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Mathematische Annalen

Oscillating asymptotics and conjectures of Andrews

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Abstract

In 1986, Andrews (Am Math Mon 93(9):708–711, 1986; Adv Math 61(2):156–164, 1986) studied the function $\sigma(q)$ from Ramanujan's "Lost" Notebook, and made several conjectures on its Fourier coefficients $S(n)$, which count certain partition ranks. In 1988, Andrews et al. (Invent Math 91(3):391–407, 1988) famously resolved these conjectures, relating the coefficients $S(n)$ to the arithmetic of $\mathbb{Q}(\sqrt{6})$; this relationship was further expounded upon by Cohen (Invent Math 91(3):409–422, 1988) in his work on Maass waveforms, and was more recently extended by Zwegers (Q J Math 63(3):753–770, 2012) and by Li and Roehrig (Q J Math 76:367–380, 2025). A closer inspection of Andrews' original work on $\sigma(q)$ reveals additional related functions and conjectures, which we study in this paper. In particular, we study the function $v_1(q)$, also from Ramanujan's "Lost" Notebook, a q-hypergeometric series with partition-theoretic Fourier coefficients $V_1(n)$, and prove two of Andrews' conjectures on $V_1(n)$ which are parallel to his original conjectures on $S(n)$. Our methods differ from those used in Andrews et al. (Invent Math 91(3):391–407, 1988), and require a blend of novel techniques inspired by Garoufalidis' and Zagier's recent work on asymptotics of Nahm sums (Garoufalidis and Zagier in Ramanujan J 55:1, 2021; SIGMA Symmetry Integr Geom Methods Appl 19:Paper No. 082, 2023), with classical techniques including the Circle Method in Analytic Number Theory; our methods may also be applied to determine the asymptotic behavior of similar q-hypergeometric series of interest which are not amenable to classical techniques. We also offer explanations of additional

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related conjectures of Andrews, ultimately connecting the asymptotics of $V_1(n)$ to the arithmetic of $\mathbb{Q}(\sqrt{-3})$.

Mathematics Subject Classification 11P82 · 33B30 · 33D99

1 Introduction and statement of results

In [5], Andrews, Dyson, and Hickerson famously studied the q-hypergeometric series

$$\sigma(q) := \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n} =: \sum_{n=0}^{\infty} S(n)q^n,$$

found in Ramanujan's "Lost" Notebook, along with its companion

$$\sigma^*(q) := 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n},$$

which was discovered later. Here and throughout, we let $q := e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, the upper half-plane. Moreover, we define the q-Pochhammer symbol by $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ for $n \in \mathbb{N}_0 \cup \{\infty\}$. On one hand, these functions can be interpreted combinatorially, e.g., when expanded as a q-series, the coefficients $S(n)$ of $\sigma(q)$ count the difference between the number of partitions into distinct parts with even and odd rank. The authors of [5] were in part motivated to study these functions following Andrews' earlier conjectures [3]:

Conjecture (Conjecture 1 [3]). $\limsup |S(n)| = +\infty$.

Conjecture (Conjecture 2 [3]). $S(n) = 0$ *for infinitely many* n.

By showing a deep connection between these hypergeometric series and the arithmetic of $\mathbb{Q}(\sqrt{6})$, extending beyond their combinatorial interpretations, Andrews-Dyson-Hickerson [5] succeeded in proving Andrews' two conjectures on $\sigma(q)$ above. For example, we now know that the coefficients of $\sigma(q)$ may also be defined by a Hecke L-function which is a certain sum over ideals in $\mathbb{Z}[\sqrt{6}]$ [5, 11]. In an extension of the work in [5], Cohen [11] further constructed a Maass waveform from (the coefficients of) σ and σ^*, which Zwegers generalized in [50], by constructing a family of what he termed mock Maass theta functions associated to indefinite binary quadratic forms. This was used to provide further examples of similar q-series related to the Maass waveforms in [8, 26, 28]. It has very recently been reconstructed using theta integrals by Li and Roehrig [29], who used this as a motivational example to discover new real-analytic modular forms whose Fourier coefficients are given by logarithms of real quadratic numbers.

In the same paper [3], Andrews made further conjectures like those above for $\sigma(q)$ for another function $v_1(q)$ also from Ramanujan's "Lost" Notebook, defined by

$$v_1(q) := \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} =: \sum_{n\geq 0} V_1(n)q^n. \tag{1}$$

The function $v_1(q)$ admits a similar combinatorial interpretation to $\sigma(q)$. A partition is called odd-even if the parity of the parts alternates, starting with the smallest part odd. The rank, defined as the difference between the largest part and the number of parts, of an odd-even partition is even. The coefficients $V_1(n)$ count the difference between the number of odd-even partitions of n with rank congruent to 0 (mod 4) and congruent to 2 (mod 4) (see also [2]).

While he noted that the growth of $|V_1(n)|$ "is not very smooth," Andrews conjectured that there "appear[s] to be great sign regularity". More precisely, he states the following conjectures.

Conjecture (Conjecture 3 [3]). *We have that* $|V_1(n)| \to \infty$ *as* $n \to \infty$.

Conjecture (Conjecture 4 [3]). *For almost all* n, $V_1(n)$, $V_1(n+1)$, $V_1(n+2)$ *and* $V_1(n+3)$ *are two positive and two negative numbers.*

Conjecture (Conjecture 5 [3]). *For* $n \geq 5$ *there is an infinite sequence* $N_5 = 293$, $N_6 = 410$, $N_7 = 545$, $N_8 = 702$, ..., $N_n > 10n^2$, ... *such that* $V_1(N_n)$, $V_1(N_n+1)$, $V_1(N_n+2)$ *all have the same sign.*

Conjecture (Conjecture 6 [3]). *With reference to Conjecture 3, the numbers* $|V_1(N_n)|$, $|V_1(N_n+1)|$, $|V_1(N_n+2)|$ *contain a local minimum of the sequence* $|V_1(j)|$.

While of interest in their own right, we now know thanks in particular to work in [5, 11, 50] that $\sigma(q)$ and related functions are also of interest due to connections to real quadratic fields and Maass waveforms; Andrews himself also put $\sigma(q)$, $v_1(q)$ and related functions into a broader context by asking more general questions about q-series with bounded and unbounded coefficients in [3]. Our main results in this paper are as follows.

Theorem 1.1 *Andrews' Conjecture 4 is true. Moreover, Andrews' Conjecture 3 is true in the following sense:* $|V_1(n)| \to \infty$ *as* $n \to \infty$ *away from a set of density* 0.

Remark After computational and theoretical investigations, we have modified Andrews' Conjecture 3 to say "$|V_1(n)| \to \infty$ as $n \to \infty$ away from a set of density 0" as above. This refined conjecture will follow from the arguments needed to prove Andrews' Conjecture 4 (see Sect. 5).

Remark See Sect. 6 for our explanations of Andrews' Conjecture 5 and Conjecture 6, ultimately relating the asymptotics of $V_1(n)$ to the arithmetic of $\mathbb{Q}(\sqrt{-3})$.

Figure 1 shows the first 1000 values for $V_1(n)$ together with their sign patterns; one can see that the asymptotics of $V_1(n)$ appears to depend on n (mod 4). Moreover, the sequence can be divided into 4 reoccurring sections with the following patterns for $\text{sign}(V_1(n))$, $n \equiv n_0 \pmod 4$.

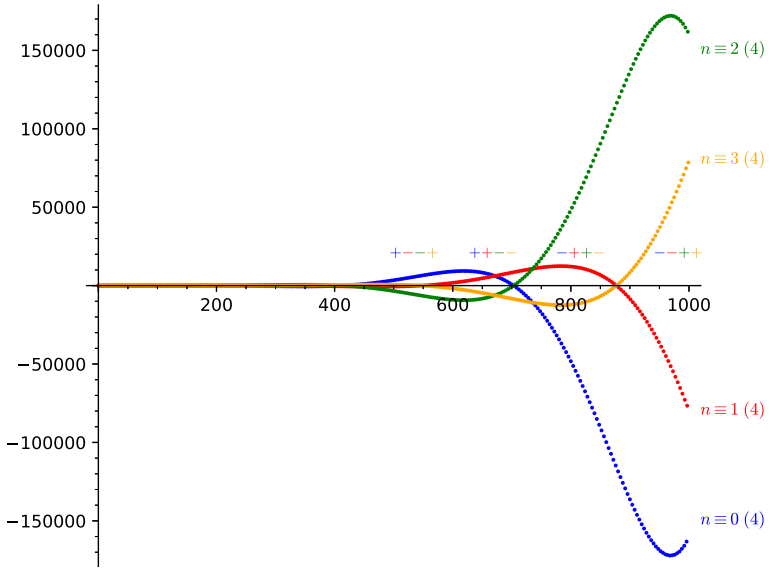


Fig. 1 $V_1(n)$ for $n = 1, \ldots, 1000$

n_0	0	1	2	3
Section 1	+	−	−	+
Section 2	+	+	−	−
Section 3	−	+	+	−
Section 4	−	−	+	+

For example, in Fig. 1, we see the sign pattern $+--+$ between $n = 546$ and $n = 702$ (Sect. 1), the sign pattern $++--$ between $n = 703$ and $n = 877$ (Sect. 2), etc. We ultimately establish the sign regularity of $V_1(n)$ in Theorem 1.1 after establishing the asymptotic behavior of $v_1(q)$, our second main result.

1.1 The asymptotics of v_1

Here and throughout, we let $e(u) := e^{2\pi i u}$. We make extensive use of the dilogarithm, which is a natural extension of the usual logarithm function, and can be defined by the power series

$$\mathrm{Li}_2(z) := \sum_{n \geq 1} \frac{z^n}{n^2}$$

for $|z| < 1$, and naturally extended to the cut plane $\mathbb{C}\setminus[1,\infty)$ by analytic continuation, yielding

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u}$$

for $z \in \mathbb{C}\setminus[1,\infty)$. We note that we also use a slightly different branching of Li_2 for the proof of Theorem 1.2, see Sect. 3.

To state our results, we require the Bloch–Wigner dilogarithm (see e.g. [48])

$$\mathrm{D}(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z|.$$

Theorem 1.2 *Let $\zeta = e(\alpha) \in \mathbb{C}$ be a root of unity of order $m \in \mathbb{N}_0$.*

(1) *If $4 \nmid m$, then $v_1(\zeta e^{-z}) = O(1)$ as $z \to 0$ along any ray in the right half-plane.*
(2) *If $4|m$, then as $z \to 0$, on a ray in the right half-plane with $0 \neq |\arg z| < \frac{\pi}{2}$, we have*

$$\begin{aligned} v_1(\zeta e^{-z}) &= \exp\left(\frac{16V}{zm^2}\right)\sqrt{\frac{2\pi i}{z}}\gamma^+_{(\alpha)}(z)(1+O(|z|^L)) \\ &+ \exp\left(\frac{-16V}{zm^2}\right)\sqrt{\frac{2\pi i}{-z}}\gamma^-_{(\alpha)}(z)(1+O(|z|^L)) \end{aligned} \tag{2}$$

for all $L > 0$, where V is given in terms of the Bloch–Wigner dilogarithm D by

$$V = \mathrm{D}(e(1/6))\frac{i}{8} = 0.1268677\ldots i,$$

and the power series $\gamma^{\pm}_{(\alpha)}(z) \in \mathbb{C}[[z]]$ are defined in Sect. 3.3.

1. *In particular, for $\zeta = \pm i$, the power series are defined in (25), resp. (27) and the first terms are given by*

$$\begin{aligned} \gamma^+_{(1/4)}(z) &= \overline{\gamma^-_{(3/4)}(z)} = \gamma^+\left(1+\left(\frac{1}{3}-\frac{77}{216}\sqrt{3}\right)iz \right. \\ &\left. +\left(-\frac{89449}{31104}+\frac{647}{648}\sqrt{3}\right)z^2+O(|z|^3)\right), \\ \gamma^-_{(1/4)}(z) &= \overline{\gamma^+_{(3/4)}(z)} = \gamma^-\left(1+\left(\frac{1}{3}+\frac{77}{216}\sqrt{3}\right)iz \right. \\ &\left. +\left(-\frac{89449}{31104}-\frac{647}{648}\sqrt{3}\right)z^2+O(|z|^3)\right), \end{aligned}$$

where

$$\gamma^+ := \gamma^+_{(1/4)}(0) = \gamma^-_{(3/4)}(0) = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}} = 0.5280518\ldots,$$

$$\gamma^- := \gamma^-_{(1/4)}(0) = \gamma^+_{(3/4)}(0) = \frac{1}{2\sqrt[4]{3(2+\sqrt{3})}} = 0.2733397\ldots.$$

Remark (1) We have that $V = \frac{\mathcal{G}i}{8}$, where $\mathcal{G} = 1.0149...$ is the maximum of the Bloch–Wigner dilogarithm function $\mathrm{D}(z)$, attained at $e(1/6)$ [48]. We note that $\mathcal{G}$ is also the minimal hyperbolic volume of non-compact hyperbolic 3-manifolds. Since this volume is attained by the Gieseking manifold [12], $\mathcal{G}$ is also known as Gieseking's constant.

(2) Note that when z approaches zero on a given ray in the right half-plane with $\arg(z) > 0$, the first term in (2) will be exponentially large while the other one will be exponentially small and vice versa if $\arg(z) < 0$.

1.2 The asymptotics of V_1

We use Theorem 1.2 to ultimately establish the following theorem giving the first term asymptotic approximation for $V_1(n)$.

Theorem 1.3 *As $n \to \infty$ we have*

$$V_1(n) = (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{e^{\sqrt{2|V|n}}}{\sqrt{n}} (\gamma^+ + (-1)^n \gamma^-) \left(\cos(\sqrt{2|V|n}) - (-1)^n \sin(\sqrt{2|V|n})\right) \times \left(1 + O\left(n^{-\frac{1}{2}}\right)\right) + O\left(n^{-\frac{1}{2}} e^{\sqrt{\frac{|V|n}{2}}}\right). \tag{3}$$

The sequence $V_1(n)e^{-\sqrt{2|V|n}}\sqrt{n}$ is plotted in Fig. 2.

From Figs. 1 and 2 it appears that $V_1(n)$ is arbitrarily small infinitely often. We discuss this observation alongside Andrews' Conjecture 5 and Conjecture 6 in Sect. 6, giving candidates for these points and providing strong numerical and heuristic evidence.

In fact, our methods give a stronger asymptotic for $V_1(n)$, namely, the (non-convergent) asymptotic formula

$$V_1(n) = \sum_{m \geq 1} \sum_{\substack{k=1 \\ \gcd(k,4m)=1}}^{4m} \frac{\zeta_{4m}^{-kn}}{\sqrt{2i}\sqrt{n}} \exp\left(\frac{2\sqrt{Vn}}{m}\right) \gamma^+_{\left(\frac{k}{4m}\right)} \left(\frac{1}{m}\sqrt{\frac{V}{n}}\right) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right) + \frac{\zeta_{4m}^{-kn}}{\sqrt{-2i}\sqrt{n}} \exp\left(\frac{2\sqrt{-Vn}}{m}\right) \gamma^-_{\left(\frac{k}{4m}\right)} \left(\frac{1}{m}\sqrt{\frac{-V}{n}}\right) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right). \tag{4}$$

Our (non-convergent) asymptotic in (4) is reminiscent of the original formula of Hardy and Ramanujan for $p(n)$ before it was extended to Rademacher's celebrated exact formula for $p(n)$, as well as the famous Andrews-Dragonette formula for the coefficients of the mock theta function $f(q)$ [1], and their conjectured exact formula which was proved by Bringmann and Ono in their breakthrough paper [9]. Rademacher's beautiful exact formula for $p(n)$, as it turns out, is intimately connected to the theory

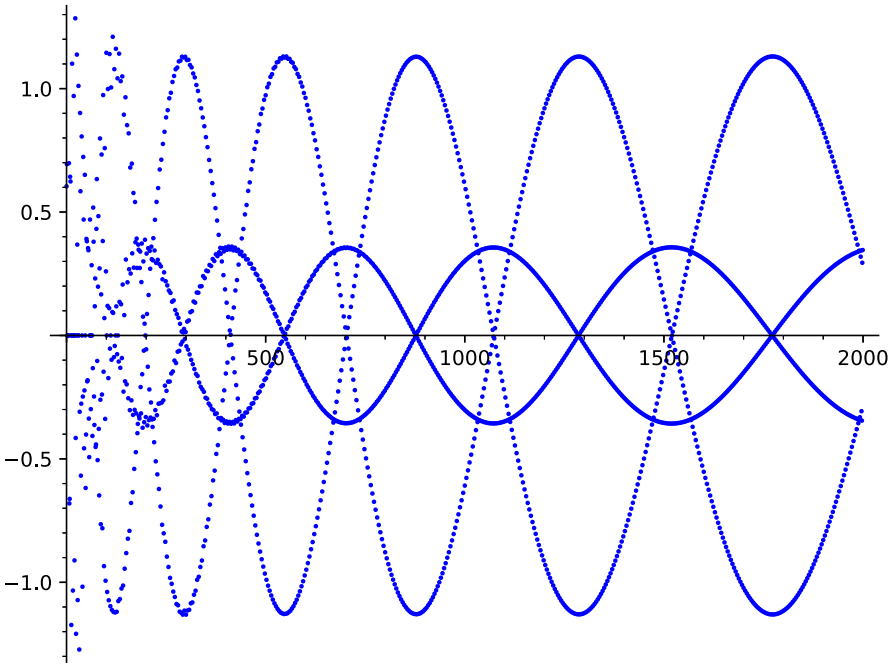


Fig. 2 $V_1(n)e^{-\sqrt{2|V|n}}\sqrt{n}$ for $0 \le n \le 2000$

of Maass–Poincaré series. It would be of interest to explore similar formulae and connections for $V_1(n)$ related to (4).

