A modified equatorial $\beta-$plane approximation modelling nonlinear wave-current interactions

David Henry
School of Mathematical Sciences,
University College Cork, Cork, Ireland
d.henry@ucc.ie

Abstract

A modification of the standard geophysical equatorial $\beta-$plane model equations, incorporating a gravitational-correction term in the tangent plane approximation, is derived. We present an exact solution satisfying the modified equations, whose form is explicit in the Lagrangian framework, and which represents three-dimensional, nonlinear oceanic wave-current interactions. It is rigorously established, by way of analytical and degree-theoretical considerations, that the solution is dynamically possible, in the sense that the mapping it prescribes from Lagrangian to Eulerian coordinates is a global diffeomorphism.

Keywords: Exact solution; global diffeomorphism; Lagrangian variables; wave-current interactions; $\beta-$plane.
MSC (2010): 76B15; 74G05; 37N10.

1 Introduction

The modelling of geophysical fluid dynamics in the equatorial region is a highly complex subject of vast importance which has recently witnessed a number of interesting mathematical developments. Geophysical fluid dynamics is the study of fluid motion where the Earth's rotation plays a significant role in the resulting dynamics, and accordingly Coriolis forces are incorporated into the governing Euler equation. The ensuing governing equations are applicable for a wide range of oceanic and atmospheric flows [11, 14, 28], thereby encapsulating the necessarily high-level of mathematical sophistication required to model such a rich variety of phenomena. This level of complexity leads to an inherent mathematical intractability in the model equations, and in order to mitigate this one typically employs oceanographical considerations in order to derive simpler approximate models.

A classical example which is typically employed in oceanographic considerations is the $\beta-$plane approximation, whereby the earth's curved surface is approximated (locally) by a tangent plane. In the context of modelling equatorial flows this approach is applicable when we restrict our focus to regions of relatively small latitudinal variation (to the order of $2°$) about the equator; physically, the equator acts as a natural waveguide leading to equatorially-trapped zonally propagating waves which decay exponentially away from the equator, cf. [12]. We note that there has been an abundance of recent mathematical progress in deriving, and analysing, exact solutions to the $\beta-$plane equations modelling equatorial oceanic water waves [3, 4, 5, 6, 13, 15, 17, 18, 19, 20, 21]— an interesting reflection on the relevance of exact solutions in physical oceanography may be found in [9].

However, we remark that while the $\beta-$plane approximation is regarded as reasonable for large-scale oceanographical considerations, nevertheless from a mathematical modelling perspective it is lamentable that an appreciable level of mathematical detail and structure is lost from the model equations as a result of the 'flattening out' of the earth's surface. A number of interesting mathematical approaches have been recently instigated which aim to retain some of this structure in modelling equatorial water waves, cf. [7, 8, 10, 16]. The primary aim of this paper is to address this matter with a view to retaining artefacts of the geometry of the earth's curvature by way of incorporating a gravitational-correction term into the standard $\beta-$tangent plane model, resulting in the modified governing equations (3).

Following the derivation in Section 2, we present a mapping (4) which we claim is an exact solution to the modified equations (3) representing three-dimensional, nonlinear wave-current interactions; the zonally-periodic wavelike term is equatorially-trapped (exhibiting exponentially strong meridional decay) and propagates eastwards above a flow which accommodates a depth-invariant mean current— either following or adverse— of any physically plausible (as defined by (5)) magnitude. In Section 3 we prove by direct computation that the mapping (4), explicit in terms of Lagrangian labelling variables, is compatible with the governing equations (3), and that it maps the Lagrangian labelling domain to a fluid domain bounded above by the free-surface interface. We note that while the underlying current in the exact solution (4) assumes an apparently simple form in the Lagrangian framework, it greatly increases the complexity, both mathematically and physically, of the resulting fluid motion [13, 18] in the Eulerian setting.

From an oceanographic perspective, large-scale currents and wave-current interactions play a major role in the geophysical dynamics of the equatorial region [7, 8, 11, 12, 22]. Aside from being physical important [2, 26], the consideration of underlying currents, and wave-current interactions, is a compelling subject in its own right from a purely mathematical viewpoint. The robustness of the modified governing equations in admitting such a general range of underlying currents in our exact solution is attributable precisely to the gravitational-correction terms, and contrasts strongly with the situation in [15] where the range of admissible adverse currents is greatly restricted.

We complete our analysis in Section 4 by employing analytical considerations to establish that (4) defines a local diffeomorphism from the Lagrangian labelling domain to the fluid domain, and that this mapping is globally injective. Further degree-theoretical considerations then enable us to prove that (4) is, in fact, a global diffeomorphism: these deliberations establish rigorously that the (highly physically-complex!) motion prescribed by the mapping (4), which represents three-dimensional, nonlinear wave-current interactions, is dynamically possible and globally justified. A discussion of some physical characteristics of the flow is presented in Section 5.

