QUANTUM ENTANGLEMENT FOR SYSTEMS OF IDENTICAL BOSONS. II. SPIN SQUEEZING AND OTHER ENTANGLEMENT TESTS: APPENDICES

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Appendix A Spin Operators - Multi-Mode Case

As well as spin operators for the simple case of two modes we can also define spin operators in multimode cases involving two sub-systems A and B. For example, there may be two types of bosonic particle involved, each *component* distinguished from the other by having different hyperfine internal states $|A\rangle\,,|B\rangle$. Each component may be associated with a complete orthonormal set of *spatial mode functions* $\phi_{ai}(\mathbf{r})$ and $\phi_{bi}(\mathbf{r})$, so there will be two sets of modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$, where in the $|\mathbf{r}\rangle$ representation we have $\langle\mathbf{r}\,|\phi_{ai}\rangle = \phi_{ai}(\mathbf{r})\,|A\rangle$ and $\langle\mathbf{r}\,|\phi_{bi}\rangle = \phi_{bi}(\mathbf{r})\,|B\rangle$. Mode orthogonality between A and B modes arises from $\langle A|B\rangle = 0$ rather from the spatial mode functions being orthogonal.

We can define *spin operators* for the combined *multimode* A and B subsystems [43] via

$$\begin{aligned}
\widehat{S}_x &= \frac{1}{2}\int d\mathbf{r}\,\left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) + \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})\right)\\
\widehat{S}_y &= \frac{1}{2i}\int d\mathbf{r}\,\left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) - \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})\right)\\
\widehat{S}_z &= \frac{1}{2}\int d\mathbf{r}\,\left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) - \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r})\right) \qquad (193)
\end{aligned}$$

where the field operators satisfy the non-zero commutation rules

$$[\widehat{\Psi}_c(\mathbf{r}), \widehat{\Psi}_d^\dagger(\mathbf{r}')] = \delta_{cd}\,\delta(\mathbf{r}-\mathbf{r}') \qquad c,d = a,b \qquad (194)$$

It is then easy to show that the standard spin angular momentum commutation rules are satisfied. $[\widehat{S}_\xi\ ,\widehat{S}_\mu\ ] = i\epsilon_{\xi\mu\lambda}\widehat{S}_\lambda$.

For convenience we can expand the field operators in terms of an orthonormal set of spatial mode functions $\phi_i(\mathbf{r})$. We can choose the spatial mode functions to be the same $\phi_{ai}(\mathbf{r}) = \phi_{bi}(\mathbf{r}) = \phi_i(\mathbf{r})$ (these might be momentum eigenfunctions) and then the field annihilation operators for each component are

$$\widehat{\Psi}_a(\mathbf{r}) = \sum_i \widehat{a}_i\ \phi_i(\mathbf{r}) \qquad \widehat{\Psi}_b(\mathbf{r}) = \sum_i \widehat{b}_i\ \phi_i(\mathbf{r}) \qquad (195)$$

These expansions are consistent with the field operator commutation rules (194) based on the usual non-zero mode operator commutation rules $[\widehat{a}_i, \widehat{a}_j^\dagger] = [\widehat{b}_i, \widehat{b}_j^\dagger] = \delta_{ij}$.

By substituting for the field operators we can then express the spin operators in terms of mode operators as

$$\widehat{S}_x = \frac{1}{2}\sum_i \left(\widehat{b}_i^\dagger\widehat{a}_i + \widehat{a}_i^\dagger\widehat{b}_i\right) \quad \widehat{S}_y = \frac{1}{2i}\sum_i \left(\widehat{b}_i^\dagger\widehat{a}_i - \widehat{a}_i^\dagger\widehat{b}_i\right) \quad \widehat{S}_z = \frac{1}{2}\sum_i \left(\widehat{b}_i^\dagger\widehat{b}_i - \widehat{a}_i^\dagger\widehat{a}_i\right) \qquad (196)$$

and it is then easy to confirm that the standard spin angular momentum commutation rules are satisfied. $[\widehat{S}_\xi\ ,\widehat{S}_\mu\ ] = i\epsilon_{\xi\mu\lambda}\widehat{S}_\lambda$. We now have both field and mode expressions for spin operators in multimode cases involving two sub-systems A and B.

Finally, the *total number of particles* is given by

$$\begin{aligned}
\widehat{N} &= \int d\mathbf{r}\,\left(\widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})+\widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r})\right)\\
&= \sum_i\left(\widehat{b}_i^\dagger\widehat{b}_i+\widehat{a}_i^\dagger\widehat{a}_i\right)\\
&= \widehat{N}_b+\widehat{N}_a \qquad (197)
\end{aligned}$$

in an obvious notation.

Appendix B Alternative Spin Squeezing Criteria

B.1 Two Perpendicular Components

Other criteria for spin squeezing are also used, for example in the article by Wineland et al [17]. To focus on spin squeezing for $\widehat{S}_z$compared to *any* orthogonal spin operators we can combine the second and third Heisenberg uncertainty principle relationships to give

$$\left\langle \Delta\widehat{S}_z{}^2 \right\rangle \left(\left\langle \Delta\widehat{S}_x{}^2 \right\rangle + \left\langle \Delta\widehat{S}_y{}^2 \right\rangle \right) \geq \frac{1}{4} \left(| \left\langle \widehat{S}_x \right\rangle |^2 + | \left\langle \widehat{S}_y \right\rangle |^2 \right) \tag{198}$$

Then we may define two new spin operators via

$$\widehat{S}_{\perp 1} = \cos\theta \; \widehat{S}_x + \sin\theta \; \widehat{S}_y \qquad \widehat{S}_{\perp 2} = -\sin\theta \; \widehat{S}_x + \cos\theta \; \widehat{S}_y \tag{199}$$

where θ corresponds to a rotation angle in the xy plane, and which satisfy the standard angular momentum commutation rules $[\widehat{S}_{\perp 1}, \widehat{S}_{\perp 2}] = i\widehat{S}_z$, $[\widehat{S}_{\perp 2}, \widehat{S}_z] = i\widehat{S}_{\perp 1}$, $[\widehat{S}_z, \widehat{S}_{\perp 1}] = i\widehat{S}_{\perp 2}$. It is straightforward to show that $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle + \left\langle \Delta\widehat{S}_y{}^2 \right\rangle = \left\langle \Delta\widehat{S}_{\perp 1}{}^2 \right\rangle + \left\langle \Delta\widehat{S}_{\perp 2}{}^2 \right\rangle$ and $\left| \left\langle \widehat{S}_{\perp 1} \right\rangle \right|^2 + \left| \left\langle \widehat{S}_{\perp 2} \right\rangle \right|^2 = \left| \left\langle \widehat{S}_x \right\rangle \right|^2 + \left| \left\langle \widehat{S}_y \right\rangle \right|^2$ so that

$$\left\langle \Delta\widehat{S}_z{}^2 \right\rangle \left(\left\langle \Delta\widehat{S}_{\perp 1}{}^2 \right\rangle + \left\langle \Delta\widehat{S}_{\perp 2}{}^2 \right\rangle \right) \geq \frac{1}{4} \left(\left| \left\langle \widehat{S}_{\perp 1} \right\rangle \right|^2 + \left| \left\langle \widehat{S}_{\perp 2} \right\rangle \right|^2 \right) \tag{200}$$

so that *spin squeezing* for $\widehat{S}_z$compared to *any two* orthogonal spin operators such as $\widehat{S}_{\perp 1}$ or $\widehat{S}_{\perp 2}$ would be defined as

$$\begin{aligned} \left\langle \Delta\widehat{S}_z{}^2 \right\rangle &< \frac{1}{2}\sqrt{\left(\left| \left\langle \widehat{S}_{\perp 1} \right\rangle \right|^2 + \left| \left\langle \widehat{S}_{\perp 2} \right\rangle \right|^2 \right)} \\ & \textit{and} \\ \left\langle \Delta\widehat{S}_{\perp 1}{}^2 \right\rangle + \left\langle \Delta\widehat{S}_{\perp 2}{}^2 \right\rangle &> \frac{1}{2}\sqrt{\left(\left| \left\langle \widehat{S}_{\perp 1} \right\rangle \right|^2 + \left| \left\langle \widehat{S}_{\perp 2} \right\rangle \right|^2 \right)} \end{aligned} \tag{201}$$

For spin squeezing in $\left\langle \Delta\widehat{S}_z{}^2 \right\rangle$ we require the *spin squeezing parameter* ξ to satisfy an inequality

$$\xi^2 = \frac{\left\langle \Delta\widehat{S}_z{}^2 \right\rangle}{\left(\left| \left\langle \widehat{S}_{\perp 1} \right\rangle \right|^2 + \left| \left\langle \widehat{S}_{\perp 2} \right\rangle \right|^2 \right)} < \frac{1}{2\sqrt{\left(\left| \left\langle \widehat{S}_{\perp 1} \right\rangle \right|^2 + \left| \left\langle \widehat{S}_{\perp 2} \right\rangle \right|^2 \right)}} \sim \frac{1}{N} \tag{202}$$

The last step is an approximation for an N particle state based on the assumption that the Bloch vector lies in the xy plane and close to the Bloch sphere, this situation being the most conducive to detecting the fluctuation $\left\langle \Delta\widehat{S}_z{}^2 \right\rangle$.

In this situation $\sqrt{\left(\left|\left\langle \widehat{S}_{\perp 1}\right\rangle\right|^2+\left|\left\langle \widehat{S}_{\perp 2}\right\rangle\right|^2\right)}$ is approximately $N/2$. The condition $N\xi^2<1$ is sometimes taken as the condition for spin squeezing [28], but it should be noted that this is approximate and Eq. (201) gives the correct expression.

B.2 Planar Spin Squeezing

A special case of recent interest is that referred to as *planar squeezing* [18] in which the Bloch vector for a suitable choice of spin operators lies in a *plane* and along one of the *axes*. If this plane is chosen to be the xy plane and the x axis is chosen then $\left\langle \widehat{S}_z\right\rangle=0$ and $\left\langle \widehat{S}_y\right\rangle=0$, resulting in only one Heisenberg uncertainty principle relationship where the right side is non-zero, namely $\left\langle \Delta\widehat{S}_y{}^2\right\rangle\left\langle \Delta\widehat{S}_z{}^2\right\rangle\geq\frac{1}{4}\left|\left\langle \widehat{S}_x\right\rangle\right|^2$. Combining this with $\left\langle \Delta\widehat{S}_x{}^2\right\rangle\left\langle \Delta\widehat{S}_y{}^2\right\rangle\geq 0$ gives $\left(\left\langle \Delta\widehat{S}_y{}^2\right\rangle+\left\langle \Delta\widehat{S}_x{}^2\right\rangle\right)\left\langle \Delta\widehat{S}_z{}^2\right\rangle\geq\frac{1}{4}\left|\left\langle \widehat{S}_x\right\rangle\right|^2$. So the total spin fluctuation *in* the xy plane defined as $\left\langle \Delta\widehat{S}_{para}{}^2\right\rangle=\left\langle \Delta\widehat{S}_y{}^2\right\rangle+\left\langle \Delta\widehat{S}_x{}^2\right\rangle$ will be squeezed compared to the spin fluctuation perpendicular to the xy plane given by $\left\langle \Delta\widehat{J}_{perp}{}^2\right\rangle=\left\langle \Delta\widehat{S}_z{}^2\right\rangle$ if

$$\left\langle \Delta\widehat{S}_{para}{}^2\right\rangle<\frac{1}{2}\left|\left\langle \widehat{S}_x\right\rangle\right|\ and\ \left\langle \Delta\widehat{S}_{perp}{}^2\right\rangle>\frac{1}{2}\left|\left\langle \widehat{S}_x\right\rangle\right| \tag{203}$$

By minimizing $\left\langle \Delta\widehat{S}_{para}{}^2\right\rangle$ whilst satisfying the constraints $\left\langle \widehat{S}_z\right\rangle=\left\langle \widehat{S}_y\right\rangle=0$ a spin squeezed state is found that satisfies (203) with $\left\langle \Delta\widehat{S}_{para}{}^2\right\rangle\ \tilde{}\ J^{2/3}$, $\left\langle \Delta\widehat{S}_{perp}{}^2\right\rangle\sim J^{4/3}$, $\left|\left\langle \widehat{S}_x\right\rangle\right|\sim J$ for large $J=N/2$ [18]. The Bloch vector is on the Bloch sphere.

B.3 Spin Squeezing in Multi-Mode Cases

Since the multi-mode spin operators defined in Eq. (193) satisfy the standard angular momentum operator commutation rules, the usual Heisenberg Uncertainty rules analogous to (5) apply, so that spin squeezing can also exist in the multi-mode case as well. Thus

$$\begin{aligned}
\left\langle \Delta\widehat{S}_x{}^2\right\rangle &< \frac{1}{2}\left|\left\langle \widehat{S}_z\right\rangle\right|\ and\ \left\langle \Delta\widehat{S}_y{}^2\right\rangle>\frac{1}{2}\left|\left\langle \widehat{S}_z\right\rangle\right| \\
\left\langle \Delta\widehat{S}_y{}^2\right\rangle &< \frac{1}{2}\left|\left\langle \widehat{S}_x\right\rangle\right|\ and\ \left\langle \Delta\widehat{S}_z{}^2\right\rangle>\frac{1}{2}\left|\left\langle \widehat{S}_x\right\rangle\right| \\
\left\langle \Delta\widehat{S}_z{}^2\right\rangle &< \frac{1}{2}\left|\left\langle \widehat{S}_y\right\rangle\right|\ and\ \left\langle \Delta\widehat{S}_x{}^2\right\rangle>\frac{1}{2}\left|\left\langle \widehat{S}_y\right\rangle\right|
\end{aligned} \tag{204}$$

for $\widehat{S}_x$being squeezed compared to $\widehat{S}_y$, and so on.

Similar alternative criteria to (201) can also be obtained, for example for $\widehat{S}_z$ being squeezed compared to *any two* orthogonal spin operators such as $\widehat{S}_{\perp 1}$ or $\widehat{S}_{\perp 2}$ defined similarly to (199).

Appendix C Significance of Spin Squeezing Test

The spin squeezing test for two mode systems was based on the general form (15) for all *separable* states together with the requirement that the sub-system density operators $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ were compliant with the *local* particle number *SSR*. From the point of view of a *supporter* for applying the local particle number SSR if the result of an experiment is that spin squeezing has occurred, the immediate conclusion is that the state is entangled. On the other hand from the point of view of a *sceptic* about being required to apply the local particle number SSR for the sub-system states, such a sceptic would draw different conclusions from an experiment that demonstrated spin squeezing. They would immediately point out that in this case spin squeezing is *not* a test for entanglement. However, as we will now see these conclusions are still of some interest.

To discuss this it is convenient to divide possible *mathematical* forms for the density operator into categories. Considering *all* general two mode quantum states that are compliant with the *global* particle number SSR, we may first divide such quantum states into three categories, as set out in Table 1.

REGION	**OVERALL**	**SUB-SYSTEM**	**CATEGORY**
	QUANTUM STATE	**QUANTUM STATE**	
A	$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$	Both $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are local	* **Separable**
		particle number SSR compliant	
B	$\widehat{\rho} = \sum_R P_R \, \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$	Neither $\widehat{\rho}_R^A$ nor $\widehat{\rho}_R^B$ is local	* **Separable but**
		particle number SSR compliant	**non-local** [4];
			* **Entangled** [2]
C	$\widehat{\rho} \neq \sum_R P_R \, \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$	N/A	* **Entangled**

Table I. Categories of two mode quantum states.

The regions referred to are shown in Figure 2. All authors would regard the quantum states in Region A as being separable and those in Region C as being entangled - it is only those in Region B where the category is disputed. Those such as [2] (local SSR supporters) who require local particle number SSR compliance for each sub-system state would classify the overall state as entangled, those who do not require this (local SSR sceptics) such as [4] would classify the overall state as separable but non-local. Note that no further sub-classification is needed.

In Appendix N of paper I we show that if states of the form (44) are globally SSR compliant, then *both* the sub-system states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are local particle number SSR compliant *in general*. However, we point out that there are special matched choices for both $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ along with the P_R, where neither $\widehat{\rho}_R^A$ nor $\widehat{\rho}_R^B$ is local particle number SSR compliant even though $\widehat{\rho}$ is global particle number SSR compliant. But the case where just *one* of $\widehat{\rho}_R^A$ or $\widehat{\rho}_R^B$ is non SSR compliant does not occur, so Region B does not need to be sub-divided along these lines.

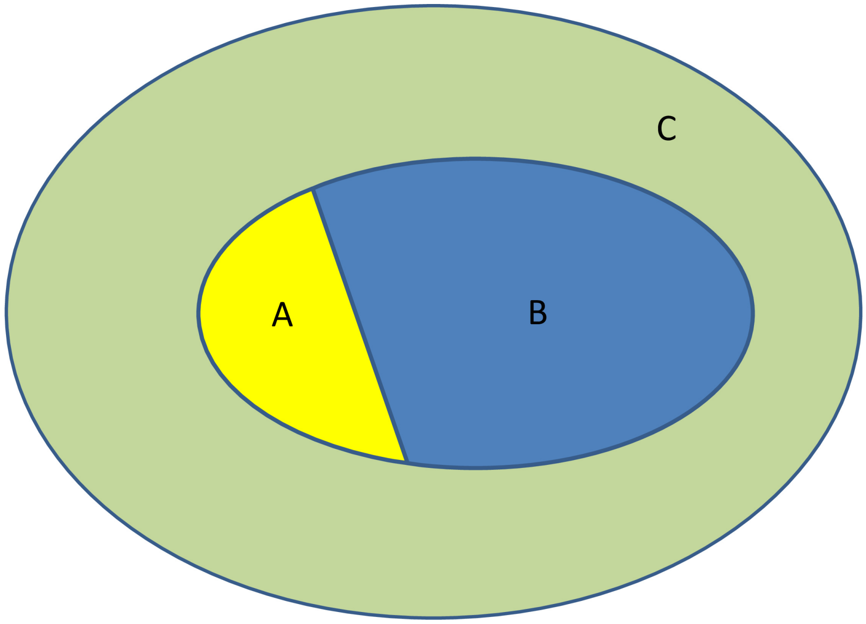

Figure 2. Categories of two mode quantum states that are global particle number SSR compliant. The regions A, B and C are described in Table I and represent the sets of separable, separable but non-local and entangled states respectively.

Now let us consider quantum states for which $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$ and $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$. Such states are clearly not spin squeezed. Firstly, we know that *all* states in Region A satisfy these inequalities. However, *some* states in Region B and *some* states in Region C may also satisfy these inequalities. In Figure 3 the quantum states in Region B that satisfy these inequalities are depicted as lying in Region D, those in Region C that do so are depicted as lying in Region F. Hence, if we find that the quantum state is such that spin squeezing *does* occur (as in the test of (47)) we can definitely say that it does *not* lie in Regions A, D or F. It must therefore be located in Regions E or L. The question is - Does this determine whether the state is entangled or not according to the *supporters* of applying the local SSR as in the definition of entanglement used in the present paper? The answer is that it does. This is because the quantum state must be located within either of Regions B or C, since these regions include E and L respectively. In both cases it would be *entangled* according to the definition used here [2] (see Table 1).

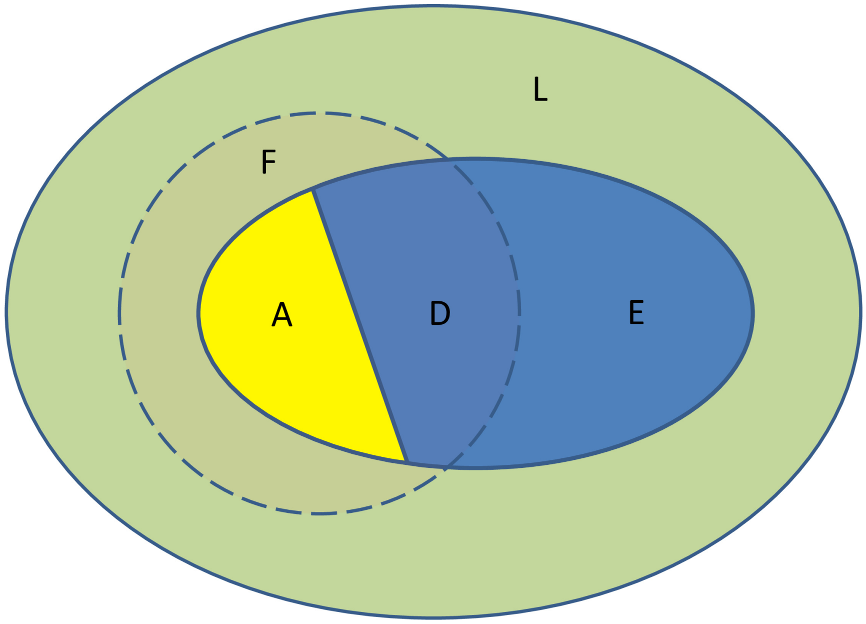

Figure 3. Categories of two mode quantum states satisfying inequalities $\left\langle \Delta\widehat{S}_x{}^2 \right\rangle_\rho \geq \frac{1}{2} | \left\langle \widehat{S}_z \right\rangle_\rho |$ and $\left\langle \Delta\widehat{S}_y{}^2 \right\rangle_\rho \geq \frac{1}{2} | \left\langle \widehat{S}_z \right\rangle_\rho |$ Such states are represented as being in regions are A, D and F, where A includes all the separable states, D includes some of the separable but non-local states and F includes some of the entangled states (apart from those in D). The region $A \oplus D \oplus F$ includes all the unsqueezed states. Referring to Figure 2, $B = D \oplus E$ and $C = F \oplus L$.

However, the *sceptics* of applying the local SSR would draw a different conclusion from the experiment that demonstrated spin squeezing (as in the test of (47)). They would agree that the mathematics shows that a state in Region A could not demonstrate spin squeezing. Nor by assumption could states in Regions D or F. This means that the state must lie in either Region L or Region E. So from the point of view of the sceptic, *either* the state is *entangled* (if it lies in Region L) *or* the sub-system states in *all* separable states (Region E) do not comply with the local particle number *SSR*. The sceptic's conclusion is clearly interesting - in the first case the quantum state is entangled, and hence may demonstrate other non-classical features, and in the second case the possibility exists of finding sub-systems in states that have the unexpected feature in non-relativistic many body physics of having coherences between states with differing particle number. If there was a *second* experimental test that could show that the state was not entangled, then this would demonstrate the existence of quantum states (sub-systems are themselves possible quantum systems) in which the particle number SSR breaks down.

