) level) and for each of these schemes the value of IANNS(p_r) was found. The 50 values of IANNS(p_r) were checked to ensure that the k value ($k_{\min}$ say) giving the minimum value of IANNS(p_r) was contained well within the sequence of k values considered, and this k value, together with its associated h value, was noted. In the next Section, five tables are presented listing the recommended values of parameters k, h arising from this investigation.

It was found that in all 810 cases, the recommended value for k (i.e. $k_{\min}$) was less than k_{SPR}, the value arising from equation (3). The percentage reduction in ANNS(p_r) value when using $k_{\min}$ rather than k_{SPR} is quite variable, and depends mainly on the level of ANNS(p_a), but also on the shift size to be detected. This reduction is greatest for low levels of ANNS(p_a) and for detection of small shift-sizes. For schemes with ANNS(p_a) = 25, the percentage reductions in the value of ANNS(p_r) are about 4.5, 2.4, 1.5, 0.9 and 0.7 for the five shift-sizes, while the corresponding percentages for schemes with ANNS(p_a) = 100 are approximately 2.5, 1.6, 1.0, 0.65 and 0.5.

Tables of *k, h* values for a Geometric CUSUM Chart

For a quality engineer, the task of finding appropriate values for the parameters of a geometric CUSUM chart is not an easy one, and the objective here is to provide guidance for potential users. To this end, tables of recommended k, h values have been developed, using the investigation described in the previous Section. The range and structure of these tables is as follows:

In Table 1, recommended (k, h) values are listed for detecting a downward shift from p_a to p_r = (p_a/1.5) for each of the following levels of ANNS(p_a): 25, 50, 100, 200, 300 and 500, and for 27 values of p_a ranging from 0.20 to 0.001. Tables 2, 3, 4 and 5 are organized similarly to Table 1, and refer to shift sizes of p_r = (p_a/2), (p_a/3), (p_a/4) and (p_a/5) respectively. In these tables, it was decided not to exclude those entries for which ANNS(p_r) may be considered quite large (e.g. for shifts of size p_a/1.5 with ANNS(p_a) ≥ 200).

Bearing in mind that the evaluations of ANNS(p_r) were done for curtailed CUSUM schemes, it is not surprising that for detection of large shifts (in Tables 4 and 5), some of the ANNS(p_r) values are less than one. This arises from the fact that, for large shifts, there is likely to be a long CRL (conforming run-length) immediately following a shift, and this (incomplete) CRL can produce a signal, through the operation of Procedure B.

Table 1: Proposed values of parameters (k, h) for geometric CUSUM schemes for detecting a downward shift from p_a to $p_r = p_a/1.5$ for six levels of ANNS(p_a), with p_a ranging from 0.20 to 0.001. *ANNSr* denotes ANNS(p_r).

	ANNS(p_a) = 25			ANNS(p_a) = 50			ANNS(p_a) = 100			ANNS(p_a) = 200			ANNS(p_a) = 300			ANNS(p_a) = 500		
p_a	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*
0.20	3.7	22.7	5.12	4.2	26.2	7.85	4.4	33.2	11.32	4.6	39.4	15.44	4.6	45.4	18.11	4.8	47.4	21.61
0.18	4.3	24.3	5.14	4.8	29.2	7.93	5.0	37.2	11.50	5.2	45.0	15.65	5.4	47.0	18.36	5.4	54.8	21.94
0.16	4.9	28.5	5.18	5.6	32.4	8.05	5.8	41.8	11.58	6.2	46.8	15.85	6.2	53.6	18.55	6.2	62.8	22.30
0.14	5.7	33.6	5.22	6.4	39.4	8.09	7.0	44.2	11.70	7.2	54.8	16.02	7.2	63.0	18.83	7.2	73.6	22.56
0.12	6.7	41.4	5.25	7.6	47.0	8.15	8.2	55.0	11.84	8.4	68.2	16.18	8.6	73.8	19.05	8.6	86.4	22.89
0.10	8.4	47.5	5.29	9.6	53.0	8.22	10.0	67.2	11.95	10.4	80.6	16.39	10.4	92.8	19.29	10.6	103.0	23.18
0.08	10.4	65.7	5.32	11.8	73.8	8.29	12.8	84.8	12.08	13.2	103.2	16.56	13.4	114.0	19.51	13.6	128.0	23.48
0.06	14.1	90.2	5.35	16.0	100.2	8.36	17.4	114.4	12.18	17.6	147.2	16.76	18.2	154.2	19.74	18.4	175.0	23.77
0.05	17.6	100.1	5.36	19.8	115.0	8.39	20.8	143.2	12.24	21.8	167.0	16.85	21.8	192.4	19.86	22.2	213.8	23.92
0.04	18.1	259.3	5.37	25.2	141.6	8.43	26.6	173.8	12.30	27.2	216.6	16.93	27.8	234.6	19.97	27.8	275.0	24.06
0.03	24.6	338.9	5.39	33.6	194.8	8.46	35.2	244.2	12.35	36.6	290.8	17.02	37.4	315.0	20.08	37.6	363.4	24.20
0.025	29.6	412.5	5.39	40.6	233.2	8.48	43.2	280.0	12.38	44.2	348.4	17.07	44.8	386.2	20.13	45.2	441.2	24.28
0.020	37	529	5.41	51	293	8.50	54	356	12.40	55	449	17.10	56	492	20.20	57	547	24.40
0.018	42	549	5.40	57	323	8.52	60	398	12.43	62	482	17.15	62	556	20.24	63	622	24.41
0.016	47	637	5.41	64	367	8.52	67	460	12.44	70	540	17.17	70	623	20.26	71	700	24.42
0.014	54	719	5.41	73	424	8.52	77	521	12.46	80	621	17.18	81	690	20.26	82	780	24.45
0.012	63	847	5.41	85	501	8.53	90	609	12.48	93	737	17.21	94	823	20.28	95	936	24.48
0.010	75	1055	5.41	102	604	8.52	108	735	12.48	112	880	17.21	113	989	20.31	115	1101	24.51
0.009	84	1145	5.41	114	663	8.53	120	819	12.48	124	992	17.23	126	1091	20.32	128	1221	24.53
0.008	94	1319	5.41	128	752	8.53	136	905	12.48	140	1107	17.22	142	1225	20.33	144	1378	24.54
0.007	108	1487	5.42	147	851	8.54	155	1046	12.49	160	1270	17.24	162	1412	20.34	164	1597	24.55
0.006	126	1743	5.42	171	1003	8.54	181	1221	12.50	187	1478	17.24	190	1627	20.36	192	1849	24.56
0.005	152	2061	5.42	206	1195	8.54	217	1473	12.50	225	1766	17.26	228	1958	20.36	230	2238	24.57
0.004	189	2637	5.42	257	1506	8.54	271	1852	12.51	281	2220	17.26	285	2456	20.38	288	2792	24.59
0.003	253	3483	5.42	343	2009	8.55	362	2464	12.51	375	2961	17.27	380	3285	20.38	385	3705	24.60
0.002	379	5273	5.42	516	2998	8.55	544	3688	12.51	564	4422	17.28	571	4917	20.39	577	5591	24.61
0.001	759	10543	5.42	1033	5999	8.55	1088	7398	12.52	1129	8848	17.29	1142	9866	20.40	1155	11188	24.62

Table 2: Proposed values of parameters (k, h) for geometric CUSUM schemes for detecting a downward shift from p_a to $p_r = p_a/2$ for six levels of ANNS(p_a), with p_a ranging from 0.20 to 0.001. *ANNSr* denotes ANNS(p_r).