2 Modified equatorial $\beta-$plane equations

We consider geophysical waves propagating in the Equatorial region on an incompressible, inviscid fluid, assuming that the earth is a perfect sphere of radius $R = 6378$ km. Our deliberations concern the $\beta-$plane regime whereby the earth's curved surface is approximated by a tangent plane, a simplification typically implemented in oceanographical considerations, enabling us to choose a rectangular coordinate system rotating with the earth whose origin is fixed at a point on the equator lying on the earth's surface. We choose the $\{x, y, z\}$-coordinates so that, in this approximation, the x-axis is pointing horizontally due east (zonal direction), the y-axis is due north (meridional direction), and the z-axis is pointing vertically upwards and perpendicular to the earth's surface. The full governing equations for geophysical fluid dynamics are given as follows. Firstly we have the Euler equation

$$\frac{D\boldsymbol{u}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho}\nabla P + \boldsymbol{F}, \tag{1}$$

where $\boldsymbol{u} = (u, v, w)$ is the fluid velocity, $\boldsymbol{\Omega}$ is the angular velocity vector of the earth's rotation (with $\Omega = 73 \times 10^{-6}$ rad/s the (constant) rotational speed), $\boldsymbol{F}$ is the external body force (in our setting due to gravity), ρ is the water density, and P is the pressure. The 'β-plane approximation' involves linearising the Coriolis force terms in (1), and since we are interested in Equatorial wave-current interactions it is applicable for describing geophysical ocean waves in a region which is within 2^o latitude either side of the Equator. The linearisation is achieved through invoking the approximations $\sin\Phi \approx \Phi$, $\cos\Phi \approx 1$ (where the latitude Φ is small), leading to

$$2\boldsymbol{\Omega} \times \boldsymbol{u} \sim 2\Omega \left(w - \frac{yv}{R}, \frac{yu}{R}, -u \right). \tag{2}$$

We note that the geophysical parameter $\beta = 2\Omega/R = 2.28 \cdot 10^{-11}$ m^{-1}s^{-1} makes a natural appearance at this stage of proceedings. To this point we have followed the standard $\beta-$plane approach for simplifying the full governing equations of geophysical fluid dynamics (1). In considering the form that the gravitational body force $\boldsymbol{F}$ takes in our approximation, we accommodate a correction term which incorporates the deviation of the tangent plane from the earth's curved surface as follows. We consider the point $\boldsymbol{P}$ in figure 1, and note that its distance from the earth's centre $\boldsymbol{O}$ is $R + H = \sqrt{(R+z)^2 + y^2}$ where the plane is aligned with the $x-$coordinate.

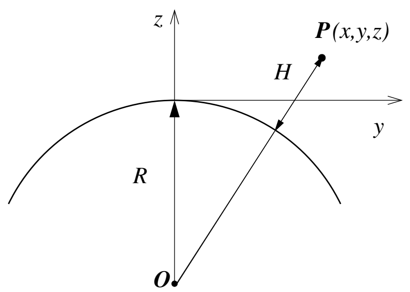

Figure 1: Schematic of tangent plane approximation.

As R is significantly larger than either y or z, we can approximate the gravitational potential $\mathcal{V}$ at P by

$$\mathcal{V}(x,y,z) = Hg = \left(\sqrt{(R+z)^2 + y^2} - R \right) g \approx \left(z + \frac{y^2}{2R} \right) g,$$

where $g = 9.8$ m/s^{-2} is the standard gravitational constant. The associated gravitational field then takes the form $\boldsymbol{F} = -\nabla\mathcal{V} = (0, -y/R, -1)g$, and together with (2) the full governing equations (1) reduce to

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v &= -\frac{1}{\rho}P_x \\ v_t + uv_x + vv_y + wv_z + \beta y u &= -\frac{1}{\rho}P_y - \frac{g}{R}y \\ w_t + uw_x + vw_y + ww_z - 2\Omega u &= -\frac{1}{\rho}P_z - g, \end{aligned} \tag{3a}$$

where the gy/R term is the gravitational correction term which arises when we accommodate the direction that gravity acts in for the tangent $\beta-$plane model. The equations of fluid motion (3a) are supplemented by the equation of mass conservation

$$\frac{D\rho}{Dt} = 0 \tag{3b}$$

and the equation of incompressibility

$$\nabla \cdot \boldsymbol{u} = 0. \tag{3c}$$

For convenience, unless otherwise stated, in this paper we assume that the fluid density is constant, in which case (3b) holds trivially; however, in section 5.3 we show that the exact solution we consider may, upon slight modifications, incorporate stratification in the fluid flow. The boundary conditions for surface water waves are the kinematic and dynamic boundary conditions