To establish (4), our calculations are as in Sect. 4.2 using the Circle Method and collecting contributions from all roots of unity as we now explain. Let $\zeta_{4m}^k := e^{\frac{2\pi ik}{4m}}$. First note that as $m \to \infty$ the $4m$-th roots of unity become dense on the unit circle (and the other roots of unity do not give any growing asymptotic contribution by Theorem 1.2), and that the asymptotic formulae given in Theorem 1.2 hold in cones across each root of unity. In turn, this means that we simply need to sum the contributions from each $4m$-th root of unity to obtain the claimed asymptotic behavior, and there is no error term being bound.

At each root of unity ζ_{4m}^k with $\gcd(k, 4m) = 1$ (so that it is primitive), we simply repeat the arguments given in Sect. 4.2, but instead considering an arc around ζ_{4m}^k parametrized by $q = \zeta_{4m}^k e^{-\lambda+i\theta}$ with $\theta \in (-\delta, \delta)$. After plugging in the asymptotic for $v_1(\zeta_{4m}^k e^{-z})$ given in Theorem 1.2, the same calculations using the saddle-point method then give (4).

Extracting the $m = 1$ term and combining the coefficients yields the first-term approximation given in Theorem 1.3. Of course, since $\gamma_\cdot^\pm$ are power series, the error terms make this asymptotic lose some meaning. This could be remedied by developing techniques to control the radial asymptotic to arbitrary precision. In particular one would need much stronger methods than the saddle point method provides us.

Returning to our main results, we remark that our methods used to prove Theorem 1.1 differ from the methods used in [5] to prove Andrews' related conjectures on $\sigma(q)$, which are not obviously applicable; however, it would be of interest to find a q-series identity for $v_1(q)$ using Bailey pairs or other methods which lends itself to revealing more information about the behavior of $V_1(n)$ in an analogous way. Instead, our methods are inspired by both newer methods of Garoufalidis and Zagier on asymptotics of Nahm sums [17, 18], and older methods in Analytic Number Theory including Wright's Circle Method and the saddle-point method. In particular, standard techniques in the literature are not well-suited to studying the radial asymptotic behavior of v_1. Instead, in order to prove Theorem 1.2 we determine an integral representation of v_1 which is similar to Watson's contour integral [19, 43]. Our novel approach has applications beyond simply the study of the function v_1; in particular it can be used to compute the asymptotic behavior of similar q-hypergeometric series which are not amenable to classical techniques.

The aforementioned sums named after Nahm were introduced in [32] in relation to characters of rational conformal field theories. They are a special class of q-hypergeometric series [19] given in the one-dimensional case by $\sum_{n\geq 0} q^{An^2/2+bn+c}/(q;q)_n$, for $A \in \mathbb{Q}_{>0}$, $b, c \in \mathbb{Q}$. In part motivated by the famous Rogers-Ramanujan identities [4], Zagier classified all one-dimensional modular Nahm sums in [48], thereby proving a conjecture of Nahm in the one-dimensional case.

These sums, in addition to having natural applications elsewhere, have previously-studied asymptotic properties which inform our study of Andrews' series $v_1(q)$ here. For example, Nahm sums come equipped with a Nahm equation; in the one-dimensional case the equation $1 - X = X^A$, whose solutions are conjecturally connected to the modularity of the Nahm sum at hand [10, 42, 48]. In particular, the dilogarithm of the unique solution in $(0, 1)$ determines the asymptotic behavior of the Nahm sum as q approaches a root of unity.

The q-hypergeometric series $v_1(q)$ studied here is nearly a Nahm sum, up to the change in sign appearing in the q-Pochhammer symbols $(-q^2;q^2)_n$ in the denominators of its summands. For $v_1(q)$, the analogue of the Nahm-equation becomes $(1-Q)^2 = -Q$, see (24) with $Q = e^{-4iv_0}$, and the dilogarithms of its solutions $e(\pm 1/6)$ appear in the asymptotics of $v_1(q)$ in Theorem 1.2. In particular, Theorem 1.2 implies that $v_1(q)$ is not a modular form. For more on these q-series and related recent work, see the previously mentioned [17, 18, 41, 48] and [32], as well as work of Calegari et al. [10] and Vlasenko and Zwegers [42], and their references.

We further note that Andrews also conjectures in [3] that the coefficients of three additional functions have behavior similar to that of the sequence $V_1(n)$. It is natural to expect that the analysis and methods of proof given here for $v_1(q)$ and $V_1(n)$ could also be applied to study the three additional functions in [3], as well as other similar functions (e.g., see [30]). Also note that v_1 is closely related to the function $\mathscr{O}$ studied by Jang in [23].

The remainder of the paper is structured as follows. In Sect. 2, we provide some background on Wright's Circle Method and the saddle-point method used later in the paper, and also establish some preliminary asymptotic results in the spirit of recent work of Garoufalidis and Zagier. In Sect. 3 we prove Theorem 1.2 on the asymptotics of

$v_1(q)$. In Sect. 4 we prove Theorem 1.3 on the asymptotics of $V_1(n)$. In Sect. 5, we prove Theorem 1.1 (Andrews' Conjectures 3 and 4). Finally, in Sect. 6 we offer explanations for Andrews' Conjectures 5 and 6 regarding the coefficients $V_1(n)$, ultimately relating them to the arithmetic of $\mathbb{Q}(\sqrt{-3})$.

2 Preliminaries

2.1 The saddle-point method

To asymptotically estimate certain integrals that appear in our work, we make use of the saddle-point method. We use the ideas presented by Olver [35], O'Sullivan [34], as well as the notes [24]. For convenience, we recall its essence here. Consider an integral of the shape

$$\int_\Gamma f(z)e^{Ag(z)}dz \tag{5}$$

where f, g are complex analytic functions and Γ is a contour in the complex plane. We wish to approximate the integral as $A \to \infty$.

Since f, g are analytic, we are able to continuously deform the contour Γ without changing the value of the integral (if one instead deforms the contour over poles of the integrand, one simply needs to take care to include residues). The points where the real part of $g(z)$ is maximized and the imaginary part of $g(z)$ is constant are called *saddle-points*, and are zeros of $g'(z)$.

By shifting the path Γ to a path running through the saddle-point and making appropriate shifts of the integration variable to center on the zero of $g(z)$, the integral (5) may be rewritten in terms of Gaussian-like integrals. These integrals may then be approximated by well-known means for large values of A.

2.2 Wright's Circle Method

Wright [45, 46] developed a modified version of Hardy and Ramanujan's Circle Method. Wright's work provides a very general approach to obtaining the asymptotic behavior of Fourier coefficients of generating functions whose radial asymptotic behavior towards roots of unity is known. We recall the essence of Wright's Circle Method here for the convenience of the reader.

Consider a generating function $f(q) := \sum_{n\geq 0} A(n)q^n$ with radius of convergence equal to one. The central idea is to use Cauchy's integral theorem to recover the coefficients $A(n)$ as

$$A(n) = \frac{1}{2\pi i}\int_C \frac{f(q)}{q^{n+1}}dq,$$

where C is a circular contour of radius less than one, transversed precisely once in the anticlockwise direction. Since we are able to choose the radius of C, we pick a radius

that tends to 1 as $n \to \infty$ (meaning that C tends to the unit circle). Now assume that in arbitrarily wide cones inside the unit disk toward roots of unity the generating function $f(q)$ has known asymptotic behavior. We then choose to place so-called major arcs around the roots of unity where $f(q)$ has the largest growth, and so-called minor arcs everywhere else. Using varying asymptotic methods, one is able to precisely determine the contribution of the major arcs to the asymptotic growth of the coefficients $A(n)$, which we call the main term and denote by $M(n)$. In the present paper, we use the saddle-point method. On the other hand, contributions from the minor arc are bounded more crudely in an error term, denoted $E(n)$. Overall this yields an asymptotic of the form $A(n) = M(n) + E(n)$.

Although Wright's Circle Method gives weaker bounds than the original version of Hardy and Ramanujan (and loses the possibility of exact formulae), it is much more flexible for dealing with non-modular generating functions, and has seen extensive use in the literature in recent years.

2.3 Asymptotics

Using the usual Bernoulli numbers $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}, \ldots$, the Bernoulli polynomials are defined for $n \in \mathbb{N}_0$ by

$$B_n(X) := \sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k}.$$

For $\varphi \in \mathbb{C}$, $|\varphi| = 1$, $|\arg \varphi| < \frac{\pi}{2}$ let $\widetilde{\log}$ be the logarithm such that $\mathrm{Li}_1^{\varphi}(e^{-iv}) = -\widetilde{\log}(1 - e^{-iv})$ has branch cuts whenever $\mathrm{Re}((v + 2\pi n)/\varphi) = 0$ for some $n \in \mathbb{Z}$ and $\mathrm{Re}(v) > 0$. In other words, the principal branch cuts are rotated by φ, cf. Fig. 3. For $v \in \mathbb{C}$ not on a branch cut we define the dilogarithm

$$\mathrm{Li}_2^{\varphi}(e^{-iv}) = \int_0^{\infty} \mathrm{Li}_1^{\varphi}\left(e^{-\varphi x - iv}\right) dx$$

where we avoid the branch cuts of $\mathrm{Li}_1^{\varphi}\left(1 - e^{-\varphi x - iv}\right)$ (cf. Fig. 3). Then $\mathrm{Li}_2^{\varphi}(e^{-iv})$ has the same branch cuts as $\mathrm{Li}_1^{\varphi}(e^{-iv})$ and jumps by $2\pi v$ when v crosses a branch cut (cf. [48]).

Moreover, for $m \in \mathbb{Z}_{\leq 0}$ the polylogarithm Li_m is defined inductively by

$$\mathrm{Li}_m^{\varphi}(z) := \frac{1}{z} \frac{d}{dz} \mathrm{Li}_{m+1}^{\varphi}(z) \tag{6}$$

where we note that Li_m^{φ} for $m \leq 0$ is independent of φ and the branching of Li_1^{φ}. Therefore, we will omit the index for $m \leq 0$ sometimes.

With this setup, we require some basic asymptotic estimates.

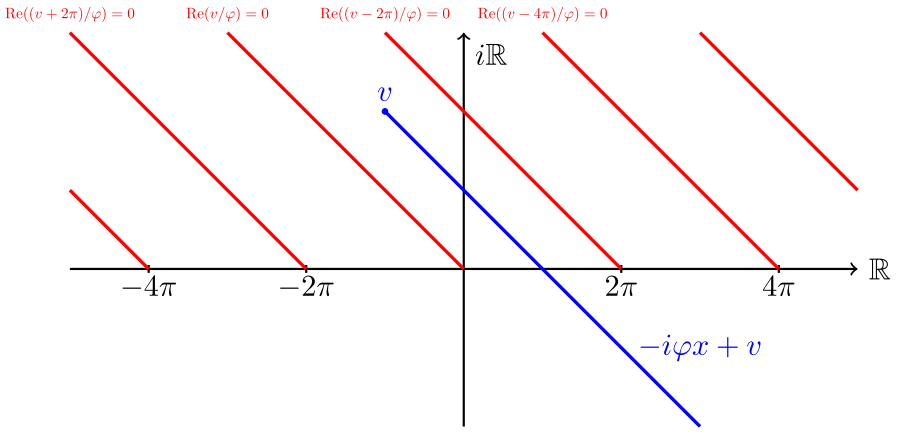


Fig. 3 The branch cuts of $\mathrm{Li}_s^{\varphi}(e^{-iv})$, $s = 1, 2$ and $-i\varphi x + v$ for $x \geq 0$

Lemma 2.1 *The following are true.*

(1) *Let $\alpha \in \mathbb{R}$. As $|t| \to \infty$ for t on a ray in $\mathbb{C}$ we have with $\pm = \mathrm{sign}(\mathrm{Re}(t))$*

$$\sin(\alpha(it - n_0)) = \pm\frac{1}{2i}\exp(\pm\alpha(t - n_0 i))(1 + o(|t|^{-L}))$$

for all $L \in \mathbb{N}$.

(2) *Let $\zeta = e(\frac{r}{m})$ be a root of unity of order $m \in \mathbb{N}$. As $z \to 0$ in the right half-plane, i.e. $q = \zeta e^{-z} \to \zeta$, we have*

$$(q;q)_\infty = \exp\left(-\frac{\pi^2}{6m^2 z} + \frac{z}{24}\right)\sqrt{\frac{2\pi}{mz}}\, e\left(\frac{s(-r,m)}{2}\right)(1 + o(|z|^L))$$

for all $L \in \mathbb{N}$. Here, $s(r,m)$ is the Dedekind sum defined by

$$s(r,m) := \sum_{l=1}^{m-1}\frac{l}{m}\left(\frac{rl}{m} - \left\lfloor\frac{rl}{m}\right\rfloor - \frac{1}{2}\right).$$

(3) *Assuming the notation above, if m is even we have*

$$(-q;q)_\infty = e^{-\frac{\pi^2}{6m^2 z} + \frac{z}{12}} Q(\zeta)(1 + o(|z|^L))$$

as $z \to 0$ in the right half-plane for all $L \in \mathbb{N}$ where

$$Q(\zeta) = e\left(\frac{s(-r, \frac{m}{2}) - s(-r,m)}{2}\right). \tag{7}$$

Proof We prove each part of the lemma separately as follows.

(1) We have

$$\sin(\alpha(it-n_0)) = \frac{e^{-\alpha(t+in_0)} - e^{\alpha(t+in_0)}}{2i}$$
$$= \begin{cases} \frac{\exp(\alpha(t+in_0))}{2i}\left(1-e^{-\alpha(2t+2in_0)}\right), & \text{if } \mathrm{Re}(t) > 0, \\ -\frac{\exp(-\alpha(t+in_0))}{2i}\left(1-e^{\alpha(2t+2in_0)}\right), & \text{if } \mathrm{Re}(t) < 0. \end{cases}$$

In each case the second exponential becomes exponentially small as $t \to \infty$.

(2) It is well known that the eta function

$$\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathbb{H}$ satisfies the modular transformation formula

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = e\left(\frac{a+d}{24c} - \frac{s(d,c)}{2} - \frac{1}{8}\right)(c\tau+d)^{\frac{1}{2}}\eta(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $c > 0$. Hence, for $\frac{r}{m} \in \mathbb{Q}$ with $(r,m) = 1$ we choose $a, b \in \mathbb{Z}$ such that $\left(\begin{smallmatrix} -a & -b \\ m & -r \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then we obtain for all $\tau \in \mathbb{H}$ with $q = e(\tau)$

$$(q;q)_\infty = q^{-\frac{1}{24}}\eta(\tau)$$
$$= q^{-\frac{1}{24}}(m\tau - r)^{-\frac{1}{2}} e\left(\frac{a+r}{24m} + \frac{s(-r,m)}{2} + \frac{1}{8}\right)\eta\left(\frac{a\tau+b}{-m\tau+r}\right).$$

Setting $\tau = \frac{r}{m} - \frac{z}{2\pi i}$ we have $m\tau - r = -\frac{mz}{2\pi i}$ and

$$\frac{a\tau+b}{-m\tau+r} = 2\pi i\frac{ar+bm}{m^2 z} - \frac{a}{m} = \frac{2\pi i}{m^2 z} - \frac{a}{m}.$$

Because z is in the right half-plane we have $\tau \in \mathbb{H}$ and we obtain

$$\eta\left(\frac{a\tau+b}{-m\tau+r}\right) = \eta\left(\frac{2\pi i}{m^2 z} - \frac{a}{m}\right)$$
$$= e\left(\frac{1}{24}\left(\frac{2\pi i}{m^2 z} - \frac{a}{m}\right)\right)\prod_{n\geq 1}\left(1 - e\left(n\left(\frac{2\pi i}{m^2 z} - \frac{a}{m}\right)\right)\right)$$
$$= \exp\left(-\frac{\pi^2}{6m^2 z}\right) e\left(\frac{-a}{24m}\right)\prod_{n\geq 1}\left(1 - \exp\left(\frac{-4\pi^2 n}{m^2 z} - \frac{2\pi i a n}{m}\right)\right).$$

As $z \to 0$ in the right half-plane we have

$$\prod_{n\geq 1}\left(1 - \exp\left(\frac{-4\pi^2 n}{m^2 z} - \frac{2\pi i a n}{m}\right)\right) = (1 + o(|z|^L))$$

for all $L \in \mathbb{N}$. Hence, we obtain with $q = e^{2\pi i\tau} = e(\frac{r}{m})e^{-z}$

$$(q;q)_\infty \;=\; \exp\left(-\frac{\pi^2}{6m^2z}+\frac{z}{24}\right)\sqrt{\frac{2\pi}{mz}}e\left(\frac{s(-r,m)}{2}\right)(1+o(|z|^L))$$

as claimed.

(3) The claim follows from the previous statement and the identity $(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}$.

□

Moreover, we will use a refinement of Lemma 2.1 in [17] following Lemma 4 in [44].