	ANNS(p_a) = 25			ANNS(p_a) = 50			ANNS(p_a) = 100			ANNS(p_a) = 200			ANNS(p_a) = 300			ANNS(p_a) = 500		
p_a	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*
0.20	4.5	15.1	2.67	5.1	18.2	3.83	5.3	23.1	5.19	5.3	29.4	6.71	5.5	31.1	7.65	5.6	34.9	8.91
0.18	5.3	16.3	2.69	5.4	22.9	3.88	5.9	26.7	5.25	6.1	32.4	6.80	6.2	35.5	7.75	6.3	39.8	9.02
0.16	6.3	17.6	2.71	6.3	25.5	3.91	6.8	30.3	5.32	6.9	37.7	6.89	7.2	39.7	7.86	7.3	44.6	9.14
0.14	6.9	22.5	2.74	7.6	27.9	3.96	7.9	35.3	5.37	8.1	43.2	6.97	8.1	48.8	7.96	8.3	53.8	9.26
0.12	8.3	26.3	2.77	8.8	34.5	3.99	9.1	44.0	5.44	9.8	49.5	7.06	9.7	57.0	8.07	9.8	64.3	9.39
0.10	10.1	32.1	2.78	10.9	41.2	4.03	11.3	52.2	5.50	11.7	62.7	7.14	11.9	68.9	8.17	12.2	75.7	9.51
0.08	12.9	40.6	2.81	13.7	53.3	4.07	14.1	68.6	5.57	14.9	79.4	7.23	15.1	87.7	8.27	15.4	97.3	9.64
0.06	17.9	53.0	2.82	19.3	67.6	4.11	19.7	87.9	5.62	20.1	108.4	7.31	20.7	116.3	8.37	20.9	131.4	9.76
0.05	21.4	65.3	2.84	23.1	83.3	4.13	23.9	105.7	5.65	24.6	128.3	7.36	24.9	142.1	8.42	25.2	159.7	9.82
0.04	26.9	82.7	2.85	28.7	107.5	4.15	29.9	134.9	5.68	30.7	164.4	7.40	31.4	178.5	8.47	31.6	202.9	9.88
0.03	36.2	110.9	2.86	38.7	143.6	4.17	40.3	180.3	5.71	41.6	217.5	7.44	41.8	244.1	8.52	42.8	268.7	9.95
0.025	43.7	133.3	2.86	46.6	173.1	4.17	49.3	211.4	5.72	50.1	262.2	7.46	50.8	289.3	8.55	51.6	323.1	9.97
0.020	55	167	2.87	58	221	4.19	61	273	5.75	63	329	7.5	64	361	8.58	64	415	10.01
0.018	61	187	2.87	65	243	4.19	68	303	5.76	70	367	7.51	71	404	8.59	72	453	10.04
0.016	69	209	2.87	73	275	4.19	77	338	5.76	79	412	7.51	80	455	8.6	81	511	10.03
0.014	79	239	2.87	85	306	4.2	88	387	5.75	90	475	7.51	91	527	8.61	93	582	10.05
0.012	92	281	2.88	98	366	4.21	103	451	5.76	106	547	7.52	107	608	8.61	108	687	10.05
0.010	111	336	2.88	119	431	4.2	123	549	5.78	127	661	7.53	128	737	8.62	130	823	10.06
0.009	123	375	2.88	132	482	4.21	137	608	5.77	141	737	7.53	143	813	8.64	145	910	10.07
0.008	138	425	2.88	149	540	4.21	154	686	5.77	159	827	7.53	161	915	8.64	163	1028	10.09
0.007	158	485	2.88	170	620	4.21	177	778	5.78	182	944	7.53	184	1047	8.63	186	1180	10.08
0.006	185	564	2.88	197	733	4.21	206	913	5.78	212	1107	7.54	215	1221	8.64	217	1380	10.09
0.005	221	683	2.88	237	878	4.21	248	1092	5.78	255	1325	7.54	258	1468	8.65	261	1652	10.10
0.004	277	852	2.89	298	1088	4.22	310	1367	5.78	319	1658	7.55	323	1833	8.65	326	2072	10.10
0.003	369	1140	2.89	397	1455	4.22	413	1829	5.79	425	2218	7.55	431	2446	8.66	436	2753	10.11
0.002	557	1693	2.89	594	2197	4.22	620	2745	5.79	639	3319	7.55	646	3682	8.66	654	4137	10.11
0.001	1115	3388	2.89	1190	4389	4.22	1241	5493	5.79	1279	6642	7.56	1293	7368	8.66	1309	8280	10.12

Table 3: Proposed values of parameters (k, h) for geometric CUSUM schemes for detecting a downward shift from p_a to $p_r = p_a/3$ for six levels of ANNS(p_a), with p_a ranging from 0.20 to 0.001. *ANNSr* denotes ANNS(p_r).

	ANNS(p_a) = 25			ANNS(p_a) = 50			ANNS(p_a) = 100			ANNS(p_a) = 200			ANNS(p_a) = 300			ANNS(p_a) = 500		
p_a	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*
0.20	5.9	10.3	1.29	6.1	14.5	1.77	6.2	18.7	2.31	6.4	22.7	2.89	6.4	25.5	3.23	6.6	28.1	3.68
0.18	6.5	12.1	1.31	6.7	16.7	1.79	7.4	19.7	2.34	7.2	25.7	2.92	7.1	29.5	3.27	7.4	31.9	3.73
0.16	7.4	14.1	1.32	7.8	18.7	1.81	8.2	23.5	2.36	8.4	28.7	2.96	8.7	31.0	3.32	8.8	34.9	3.78
0.14	8.9	15.4	1.33	9.1	21.8	1.83	9.6	26.9	2.39	9.7	33.6	2.99	9.9	36.8	3.36	10.0	41.1	3.83
0.12	9.6	20.9	1.34	10.9	25.3	1.85	11.3	32.2	2.42	11.6	39.3	3.03	11.4	45.1	3.40	11.9	48.9	3.88
0.10	12.7	22.9	1.35	13.9	29.2	1.87	13.7	39.4	2.44	13.9	48.8	3.06	14.3	53.0	3.44	14.4	59.9	3.92
0.08	15.3	31.4	1.36	16.8	39.3	1.88	17.4	49.9	2.47	17.4	62.9	3.10	17.7	69.3	3.48	17.9	78.1	3.98
0.06	20.3	43.5	1.37	22.9	52.7	1.90	23.7	67.1	2.49	23.9	83.5	3.13	24.3	92.0	3.52	24.9	102.0	4.02
0.05	25.9	48.7	1.38	27.2	65.3	1.91	27.9	83.9	2.51	29.0	100.1	3.15	29.1	112.6	3.54	29.7	125.2	4.05
0.04	32.7	61.3	1.38	34.3	82.0	1.92	35.6	103.7	2.52	36.9	124.9	3.17	37.1	139.5	3.56	37.2	158.5	4.07
0.03	43.6	82.9	1.39	45.8	111.1	1.93	48.0	138.1	2.53	48.5	172.1	3.18	49.9	186.7	3.58	50.2	211.5	4.09
0.025	52.6	99.7	1.39	54.7	135.3	1.93	57.7	167.3	2.54	58.6	206.5	3.19	59.7	226.7	3.59	60.3	255.5	4.11
0.02	67	123	1.40	69	169	1.94	72	212	2.55	74	258	3.21	75	285	3.61	76	319	4.12
0.018	73	141	1.40	77	187	1.94	80	236	2.55	83	284	3.21	84	314	3.61	85	353	4.13
0.016	82	160	1.40	86	214	1.95	91	263	2.56	93	322	3.21	94	357	3.62	95	402	4.14
0.014	95	180	1.40	100	239	1.95	104	301	2.56	107	366	3.22	108	406	3.62	109	458	4.14
0.012	111	210	1.40	116	282	1.95	121	354	2.56	124	432	3.22	126	475	3.62	128	531	4.14
0.010	132	256	1.40	141	333	1.95	145	426	2.56	150	514	3.22	151	573	3.63	153	643	4.15
0.009	147	283	1.40	156	373	1.95	162	471	2.56	166	575	3.22	169	631	3.62	170	715	4.15
0.008	166	318	1.40	174	425	1.95	182	532	2.57	187	646	3.22	190	712	3.63	192	802	4.15
0.007	189	365	1.40	201	480	1.95	208	608	2.56	214	739	3.23	217	815	3.63	219	920	4.15
0.006	221	426	1.40	233	565	1.95	243	709	2.56	250	861	3.23	253	953	3.63	256	1072	4.15
0.005	266	509	1.40	282	671	1.95	292	851	2.57	300	1035	3.23	304	1144	3.63	308	1284	4.16
0.004	331	642	1.40	351	845	1.95	366	1061	2.57	375	1296	3.23	380	1432	3.64	385	1607	4.16
0.003	441	858	1.40	469	1125	1.95	488	1417	2.57	501	1725	3.23	507	1910	3.64	513	2147	4.16
0.002	666	1276	1.40	702	1696	1.95	731	2132	2.57	752	2590	3.23	760	2871	3.64	771	3217	4.16
0.001	1333	2552	1.40	1405	3392	1.95	1463	4267	2.57	1505	5182	3.23	1521	5746	3.64	1543	6439	4.16