$$w = \eta_t + u\eta_x + v\eta_y \text{ on } z = \eta(x, y, t), \tag{3d}$$

$$P = P_{atm} \text{ on } z = \eta(x, y, t), \tag{3e}$$

where P_{atm} is the (constant) atmospheric pressure, and $\eta(x, y, t)$ is the free surface. The boundary condition (3d) states that all the particles located at the wave surface remain on the surface for all time t, and the boundary condition (3e) decouples the water flow from the motion of the air above. Finally, we assume the water to be infinitely deep and request that the wave motion be insignificant at great depths; this corresponds to the flow converging rapidly with depth to a uniform zonal current,

$$(u, v, w) \to (-c_0, 0, 0) \text{ as } z \to -\infty. \tag{3f}$$

The system of equations (3) comprises the $\beta-$plane approximation (incorporating gravitational correction terms) of the full governing equations for geophysical ocean waves with a constant underlying current.

3 Exact solution

We now prove that the system

$$x = q - c_0 t - \frac{1}{k} e^{k[r-f(s)]} \sin[k(q-ct)], \tag{4a}$$

$$y = s, \tag{4b}$$

$$z = r + \frac{1}{k} e^{k[r-f(s)]} \cos[k(q-ct)], \tag{4c}$$

defines an exact solution of the $\beta-$plane governing equations (3), where the Eulerian coordinates (x, y, z) of the fluid particles are prescribed explicitly in terms of the Lagrangian labelling variables $(q, s, r) \in \mathcal{D} = \mathbb{R} \times [-s_0, s_0] \times (-\infty, r_0)$, for $r_0 < 0$, and time t. We note that the restriction on the latitudinal parameter s to a bounded interval is dictated solely by geophysical considerations relating to the scale of applicability of the $\beta-$plane model; the typical value $s_0 \approx 250$km may be taken for the equatorial radius of deformation, cf. [11]. Mathematically, (4) prescribes a solution of (3) for all $s \in \mathbb{R}$, as we see below. The c_0 term represents a constant underlying mean-current, where for $c_0 > 0$ the current is adverse, while for $c_0 < 0$ the current is following. For subsequent considerations we remark that, for all physically plausible values of c_0, we have

$$|c_0| < \frac{g}{2\Omega} \approx 6.7 \times 10^4 \text{m/s}. \tag{5}$$

The remaining terms in (4) consist of the wavenumber $k = 2\pi/L$, where L is the wavelength, and the 'decay function' $f(s)$ given by

$$f(s) = \frac{c\beta}{2\mathfrak{g}} s^2, \tag{6}$$

where the constant wave phasespeed $c > 0$ is taken to be positive in order for (6) to produce a decay in fluid particle motion moving away from the equator, with

$$\mathfrak{g} = g + 2\Omega c_0 \quad (> 0) \tag{7}$$

a perturbation of the usual gravitational constant of acceleration due to Coriolis effects and the underlying current; the positivity of relation (7) is an immediate consequence of the physical assumption (5).

Main Result 3.1. *For all physical plausible (such that* (5) *holds) values of the mean zonal current* c_0*, the fluid motion prescribed by* (4) *is an exact solution of the governing equations* (3)*. This solution represents three-dimensional, nonlinear geophysical wave-current interactions; the wave terms* *are equatorially-trapped steady periodic waves, propagating zonally eastward with constant wave phasespeed c, with insignificant motion at great depths.*

A detailed exposition of the precise nature of the wave phasespeed c, and other physical characteristics of the flow induced by (4), will be presented in section 5 below. The explicitness in terms of Lagrangian coordinates of the solution formulated in (4) is advantageous for a number of reasons, prime among them being the ease of calculation of the fluid kinematics, which we compute directly to get the velocity field

$$(u, v, w) = \left(\frac{Dx}{Dt}, \frac{Dy}{Dt}, \frac{Dz}{Dt}\right) = \left(ce^{\xi}\cos\theta - c_0, 0, ce^{\xi}\sin\theta\right), \tag{8}$$

and the fluid acceleration

$$\left(\frac{Du}{Dt}, \frac{Dv}{Dt}, \frac{Dw}{Dt}\right) = \left(kc^2e^{\xi}\sin\theta, 0, -kc^2e^{\xi}\cos\theta\right), \tag{9}$$

where D/Dt is the material (or Lagrangian) derivative, and for convenience we define $\xi(r, s) = k\,(r - f(s))$, $\theta(q, t) = k(q - ct)$. It follows directly from (8) that the wave motion described by (4) is insignificant at great depths, and hence the limiting relation (3f) holds for the velocity field. Geophysically, the attenuation of the meridional component of the Coriolis force at the equator has the effect that the equator works as a (fictitious) natural boundary, resulting in equatorially-trapped waves. The equatorially-trapped nature of the wave solution prescribed by (4) is captured by $v \equiv 0$ in (8), which implies no meridional fluid motion. This is consonant with equatorial field data [23], which confirms that meridional speeds near the equator are much smaller than the zonal speeds, and neglecting them therefore has an insignificant dynamical effect. Consequently, the governing equations (3a) reduce to