Lemma 2.2 *Let $w = e^{iv} \in \mathbb{C}$ such that if $\mathrm{Re}(v) > 0$ then $\mathrm{Re}((v+2\pi n)/\varphi) \neq 0$ for all $n \in \mathbb{Z}$. Moreover, let $\zeta \in \mathbb{C}$ be a root of unity of order $m \in \mathbb{N}$. Then we have as $z \to 0$ in the right half-plane, i.e. $q = \zeta e^{-z/m} \to \zeta$*

$$(wq;q)_\infty \;=\; \exp\left(-\frac{\mathrm{Li}_2^\varphi(w^m)}{mz}-\frac{1}{2}\mathrm{Li}_1^\varphi(w^m)+\sum_{t=1}^{m}\frac{t}{m}\mathrm{Li}_1^\varphi(\zeta^t w)+\psi_\zeta(z;w)\right)$$

where $\psi_\zeta(z;w) \in \mathbb{C}[[z]]$ has an asymptotic expansion as $z \to 0$

$$\psi_\zeta(z;w) = -\sum_{s=2}^{N}\sum_{t=1}^{m}B_s\left(1-\frac{t}{m}\right)\mathrm{Li}_{2-s}(\zeta^t w)\frac{z^{s-1}}{s!}+O(|z|^N)$$

for all $N \in \mathbb{N}$.

Proof Throughout the proof we write $z = \varphi h$ where $h \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{C}$ with $|\varphi| = 1$, $|\arg(\varphi)| < \frac{\pi}{2}$. We have

$$\widetilde{\log}(wq;q)_\infty \;=\; \sum_{n\geq 1}\widetilde{\log}(1-wq^n) \;=\; \sum_{t=0}^{m-1}\sum_{k\geq 1}\widetilde{\log}\left(1-\zeta^{-t}we^{-\varphi h(km-t)/m}\right)$$

and apply the Euler–Maclaurin summation formula [49, p.13] to obtain for every $N \in \mathbb{N}$

$$\begin{aligned}\widetilde{\log}(wq;q)_\infty = &\sum_{t=0}^{m-1}\frac{1}{h}\int_0^\infty \widetilde{\log}\left(1-\zeta^{-t}we^{-\varphi(x-th/m)}\right)dx\\ &+\sum_{n=0}^{N}\frac{(-1)^n B_{n+1}}{(n+1)!}\frac{d^n}{dx^n}\widetilde{\log}\left(1-\zeta^{-t}we^{-\varphi(x-th/m)}\right)dx|_{x=0}\,h^n+\mathcal{E}_{t,N}\end{aligned} \tag{8}$$

with

$$\mathcal{E}_{t,N} := h^N \int_0^\infty \mathrm{Li}_{1-N}\left(\zeta^{-t} w e^{-\varphi(x-th/m)}\right) \frac{\overline{B}_N(x)}{N!} dx.$$

The Euler–Maclaurin summation formula applies in this case, as the function defined by $x \mapsto \widetilde{\log}\left(1-\zeta^{-t} w e^{-\varphi(x-th/m)}\right)$ and all of its derivatives

$$\frac{d^n}{dx^n} \widetilde{\log}\left(1-\zeta^{-t} w e^{-\varphi(x-th/m)}\right) = -(-1)^n \varphi^n \mathrm{Li}^{\varphi}_{1-n}(\zeta^{-t} w e^{-\varphi x + zt/m}) \tag{9}$$

are of rapid decay as $x \to \infty$.

We have

$$\begin{aligned} \frac{1}{h}\int_0^\infty \widetilde{\log}\left(1-\zeta^{-t} w e^{-\varphi(x-th/m)}\right) dx &= -\frac{1}{z}\mathrm{Li}^{\varphi}_2(e^{zt/m}\zeta^{-t} w) \\ &= -\frac{1}{z}\sum_{l\geq 0} \frac{\mathrm{Li}^{\varphi}_{2-l}(\zeta^{-t} w)}{l!}\left(\frac{zt}{m}\right)^l \end{aligned}$$

where $\widetilde{\mathrm{Li}}^{\varphi}_s$ are the polylogarithms defined in (6). Evaluating the derivatives (9) at $x = 0$ gives

$$\begin{aligned} \frac{d^n}{dx^n} \widetilde{\log}\left(1-\zeta^{-t} w e^{-\varphi(x-th/m)}\right)|_{x=0} &= -(-1)^n \varphi^n \mathrm{Li}^{\varphi}_{1-n}(\zeta^{-t} w e^{zt/m}) \\ &= -(-1)^n \varphi^n \sum_{k\geq 0} \frac{\mathrm{Li}^{\varphi}_{1-n-k}(\zeta^{-t} w)}{k!}\left(\frac{zt}{m}\right)^k. \end{aligned}$$

Hence, (8) can be written as

$$\begin{aligned} \log(wq;q)_\infty = &-\sum_{t=0}^{m-1} \frac{1}{z}\sum_{l\geq 0} \frac{\mathrm{Li}^{\varphi}_{2-l}(\zeta^{-t} w)}{l!}\left(\frac{zt}{m}\right)^m \\ &-\sum_{t=0}^{m-1}\sum_{n=0}^{N} \frac{B_{n+1}}{(n+1)!}\sum_{k\geq 0} \frac{\mathrm{Li}^{\varphi}_{1-n-k}(\zeta^{-t} w)}{k!}\left(\frac{t}{m}\right)^k z^{k+n} + \mathcal{E}_{t,N} \end{aligned}$$

by using the distribution property of the dilogarithm [48, p.9]. Moreover, shifting $n \mapsto n-1$ and summing over $s = n+k = 1 \ldots N+1$ shows that $\log(wq;q)_\infty$ is equal to

$$\begin{aligned} &-\frac{\mathrm{Li}^{\varphi}_2(w^m)}{mz} - \sum_{t=0}^{m-1}\sum_{s=1}^{N+1}\sum_{n=0}^{s}\binom{s}{n} B_n\left(\frac{t}{m}\right)^{s-n} \mathrm{Li}^{\varphi}_{2-s}(\zeta^{-t} w)\frac{z^{s-1}}{s!} + \mathcal{E}_{t,N} + O(|z|^N) \\ &= -\frac{\mathrm{Li}^{\varphi}_2(w^m)}{mz} - \sum_{t=0}^{m-1} \mathrm{Li}^{\varphi}_1(\zeta^{-t} w)\left(\frac{t}{m}-\frac{1}{2}\right) \end{aligned}$$

$$-\sum_{t=0}^{m-1}\sum_{s\geq 2} B_s\left(\frac{t}{m}\right)\operatorname{Li}_{2-s}(\zeta^{-t}w)\frac{z^{s-1}}{s!}+\mathcal{E}_{t,N}+O(|z|^N)$$

where we collect all terms with $s \geq N$ in $O(|z|^N)$. Replacing t by $m-t \in \{1,\dots,m\}$ yields

$$\begin{aligned}
&-\frac{\operatorname{Li}_2^{\varphi}(w^m)}{mz}+\sum_{t=1}^{m}\operatorname{Li}_1^{\varphi}(\zeta^t w)\left(\frac{t}{m}-\frac{1}{2}\right)\\
&\quad-\sum_{t=0}^{m-1}\sum_{s\geq 2} B_s\left(1-\frac{t}{m}\right)\operatorname{Li}_{2-s}(\zeta^{t}w)\frac{z^{s-1}}{s!}+\mathcal{E}_{t,N}+O(|z|^N)\\
&=-\frac{\operatorname{Li}_2^{\varphi}(w^m)}{mz}-\frac{1}{2}\operatorname{Li}_1^{\varphi}(w^m)+\sum_{t=0}^{m-1}\frac{t}{m}\operatorname{Li}_1^{\varphi}(\zeta^t w)+\psi_{w,\zeta}(z),
\end{aligned}$$

where we used

$$\sum_{t=1}^{m}\operatorname{Li}_1^{\varphi}(\zeta^t w)\;=\;\sum_{t=1}^{m}\widetilde{\log}(1-\zeta^t w)\;=\;\widetilde{\log}\left(\prod_{t=1}^{m}1-\zeta^t w\right)\;=\;\widetilde{\log}(1-w^m)\;=\;\operatorname{Li}_1^{\varphi}(w^m).$$

Note that $\operatorname{Li}_{1-N}(z) \in (1-z)^N\mathbb{C}[z]$ for $N > 0$. Our assumption implies that $we^{-\varphi xh} \neq 1$, hence there exists $C > 0$ such that $|\operatorname{Li}_{1-N}(we^{-\varphi(hx+th/m)})| < C|we^{-\varphi(hx+th/m)}| < C|we^{-\varphi xh)}|$. Hence, we obtain for some $D > 0$

$$\begin{aligned}
|\mathcal{E}_{N,t}| \;&\leq\; \frac{h^N}{N!}C\int_0^{\infty}|we^{-\varphi xh}|\overline{B}_N(x)dx\\
&=\frac{h^N}{N!}C\sum_{x_0=0}^{\infty}\int_0^{\infty}|we^{-\varphi h(x_0+x)}|B_N(x)dx\\
&=\frac{h^N}{N!}C|w|\sum_{x_0=0}^{\infty}|e^{-\varphi hx_0}|\int_0^{1}|e^{-\varphi hx}|B_N(x)dx\\
&=\frac{h^N}{N!}\frac{C|w|}{1-|e^{-\varphi h}|}D\;=\;O(h^N),
\end{aligned}$$

since $|e^{-\varphi h}| < 1$. In particular, $\psi_\zeta(z;w)$ has the claimed asymptotic expansion. This completes the proof. □

3 Proof of Theorem 1.2

3.1 Proof of Theorem 1.2 (1)

We begin by showing that at any root of unity with order not divisible by 4, $v_1(q)$ converges.

Lemma 3.1 *Let $\zeta_N := e^{\frac{2\pi i}{N}}$. For any root of unity ζ_m^ℓ with $\gcd(\ell, m) = 1$ and $4 \nmid m$, we have that*

$$\lim_{z\to 0} v_1(\zeta_m^\ell e^{-z}) \;=\; v_1(\zeta_m^\ell) \;=\; \sum_{s=0}^{m-1} \frac{\zeta_{2m}^{\ell s(s+1)}}{(-\zeta_m^{2\ell};\, \zeta_m^{2\ell})_s} \times \begin{cases} 2 & \text{if } m \text{ is odd,} \\ \frac{4}{5} & \text{if } m \equiv 2 \bmod 4 \end{cases}$$

as $z \to 0$ on a ray in the right half-plane.

Proof Let $z = he^{i\vartheta}$ for $\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. With $q = \zeta_m^\ell e^{-z}$, we have

$$\frac{q^{n(n+1)/2}}{(-q^2;\, q^2)_n} \;=\; \prod_{j=1}^{n} \frac{1}{q^j + q^{-j}} \;=\; \frac{1}{2^n} \prod_{j=1}^{n} \frac{1}{\cosh\left(2\pi i j\ell/m + jz\right)}.$$

For fixed $\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\cosh\left(2\pi i j\ell/m + xe^{i\vartheta}\right)$ only depends on $\ell j \bmod m$ and never vanishes for $x \geq 0$ because $4 \nmid m$. Moreover, for each $j = 0, \ldots, m-1$, we have $\log|\cosh(2\pi i j\ell/m + xe^{i\vartheta})| \to \log|\cos(2\pi j\ell/m)|$ as $x \to 0$ and $\log|\cosh(2\pi i j\ell/m + xe^{i\vartheta})| \sim A_j x$ as $x \to \infty$ for some $A_j > 0$. Using this, as well the fact that there are only finitely many congruence classes modulo m, a calculus argument (e.g., using the Taylor expansion around $x = 0$) reveals that there exist $a, b > 0$ that only depend on ϑ and m such that

$$\log|\cosh(2\pi i j\ell/m + xe^{i\vartheta})| > ax - b\sqrt{x} + \log|\cos(2\pi j\ell/m)|$$

for all $x \geq 0$, $j \in \mathbb{Z}_{\geq 0}$. Thus, we have that

$$\begin{aligned} \left|\frac{q^{n(n+1)/2}}{(-q^2;\, q^2)_n}\right| &< \frac{1}{2^n} \prod_{j=1}^{n} \frac{\exp(-ajh + b\sqrt{jh})}{|\cos(2\pi j\ell/m)|} \\ &\leq \frac{1}{2^n} \exp(-an(n+1)h/2 + bn^{3/2}\sqrt{h}) \prod_{j=1}^{n} \frac{1}{|\cos(2\pi j\ell/m)|}, \end{aligned}$$

where we have used the inequality $\sum_{j=1}^{n} \sqrt{j} \leq n^{3/2}$. There exists a constant $C > 0$ such that

$$\exp(-an(n+1)h/2 + bn^{3/2}\sqrt{h}) < C$$

for all $h \geq 0$ and $n \in \mathbb{Z}_{\geq 0}$. Because $4 \nmid m$, we have $\prod_{j=1}^{m} |\cos(2\pi j\ell/m)| = 2^{\rho_m - m}$, where $\rho_m = 1$ if m is odd, and $\rho_m = 2$ if $m \equiv 2 \pmod 4$. Thus, we see that

$$
\begin{aligned}
|v_1(\zeta_m^\ell e^{-z})| &\le \sum_{n=0}^{\infty} \left| \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \right| \\
&\le C \sum_{n=0}^{\infty} \frac{1}{2^n} \prod_{j=1}^{n} \frac{1}{|\cos(2\pi j\ell/m)|} \\
&= C \sum_{s=0}^{m-1} \sum_{n=0}^{\infty} \frac{1}{2^{s+mn}} \prod_{j=1}^{s+mn} \frac{1}{|\cos(2\pi j\ell/m)|} && (n \mapsto s+mn) \\
&= C \sum_{s=0}^{m-1} \sum_{n=0}^{\infty} \frac{1}{2^{s+mn}} \prod_{l=0}^{n} \prod_{k=1}^{m} \frac{1}{|\cos(2\pi k\ell/m)|} \prod_{j=1}^{s} \frac{1}{|\cos(2\pi j\ell/m)|} && (j \mapsto lm+k) \\
&= C \sum_{s=0}^{m-1} \prod_{j=1}^{s} \frac{1}{2^s |\cos(2\pi j\ell/m)|} \sum_{n=0}^{\infty} 2^{(m-\rho_m)(n+1)-mn} \\
&= \frac{C}{2^{\rho_m}} \sum_{s=0}^{m-1} \prod_{j=1}^{s} \frac{2^m}{2^s |\cos(2\pi j\ell/m)|} \sum_{n=0}^{\infty} \frac{1}{2^{\rho_m n}} \\
&= \frac{C}{2^{\rho_m}(1-2^{-\rho_m})} \sum_{s=0}^{m-1} \prod_{j=1}^{s} \frac{2^m}{2^s |\cos(2\pi j\ell/m)|}
\end{aligned}
$$

is finite. Hence, the theorem of dominated convergence implies that we can interchange the summation over n and the limit $q \to \zeta_m^\ell$. In other words, we have

$$\lim_{h \to 0} v_1(\zeta_m^\ell e^{-h}) = v_1(\zeta_m^\ell).$$

More precisely, with the notation from above we have

$$
\begin{aligned}
v_1(\zeta_m^\ell) &= \sum_{n=0}^{\infty} \frac{\zeta_{2m}^{\ell n(n+1)}}{(-\zeta_m^{2\ell};\zeta_m^{2\ell})_n} = \sum_{s=0}^{m-1} \sum_{n=0}^{\infty} \frac{\zeta_{2m}^{\ell(s+mn)(s+mn+1)}}{(-\zeta_m^{2\ell};\zeta_m^{2\ell})_{s+mn}} \\
&= \sum_{s=0}^{m-1} \sum_{n=0}^{\infty} \frac{(-1)^{(m+1)n}}{2^{\rho_m n}} \frac{\zeta_{2m}^{\ell s(s+1)}}{(-\zeta_m^{2\ell};\zeta_m^{2\ell})_s} = \left(\sum_{n=0}^{\infty} \frac{(-1)^{(m+1)n}}{2^{\rho_m n}} \right) \left(\sum_{s=0}^{m-1} \frac{\zeta_{2m}^{\ell s(s+1)}}{(-\zeta_m^{2\ell};\zeta_m^{2\ell})_s} \right) \\
&= \sum_{s=0}^{m-1} \frac{\zeta_{2m}^{\ell s(s+1)}}{(-\zeta_m^{2\ell};\zeta_m^{2\ell})_s} \times \begin{cases} 2 & \text{if } m \text{ is odd,} \\ \frac{4}{5} & \text{if } m \equiv 2 \bmod 4. \end{cases}
\end{aligned}
$$

□

3.2 Proof of Theorem 1.2, (3)

Throughout we assume that $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ and $0 \neq |\arg(\varphi)| < \frac{\pi}{2}$ and write $z = \varphi h$. We will present the case $q = ie^{-z} \to i$ for $z = \varphi h \to 0$ in a fixed ray in the right half-plane in detail. The case $q \to -i$ is analogous, and so we omit the proof for brevity.