Table 4: Proposed values of parameters (k, h) for geometric CUSUM schemes for detecting a downward shift from p_a to $p_r = p_a/4$ for six levels of ANNS(p_a), with p_a ranging from 0.20 to 0.001. *ANNSr* denotes ANNS(p_r).

	ANNS(p_a) = 25			ANNS(p_a) = 50			ANNS(p_a) = 100			ANNS(p_a) = 200			ANNS(p_a) = 300			ANNS(p_a) = 500		
p_a	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*
0.20	6.5	9.1	0.83	6.6	13.1	1.12	6.9	16.4	1.44	7.0	20.1	1.77	7.2	22.5	1.98	7.3	25.3	2.23
0.18	7.4	10.5	0.85	8.1	13.6	1.14	7.9	18.3	1.46	8.3	22.1	1.79	8.1	25.6	2.00	8.7	27.0	2.26
0.16	8.1	12.7	0.85	8.1	17.9	1.15	8.8	21.5	1.47	9.1	26.1	1.82	9.5	28.1	2.02	9.5	32.1	2.29
0.14	9.9	13.3	0.86	10.6	17.9	1.16	10.6	24.1	1.49	10.7	30.2	1.84	11.1	32.7	2.05	10.9	37.8	2.32
0.12	10.8	18.1	0.87	11.9	22.7	1.17	12.2	29.5	1.51	13.1	34.5	1.86	13.3	38.0	2.07	12.9	44.7	2.35
0.10	14.3	19.5	0.87	15.1	26.3	1.18	14.9	35.6	1.52	16.1	41.6	1.88	16.2	46.3	2.10	15.9	53.8	2.38
0.08	17.6	25.7	0.88	18.6	34.5	1.19	19.3	44.2	1.54	19.7	54.5	1.90	20.1	59.9	2.12	20.4	67.3	2.40
0.06	23.7	35.3	0.89	24.9	47.3	1.20	26.3	59.3	1.55	27.1	72.2	1.92	27.1	81.3	2.15	27.6	90.9	2.43
0.05	28.1	43.8	0.89	30.1	57.5	1.21	32.4	69.9	1.56	32.4	88.3	1.93	32.3	99.7	2.16	33.1	110.8	2.45
0.04	37.1	51.5	0.89	38.3	71.2	1.22	39.3	91.8	1.57	40.9	110.9	1.94	41.3	123.3	2.17	41.9	138.4	2.46
0.03	48.9	70.7	0.90	51.0	96.1	1.22	53.8	120.5	1.58	54.9	148.7	1.95	56.1	163.5	2.18	56.2	185.9	2.47
0.025	58.7	85.4	0.90	63.1	112.6	1.22	64.3	146.5	1.58	65.6	180.5	1.96	66.8	199.1	2.19	68.2	222.1	2.48
0.02	73	110	0.91	77	147	1.23	81	184	1.59	83	225	1.97	84	250	2.20	85	281	2.49
0.018	82	120	0.90	87	160	1.23	89	207	1.59	92	252	1.97	93	279	2.20	95	311	2.49
0.016	92	136	0.90	97	183	1.23	101	231	1.59	104	282	1.97	105	314	2.20	106	354	2.49
0.014	105	156	0.90	111	209	1.23	116	264	1.59	119	323	1.97	120	359	2.20	122	403	2.50
0.012	123	182	0.91	129	246	1.24	135	309	1.59	139	377	1.97	141	417	2.20	142	472	2.50
0.010	148	218	0.91	155	295	1.23	163	369	1.59	167	453	1.98	168	506	2.21	171	566	2.50
0.009	164	243	0.91	175	322	1.24	180	414	1.60	185	505	1.97	187	561	2.21	190	629	2.50
0.008	185	273	0.91	197	362	1.24	202	467	1.59	208	570	1.98	211	630	2.21	214	708	2.51
0.007	211	313	0.91	222	421	1.23	232	532	1.60	239	648	1.98	241	722	2.21	245	809	2.51
0.006	247	364	0.91	261	488	1.24	272	617	1.59	278	759	1.98	282	840	2.21	285	947	2.51
0.005	297	436	0.91	315	582	1.24	325	746	1.60	335	908	1.98	339	1007	2.21	343	1135	2.51
0.004	372	544	0.91	393	730	1.24	406	934	1.60	419	1136	1.98	424	1260	2.21	429	1419	2.51
0.003	494	731	0.91	525	972	1.24	544	1239	1.60	558	1518	1.98	566	1680	2.21	573	1891	2.51
0.002	741	1097	0.91	784	1467	1.24	815	1864	1.60	839	2273	1.98	848	2525	2.21	859	2842	2.51
0.001	1484	2193	0.91	1570	2933	1.24	1631	3729	1.60	1677	4555	1.98	1697	5054	2.22	1721	5679	2.51

Table 5: Proposed values of parameters (k, h) for geometric CUSUM schemes for detecting a downward shift from p_a to $p_r = p_a/5$ for six levels of ANNS(p_a), with p_a ranging from 0.20 to 0.001. *ANNSr* denotes ANNS(p_r).