The second experiment would seem to require a test for entanglement which is *necessary* as well as being sufficient - the latter alone being usually the case for entanglement tests. Such criteria and measurements are a challenge, but not impossible even though we have not met this challenge in these two papers. Thus, *in principle* there could be a *pair* of experiments that give evidence of entanglement, *or* failure of the super-selection rule. For such investigations to be possible, the use of entanglement criteria that *do* invoke the local super-selection rules is *also* required. Such tests are the focus of these two papers, though here our primary reason is because we consider applying the local particle number SSR is required by the physics of non-relativistic quantum many body systems involving massive particles.

Appendix D Spin Squeezing and Entanglement - Multi-Mode Case

In this Appendix we discuss three cases of separable states for multimode systems and then examine whether spin squeezing tests demonstrate entanglement for each of these cases.

D.1 Multi-Mode Separable States - Three Cases

As we have seen the multi-mode case involves a set of n modes with annihilation operators $\widehat{a}_i$ for bosons with hyperfine component A, and another set of n modes with annihilation operators $\widehat{b}_i$ for bosons with hyperfine component B. Since entanglement implies a clear choice of what sub-systems are to be entangled, there are numerous choices possible here for the present multi-mode case. *Case 1* involves two sub-systems, one consisting of all the $\widehat{a}_i$ modes as sub-system A and the other consisting of all the $\widehat{b}_i$ modes as sub-system B. *Case 2* involves $2n$ sub-systems, the Ai th containing the mode $\widehat{a}_i$ and the Bi th containing the mode $\widehat{b}_i$. *Case 3* involves n sub-systems, the ith containing the two modes $\widehat{a}_i$ and $\widehat{b}_i$. These three cases relate to entanglement causing interactions in differing circumstances. Case 1 might apply to cases where separable states can be created with all the $\widehat{a}_i$ modes coupled together to produce states $\widehat{\rho}_R^A$ and the $\widehat{b}_i$ modes coupled together to produce states $\widehat{\rho}_R^B$. Case 2 might apply to cases where separable states can be created with the $\widehat{a}_i$ and all the $\widehat{b}_i$ modes independent of each together to produce states $\widehat{\rho}_R^{a(i)} \otimes \widehat{\rho}_R^{b(i)}$. Case 3 might apply to cases where separable states can be created with the $\widehat{a}_i$ and the matching $\widehat{b}_i$ modes coupled together to produce states $\widehat{\rho}_R^{ab(i)}$. Cases 2 and 3 will be discussed further in SubSection 4.4 dealing with the entanglement test introduced by Sørensen et al [14].

The density operators for *separable* states in the three cases will be of the form

$$\widehat{\rho}_{sep} = \sum_R P_R \, \widehat{\rho}_R \, ,$$

$$\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B \qquad \text{Case 1} \tag{205}$$

$$\widehat{\rho}_R = \widehat{\rho}_R^{a(1)} \otimes .. \otimes \widehat{\rho}_R^{a(i)} .. \otimes \widehat{\rho}_R^{a(n)} \otimes \widehat{\rho}_R^{b(1)} \otimes .. \otimes \widehat{\rho}_R^{b(n)} \qquad \text{Case 2} \tag{206}$$

$$\widehat{\rho}_R = \widehat{\rho}_R^{ab(1)} \otimes \widehat{\rho}_R^{ab(2)} \otimes .. \otimes \widehat{\rho}_R^{ab(i)} .. \otimes \widehat{\rho}_R^{ab(n)} \qquad \text{Case 3} \tag{207}$$

$$\tag{208}$$

Discussion of whether there is a spin squeezing test for Case 1 in the multi-mode case involves a generalization of the theory set out in SubSection 3.1. There is a *Bloch vector* entanglement test, in that if either of $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ is non-zero, then the state is entangled. We also find that spin squeezing in *any* spin component requires the state to be entangled, thus generalizing the spin squeezing test to the *multi-mode* case, for *two* sub-systems consisting of all the modes $\widehat{a}_i$ and all the modes $\widehat{b}_i$ The details are covered in SubSection D.2.

For Case 2 a spin squeezing test for entanglement also be obtained. The test is again that *spin squeezing* in *any* spin component $\widehat{S}_x, \widehat{S}_y$ or $\widehat{S}_z$ confirms entanglement of the $2n$ sub-systems consisting of *single* modes $\widehat{a}_i$ and $\widehat{b}_i$. Furthermore, there is also a *Bloch vector* entanglement test, in that if either of $\left\langle \widehat{S}_x \right\rangle$ or $\left\langle \widehat{S}_y \right\rangle$ is non-zero, then the state is entangled. As these systems can have quantum *states* with large numbers N of bosonic particles, it can be said that entanglement in an N particle *system* has occurred if spin squeezing is found. The proof of these tests is set out in SubSection D.3.

For Case 3 there is also a spin squeezing test for entanglement, but it is restricted. Here the test is that *spin squeezing* in $\widehat{S}_z$ confirms entanglement of the n sub-systems consisting of *pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$, but the test is *restricted* to the situation where exactly *one boson* occupies each mode pair. No spin squeezing test was found for the other spin operators, nor was a Bloch vector entanglement test obtained. The proof of this result is set out in SubSection D.4. That no general spin squeezing test for entanglement exists can be shown by a counter-example. If all the N bosons occupied one mode pair $\widehat{a}_i$ and $\widehat{b}_i$, and the quantum state $\widehat{\rho}_R^{ab(i)}$ for this pair corresponded to the *relative phase eigenstate* with phase $\theta_p = 0$ (see SubSection 3.7) then although the overall state is separable, spin squeezing in $\widehat{S}_y$ compared to $\widehat{S}_z$ occurs (with $\left\langle \Delta\widehat{S}_y^2 \right\rangle = \frac{1}{4} + \frac{1}{8}\ln N$, $\left\langle \Delta\widehat{S}_z^2 \right\rangle = \left(\frac{1}{6} - \frac{\pi^2}{64}\right) N^2$ and $\left\langle \widehat{S}_x \right\rangle = N\,\frac{\pi}{8}$. Thus there is a situation where a *non-entangled* state for sub-systems consisting of *mode pairs* is spin squeezed, so spin squeezing does *not* always confirm entanglement.

As in the previous two mode cases, having established in multi-mode cases that spin squeezing requires entanglement a further question then is: Does entanglement automatically lead to spin squeezing? The answer is no, since cases where the quantum state is entangled but not spin squeezed can be found - an example is given in the previous paragraph. Thus in general, spin squeezing and entanglement are *not equivalent* - they do not occur *together* for all states. Spin squeezing is a *sufficient* condition for entanglement, it is not a *necessary* condition.

D.2 Spin Squeezing Test for Bipartite System (Case 1)

We now consider spin squeezing for the multi-mode spin operators given in Eqs. (193) and (196) in Appendix A. We consider separable states for *Case 1*, the density operator being given in Eq. (205). In this *bipartite case* the two subsystems consist of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$. The development involves expressions such as $\left\langle \widehat{\Psi}_c(\mathbf{r}) \right\rangle_R^C = Tr_C(\widehat{\Psi}_c(\mathbf{r})\widehat{\rho}_R^C)$, $\left\langle \widehat{\Psi}_c^\dagger(\mathbf{r}) \right\rangle_R^C = Tr_C(\widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\rho}_R^C)$ and $\left\langle \widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c^\dagger(\mathbf{r}') \right\rangle_R^C = Tr_C(\widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c^\dagger(\mathbf{r}')\widehat{\rho}_R^C)$, $\left\langle \widehat{\Psi}_c(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}') \right\rangle_R^C = Tr_C(\widehat{\Psi}_c(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}')\widehat{\rho}_R^C)$, $\left\langle \widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}') \right\rangle_R^C = Tr_C(\widehat{\Psi}_c^\dagger(\mathbf{r})\widehat{\Psi}_c(\mathbf{r}')\widehat{\rho}_R^C)$, where $C = A, B$.

Firstly, we have

$$\left\langle \widehat{S}_x \right\rangle_R = \frac{1}{2}\int d\mathbf{r}\,\left(\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}) \right\rangle_R^A \left\langle \widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B \right) = 0 \quad (209)$$

since from the local particle number SSR for sub-systems A and B we have $\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r}) \right\rangle_R^B = \left\langle \widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A = 0$. A similar result applies to $\left\langle \widehat{S}_y \right\rangle_R$ so it then follows that

$$\left\langle \widehat{S}_x \right\rangle = \left\langle \widehat{S}_y \right\rangle = 0 \quad (210)$$

This immediately yields the *Bloch vector* entanglement test. It also leads to the *spin squeezing* in $\widehat{S}_z$ entanglement test, namely if $\widehat{S}_z$ is squeezed with respect to $\widehat{S}_x$ or $\widehat{S}_y$ (or vice versa), then the state must be entangled. The question then is: Does spin squeezing in $\widehat{S}_x$ with respect to $\widehat{S}_y$ (or vice versa) require the state to be entangled for the two n mode sub-systems A and B?

To obtain an inequality for the variance in $\widehat{S}_x$, we see that

$$\begin{aligned} \left\langle \widehat{S}_x^2 \right\rangle_R &= \frac{1}{4}\iint d\mathbf{r}\,d\mathbf{r}' \times \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b^\dagger(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}') \right\rangle_R^A \\ &\quad + \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r})\widehat{\Psi}_a^\dagger(\mathbf{r}') \right\rangle_R^A + \left\langle \widehat{\Psi}_b(\mathbf{r})\widehat{\Psi}_b^\dagger(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}') \right\rangle_R^A \\ &\quad + \left\langle \widehat{\Psi}_b(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a^\dagger(\mathbf{r}') \right\rangle_R^A \} \end{aligned} \quad (211)$$

From the local particle number SSR for sub-systems A and B we have $\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b^\dagger(\mathbf{r}') \right\rangle_R^B = \left\langle \widehat{\Psi}_a(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}') \right\rangle_R^A = 0$, so the first and fourth terms are zero. Using the field operator commutation rules we then obtain

$$\begin{aligned} \left\langle \widehat{S}_x^2 \right\rangle_R &= \frac{1}{2}\iint d\mathbf{r}\,d\mathbf{r}'\, \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \\ &\quad + \frac{1}{4}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \end{aligned} \quad (212)$$

so that

$$\begin{aligned} \left\langle \Delta\widehat{S}_x^2 \right\rangle_R &= \frac{1}{2}\iint d\mathbf{r}\,d\mathbf{r}'\, \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \\ &\quad + \frac{1}{4}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \end{aligned} \quad (213)$$

Hence from (20)

$$\begin{aligned} &\left\langle \Delta\widehat{S}_x^2 \right\rangle \\ \geq\ & \sum_R P_R \{ \frac{1}{2}\iint d\mathbf{r}\,d\mathbf{r}'\, \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \\ &+ \frac{1}{4}\int d\mathbf{r}\, \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \end{aligned} \quad (214)$$

The same result applies to $\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle$.

Now we can easily show that

$$\left\langle \widehat{S}_{z}\right\rangle = \sum_{R} P_R \frac{1}{2} \int d\mathbf{r} \, \{ \left\langle \widehat{\Psi}_b^{\dagger}(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})\right\rangle_R^B - \left\langle \widehat{\Psi}_a^{\dagger}(\mathbf{r})\widehat{\Psi}_a(\mathbf{r})\right\rangle_R^A \} \tag{215}$$

so that

$$\frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle| \leq \sum_{R} P_R \frac{1}{4} \int d\mathbf{r} \, \{ \left\langle \widehat{\Psi}_b^{\dagger}(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})\right\rangle_R^B + \left\langle \widehat{\Psi}_a^{\dagger}(\mathbf{r})\widehat{\Psi}_a(\mathbf{r})\right\rangle_R^A \} \tag{216}$$

as $\left\langle \widehat{\Psi}_b^{\dagger}(\mathbf{r})\widehat{\Psi}_b(\mathbf{r})\right\rangle_R^B$ and $\left\langle \widehat{\Psi}_a^{\dagger}(\mathbf{r})\widehat{\Psi}_a(\mathbf{r})\right\rangle_R^A$ are real and positive.

Hence we find that

$$\begin{aligned}
& \left\langle \Delta\widehat{S}_{x}^{2}\right\rangle - \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle| \\
\geq \; & \sum_{R} P_R \frac{1}{2} \iint d\mathbf{r}\, d\mathbf{r}' \, \left\langle \widehat{\Psi}_b^{\dagger}(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}')\right\rangle_R^B \left\langle \widehat{\Psi}_a^{\dagger}(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r})\right\rangle_R^A && (217) \\
= \; & \sum_{R} P_R \frac{1}{2} \iint d\mathbf{r}\, d\mathbf{r}' \, Tr_B\{\widehat{\Psi}_b(\mathbf{r}')\, \widehat{\rho}_R^B \, \widehat{\Psi}_b^{\dagger}(\mathbf{r})\} \, Tr_A\{\widehat{\Psi}_a(\mathbf{r})\, \widehat{\rho}_R^A \, \widehat{\Psi}_a^{\dagger}(\mathbf{r}')\} && (218) \\
= \; & \frac{1}{2} \iint d\mathbf{r}\, d\mathbf{r}' \, Tr\left\{ \widehat{\Psi}_a(\mathbf{r})\, \widehat{\Psi}_b(\mathbf{r}')\, \widehat{\rho}_{sep}\, \widehat{\Psi}_a^{\dagger}(\mathbf{r}')\, \widehat{\Psi}_b^{\dagger}(\mathbf{r})\right\} && (219)
\end{aligned}$$

giving three forms that the inequality for $\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle - \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle|$ has to satisfy in the case of a separable state. The last form involves a double space integral of a quantum correlation function. Note the order of $\mathbf{r}$ and $\mathbf{r}'$. It is straightforward to show that the right side of the inequality is real, but to achieve an entanglement test involving spin squeezing for $\widehat{S}_{x}$ we need to show that it is non-negative. Identical inequalities can be found for $\left\langle \Delta\widehat{S}_{y}^{2}\right\rangle - \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle|$.

D.2.1 Mode Expansions

If we use Eq. (195) to expand the field operators then using Eq. (218) we have

$$\begin{aligned}
& \left\langle \Delta\widehat{S}_{x}^{2}\right\rangle - \frac{1}{2}|\left\langle \widehat{S}_{z}\right\rangle| \\
\geq \; & \sum_{R} P_R \frac{1}{2} \sum_{ij} \sum_{kl} \iint d\mathbf{r}\, d\mathbf{r}' \\
& \times \{\phi_i(\mathbf{r})\phi_j^{*}(\mathbf{r}')\phi_k(\mathbf{r}')\phi_l^{*}(\mathbf{r})\} \\
& \times \left\{ Tr_A\{\widehat{a}_i \, \widehat{\rho}_R^A \, \widehat{a}_j^{\dagger}\} \; Tr_B\{\widehat{b}_k \, \widehat{\rho}_R^B \, \widehat{b}_l^{\dagger}\}\right\} \\
= \; & \sum_{R} P_R \frac{1}{2} \sum_{ij} \left\{ Tr_A\{\widehat{a}_i \, \widehat{\rho}_R^A \, \widehat{a}_j^{\dagger}\} \; Tr_B\{\widehat{b}_j \, \widehat{\rho}_R^B \, \widehat{b}_i^{\dagger}\}\right\} \\
= \; & \sum_{R} P_R \frac{1}{4} \sum_{ij} (A_{ij}^R \, B_{ji}^R + B_{ij}^R \, A_{ji}^R) && (220) \\
= \; & \sum_{R} P_R \frac{1}{4} Tr\{A^R B^R + B^R A^R\} && (221)
\end{aligned}$$

where mode orthogonality has been used and we have introduced *matrices* A^R and B^R whose elements are

$$A_{ij}^R = Tr_A\{\widehat{a}_i\,\widehat{\rho}_R^A\,\widehat{a}_j^\dagger\} \qquad B_{ji}^R = Tr_B\{\widehat{b}_j\,\widehat{\rho}_R^B\,\widehat{b}_i^\dagger\} \tag{222}$$

It is easy to show that $A_{ij}^R = (A_{ji}^R)^*$ and $B_{ij}^R = (B_{ji}^R)^*$ showing that the matrices A^R and B^R are *Hermitian*, as is $A^R B^R + B^R A^R$. The quantity $\sum\limits_{ij}(A_{ij}^R\ B_{ji}^R + B_{ij}^R\ A_{ji}^R)$ is *real*. The question is: Is it also *positive* ?

For the simple case where there is only *one* spatial mode for each component the right side of the inequality is just equal to $\sum\limits_R P_R \frac{1}{2}\left\{Tr_A\{\widehat{a}\,\widehat{\rho}_R^A\,\widehat{a}^\dagger\}\ Tr_B\{\widehat{b}\,\widehat{\rho}_R^B\,\widehat{b}^\dagger\}\right\} = \sum\limits_R P_R \frac{1}{2} N_R^A\ N_R^B$, where N_R^A and N_R^B give the mean numbers of bosons in sub-systems A and B for the states $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$. The right side of the inequality is positive, showing that the separable state is not spin squeezed for $\widehat{S}_x$ with respect to $\widehat{S}_y$ (or vice versa), leading as before to the test that such spin squeezing requires entanglement.

D.2.2 Positive Definiteness

For the multi-mode case we now take into account that the sub-system density operators $\widehat{\rho}_R^A$ and $\widehat{\rho}_R^B$ are *positive-definite*. Their eigenvalues π_λ^{AR} and π_μ^{BR} are real and non-negative as well as summing to unity, and we can write the density operators in terms of their orthonormal eigenvectors $|AR,\lambda\rangle$ and $|BR,\mu\rangle$ as

$$\widehat{\rho}_R^A = \sum_\lambda \pi_\lambda^{AR}\ |AR,\lambda\rangle\,\langle AR,\lambda| \qquad \widehat{\rho}_R^B = \sum_\mu \pi_\mu^{BR}\ |BR,\mu\rangle\,\langle BR,\mu| \tag{223}$$

Then from (222)

$$A_{ij}^R = \sum_\lambda \pi_\lambda^{AR}\,\langle AR,\lambda|\ \widehat{a}_j^\dagger \widehat{a}_i\ |AR,\lambda\rangle \qquad B_{ji}^R = \sum_\mu \pi_\mu^{BR}\,\langle BR,\mu|\ \widehat{b}_i^\dagger \widehat{b}_j\ |BR,\mu\rangle \tag{224}$$

Consider a $1\times n$ row matrix $\xi^\dagger = \{\xi_1^*, \xi_2^*, .., \xi_n^*\ \}$

$$\begin{aligned}
\xi^\dagger A^R \xi &= \sum_{ij} \xi_i^*\,A_{ij}^R\,\xi_j \\
&= \sum_\lambda \pi_\lambda^{AR} \sum_{ij} \xi_i^*\ \langle AR,\lambda|\ \widehat{a}_j^\dagger \widehat{a}_i\ |AR,\lambda\rangle\ \xi_j \\
&= \sum_\lambda \pi_\lambda^{AR}\ \langle AR,\lambda|\ \widehat{\Omega}_A^\dagger\,\widehat{\Omega}_A\ |AR,\lambda\rangle
\end{aligned} \tag{225}$$

where we have introduced the operator $\widehat{\Omega}_A = \sum_i \xi_i^*\,\widehat{a}_i$. Since $\xi^\dagger A^R \xi$ is always non-negative for all ξ, this shows that A^R is a *positive definite* matrix. Similarly, considering a $1\times n$ row matrix $\eta^\dagger = \{\eta_1^*, \eta_2^*, .., \eta_n^*\ \}$ and introducing the operator $\widehat{\Omega}_B = \sum_i \eta_i\,\widehat{b}_i$ we find that

$$\eta^\dagger B^R \eta = \sum_\mu \pi_\mu^{BR}\ \langle BR,\mu|\ \widehat{\Omega}_B^\dagger\,\widehat{\Omega}_B\ |BR,\mu\rangle \tag{226}$$

which is also always non-negative, showing that B^R is also a *positive definite* matrix.

We can then express the positive definite Hermitian matrices A^R and B^R in terms of their normalized column eigenvectors θ_α^A and ζ_β^B respectively, where the corresponding real, positive eigenvalues are ν_α and σ_β Thus we have (for ease of notation R will be left understood)

$$\begin{aligned} A^R\,\theta_\alpha^A &= \nu_\alpha\theta_\alpha^A \qquad (\theta_\alpha^A)^\dagger\theta_\gamma^A = \delta_{\alpha\gamma} \qquad A^R = \sum_\alpha \nu_\alpha\theta_\alpha^A\,(\theta_\alpha^A)^\dagger \\ B^R\,\zeta_\beta^B &= \sigma_\beta\zeta_\beta^B \qquad (\zeta_\beta^B)^\dagger\zeta_\epsilon^B = \delta_{\beta\epsilon} \qquad B^R = \sum_\beta \sigma_\beta\zeta_\beta^B\,(\zeta_\beta^B)^\dagger \end{aligned} \tag{227}$$

Then

$$\begin{aligned} & Tr\{A^R B^R + B^R A^R\} \\ = \;& Tr\{\sum_\alpha\sum_\beta \nu_\alpha\sigma_\beta\,\theta_\alpha^A\,(\theta_\alpha^A)^\dagger\zeta_\beta^B\,(\zeta_\beta^B)^\dagger\} \\ & +Tr\{\sum_\alpha\sum_\beta \nu_\alpha\sigma_\beta\,\zeta_\beta^B\,(\zeta_\beta^B)^\dagger\,\theta_\alpha^A\,(\theta_\alpha^A)^\dagger\} \\ = \;& \sum_\alpha\sum_\beta \nu_\alpha\sigma_\beta\,[(\theta_\alpha^A)^\dagger\zeta_\beta^B]\,[(\zeta_\beta^B)^\dagger\theta_\alpha^A] \\ & +\sum_\alpha\sum_\beta \nu_\alpha\sigma_\beta\,[(\zeta_\beta^B)^\dagger\theta_\alpha^A]\,[(\theta_\alpha^A)^\dagger\zeta_\beta^B] \\ = \;& 2\sum_\alpha\sum_\beta \nu_\alpha\sigma_\beta\,|\,[(\theta_\alpha^A)^\dagger\zeta_\beta^B]\,|^2 \end{aligned} \tag{228}$$

Hence we have using (221)

$$\begin{aligned} & \left\langle \Delta\widehat{S}_x^2 \right\rangle - \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle| \\ \geq \;& \sum_R P_R \frac{1}{2}\sum_\alpha\sum_\beta \nu_\alpha\sigma_\beta\,|\,[(\theta_\alpha^A)^\dagger\zeta_\beta^B]\,|^2 \end{aligned} \tag{229}$$

where the right side of the inequality is non-negative. The same result applies to $\left\langle \Delta\widehat{S}_y^2 \right\rangle - \frac{1}{2}\left|\left\langle \widehat{S}_z \right\rangle\right|$. Thus separable states are *not* spin squeezed in $\widehat{S}_x$ or in $\widehat{S}_y$.