We split up the sum defining $v_1(q)$ depending on $n \pmod 2$, i.e., consider separately

$$v_1^{[0]}(q) = \sum_{n\geq 0 \text{ even}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} = \frac{1}{(-q^2;q^2)_\infty} \sum_{n\geq 0 \text{ even}} (-i)^{n/2} e^{-zn(n+1)/2} (-e^{-2nz}q^2;q^2)_\infty,$$
$$v_1^{[1]}(q) = \sum_{n\geq 0 \text{ odd}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} = \frac{1}{(-q^2;q^2)_\infty} \sum_{n\geq 0 \text{ odd}} i^{(n+1)/2} e^{-zn(n+1)/2} (e^{-2nz}q^2;q^2)_\infty, \tag{10}$$

as we have

$$i^{n(n+1)/2} = \begin{cases} (-i)^{n/2}, & \text{if } n \text{ is even,} \\ i^{(n+1)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

To state our next result, recall that for $n \geq 0$ the q-Pochhammer symbol satisfies the classical formula

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

meaning that we may extend the definition of the q-Pochhammer symbol to $-n$ by defining

$$(a;q)_{-n} := \frac{(a;q)_\infty}{(aq^{-n};q)_\infty} = \frac{1}{(aq^{-n};q)_n}.$$

Then the q-Pochhammer symbol satisfies

$$(a;q)_{-n} = q^{n(n-1)/2} \frac{(-q/a)^n}{(q/a;q)_n}.$$

We will prove the following proposition which implies (3) in Theorem 1.2.

Proposition 3.2 *As $z \to 0$ in the right half-plane on a ray with $\arg z \neq 0$, we have with $q = ie^{-z}$*

$$v_1^{[0]}(q) = e^{-\frac{V}{z}} \sqrt{\frac{2\pi i}{-z}}\, \gamma^-_{(1/4)}(z)(1 + O(|z|^L)) + \phi^{[0]}_{(1/4)}(z)(1 + O(|z|^L)), \tag{11}$$
$$v_1^{[1]}(q) = e^{\frac{V}{z}} \sqrt{\frac{2\pi i}{z}}\, \gamma^+_{(1/4)}(z)(1 + O(|z|^L)) + \phi^{[1]}_{(1/4)}(z)(1 + O(|z|^L)) \tag{12}$$

for all $L > 0$, where $\gamma^{\pm}_{(1/4)}(z) \in \mathbb{C}[[z]]$ is defined in (25), resp. (27) and $\phi^{[0]}_{(1/4)}(z), \phi^{[1]}_{(1/4)}(z) \in \mathbb{C}[[z]]$ with

$$\phi^{[0]}_{(1/4)}(z) = \sum_{\substack{n<0: \\ n\equiv 0 \pmod 2}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} = -4i\, z - 48\, z^2 + \frac{2878}{3} i\, z^3 + 26704\, z^4 + O(|z|^5),$$
$$\phi^{[1]}_{(1/4)}(z) = \sum_{\substack{n<0: \\ n\equiv 1 \pmod 2}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} = 2 + 8i\, z - 96\, z^2 - \frac{5708}{3} i\, z^3 + 52640\, z^4 + O(|z|^5).$$

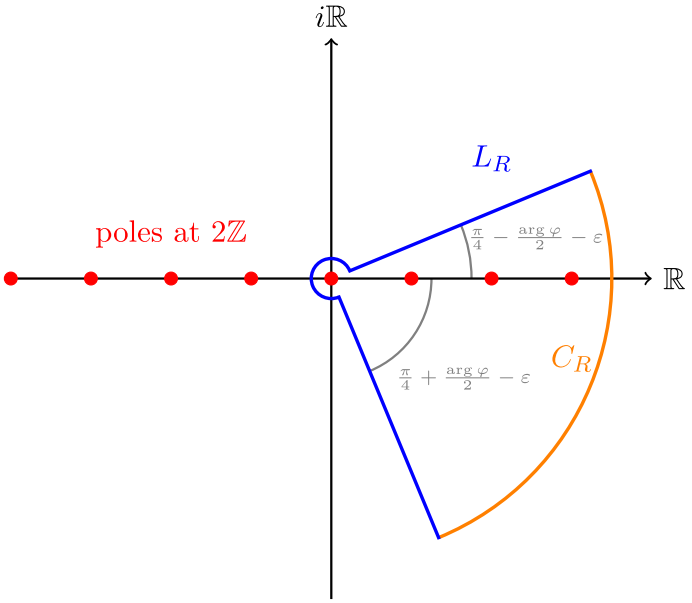


Fig. 4 Contours L_R and C_R

3.2.1 Watson's contour integral

Using Watson's contour integral ([19, 4.2], [43]), we establish the following integral representation for $v_1^{[0]}(q)$ and $v_1^{[1]}(q)$.

Lemma 3.3 *For $q = ie^{-z}$ with $\mathrm{Re}(z) > 0$, we have*

$$v_1^{[0]}(q) = \frac{-1}{2i}\frac{1}{(-q^2;q^2)_\infty}\int_{L_\infty} e^{\pi i s/4}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty \frac{1}{2\sin(\pi s/2)}ds, \tag{13}$$

$$v_1^{[1]}(q) = \frac{e^{\pi i 3/4}}{2}\frac{1}{(-q^2;q^2)_\infty}\int_{L_\infty} e^{-\pi i s/4}e^{-zs(s+1)/2}(e^{-2sz}q^2;q^2)_\infty \frac{1}{2\cos(\pi s/2)}ds \tag{14}$$

where L_∞ is the contour depicted in Fig. 4 as $R \to \infty$ for some small $\varepsilon > 0$.

Proof We will prove the statement for $v_1^{[0]}(q)$ in detail. The proof for $v_1^{[1]}(q)$ follows analogously.

As mentioned above, we write $z = \varphi h \in \mathbb{C}$ where $h \in \mathbb{R}_{>0}$ and $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ and $|\arg\varphi| < \frac{\pi}{2}$. The function $\frac{1}{\sin(\pi s/2)}$ has poles at $s \in 2\mathbb{Z}$ with residues $(-1)^{s/2}\,\frac{2}{\pi}$.

Hence, with $(-i)^{s/2}(-1)^{s/2} = i^{s/2} = e^{\pi i s/4}$ for $s \in 2\mathbb{Z}$, we obtain by Cauchy's theorem - if the subsequent integrals are convergent - using the contours from Fig. 4

$$\begin{aligned}
&\frac{-1}{2i}\frac{1}{(-q^2;q^2)_\infty}\lim_{R\to\infty}\int_{L_R+C_R} e^{\pi i s/4}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty\frac{1}{2\sin(\pi s/2)}ds\\
&\quad=\frac{1}{(-q^2;q^2)_\infty}\sum_{n\geq 0 \text{ even}}\operatorname*{Res}_{s=n}\left((-i)^{s/2}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty\frac{\pi(-1)^{s/2}}{2\sin(\pi s/2)}\right)\\
&\quad=\frac{1}{(-q^2;q^2)_\infty}\sum_{n\geq 0 \text{ even}}(-i)^{n/2}e^{-zn(n+1)/2}\,(-e^{-2nz}q^2;q^2)_\infty\\
&\quad=v_1^{[0]}(q).
\end{aligned}$$

It remains to prove the following 2 claims.

(1) The integral over L_∞ converges.
(2) The integral over the arc C_R vanishes as $R \to \infty$.

Before proving (1) and (2) we make some initial observations. If we parameterize L_R and C_R away from the indention around 0 by $s = re^{i\theta}$ with $0 < r \leq R$ and $\theta \in \left[-\frac{\pi}{4} - \frac{\arg\varphi}{2} + \varepsilon, \frac{\pi}{4} - \frac{\arg\varphi}{2} - \varepsilon\right]$. Then we have $-zs^2 = -hr^2e^{i(\arg\varphi+2\theta)}$ with $\arg\varphi + 2\theta \in [-\frac{\pi}{2} + 2\varepsilon, \frac{\pi}{2} - 2\varepsilon]$ for $\varepsilon > 0$, i.e. $\mathrm{Re}(-zs^2) < 0$. Similarly, one checks $\mathrm{Re}(-zs) < 0$.

In particular, the Pochhammer symbol can be uniformly bounded by

$$\begin{aligned}
\left|(-e^{-2sz}q^2;q^2)_\infty\right| &\leq \prod_{j\geq 1}1+|e^{-2sz}||q^{2j}| \;=\; \prod_{j\geq 1}1+|e^{-2\mathrm{Re}(sz)}||e^{-2j\mathrm{Re}(z)}|\\
&< \prod_{j\geq 1}1+|e^{-2j\mathrm{Re}(z)}|,
\end{aligned}\tag{15}$$

since $-\mathrm{Re}(sz) < 0$. Hence, we have

$$\begin{aligned}
&\left|\int_{L_R'+C_R} e^{\pi i s/4}e^{-zs(s+1)/2}(-e^{-2sz}q^2;q^2)_\infty\frac{1}{2\sin(\pi s/2)}\right|\\
&\quad\leq\prod_{j\geq 1}(1+|e^{-2j\mathrm{Re}(z)}|)\int_{L_R'+C_R} e^{\mathrm{Re}(\pi i s/4-zs(s+1)/2)}\left|\frac{1}{2\sin(\pi s/2)}\right|ds
\end{aligned}\tag{16}$$

where L_R' denotes the part of L_R away from the indention around 0.

As $\mathrm{Im}|s| \to \infty$, we have for all $L \in \mathbb{N}$

$$|\sin(\pi s/2)| = \frac{1}{2}e^{\pi|\mathrm{Im}(s)|/2}(1+o(|s|^{-L}))$$

and with $s = re^{i\theta}$ we compute

$$\begin{aligned}
&\operatorname{Re}\left(\frac{\pi i s}{4} - \frac{zs(s+1)}{2}\right) - \frac{\pi|\operatorname{Im}(s)|}{2} \\
&= \operatorname{Re}\left(\frac{\pi i r e^{i\theta}}{4} - \frac{h e^{i\arg\varphi} r^2 e^{2i\theta}}{2} - \frac{h e^{i\arg\varphi} r e^{i\theta}}{2}\right) - \frac{\pi|\operatorname{Im}(s)|}{2} \\
&= -\frac{\pi r\sin(\theta)}{4} + \operatorname{Re}\left(-\frac{hr^2 e^{i(\arg\varphi+2\theta)}}{2} - \frac{hre^{i(\arg\varphi+\theta)}}{2}\right) - \frac{\pi|\operatorname{Im}(s)|}{2} \\
&= -\frac{\pi r\sin(\theta)}{4} - \frac{r^2h^2\cos(\arg\varphi+2\theta)}{2} - \frac{rh\cos(\arg\varphi+\theta)}{2} - \frac{r\pi|\sin\theta|}{2} \\
&= -\frac{r^2h^2\cos(\arg\varphi+2\theta)}{2} - r\left(\frac{\pi\sin(\theta)}{4} + \frac{h\cos(\arg\varphi+\theta)}{2} - \frac{\pi|\sin\theta|}{2}\right).
\end{aligned}$$

Hence, the exponent in the integrand in (16) is eventually negative, since $\arg\varphi + 2\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and thus $\cos(\arg\varphi + 2\theta) > \delta > 0$ for some δ.

More precisely, we have for some constant $M > 0$, uniformly in θ,

$$\begin{aligned}
&-\frac{r^2h^2\cos(\arg\varphi+2\theta)}{2} - r\left(\frac{\pi\sin(\theta)}{4} + \frac{h\cos(\arg\varphi+\theta)}{2} - \frac{\pi|\sin\theta|}{2}\right) \\
&< -\frac{r^2h^2\delta}{2} + r\left(\frac{\pi}{4} + \frac{h}{2} - \frac{\pi}{2}\right) < -Mr^2
\end{aligned}$$

for R and r large enough.

Therefore, it is sufficient to prove both claims for the integral

$$\int e^{-Mr^2}\, ds.$$

Claim (1): The integral over L_∞ converges.
We consider the integral along the contour $\{re^{i\theta_\pm}, r \in \mathbb{R}_{>0}\}$ with $\theta_\pm = \pm\frac{\pi}{4} - \frac{\arg\varphi}{2}$. By the discussion above, the integral is

$$O\left(\int_0^R e^{-Mr^2}\, dr\right),$$

which converges as $R \to \infty$.

Claim (2): The integral over the arc C_R vanishes as $R \to \infty$.
Similarly, we see that the integral over C_R is eventually bounded by a constant times

$$\int_{(-\frac{\pi}{4}-\frac{\arg\varphi}{2}, \frac{\pi}{4}-\frac{\arg\varphi}{2})} e^{-MR^2}\, d\theta \to 0$$

as $R \to \infty$. □

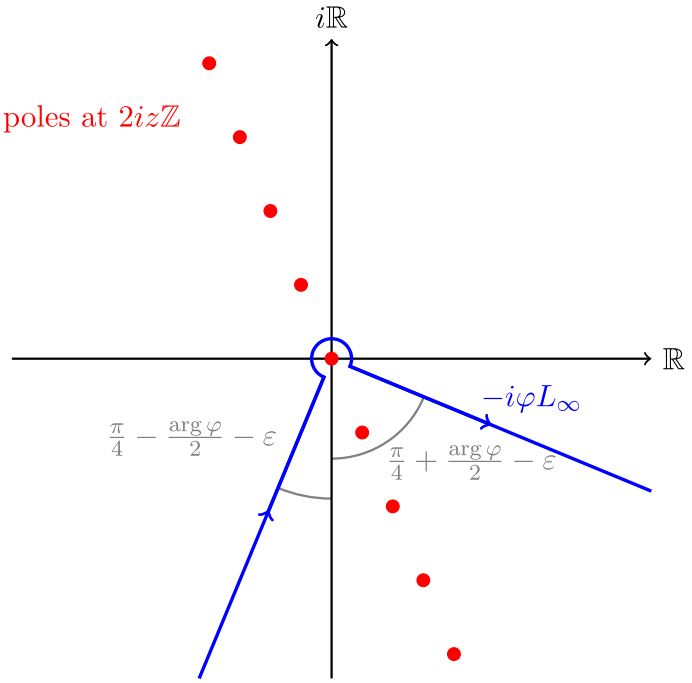


Fig. 5 The contour $-i\varphi L_\infty$

3.2.2 Sum over even integers

We will prove the statement for $v_1^{[0]}$ in detail. As the proof for $v_1^{[1]}$ follows *mutatis mutandis*, we will only sketch it in Sect. 3.2.3.

Proof of (11) We use the integral representation from Lemma 3.3 with $q = ie^{-z}$ and substitute $s = iv/z$ to obtain

$$v_1^{[0]}(q) = \frac{-1}{2z(-q^2;q^2)_\infty} \int_{-izL_\infty} e^{-\pi v/4z} e^{v^2/2z - iv/2} (-e^{-2iv} q^2; q^2)_\infty \frac{1}{\sin\left(\frac{\pi iv}{2z}\right)} dv, \tag{17}$$

where the contour $-izL_\infty$ is depicted in Fig. 5.

We consider the integral representation (17) of $v_1^{[0]}$ and change the contour $-izL_\infty$ to a contour $\mathcal{S}$ with fixed minimum distance from 0 and passing through $-\frac{\pi}{12}$. The poles of the integrand in (17) are at $v \in i2z\mathbb{Z}$, and as $z \to 0$ they accumulate at 0. Hence, if we integrate along the contour $\mathcal{S}$, all poles at $i2z\mathbb{Z}_{<0}$ eventually get shifted to the other side of the contour (cf. Fig. 6). In other words the integral (17) can be written as

$$\frac{-1}{2z(-q^2;q^2)_\infty} \int_{\mathcal{S}} e^{-\pi v/4z} e^{v^2/2z - iv/2} (-e^{-2iv} q^2; q^2)_\infty \frac{1}{\sin\left(\frac{\pi iv}{2z}\right)} dv$$

$$+\frac{-1}{2z(-q^2;q^2)_\infty}\sum_{\substack{n<0:\\ |2zn|<d_0}}\operatorname*{Res}_{v=-2izn}\left(e^{-\pi v/4z}e^{v^2/2z-iv/2}(-e^{-2iv}q^2;q^2)_\infty\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}\right) \tag{18}$$

for some $d_0 > 0$, which is the distance from 0 to where the contour crosses the line of poles. The residue of $\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}$ at $v = -2izn$ is given by $-(-1)^n 2z$ and thus

$$\begin{aligned}&\operatorname*{Res}_{v=-2izn}\left(e^{-\pi v/4z}e^{v^2/2z-iv/2}(-e^{-2iv}q^2;q^2)_\infty\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}\right)\\ &\quad=(-i)^n e^{-(2n^2-n)z}(-e^{-4nz}q^2;q^2)_\infty.\end{aligned}$$

As $z \to 0$ the residues can be collected in

$$\begin{aligned}\phi^{[0]}_{(1/4)}(z) &:= \frac{-1}{2z(-q^2;q^2)_\infty}\sum_{n<0}\operatorname*{Res}_{v=-2izn}\left(e^{-\pi v/4z}e^{v^2/2z-iv/2}(-e^{-2iv}q^2;q^2)_\infty\frac{1}{\sin\left(\frac{\pi iv}{2z}\right)}\right)\\ &= \frac{1}{(-q^2;q^2)_\infty}\sum_{n<0}(-i)^n e^{-(2n^2-n)z}(-e^{-4nz}q^2;q^2)_\infty.\end{aligned}$$

Using the q-Pochhammer symbol for negative indices this implies with $q = ie^{-z}$

$$\begin{aligned}\phi^{[0]}_{(1/4)}(z) &= \sum_{\substack{n<0:\\ n\equiv 0 \bmod 2}}\frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} = \sum_{\substack{l>0:\\ l\equiv 0 \bmod 2}} q^{-l(l-1)/2}(-1;q^2)_l\\ &= 2\sum_{\substack{m>0:\\ m\equiv 1 \bmod 2}} q^{-m(m+1)/2}(-q^2;q^2)_m.\end{aligned}$$