	ANNS(p_a) = 25			ANNS(p_a) = 50			ANNS(p_a) = 100			ANNS(p_a) = 200			ANNS(p_a) = 300			ANNS(p_a) = 500		
p_a	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*	k	h	*ANNSr*
0.20	6.5	9.1	0.61	6.6	13.1	0.82	7.4	15.5	1.04	7.4	19.5	1.27	7.7	21.0	1.40	8.7	21.4	1.58
0.18	7.4	10.5	0.62	8.1	13.6	0.83	8.9	16.2	1.05	8.7	21.2	1.28	9.4	21.9	1.42	9.2	25.7	1.60
0.16	8.1	12.7	0.62	9.4	14.9	0.83	10.4	18.1	1.06	10.2	23.5	1.30	9.9	27.2	1.44	10.7	28.7	1.62
0.14	9.9	13.3	0.63	10.6	17.9	0.84	11.9	21.2	1.07	11.6	27.7	1.31	12.1	29.9	1.45	11.8	35.1	1.64
0.12	13.1	13.9	0.63	12.7	21.3	0.85	13.9	25.3	1.08	14.1	31.9	1.33	14.1	35.9	1.47	14.3	40.5	1.66
0.10	14.3	19.5	0.64	16.1	24.6	0.86	16.6	31.5	1.09	16.7	39.9	1.34	17.6	42.5	1.49	17.7	48.5	1.68
0.08	18.3	24.5	0.65	20.1	31.8	0.87	20.7	40.9	1.10	21.3	50.2	1.35	21.6	55.7	1.50	22.1	62.1	1.70
0.06	25.3	31.8	0.65	27.9	41.2	0.87	27.7	55.9	1.11	28.8	67.7	1.37	29.3	74.9	1.52	30.3	83.0	1.72
0.05	31.4	37.5	0.65	33.9	49.3	0.88	34.6	65.1	1.12	35.6	79.9	1.38	35.7	89.9	1.53	36.4	100.7	1.73
0.04	41.1	44.6	0.65	40.7	66.3	0.88	43.6	81.7	1.13	43.9	103.1	1.38	44.8	113.3	1.54	45.9	126.4	1.74
0.03	52.2	64.7	0.66	55.7	86.6	0.88	57.6	111.5	1.13	59.3	136.9	1.39	59.8	153.1	1.55	61.7	169.6	1.75
0.025	63.1	77.5	0.66	67.7	103.3	0.89	70.7	131.4	1.13	72.1	163.1	1.39	72.6	182.5	1.55	73.4	206.5	1.75
0.02	79	98	0.66	84	132	0.89	87	169	1.14	90	207	1.40	91	230	1.56	92	260	1.76
0.018	89	107	0.66	93	148	0.89	97	188	1.14	100	230	1.40	101	256	1.56	103	287	1.76
0.016	101	119	0.66	105	166	0.89	110	210	1.14	113	258	1.40	114	288	1.56	116	324	1.76
0.014	115	138	0.66	120	190	0.89	126	240	1.14	129	296	1.40	130	331	1.56	132	372	1.76
0.012	135	159	0.66	141	220	0.89	146	283	1.14	150	348	1.41	153	383	1.56	154	435	1.76
0.010	161	194	0.66	169	265	0.89	176	338	1.14	181	416	1.41	183	463	1.56	186	520	1.77
0.009	180	213	0.66	188	294	0.89	195	378	1.14	202	460	1.41	203	515	1.56	207	578	1.77
0.008	203	239	0.66	212	330	0.89	221	422	1.14	226	521	1.41	229	579	1.57	232	653	1.77
0.007	230	277	0.66	243	377	0.90	252	484	1.14	258	597	1.41	262	661	1.56	265	747	1.77
0.006	271	319	0.66	283	441	0.89	293	567	1.14	301	697	1.41	306	772	1.57	309	873	1.77
0.005	323	388	0.66	340	529	0.89	354	676	1.14	362	836	1.41	367	927	1.57	372	1046	1.77
0.004	400	492	0.66	424	664	0.90	441	849	1.14	455	1040	1.41	460	1157	1.57	466	1306	1.77
0.003	536	651	0.66	567	883	0.90	589	1131	1.15	605	1392	1.41	613	1546	1.57	621	1744	1.77
0.002	808	971	0.66	849	1328	0.90	886	1694	1.15	910	2084	1.41	920	2320	1.57	931	2620	1.77
0.001	1618	1942	0.66	1699	2658	0.90	1773	3389	1.15	1820	4172	1.41	1841	4642	1.57	1863	5242	1.77

Detecting a Range of Sizes of Downward Shift

There are likely to be situations where one would prefer to specify a *range* of sizes of downward shift, with the intention of achieving reasonable detection performance for any shift sizes in the range. We assume that an interval of possible values for p_r can be specified, and we pose the question: is it possible to find a geometric CUSUM scheme that will perform reasonably well in detecting shifts from p_a to any level in this interval?

In this investigation, we compared the performance of ten matched CUSUM schemes each designed to detect a specified shift-size, for the case of $p_a = 0.01$, the schemes being matched by having the in-control level ANNS(p_a) = 50. The specified shift-sizes are denoted by p_a/m, where the ten values of m are 1.5, 2.0, 2.5, 3.0, …4.5, 5, 6, 7. The ten schemes and their parameters are listed in Table 6. Each row refers to a specific downshifted value of p and each row-entry gives, for each scheme, the percentage by which the ANNS(p) value exceeds the lowest of the ANNS(p) values for the ten schemes. For example, the value 4 in the row for $p = p_a/5$ for the scheme designed to detect shifts to $(p_a)/2.5$ gives the rounded percentage by which ANNS($p = p_a/5$) for the latter scheme exceeds the ANNS($p = p_a/5$) for the best of the 10 schemes.

Here are some observations on detection performance of the schemes considered. A more detailed version of Table 6 (with a larger number of rows) has also been evaluated, and the observations given here are also based on this more detailed table.

(i) The first scheme (for $m = 1.5$) is good for detecting small shifts (down to about $p_a/2.25$) but is relatively poor for detecting larger shifts.

(ii) The scheme for ($m = 2$) has a broader range of effectiveness than that for ($m = 1.5$), and similarly the scheme for ($m = 2.5$) has a broader range of effectiveness than that for ($m = 2$). This pattern continues for the later schemes.

(iiii) Schemes designed for detecting large downward shifts such as for $m = 3.5$ or $m = 4$ work well for even larger shifts, but not for detecting small shifts such as $p_a/1.25$ and $p_a/1.5$.

(iv) The third ($m = 2.5$) and fourth ($m = 3$) schemes appear to have the best overall performance in this instance, apart from performing less well at detecting the smallest downward shift considered.

(v) If one has chosen a range of downward shift-sizes (say p_{upper} down to p_{lower}) for fast detection, the choice of the p_r value to use in equation (3) should *not* be in the middle of the range, but rather towards the upper end of the range. For example, for detecting shifts in the range $p_a/2$ to $p_a/5$, the scheme for the shift size $p_a/3$ looks better than the neighbouring alternatives.

A similar investigation for the case of $p_a = 0.01$, with ANNS(p_a) = 100, led to broadly similar results to those presented in Table 6.

Table 6: Comparisons of CUSUM Schemes for Detecting Shifts from $p_a = 0.01$ to $p_r = (p_a)/m$
All schemes have ANNS(p_a) ≥ 50, and the values of m are 1.5, 2.0, 2.5...,4.5, 5.0, 6.0, 7.0.

For values of p below $p_a = 0.01$, each table-entry is the rounded percentage by which ANNS(p) for a CUSUM scheme exceeds the lowest value of ANNS(p) for all the schemes.
(*The row for $p_a = 0.01$ contains the ANNS(p_a) values.*)

Values of m		1.5	2	2.5	3	3.5	4	4.5	5	6	7
CUSUM	*k*	102	119	129	141	150	160	160	174	182	192
parameters	*h*	604	431	378	333	307	283	283	255	241	225
Values of p											
p_a = 0.01		**50.01**	**50.03**	**50.02**	**50.02**	**50.02**	**50.03**	**50.03**	**50.03**	**50.02**	**50.02**
p_a/1.25		***0***	6	10	13	16	18	18	20	21	23
p_a/1.5		***0***	3	6	9	12	15	15	18	20	21
p_a/1.75		1	***0***	2	5	7	9	9	12	14	16
p_a/2		3	***0***	1	3	4	6	6	9	11	12
p_a/2.25		5	0	***0***	1	2	4	4	6	8	9
p_a/2.5		8	1	***0***	0	1	3	3	5	6	7
p_a/2.75		10	1	0	***0***	1	2	2	3	4	5
p_a/3		11	2	1	***0***	0	1	1	2	3	4
p_a/3.25		13	3	1	***0***	0	1	1	2	2	3
p_a/3.5		15	4	1	0	***0***	0	0	1	2	3
p_a/3.75		16	5	2	0	***0***	0	0	1	1	2
p_a/4		17	5	2	1	0	***0***	***0***	0	1	1
p_a/4.25		18	6	3	1	0	***0***	***0***	0	1	1
p_a/4.5		19	7	3	1	0	***0***	***0***	0	0	1
p_a/4.75		20	7	4	1	0	***0***	***0***	0	0	1
p_a/5		21	8	4	2	1	0	0	***0***	0	0
p_a/5.25		22	8	4	2	1	0	0	***0***	0	0
p_a/5.5		23	9	5	2	1	0	0	***0***	0	0
p_a/6		24	10	5	3	1	1	1	0	***0***	0
p_a/6.5		26	10	6	3	2	1	1	0	***0***	0
p_a/7		27	11	7	3	2	1	1	0	0	***0***
p_a/7.5		28	12	7	4	2	1	1	0	0	***0***
p_a/8		29	12	8	4	3	1	1	0	0	***0***
p_a/8.5		29	13	8	4	3	2	2	1	0	***0***
p_a/9		30	13	8	5	3	2	2	1	0	***0***
p_a/10		31	14	9	5	3	2	2	1	0	***0***