Thus we have established the *spin squeezing* test for the multi–mode Case 1 - states that are spin squeezed in $\widehat{S}_x$ compared to $\widehat{S}_y$ (or vice versa) must be entangled states for the two subsystems consisting of all modes $\widehat{a}_i$ and all modes $\widehat{b}_i$.

For the other spin components, the Bloch vector result in (210) that $\left\langle \widehat{S}_x \right\rangle = \left\langle \widehat{S}_y \right\rangle = 0$ for *separable* states enables us to show that if $\widehat{S}_z$ is squeezed compared to $\widehat{S}_x$ (or vice versa) or if $\widehat{S}_z$ is squeezed compared to $\widehat{S}_y$ (or vice versa) then the state must be entangled. Thus spin squeezing in *any* spin component requires the state to be entangled, just as for the two mode case.

D.3 Spin Squeezing Tests for Single Mode Sub-Systems (Case 2)

We now consider spin squeezing for the multi-mode spin operators given in Eqs. (193) and (196) in Appendix A. We consider separable states for *Case 2*, the density operator being given in Eq. (206). In this *single mode sub-system case* there are $2n$ subsystems consist of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$.

This case is that involved in the modified approach to Sørensen et al and we will see that it leads to a useful inequality for $\left\langle \Delta\widehat{S}_x^{\,2}\right\rangle$ or $\left\langle \Delta\widehat{S}_y^{\,2}\right\rangle$ that applies when non-entangled states are those when *all* the separate modes $\widehat{a}_i$ and $\widehat{b}_i$ are the sub-systems. We will follow the approach used for the simple two mode case in Section 3.

Firstly, the *variance* for a Hermitian operator $\widehat{\Omega}$ in a mixed state

$$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R \tag{230}$$

is always greater than or equal to the the average of the variances for the separate components

$$\left\langle \Delta\widehat{\Omega}^{\,2}\right\rangle \geq \sum_R P_R \left\langle \Delta\widehat{\Omega}^2\right\rangle_R \tag{231}$$

where $\left\langle \Delta\widehat{\Omega}^{\,2}\right\rangle = Tr(\widehat{\rho}\,\Delta\widehat{\Omega}^{\,2})$ with $\Delta\widehat{\Omega} = \widehat{\Omega} - \left\langle \widehat{\Omega}\right\rangle$ and $\left\langle \Delta\widehat{\Omega}^{\,2}\right\rangle_R = Tr(\widehat{\rho}_R\,\Delta\widehat{\Omega}_R^{\;2})$ with $\Delta\widehat{\Omega}_R = \widehat{\Omega} - \left\langle \widehat{\Omega}\right\rangle_R$. The proof is straight-forward and given in Ref. [19].

Next we calculate $\left\langle \Delta\widehat{S}_x^{\,2}\right\rangle_R$, $\left\langle \Delta\widehat{S}_y^{\,2}\right\rangle_R$ and $\left\langle \widehat{S}_x\right\rangle_R$, $\left\langle \widehat{S}_y\right\rangle_R$, $\left\langle \widehat{S}_z\right\rangle_R$ for the case where

$$\widehat{\rho} = \sum_R P_R \left(\widehat{\rho}_R^{\,a\,1} \otimes \widehat{\rho}_R^{\,b\,1}\right) \otimes \left(\widehat{\rho}_R^{\,a\,2} \otimes \widehat{\rho}_R^{\,b\,2}\right) \otimes \left(\widehat{\rho}_R^{\,a\,3} \otimes \widehat{\rho}_R^{\,b\,3}\right) \otimes .... \tag{232}$$

as is required for a *general non-entangled* state *all* $2n$ modes. This situation is that of Case 2 for the sub-systems, as described in SubSection D.1. As the density operators for the individual modes must represent possible physical states for such modes, so the super-selection rule for atom number applies and we have

$$\begin{aligned}
\langle (\widehat{a}_i)^p \rangle_{a\,i} &= Tr(\widehat{\rho}_R^{\,a\,i}(\widehat{a}_i)^p) = 0 \qquad \left\langle (\widehat{a}_i^\dagger)^p \right\rangle_{a\,i} = Tr(\widehat{\rho}_R^{\,a\,i}(\widehat{a}_i^\dagger)^p) = 0 \\
\left\langle (\widehat{b}_i)^m \right\rangle_{b\,i} &= Tr(\widehat{\rho}_R^{\,b\,i}(\widehat{b}_i)^m) = 0 \qquad \left\langle (\widehat{b}_i^\dagger)^m \right\rangle_{b\,i} = Tr(\widehat{\rho}_R^{\,b\,i}(\widehat{b}_i^\dagger)^m) = 0
\end{aligned} \tag{233}$$

The Schwinger spin operators are

$$\begin{aligned}
\widehat{S}_x &= \sum_i (\widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i)/2 = \sum_i \widehat{S}_x^i \\
\widehat{S}_y &= \sum_i (\widehat{b}_i^\dagger \widehat{a}_i - \widehat{a}_i^\dagger \widehat{b}_i)/2i = \sum_i \widehat{S}_y^i \\
\widehat{S}_z &= \sum_i (\widehat{b}_i^\dagger \widehat{b}_i - \widehat{a}_i^\dagger \widehat{a}_i)/2 = \sum_i \widehat{S}_z^i
\end{aligned} \tag{234}$$

where $\widehat{a}_i$, $\widehat{b}_i$ and $\widehat{a}_i^\dagger$, $\widehat{b}_i^\dagger$ respectively are mode annihilation, creation operators. Note that this expression for the spin operators is the same as (196) for the multi-mode case treated in Appendix A. From Eqs. (234) we find that

$$\widehat{S}_x^{\,2} = \sum_i (\widehat{S}_x^i)^2 + \sum_{i \neq j} \widehat{S}_x^i \widehat{S}_x^j \tag{235}$$

so that on taking the trace with $\widehat{\rho}_R$ and using Eqs. (232) we get after applying the commutation rules $[\widehat{e}, \widehat{e}^\dagger] = \widehat{1}$ ($\widehat{e} = \widehat{a}$ or $\widehat{b}$)

$$\left\langle \widehat{S}_x^{\,2} \right\rangle_R = \sum_i \left\langle (\widehat{S}_x^i)^2 \right\rangle_R + \sum_{i \neq j} \left\langle \widehat{S}_x^i \right\rangle_R \left\langle \widehat{S}_x^j \right\rangle_R \tag{236}$$

As we also have

$$\left\langle \widehat{S}_x \right\rangle_R = \sum_i \left\langle \widehat{S}_x^i \right\rangle_R \qquad \left\langle \widehat{S}_x \right\rangle_R^2 = \sum_i \left\langle \widehat{S}_x^i \right\rangle_R^2 + \sum_{i \neq j} \left\langle \widehat{S}_x^i \right\rangle_R \left\langle \widehat{S}_x^j \right\rangle_R \tag{237}$$

using Eqs. (232) and we see finally that the variance $\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle_R$ is

$$\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle_R = \sum_i \left\langle (\widehat{S}_x^i)^2 \right\rangle_R - \sum_i \left\langle \widehat{S}_x^i \right\rangle_R^2 \tag{238}$$

all the terms with $i \neq j$ cancelling out. and therefore from Eq. (231)

$$\left\langle \Delta \widehat{S}_x^{\,2} \right\rangle \geq \sum_R P_R \sum_i \left(\left\langle (\widehat{S}_x^i)^2 \right\rangle_R - \left\langle \widehat{S}_x^i \right\rangle_R^2 \right) \tag{239}$$

An analogous result applies for $\left\langle \Delta \widehat{S}_y^{\,2} \right\rangle$.

But using (233)

$$\begin{aligned}
(\widehat{S}_x^i)^2 &= \frac{1}{4}(\widehat{b}_i^\dagger \widehat{a}_i \widehat{b}_i^\dagger \widehat{a}_i + \widehat{b}_i^\dagger \widehat{a}_i \widehat{a}_i^\dagger \widehat{b}_i + \widehat{a}_i^\dagger \widehat{b}_i \widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i \widehat{a}_i^\dagger \widehat{b}_i) \\
\left\langle (\widehat{S}_x^i)^2 \right\rangle_R &= \frac{1}{4}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R) + \frac{1}{2}(\langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R)
\end{aligned} \tag{240}$$

and

$$\left\langle \widehat{S}_x^i \right\rangle_R = 0 \tag{241}$$

It then follows that

$$\left\langle \widehat{S}_x \right\rangle = \sum_R P_R \left\langle \widehat{S}_x \right\rangle_R = 0 \qquad \left\langle \widehat{S}_y \right\rangle = \sum_R P_R \left\langle \widehat{S}_y \right\rangle_R = 0 \tag{242}$$

so that

$$\left\langle \Delta \widehat{S}_x^2 \right\rangle \geq \sum_R P_R \sum_i \left(\frac{1}{4}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R) + \frac{1}{2}(\langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R) \right) \tag{243}$$

The same result applies for $\left\langle \Delta \widehat{S}_y^2 \right\rangle$.

Now using (233)

$$\left\langle \widehat{S}_z^i \right\rangle_R = \frac{1}{2}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R - \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R)) \tag{244}$$

$$\begin{aligned}
\left\langle \widehat{S}_z \right\rangle &= \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R \\
\frac{1}{2}\left| \left\langle \widehat{S}_z \right\rangle \right| &= \frac{1}{2}\sum_R P_R \left| \sum_i \frac{1}{2}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R - \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R)) \right| \\
&\leq \sum_R P_R \frac{1}{4} \sum_i \left| (\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R - \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R)) \right| \\
&\leq \sum_R P_R \frac{1}{4} \sum_i (\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R))
\end{aligned} \tag{245}$$

and thus

$$\begin{aligned}
&\left\langle \Delta \widehat{S}_x^2 \right\rangle - \frac{1}{2}\left| \left\langle \widehat{S}_z \right\rangle \right| \\
\geq\ & \sum_R P_R \sum_i \left(\frac{1}{4}(\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R) + \frac{1}{2}(\langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R) \right) \\
& - \sum_R P_R \frac{1}{4} \sum_i (\left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R + \langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R)) \\
=\ & \sum_R P_R \frac{1}{2} \sum_i (\langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R) \\
\geq\ & 0
\end{aligned} \tag{246}$$

A similar proof shows that $\left\langle \Delta \widehat{S}_y^2 \right\rangle - \frac{1}{2}\left| \left\langle \widehat{S}_z \right\rangle \right| \geq 0$ for the non-entangled state of all $2n$ modes.

This shows that for the general non-entangled state with all modes $\widehat{a}_i$ and $\widehat{b}_i$ as the sub-systems, the variances for two of the spin fluctuations $\left\langle \Delta \widehat{S}_x^2 \right\rangle$ and $\left\langle \Delta \widehat{S}_y^2 \right\rangle$ are both greater than $\frac{1}{2}\left| \left\langle \widehat{S}_z \right\rangle \right|$, and hence there is no spin squeezing for $\widehat{S}_x$ or $\widehat{S}_y$. Note that as $\left|\left\langle \widehat{S}_{\,y} \right\rangle\right| = 0$, the quantity $\sqrt{\left(\left|\left\langle \widehat{S}_{\perp\,1} \right\rangle\right|^2 + \left|\left\langle \widehat{S}_{\perp\,2} \right\rangle\right|^2\right)}$ is the same as $\left|\left\langle \widehat{S}_{\,z} \right\rangle\right|$, so the alternative criterion in Eq. (201) is the same as that in Eq. (6) which is used here.

Hence we have shown that for a *non-entangled* physical state for all the $2n$ modes $\widehat{a}_i$ and $\widehat{b}_i$

$$\left\langle \Delta\widehat{S}_{\,x}^{\,2} \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_{\,z} \right\rangle| \quad and \quad \left\langle \Delta\widehat{S}_{\,y}^{\,2} \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_{\,z} \right\rangle| \tag{247}$$

so that spin squeezing in either $\widehat{S}_x$ or $\widehat{S}_y$ requires entanglement.

From (242) we see that $\left\langle \widehat{S}_{\,x} \right\rangle = \left\langle \widehat{S}_{\,y} \right\rangle = 0$ for the general separable state, showing there is a *Bloch vector test* for entanglement such that if either $\left\langle \widehat{S}_{\,x} \right\rangle$ or $\left\langle \widehat{S}_{\,y} \right\rangle$ is non-zero, then the state must be entangled.

Finally, if there is spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_x$ or vice versa, or spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_y$ or vice versa, it follows that one of $\left\langle \widehat{S}_{\,x} \right\rangle$ or $\left\langle \widehat{S}_{\,y} \right\rangle$ is non-zero. But as both these quantities are zero for a non-entangled state, if follows that spin squeezing in $\widehat{S}_z$ also requires entanglement.

Thus, spin squeezing in *any* spin operator $\widehat{S}_x, \widehat{S}_y$ or $\widehat{S}_z$ is a sufficiency test for entanglement of all the separate mode sub-systems.

D.4 Spin Squeezing Tests for Two Mode Sub-Systems (Case 3)

We now consider spin squeezing for the multi-mode spin operators given in Eqs. (193) and (196) in Appendix A. We consider separable states for *Case 3*, the density operator being given in Eq. (207). In this *mode pair sub-system case* there are n subsystems consist of *all pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$.

This case is also involved in a modified approach to Sørensen et al and we show a useful inequality for $\left\langle \Delta\widehat{S}_{\,z}^{\,2} \right\rangle$ applies when non-entangled states are those when the *pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$ are the separate sub-systems, but only in restricted situations. The pairs of modes corresponding to localized modes on different lattice sites or pairs of modes with the same momenta does represent the closest way of simulating the approach used by Sørensen et al where identical particles i were regarded as the sub-systems.

Now the general non-entangled state will be

$$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R^{1} \otimes \widehat{\rho}_R^{2} \otimes \widehat{\rho}_R^{3} \otimes ... \tag{248}$$

where the $\widehat{\rho}_R^{i}$ are now the density operators for sub-system i consisting of the pair of modes $\widehat{a}_i$ and $\widehat{b}_i$ (which are of the form given in Eq. (303)) and the conditions in Eq. (233) no longer apply. The Fock states are of the form $|N_{ia}\rangle\otimes|N_{ib}\rangle$ for the pair of modes $\widehat{a}_i$ and $\widehat{b}_i$, and for this Fock state the total occupancy of the pair of modes is $N_i = N_{ia} + N_{ib}$. From the super-selection rule the density operator $\widehat{\rho}_R^i$ for the ith pair of modes $\widehat{a}_i$ and $\widehat{b}_i$ is diagonal in the total occupancy. For $N_i = 0$ there is one non zero matrix element $(\langle 0|_{ia}\otimes\langle 0|_{ib})\,\widehat{\rho}_R^i\,(|0\rangle_{ia}\otimes|0\rangle_{ib})$. For $N_i = 1$ there are four non zero matrix elements, which may be written

$$\begin{aligned}
(\langle 1|_{ia}\otimes\langle 0|_{ib})\,\widehat{\rho}_R^i\,(|1\rangle_{ia}\otimes|0\rangle_{ib}) &= \rho_{aa}^i \\
(\langle 1|_{ia}\otimes\langle 0|_{ib})\,\widehat{\rho}_R^i\,(|0\rangle_{ia}\otimes|1\rangle_{ib}) &= \rho_{ab}^i \\
(\langle 0|_{ia}\otimes\langle 1|_{ib})\,\widehat{\rho}_R^i\,(|1\rangle_{ia}\otimes|0\rangle_{ib}) &= \rho_{ba}^i \\
(\langle 0|_{ia}\otimes\langle 1|_{ib})\,\widehat{\rho}_R^i\,(|0\rangle_{ia}\otimes|1\rangle_{ib}) &= \rho_{bb}^i
\end{aligned} \tag{249}$$

For $N_i = 2$ there are nine non zero matrix element $(\langle 2|_{ia}\otimes\langle 0|_{ib})\,\widehat{\rho}_R^i\,(|2\rangle_{ia}\otimes|0\rangle_{ib})$, ..., $(\langle 0|_{ia}\otimes\langle 2|_{ib})\,\widehat{\rho}_R^i\,(|0\rangle_{ia}\otimes|2\rangle_{ib})$ and the number increases with N_i.

If we restrict ourselves to general entangled states for *one particle*, where $N_i = 1$ for all pairs of modes, then the density operator $\widehat{\rho}_R^i$ is of then form

$$\begin{aligned}
\widehat{\rho}_R^i &= \rho_{aa}^i(|1\rangle_{ia}\langle 1|_{ia}\otimes|0\rangle_{ib}\langle 0|_{ib}) + \rho_{ab}^i(|1\rangle_{ia}\langle 0|_{ia}\otimes|0\rangle_{ib}\langle 1|_{ib}) \\
&\quad+\rho_{ba}^i(|0\rangle_{ia}\langle 1|_{ia}\otimes|1\rangle_{ib}\langle 0|_{ib}) + \rho_{bb}^i(|0\rangle_{ia}\langle 0|_{ia}\otimes|1\rangle_{ib}\langle 1|_{ib})
\end{aligned} \tag{250}$$

In addition Hermitiancy, positivity, unit trace $Tr(\widehat{\rho}_R^i) = 1$ and $Tr(\widehat{\rho}_R^i)^2 \leq 1$ can be used as in Eq (295) to parameterize the matrix elements in (249).

$$\begin{aligned}
\rho_{aa}^i &= \sin^2\alpha_i \qquad \rho_{bb}^i = \cos^2\alpha_i \\
\rho_{ab}^i &= \sqrt{\sin^2\alpha_i\,\cos^2\alpha_i}\,\sin^2\beta_i\,\exp(+i\phi_i) \qquad \rho_{ba}^i = \sqrt{\sin^2\alpha_i\,\cos^2\alpha_i}\,\sin^2\beta_i\,\exp(-i\phi_i)
\end{aligned} \tag{251}$$

The expectation values for the spin operators $\widehat{S}_x^i$, $\widehat{S}_y^i$ and $\widehat{S}_z^i$ associated with the ith pair of modes are then

$$\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R &= Tr(\widehat{\rho}_R^i\,\frac{1}{2}(\widehat{b}_i^\dagger\widehat{a}_i + \widehat{a}_i^\dagger\widehat{b}_i) \\
&= \frac{1}{2}\left(\rho_{ab}^i + \rho_{ba}^i\right) \\
\left\langle \widehat{S}_y^i \right\rangle_R &= \frac{1}{2i}\left(\rho_{ab}^i - \rho_{ba}^i\right) \\
\left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\left(\rho_{bb}^i - \rho_{aa}^i\right)
\end{aligned} \tag{252}$$

which are of exactly the same form as in Eq. (294) as in the Appendix G derivation of the original Sørensen et al [14] results based on treating identical particles as the sub-systems. The proof however is now different and rests on restricting the states $\widehat{\rho}_R^i$ to each containing exactly one boson.

The remainder of the proof is exactly the same as in Appendix G and we find that

$$\left\langle \Delta\widehat{S}_z^{\,2} \right\rangle \geq \frac{1}{N}\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2\right) \tag{253}$$

for non-entangled *pairs* of modes $\widehat{a}_i$ and $\widehat{b}_i$. Thus when the interpretation is changed so that are the separate sub-systems are these pairs of modes, it follows that spin squeezing in $\widehat{S}_z$ with respect to $\widehat{S}_x$ or $\widehat{S}_y$ requires entanglement of all the mode pairs, but only if there is one particle in each mode pair.

In general, spin squeezing in either $\widehat{S}_x$ or $\widehat{S}_y$ is not linked to entanglement for Case 3 sub-systems, as has been pointed out in SubSection D.1 by a counter-example involving the relative phase state. Also there is no Bloch vector entanglement test. For we have in general

$$\begin{aligned} \left\langle \widehat{S}_x^i \right\rangle_R &= Tr(\widehat{\rho}_R^i \frac{1}{2}(\widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i) \\ \left\langle \widehat{S}_y^i \right\rangle_R &= Tr(\widehat{\rho}_R^i \frac{1}{2i}(\widehat{b}_i^\dagger \widehat{a}_i - \widehat{a}_i^\dagger \widehat{b}_i) \end{aligned} \tag{254}$$

and the local particle number SSR does not require these quantities to be zero for sub-systems consisting of pairs of modes $\widehat{a}_i$ and $\widehat{b}_i$. Thus in general $\left\langle \widehat{S}_x \right\rangle$ and $\left\langle \widehat{S}_y \right\rangle$ can be non-zero for a separable state, so the Bloch vector entanglement test does not apply.

Appendix E Hillery Spin Variance - Multi-Mode

It turns out that the Hillery spin variance test can also be applied in multi-mode situations, where the spin operators are defined as in Appendix A. As explained in Appendix D three cases occur in regard to specifying the sub-systems. For *Case 1*, where there are *two sub-systems* each consisting of all the modes $\widehat{a}_i$ or all the modes $\widehat{b}_i$. the Hillery spin variance test as in (84) applies. The proof set out below again does not require the sub-system density operators to be local SSR compliant. Also, for *Case 2* where there are $2n$ subsystems consisting of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$ the Hillery spin variance test as in (84) applies. The proof is set out below again does not require the sub-system density operators to be local SSR compliant. However, for *Case 3* where there are n sub-systems consisting of *all* mode pairs $\widehat{a}_i$ and $\widehat{b}_i$ the Hillery spin variance test does *not* apply. Basically, this is because specific sub-system density operators $\widehat{\rho}_R^{ab(i)}$ (see (207)) could be *entangled* states of the modes $\widehat{a}_i$ and $\widehat{b}_i$ all of which *do* satisfy the Hillery test involving $\left\langle \widehat{N}_i \right\rangle_R$ for this ith sub-system. If we choose a special separable state of the form (207) with just *one* term (no sum over R), it is easy to see that the Hillery test will be satisfied for the full system. However, the full system state involving these sub-systems is still a *separable* state, showing that satisfying the Hillery spin variance test does not always require the state to be entangled.