We have

$$(-q^2;q^2)_m = \prod_{j=1}^{m}\left(1+(-1)^j e^{-2jz}\right) \in z^{\lceil m/2\rceil}\mathbb{C}[[z]]$$

since $1+(-1)^j e^{-2jz} \in z\mathbb{C}[[z]]$ for $j = 1, \ldots, m$ odd. Hence, $\phi^{[0]}_{(1/4)}(z) \in \mathbb{C}[[z]]$ and the first terms are given by

$$\begin{aligned}\phi^{[0]}_{(1/4)}(z) &= -4iz - 48z^2 + \frac{2878}{3}iz^3 + 26704z^4 - \frac{28574401}{30}iz^5 - \frac{207245984}{5}z^6\\ &\quad + \frac{2683138049759}{1260}iz^7 + \frac{13245720939344}{105}z^8 - \frac{768005626895809921}{90720}iz^9\\ &\quad + O(z^{10}).\end{aligned}$$

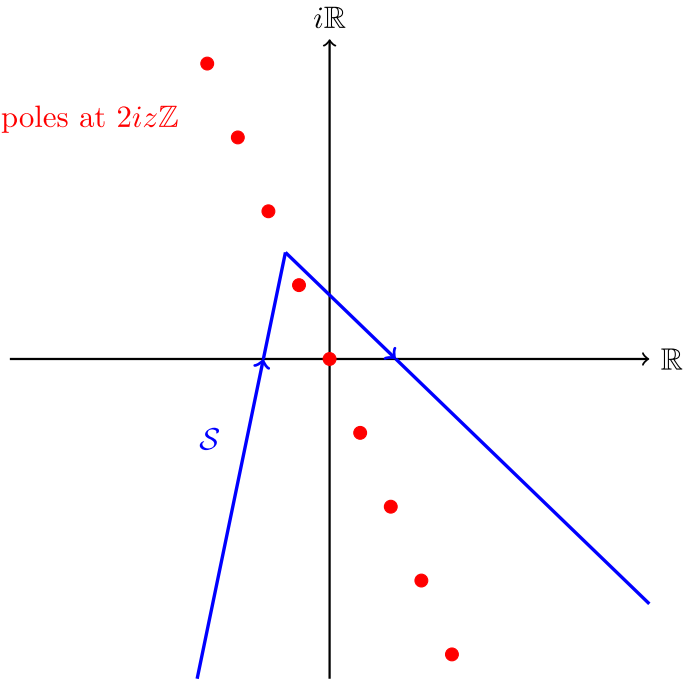


Fig. 6 A contour $\mathcal{S}$ (before applying Lemma 2.1)

We apply Lemmas 2.1 and 2.2 to the integrands in (18), to obtain that $v_1^{[0]}(q)$ is asymptotically equal to

$$\frac{ie^{\pi^2/48z-z/12}}{2z}\int_{\mathcal{S}} \operatorname{sign}(\operatorname{Re}(v/\varphi))\exp\left(\frac{-\operatorname{Li}_2^{\varphi}(e^{-4iv})+4v^2-2\pi v-4\operatorname{sign}(\operatorname{Re}(v/\varphi))\pi v}{8z}\right) \tag{19}$$

$$\exp\left(-\frac{iv}{2}+\frac{\operatorname{Li}_1^{\varphi}(-e^{-2iv})}{2}+\psi_{-1}(4z;-e^{-2iv})\right)dv \;+\; \phi^{[0]}_{(1/4)}(z) \tag{20}$$

which can be rewritten with

$$\begin{aligned} f(v) &= -\frac{\operatorname{Li}_2^{\varphi}(e^{-4iv})}{8}+\frac{v^2}{2}-\frac{\pi v}{4}-\operatorname{sign}(\operatorname{Re}(v/\varphi))\frac{\pi v}{2}, \\ g(z;v) &= \operatorname{sign}(\operatorname{Re}(v/\varphi))\exp\left(-\frac{iv}{2}+\frac{\operatorname{Li}_1^{\varphi}(-e^{-2iv})}{2}+\psi_{-1}(4z;-e^{-2iv})\right), \end{aligned} \tag{21}$$

as

$$v_1^{[0]}(q) \;=\; \frac{i}{2z}e^{\pi^2/48z-z/12}\int_{\mathcal{S}} e^{f(v)/z}g(z;v)dv\,(1+O(|z|^L)) \;+\; \phi^{[0]}_{(1/4)}(z)(1+O(|z|^L)) \tag{22}$$

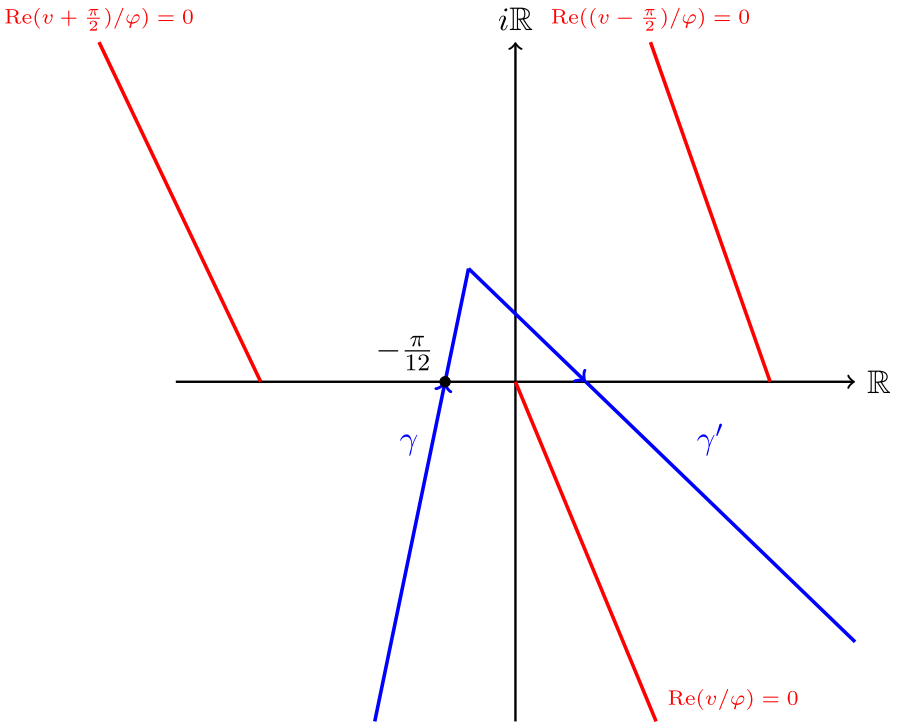


Fig. 7 The contours $\mathcal{S}$, parameterized by γ, γ' (after applying Lemma 2.1)

for all $L > 0$. Note that both f and g are holomorphic functions on the domain

$$\mathbb{C} \setminus \left(\{\varphi i \mathbb{R}_{<0}\} \bigcup_{0 \neq n \in \mathbb{Z}} \{n + \varphi i \mathbb{R}_{>0}\} \right) \tag{23}$$

as $\mathrm{Li}_2^{\varphi}(z)$ jumps by $2\pi i \log z$ when z crosses the cut on $\mathrm{Re}(z/\varphi) = 0, \mathrm{Re}(z) > 0$. Moreover, $e^{\frac{\mathrm{Li}_1(-e^{-2iv})}{2}}$ changes the sign when v crosses the line $\mathrm{Re}(z/\varphi) = 0,\ \mathrm{Re}(z) > 0$. The contour and the branch cuts of f are plotted in Fig. 7.

We compute

$$\begin{aligned} f'(v) &= -\frac{i}{2}\log(1-e^{-4iv}) + v - \frac{\pi}{4} - \mathrm{sign}(\mathrm{Re}(v/\varphi))\frac{\pi}{2} \\ f''(v) &= \frac{1+e^{-4iv}}{1-e^{-4iv}} \end{aligned}$$

and thus the critical points v_0 of f satisfy

$$(1-e^{-4iv_0})^2 = -e^{-4iv_0}, \tag{24}$$

in other words $e^{-4iv_0} = e(\pm\frac{1}{6})$. Using this along with the condition $f'(v) = 0$ at saddle points, one may easily classify all saddle points of f.

Since we are interested in saddle points between the two branch cuts in the upper half-plane emanating from $-\pi/2$ and 0 we easily check that there are two saddle points which maximize the real part of f, given by $-5\pi/12$ and $-\pi/12$. Away from the branch cuts, the function f is holomorphic in the complex plane, and so in particular we may deform the contour of integration by Cauchy's theorem. We choose the contour passing through the saddle point $-\pi/12$ given by the union of two pieces - again noting that by Cauchy's theorem we are free to choose our path of integration as long as it remains in the region where the integrand is holomorphic. The first piece is given by $\gamma(s) = -\pi/12 - s\sqrt{i\varphi}$ for $s \in (-c_0, \infty)$ where $c_0 = \pi/2 - \varepsilon$ for some fixed choice of $\varepsilon > 0$. Let $\varphi = -ie^{it}$ and $d(x) := -\frac{\pi}{12} + \frac{x}{\mathrm{Re}(\sqrt{i\varphi})}\sqrt{i\varphi}$, noting that $d(\pi/12)$ is the point at which $\gamma(s)$ intersects the line $i\mathbb{R}\varphi$. The second piece is given piecewise by

$$\gamma'(s) = \begin{cases} d(\pi/12) - is\sqrt{i\varphi}, & \text{if } 0 < t < \pi/2, \\ d(\pi/12) - is\sqrt{-i/\varphi}, & \text{if } \pi/2 \le t < 3\pi/4, \\ d(\pi/4) - is\sqrt{-i/\varphi}, & \text{if } 3\pi/4 \le t < \pi. \end{cases}$$

We now aim to show that $\mathrm{Re}(f(\gamma(s))/\varphi)$ is maximized at $s = 0$, i.e. at the saddle point, so that $\gamma(s)$ is a stationary contour. For this, we first rewrite

$$\mathrm{Re}\left(\frac{f(\gamma(s))}{\varphi}\right) = s^2\mathrm{Re}\left(\frac{f(\gamma(s))}{s^2\varphi}\right).$$

Now making the change of variable $s\sqrt{i\varphi} = \mathfrak{z}$ we obtain

$$s^2\mathrm{Re}\left(\frac{if(-\pi/12 - \mathfrak{z})}{\mathfrak{z}}\right).$$

The function inside the real part is holomorphic and non-constant on the open domain $D_T := \{\mathfrak{z} \in \mathbb{C} \colon \mathrm{Re}(\mathfrak{z} > 0, \mathrm{Im}(\mathfrak{z}) > 0, |z| < T)\}$ by construction, and so the maximum modulus principal for harmonic functions implies that it takes its maximal real part on the boundary of D_T. This occurs when $\varphi = \pm i$, when $|\mathfrak{z}| = s = T$ (and we take a limit $T \to \infty$), or when $s = 0$.

Firstly, using e.g. SageMath [51], we may explicitly compute a (complicated) expression for $\mathrm{Re}(\frac{f(\gamma(s))}{\varphi})$. Using this along with the fact that $\mathrm{Li}_2(z) \sim z$ as $|z| \to 0$ we have that $\mathrm{Re}(\frac{f(\gamma(s))}{\varphi}) \sim -cs$ for some positive constant c as $s \to \infty$. This means that on the boundary piece $|\mathfrak{z}| = s = T$ as $T \to \infty$ that the real part is strictly smaller than at the saddle point.

Now, when $s = 0$ for any φ we have equality, since we are at the saddle point itself.

Finally, we consider the cases where $\varphi = \pm i$. First let $\varphi = -i$. A direct calculation (again using SageMath) shows that there is equality of maximal real part to that at the saddle point for certain $s > 0$ (but it is never exceeded). Another direct calculation shows that on $\varphi = i$ and $s > 0$ the real part is strictly smaller than at the saddle point.

Noting that we do not allow these two boundary cases in our application shows that $\gamma(s)$ is a stationary contour for $s \in (0, \infty)$.

For $s \in (-c_0, 0)$ the argument is similar, simply noting that a direct calculation for $s = -c_0$ yields a strictly smaller real part than at the saddle point, and so overall we obtain that $\gamma(s)$ is a stationary contour for $s \in (-c_0, \infty)$. The calculations for the second piece of the contour are similar, using the maximum modulus principal on each piece.

We write $Q := e^{-4iv_0} = e(1/6) = \frac{1+\sqrt{3}i}{2}$. If we parameterize $\mathcal{S}$ as $v = -\frac{\pi}{12} + is\sqrt{z}$ in a small neighborhood around v_0 we obtain that the contribution corresponding to the stationary point $v_0 = -\frac{\pi}{12}$ is given by

$$\begin{aligned}&\frac{-1}{2\sqrt{z}} e^{\pi^2/48z + f(v_0)/z - z/12} \int \exp\left(-f''(-\tfrac{\pi}{12})\frac{s^2}{2} + \sum_{l\geq 3} \frac{f^{(l)}(Q)}{l!}(is)^l z^{l/2-1}\right) \\ &\quad g\left(-\frac{\pi}{12} + is\sqrt{z}; z\right) ds \\ &= \sqrt{\frac{2\pi i}{-z}} e^{\pi^2/48z + f(v_0)/z}\, \gamma^-_{(1/4)}(z)\end{aligned}$$

where

$$\begin{aligned}\frac{\pi^2}{48} + f(v_0) &= \frac{\pi^2}{48} - \frac{\mathrm{Li}_2(e^{-4iv_0})}{8} + \frac{v_0^2}{2} + \frac{\pi v_0}{4} = \frac{D(e(1/6))i}{8} \\ &= -V = -0.1268677\ldots i.\end{aligned}$$

With the definition of $\psi_{-1}(4z; -e^{-2iv})$ from Lemma 2.2 the power series $\gamma^-_{(1/4)}(z) \in \mathbb{C}[[z]]$ is defined as a formal Gaussian integration by

$$\begin{aligned}\gamma^-_{(1/4)}(z) = &\frac{1}{2\sqrt{-2\pi i}} e^{-\pi i/24 - z/12} \\ &\times \int \exp\left(-\sqrt{3}i\frac{s^2}{2} + \frac{is\sqrt{z}}{2} + \frac{1}{8}\sum_{l\geq 3} \mathrm{Li}_{2-l}(Q)\frac{(4s\sqrt{z})^l}{zl!}\right. \\ &\left. - \sum_{t=1}^{2} \sum_{k\geq 1, r\geq 0} B_k\left(1 - \frac{t}{2}\right) \mathrm{Li}_{2-k-r}(-(-1^t)\sqrt{Q}) \frac{(2s\sqrt{z})^r (4z)^{k-1}}{r!k!}\right) ds.\end{aligned} \tag{25}$$

The first coefficients are given by

$$\gamma^-_{(1/4)}(z) = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}} \left(1 + \left(\frac{1}{3} + \frac{77}{216}\sqrt{3}\right) iz - \left(\frac{89449}{31104} + \frac{647}{648}\sqrt{3}\right) z^2 + O(|z|^3)\right). \tag{26}$$

Putting everything together, we obtain the asymptotic expansion

$$v_1^{[0]}(q) = \sqrt{\frac{2\pi i}{-z}} e^{-V/z}\, \gamma^{-}_{(1/4)}(z)(1 + O(|z|^L)) \; + \phi^{[0]}_{(1/4)}(z)(1 + O(|z|^L))$$

for all $L > 0$. If $\arg\varphi > 0$, the exponential contribution is the biggest term in (11) and for $\arg\varphi < 0$, the power series $\phi^{[0]}_{(1/4)}(z)$ has the largest contribution. □

3.2.3 Sum over odd parts

Proof of (12) The asymptotics of $v_1^{[1]}(q)$ as defined in (29) as $q \to i$ is similar.