Note 1: In each row, any true zero is indicated by italic boldface. The other "zeros" denote values less than 0.5%.
Note 2: CUSUM schemes 6 and 7 (for m = 4 and 4.5) were found to be the same.
Note 3: For each scheme, the boxed section indicates those values of p for which ANNS(p) is within 5% of the lowest value of ANNS(p) across the ten schemes.

Finding Geometric CUSUM Schemes by Interpolation for p_a in the Range 0.001 to 0.10

In this Section we investigate the possibility of interpolating values for the parameters (k, h) for levels of p_a in the range 0.001 to 0.10. We begin by considering how to get a suitable value for k. It is clear from the investigation in an earlier Section that the k value should be less than the corresponding value of k_{SPR}. To explore this further, the values of the ratio k_{min}/k_{SPR} were recorded for all 810 CUSUM schemes listed in Tables 1 to 5, where k_{min} is the k value recommended for a particular level of p_a, for a specified ANNS(p_a) and specified shift-size. The main influence on these ratio values is the level of ANNS(p_a) and there is also a lesser dependence on the size of shift. These ratio values show increasing stability (for a specific ANNS(p_a) level and shift size) for the smaller values of p_a. To illustrate this stability, examples of the ratio values are presented in Table 7 for a number of levels of ANNS(p_a) and shift size. But it is noticeable that for the shift size m = 1.5 with ANNS(p_a) specified at 25, there is a step-change in these ratio values between p_a = 0.05 and p_a = 0.04. Apart from this, the ratio values are quite stable. Summarizing information about the values of the ratios k_{min}/k_{SPR} for use in both interpolations and extrapolations is presented in Table 8. The manner in which this information is used will be described later.

Next, we consider how to interpolate values for the parameter h. It was noticed that, when both ANNS(p_a) and the shift size are fixed, the quantities ($h \times p_a$), and (h/k) for the CUSUM schemes listed in Tables 1, 2, 3, 4 and 5 show a degree of stability across the range of p_a values in the tables, particularly for p_a values less than 0.06. There is one exception to this for values of p_a above 0.04 when m = 1.5 with ANNS(p_a) specified at 25. Using the coefficient of variation (SD/mean), it was found that the (h/k) values have slightly less variability than the ($h \times p_a$) values, and it was decided to use the stability of (h/k) in the interpolation of values for the parameter h.

The interpolations to find values for parameters (k, h) were carried out as follows: Suppose the value of p_a is 0.038 and we wish to find a CUSUM scheme having ANNS(p_a) $\geq$ 100, for detecting a shift to $p_r = p_a/2$. First, we use equation (3) to get k_{SPR}, and then we refer to Table 8 to get the multiplier which turns out to be 0.895. The product of k_{SPR} (unrounded) and 0.895 gives us 31.72, and we take 31.7 as our value for k. Next, we refer to Table 2 (because it contains k, h values for a shift from p_a to $p_a/2$) and we pick out six sets of values for (k, h): three for those p_a levels just above 0.038, and three for those p_a levels just below 0.038. These values for (k, h) are (61, 273), (49.3, 211.4), (40.3, 180.3), (29.9, 134.9), (23.9, 105.7) and (19.7, 87.9). Using these we calculate the consequent six values of (h/k). Consideration was given to several ways of using these six values for (h/k) to find an estimate for the quantity (h/k) to be associated with the value p_a = 0.038. The procedure of simply using the average of the six values seems to work as well as more elaborate methods that use weighted averages, and it was decided to use this method, which in this example gave a value of 4.439. The product of 31.7 (our value for k) and the estimated (h/k) ratio gave us a value of

140.7 for h. Then we used the transition-matrix approach in Appendix 1 to evaluate ANNS(p_a) for the scheme k = 31.7, h =140.7 , and it was found to be 99.9, which is very close to the target value of 100 for ANNS(p_a). If one preferred to use integer values for k and h, the scheme with k = 32 and h = 141 yields ANNS(p_a) = 104.0, which is also acceptably close to the target of 100 for ANNS(p_a).

The results from other interpolations are given in Table 9 which presents the (k, h) parameter values found by interpolation for four values of p_a (0.093, 0.046, 0.022, 0.0037), for a range of levels of ANNS(p_a) and for all five shift-sizes. Table 9 also contains the evaluations of ANNS(p_a) for the CUSUM schemes, and it is instructive to compare each evaluated ANNS(p_a) with the corresponding target-value. The first value used for p_a (0.093) is toward the upper end of the range (0.10 to 0.001), and the evaluations of ANNS(p_a) differ from the target values by (on average) just over 5%, with the greatest differences being 10% and 11%.

For the second value (0.046) used for p_a in Table 9, the average difference from the target values for ANNS(p_a) is about 4%, with the greater percentage differences occurring when the target value is 200. One can see from Table 9 that the greater accuracy (in coming close to the desired value for ANNS(p_a)) is achieved for the two lowest values of p_a, and this is due to the increasing stability of the h/k ratio for lower values of p_a. There is one instance in Table 9 where interpolation is not recommended: the case of ANNS(p_a) = 25, for detection of shifts with m = 1.5. The lack of stability in the h/k ratio for p_a values greater than 0.04 has been noted earlier. In this instance (with p_a = 0.046), the accuracy of the interpolation is poor, but interpolations are still effective provided p_a is less than 0.025.

Other interpolation checks (not reported here) have also been done. To provide guidance to quality engineers, the conclusions are as follows:

Finding values for (k, h) *using interpolation for values of* p_a *in the interval* [0.10, 0.06] *are marginal in accuracy, especially for detecting the two smaller shifts, and when the required ANNS*(p_a) *is at least* 200. *Interpolations for values of* p_a *in the interval* [0.06, 0.03] *are quite acceptable as regards accuracy in coming close to the target value for ANNS*(p_a). (*There is one exception to this: detection of the smallest of the five shift-sizes, with the ANNS*(p_a) *specification at* 25.) *The accuracy achieved when* p_a *is in the interval* [0.03, 0.01] *is quite good, and it is very good for* p_a *values in the interval* [0.01, 0.001].