E.1 Hillery Spin Variance Test for Bipartite System (Case 1)

We first consider *Case 1* where there are *two sub-systems* each consisting of all the modes $\widehat{a}_i$ or all the modes $\widehat{b}_i$. We use the results from (214) to find that for a separable state

$$\begin{aligned} & \left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \\ \geq\; & \sum_R P_R \{ \iint d\mathbf{r}\, d\mathbf{r}' \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \\ & + \frac{1}{2} \int d\mathbf{r}\ \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \} \end{aligned} \tag{255}$$

The same result would have occurred if the local sub-system SSR had been disregarded, the terms such as $\frac{1}{4}\iint d\mathbf{r}\, d\mathbf{r}' \times \{ \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b^\dagger(\mathbf{r}') \right\rangle_R^B \left\langle \widehat{\Psi}_a(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}') \right\rangle_R^A$ cancelling out.

The mean number of bosons is obtained from (197) and hence

$$\frac{1}{2}\left\langle \widehat{N} \right\rangle = \frac{1}{2}\sum_R P_R \int d\mathbf{r}\ \left(\left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}) \right\rangle_R^B + \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r})\widehat{\Psi}_a(\mathbf{r}) \right\rangle_R^A \right) \tag{256}$$

Thus we have

$$\begin{aligned}
&\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle - \frac{1}{2}\left\langle \widehat{N}\right\rangle \\
\geq\ & \sum_R P_R \iint d\mathbf{r}\, d\mathbf{r}'\, \left\langle \widehat{\Psi}_b^\dagger(\mathbf{r})\widehat{\Psi}_b(\mathbf{r}')\right\rangle_R^B \left\langle \widehat{\Psi}_a^\dagger(\mathbf{r}')\widehat{\Psi}_a(\mathbf{r})\right\rangle_R^A
\end{aligned} \tag{257}$$

Using the mode expansion (195) we then get

$$\begin{aligned}
&\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle - \frac{1}{2}\left\langle \widehat{N}\right\rangle \\
\geq\ & \sum_R P_R \sum_{ij}\sum_{kl} \iint d\mathbf{r}\, d\mathbf{r}'\, \phi_i^*(\mathbf{r})\phi_j(\mathbf{r}')\phi_k^*(\mathbf{r}')\phi_l(\mathbf{r}) \left\langle \widehat{b}_i^\dagger\widehat{b}_j\right\rangle_R^B \left\langle \widehat{a}_k^\dagger\widehat{a}_l\right\rangle_R^A \\
=\ & \sum_R P_R \sum_{ij} Tr_A\{\widehat{a}_i\,\widehat{\rho}_R^A\,\widehat{a}_j^\dagger\} Tr_B\{\widehat{b}_j\,\widehat{\rho}_R^B\,\widehat{b}_i^\dagger\} && (258) \\
=\ & \sum_R P_R \frac{1}{2} Tr(A^R B^R + B^R A^R) && (259)
\end{aligned}$$

after orthogonality is used and the matrix elements A_{ij}^R and B_{ji}^R are introduced from (222).

Since we have shown in Appendix D.2 that the right side of the last inequality is always non-negative, the *Hillery spin variance* entanglement test follows that if

$$\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle < \frac{1}{2}\left\langle \widehat{N}\right\rangle \tag{260}$$

then the quantum state must be an entangled state for the case of two sub-systems each consisting of all the modes $\widehat{a}_i$ or all the modes $\widehat{b}_i$.

E.2 Hillery Spin Variance Test for Single Mode Sub-Systems (Case 2)

We now consider separable states for *Case 2*, the density operator being given in Eq. (206). In this *single mode sub-system case* there are $2n$ subsystems consisting of *all* modes $\widehat{a}_i$ and *all* modes $\widehat{b}_i$. We use the results from (243) to find that for a separable state

$$\begin{aligned}
&\left\langle \Delta\widehat{S}_x^2\right\rangle + \left\langle \Delta\widehat{S}_y^2\right\rangle \\
\geq\ & \sum_R P_R \sum_i \left(\frac{1}{2}(\left\langle (\widehat{b}^\dagger\widehat{b})_i\right\rangle_R + \left\langle (\widehat{a}^\dagger\widehat{a})_i\right\rangle_R) + (\left\langle (\widehat{a}^\dagger\widehat{a})_i\right\rangle_R \left\langle (\widehat{b}^\dagger\widehat{b})_i\right\rangle_R)\right)
\end{aligned} \tag{261}$$

The same result would have occurred if the local sub-system SSR had been disregarded, the terms such as $\frac{1}{4}\left\langle \widehat{b}_i^\dagger\widehat{b}_i^\dagger\right\rangle_{Ri} \langle \widehat{a}_i\widehat{a}_i\rangle_R$ cancelling out.

The mean number of bosons is obtained from (197)

$$\frac{1}{2}\left\langle \widehat{N}\right\rangle = \frac{1}{2}\sum_R P_R \sum_i (\left\langle (\widehat{b}^\dagger\widehat{b})_i\right\rangle_R + \left\langle (\widehat{a}^\dagger\widehat{a})_i\right\rangle_R) \tag{262}$$

Thus we have

$$\begin{aligned} & \left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle - \frac{1}{2}\left\langle \widehat{N} \right\rangle \\ \geq \; & \sum_R P_R \sum_i \left(\langle (\widehat{a}^\dagger \widehat{a})_i \rangle_R \left\langle (\widehat{b}^\dagger \widehat{b})_i \right\rangle_R \right) \end{aligned} \tag{263}$$

which is always non-negative.

The *Hillery spin variance* entanglement test follows that if the inequality in (260) occurs then the quantum state must be an entangled state for the case of $2n$ sub-systems consisting of all the modes $\widehat{a}_i$ and all the modes $\widehat{b}_i$.

E.3 Hillery Spin Variance Test for Two Mode Sub-Systems (Case 3)

We now consider separable states for *Case 3*, the density operators being given in Eq. (207). In this *two mode sub-system case* there are n subsystems consisting of *all* mode pairs $\widehat{a}_i$ and $\widehat{b}_i$. We consider a *special* separable state with just one term where

$$\widehat{\rho}_{sep} = \widehat{\rho}^{ab(1)} \otimes \widehat{\rho}^{ab(2)} \otimes .. \otimes \widehat{\rho}^{ab(i)} .. \otimes \widehat{\rho}^{ab(n)} \tag{264}$$

We use the results from (238) to find that

$$\begin{aligned} & \left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \\ = \; & \sum_i \left(\left\langle (\Delta\widehat{S}_x^i)^2 \right\rangle + \left\langle (\Delta\widehat{S}_y^i)^2 \right\rangle \right) \end{aligned} \tag{265}$$

where $\Delta\widehat{S}_\alpha^i = \widehat{S}_\alpha^i - \left\langle \widehat{S}_\alpha^i \right\rangle_R$ for $\alpha = x, y$. This result did not depend on applying the local SSR.

Now suppose each of the two mode states $\widehat{\rho}^{ab(i)}$ is an entangled state of the modes $\widehat{a}_i$ and $\widehat{b}_i$ in which the Hillery spin variance test is satisfied. Then

$$\left\langle (\Delta\widehat{S}_x^i)^2 \right\rangle + \left\langle (\Delta\widehat{S}_y^i)^2 \right\rangle < \frac{1}{2} \langle \widehat{n}_i \rangle$$

Hence

$$\begin{aligned} & \left\langle \Delta\widehat{S}_x^2 \right\rangle + \left\langle \Delta\widehat{S}_y^2 \right\rangle \\ < \; & \sum_i \frac{1}{2} \left\langle \widehat{N}_i \right\rangle \\ = \; & \frac{1}{2} \left\langle \widehat{N} \right\rangle \end{aligned} \tag{266}$$

where $\widehat{N} = \sum_i \widehat{N}_i$ is the total number operator and $\widehat{N}_i = \widehat{b}_i^\dagger \widehat{b}_i + \widehat{a}_i^\dagger \widehat{a}_i$

Thus the Hillery spin variance test is satisfied even though the state (264) is separable, shoeing that the test cannot be applied for multi-mode Case 3.

Appendix F Raymer Entanglement Test

In this Appendix the proof of the Raymer entanglement test in SubSection 4.3 is presented.

With Hermitian operators $\widehat{\Omega}_A, \widehat{\Lambda}_A$ and $\widehat{\Omega}_B, \widehat{\Lambda}_B$ for the two sub-systems we consider

$$\widehat{U} = \alpha\widehat{\Omega}_A + \beta\widehat{\Omega}_B \qquad \widehat{V} = \alpha\widehat{\Lambda}_A - \beta\widehat{\Lambda}_B \tag{267}$$

where α, β are real. Then with $\widehat{\rho} = \sum_R P_R\,\widehat{\rho}_R$ and $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$ and using (20) it can first be shown that

$$\left\langle \Delta\widehat{U}^2 \right\rangle \geq \sum_R P_R \left\langle \Delta\widehat{U}_R^2 \right\rangle \qquad \left\langle \Delta\widehat{V}^2 \right\rangle \geq \sum_R P_R \left\langle \Delta\widehat{V}_R^2 \right\rangle \tag{268}$$

where $\Delta\widehat{U}_R = \widehat{U} - \left\langle \widehat{U} \right\rangle_R$, $\Delta\widehat{V}_R = \widehat{V} - \left\langle \widehat{V} \right\rangle_R$ with $\left\langle \widehat{U} \right\rangle_R = Tr(\widehat{U}\,\widehat{\rho}_R)$, $\left\langle \widehat{V} \right\rangle_R = Tr(\widehat{V}\,\widehat{\rho}_R)$.

Substituting for $\widehat{U}$ and $\widehat{V}$ from (267) and using $\widehat{\rho}_R = \widehat{\rho}_R^A \otimes \widehat{\rho}_R^B$ we can then evaluate the various terms as follows.

$$\begin{aligned}
\left\langle \widehat{U}^2 \right\rangle_R &= \alpha^2 \left\langle \widehat{\Omega}_A^2 \right\rangle_A^R + \beta^2 \left\langle \widehat{\Omega}_B^2 \right\rangle_B^R + 2\alpha\beta \left\langle \widehat{\Omega}_A \right\rangle_A^R \left\langle \widehat{\Omega}_B \right\rangle_B^R \\
\left\langle \widehat{U} \right\rangle_R &= \alpha \left\langle \widehat{\Omega}_A \right\rangle_A^R + \beta \left\langle \widehat{\Omega}_B \right\rangle_B^R \\
\left(\left\langle \widehat{U} \right\rangle_R \right)^2 &= \alpha^2 \left(\left\langle \widehat{\Omega}_A \right\rangle_A^R \right)^2 + \beta^2 \left(\left\langle \widehat{\Omega}_B \right\rangle_B^R \right)^2 + 2\alpha\beta \left\langle \widehat{\Omega}_A \right\rangle_A^R \left\langle \widehat{\Omega}_B \right\rangle_B^R \\
\left\langle \Delta\widehat{U}_R^2 \right\rangle &= \alpha^2 \left(\left\langle \widehat{\Omega}_A^2 \right\rangle_A^R - \left(\left\langle \widehat{\Omega}_A \right\rangle_A^R \right)^2 \right) + \beta^2 \left(\left\langle \widehat{\Omega}_B^2 \right\rangle_B^R - \left(\left\langle \widehat{\Omega}_B \right\rangle_B^R \right)^2 \right)
\end{aligned} \tag{269}$$

with a similar result for $\left\langle \Delta\widehat{V}_R^2 \right\rangle$. Here for sub-system A we define $\left\langle \widehat{\Omega}_A^2 \right\rangle_A^R = Tr(\widehat{\Omega}_A^2\,\widehat{\rho}_R^A)$, $\left\langle \widehat{\Omega}_A \right\rangle_A^R = Tr(\widehat{\Omega}_A\,\widehat{\rho}_R^A)$ and $\left\langle \widehat{\Lambda}_A^2 \right\rangle_A^R = Tr(\widehat{\Lambda}_A^2\,\widehat{\rho}_R^A)$, $\left\langle \widehat{\Lambda}_A \right\rangle_A^R = Tr(\widehat{\Lambda}_A\,\widehat{\rho}_R^A)$ with analogous expressions for sub-system B.

We thus have

$$\begin{aligned}
\left\langle \Delta\widehat{U}^2 \right\rangle &\geq \alpha^2 \sum_R P_R \left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \beta^2 \sum_R P_R \left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R \\
\left\langle \Delta\widehat{V}^2 \right\rangle &\geq \alpha^2 \sum_R P_R \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R + \beta^2 \sum_R P_R \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R
\end{aligned} \tag{270}$$

where $\Delta\widehat{\Omega}_{AR} = \widehat{\Omega}_A - \left\langle \widehat{\Omega}_A \right\rangle_A^R$, $\Delta\widehat{\Omega}_{BR} = \widehat{\Omega}_B - \left\langle \widehat{\Omega}_B \right\rangle_B^R$, $\Delta\widehat{\Lambda}_{AR} = \widehat{\Lambda}_A - \left\langle \widehat{\Lambda}_A \right\rangle_A^R$ and $\Delta\widehat{\Lambda}_{BR} = \widehat{\Lambda}_B - \left\langle \widehat{\Lambda}_B \right\rangle_B^R$.

Adding the two results gives

$$
\begin{aligned}
& \left\langle \Delta\widehat{U}^2 \right\rangle + \left\langle \Delta\widehat{V}^2 \right\rangle \\
\geq \quad & \alpha^2 \sum_R P_R \left(\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R \right) \\
& + \beta^2 \sum_R P_R \left(\left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R + \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R \right) \qquad (271)
\end{aligned}
$$

a general variance inequality for separable states.

This last result can be developed further based on the *commutation rules*

$$
[\widehat{\Omega}_A, \widehat{\Lambda}_A] = i\widehat{\Theta}_A \qquad [\widehat{\Omega}_B, \widehat{\Lambda}_B] = i\widehat{\Theta}_B \qquad (272)
$$

The *Schwarz* inequalities - valid for all real λ_A and λ_B

$$
\begin{aligned}
\left\langle (\Delta\widehat{\Omega}_{AR} - i\lambda_A \Delta\widehat{\Lambda}_{AR})\, \widehat{\rho}_R^A \, (\Delta\widehat{\Omega}_{AR} + i\lambda_A \Delta\widehat{\Lambda}_{AR}) \right\rangle_A^R &\geq 0 \\
\left\langle (\Delta\widehat{\Omega}_{BR} - i\lambda_B \Delta\widehat{\Lambda}_{BR})\, \widehat{\rho}_R^B \, (\Delta\widehat{\Omega}_{BR} + i\lambda_B \Delta\widehat{\Lambda}_{BR}) \right\rangle_B^R &\geq 0 \qquad (273)
\end{aligned}
$$

lead to the following inequalities

$$
\begin{aligned}
\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \lambda_A \left\langle \widehat{\Theta}_A \right\rangle_A^R + \lambda_A^2 \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R &\geq 0 \\
\left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R + \lambda_B \left\langle \widehat{\Theta}_B \right\rangle_B^R + \lambda_B^2 \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R &\geq 0 \qquad (274)
\end{aligned}
$$

so by taking $\lambda_{A,B} = 1$ or -1 we have

$$
\begin{aligned}
\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R + \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R &\geq |\left\langle \widehat{\Theta}_A \right\rangle_A^R| \\
\left\langle \Delta\widehat{\Omega}_{BR}^2 \right\rangle_B^R + \left\langle \Delta\widehat{\Lambda}_{BR}^2 \right\rangle_B^R &\geq |\left\langle \widehat{\Theta}_B \right\rangle_B^R| \qquad (275)
\end{aligned}
$$

The Heisenberg Uncertainty principle results $\left\langle \Delta\widehat{\Omega}_{AR}^2 \right\rangle_A^R \left\langle \Delta\widehat{\Lambda}_{AR}^2 \right\rangle_A^R \geq |\left\langle \widehat{\Theta}_A \right\rangle_A^R|^2/4$ etc also follow from (274).

Noting that $\sum_R P_R |\left\langle \widehat{\Theta}_A \right\rangle_A^R| \geq |\sum_R P_R \left\langle \widehat{\Theta}_A \right\rangle_A^R| = |\left\langle \widehat{\Theta}_A \right\rangle|$ and $\sum_R P_R |\left\langle \widehat{\Theta}_B \right\rangle_B^R| \geq |\sum_R P_R \left\langle \widehat{\Theta}_B \right\rangle_A^R| = |\left\langle \widehat{\Theta}_B \right\rangle|$ since the modulus of a sum is never greater than the sum of the moduli, we finally arrive at the final inequality for *separable* states

$$
\left\langle \Delta(\alpha\widehat{\Omega}_A + \beta\widehat{\Omega}_B)^2 \right\rangle + \left\langle \Delta(\alpha\widehat{\Lambda}_A - \beta\widehat{\Lambda}_B)^2 \right\rangle \geq \alpha^2 |\left\langle \widehat{\Theta}_A \right\rangle| + \beta^2 |\left\langle \widehat{\Theta}_B \right\rangle| \qquad (276)
$$

This leads to the following test for *entanglement*

$$
\left\langle \Delta(\alpha\widehat{\Omega}_A + \beta\widehat{\Omega}_B)^2 \right\rangle + \left\langle \Delta(\alpha\widehat{\Lambda}_A - \beta\widehat{\Lambda}_B)^2 \right\rangle < \alpha^2 |\left\langle \widehat{\Theta}_A \right\rangle| + \beta^2 |\left\langle \widehat{\Theta}_B \right\rangle| \qquad (277)
$$

which is usually based on choices where $\alpha^2 = \beta^2 = 1$.

Appendix G Derivation of Sørensen et al Results

Sørensen et al [14] derive a number of inequalities from which they deduce a further inequality for the spin squeezing parameter in the case of a non-entangled state. From this result they conclude that spin squeezing implies entanglement. The final inequality they obtain for a non-entangled state is

$$\left\langle \Delta\widehat{S}\,_z^2 \right\rangle \geq \frac{1}{N}\left(\left\langle \widehat{S}\,_x \right\rangle^2 + \left\langle \widehat{S}\,_y \right\rangle^2 \right) \tag{278}$$

Their approach is based on writing the density operator for a non-entangled state of N identical particles as in Eq. (105)

$$\widehat{\rho} = \sum_R P_R\, \widehat{\rho}_R^1 \otimes \widehat{\rho}_R^2 \otimes \widehat{\rho}_R^3 \otimes ... = \sum_R P_R\, \widehat{\rho}_R \tag{279}$$

The spin operators are defined as

$$\begin{aligned}
\widehat{S}_x &= \textstyle\sum_i \widehat{S}_x^i = \sum_i (|\phi_b(i)\rangle\,\langle\phi_a(i)| + |\phi_a(i)\rangle\,\langle\phi_b(i)|)/2 \\
\widehat{S}_y &= \textstyle\sum_i \widehat{S}_y^i = \sum_i (|\phi_b(i)\rangle\,\langle\phi_a(i)| - |\phi_a(i)\rangle\,\langle\phi_b(i)|)/2i \\
\widehat{S}_z &= \textstyle\sum_i \widehat{S}_z^i = \sum_i (|\phi_b(i)\rangle\,\langle\phi_b(i)| - |\phi_a(i)\rangle\,\langle\phi_a(i)|)/2
\end{aligned} \tag{280}$$

where the sum i is over the identical atoms and each atom is associated with two states $|\phi_a\rangle$ and $|\phi_b\rangle$. Clearly, the spin operators satisfy the standard commutation rules for angular momentum operators.

Sørensen et al [14] state that the variance for $\widehat{S}_z$ satisfies the result

$$\left\langle \Delta\widehat{S}\,_z^2 \right\rangle = \frac{N}{4} - \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 + \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 - \left\langle \widehat{S}_z \right\rangle^2 \tag{281}$$

To prove this we have

$$\begin{aligned}
\left\langle \widehat{S}\,_z^2 \right\rangle &= \sum_R P_R\, Tr(\widehat{\rho}_R \textstyle\sum_i \sum_j \widehat{S}_z^i \widehat{S}_z^j) \\
&= \sum_R P_R \left(\sum_i \left\langle \left(\widehat{S}_z^i\right)^2 \right\rangle_R + \sum_{i\neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right) \\
&= \frac{N}{4} + \sum_R P_R \left(\sum_{i\neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right)
\end{aligned} \tag{282}$$

where we have used

$$\begin{aligned}
\left(\widehat{S}_z^i\right)^2 &= \frac{1}{4}(|\phi_b(i)\rangle\langle\phi_b(i)| - |\phi_a(i)\rangle\langle\phi_a(i)|)^2 \\
&= \frac{1}{4}(|\phi_b(i)\rangle\langle\phi_b(i)|\phi_b(i)\rangle\langle\phi_b(i)| - (|\phi_b(i)\rangle\langle\phi_b(i)|\phi_a(i)\rangle\langle\phi_a(i)|) \\
&\quad + \frac{1}{4}(-(|\phi_a(i)\rangle\langle\phi_a(i)|\phi_b(i)\rangle\langle\phi_b(i)| + (|\phi_a(i)\rangle\langle\phi_a(i)|\phi_a(i)\rangle\langle\phi_a(i)|) \\
&= \frac{1}{4}((|\phi_b(i)\rangle\langle\phi_b(i)| + (|\phi_a(i)\rangle\langle\phi_a(i)|) \\
&= \frac{1}{4}\widehat{1}_i
\end{aligned} \tag{283}$$

a result based on the orthogonality, normalization and completeness of the states $|\phi_a(i)\rangle, |\phi_b(i)\rangle$. Also

$$\begin{aligned}
\left\langle \widehat{S}_z \right\rangle_R &= Tr(\widehat{\rho}_R \sum_i \widehat{S}_z^i) \\
&= \sum_i \left\langle \widehat{S}_z^i \right\rangle_R \\
\sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 &= \sum_R P_R \left(\sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 + \sum_{i\neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right)
\end{aligned} \tag{284}$$

so eliminating the term $\sum_R P_R \left(\sum_{i\neq j} \left\langle \widehat{S}_z^i \right\rangle_R \left\langle \widehat{S}_z^j \right\rangle_R \right)$ gives the required expression for $\left\langle \Delta \widehat{S}_z^{\,2} \right\rangle = \left\langle \widehat{S}_z^{\,2} \right\rangle - \left\langle \widehat{S}_z \right\rangle^2$.