We change the contour in the integral representation in Lemma 3.3 to a stationary contour $\mathcal{S}$. After applying the asymptotics from Lemma 2.1 and Lemma 2.2 and following a similar argument as in Sect. 3.2.2 we obtain that $v_1^{[1]}(q)$ is asymptotically equal to

$$\begin{aligned}
&\frac{-e^{\pi i 3/4}}{2z} e^{\pi^2/48z - z/12} \int_{\mathcal{S}} \operatorname{sign}(\operatorname{Re}(v/\varphi)) \\
&\quad \times \exp\left(\frac{-\operatorname{Li}_2(e^{-4iv}) + 4v^2 + 2\pi v - 4\operatorname{sign}(\operatorname{Re}(v/\varphi))\pi v}{8z} \right) \\
&\quad \times \exp\left(-\frac{iv}{2} + \frac{\operatorname{Li}_1^{\varphi}(e^{-2iv})}{2} + \psi_{-1}(4z; e^{-2iv}) \right) dv + \phi^{[1]}_{(1/4)}(z),
\end{aligned}$$

where

$$\begin{aligned}
\phi^{[1]}_{(1/4)}(z) \; &= \sum_{\substack{n<0 \\ n\equiv 1 \bmod 2}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \; = \sum_{\substack{l>0: \\ l\equiv 1 \bmod 2}} q^{-l(l-1)/2}(-1;q^2)_l \\
&= 2 \sum_{\substack{m\geq 0: \\ m\equiv 0 \bmod 2}} q^{-m(m+1)/2}(-q^2;q^2)_m \\
&= 2 + 8iz - 96z^2 - \frac{5708}{3} i z^3 + 52640 z^4 + \frac{28056121}{15} i z^5 - \frac{405909568}{5} z^6 \\
&\quad - \frac{2622584263067}{630} i z^7 + \frac{5171242573856}{21} z^8 \\
&\quad + \frac{748741881749741041}{45360} i z^9 + O(z^{10}).
\end{aligned}$$

We define

$$\begin{aligned}
f(v) \; &= \; -\frac{\operatorname{Li}_2^{\varphi}(e^{-4iv})}{8} + \frac{v^2}{2} + \frac{\pi v}{4} - \operatorname{sign}(\operatorname{Re}(v/\varphi)) \frac{\pi v}{2}, \\
g(z; v) \; &= \; \operatorname{sign}(\operatorname{Re}(v/\varphi)) \exp\left(-\frac{iv}{2} + \frac{\operatorname{Li}_1^{\varphi}(e^{-2iv})}{2} + \psi_{-1}(4z; e^{-2iv}) \right)
\end{aligned}$$

where f and g are holomorphic functions on the domain defined in (23) for the same reason as in Sect. 3.2.2. Then

$$v_1^{[1]}(q) = \frac{-e^{\pi i 3/4}}{2z} e^{\pi^2/48z - z/12} \int_{\mathcal{S}} e^{f(v)/z} g(z; v) dv (1 + O(|z|^L)) + \phi_{(1/4)}^{[0]}(z)(1 + O(|z|^L))$$

for all $L > 0$ and we compute

$$\begin{aligned} f'(v) &= -\frac{i}{2}\log(1 - e^{-4iv}) + v + \frac{\pi v}{4} - \text{sign}\,(\text{Re}\,(v/\varphi))\,\frac{\pi v}{2} \\ f''(v) &= \frac{1 + e^{-4iv}}{1 - e^{-4iv}} \end{aligned}$$

such that the unique stationary point of f is $v_0 = \frac{\pi}{12}$. Similar arguments as above using the saddle-point method imply that the contribution corresponding to the stationary point $v_0 = \frac{\pi}{12}$ is given by

$$\begin{aligned} & e^{\pi^2/48z + f(v_0)/z - z/12} \int \exp\Big(-f''(\tfrac{\pi}{12})\frac{s^2}{2} + \sum_{l\geq 3} \frac{f^{(l)}(Q)}{l!}(is)^l z^{l/2-1}\Big) g\Big(\frac{\pi}{12} + is\sqrt{z}; z\Big) ds \\ &= e^{V/z}\sqrt{\frac{2\pi i}{z}}\, \gamma_{(1/4)}^{+}(z) \end{aligned}$$

where

$$\begin{aligned} \gamma_{(1/4)}^{+}(z) = \frac{1}{2\sqrt{2\pi}} e^{-\pi i/24 - z/12} & \\ \times \int \exp\Big(\sqrt{3} i \frac{s^2}{2} + \frac{is\sqrt{z}}{2} + \frac{1}{8}\sum_{l\geq 3} \text{Li}_{2-l}(Q)\frac{(4s\sqrt{z})^l}{zl!} & \\ - \sum_{t=1}^{2} \sum_{k\geq 1, r\geq 0} B_k\Big(1 - \frac{t}{2}\Big) \text{Li}_{2-k-r}((-1^t)\sqrt{Q}) \frac{(2s\sqrt{z})^r (4z)^{k-1}}{r!k!}\Big) ds. & \end{aligned} \tag{27}$$

The first coefficients are given by

$$\gamma_{(1/4)}^{+}(z) = \frac{1}{2\sqrt[4]{3(2-\sqrt{3})}} \left(1 + \left(\frac{1}{3} - \frac{77}{216}\sqrt{3}\right) iz + \left(-\frac{89449}{31104} + \frac{647}{648}\sqrt{3}\right) z^2 + O(|z|^3)\right).$$

Putting everything together, we obtain the asymptotic expansion

$$v_1^{[1]}(q) = \sqrt{\frac{2\pi i}{z}} e^{V/z}\, \gamma_{(1/4)}^{+}(z)(1 + O(|z|^L)) + \phi_{(1/4)}^{[1]}(z)(1 + O(|z|^L))$$

for all $L > 0$ which completes the proof. □

3.3 Proof of Theorem 1.2, (2)

Throughout let $\zeta = e(\alpha) = e(r/m)$ be a root of unity of order m divisible by 4. As in Sect. 3.2.2 we split up the sum defining $v_1(q)$ depending on $n \bmod m$

$$v_1(q) = \sum_{n_0=0}^{\frac{m}{2}-1} v_1^{[n_0]}(q), \tag{28}$$

where

$$v_1^{[n_0]}(q) = \sum_{\substack{n\geq 0 \\ n\equiv n_0 \bmod \frac{m}{2}}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \tag{29}$$

for $n_0 \in \{0, \ldots, \frac{m}{2} - 1\}$. We start with the following lemma.

Lemma 3.4 *Let $\zeta = e(\alpha) = e\left(\frac{r}{m}\right)$ be as above.*

(1) *Choose $\overline{r} \in \mathbb{Z}$ with $\overline{r} = r \bmod 4$. If $n = n_0 \bmod \frac{m}{2}$ then*

$$\zeta^{n(n+1)/2} = \zeta^{n_0(n_0+1)/2} e\left(\frac{n-n_0}{4} + (-1)^{n_0}\frac{\overline{r}}{2}\frac{n-n_0}{m}\right).$$

(2) *Let $\overline{\overline{r}} = \pm 1$ such that $\overline{\overline{r}} = r \bmod 4$. Then for all $n = n_0 \bmod \frac{m}{2}$*

$$\zeta^{n(n+1)/2}(-1)^{2(n-n_0)/m} = \zeta^{n_0(n_0+1)/2} e\left((-1)^{m/4+n_0+1}\,\overline{\overline{r}}\,\frac{n-n_0}{2m}\right).$$

Proof We begin by proving part (1).

(1) We write with $n = km/2 + n_0$, $k = 2\frac{n-n_0}{m}$,

$$\begin{aligned}
\zeta^{n(n+1)/2} &= e\left(\alpha n(n+1)/2\right) \\
&= e\left(\alpha\left(\tfrac{km}{2}+n_0\right)\left(\tfrac{km}{2}+n_0+1\right)/2\right) \\
&= e\left(\alpha\left(\tfrac{km}{2}+n_0\right)^2/2 + \alpha\left(\tfrac{km}{2}+n_0\right)/2\right) \\
&= e\left(\alpha\left(\tfrac{k^2m^2}{8} + \tfrac{kmn_0}{2} + \tfrac{n_0^2}{2} + \tfrac{km}{4} + \tfrac{n_0}{2}\right)\right).
\end{aligned}$$

Note that $\frac{\alpha m^2}{8} \in \frac{1}{2}\mathbb{Z}$ and thus $e(\alpha\frac{k^2m^2}{8}) = e(\alpha\frac{km^2}{8})$ as $k^2 = k \bmod 2$. Hence, we obtain

$$\begin{aligned}
\zeta^{n(n+1)/2} &= e\left(\alpha k\left(\frac{m^2}{8} + \frac{mn_0}{2} + \frac{m}{4}\right)\right) e\left(\alpha\frac{n_0(n_0+1)}{2}\right) \\
&= e\left(\alpha km\left(\frac{m}{8} + \frac{n_0}{2} + \frac{1}{4}\right)\right) \zeta^{n_0(n_0+1)/2}.
\end{aligned}$$

Moreover, we compute

$$e\left(\alpha km\left(\frac{m}{8}+\frac{n_0}{2}+\frac{1}{4}\right)\right) = e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+2n_0+1\right)\right)$$

and note that the denominator of $\alpha\frac{km}{4}$ is either 1 or 4. If n_0 is even, $2n_0$ is divisible by 4 and $e(\alpha\frac{km}{4}2n_0)) = 1$. Otherwise, $2n_0+2$ is divisible by 4 and thus $e(\alpha\frac{km}{4}(2n_0+1)) = e(-\alpha\frac{km}{4})$. In other words,

$$\begin{aligned} e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+2n_0+1\right)\right) &= \begin{cases} e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+1\right)\right), & \text{if } n_0 \text{ is even,} \\ e\left(\alpha\frac{km}{4}\left(\frac{m}{2}-1\right)\right), & \text{if } n_0 \text{ is odd,} \end{cases} \\ &= e\left(\alpha\frac{km}{4}\left(\frac{m}{2}+(-1)^{n_0}1\right)\right) \\ &= e\left(\alpha\frac{km^2}{8}\right)\, e\left((-1)^{n_0}\alpha\frac{mk}{4}\right). \end{aligned}$$

Note that $\alpha\frac{km^2}{8} \in \frac{1}{2}\mathbb{Z}$, hence

$$e\left(\alpha\frac{km^2}{8}\right) = e\left(\frac{km}{8}\right) = e\left(\frac{n-n_0}{4}\right).$$

Moreover, we note that $\alpha\frac{mk}{4}$ has denominator 4 such that

$$e\Big((-1)^{n_0}\alpha\frac{mk}{4}\Big) = e\Big((-1)^{n_0}\overline{r}\frac{k}{4}\Big) = e\Big((-1)^{n_0}\frac{\overline{r}}{2}\frac{n-n_0}{m}\Big).$$

This proves the first part of the lemma.

(2) To prove part (2), we choose $\overline{r}$ with $\overline{r} = \overline{\overline{r}} + 4l$, $l \in \mathbb{Z}$ such that

$$(-1)^{n_0}2l + \frac{m}{4} - 1 = \begin{cases} 0, & \text{if } \frac{m}{4} \text{ is odd,} \\ -(-1)^{n_0}\overline{\overline{r}}, & \text{if } \frac{m}{4} \text{ is even.} \end{cases}$$

We continue with the notation from above with $(-1)^{2(n-n_0)/m} = e(-\frac{k}{2})$:

$$\begin{aligned} e\left(\tfrac{km}{8}\right) e\left((-1)^{n_0}\tfrac{\overline{r}k}{4}\right) e\left(-\tfrac{k}{2}\right) &= e\left(\tfrac{km}{8}+(-1)^{n_0}\tfrac{\overline{r}k}{4}-\tfrac{k}{2}\right) \\ &= e\left(k\,\left(\tfrac{m}{4}+(-1)^{n_0}\tfrac{\overline{r}}{2}-1\right)/2\right) \\ &= e\left(k\,\left(\tfrac{m}{4}+(-1)^{n_0}\tfrac{\overline{\overline{r}}+4l}{2}-1\right)/2\right) \\ &= e\left(k\,\left(\tfrac{m}{4}+(-1)^{n_0}\tfrac{\overline{\overline{r}}}{2}+(-1)^{n_0}2l-1\right)/2\right). \end{aligned}$$

Using the choice of $\overline{r}$, the last expression becomes

$$\begin{cases} e\left(k\ (-1)^{n_0}\frac{\overline{\overline{r}}}{4}\right), & \text{if } \frac{m}{4} \text{ is odd,} \\ e\left(k\ \left((-1)^{n_0}\frac{\overline{\overline{r}}}{2} - (-1)^{n_0}\overline{r}\right)/2\right), & \text{if } \frac{m}{4} \text{ is even,} \end{cases}$$

which in both cases is equal to

$$e\left(k\ (-1)^{m/4-1+n_0}\ \frac{\overline{\overline{r}}}{4}\right) = e\left((-1)^{m/4-1+n_0}\ \overline{\overline{r}}\frac{n-n_0}{2m}\right).$$

□

From now on we write

$$\delta := (-1)^{m/4+n_0+1}\ \overline{\overline{r}}$$

with $\overline{\overline{r}} = \pm 1$ as in the previous lemma.

3.3.1 Integral representation

Lemma 3.5 *Let L_∞ be the contour depicted in Fig. 4. Then with $q = \zeta e^{-z}$,*

$$v_1^{[n_0]}(q) = \frac{-\zeta^{n_0(n_0+1)/2} e(\delta\frac{n_0}{2m})}{m(-q^2;q^2)_\infty}\int_{L_\infty} e^{\pi\delta t/m} e^{zt^2/2} e^{-izt/2}\frac{(-\zeta^{2n_0}e^{-2zit}q^2;q^2)_\infty}{\sin(\pi 2(s-n_0)/m)}dt.$$

Proof First, note that we have

$$v_1^{[n_0]}(q) = \frac{\zeta^{n_0(n_0+1)/2}}{(-q^2;q^2)} \sum_{\substack{n\geq 0 \\ n=n_0 \bmod \frac{m}{2}}} e(\delta\tfrac{n-n_0}{2m}) e^{-hn(n+1)/2}\ (-\zeta^{2n_0}e^{-h2n}q^2;q^2)_\infty$$

using Lemma 3.4, (2). The function $\frac{1}{\sin(2\pi(s-n_0)/m)}$ has poles at $s \in \mathbb{Z}$ with $s = n_0 \bmod \frac{m}{2}$ and residues $(-1)^{2(n-n_0)/m}\ \frac{m}{2\pi}$. Hence, we write with Cauchy's residue theorem

$$\begin{aligned} v_1^{[n_0]}(q) &= \frac{-1}{2\pi i}\frac{1}{(-q^2;q^2)_\infty}\lim_{R\to\infty}\int_{L_R+C_R}\zeta^{s(s+1)/2}e^{-hs(s+1)/2}(-\zeta^{2n_0}e^{-2hs}q^2;q^2)_\infty \\ &\quad \frac{2\pi(-1)^{2(s-n_0)/m}}{m\sin(\pi 2(s-n_0)/m)}ds \\ &= \frac{-\zeta^{n_0(n_0+1)/2}e(\delta\frac{n_0}{2m})}{im(-q^2;q^2)_\infty}\lim_{R\to\infty}\int_{L_R+C_R} e^{\pi i\delta s/m}e^{-hs(s+1)/2}\frac{(-\zeta^{2n_0}e^{-2hs}q^2;q^2)_\infty}{\sin(\pi 2(s-n_0)/m)}ds. \end{aligned}$$

The convergence follows analogously to Lemma 3.3. □

3.3.2 Proof of Theorem 1.2, (2)

We will prove the following result, Proposition 3.6 below. The proof of Theorem 1.2, (2) and the asymptotics of $v_1(q)$ follow then by summing $v_1^{[n_0]}(q)$ over $n_0 = 0, \dots, \frac{m}{2}-1$ as in (28) and setting

$$\gamma_{(\alpha)}^{\pm} = \sum_{\substack{n_0=0,\dots,\frac{m}{2} \\ \delta=\pm 1}} \gamma_{(\alpha)}^{[n_0]}.$$

Proposition 3.6 *Let $\zeta = e(\alpha)$ be a root of unity of order m. For $n_0 \in \{0, \dots, \frac{m}{2}-1\}$ we have*

$$v_1^{[n_0]}(q) = e^{\delta \frac{16V}{zm^2}} \left(\frac{z}{2\pi i}\right)^{-1/2} \gamma_{(\alpha)}^{[n_0]}(z)(1+O(|z|^L)) + \phi_{(\alpha)}^{[n_0]}(z)(1+O(|z|^L))$$

for all $L > 0$ as $q = \zeta e^{-z} \to \zeta$, where $\gamma_{(\alpha)}^{[n_0]}(z) \in \mathbb{C}[[z]]$ is defined in (30) and

$$\phi_{(\alpha)}^{[n_0]}(z) = \sum_{\substack{n<0 \\ n\equiv n_0 \bmod \frac{m}{2}}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}.$$

Proof Throughout we write $z = \varphi h$ where $\varphi \in \mathbb{C}$ with $|\varphi| = 1$ and $0 \neq |\arg(\varphi)| < \frac{\pi}{2}$. We substitute $s = iv/z$ in the integral representation from Lemma 3.5 to obtain

$$v_1^{[n_0]}(q) = \frac{-\zeta^{n_0(n_0+1)/2} e(\delta \frac{n_0}{2m})}{mh(-q^2;q^2)_\infty} \int_{-ihL_\infty} e^{\pi\delta v/mz} e^{v^2/2z} e^{-iv/2} \frac{(-\zeta^{2n_0} e^{-2iv} q^2;q^2)_\infty}{\sin(\pi 2(iv/z - n_0)/m)} dv.$$

Changing the contour of integration to a stationary contour $\mathcal{S}$, we include the poles at $-2iz\mathbb{Z}_{<0}$ whose residues give a power series

$$\phi_{(\alpha)}^{[n_0]}(z) = \sum_{\substack{n<0 \\ n\equiv n_0 \bmod \frac{m}{2}}} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n}.$$

Applying the asymptotics from Lemma 2.1 and Lemma 2.2 to the integrands, we obtain that $v_1^{[n_0]}(q)$ is equal to

$$\frac{-\zeta^{n_0(n_0+1)/2} e(\delta \frac{n_0}{2m})}{mhQ(\zeta^2)} \int_{\mathcal{S}} e^{f(v)/z} g(z;v) dv + \phi_{(\alpha)}^{[n_0]}(z)$$

to all orders where $Q(\zeta^2)$ is defined in (7) and

$$f(v) = -\frac{2\,\mathrm{Li}_2^{\varphi}(e^{-miv})}{m^2} + \frac{v^2}{2} - \frac{\pi\delta v - \mathrm{sign}\,(\mathrm{Re}\,(v/\varphi))\,2\pi v}{m},$$

$$g(z;v) = \mathrm{sign}(\mathrm{Re}(v/\varphi))\exp\Bigg(-\frac{iv}{2} - \mathrm{sign}(\mathrm{Re}(v/\varphi))2\pi imn_0 - \frac{\mathrm{Li}_1^{\varphi}(e^{-miv})}{2} + \sum_{t=1}^{m/2}\frac{2t}{m}\,\mathrm{Li}_1^{\varphi}(-\zeta^{2t+2n_0}e^{-2iv}) + \psi_{\zeta^2}(mz;-\zeta^{2n_0}e^{-2iv})\Bigg).$$

We recall that $\mathrm{Li}_2^{\varphi}(e^{-miv})$ jumps by $2\pi mv$ when v crosses the branch cut at $\mathrm{Re}(v/\varphi) = 0$. Hence, the function $f(v)$ is holomorphic on the domain defined in (23). A similar argument shows that $g(v)$ is holomorphic on the same domain.