Table 7: Illustrations of the Stability of the Ratio of k_{min} to k_{SPR} for Specific ANNS(p_a) and Shift size (m) for a Range of Values of p_a.

p_a	ANNS(p_a) = 25	ANNS(p_a) = 25	ANNS(p_a) = 50	ANNS(p_a) = 100	ANNS(p_a) = 200	ANNS(p_a) = 500
	m = 1.5	m = 2	m = 3	m = 2	m = 3	m = 1.5
0.10	0.753	0.788	0.904	0.881	0.904	0.951
0.08	0.733	0.792	0.862	0.866	0.893	0.958
0.06	0.732	0.811	0.869	0.893	0.907	0.955
0.05	0.755	0.802	0.854	0.896	0.910	0.952
0.04	0.616	0.800	0.855	0.889	0.920	0.946
0.03	0.622	0.801	0.851	0.892	0.901	0.951
0.025	0.621	0.803	0.844	0.906	0.904	0.949
0.020	0.619	0.806	0.849	0.893	0.910	0.953
0.018	0.631	0.803	0.851	0.895	0.918	0.946
0.016	0.627	0.806	0.844	0.899	0.913	0.947
0.014	0.629	0.806	0.858	0.898	0.918	0.955
0.012	0.628	0.804	0.852	0.900	0.910	0.947
0.010	0.622	0.807	0.861	0.894	0.916	0.953
0.009	0.626	0.804	0.857	0.895	0.912	0.954
0.008	0.622	0.801	0.849	0.894	0.913	0.953
0.007	0.625	0.802	0.858	0.898	0.913	0.949
0.006	0.625	0.804	0.852	0.896	0.914	0.952
0.005	0.627	0.800	0.858	0.898	0.913	0.949
0.004	0.624	0.802	0.854	0.897	0.913	0.950
0.003	0.626	0.800	0.856	0.896	0.914	0.952
0.002	0.624	0.805	0.853	0.896	0.914	0.950
0.001	0.624	0.805	0.853	0.896	0.914	0.950

Table 8: Choosing a Value for k: Multipliers to be applied to the Value given by the Sequential Probability Ratio Formula.

	Shift-size to be detected				
	(p_a)/1.5	(p_a)/2	(p_a)/3	(p_a)/4	(p_a)/5
ANNS(p_a) = 25	0.625	0.805	0.81	0.805	0.805
ANNS(p_a) = 50	0.85	0.86	0.855	0.85	0.845
ANNS(p_a) = 100	0.895	0.895	0.89	0.885	0.88
ANNS(p_a) = 200	0.925	0.92	0.915	0.91	0.905
ANNS(p_a) = 300	0.94	0.935	0.925	0.92	0.915
ANNS(p_a) = 500	0.95	0.945	0.935	0.93	0.93

Table 9: Results of Interpolation to find Parameters of Geometric CUSUM Schemes

Parameter values (*k*, *h*) *found by interpolation for various target levels of ANNS*(p_a), *for detecting five shift-sizes, for each of four values of* p_a.

p_a =0.093	Target ANNS(p_a) = 50			Target ANNS(p_a) = 100		
Shift p_a/m	*k*	*h*	Evaluated ANNS(p_a)	*k*	*h*	Evaluated ANNS(p_a)
m						
1.5	10.3	62.1	55.7	10.8	71.7	95.3
2	11.9	44.4	51.5	12.4	57.2	110.4
3	14.2	32.8	46.3	14.8	42.5	99.6
4	15.9	29.2	49.3	16.6	38.1	105.9
5	17.3	27.1	52.1	18.0	34.1	103.7

p_a = 0.046	Target ANNS(p_a) = 25			Target ANNS(p_a) = 200		
Shift p_a/m	*k*	*h*	Evaluated ANNS(p_a)	*k*	*h*	Evaluated ANNS(p_a)
m						
1.5	*not recommended*			23.5	186.6	196.4
2	23.4	71.6	25.3	26.8	141.8	207.7
3	28.1	55.0	26.4	31.8	111.4	216.2
4	31.4	46.0	25.1	35.5	96.6	207.6
5	34.2	41.9	26.5	38.5	89.0	207.1

p_a = 0.022	Target ANNS(p_a) = 25			Target ANNS(p_a) = 500		
Shift p_a/m	*k*	*h*	Evaluated ANNS(p_a)	*k*	*h*	Evaluated ANNS(p_a)
m						
1.5	34	469	25.6	51.6	504.4	517.3
2	50	152	25.2	58.6	371.6	508.3
3	60	114	25.1	69.0	290.9	510.1
4	67	97	24.8	77.1	254.3	495.6
5	73	87	25.2	83.9	233.8	496.5

p_a = 0.0037	Target ANNS(p_a) = 50			Target ANNS(p_a) = 300		
Shift p_a/m	*k*	*h*	Evaluated ANNS(p_a)	*k*	*h*	Evaluated ANNS(p_a)
m						
1.5	279	1625	50.8	308	2653	297.7
2	321	1185	50.1	349	1986	300.6
3	380	914	50.2	411	1550	302.0
4	424	788	49.7	458	1364	300.7
5	458	716	49.7	496	1252	297.9

Numerical Example

We consider a manufacturing process that has been in operation for three years in stable conditions so that we have a reliable value for the proportion of nonconforming items produced. Let us suppose that this value is 4.5% (so that the yield is 95.5%). Suppose that in recent months a possible improvement in the process has been put in place, but it is not known how long it may take for the improvement to take effect.

Here, we consider the design of a geometric CUSUM scheme to detect a downward shift in the proportion. Suppose we are interested in detecting whether there has been a downward shift from p_a =0.045 to p_r = 0.03 (which is p_a/1.5). Table 1 does not contain recommended CUSUM parameter-values for the p_a value of interest here (0.045), so that interpolation is necessary. Suppose we wish to control the rate of false-alarms by requiring that ANNS(p_a = 0.045) be at least 50.

In finding a value for *k*, we start by putting values (0.045 and 0.03 for p_a and p_r) into equation (3) which gives 26.017, the SPR-based value. Applying the results of the investigation in the Section "Finding Values for the Parameters *k, h*", we reduce this value (26.017) using the relevant multiplier from Table 8 (which is 0.85 for ANNS(p_a) = 50 and *m* = 1.5) and this gives the value 22.1 for *k*.

Next, interpolation is carried out to get a value for *h*, using the stability of the ratio (*h/k*). In Table 1, we pick out the (*k, h*) values for the six schemes that are closest to the value p_a = 0.045, in the column for ANNS(p_a) = 50. The average of the six values for *h/k* is 5.9142, and this is multiplied by the *k* value (22.1) giving the interpolated value for *h* as 130.7.

Using the evaluation method in Appendix 1, one finds ANNS(p_a) = 51.7, which is quite close to the desired value of 50. If we had chosen 100 for ANNS(p_a) as our desired rate of false-alarms, corresponding evaluations would have led to the scheme *k* = 23.3, *h* = 155.5, for which the evaluated ANNS(p_a) turns out to be 96.5. In starting either of these schemes, the initial value of the CUSUM is set to *h*/2, rounded to the nearest tenth, bearing in mind that in this example *h* is an integer multiple of 0.1.

Finding Geometric CUSUM Schemes by Extrapolation for p_a Values below 0.001.

In this Section, the stability of the quantity hp_a is used to find the parameters of CUSUM schemes for values of p_a below 0.001. Unlike the case of interpolation, the use of this quantity yields schemes that have

slightly closer agreement (than the alternative h/k quantity) with the target ANNS(p_a) value. As in the case of interpolations, the user must decide on which of the five specified shift-sizes is to be detected, and on which of the six levels for ANNS(p_a) is suitable. The next step in the procedure is to find a value for k, and this is done as described earlier for interpolations, using equation (3) and the multipliers in Table 8.

A simple extrapolation procedure is then used to find a value for h. For each shift-size and chosen level for ANNS(p_a) the quantities hp_a become increasingly stable for values of p_a below 0.01. The set of eight values for hp_a associated with the p_a values 0.008, 0.007,…,0.001 were used in each extrapolation. The largest and smallest of these values were discarded and the average of the six remaining values was taken as the estimated value for hp_a to be used for extrapolations for p_a values below 0.001. These 30 estimates for hp_a are presented in Table 10, for the six levels of ANNS(p_a) and the five shift-sizes.