Next, Sørensen et al [14] state that

$$\left\langle \widehat{S}_x \right\rangle^2 \leq N \sum_R P_R \sum_i \left\langle \widehat{S}_x^i \right\rangle_R^2 \qquad \left\langle \widehat{S}_y \right\rangle^2 \leq N \sum_R P_R \sum_i |\left\langle \widehat{S}_y^i \right\rangle_R|^2 \tag{285}$$

To prove this we have

$$\begin{aligned}
\left\langle \widehat{S}_x \right\rangle &= \sum_R P_R \, Tr(\widehat{\rho}_R \sum_i \widehat{S}_x^i) \\
&= \sum_R P_R \sum_i \left\langle \widehat{S}_x^i \right\rangle_R \\
|\left\langle \widehat{S}_x \right\rangle| &\leq \sum_R P_R \sum_i |\left\langle \widehat{S}_x^i \right\rangle_R|
\end{aligned} \tag{286}$$

since the modulus of a sum is less than or equal to the sum of the moduli. Now

$$\begin{aligned}
\left\langle \widehat{S}_x \right\rangle^2 &= |\left\langle \widehat{S}_x \right\rangle|^2 \leq \left(\sum_R P_R \sum_i |\left\langle \widehat{S}_x^i \right\rangle_R| \right)^2 \\
&\leq \sum_R P_R \left(\sum_i |\left\langle \widehat{S}_x^i \right\rangle_R| \right)^2
\end{aligned} \tag{287}$$

using the general result that $\left(\sum_R P_R \sqrt{C_R}\right)^2 \leq \sum_R P_R\, C_R$, where $\sum_R P_R = 1$ with here $\sqrt{C_R} = \sum_i |\left\langle \widehat{S}^i_x \right\rangle_R|$. Next consider

$$\begin{aligned}
y &= N \sum_i |\left\langle \widehat{S}^i_x \right\rangle_R|^2 \\
z &= \left(\sum_i |\left\langle \widehat{S}^i_x \right\rangle_R|\right)^2 = \left(\sum_i |\left\langle \widehat{S}^i_x \right\rangle_R|\right)^2 \\
y - z &= \sum_{i<j} (|\left\langle \widehat{S}^i_x \right\rangle_R| - |\left\langle \widehat{S}^j_x \right\rangle_R|)^2 \geq 0
\end{aligned} \tag{288}$$

so that

$$\left\langle \widehat{S}_{\,x} \right\rangle^2 \leq N \sum_R P_R \sum_i |\left\langle \widehat{S}^i_x \right\rangle_R|^2 \qquad \left\langle \widehat{S}_{\,y} \right\rangle^2 \leq N \sum_R P_R \sum_i |\left\langle \widehat{S}^i_y \right\rangle_R|^2 \tag{289}$$

which is the required result. The inequality for $\left\langle \widehat{S}_{\,y} \right\rangle^2$ is proved similarly.

Another inequality is stated [14] for $\left\langle \widehat{S}_{\,z} \right\rangle^2$. This is

$$\left\langle \widehat{S}_{\,z} \right\rangle^2 \leq \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 \tag{290}$$

To show this we have

$$\begin{aligned}
\left\langle \widehat{S}_{\,z} \right\rangle &= \sum_R P_R\, Tr(\widehat{\rho}_R \sum_i \widehat{S}^i_z) \\
&= \sum_R P_R \sum_i \left\langle \widehat{S}^i_z \right\rangle_R \\
&= \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R \\
|\left\langle \widehat{S}_{\,z} \right\rangle| &\leq \sum_R P_R\, |\left\langle \widehat{S}_z \right\rangle_R|
\end{aligned} \tag{291}$$

so that

$$\begin{aligned}
\left\langle \widehat{S}_{\,z} \right\rangle^2 &= |\left\langle \widehat{S}_{\,z} \right\rangle|^2 \leq \left(\sum_R P_R\, |\left\langle \widehat{S}_z \right\rangle_R|\right)^2 \\
&\leq \sum_R P_R\, |\left\langle \widehat{S}_z \right\rangle_R|^2 \\
&= \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2
\end{aligned} \tag{292}$$

using the general result that $\left(\sum_R P_R \sqrt{C_R}\right)^2 \leq \sum_R P_R\, C_R$, where $\sum_R P_R = 1$ with here $\sqrt{C_R} = |\left\langle \widehat{S}_z \right\rangle_R|$.

Finally, we find that

$$
\begin{aligned}
\sum_R P_R \sum_i \left(\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 \right) &\leq \frac{1}{4}N \\
-\sum_R P_R \sum_i \left(\left\langle \widehat{S}_z^i \right\rangle_R^2 \right) &\geq -\frac{1}{4}N + \sum_R P_R \sum_i \left(\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 \right)
\end{aligned}
\tag{293}
$$

To show this we use the properties of the density operator $\widehat{\rho}_R^i$ for the ith particle of Hermitiancy, positiveness, unit trace $Tr(\widehat{\rho}_R^i) = 1$ and $Tr(\widehat{\rho}_R^i)^2 \leq 1$. In terms of matrix elements of the density operator $\widehat{\rho}_R^i$ between the two states $|\phi_a(i)\rangle$, $|\phi_b(i)\rangle$ the quantities $\left\langle \widehat{S}_x^i \right\rangle_R$, $\left\langle \widehat{S}_y^i \right\rangle_R$ and $\left\langle \widehat{S}_z^i \right\rangle_R$ are

$$
\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R &= Tr(\widehat{\rho}_R^i \frac{1}{2}(|\phi_b(i)\rangle \langle\phi_a(i)| + |\phi_a(i)\rangle \langle\phi_b(i)|)) \\
&= \frac{1}{2}\left(\rho_{ab}^i + \rho_{ba}^i\right) \\
\left\langle \widehat{S}_y^i \right\rangle_R &= \frac{1}{2i}\left(\rho_{ab}^i - \rho_{ba}^i\right) \\
\left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\left(\rho_{bb}^i - \rho_{aa}^i\right)
\end{aligned}
\tag{294}
$$

where $\rho_{cd}^i = \langle\phi_c(i)|\,\widehat{\rho}_R^i\,|\phi_d(i)\rangle$. The Hermitiancy and positiveness of $\widehat{\rho}_R^i$ show that ρ_{bb}^i and ρ_{aa}^i are real and positive, $\rho_{ab}^i = (\rho_{ba}^i)^*$ and $\rho_{aa}^i\rho_{bb}^i - |\rho_{ab}^i|^2 \geq 0$. The condition $Tr(\widehat{\rho}_R^i) = 1$ leads to $\rho_{aa}^i + \rho_{bb}^i = 1$, from which $Tr(\widehat{\rho}_R^i)^2 \leq 1$ follows using the previous positivity results. Taken together these conditions lead to the following useful parametrization of the density matrix elements

$$
\begin{aligned}
\rho_{aa}^i &= \sin^2\alpha_i \qquad \rho_{bb}^i = \cos^2\alpha_i \\
\rho_{ab}^i &= \sqrt{\sin^2\alpha_i\,\cos^2\alpha_i}\,\sin^2\beta_i\,\exp(+i\phi_i) \qquad \rho_{ba}^i = \sqrt{\sin^2\alpha_i\,\cos^2\alpha_i}\,\sin^2\beta_i\,\exp(-i\phi_i)
\end{aligned}
\tag{295}
$$

where α_i, β_i and ϕ_i are real. In terms of these quantities we then have

$$
\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R &= \frac{1}{2}\sin 2\alpha_i\,\sin^2\beta_i\,\cos\phi_i \\
\left\langle \widehat{S}_y^i \right\rangle_R &= \frac{1}{2}\sin 2\alpha_i\,\sin^2\beta_i\,\sin\phi_i \\
\left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\cos 2\alpha_i
\end{aligned}
\tag{296}
$$

It is then easy to show that

$$\begin{aligned}
\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 &= \frac{1}{4} - \frac{1}{4}\sin^2 2\alpha_i \,(1 - \sin^4 \beta_i\,) \\
&\leq \frac{1}{4}
\end{aligned} \tag{297}$$

and the final inequality (293) then follows by taking the sum over particles i and then using $\sum_R P_R = 1$. If only the Schwarz inequality is used instead of the more detailed consequences of Hermtiancy, positiveness etc it can be shown that $\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 \leq \frac{3}{4}$, which though correct is not useful.

Combining the inequalities in Eqs. (285), (290) and (293) into Eq. (281) shows that

$$\begin{aligned}
\left\langle \Delta \widehat{S}_z^{\,2} \right\rangle &= \frac{N}{4} - \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 + \sum_R P_R \left\langle \widehat{S}_z \right\rangle_R^2 - \left\langle \widehat{S}_z \right\rangle^2 \\
&\geq \frac{N}{4} - \sum_R P_R \sum_i \left\langle \widehat{S}_z^i \right\rangle_R^2 \\
&\geq \frac{N}{4} - \frac{1}{4}N + \sum_R P_R \sum_i \left(\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 \right) \\
&\geq \frac{1}{N} \left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right)
\end{aligned} \tag{298}$$

for the case of a non-entangled state. This result is that in Sørensen et al.[14].

Appendix H Revising Sørensen Entanglement Test

In this Appendix we consider three ways that the proof of the Sørensen et al [14] entanglement test could be revised to apply to systems of *identical* particles, rather than the systems of *distinguishable* particles assumed in the original proof.

H.1 Revision Based on Localized Modes in Position or Momentum

The work of Sørensen et al really applies only when the individual spins are distinguishable. It is possible however to modify the work of Sørensen et al [14] to apply to a system of identical bosons in accordance with the symmetrization and super-selection rules if the index i is *re-interpreted* as specifying different modes, for example modes localized on *optical lattice* sites or in different *momentum states* $i = 1, 2, .., n$. Another example would be single two state ions with each ion being trapped in a different spatial region. The revised approach draws on the results established for multi-mode cases in Appendix D. With two single particle states a, b available on each site (these could be two different internal atomic states or two distinct spatial modes localized on the site) the modes would then be labelled $|\phi_{\alpha i}\rangle$ with $\alpha = a, b$. The mode orthogonality and completeness relations would then be

$$\begin{aligned} \langle \phi_{\alpha\, i} | \phi_{\beta\, j} \rangle &= \delta_{\alpha\beta}\delta_{ij} \\ \sum_{\alpha i} |\phi_{\alpha\, i}\rangle \langle \phi_{\alpha\, i}| &= \widehat{1} \end{aligned} \tag{299}$$

With the particles now labelled $K = 1, 2, 3, ...$one can define spin operators in first quantization via

$$\begin{aligned} \widehat{S}_x &= \sum_K \sum_i (|\phi_{b\, i}(K)\rangle \langle \phi_{a\, i}(K)| + |\phi_{a\, i}(K)\rangle \langle \phi_{b\, i}(K)|)/2 \\ \widehat{S}_y &= \sum_K \sum_i (|\phi_{b\, i}(K)\rangle \langle \phi_{a\, i}(K)| - |\phi_{a\, i}(K)\rangle \langle \phi_{b\, i}(K)|)/2i \\ \widehat{S}_z &= \sum_K \sum_i (|\phi_{b\, i}(K)\rangle \langle \phi_{b\, i}(K)| - |\phi_{a\, i}(K)\rangle \langle \phi_{a\, i}(K)|)/2 \end{aligned} \tag{300}$$

In second quantization if the annihilation, creation operators for the modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ are $\widehat{a}_i$, $\widehat{b}_i$ and $\widehat{a}_i^\dagger$, $\widehat{b}_i^\dagger$ respectively, then the Schwinger spin operators are just

$$\begin{aligned} \widehat{S}_x &= \sum_i (\widehat{b}_i^\dagger \widehat{a}_i + \widehat{a}_i^\dagger \widehat{b}_i)/2 = \sum_i \widehat{S}_x^i \\ \widehat{S}_y &= \sum_i (\widehat{b}_i^\dagger \widehat{a}_i - \widehat{a}_i^\dagger \widehat{b}_i)/2i = \sum_i \widehat{S}_y^i \\ \widehat{S}_z &= \sum_i (\widehat{b}_i^\dagger \widehat{b}_i - \widehat{a}_i^\dagger \widehat{a}_i)/2 = \sum_i \widehat{S}_z^i \end{aligned} \tag{301}$$

It is easy to confirm that the overall spin operators $\widehat{S}_{\alpha}$ and the spin operators $\widehat{S}_{\alpha}^{i}$ for the separate *pairs* of *modes* $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ (or $\widehat{a}_i$, $\widehat{b}_i$ for short) satisfy the same commutation rules as Sørensen et al [14] have for the overall spin operators and those for the separate *particles*. With this modification the non-entangled state in Eq. (105) could be interpreted as being a non-entangled state where the subsystems are actually *pairs* of *modes* $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ and the density operators $\widehat{\rho}_{R}^{i}$ would then refer to a subsystem consisting of these pairs of modes. This corresponds to Case 3 discussed in SubSection D.4. It is to be noted that entanglement of *pairs* of modes is different to entanglement of all *separate* modes - Case 2 discussed in SubSection D.3. It is an example of a special kind of *multimode entanglement* - since the modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ may themselves be entangled we may have "entanglement of entanglement". In terms of the present paper the density operators $\widehat{\rho}_{R}^{i}$ would be restricted by the super-selection rule to statistical mixtures of states with specific total numbers N_i of bosons in the pair of modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$. In terms of Fock states $|n_{a\,i}\rangle, |n_{b\,i}\rangle$ for this pair of modes the allowed quantum states for the sub-system will be

$$|\Phi_{N_i}\rangle = \sum_{k=0}^{N_i} A_k^{N_i}\ |k\rangle_{a\,i} |N_i - k\rangle_{b\,i} \tag{302}$$

so at this stage the general mixed physical state for the two mode system *could* be

$$\widehat{\rho}_{R}^{i} = \sum_{N_i=0}^{\infty} \sum_{\Phi} P_{\Phi N_i} \sum_{k=0}^{N_i} \sum_{l=0}^{N_i} A_k^{N_i}\,(A_l^{N})^{*}\,|k\rangle_{a\,i}\,\langle l|_{a\,i} \otimes |N_i - k\rangle_{b\,i}\,\langle N_i - l|_{b\,i} \tag{303}$$

This state has no coherences between states of the two mode subsystem with differing total boson number N_i for the pair of modes. However this is still an entangled states for the two modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$, so the overall state in Eq. (303) is not a separable state if the subsystems were to consist of *all* the distinct modes.

H.2 Revision Based on Separable State of Single Modes

It is possible however to link spin squeezing and entanglement in the case where the sub-systems consist of *all* the distinct modes (Case 2 in Appendix D). To obtain a *fully non-entangled state* of *all* the modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ the density operator $\widehat{\rho}_{R}^{i}$ must then be a product of density operators for modes $|\phi_{ai}\rangle$ and $|\phi_{bi}\rangle$

$$\widehat{\rho}_{R}^{i} = \widehat{\rho}_{R}^{a\,i} \otimes \widehat{\rho}_{R}^{b\,i} \tag{304}$$

giving the full density operator as

$$\widehat{\rho} = \sum_{R} P_R \left(\widehat{\rho}_{R}^{a\,1} \otimes \widehat{\rho}_{R}^{b\,1}\right) \otimes \left(\widehat{\rho}_{R}^{a\,2} \otimes \widehat{\rho}_{R}^{b\,2}\right) \otimes \left(\widehat{\rho}_{R}^{a\,3} \otimes \widehat{\rho}_{R}^{b\,3}\right) \otimes \dots \tag{305}$$

as is required for a general non-entangled state all $2N$ modes. Furthermore, as previously the density operators for the individual modes must represent possible physical states for such modes, so the super-selection rule for atom number will apply and we have

$$
\begin{array}{rclcl}
\langle(\widehat{a}_i)^n\rangle_{a\,i} & = & Tr(\widehat{\rho}_R^{a\,i}(\widehat{a}_i)^n) = 0 & & \left\langle(\widehat{a}_i^\dagger)^n\right\rangle_{a\,i} = Tr(\widehat{\rho}_R^{a\,i}(\widehat{a}_i^\dagger)^n) = 0 \\
\left\langle(\widehat{b}_i)^m\right\rangle_{b\,i} & = & Tr(\widehat{\rho}_R^{b\,i}(\widehat{b}_i)^m) = 0 & & \left\langle(\widehat{b}_i^\dagger)^m\right\rangle_{b\,i} = Tr(\widehat{\rho}_R^{b\,i}(\widehat{b}_i^\dagger)^m) = 0
\end{array}
\tag{306}
$$

The question is whether this reformulation will lead to a useful inequality for the spin variances such as $\left\langle\Delta\widehat{S}_x^2\right\rangle$. This issue is dealt with in Appendix D and it is found that we can indeed show for the general *fully non-entangled* state (305) that

$$
\left\langle\Delta\widehat{S}_x^2\right\rangle \geq \frac{1}{2}|\left\langle\widehat{S}_z\right\rangle| \quad and \quad \left\langle\Delta\widehat{S}_y^2\right\rangle \geq \frac{1}{2}|\left\langle\widehat{S}_z\right\rangle| \tag{307}
$$

This shows that if there is spin squeezing in *either* $\widehat{S}_x$ *or* $\widehat{S}_y$ then the state must be entangled. Note that this result depends on the general non-entangled state being non-entangled for *all* modes and that the density operator for each mode $\widehat{a}_i$ or $\widehat{b}_i$ being a physical state with no coherences between mode Fock states with differing atom numbers. In terms of the revised interpretation of the density operator to refer to a multi-mode system with modes $|\phi_{ai}\rangle$, $|\phi_{bi}\rangle$ the statement that spin squeezing for systems of identical massive bosons requires all the modes to be entangled is correct. However superposition states of the form (302) that are consistent with the super-selection rule applying to pure states of a two mode system are precluded, and such states ought to be allowed if entanglement of *pairs* of modes rather than of *separate* modes is to be considered.

In addition, we can show that if either $\left\langle\widehat{S}_x\right\rangle$ or $\left\langle\widehat{S}_y\right\rangle$ is non-zero, then the state must be entangled - the *Bloch vector* test. Finally, if it is found that if there is spin squeezing in $\widehat{S}_z$ then the state must be entangled. Thus spin squeezing in *any* spin component confirms entanglement of the $2n$ individual modes.

H.3 Revision Based on Separable State of Pairs of Modes with One Boson Occupancy

It is also possible however to link spin squeezing and entanglement in the case where the subsystems consist of *pairs* of modes (Case 3 in Appendix D), but only if *further restrictions* are applied. The general *non-entangled* state of the *pairs* of *modes* would actually be of the form (see (207), here the ab dropped for simplicity)

$$
\widehat{\rho} = \sum_R P_R\,\widehat{\rho}_R^1 \otimes \widehat{\rho}_R^2 \otimes \widehat{\rho}_R^3 \otimes ... \tag{308}
$$

where the $\widehat{\rho}_R^i$ are now of the form given in Eq. (303) and no longer are density operators for the ith identical particle. Unlike in (306) we now have expectation values $\langle(\widehat{a}_i)^n\rangle_i = Tr(\widehat{\rho}_R^i(\widehat{a}_i)^n)$ etc that are non-zero, so considerations of the link between spin squeezing and entanglement - now entanglement of pairs of modes, will be different.