The stationary points v_0 of f are given by

$$f'(v_0) \;=\; -\frac{2i\log(1-e^{-miv_0})}{m} + v_0 - \frac{\pi\delta - \mathrm{sign}\,(\mathrm{Re}\,(v_0/\varphi))\,2\pi}{m} \;=\; 0.$$

This implies in particular $(1-e^{-iv_0m})^2 = -e^{-iv_0m}$, i.e. $e^{-miv_0} = e(\pm 1/6)$ and it can be checked that $v_0 = \delta\frac{\pi}{3m}$ is the unique stationary point.

Applying the saddle-point method and using $\frac{\pi^2}{3m^2} + f(v_0) \;=\; \delta\frac{16V}{m^2}$ implies that the contribution coming from the saddle point is given by

$$\frac{-\zeta^{n_0(n_0+1)/2}e(\delta\frac{n_0}{2m})}{mhQ(\zeta^2)\sqrt{z}}e^{\delta 16V/m^2 z - z/12} \times \int \exp\Bigg(-f''(v_0)\frac{s^2}{2} + \sum_{l\geq 3}\frac{f^{(l)}(e^{-miv_0})}{l!}(is)^l z^{l/2-1}\Bigg)g\Big(v_0 + is\sqrt{z};z\Big)ds = e^{\frac{\delta V}{zm^2}}\sqrt{\frac{2\pi}{\delta z}}\gamma_{(\alpha)}^{[n_0]}(z)$$

where the integral goes through a small neighborhood of v_0. In other words, we have the asymptotic expansion

$$v_1^{[n_0]}(q) \;=\; e^{\frac{\delta V}{zm^2}}\left(\frac{\delta z}{2\pi}\right)^{-1/2}\gamma_{(\alpha)}^{[n_0]}(z)(1+O(|z|^L)) + \phi_{(\alpha)}^{[n_0]}(z)(1+O(|z|^L))$$

for all $L > 0$ where

$$\gamma_{(\alpha)}^{[n_0]} \;=\; -\frac{\zeta^{n_0(n_0+1)/2}e(\delta\frac{n_0}{2m})2g(v_0)}{mQ(\zeta^2)\sqrt{f''(v_0)}} \tag{30}$$

with

$$f''(v_0) \;=\; \frac{1+e^{-miv_0}}{1-e^{-miv_0}},$$

which completes the proof. □

4 Proof of Theorem 1.3

4.1 Wright's circle method

In this section we prove Theorem 1.3 using Theorem 1.2. We follow the idea of Wright's Circle Method (see Sect. 2.2), and also make use of the saddle-point method (see Sect. 2.1). We label two major arcs near $\pm i$ as C_1 and C_2, where C_1 is the arc near i, and C_2 is the arc near $-i$. Using Cauchy's theorem, we recover our coefficients as

$$V_1(n) = \frac{1}{2\pi i}\int_C \frac{v_1(q)}{q^{n+1}}dq,$$

where C is a circle of radius less than 1 traversed exactly once in the counter-clockwise direction. We split the integral above into three pieces,

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C-\cup C_j}.$$

We denote the asymptotic contribution of the first two integrals $M(n)$ (the main term), and the contribution of the final integral $E(n)$ (the error term).

4.2 Major arc estimates

In this section we calculate the asymptotic contribution arising from the major arcs. We elucidate explicitly the case of one of the contributing terms on the major arc - the calculations for all other contributions are very similar.

Consider the term $M_1(n) := \frac{1}{2\pi i}\int_{C_1}\frac{v_1(q)}{q^{n+1}}dq$. Choose the radius of the circle C to be $e^{-\lambda}$ with $\lambda := \sqrt{\frac{|V|}{n}}$. Then the arc C_1 is described by $ie^{-\lambda+i\theta}$ with $\theta \in (-\delta, \delta)$ for some parameter $\delta > 0$. Therefore we make the change of variable $q = ie^{-z}$ and parameterize where z runs from $\lambda + i\delta$ to $\lambda - i\delta$, to obtain

$$M_1(n) = -\frac{(-i)^n}{2\pi i}\int_{\lambda+i\delta}^{\lambda-i\delta}\frac{v_1\left(ie^{-z}\right)}{e^{-zn}}dz = \frac{(-i)^n}{2\pi i}\int_{\lambda-i\delta}^{\lambda+i\delta}\frac{v_1\left(ie^{-z}\right)}{e^{-zn}}dz. \tag{31}$$

From Theorem 1.2 we have that

$$v_1(ie^{-z}) = e^{\frac{V}{z}}\left(\frac{z}{2\pi i}\right)^{-\frac{1}{2}}\gamma^+ + e^{-\frac{V}{z}}\left(\frac{-z}{2\pi i}\right)^{-\frac{1}{2}}\gamma^- + e^{\frac{V}{z}}O(|z|^{\frac{1}{2}}) + e^{-\frac{V}{z}}O(|z|^{\frac{1}{2}}). \tag{32}$$

The first main term of the asymptotics in (32) yields the following contribution to (31):

$$\frac{(-i)^n\gamma^+}{\sqrt{2\pi i}}\int_{\lambda-i\delta}^{\lambda+i\delta} e^{\frac{V}{z}+nz}z^{-\frac{1}{2}}dz \;=\; \frac{(-i)^n\gamma^+}{n^{1/4}\sqrt{2\pi i}}\int_{\sqrt{|V|}(1-i)}^{\sqrt{|V|}(1+i)} e^{\sqrt{n}\left(\frac{V}{z}+z\right)}z^{-\frac{1}{2}}dz, \quad (33)$$

where we let $\delta=\lambda$ and made the change of variable $z\mapsto\frac{z}{\sqrt{n}}$.

The integral in (33) is of a shape to which we may apply the saddle-point method. We move the contour in (33) through the saddle point $\sqrt{V}$, a zero of the derivative of the function $g(z):=\frac{V}{z}+z$ in the exponential in the integrand. We label this transformed contour Γ so that (33) becomes

$$\frac{(-i)^n\gamma^+}{n^{1/4}\sqrt{2\pi i}}\int_{\Gamma} e^{\sqrt{n}\left(\frac{V}{z}+z\right)}z^{-\frac{1}{2}}dz. \quad (34)$$

We next make the change of variable $z=\sqrt{V}+iwn^{-\frac{1}{4}}$, and expand relevant functions around the saddle point:

$$\begin{aligned}\sqrt{n}g(z) \;&=\; \sqrt{n}\sum_{r=0}^{\infty}\frac{g^{(r)}(\sqrt{V})}{r!}(iw)^r n^{-\frac{r}{4}}\\ &=\; 2\sqrt{nV}-V^{-\frac{1}{2}}w^2+\sum_{r=3}^{\infty}(-i)^r V^{\frac{1}{2}(1-r)}w^r n^{\frac{2-r}{4}}\end{aligned}$$

where we have used that $g(\sqrt{V})=2\sqrt{V}$, $g'(\sqrt{V})=0$, and $g^{(n)}(\sqrt{V})=(-1)^n n!(\sqrt{V})^{1-n}$ for $n\geq 2$. Thus, we have that

$$e^{\sqrt{n}g(\sqrt{V})} \;=\; e^{2\sqrt{nV}}e^{-V^{-\frac{1}{2}}w^2}\left(1+\sum_{r=1}^{\infty}n^{-\frac{r}{4}}\widetilde{p}_r(w)\right), \quad (35)$$

where each $\widetilde{p}_r(w)\in\mathbb{C}[w]$. Similarly, we have that

$$z^{-\frac{1}{2}} \;=\; V^{-\frac{1}{4}}\left(1+\sum_{r=1}^{\infty}\frac{\left(-\frac{iw}{2}\right)^r(2r-1)!!}{r!}V^{-\frac{r}{2}}n^{-\frac{r}{4}}\right). \quad (36)$$

We use (35) and (36) in (34) (with $z=\sqrt{V}+iwn^{-\frac{1}{4}}$) to obtain

$$\frac{i(-i)^n\gamma^+}{n^{\frac{1}{2}}\sqrt{2\pi i}}e^{2\sqrt{nV}}V^{-\frac{1}{4}}\int_{\Gamma'}e^{-V^{-\frac{1}{2}}w^2}\left(1+\sum_{r=1}^{\infty}n^{-\frac{r}{4}}p_r(w)\right)dw, \quad (37)$$

where each $p_r(w)\in\mathbb{C}[w]$ arises from multiplying the corresponding polynomials in (35) and (36). Here, because the contour Γ was chosen to run through the saddle point $\sqrt{V}$, the contour Γ' runs through the origin. Moreover, we choose Γ' so that it also

has a horizontal tangent at 0. Then, in the limit as $n \to \infty$, we have that $\Gamma' \to \mathbb{R}$. Thus, the expression in (37) is asymptotic to

$$\frac{i(-i)^n\gamma^+}{n^{\frac{1}{2}}\sqrt{2\pi i}}e^{2\sqrt{nV}}V^{-\frac{1}{4}}\int_{-\infty}^{\infty}e^{-V^{-\frac{1}{2}}w^2}\left(1+\sum_{r=1}^{\infty}n^{-\frac{r}{4}}p_r(w)\right)dw \tag{38}$$

$$= \frac{i(-i)^n\gamma^+}{\sqrt{2in}}e^{2\sqrt{nV}}\left(1+O\left(n^{-\frac{1}{2}}\right)\right), \tag{39}$$

where in the final step we use that the polynomials arising from (35) and (36) are odd (resp. even) for r odd (resp. even), and that when multiplied they begin with a term of order $n^{-\frac{1}{2}}$.

The calculations for the contributions of the other terms in (32) along with the contributions arising from the major arc around $-i$ are very similar and so we omit them for brevity. Collecting all of the contributions yields

$$\begin{aligned} M(n) \;=\; & \left(\frac{(-i)^n\sqrt{i}\gamma^+}{\sqrt{2n}}e^{2\sqrt{nV}}+\frac{i^{n-1}\sqrt{i}\gamma^+}{\sqrt{2n}}e^{2\sqrt{-nV}}+\frac{(-i)^n\sqrt{i}\gamma^-}{\sqrt{2n}}e^{2\sqrt{-nV}}\right.\\ & \left.+\frac{i^{n+1}\sqrt{i}\gamma^-}{\sqrt{2n}}e^{2\sqrt{nV}}\right)\\ & \times\left(1+O\left(n^{-\frac{1}{2}}\right)\right). \end{aligned}$$

Simplifying this yields that

$$\begin{aligned} M(n) \;=\; & (-1)^{\lfloor\frac{n}{2}\rfloor}\,\mathrm{Re}(\sqrt{2i}(\gamma^+-(-1)^n i\gamma^-))\frac{e^{\sqrt{2|V|n}}}{\sqrt{n}}\left(\cos(\sqrt{2|V|n})+(-1)^{n+1}\sin(\sqrt{2|V|n})\right)\\ & \times\left(1+O\left(n^{-\frac{1}{2}}\right)\right). \end{aligned} \tag{40}$$

Using that $\sqrt{i}=\frac{1+i}{\sqrt{2}}$ we see that $M(n)$ gives the first term of Theorem 1.3. It remains to estimate the contribution from the minor arcs.

4.3 Minor arc estimates

In this section we bound the asymptotic contribution of the minor arcs, which turn out to almost always be exponentially smaller than those from the major arcs (a fact we prove in Sect. 5).

We begin by noting that the asymptotic formula for v_1 toward all roots of unity provided by Theorem 1.3 is valid for all z in any cone contained in the right half-plane. In particular, this means that we have the asymptotic behavior of v_1 in a punctured neighborhood inside the unit disk of any root of unity. Since the roots of unity are dense on the unit circle, we thus have asymptotic estimates covering the entire unit circle. This is in essence the estimate one requires for the Circle Method of Hardy and

Ramanujan, extended by Rademacher, when taking the path of integration on Farey arcs; see [4, 20, 37].

Recalling Lemma 3.1, it suffices to consider primitive roots of unity whose order is divisible by 4, but which are not $\pm i$. Consider the error term, given by

$$E(n) := \frac{1}{2\pi i} \int_{C - \cup C_j} \frac{v_1(q)}{q^{n+1}} dq.$$

We have

$$|E(n)| \ = \ \frac{1}{2\pi} \left| \int_{C - \cup C_j} \frac{v_1(q)}{q^{n+1}} dq \right|.$$

By Theorem 1.2 we see that the largest contribution to the error arc is given by the 8-th order roots of unity. So we may bound the entire error term $E(n)$ by the contribution from the 8-th order roots of unity multiplied by the length of the integral, which is less than 2π.

Let ζ be an mth root of unity, where $4|m$ and $m > 1$. Following the saddle-point method in the same way as for the major arcs, and using Theorem 1.2, we obtain a finite sum of integrals of the form

$$\frac{\zeta^{-n} K_\zeta}{\sqrt{2\pi i}} \int_\Gamma e^{\frac{\pm 16V}{m^2 z} + nz} z^{-\frac{1}{2}} dz \ = \ \frac{\zeta^{-n} K_\zeta}{\sqrt{2\pi i}} n^{-\frac{1}{4}} \int_{\Gamma'} e^{\sqrt{n}\left(\frac{\pm 16V}{m^2 z} + z\right)} z^{-\frac{1}{2}} dz.$$

Here, $K_\zeta \in \mathbb{C}$ is some constant depending on the root of unity ζ (and also on which term from Theorem 1.2 we are adding). Following the saddle-point method, this yields the contribution

$$K'_\zeta \frac{e^{2\sqrt{\pm \frac{16nV}{m^2}}}}{n^{\frac{1}{2}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right), \tag{41}$$

for some constant $K'_\zeta \in \mathbb{C}$.

By (41) we get (setting $K'_{\zeta,8}$ as the constant arising from the $m = 8$ term there, i.e. 8-th order roots of unity)

$$|E(n)| \ll \left| K'_{\zeta,8} e^{\sqrt{\pm nV}} n^{-\frac{1}{2}} \right| \ = \ O\left(n^{-\frac{1}{2}} e^{\sqrt{\frac{n|V|}{2}}}\right). \tag{42}$$

Now combining (40) along with (42) finishes the proof of Theorem 1.3.

5 Proof of Theorem 1.1

5.1 Proof of Andrews' Conjecture 3

Andrews' Conjecture 3 states that $|V_1(n)| \to \infty$ as $n \to \infty$. As remarked in Sect. 1, after computational and theoretical investigations, we believe that this conjecture can be refined to say "We have that $|V_1(n)| \to \infty$ as $n \to \infty$ away from a set of density 0." This refined conjecture will follow from the arguments needed to prove Andrews' Conjecture 4 below.

5.2 Proof of Andrews' Conjecture 4

Recall that Andrews' Conjecture 4 states that four consecutive values of V_1 come with two positive and two negative signs almost always.

From Theorem 1.3, this reduces to studying the function

$$(-1)^{\lfloor \frac{n}{2} \rfloor} \left(\cos\left(\sqrt{2|V|n}\right) + (-1)^{n+1} \sin\left(\sqrt{2|V|n}\right)\right).$$

Note that we have the following table of signs for $(-1)^{\lfloor \frac{n}{2} \rfloor}$:

$n \pmod 4$	$(-1)^{\lfloor \frac{n}{2} \rfloor}$
0	+
1	+
2	−
3	−

Thus, it is enough to study the function (of n)

$$\cos\left(\sqrt{2|V|n}\right) + (-1)^{n+1} \sin\left(\sqrt{2|V|n}\right)$$

at n, $n+1$, $n+2$, and $n+3$.

Heuristically, when n gets large the values $\cos(\sqrt{2|V|(n+j)})$ (resp. $\sin(\sqrt{2|V|(n+j)})$) for $j \in \{0, 1, 2, 3\}$ are close to each other. To see this, for $a \in \mathbb{R}$ consider

$$\begin{aligned}
&\lim_{x\to\infty} \cos(a\sqrt{x+1}) - \cos(a\sqrt{x}) \\
&= \lim_{x\to\infty} -2\sin\left(\frac{a(\sqrt{x+1}-\sqrt{x})}{2}\right)\sin\left(\frac{a(\sqrt{x+1}+\sqrt{x})}{2}\right) \\
&= \lim_{x\to\infty} -2\sin\left(\frac{a}{2(\sqrt{x+1}+\sqrt{x})}\right)\sin\left(\frac{a(\sqrt{x+1}+\sqrt{x})}{2}\right) \\
&= 0,
\end{aligned}$$

where the final step arises from the expansion at infinity, which is

$$\left|\cos(a\sqrt{x+1}) - \cos(a\sqrt{x})\right| \leq 2\left(\frac{a}{4\sqrt{x}} + O\left(x^{-\frac{3}{2}}\right)\right). \tag{43}$$

Note that a similar calculation holds for the sin term.