Table 10: Estimated Values of hp_a to be used in Extrapolations for p_a values less than 0.001, for each of six levels of ANNS(p_a), and five levels of downward shift-size p_a/m

m	ANNS(p_a) = 25	ANNS(p_a) = 50	ANNS(p_a) = 100	ANNS(p_a) = 200	ANNS(p_a) = 300	ANNS(p_a) = 500
1.5	10.4922	6.0047	7.3632	8.8632	9.8282	11.1543
2	3.3987	4.3717	5.4785	6.6325	7.3382	8.2688
3	2.5547	3.3815	4.2560	5.1755	5.7238	6.4322
4	2.1875	2.9235	3.7256	4.5488	5.0420	5.6748
5	1.9383	2.6487	3.3890	4.1738	4.6342	5.2338

As an illustration, suppose we wish to find the (k, h) parameters of a geometric CUSUM scheme for detecting a downward shift from $p_a = 0.0001$ to $p_r = (0.0001)/3$, with the requirement that ANNS(p_a) should be at least 100. We begin by finding the value for k using equation (3) and the relevant multiplier from Table 8, and this gives $k = 14665$. Next, we pick out the value for hp_a from Table 10 (for ANNS(p_a) = 100, and m=3), and this value is 4.2560. Dividing this by p_a gives $h = 42560$. Simulation with 25 million repetitions was used to evaluate ANNS(p_a) for this CUSUM scheme, and it was estimated that ANNS(p_a) was 99.8, which is very close to the target of 100. The estimated standard error of this simulation-based estimate was quite small at 0.021.

Some further extrapolation results are presented in Table 11, for two levels of p_a (0.0005 and 0.0002), for several levels of ANNS(p_a), and for all five shift-sizes. It was possible to use the transition-matrix method given in Appendix 1 for some of the evaluations of ANNS(p_a), but for larger values of h, simulation was used, with 25 million repetitions. One can see that the ANNS(p_a) specifications for all 20 CUSUM schemes in Table 11 have been closely met .

Table 11: Results of Extrapolation to find Parameters of Geometric CUSUM Schemes

Parameter values (k, h) *found by extrapolation for various target levels of ANNS*(p_a) *and five shift-sizes. Ten CUSUM schemes are for* $p_a = 0.0005$, *and ten for* $p_a = 0.0002$. *For each scheme, ANNS*(p_a) *is evaluated.*

$p_a = 0.0005$	Target ANNS(p_a) = 50			Target ANNS(p_a) = 200		
Shift p_a/m	k	h	Evaluated ANNS(p_a)	k	h	Evaluated ANNS(p_a)
m						
1.5	2067	12009	50.1	2249	17726	193.9
2	2384	8743	49.8	2550	13265	195.0
3	2817	6763	49.9	3015	10351	200.2
4	3141	5847	49.6	3363	9098	200.7
5	3399	5297	49.6	3640	8348	200.1

$p_a = 0.0002$	Target ANNS(p_a) = 25			Target ANNS(p_a) = 100		
Shift p_a/m	k	h	Evaluated ANNS(p_a)	k	h	Evaluated ANNS(p_a)
m						
1.5	3801	52461	24.9	5442	36816	98.7
2	5579	16993	25.1	6203	27393	98.8
3	6673	12773	25.1	7332	21280	99.8
4	7439	10938	25.0	8178	18628	100.4
5	8097	9692	24.9	8851	16945	99.3

Summary and Discussion

The detection of a sudden downward shift from a known value (p_a) of a process-proportion, using a geometric CUSUM scheme with a Fast Initial Response (FIR) feature, has been considered. The observations to be used by this CUSUM are the ordered values of Conforming Run-lengths (CRLs), and the value of the CUSUM is updated whenever a nonconforming item is observed. It is possible for the CUSUM to signal during a run of conforming items, without having to wait for the updating of the CUSUM at the next nonconforming item, and this has been termed a curtailed geometric CUSUM chart. Curtailment of a geometric CUSUM is relevant for detection of *downward* shifts, because if the value of the CUSUM at a *nonconforming* item has reached or exceeded h, then a signal could have been produced by an earlier conforming item

The focus of the paper is the provision of guidance to quality engineers in choosing values for the parameters (k, h) of a geometric CUSUM chart, following the specification (by the quality engineer) of three quantities:

(1) the in-control value of the proportion p_a,

(2) the size of downward shift to be detected, and

(3) an acceptable rate for false signals from the CUSUM chart. The acceptable rate of false signals is specified by choosing a desired minimum level for ANNS(p_a) which is the average number of nonconforming items observed from the start-up of the CUSUM until the first signal, assuming that there has been no shift from the level p_a.

The detection performance of the CUSUM scheme is quantified by a steady-state evaluation of ANNS(p_r), the average number of nonconforming items from the shift until the CUSUM signals. Because of the steady-state evaluation, the Sequential Probability Ratio (SPR) formula does not give an optimal choice for the parameter k. Values for multipliers to be applied to the SPR-based values for k have been estimated, and these depend mainly on the level chosen for ANNS(p_a), but also on the size of shift to be detected.

In developing tables of recommended parameter values for k and h, searches were conducted to find the best (k, h) values for each of 27 levels of p_a in the range [0.20 to 0.001] for detection of each of five sizes of downward shift, for each of six levels of ANNS(p_a), so that 810 CUSUM schemes were found, and these schemes were organized into five tables. The possibility of using the values in these tables for interpolation and extrapolation has been investigated, and illustrative examples have been given. The interpolations are quite accurate in finding (k, h) values that come close to the ANNS(p_a) specification, for p_a values in the interval [0.06 to 0.001], with increasing accuracy for the lower levels of p_a. The extrapolations are shown to be very effective for the p_a values of 0.0005 and 0.0002 . Given that the geometric distribution converges to an exponential distribution for quite small values of the proportion p, it is also possible to use this extrapolation approach to find the parameters of exponential CUSUM schemes for detecting an increase in the mean time between events.

While it is assumed throughout the article that one has accurate information on the in-control value (p_a) of the proportion being monitored, an investigation has been carried out here on the performance of geometric CUSUM schemes for detecting a range of downward shift-sizes. This is related to investigations on CRL monitoring schemes where the parameters (p_a and p_r) are estimated (Zhang et al. 2017).

There is an equivalence between the geometric CUSUM chart and the Bernoulli CUSUM chart, and the correspondence between the two sets of parameters for these charts has been discussed in several articles, most recently in (Szarka and Woodall 2012). Thus, it is possible to find the parameters of a Bernoulli CUSUM chart for detecting a specified downward shift in a proportion by first finding the parameters of the corresponding geometric CUSUM scheme, using the methods presented here.

While it is assumed in this article that all items are inspected, the results may also be useful when some fraction of items (say, every tenth or every hundredth) is inspected. A treatment of sampling inspection for a geometric CUSUM chart was provided in (Bourke 2001b). Another area of current investigation is the design of a double geometric CUSUM scheme which would broaden the range of downward shift-sizes that can be detected quickly, but at the expense of greater complexity in the monitoring scheme.