If the density operators $\widehat{\rho}_R^i$ associated with the *pair* of modes $\widehat{a}_i$, $\widehat{b}_i$ are all *restricted* to be associated with *one boson states* then this density operator is of the form

$$\begin{aligned}\widehat{\rho}_R^i &= \rho_{aa}^i(|1\rangle_{ia}\langle 1|_{ia}\otimes|0\rangle_{ib}\langle 0|_{ib}) + \rho_{ab}^i(|1\rangle_{ia}\langle 0|_{ia}\otimes|0\rangle_{ib}\langle 1|_{ib}) \\ &\quad +\rho_{ba}^i(|0\rangle_{ia}\langle 1|_{ia}\otimes|1\rangle_{ib}\langle 0|_{ib}) + \rho_{bb}^i(|0\rangle_{ia}\langle 0|_{ia}\otimes|1\rangle_{ib}\langle 1|_{ib})\end{aligned} \tag{309}$$

where the ρ_{ef}^i are density matrix elements. With this restriction the pair of modes $\widehat{a}_i$, $\widehat{b}_i$ behave like *distinguishable* two state particles, essentially the case that Sørensen et al [14] implicitly considered. The expectation values for the spin operators $\widehat{S}_x^i$, $\widehat{S}_y^i$ and $\widehat{S}_z^i$ associated with the ith pair of modes are then

$$\begin{aligned}\left\langle \widehat{S}_x^i \right\rangle_R &= \frac{1}{2}\left(\rho_{ab}^i + \rho_{ba}^i\right) \qquad \left\langle \widehat{S}_y^i \right\rangle_R = \frac{1}{2i}\left(\rho_{ab}^i - \rho_{ba}^i\right) \\ \left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\left(\rho_{bb}^i - \rho_{aa}^i\right)\end{aligned} \tag{310}$$

If in addition Hermitiancy, positivity, unit trace $Tr(\widehat{\rho}_R^i) = 1$ and $Tr(\widehat{\rho}_R^i)^2 \leq 1$ are used (see Appendix G) then we can show that ρ_{bb}^i and ρ_{aa}^i are real and positive, $\rho_{ab}^i = (\rho_{ba}^i)^*$ and $\rho_{aa}^i\rho_{bb}^i - |\rho_{ab}^i|^2 \geq 0$. The condition $Tr(\widehat{\rho}_R^i) = 1$ leads to $\rho_{aa}^i + \rho_{bb}^i = 1$, from which $Tr(\widehat{\rho}_R^i)^2 \leq 1$ follows using the previous positivity results. These results enable the matrix elements in (309) to be parameterized in the form

$$\begin{aligned}\rho_{aa}^i &= \sin^2\alpha_i\,, \qquad \rho_{bb}^i = \cos^2\alpha_i \\ \rho_{ab}^i &= \sqrt{\sin^2\alpha_i\,\cos^2\alpha_i}\,\sin^2\beta_i\,\exp(+i\phi_i) \\ \rho_{ba}^i &= \sqrt{\sin^2\alpha_i\,\cos^2\alpha_i}\,\sin^2\beta_i\,\exp(-i\phi_i)\end{aligned} \tag{311}$$

where α_i, β_i and ϕ_i are real. In terms of these quantities we then have

$$\begin{aligned}\left\langle \widehat{S}_x^i \right\rangle_R &= \frac{1}{2}\sin 2\alpha_i\,\sin^2\beta_i\,\cos\phi_i\,, \qquad \left\langle \widehat{S}_y^i \right\rangle_R = \frac{1}{2}\sin 2\alpha_i\,\sin^2\beta_i\,\sin\phi_i \\ \left\langle \widehat{S}_z^i \right\rangle_R &= \frac{1}{2}\cos 2\alpha_i\end{aligned} \tag{312}$$

and then a key inequality

$$\left\langle \widehat{S}_x^i \right\rangle_R^2 + \left\langle \widehat{S}_y^i \right\rangle_R^2 + \left\langle \widehat{S}_z^i \right\rangle_R^2 = \frac{1}{4} - \frac{1}{4}\sin^2 2\alpha_i\,(1-\sin^4\beta_i\,) \leq \frac{1}{4} \tag{313}$$

follows. This result depends on the density operators $\widehat{\rho}_R^i$ being for one boson states, as in (309). The same steps as in Sørensen et al [14] (see Appendix G) leads to the result

$$\left\langle \Delta\widehat{S}_z^2 \right\rangle \geq \frac{1}{N}\left(\left\langle \widehat{S}_x \right\rangle^2 + \left\langle \widehat{S}_y \right\rangle^2 \right) \tag{314}$$

for non-entangled *pair* of modes $\widehat{a}_i$, $\widehat{b}_i$. Thus when the interpretation is changed so that are the separate sub-systems are these pairs of modes *and* the sub-systems are in one boson states, it follows that spin squeezing requires entanglement of all the mode pairs.

A similar proof extending the test of Sørensen et al [14] to apply to systems of identical bosons is given by Hyllus et al [27] based on a particle entanglement approach. In their approach bosons in differing external modes (analogous to differing i here) are treated as distinguishable, and the symmetrization principle is ignored for such bosons.

Appendix I Benatti Entanglement Tests

In this Appendix the entanglement tests introduced by Benatti et al [29] are examined.

In the first case, for *separable* states they found (see Eq. (11)) that for three orthogonal spin operators $\widehat{J}_{n1}$, $\widehat{J}_{n2}$ and $\widehat{J}_{n3}$

$$\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle \geq \frac{N}{2} \tag{315}$$

so that if

$$\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle < \frac{N}{2} \tag{316}$$

then the state must be entangled. This test is an extended form of the Hillery spin variance test (84). To prove this result we note that $\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle = \left\langle \Delta\widehat{S}_{x}^{2}\right\rangle + \left\langle \Delta\widehat{S}_{y}^{2}\right\rangle + \left\langle \Delta\widehat{S}_{z}^{2}\right\rangle = \left\langle \widehat{S}_{x}^{2}\right\rangle + \left\langle \widehat{S}_{y}^{2}\right\rangle + \left\langle \widehat{S}_{z}^{2}\right\rangle - \left\langle \widehat{S}_{x}\right\rangle^{2} - \left\langle \widehat{S}_{y}\right\rangle^{2} - \left\langle \widehat{S}_{z}\right\rangle^{2} = N(N+2)/4 - \left\langle \widehat{S}_{x}\right\rangle^{2} - \left\langle \widehat{S}_{y}\right\rangle^{2} - \left\langle \widehat{S}_{z}\right\rangle^{2}$ for all states with N bosons. For separable states we have $\left\langle \widehat{S}_{x}\right\rangle = \left\langle \widehat{S}_{y}\right\rangle = 0$ so that $\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle = N(N+2)/4 - \left\langle \widehat{S}_{z}\right\rangle^{2}$. As the eigenvalues for $\widehat{S}_{z}$ lie between $-N/2$ and $+N/2$ we have $\left\langle \widehat{S}_{z}\right\rangle^{2} \leq N^{2}/4$. Thus $\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n3}^{2}\right\rangle \geq \frac{N}{2}$ as required.

In the second case, for *separable* states they also found (see Eq. (13)) that for three orthogonal spin operators $\widehat{J}_{n1}$, $\widehat{J}_{n2}$ and $\widehat{J}_{n3}$

$$(N-1)\left(\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle\right) - \left\langle \widehat{J}_{n3}^{2}\right\rangle \geq \frac{N(N-2)}{4} \tag{317}$$

so that if

$$(N-1)\left(\left\langle \Delta\widehat{J}_{n1}^{2}\right\rangle + \left\langle \Delta\widehat{J}_{n2}^{2}\right\rangle\right) - \left\langle \widehat{J}_{n3}^{2}\right\rangle < \frac{N(N-2)}{4} \tag{318}$$

then the state must be entangled. To prove this result for $n1 = \overrightarrow{x}$, $n2 = \overrightarrow{y}$ and $n3 = \overrightarrow{z}$, we use the result (29) for separable states that $\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle + \left\langle \Delta\widehat{S}_{y}^{2}\right\rangle \geq \sum_{R} P_{R}\,\frac{1}{2}(\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R} + \langle \widehat{a}^{\dagger}\widehat{a}\rangle_{R}) + \sum_{R} P_{R}\,(\langle \widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R}) = N/2 + \sum_{R} P_{R}\,(\langle \widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R})$. It is straightforward to show that $\widehat{S}_{z}^{2} = (\widehat{b}^{\dagger}\widehat{b}+\widehat{a}^{\dagger}\widehat{a})^{2}/4 - \widehat{b}^{\dagger}\widehat{b}.\widehat{a}^{\dagger}\widehat{a}$, so that $\left\langle \widehat{S}_{z}^{2}\right\rangle = N^{2}/4 - \sum_{R} P_{R}\,(\langle \widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R})$. Hence for separable states $(N-1)(\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle + \left\langle \Delta\widehat{S}_{y}^{2}\right\rangle) - \left\langle \widehat{S}_{z}^{2}\right\rangle \geq (N-1)N/2 + (N-1)\sum_{R} P_{R}\,(\langle \widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R}) - N^{2}/4 + \sum_{R} P_{R}\,(\langle \widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R})$. Thus $(N-1)(\left\langle \Delta\widehat{S}_{x}^{2}\right\rangle + \left\langle \Delta\widehat{S}_{y}^{2}\right\rangle) - \left\langle \widehat{S}_{z}^{2}\right\rangle \geq \frac{N(N-2)}{4} + N\sum_{R} P_{R}\,(\langle \widehat{a}^{\dagger}\widehat{a}\rangle_{R}\left\langle \widehat{b}^{\dagger}\widehat{b}\right\rangle_{R})$. As the second term on the right side is always positive the required inequality follows.

Finally, they considered another inequality (see Eq. (12)) found to apply for *separable* states involving *distinguishable* particles in Ref. [28].

$$\left(\left\langle \widehat{J}_{n1}^2 \right\rangle + \left\langle \widehat{J}_{n2}^2 \right\rangle\right) - \frac{N}{2} - (N-1)\left\langle \Delta\widehat{J}_{n3}^2 \right\rangle \leq 0 \tag{319}$$

so the question is whether an entanglement test $\left(\left\langle \widehat{J}_{n1}^2 \right\rangle + \left\langle \widehat{J}_{n2}^2 \right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{J}_{n3}^2 \right\rangle > 0$ applies for the case of indistinguishable particles. For the case where $n1 = \overrightarrow{x}$, $n2 = \overrightarrow{y}$ and $n3 = \overrightarrow{z}$, $\left(\left\langle \widehat{S}_x^2 \right\rangle + \left\langle \widehat{S}_y^2 \right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{S}_z^2 \right\rangle = \left(\left\langle \widehat{S}_x^2 \right\rangle + \left\langle \widehat{S}_y^2 \right\rangle + \left\langle \widehat{S}_z^2 \right\rangle\right) - N/2 - (N)\left\langle \widehat{S}_z^2 \right\rangle + (N-1)\left\langle \widehat{S}_z \right\rangle^2 = N(N+2)/4 - N/2 - N\left(N^2/4 - \sum_R P_R\left(\left\langle \widehat{a}^\dagger\widehat{a} \right\rangle_R \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R\right)\right) + (N-1)\left\langle \widehat{S}_z \right\rangle^2$. As $\left\langle \widehat{S}_z \right\rangle^2 \leq N^2/4$ we see that $\left(\left\langle \widehat{S}_x^2 \right\rangle + \left\langle \widehat{S}_y^2 \right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{S}_z^2 \right\rangle \leq N\sum_R P_R\left(\left\langle \widehat{a}^\dagger\widehat{a} \right\rangle_R \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R\right)$, which is certainly ≥ 0 and not ≤ 0 as required. However, perhaps an entanglement test such that if is could be shown that

$$\left(\left\langle \widehat{S}_x^2 \right\rangle + \left\langle \widehat{S}_y^2 \right\rangle\right) - N/2 - (N-1)\left\langle \Delta\widehat{S}_z^2 \right\rangle > N\sum_R P_R\left(\left\langle \widehat{a}^\dagger\widehat{a} \right\rangle_R \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R\right) \tag{320}$$

always applies then it could be included that the state is entangled. Unfortunately the right side could be too large for the left side to always exceed the right side for some separable states. Noting that $\left\langle \widehat{a}^\dagger\widehat{a} \right\rangle_R + \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R = N$ for the N bosons states being considered we find that the right side is maximized when $\left\langle \widehat{a}^\dagger\widehat{a} \right\rangle_R = \left\langle \widehat{b}^\dagger\widehat{b} \right\rangle_R = N/2$ for all P_R, giving a maximum for the right side of $N^3/2$ - and this can occur for some separable states. To show that the state is entangled the left side must exceed this value, otherwise the state might be one of the separable states. However, the left side is at most of order N^2 from the first two terms and the negative terms only make the left side smaller. Hence there is no entanglement test of the form (320).

Appendix J Heisenberg Uncertainty Principle Results

J.1 Derivation of Inequalities

Here we derive the results in SubSection 4.6 leading to inequalities for the variance $\left\langle \Delta\widehat{J}_x^2 \right\rangle$ considered as a function of $|\left\langle \widehat{J}_z \right\rangle|$ for states where the spin operators are chosen such that $\left\langle \widehat{J}_x \right\rangle = \left\langle \widehat{J}_y \right\rangle = 0$.

From the Schwarz inequality $\left\langle \widehat{J}_z \right\rangle^2 \leq \left\langle \widehat{J}_z^2 \right\rangle$ so that

$$\left\langle \widehat{J}_x^2 \right\rangle + \left\langle \widehat{J}_y^2 \right\rangle + \left\langle \widehat{J}_z \right\rangle^2 \leq \left\langle \widehat{J}_x^2 \right\rangle + \left\langle \widehat{J}_y^2 \right\rangle + \left\langle \widehat{J}_z^2 \right\rangle = J(J+1) \tag{321}$$

giving Eq. (113). Subtracting $\left\langle \widehat{J}_x \right\rangle^2 = \left\langle \widehat{J}_y \right\rangle^2 = 0$ from each side gives

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle + \left\langle \Delta\widehat{J}_y^2 \right\rangle + \left\langle \widehat{J}_z \right\rangle^2 \leq J(J+1) \tag{322}$$

Substituting for $\left\langle \Delta\widehat{J}_y^2 \right\rangle$ from the Heisenberg uncertainty principle result in Eq. (114) gives

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle^2 - \left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right) \left\langle \Delta\widehat{J}_x^2 \right\rangle + \frac{1}{4}\xi \left\langle \widehat{J}_z \right\rangle^2 \leq 0 \tag{323}$$

The left side is a parabolic function of $\left\langle \Delta\widehat{J}_x^2 \right\rangle$ and for this to be negative requires $\left\langle \Delta\widehat{J}_x^2 \right\rangle$ to lie between the two roots of this function, giving

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle \geq \frac{1}{2}\left\{ \left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right) - \sqrt{\left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{J}_z \right\rangle^2} \right\} \tag{324}$$

$$\left\langle \Delta\widehat{J}_x^2 \right\rangle \leq \frac{1}{2}\left\{ \left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right) + \sqrt{\left(J(J+1) - \left\langle \widehat{J}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{J}_z \right\rangle^2} \right\} \tag{325}$$

which are the required inequalities in Eq. (115) and (116).

J.2 Numerical Study of Inequalities

Here we consider the question: Is it possible to find values for $\left\langle \Delta\widehat{S}_x^2 \right\rangle$ and $|\left\langle \widehat{S}_z \right\rangle|$ in which all three inequalities (115), (116) and (117) are satisfied?

Results showing the regions in the $\left\langle \Delta\widehat{S}_x^2 \right\rangle$ versus $|\left\langle \widehat{S}_z \right\rangle|$ plane corresponding to the three inequalities are shown in Figures 4 and 5 for the cases where $J = 1000$ and with $\xi = 1.0$ and $\xi = 10.0$ respectively. The quantities for which the regions are shown are the scaled variance and mean $\left\langle \Delta\widehat{S}_x^2 \right\rangle /J$ and $|\left\langle \widehat{S}_z \right\rangle|/J$, with $\left\langle \Delta\widehat{S}_x^2 \right\rangle$ given as a function of $|\left\langle \widehat{S}_z \right\rangle|$ via (115), (116) and (117). That regions exist where the quantity $\left(J(J+1) - \left\langle \widehat{S}_z \right\rangle^2 \right)^2 - \xi \left\langle \widehat{S}_z \right\rangle^2$ then becomes negative is seen in Figure 4. The spin squeezing region is always consistent with the second Heisenberg inequality (116) and for large $J = 1000$ there is a large region of overlap with the first inequality (115). For small J and large ξ the region of overlap becomes much smaller, as the result in Figure 6 for $J = 1$ and with $\xi = 10.0$ shows.

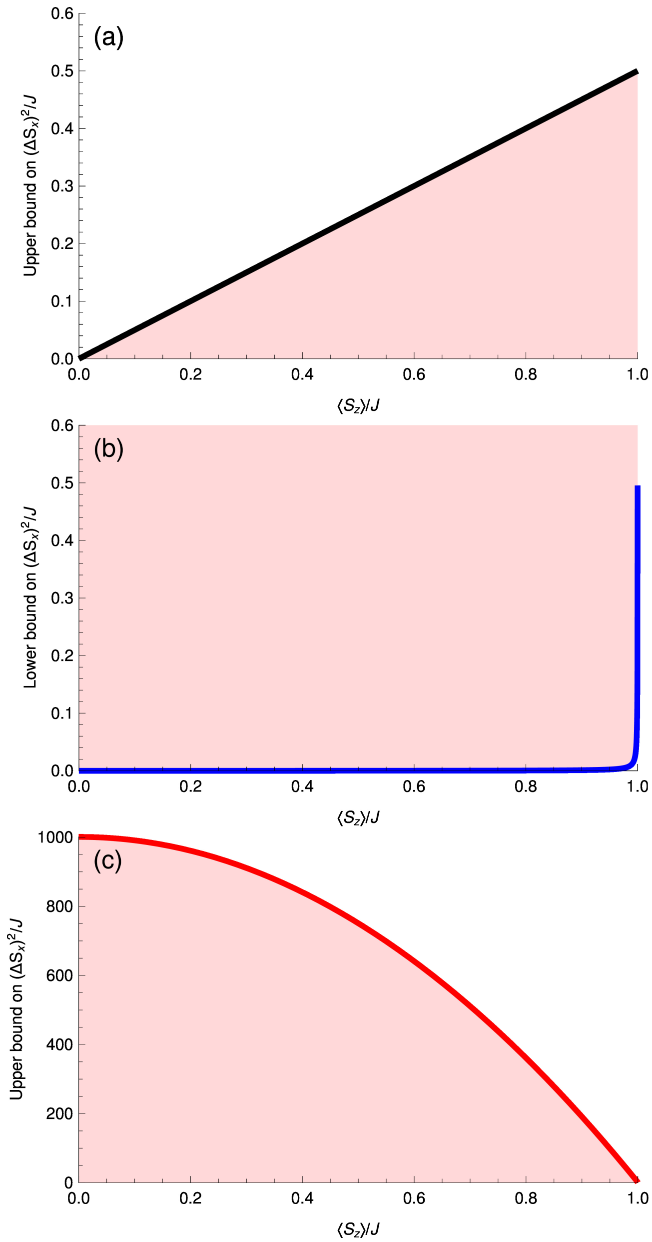

Figure 4. Regions in the $<\Delta\widehat{S}_x^2>$ versus $|<\widehat{S}_z>|$ plane (shown shaded) for states that satisfy (a) the spin squeezing inequality Eq. (117) (b) the smaller Heisenberg uncertainty principle inequality Eq. (115) and (c) the larger HUP inequality Eq. (116). The case shown is for $J=1000$ and HUP factor $\xi=1$. Both $<\Delta\widehat{S}_x^2>$ and $|<\widehat{S}_z>|$ are in units of J. The spin operators are chosen so that $<\widehat{S}_x>=<\widehat{S}_y>=0$.

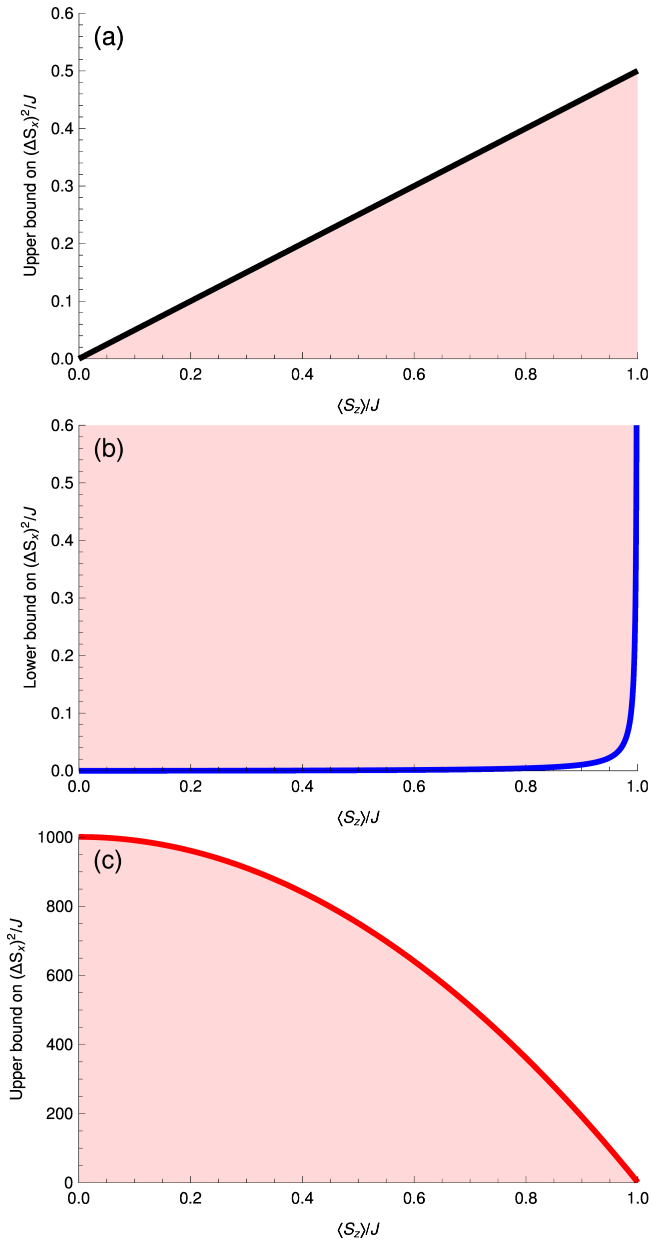

Figure 5. As in Figure 4, but with $J = 1000$ and HUP factor $\xi = 10.0$.

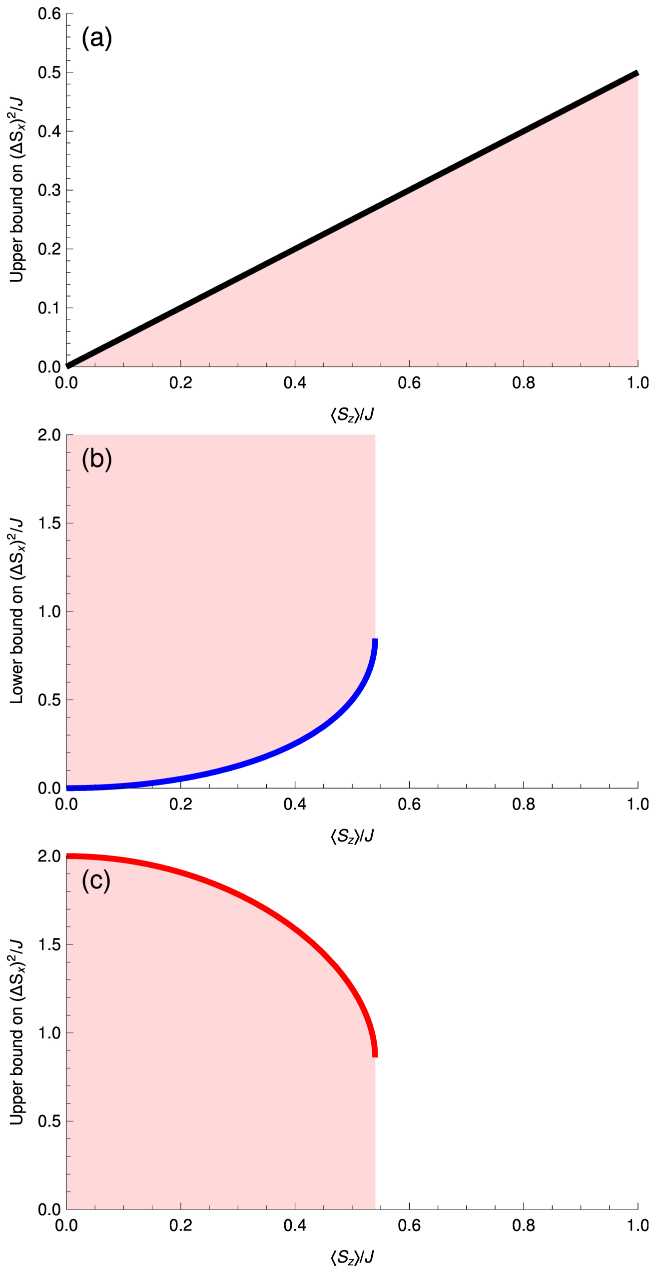

Figure 6. As in Figure 4, but with $J = 1$ and HUP factor $\xi = 10.0$.