By Theorem 1.3 we have that

$$\begin{aligned} V_1(n) &= M(n) + E(n) \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{e^{\sqrt{2|V|n}}}{\sqrt{n}} (\gamma^+ + (-1)^n \gamma^-) \left(\cos(\sqrt{2|V|n}) - (-1)^n \sin(\sqrt{2|V|n})\right) \left(1 + O\left(n^{-\frac{1}{2}}\right)\right) \\ &\quad + O\left(n^{-\frac{1}{2}} e^{\sqrt{\frac{|V|n}{2}}}\right). \end{aligned}$$

We want to prove that almost all of the time the main term $M(n)$ is asymptotically larger than the error $E(n)$. The only time this could not happen is when the factor $\cos(\sqrt{2|V|n}) + (-1)^{n+1}\sin(\sqrt{2|V|n})$ is exponentially small, which in turn can only happen near to roots of

$$\mathcal{F}_{\pm}(x) := \cos(x) \pm \sin(x).$$

Such roots occur at $\pi\left(\ell \pm \frac{1}{4}\right)$ for any $\ell \in \mathbb{Z}$. In the interval $[0, 2\pi]$ we thus have four different roots, which we label by ϑ_j with $1 \leq j \leq 4$. Then taking the Taylor series of $\mathcal{F}_{\pm}$ about ϑ_j gives

$$(x - \vartheta_j)\mathcal{F}'_{\pm}(\vartheta_i) + O\left((x - \vartheta_j)^2\right),$$

where $|\mathcal{F}'_{\pm}(\vartheta_i)| = \sqrt{2}$.

Then the part where $M(n) \gg E(n)$ occurs when we have $|\mathcal{F}_{\pm}(x)| > e^{-\kappa\sqrt{n}}$ with $\kappa := \sqrt{\frac{|V|}{2}}$. Equivalently, we want the argument of $\mathcal{F}$ to stay $e^{-\kappa\sqrt{n}+\varepsilon\sqrt{n}}$ with $\varepsilon > 0$ away from each ϑ_j, since $|\cos(x_n) \pm \sin(x_n)| > e^{-\kappa\sqrt{n}}$ if

$$|x - \vartheta_j| + O\left((x - \vartheta_j)^2\right) = e^{-\kappa\sqrt{n}+\varepsilon\sqrt{n}} + O\left(\frac{e^{-2\kappa\sqrt{n}+2\varepsilon\sqrt{n}}}{2}\right) > e^{-\kappa\sqrt{n}}$$

for all j.

In what follows, we use an argument based on the equidistribution of sequences modulo 1 to show that almost always the points $x_n := \sqrt{2|V|n}$ are more than $e^{-\kappa\sqrt{n}+\varepsilon\sqrt{n}}$ away from each ϑ_j. To begin, we rescale the interval $[0, 2\pi]$ to the interval $[0, 1]$, and correspondingly consider $\vartheta'_j := \frac{\vartheta_j}{2\pi}$ and $x'_n := \frac{1}{2\pi}\sqrt{2|V|n}$.

Recall that the discrepancy D_N for a sequence $(s_1, \ldots, s_N)$ over an interval $[a, b]$ is defined to be

$$D_N := \sup_{a \le c \le d \le b} \left| \frac{|\{s_1, \ldots, s_N\} \cap [c, d]|}{N} - \frac{d-c}{b-a} \right|,$$

and is a quantitative measure of how far the given sequence is from equidistribution on the interval $[a, b]$.

A result of Schoißengeier [38], which follows straightforwardly from the Erdös–Turán inequality, states that for a sequence $a\sqrt{n}$ with $a \in \mathbb{R}^+$ one has the bound

$$D_N \ll O\left(N^{-\frac{1}{2}}\right). \tag{44}$$

Now we take the sequence $(x'_1, \ldots x'_N)$ along with $a = 0, b = 1$. Place an interval I_j of length $e^{-\kappa\sqrt{n}+\varepsilon\sqrt{n}}/2\pi$ centered at ϑ'_j. Then using (44) we see that the number of points which lie in the set $[0, 1) - \cup_j I_j$ is bounded below by

$$G(n) := 1 - O\left(n^{-\frac{1}{2}}\right). \tag{45}$$

We therefore see that the proportion of values for which $|V(n)| \to \infty$ is at least $G(n)$. Along with the fact that $\lim_{n\to\infty} G(n) = 1$, this proves a refined version of Andrews' Conjecture 3.

Using (45) along with the fact that $\mathcal{F}_{\pm}(\sqrt{2|V|(n+j)}) = \mathcal{F}_{\pm}(\sqrt{2|V|n}) + O(n^{-\frac{1}{2}})$ by (43), it is clear that almost all 4-tuples $V(n)$, $V(n+1)$, $V(n+2)$, $V(n+3)$ will all have an exponentially dominant main term M. In turn, this means that we automatically obtain the two plus and two minus signs almost always.

6 Andrews' Conjectures 5 and 6

In this section we discuss Andrews' Conjectures 5 and 6 regarding the coefficients $V_1(n)$. While our methods below do not lead to complete proofs of these two conjectures, they do explain them, ultimately relating $V_1(n)$ to the arithmetic of $\mathbb{Q}(\sqrt{-3})$. We restate these conjectures (as in Sect. 1) here for convenience.

Conjecture (Conjecture 5 [3]). *For $n \ge 5$ there is an infinite sequence $N_5 = 293$, $N_6 = 410$, $N_7 = 545$, $N_8 = 702, \ldots, N_n \ge 10n^2, \ldots$ such that $V_1(N_n)$, $V_1(N_n+1)$, $V_1(N_n+2)$ all have the same sign.*

Conjecture (Conjecture 6 [3]). *The numbers $|V_1(N_n)|$, $|V_1(N_n+1)|$, $|V_1(N_n+2)|$ contain a local minimum of the sequence $|V_1(j)|$.*

Conjecture 6 of Andrews is seen to be essentially explained by Conjecture 5 alongside the asymptotic of $V_1(n)$ given by our Theorem 1.3, as it is apparent that for a sign pattern disruption, one must have a local minimum of the sequence $|V_1(j)|$. There are several possibilities for how the sign pattern may fail. Each would rely on determining

more concrete information on the error term $E(n)$, which we discuss at the end of this section.

We remark that for the sign pattern change, one needs that the main term is arbitrarily small infinitely often. A natural related question to pose is whether in fact $V_1(n) = 0$ for infinitely many n in analogy to σ. Numerical computations suggest that $V_1(n)$ only vanishes for finitely many values of n. Checking the first five million coefficients of $v_1(q)$ it appears that $V_1(n) = 0$ if and only if

$$n \in \{2,\ 3,\ 4,\ 5,\ 11,\ 13,\ 15,\ 17,\ 19,\ 21,\ 25,\ 29,\ 31,\ 39,\ 47,\ 58,\ 60,\ 62,\ 64,\ 101,\ 111,\ 123,\ 129,\ 198\}.$$

Our techniques are not amenable to proving that only a finite number of coefficients of v_1 vanish since, as discussed below, one would require irrationality results on $\pi^2/|V|$.

6.1 $M(n)$ and $E(n)$ have the same sign

Assume that $M(n)$ and $E(n)$ have the same sign. Then we see that in order for each of Andrews' Conjecture 5 and 6 to hold, a necessary (but not sufficient) condition is that the main term $M(n)$ must be arbitrarily small infinitely often. In turn, we want to find infinite families of integers that are close to the roots ϑ_j. Solving directly, we want to choose infinitely many $n \in \mathbb{N}$ to be arbitrarily close to

$$\frac{\pi^2\left(\ell \pm \frac{1}{4}\right)^2}{2|V|}, \qquad \ell \in \mathbb{Z}.$$

To determine whether such choices exist requires more concrete knowledge regarding $\frac{\pi^2}{|V|}$. By results of Milnor [31] we have that

$$|V| = \frac{9\sqrt{3}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)}{16\pi^2},$$

where ζ_K is the usual Dedekind zeta function associated with the field K.

The question of whether a given value of a Dedekind zeta value, say $\zeta_K(2)$ with K a number field, is rational or irrational is a particularly deep question that has been investigated by many authors. In the case where K is totally real, Klingen [25] and Siegel [40] used powerful techniques within the theory of Hilbert modular forms to provide the celebrated Siegel–Klingen theorem,[1] see e.g. [39], which states that the values $\zeta_K(2n) \in |\mathrm{disc}(K)|^{-\frac{1}{2}}\pi^{2kN}\mathbb{Q}$ with $N = [K : \mathbb{Q}]$ and $n \in \mathbb{N}$.

However, we are interested in the case where K is imaginary quadratic, and thus lie outside of the scope of Siegel–Klingen. In fact, current methods are unable to determine an analogue of Siegel–Klingen for imaginary quadratic fields, and we reach an impasse. Zagier [47] investigated the values $\zeta_K(2)$ for arbitrary number fields, and determined a representation for them as a multiple of powers of π, $\sqrt{\mathrm{disc}(K)}$ and

[1] The case where K is real quadratic was already known to Hecke [21].

integrals of the shape

$$A(x) = \int_0^x \frac{1}{1+t^2} \log \frac{4}{1+t^2} dt$$

evaluated at certain points (see [47, Theorem 1]). In fact, for imaginary quadratic fields, Zagier gave a refined sharper theorem in [47, Theorem 3]. Despite these beautiful results, we are still unable to determine rationality properties of the zeta values.

A further example of the depth of such questions is that of the algebraic dependence of $\log(2)$, π, $\zeta(3)$, with ζ the usual Riemann zeta function. This was originally conjectured by Euler [16] in 1785. Very recently, Eskandari and Murty [15] determined a certain motive with periods given precisely by these three values (along with a fourth period). Conditional on the Grothendieck conjecture, this then proved that in fact this triple is algebraically independent (in opposition to Euler's conjecture). This perhaps leads to a pathway to (conditionally) prove that $\zeta_K(2)$, π, $\sqrt{\text{disc}(K)}$ for K imaginary quadratic are algebraically independent by constructing a motive with these periods in much the same spirit as [15].

Thus asking for rationality properties of

$$\frac{\pi^2}{|V|} = \frac{16\pi^4}{9\sqrt{3}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)}$$

seems out-of-reach of current mathematics. However, we are able to make slight progress in determining whether there are infinitely many values n such that the main term $M(n)$ is arbitrarily small if we make assumptions on $\frac{\pi^2}{|V|}$.

We consider three disparate cases, depending on the nature of $\frac{\pi^2}{|V|}$. In the case of $\frac{\pi^2}{|V|}$ being irrational, we show that the main term is arbitrarily small infinitely often. In the case where $\frac{\pi^2}{|V|}$ is rational with odd denominator, we show that the main term $M(n)$ vanishes infinitely often, while in the (unlikely) case where $\frac{\pi^2}{|V|}$ is rational with even denominator we show that $M(n)$ cannot be arbitrarily small infinitely often.

6.1.1 Case I: $\frac{\pi^2}{|V|}$ is irrational

Assume that $\frac{\pi^2}{|V|}$ is irrational. Then we want to determine whether there are infinitely many choices of positive integers ℓ, n such that

$$\frac{2n}{\left(\ell \pm \frac{1}{4}\right)^2} = \frac{32n}{(4\ell \pm 1)^2}$$

is arbitrarily close to $\frac{\pi^2}{|V|}$.

We use [6, Theorem 1] with $\beta = 0$, which then reads as follows, using the notation $\|\cdot\|$ to denote the distance to the nearest integer.

Theorem 6.1 *Let α be irrational and $k \geq 1$. Then there are infinitely many primes p such that*

$$\|\alpha p^k\| < p^{-\rho(k)+\varepsilon}$$

for every $\varepsilon > 0$, where $\rho(2) = \frac{3}{20}$ and $\rho(k) = (3 \cdot 2^{k-1})^{-1}$ for $k \geq 3$.

Applying Theorem 6.1 to the irrational $\frac{\pi^2}{32|V|}$ yields infinitely primes p and integers n such that

$$\left| \frac{\pi^2}{32|V|} p^2 - n \right| < p^{-\frac{3}{20}+\varepsilon}.$$

Dividing both sides by p^2 gives

$$\left| \frac{\pi^2}{32|V|} - \frac{n}{p^2} \right| < p^{-\frac{43}{20}+\varepsilon}.$$

Now simply noting that all primes $\neq 2$ are of the form $4\ell \pm 1$, we see that we obtain infinitely many integer pairs (ℓ, n) such that we are arbitrarily close to roots ϑ_j, and thus infinitely many pairs where the main term is arbitrarily small (by taking large enough p).

We see that in Case I, as one transitions through a zero of $\mathcal{F}_{\pm}$, the sign pattern of $M(n)$ is naturally disrupted. Moreover, since $E(n)$ is assumed to have the same sign as $M(n)$, these are the only places that such a transition should occur. We observe an infinite sequence of zeros of the trigonometric function $\mathcal{F}_{\pm}$, which we label ϑ_j. Taking the sequence as

$$\lfloor \vartheta_j \rfloor \tag{46}$$

yields an infinite sequence of integers around which one would expect to have three of the same sign, as predicted by Andrews. Testing numerically, of the first 715 values where V_1 has three values with the same sign (i.e. testing the first five million coefficients of $v_1(q)$), the sequence constructed in (46) is always within 2 of the sequence conjectured by Andrews. This may be explained by the fact that the sequence of consecutive values with the same sign need not begin precisely at $\lfloor \vartheta_j \rfloor$, but could begin up to two terms before this value. In Table 1 we give the first ten values of N_j for $j \geq 5$ as predicted by Andrews alongside the values of $\lfloor \vartheta_j \rfloor$.

6.1.2 Case II: $\frac{\pi^2}{32|V|}$ is rational with odd denominator

Assume that $\frac{\pi^2}{32|V|} = \frac{h}{k} \in \mathbb{Q}$ with $\gcd(h, k) = 1$ and k odd. We see that one would need to choose infinitely many positive integers n that are arbitrarily close to the points

$$\frac{h}{k} (4\ell \pm 1)^2 . \tag{47}$$

Table 1 The first 10 values of N_j and $\lfloor \vartheta_j \rfloor$ for $j \geq 5$

j	5	6	7	8	9	10	11	12	13	14
N_j	293	410	545	702	877	1072	1285	1518	1771	2044
$\lfloor \vartheta_j \rfloor$	294	410	546	702	877	1072	1286	1519	1772	2044

This is clearly true infinitely often, in particular when $(4\ell \pm 1)^2 = \alpha k$ with $\alpha \in \mathbb{Z}$. In turn, this means that the trigonometric function $\mathcal{F}_{\pm}$ arising in the main term would be evaluated precisely at one of its roots, meaning that the main term $M(n) = 0$ at these points.

However, this case would fail to explain Andrews' Conjecture 5, since $M(n)$ and $E(n)$ were assumed to have the same sign, and so numerically it seems implausible for Case II to hold.

6.1.3 Case III: $\frac{\pi^2}{32|V|}$ is rational with even denominator

As in Case II, we assume that $\frac{\pi^2}{32|V|} = \frac{h}{k} \in \mathbb{Q}$ with $\gcd(h, k) = 1$. However, if k is even, then it is clear that the right-hand side of (47) has fixed denominator k, and thus there cannot be infinitely many integers arbitrarily close to such points.

Based on numerical evidence, the sequence N_j of places where $V_1(n)$ contain three consecutive terms with the same sign appears to be infinite, beginning with the values given in Table 1. In turn, this provides strong evidence that one may discount Case III.

6.2 $M(n)$ and $E(n)$ have opposite signs

Now assume that $M(n)$ and $E(n)$ arise with different signs. Then when $\mathcal{F}_{\pm}$ is of the order $e^{-\kappa\sqrt{n}}$ the main term $M(n)$ and error term $E(n)$ have the same order of growth. Labelling these points N_j this means that there would be a sequence of points $n < N$ where $M(n)$ determines the sign of $V_1(n)$, but as n approaches the value N_j the main term and error term become close to one another. As one passes the point N, the sign of $V_1(n)$ is then dictated by $E(n)$, which dominates $M(n)$ when $\mathcal{F}_{\pm}(n) < e^{-\kappa\sqrt{n}}$. Similarly, there will be such transition points where $M(n)$ again begins to dominate $E(n)$. One sees that the sign pattern of two plus signs and two minus signs would be disrupted at these transition points, providing an infinite sequence of points explaining Andrews' Conjecture 5. However, these points would not be those given in (46) above, and so numerically it appears that this is not the case.

Overall, we see that in order to more clearly determine information on Andrews' Conjecture 5 and 6, one needs much more precise information on the error term $E(n)$. Using Theorem 1.2 it may be possible to keep track of all order $4m$ roots of unity in a similar fashion to how the problem is handled here. In fact, in a future project it is planned that we use the full Hardy–Ramanujan Circle Method to determine much more precise asymptotics for $V_1(n)$, which should yield an asymptotic as an infinite

sum over terms of a similar shape to $M(n)$. Several obstacles would then remain. In particular, the barrier of not knowing the nature of irrationality of $\frac{\pi^2}{|V|}$ remains, and at best one would be able to prove conditional theorems.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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