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Appendix 1: Evaluation of ANNS(p_a) for detecting a downward shift

Table A1: The Transition Matrix **R** for the In-Control States {0, 1, 2, …(*h*-1)} for a Geometric CUSUM Chart for Detecting a Downward Shift in a Proportion: *the Case of Integer Values for k and h.*

	CUSUM Values				
0	1	2	…	(*h*-2)	(*h*-1)
$P[X \le k]$	$P[X = k+1]$	$P[X = k+2]$	…	$P[X = k+h-2]$	$P[X = k+h-1]$
$P[X \le k-1]$	$P[X = k]$	$P[X = k+1]$	…	$P[X = k+h-3]$	$P[X = k+h-2]$
$P[X \le k-2]$	$P[X = k-1]$	$P[X = k]$	…	$P[X = k+h-4]$	$P[X = k+h-3]$
… … …					
$P[X \le k-h+2]$	$P[X = k-h+3]$	$P[X = k-h+4]$	…	$P[X = k]$	$P[X = k+1]$
$P[X \le k-h+1]$	$P[X = k-h+2]$	$P[X = k-h+3]$	…	$P[X = k-1]$	$P[X = k]$

where $P[X \le j] = 1 - (1-p)^{j+1}$ and $P[X = j] = p(1-p)^j$

Consider the following equation from Brook and Evans (1972)

$$\boldsymbol{\mu} = (\mathbf{I} - \mathbf{R})^{-1}\mathbf{1} \qquad \text{(A1.1)}$$

For the case of a geometric CUSUM chart, the elements of the vector **μ** give the average number of CRLs until a signal occurs for a CUSUM starting out from each of the possible initial states 0, 1, 2,..., (*h*-1).

Consider the value in position (1+nint[*h*/2]) of the vector **μ**, (where nint[*x*] denotes the nearest integer to *x*). This gives the average number of CRLs until a signal for a CUSUM with a starting value of nint[*h*/2]. However, this presumes that we only check the CUSUM value for a signal when a nonconforming item is observed (Procedure A in the second Section). In this article we assume that Procedure B (the curtailed geometric CUSUM) is in use, so we need to subtract 1 from the relevant value in vector **μ** to get the value of ANNS(p_a). (This adjustment is based on simulation results discussed at the end of the third Section.)

Table A2: Illustration of the Transition Matrix **R** for the In-Control States for a Geometric CUSUM Chart for Detecting a Downward Shift in a Proportion*: the Case of Rational Values for k and h.*
$k = 7/3$, $h = 10/3$

CUSUM Values

0	1/3	2/3	3/3	4/3	5/3	6/3	7/3	8/3	9/3
$P[X \le 2]$	0	$P[X = 3]$	0	0	$P[X = 4]$	0	0	$P[X = 5]$	0
$P[X \le 2]$	0	0	$P[X = 3]$	0	0	$P[X = 4]$	0	0	$P[X = 5]$
$P[X \le 1]$	$P[X = 2]$	0	0	$P[X = 3]$	0	0	$P[X = 4]$	0	0
$P[X \le 1]$	0	$P[X = 2]$	0	0	$P[X = 3]$	0	0	$P[X = 4]$	0
$P[X \le 1]$	0	0	$P[X = 2]$	0	0	$P[X = 3]$	0	0	$P[X = 4]$
$P[X = 0]$	$P[X = 1]$	0	0	$P[X = 2]$	0	0	$P[X = 3]$	0	0
$P[X = 0]$	0	$P[X = 1]$	0	0	$P[X = 2]$	0	0	$P[X = 3]$	0
$P[X = 0]$	0	0	$P[X = 1]$	0	0	$P[X = 2]$	0	0	$P[X = 3]$
0	$P[X = 0]$	0	0	$P[X = 1]$	0	0	$P[X = 2]$	0	0
0	0	$P[X = 0]$	0	0	$P[X = 1]$	0	0	$P[X = 2]$	0

where $P[X \le j] = 1 - (1-p)^{j+1}$ and $P[X = j] = p(1-p)^j$

Note 1: The 10 in-control states listed for the columns also refer to the rows.

Note 2: In the illustration, the denominator of the rational numbers is 3 (chosen here for illustration to enable the matrix to fit on the page). In the corresponding evaluations of the ANNS quantities for Tables 1 to 5, the denominators used were 5 or 10. Larger values than 10 lead to quite large transition matrices.

Note 3: Unlike the case of integer parameter-values, there are many zeros in the transition matrix, indicating that the corresponding "transitions" are impossible.

For example, if the CUSUM value is 2/3, and we observe a CRL value of 3, the next CUSUM value is 4/3, while, if the CUSUM value is 2/3, and we observe a CRL value of 4, the next CUSUM value is 7/3.

Appendix 2: Evaluation of ANNS(p_r) for Detecting a Downward Shift

In applying a transition-matrix based approach, it is necessary to assume that the CUSUM is operated according to procedure A (discussed in the second Section). Once $ANNS_A(p_r)$ has been evaluated, we subtract 1 to get $ANNS_B(p_r)$, as mentioned at the end of the third Section. To avoid notational clutter, we use $ANNS(p_r)$ in the presentation below rather than $ANNS_A(p_r)$.

When a shift from the value of p_a occurs, it can happen in either of the following ways, each requiring a separate method for evaluating ANNS:

1. A shift can occur immediately after a conforming item (because of the random-shift assumption). This gives rise to a *mixed* or *cross-over* CRL consisting of two parts: the first part from the in-control process with $p = p_a$, and the second from the shifted process with $p = p_r$. We refer to the associated evaluation of $ANNS(p_r)$ as $ANNS(p_r$, with cross-over CRL).
2. A shift can happen immediately after a non-conforming item, so that there is no cross-over CRL. We refer to the associated evaluation of $ANNS(p_r)$ as $ANNS(p_r$, no cross-over CRL).

Taking account of the methodology presented in Bourke (2001) we have

$$ANNS(p_r) = (p_r)ANNS[p_r, \text{ no cross-over CRL}] + (1 - p_r)ANNS[p_r, \text{ with cross-over CRL}] \qquad (A2.1)$$

One can see that the second term here is the main contributor to the value of $ANNS(p_r)$.

In the evaluation of each of the ANNS terms on the right-hand side of equation (A2.1), it is necessary to find the steady-state distribution of CUSUM values when $p = p_a$. In order to do this, we construct a transition matrix that causes the CUSUM value to be re-set to the head-start value $h/2$ each time a false signal occurs from the CUSUM. The methodology for doing this has been developed by Crosier (1986) and Lucas and Saccucci (1990).

We denote the resulting steady-state distribution as $\mathbf{p}_{ss}$, whose elements give the relative frequencies of the non-signalling CUSUM values 0,1,2,...(h-1) prior to the occurrence of the shift from p_a to p_r.

The first of the ANNS terms on the right-hand side of equation (A2.1) is as follows:

$$ANNS[p_r, \text{ no cross-over CRL}] = \mathbf{p}_{ss}^T\boldsymbol{\mu} \qquad (A2.2)$$

where $\boldsymbol{\mu}$ is given by equation (A1.1), with $p = p_r$.

In evaluating the second ANNS term on the right-hand side of equation (A2.1) we need to modify the steady-state distribution ($\mathbf{p}_{ss}$) over the non-signalling CUSUM states, because of the cross-over CRL. This modified distribution is as follows

$$\mathbf{r}_{ss} = \mathbf{p}_{ss}\mathbf{K} \qquad (A2\text{-}3)$$

where $\mathbf{K}$ is the transition matrix for the cross-over CRL. The matrix $\mathbf{K}$ has the same structure as given in Appendix A1, but the distribution of X (which is now the cross-over CRL) is a convolution of two geometric distributions, with parameters p_a and p_r. We then have

$$ANNS[p_r, \text{ with cross-over CRL}] = [1 - \textstyle\sum(\mathbf{r}_{ss})_i] + (\mathbf{r}_{ss})^T(1+\boldsymbol{\mu}) \qquad (A2\text{-}4)$$

The term in square brackets on the right-hand side of equation (A2-4) is the probability of a signal from the cross-over CRL, and $\boldsymbol{\mu}$ is again given by equation (A1.1), with $p = p_r$.