Inspection of the three figures shows that there are regions where all three inequalities are satisfied.

Appendix K "Separable but Non-Local" States

It is instructive to apply the various entanglement tests to the so-called separable but non-local states considered in Refs. [4], [44], for which the sub-system states are definitely *not* SSR compliant. These states should not pass the Hillery tests [22], [32] for SSR neglected entanglement, but they may pass the entanglement tests in this paper and in Ref. [2] since these states would be regarded as SSR compliant entangled. Note that these states are consistent with the global particle number SSR, so there is no dispute about whether they are possible two mode quantum states. The issue is rather whether they should be categorized as separable or entangled, and that depends on how separable (and hence entangled) states are first defined. As discussed previously, the interferometric measurements discussed here do not enable us to choose one definition over the other - that is an issue involved what types of quantum states would be allowed in the separate sub-systems.

The first example of such states is the *mixture* of *two mode coherent states* is represented by the two mode density operator

$$\begin{aligned}\widehat{\rho} &= \int \frac{d\theta}{2\pi}\, |\alpha,\alpha\rangle\,\langle\alpha,\alpha| \\ &= \int \frac{d\theta}{2\pi}\, \left(|\alpha\rangle\,\langle\alpha|\right)_a \otimes \left(|\alpha\rangle\,\langle\alpha|\right)_b \end{aligned} \tag{326}$$

where $|\alpha\rangle_C$ is a one mode coherent state for mode $c = a, b$ with $\alpha = |\alpha|\, \exp(-i\theta)$, and modes a, b are associated with bosonic annihilation operators $\widehat{a}$, $\widehat{b}$. The magnitude $|\alpha|$ is fixed. This state globally but not locally SSR compliant.

Now

$$\begin{aligned}\left\langle \widehat{a}^{\dagger}\widehat{b}\right\rangle &= Tr \int \frac{d\theta}{2\pi}\, \widehat{a}^{\dagger}\widehat{b}\, \left(|\alpha\rangle\,\langle\alpha|\right)_a \otimes \left(|\alpha\rangle\,\langle\alpha|\right)_b \\ &= Tr \int \frac{d\theta}{2\pi}\, \left(|\alpha\rangle\,\langle\alpha|\,\widehat{a}^{\dagger}\right)_a \otimes \left(\widehat{b}\,|\alpha\rangle\,\langle\alpha|\right)_b \\ &= |\alpha|^2 \end{aligned} \tag{327}$$

But

$$\begin{aligned}\left\langle \widehat{a}^{\dagger}\widehat{a}\,\widehat{b}^{\dagger}\widehat{b}\right\rangle &= Tr \int \frac{d\theta}{2\pi}\, \left(\widehat{a}^{\dagger}\widehat{a}\,|\alpha\rangle\,\langle\alpha|\right)_a \otimes \left(\widehat{b}^{\dagger}\widehat{b}\,|\alpha\rangle\,\langle\alpha|\right)_b \\ &= \int \frac{d\theta}{2\pi}\, \left(\langle\alpha|\,\widehat{a}^{\dagger}\widehat{a}\,|\alpha\rangle\right)_a \otimes \left(\langle\alpha|\,\widehat{b}^{\dagger}\widehat{b}\,|\alpha\rangle\right)_b \\ &= |\alpha|^4 \end{aligned} \tag{328}$$

Hence we have $|\left\langle \widehat{a}^{\dagger}\widehat{b}\right\rangle|^2 > 0$ and $|\left\langle \widehat{a}^{\dagger}\widehat{b}\right\rangle|^2 = \left\langle \widehat{a}^{\dagger}\widehat{a}\,\widehat{b}^{\dagger}\widehat{b}\right\rangle$. This shows the state is SSR compliant entangled. However, it fails the Hillery test for SSR neglected entanglement which is consistent with being a separable state if the local particle SSR is neglected [4, 44].

The second example of such states has an overall density operator which is a statistical mixture given by

$$\begin{aligned}\widehat{\rho} &= \frac{1}{4}(|\psi_1\rangle\langle\psi_1|)_a\otimes|\psi_1\rangle\langle\psi_1|)_b+\frac{1}{4}(|\psi_i\rangle\langle\psi_i|)_a\otimes|\psi_i\rangle\langle\psi_i|)_b \\ &+\frac{1}{4}(|\psi_{-1}\rangle\langle\psi_{-1}|)_a\otimes|\psi_{-1}\rangle\langle\psi_{-1}|)_b+\frac{1}{4}(|\psi_{-i}\rangle\langle\psi_{-i}|)_a\otimes|\psi_{-i}\rangle\langle\psi_{-i}|)_b\end{aligned} \tag{329}$$

where $|\psi_\omega\rangle = (|0\rangle + \omega\,|1\rangle)/\sqrt{2}$, with $\omega = 1, i, -, -i$. The $|\psi_\omega\rangle$ are superpositions of zero and one boson states and consequently the local particle number SSR is violated by each of the sub-system density operators $|\psi_\omega\rangle\langle\psi_\omega|)_a$ and $|\psi_\omega\rangle\langle\psi_\omega|)_b$.

Now using $\widehat{b}\,|\psi_\omega\rangle = (\omega\,|0\rangle)/\sqrt{2}$, $\langle\psi_\omega|\,\widehat{a}^\dagger = (\langle 0|\,\omega^*)/\sqrt{2}$ and $|\omega|^2 = 1$

$$\begin{aligned}\left\langle\widehat{a}^\dagger\widehat{b}\right\rangle &= Tr\frac{1}{4}\sum_\omega(\widehat{a}^\dagger\,|\psi_\omega\rangle\langle\psi_\omega|_a)\otimes(\widehat{b}\,|\psi_\omega\rangle\langle\psi_\omega|_b) \\ &= \frac{1}{4}\sum_\omega\langle\psi_\omega|\,\widehat{a}^\dagger\,|\psi_\omega\rangle_a\,\langle\psi_\omega|\,\widehat{b}\,|\psi_\omega\rangle_b \\ &= \frac{1}{4}\sum_\omega\frac{1}{2}\omega^*\frac{1}{2}\omega \\ &= \frac{1}{4}\end{aligned} \tag{330}$$

But

$$\begin{aligned}\left\langle\widehat{a}^\dagger\widehat{a}\,\widehat{b}^\dagger\widehat{b}\right\rangle &= Tr\frac{1}{4}\sum_\omega(\widehat{a}^\dagger\widehat{a}\,|\psi_\omega\rangle\langle\psi_\omega|_a)\otimes(\widehat{b}^\dagger\widehat{b}\,|\psi_\omega\rangle\langle\psi_\omega|_b) \\ &= \frac{1}{4}\sum_\omega\langle\psi_\omega|\,\widehat{a}^\dagger\widehat{a}\,|\psi_\omega\rangle_a\,\langle\psi_\omega|\,\widehat{b}^\dagger\widehat{b}\,|\psi_\omega\rangle_b \\ &= \frac{1}{4}\sum_\omega\frac{1}{2}|\omega|^2\frac{1}{2}|\omega|^2 \\ &= \frac{1}{4}\end{aligned} \tag{331}$$

Hence we have $|\left\langle\widehat{a}^\dagger\widehat{b}\right\rangle|^2 > 0$ and $|\left\langle\widehat{a}^\dagger\widehat{b}\right\rangle|^2 < \left\langle\widehat{a}^\dagger\widehat{a}\,\widehat{b}^\dagger\widehat{b}\right\rangle$. This shows the state is SSR compliant entangled. However, it again fails the Hillery test for SSR neglected entanglement which is consistent with being a separable state if the local particle SSR is neglected [4, 44]. It should be noted, however, that the density operator can also be written as

$$\begin{aligned}\widehat{\rho} &= \frac{1}{4}(|0\rangle\langle 0|)_A\otimes|0\rangle\langle 0|)_B+\frac{1}{4}(|1\rangle\langle 1|)_A\otimes|1\rangle\langle 1|)_B \\ &+\frac{1}{2}(|\Psi_+\rangle\langle\Psi_+|)_{AB}\end{aligned} \tag{332}$$

where $|\Psi_+\rangle_{AB} = (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B)/\sqrt{2}$. In this form the terms correspond to a statistical mixture of states with $0, 1, 2$ bosons. The first two terms correspond to separable states, in which the sub-system density operators are SSR compliant. The final term however is a one boson Bell state which is generally regarded as the paradigm of a two mode entangled state. Hence regarding the overall state as separable is highly questionable.

Appendix L Quadrature Squeezing Entanglement Tests

We can show that for separable states both $\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle \geq 1/2$ and $\left\langle \Delta\widehat{P}_\theta^2(+)\right\rangle \geq 1/2$, so two mode quadrature squeezing in either $\widehat{X}_\theta(+)$ or $\widehat{P}_\theta(+)$ is a test for two mode entanglement. Firstly, for SSR compliant sub-system states

$$\left\langle \widehat{X}_\theta(+)\right\rangle = \frac{1}{\sqrt{2}}\sum_R P_R(\left\langle \widehat{X}_a^\theta\right\rangle_R + \left\langle \widehat{X}_b^\theta\right\rangle_R) = 0 \tag{333}$$

since$\langle\widehat{a}\rangle_R = \left\langle\widehat{b}\right\rangle_R = 0$. Secondly,

$$\begin{aligned}
&\left\langle \widehat{X}_\theta^2(+)\right\rangle \\
&= \frac{1}{4}\sum_R P_R(\langle\widehat{a}\widehat{a}^\dagger\rangle_R + \langle\widehat{a}^\dagger\widehat{a}\rangle_R + \left\langle\widehat{b}\widehat{b}^\dagger\right\rangle_R + \left\langle\widehat{b}^\dagger\widehat{b}\right\rangle_R) \\
&= \sum_R P_R(\frac{1}{2} + \frac{1}{2}(\langle\widehat{n}_a\rangle_R + \langle\widehat{n}_b\rangle_R) \\
&= \frac{1}{2} + \frac{1}{2}\left\langle\widehat{N}\right\rangle \\
&\geq \frac{1}{2}
\end{aligned} \tag{334}$$

where for local SSR compliant states other terms involving $\langle\widehat{a}^2\rangle_R, \left\langle\widehat{b}^2\right\rangle_R, \left\langle\widehat{a}\widehat{b}\right\rangle_R = \langle\widehat{a}\rangle_R\left\langle\widehat{b}\right\rangle_R, \left\langle\widehat{a}\widehat{b}^\dagger\right\rangle_R = \langle\widehat{a}\rangle_R\left\langle\widehat{b}^\dagger\right\rangle_R$ etc. are all zero. Hence

$$\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle = \left\langle \widehat{X}_\theta^2(+)\right\rangle - \left\langle \widehat{X}_\theta(+)\right\rangle^2 \geq \frac{1}{2} \tag{335}$$

which establishes the result. Since $\widehat{P}_\theta(+) = \widehat{X}_{\theta+\pi/2}(+)$ we also have $\left\langle \Delta\widehat{P}_\theta^2(+)\right\rangle = \frac{1}{2} + \frac{1}{2}\left\langle\widehat{N}\right\rangle \geq \frac{1}{2}$ for a separable state. Hence the *two mode quadrature squeezing* test. If

$$\left\langle \Delta\widehat{X}_\theta^2(+)\right\rangle < \frac{1}{2} \qquad \mathit{or} \qquad \left\langle \Delta\widehat{P}_\theta^2(+)\right\rangle < \frac{1}{2} \tag{336}$$

then the state is entangled. Obviously $\widehat{X}_\theta(+)$ and $\widehat{P}_\theta(+)$ cannot both be squeezed for the same state.

A similar proof shows that for separable states both $\left\langle \Delta\widehat{X}_\theta^2(-)\right\rangle = \frac{1}{2} + \frac{1}{2}\left\langle\widehat{N}\right\rangle \geq 1/2$ and $\left\langle \Delta\widehat{P}_\theta^2(-)\right\rangle = \frac{1}{2} + \frac{1}{2}\left\langle\widehat{N}\right\rangle \geq 1/2$, so if

$$\left\langle \Delta\widehat{X}_\theta^2(-)\right\rangle < \frac{1}{2} \qquad \mathit{or} \qquad \left\langle \Delta\widehat{P}_\theta^2(-)\right\rangle < \frac{1}{2} \tag{337}$$

then the state is entangled. Hence *any* one of $\widehat{X}_\theta(+), \widehat{P}_\theta(+), \widehat{X}_\theta(-), \widehat{P}_\theta(-)$ being squeezed will demonstrate two mode entanglement.

The question then arises - Can two of the four two mode quadrature operators be squeezed? For simplicity we only discuss $\theta = 0$ cases in detail. Obviously pairs such as $\widehat{X}_0(+), \widehat{P}_0(+)$ or $\widehat{X}_0(-), \widehat{P}_0(-)$ cannot. Next, we consider $\widehat{X}_0(+)$ and $\widehat{P}_0(-)$. We note that for all global SSR compliant states $\left\langle \Delta \widehat{X}_0^2(+) \right\rangle + \left\langle \Delta \widehat{P}_0^2(-) \right\rangle = \frac{1}{2}\left(\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{p}_A - \widehat{p}_B)^2 \right\rangle \right) = 1 + \left\langle \widehat{N} \right\rangle$ using (140), so that if $\widehat{X}_0(+)$ is squeezed $\left\langle \Delta \widehat{X}_0^2(+) \right\rangle < \frac{1}{2}$ then $\left\langle \Delta \widehat{P}_0^2(-) \right\rangle > \frac{1}{2} + \left\langle \widehat{N} \right\rangle$, showing that both $\widehat{X}_0(+)$ and $\widehat{P}_0(-)$ cannot both be squeezed - in spite of the operators commuting. A similar conclusion applies to $\widehat{X}_0(-)$ and $\widehat{P}_0(+)$. For the pair $\widehat{X}_0(+)$ and $\widehat{X}_0(-)$ we have $\left\langle \Delta \widehat{X}_0^2(+) \right\rangle + \left\langle \Delta \widehat{X}_0^2(-) \right\rangle = \frac{1}{2}\left(\left\langle \Delta(\widehat{x}_A + \widehat{x}_B)^2 \right\rangle + \left\langle \Delta(\widehat{x}_A - \widehat{x}_B)^2 \right\rangle \right) = 1 + \left\langle \widehat{N} \right\rangle$ using (138) and (142), so the same situation as for $\widehat{X}_0(+)$ and $\widehat{P}_0(-)$ applies, and thus $\widehat{X}_0(+)$ and $\widehat{X}_0(-)$ cannot both be squeezed. A similar conclusion applies to $\widehat{P}_0(-)$ and $\widehat{P}_0(+)$. In general, only *one* of $\widehat{X}_\theta(+), \widehat{P}_\theta(+), \widehat{X}_\theta(-), \widehat{P}_\theta(-)$ can be squeezed.

Appendix M Derivation of Interferometer Results

M.1 General Theory - Two Mode Interferometer

Introducing the free and interaction evolution operators via

$$\begin{aligned} \widehat{U} &= \widehat{U}_0\,\widehat{U}_{int} \\ \widehat{U}_0 &= \exp(-i\widehat{H}_0 t/\hbar) \end{aligned} \tag{338}$$

it is straightforward to show that for

$$\widehat{M} = \frac{1}{2}(\widehat{b}^{\dagger}\widehat{b} - \widehat{a}^{\dagger}\widehat{a}) \tag{339}$$

we have

$$\begin{aligned} \left\langle \widehat{M} \right\rangle &= Tr(\widehat{M}_H\,\widehat{\rho}) \\ \left\langle \Delta\widehat{M}^2 \right\rangle &= Tr(\left\{ \widehat{M}_H - \left\langle \widehat{M}_H \right\rangle \right\}^2\,\widehat{\rho}) \end{aligned} \tag{340}$$

giving the mean and variance in terms of the input density operator and interaction picture Heisenberg operators

$$\begin{aligned} \widehat{M}_H &= \frac{1}{2}(\widehat{b}_H^{\dagger}\widehat{b}_H - \widehat{a}_H^{\dagger}\widehat{a}_H) \\ \widehat{b}_H &= \widehat{U}_{int}^{-1}\,\widehat{b}\,\widehat{U}_{int} \qquad \widehat{a}_H = \widehat{U}_{int}^{-1}\,\widehat{a}\,\widehat{U}_{int} \end{aligned} \tag{341}$$

where we have used the results $\widehat{U}_0^{-1}\,\widehat{b}\,\widehat{U}_0 = \exp(-i\omega_b t)\,\widehat{b}$ and $\widehat{U}_0^{-1}\,\widehat{a}\,\widehat{U}_0 = \exp(-i\omega_a t)\,\widehat{a}$.

The interaction picture Heisenberg operators satisfy

$$i\hbar\frac{\partial}{\partial t}\widehat{b}_H = [\widehat{b}_H, \widehat{V}_H] \qquad i\hbar\frac{\partial}{\partial t}\widehat{a}_H = [\widehat{a}_H, \widehat{V}_H] \tag{342}$$

where

$$\begin{aligned} \widehat{V}_H &= \mathcal{A}(t)\exp(-i\omega_0 t)\,\exp(i\phi)\,\widehat{b}_H^{\dagger}\widehat{a}_H\,\exp(+i\omega_{ba}t) \\ &\quad +\mathcal{A}(t)\exp(+i\omega_0 t)\,\exp(-i\phi)\,\widehat{a}_H^{\dagger}\widehat{b}_H\,\exp(-i\omega_{ba}t) \\ &= \mathcal{A}(t)\,\exp(i\phi)\,\widehat{b}_H^{\dagger}\widehat{a}_H + \mathcal{A}(t)\,\exp(-i\phi)\,\widehat{a}_H^{\dagger}\widehat{b}_H \end{aligned} \tag{343}$$

for resonance.

We then find that the Heisenberg picture operators satisfy coupled linear equations

$$i\hbar\frac{\partial}{\partial t}\widehat{b}_H = \mathcal{A}(t)\,\exp(+i\phi)\,\widehat{a}_H \qquad i\hbar\frac{\partial}{\partial t}\widehat{a}_H = \mathcal{A}(t)\,\exp(-i\phi)\,\widehat{b}_H \tag{344}$$

which after replacing the time t by the area variable s then involve time independent coefficients

$$i\frac{\partial}{\partial s}\widehat{b}_H(s) = \exp(+i\phi)\,\widehat{a}_H(s) \qquad i\frac{\partial}{\partial s}\widehat{a}_H(s) = \exp(-i\phi)\,\widehat{b}_H(s) \tag{345}$$

The equations can then be solved via Laplace transforms giving

$$\widehat{b}_H(s,\phi) = \cos s\,\widehat{b} - i\exp(i\phi)\,\sin s\,\widehat{a} \qquad \widehat{a}_H(s,\phi) = -i\exp(-i\phi)\,\sin s\,\widehat{b} + \cos s\,\widehat{a} \tag{346}$$

where now $2s$ is the area for the classical pulse.

Hence we have in general

$$\begin{aligned} \widehat{M}_H(2s,\phi) &= \frac{1}{2}(\widehat{b}_H^\dagger(s,\phi)\widehat{b}_H(s,\phi) - \widehat{a}_H^\dagger(s,\phi)\widehat{a}_H(s,\phi)) \\ &= \sin 2s\,(\sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y) + \cos 2s\,\widehat{S}_z \end{aligned} \tag{347}$$

The versatility of the measurement follows from the range of possible choices of the pulse area $2s$ and the phase ϕ.

Writing $2s = \theta$ we can then substitute into Eq. (340) to obtain results for $\left\langle \widehat{M} \right\rangle$ and $\left\langle \Delta\widehat{M}^2 \right\rangle$. These are set out in SubSection 7.3 in Eqs. (170) and (171) in terms of the mean values of the spin operators and the elements of the covariance matrix for the spin operators, evaluated for the input quantum state $\widehat{\rho}$.

M.2 Beam Splitter and Phase Changer

For the *beam splitter* we have $2s = \pi/2$ and ϕ (variable) so that

$$\widehat{M}_H(\frac{\pi}{2},\phi) = \sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y \tag{348}$$

whilst for the *phase changer* we have $2s = \pi$ and ϕ (arbitrary) so that

$$\widehat{M}_H(\pi,\phi) = -\widehat{S}_z \tag{349}$$

M.3 Other Measurables

We can also consider other choices for the measurable, which then enable us to determine other moments of the spin operators. A case of particular interest is the square of the population difference

$$\widehat{M_2} = \left(\frac{1}{2}(\widehat{b}^\dagger\widehat{b} - \widehat{a}^\dagger\widehat{a})\right)^2 \tag{350}$$

It is then straightforward to show for the beam splitter case with $2s = \pi/2$ and ϕ (variable)

$$\begin{aligned} \widehat{M}_{2H}(\frac{\pi}{2},\phi) &= \left(\sin\phi\,\widehat{S}_x + \cos\phi\,\widehat{S}_y\right)^2 \\ &= \sin^2\phi\,(\widehat{S}_x)^2 + \cos^2\phi\,(\widehat{S}_y)^2 + \sin\phi\,\cos\phi\,(\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x) \end{aligned} \tag{351}$$

Hence

$$\left\langle \widehat{M_2} \right\rangle = \sin^2\phi \left\langle (\widehat{S}_x)^2 \right\rangle + \cos^2\phi \left\langle (\widehat{S}_y)^2 \right\rangle + \sin\phi\,\cos\phi \left\langle (\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x) \right\rangle \quad (352)$$

showing that the mean for the new observable $\widehat{M_2}$ is a sinusoidal function of the BS interferometer variable ϕ with coefficients that depend on the means of $\widehat{S}_x^2$, $\widehat{S}_y^2$ and $\widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x$.

M.4 Squeezing in the xy Plane

The question is does squeezing in either $\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)$ or $\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)$ demonstrate entanglement of the modes $\widehat{a}$ and $\widehat{b}$?

For a *separable* state we have

$$\left\langle \widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi) \right\rangle_\rho = \left\langle \widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi) \right\rangle_\rho = 0 \quad (353)$$

as before, since $\left\langle \widehat{S}_{x,y}^{\#}(\frac{3\pi}{2}+\phi) \right\rangle_\rho$ are just linear combinations of the zero $\left\langle \widehat{S}_{x,y} \right\rangle_\rho$.

Since $[\ \widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi), \widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)] = i\widehat{S}_z$ the Heisenberg uncertainty principle shows that $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho \left\langle \Delta\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho \geq \frac{1}{4}|\left\langle \widehat{S}_z \right\rangle_\rho|^2$ so spin squeezing in $\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)$ with respect to $\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)$ or vice versa requires us to show that

$$\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \qquad or \qquad \left\langle \Delta\widehat{S}_y^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \quad (354)$$

Since we measure $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho$ the spin squeezing test is $\left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho < \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho|$.

Now for $\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)$ we have

$$\begin{aligned} \left\langle \Delta\widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho &= \left\langle \widehat{S}_x^{\#}(\frac{3\pi}{2}+\phi)^2 \right\rangle_\rho \\ &= \sin^2\phi \left\langle \widehat{S}_x^2 \right\rangle_\rho + \cos^2\phi \left\langle \widehat{S}_y^2 \right\rangle_\rho + \sin\phi\,\cos\phi \left\langle \widehat{S}_x\widehat{S}_y + \widehat{S}_y\widehat{S}_x \right\rangle_\rho \end{aligned} \quad (355)$$

and for a separable state we have from SubSection 2.3

$$\begin{aligned} \left\langle \widehat{S}_x^2 \right\rangle &= \left\langle \Delta\widehat{S}_x^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \\ \left\langle \widehat{S}_y^2 \right\rangle &= \left\langle \Delta\widehat{S}_y^2 \right\rangle \geq \frac{1}{2}|\left\langle \widehat{S}_z \right\rangle_\rho| \end{aligned} \quad (356)$$

whilst for the remaining term

$$\begin{aligned}\left\langle \widehat{S}_x\widehat{S}_y+\widehat{S}_y\widehat{S}_x\right\rangle_\rho &= \frac{1}{2i}\left\langle\{(\widehat{b}^\dagger)^2(\widehat{a})^2-(\widehat{a}^\dagger)^2(\widehat{b})^2\}\right\rangle \\ &= \frac{1}{2i}\sum_R P_R\{\left\langle(\widehat{b}^\dagger)^2\right\rangle_{\rho_R^B}\left\langle(\widehat{a})^2\right\rangle_{\rho_R^A}-\left\langle(\widehat{a}^\dagger)^2\right\rangle_{\rho_R^A}\left\langle(\widehat{b})^2\right\rangle_{\rho_R^B}\} \\ &= 0 \end{aligned} \tag{357}$$

using the *local particle number* SSR.

Thus as $\sin^2\phi+\cos^2\phi=1$ and applying similar considerations to $\left\langle \Delta\widehat{S}_y^\#(\frac{3\pi}{2}+\phi)^2\right\rangle_\rho$

$$\begin{aligned}\left\langle \Delta\widehat{S}_x^\#(\frac{3\pi}{2}+\phi)^2\right\rangle_\rho &\geq \frac{1}{2}|\left\langle\widehat{S}_z\right\rangle_\rho| \\ \left\langle \Delta\widehat{S}_y^\#(\frac{3\pi}{2}+\phi)^2\right\rangle_\rho &\geq \frac{1}{2}|\left\langle\widehat{S}_z\right\rangle_\rho| \end{aligned} \tag{358}$$

showing that for a separable state there is no squeezing for $\widehat{S}_x^\#(\frac{3\pi}{2}+\phi)$ compared to $\widehat{S}_y^\#(\frac{3\pi}{2}+\phi)$ or vice versa. Hence squeezing in either $\widehat{S}_x^\#(\frac{3\pi}{2}+\phi)$ or $\widehat{S}_y^\#(\frac{3\pi}{2}+\phi)$ demonstrates entanglement of the modes $\widehat{a}$ and $\widehat{b}$.

M.5 General Theory - Multi-Mode Interferometer

For the multi-mode case we consider two sets of modes $\widehat{a}_i$ and $\widehat{b}_i$ as described in Appendix A. These may be different modes associated with two different hyperfine states or they may be modes associated with two separated potential wells. The Hamiltonian analogous to that in (162) for the two mode case is

$$\begin{aligned}\widehat{H}_0 &= \sum_i \hbar(\omega_a+\omega_i)\,\widehat{a}_i^\dagger\widehat{a}_i+\sum_i\hbar(\omega_b+\omega_i)\,\widehat{b}_i^\dagger\widehat{b}_i \\ \widehat{V} &= \mathcal{A}(t)\exp(-i\omega_0 t)\,\exp(i\phi)\,\sum_i\widehat{b}_i^\dagger\widehat{a}_i+\mathcal{A}(t)\exp(+i\omega_0 t)\,\exp(-i\phi)\,\sum_i\widehat{a}_i^\dagger\widehat{b}_i \end{aligned} \tag{359}$$

where the collision terms are ignored since we are only considering the effect of the short interferometer coupling pulse. Here we have assumed that the energy for the mode $\widehat{a}_i$ is $\hbar(\omega_a+\omega_i)$, which is the sum of a basic energy for all a modes - $\hbar\omega_a$, and an energy term $\hbar\omega_i$ that distinguishes differing modes $\widehat{a}_i$ (and similarly for the mode $\widehat{b}_i$). In addition, we assume *selection rules* lead to pair-wise coupling $\widehat{a}_i\leftrightarrow\widehat{b}_i$. In the case where coupling is due to pulsed external fields (microwave and RF) we can assume that the momenta ($\sim\sqrt{m\hbar\omega_{trap}}$) associated with trapped modes $\widehat{a}_i$ and $\widehat{b}_i$ are the same, since the momenta associated with the low frequency photons ($\sim\hbar\omega_{RF}/c$) involved can be ignored. The spin operators for the multi-mode system are set out in (196) in terms of the mode operators.

As in SubSection 7.1 the *interferometer frequency* ω_0 is assumed for simplicity to be in *resonance* with the *transition frequency* $\omega_{ba} = \omega_b - \omega_a$. Following the treatment in SubSection 7.3, the choice of *measurable* is the half the total population difference between the two sets of modes

$$\widehat{M} = \frac{1}{2}\sum_i (\widehat{b}_i^\dagger \widehat{b}_i - \widehat{a}_i{}^\dagger \widehat{a}_i) = \widehat{S}_z \tag{360}$$

and we will determine its mean and variance for the state $\widehat{\rho}^\#$ given by

$$\widehat{\rho}^\# = \widehat{U}\,\widehat{\rho}\,\widehat{U}^{-1} \tag{361}$$

where the *output state* $\widehat{\rho}^\#$ has evolved from the initial *input state* $\widehat{\rho}$ due to the effect of the *multi- mode interferometer*. $\widehat{U}$ is the unitary *evolution operator* describing evolution during the time the short classical pulse is applied. Collision terms and interactions with other systems will be ignored during the short time interval involved.

As in the two mode interferometer case, the results for the mean and variance of $\widehat{M}$ depend on the *pulse area* $2s = \theta$ and the *phase* $\phi$of the interferometer coupling pulse. They have the same dependence on the *mean values* and *covariance matrix* for the *multi-mode spin operators* $\widehat{S}_x$, $\widehat{S}_y$ and $\widehat{S}_z$ for the input state $\widehat{\rho}$ as in the two mode interferometer. Thus we then find that the *general result* for the *mean* value is

$$\left\langle \widehat{M} \right\rangle = \sin\theta\,\sin\phi\,\left\langle \widehat{S}_x \right\rangle_\rho + \sin\theta\,\cos\phi\,\left\langle \widehat{S}_y \right\rangle_\rho + \cos\theta\,\left\langle \widehat{S}_z \right\rangle_\rho \tag{362}$$

and for the *variance* is

$$\begin{aligned}
&\left\langle \Delta\widehat{M}^2 \right\rangle \\
= \quad & \frac{(1-\cos 2\theta)}{2}\frac{(1-\cos 2\phi)}{2}C(\widehat{S}_x, \widehat{S}_x) + \frac{(1-\cos 2\theta)}{2}\frac{(1+\cos 2\phi)}{2}C(\widehat{S}_y, \widehat{S}_y) \\
& + \frac{(1+\cos 2\theta)}{2}C(\widehat{S}_z, \widehat{S}_z) \\
& + \frac{(1-\cos 2\theta)}{2}\sin 2\phi\,C(\widehat{S}_x, \widehat{S}_y) + \sin 2\theta\,\cos\phi\,C(\widehat{S}_y, \widehat{S}_z) + \sin 2\theta\,\sin\phi\,C(\widehat{S}_z, \widehat{S}_x)
\end{aligned} \tag{363}$$

The derivation follows the same steps as in SubSection M.1. However here we have the results $\widehat{U}_0^{-1}\,\widehat{b}_i\,\widehat{U}_0 = \exp(-i(\omega_b + \omega_i)t)\,\widehat{b}_i$ and $\widehat{U}_0^{-1}\,\widehat{a}_i\,\widehat{U}_0 = \exp(-i(\omega_a + \omega_i)t)\,\widehat{a}_i$. The factors involving $\exp(-i\omega_i)t$ cancel out in the derivation of the Heisenberg equations, which here are

$$i\hbar\frac{\partial}{\partial t}\widehat{b}_{iH} = \mathcal{A}(t)\,\exp(+i\phi)\,\widehat{a}_{iH} \qquad i\hbar\frac{\partial}{\partial t}\widehat{a}_{iH} = \mathcal{A}(t)\,\exp(-i\phi)\,\widehat{b}_{iH} \tag{364}$$

and the solutions are

$$\widehat{b}_{iH}(s,\phi) = \cos s\,\widehat{b}_i - i\exp(i\phi)\,\sin s\,\widehat{a}_i \qquad \widehat{a}_{iH}(s,\phi) = -i\exp(-i\phi)\,\sin s\,\widehat{b}_i + \cos s\,\widehat{a}_i \tag{365}$$

where now $2s$ is the area for the classical pulse.

Hence we have in general

$$\begin{aligned}\widehat{M}_H(2s,\phi) &= \frac{1}{2}\sum_i(\widehat{b}^\dagger_{iH}(s,\phi)\widehat{b}_{iH}(s,\phi)-\widehat{a}^\dagger_{iH}(s,\phi)\widehat{a}_{iH}(s,\phi)) \\ &= \sin 2s\,(\sin\phi\,\widehat{S}_x+\cos\phi\,\widehat{S}_y)+\cos 2s\,\widehat{S}_z \qquad (366)\end{aligned}$$

This leads to the same formal results (362) and (363) for the mean and variance. The versatility of the measurement again follows from the range of possible choices of the pulse area $2s$ and the phase ϕ.

Appendix N Limits on Interferometry Tests

The *tests* for entanglement in a particular quantum state are given in terms of the *mean value* and *variance* for certain physical quantities. Interferometers are used to enable these means and variances to be determined from measurements on *another* physical quantity when either the state being tested is acted upon by the interferometer or it is being unaffected. Quantum theory enables us to predict two things. Firstly, for any physical quantity $\widehat{M}$ we can predict the *possible values* that measurements could result in. Results from a succession of measurements would confirm what these values are. Secondly, for any quantum state, we can predict the *probability* that measurement leads to a specific value. A single measurement only yields one of the possible values, so *independent repetitions* of such measurements is needed to confirm what the probabilities for measuring particular values are - ideally an infinite number of repeated measurements would be required. If this was possible, the computed mean $\left\langle \widehat{M} \right\rangle$ and variance $\left\langle \Delta\widehat{M}^2 \right\rangle$ of the measurements for the physical quantity $\widehat{M}$ would confirm the quantum theory predictions for any quantum state. A finite but large number of independent measurements - each based on the *same* probability distribution for the possible results, would enable us to *estimate* the *actual* mean and variance of the measured values from the *sample* measurements. These estimates would not be precisely accurate. The question is - how *big* would the sample of repeated measurements need to be for the purpose of using the estimated mean and variance in the tests for *entanglement* ?

Statistical theory in the form of the *central limit theorem* [45] can be applied here. This tells us if the number R of repeated measurements is large, then the mean of the *sample* measurements approaches the *true* mean and the *variance* in the *sample estimation* of the *mean* is given by the *true variance* divided by R

$$\begin{aligned} \left\langle \widehat{M} \right\rangle_{sample} &\rightarrow \left\langle \widehat{M} \right\rangle \\ \left\langle \Delta \left\langle \widehat{M} \right\rangle^2 \right\rangle_{sample} &\rightarrow \frac{\left\langle \Delta\widehat{M}^2 \right\rangle}{R} \end{aligned} \tag{367}$$

We can use our theoretical estimate of the variance $\left\langle \Delta\widehat{M}^2 \right\rangle$ to get an idea of how large the sample of measurements must be in order that the standard deviation of the sample estimate for the mean is small enough that the mean can confidently be stated to exceed or be less than the quantity on the other side of the inequality in the entanglement test.

Appendix O Relative Phase State

The *relative phase eigenstate* (see [7], [46]) for an N boson two mode system has provided an important example of different outcomes for the simple spin squeezing and Hillery spin squeezing tests, so here its properties are set out in more detail. The results for interferometric measurements on the relative phase state are also presented.

The relative phase state is a globally compliant entangled state of the sub-systems a and b and is defined by

$$|N, \theta_p\rangle = \frac{1}{\sqrt{N+1}} \sum_{k=-N/2}^{N/2} \exp(ik\theta_p^N)\, |N/2-k\rangle_a\, |N/2+k\rangle_b \tag{368}$$

where $\theta_p^N = p(2\pi/(N+1))$, $p = -N/2, -N/2+1, .., +N/2$ is a quasi-continuum of $N+1$ equi-spaced phase eigenvalues, and $|N/2-k\rangle_a$, $|N/2+k\rangle_b$ are Fock states for sub-systems a and b. The Hermitian relative phase operator $\widehat{\Theta}_N$ for N boson states is then defined as

$$\widehat{\Theta}_N = \sum_p \theta_p^N\; |N, \theta_p\rangle\, \langle N, \theta_p| \tag{369}$$

and $|N, \theta_p\rangle$ is an eigenvector with eigenvalue θ_p^N.

Since these states are entangled with maximum *mode entropy*, are *spin squeezed* and are *fragmented* BEC (two modes have macroscopic occupancy) it is of some interest to examine their interferometric properties for the simple beam splitter interferometer. As shown in [7] the relative phase state has the following mean values for the spin operators when $\widehat{\rho} = |N, \theta_p\rangle\, \langle N, \theta_p|$

$$\left\langle \widehat{S}_x \right\rangle_\rho = \frac{N\pi}{8}\cos\theta_p \qquad \left\langle \widehat{S}_y \right\rangle_\rho = -\frac{N\pi}{8}\sin\theta_p \qquad \left\langle \widehat{S}_z \right\rangle_\rho = 0 \tag{370}$$

so that for the measurable

$$\left\langle \widehat{M} \right\rangle = \frac{N\pi}{8}\sin(\phi - \theta_p) \tag{371}$$

We thus have a large amplitude - proportional to N - sinusoidal dependence for the mean value of the measurable on the interferometer phase detuning $(\phi - \theta_p)$, and which goes to zero when $\phi = \theta_p$. Since we never have both $\left\langle \widehat{S}_x \right\rangle_\rho$ and $\left\langle \widehat{S}_y \right\rangle_\rho$ equal to zero the simple correlation test confirms that the relative phase eigenstate is entangled.

As mentioned above, the relative phase state is highly spin squeezed. To describe this it is convenient to introduce rotated spin operators $\widehat{J}_x$, $\widehat{J}_y$ and $\widehat{J}_z$ given by (see Ref [7], Eqn. 179)

$$\begin{aligned} \widehat{J}_x &= \widehat{S}_z \\ \widehat{J}_y &= \sin\theta_p\, \widehat{S}_x + \cos\theta_p\, \widehat{S}_y \\ \widehat{J}_z &= -\cos\theta_p\, \widehat{S}_x + \sin\theta_p\, \widehat{S}_y \end{aligned} \tag{372}$$

The new spin operators are Schwinger spin operators for *new modes* c, d where

$$\widehat{a} = -\exp(\frac{1}{2}i\theta_p)\left(\widehat{c} - \widehat{d}\right)/\sqrt{2} \quad \widehat{b} = -\exp(-\frac{1}{2}i\theta_p)\left(\widehat{c} + \widehat{d}\right)/\sqrt{2} \tag{373}$$

and the relative phase state also an *entangled* state for *new* modes. This can be shown by substituting for the $|N/2-k\rangle_a$ and $|N/2+k\rangle_b$ in terms of Fock states for the new modes c, d.

These new angular momentum operators are *principal spin operators* for which the covariance matrix is diagonal. For the mean values

$$\left\langle \widehat{J}_x \right\rangle_\rho = 0 \qquad \left\langle \widehat{J}_y \right\rangle_\rho = 0 \qquad \left\langle \widehat{J}_z \right\rangle_\rho = -\frac{N\pi}{8} \tag{374}$$

In terms of spin operators discussed above (see Eqs. (185) and (186)) we have $\widehat{J}_x = \widehat{S}_z$, $\widehat{J}_y = \widehat{S}_x^{\#}(\frac{3\pi}{2}+\theta_p)$ and $\widehat{J}_z = \widehat{S}_y^{\#}(\frac{3\pi}{2}+\theta_p)$ so the variances for $\widehat{J}_y$ and $\widehat{J}_z$ can be measured using the simple BS interferometer, and the mean for $\widehat{J}_x$ is also measurable by simply measuring the mean population difference without subjecting the relative phase eigenstate to the BS interaction.

Inverting these expressions and substituting gives the measurable in terms of the new spin operators

$$\widehat{M}_H = \cos(\phi-\theta_p)\,\widehat{J}_y - \sin(\phi-\theta_p)\,\widehat{J}_z \tag{375}$$

Hence we find for the variance of the measurable

$$\begin{aligned}\left\langle \Delta\widehat{M}^2 \right\rangle &= \cos^2(\phi-\theta_p)\,C(\widehat{J}_y,\widehat{J}_y) + \sin^2(\phi-\theta_p)\,C(\widehat{J}_z,\widehat{J}_z) \\ &\quad -2\sin(\phi-\theta_p)\,\cos(\phi-\theta_p)\,C(\widehat{J}_y,\widehat{J}_z)\end{aligned} \tag{376}$$

As $\widehat{J}_x$,$\widehat{J}_y$ and $\widehat{J}_z$ are principal spin operators $C(\widehat{J}_y,\widehat{J}_z)=0$ and substituting for the variances $C(\widehat{J}_y,\widehat{J}_y) = 1/4 + 1/8\,\ln N$ and $C(\widehat{J}_z,\widehat{J}_z) = (1/6-\pi^2/64)\,N^2$ (see [7]) we get for the variance of the measurable for an input relative phase eigenstate

$$\begin{aligned}\left\langle \Delta\widehat{M}^2 \right\rangle &= \cos^2(\phi-\theta_p)\,(\frac{1}{4}+\frac{1}{8}\ln N) + \sin^2(\phi-\theta_p)\,(\frac{1}{6}-\frac{\pi^2}{64})\,N^2 \\ &\approx \frac{1}{4} + (\phi-\theta_p)^2\,(\frac{1}{6}-\frac{\pi^2}{64})\,N^2\end{aligned} \tag{377}$$

for $\phi \approx.\ \theta_p$. The other variance is $C(\widehat{J}_x,\widehat{J}_x) = (1/12)\,N^2$. The variance for the measurable depends sinusoidally on $2(\phi-\theta_p)$. Thus the quantum noise in the measurable also goes to essentially zero at $\phi=\theta_p$, when the mean value $\left\langle \widehat{M} \right\rangle$ also goes to zero. The width $\Delta\phi$ for this low noise window scales as $1/N$ - which corresponds to the Heisenberg limit. At the zero of the mean value, the relative fluctuation varies as $1/N$ as in the Heisenberg limit. Since for $\phi=\theta_p$ we have $\widehat{M}_H = \widehat{J}_y = \widehat{S}_x^{\#}(\frac{3\pi}{2}+\theta_p)$ and $\left\langle \Delta\widehat{M}^2 \right\rangle = (\frac{1}{4}+\frac{1}{8}\ln N)$

whilst $\left\langle \widehat{S}_z \right\rangle_\rho = \left\langle \widehat{J}_z \right\rangle_\rho = -\frac{N\pi}{8}$. Thus the spin squeezing test in Eq. (354) is satisfied, confirming again that the relative phase eigenstate is an entangled state of modes a and b.

In regard to *particle entanglement* [47], [48] with $\widehat{\rho} = |N, \theta_p\rangle \langle N, \theta_p|$ and with $n_a = (N/2 - k)$, $n_b = (N/2 + k)$, the quantities in Eqs. (79) and (80) of paper 1 are given by

$$\widehat{\rho}^{(n_a n_b)} = \frac{1}{N+1} |N/2 - k\rangle_a \langle N/2 - k|_a \otimes |N/2 + k\rangle_b \langle N/2 + k|_b \quad (378)$$

$$P_{n_a n_b} = \frac{1}{N+1} \quad (379)$$

and since $\widehat{\rho}^{(n_a n_b)}$ is a separable state, it follows that $E_P(\widehat{\rho}) = 0$. Thus the measure of particle entanglement is zero for what is clearly an *entangled* state. Hence the particle entanglement measure has not detected entanglement in this example.

The relative phase state is therefore a promising candidate for use as an input state in two mode interferometry. More elaborate interferometers where the interferometric variable is associated with other systems whose parameters are to be measured might be developed. The main issue would be whether such a relative phase state could be prepared. This is an issue being dealt with elsewhere [46].

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