$$\nabla_{(x,y,z)}P = -\rho\left(\frac{Du}{Dt} + 2\Omega w,\ \beta yu + gy/R,\ \frac{Dw}{Dt} - 2\Omega u + g\right),$$

which may be expressed by way of (8) and (9) as

$$\nabla_{(x,y,z)}P = -\rho\begin{pmatrix} kc^2e^{\xi}\sin\theta + 2\Omega ce^{\xi}\sin\theta \\ \beta s[ce^{\xi}\cos\theta - c_0] + gs/R \\ -kc^2e^{\xi}\cos\theta - 2\Omega ce^{\xi}\cos\theta + \mathfrak{g} \end{pmatrix}.$$

We wish to determine $\nabla_{(q,s,r)}P$, the gradient of P in terms of the Lagrangian variables, for which we compute the Jacobian matrix

$$\mathcal{J} = \frac{\partial(x, y, z)}{\partial(q, s, r)} = \begin{pmatrix} 1 - e^{\xi}\cos\theta & f_s e^{\xi}\sin\theta & -e^{\xi}\sin\theta \\ 0 & 1 & 0 \\ -e^{\xi}\sin\theta & -f_s e^{\xi}\cos\theta & 1 + e^{\xi}\cos\theta \end{pmatrix}, \tag{10}$$

and, observing that (6) implies $\mathfrak{g} f_s = \beta s c$, and using the relation $\nabla_{(q,s,r)} P = \mathcal{J}^T \cdot \nabla_{(x,y,z)} P$, we find

$$\nabla_{(q,s,r)} P = -\rho \begin{pmatrix} (kc^2 + 2\Omega c - \mathfrak{g}) e^{\xi} \sin\theta \\ f_s e^{2\xi}(kc^2 + 2\Omega c) - \beta s c_0 + g s/R \\ -(kc^2 + 2\Omega c) e^{2\xi} - (kc^2 + 2\Omega c - \mathfrak{g}) e^{\xi} \cos\theta + \mathfrak{g} \end{pmatrix}. \tag{11}$$

We observe that the determinant of the Jacobian matrix (10) is $1 - e^{2\xi}$, which is time-independent. This corresponds [1] to incompressibility of the fluid; hence (3c) is satisfied for the fluid motion induced by (4). Furthermore, we infer that the transformation between Lagrangian and Eulerian coordinates defined by (4) is a valid change of variables since

$$\xi(r,s) = r - f(s) \leq r_0 < 0, \tag{12}$$

and accordingly the inverse of the Jacobian matrix (10) is

$$\mathcal{J}^{-1} = \begin{pmatrix} \frac{1+e^{\xi}\cos\theta}{1-e^{2\xi}} & -f_s \frac{e^{\xi}\sin\theta}{1-e^{2\xi}} & \frac{e^{\xi}\sin\theta}{1-e^{2\xi}} \\ 0 & 1 & 0 \\ \frac{e^{\xi}\sin\theta}{1-e^{2\xi}} & f_s \frac{e^{\xi}\cos\theta - e^{2\xi}}{1-e^{2\xi}} & \frac{1-e^{\xi}\cos\theta}{1-e^{2\xi}} \end{pmatrix},$$

with the velocity gradient tensor $\nabla_{(x,y,z)} \boldsymbol{u} = \left(\mathcal{J}^{-1}\right)^T \cdot \nabla_{(q,s,r)} \boldsymbol{u}$ computed as

$$\nabla_{(x,y,z)} \boldsymbol{u} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} = \frac{cke^{\chi}}{1 - e^{2\chi}} \begin{pmatrix} -\sin\theta & 0 & \cos\theta + e^{\chi} \\ f_s(e^{\chi} - \cos\theta) & 0 & -f_s \sin\theta \\ -e^{\chi} + \cos\theta & 0 & \sin\theta \end{pmatrix},$$

from which the vorticity $\boldsymbol{\omega} = (w_y - v_z, u_z - w_x, v_x - u_y)$ is computed as

$$\boldsymbol{\omega} = \left(-s \frac{kc^2\beta}{\mathfrak{g}} \frac{e^{\chi}\sin\theta}{1 - e^{2\chi}}, -\frac{2kce^{2\chi}}{1 - e^{2\chi}}, s \frac{kc^2\beta}{\mathfrak{g}} \frac{e^{\chi}\cos\theta - e^{2\chi}}{1 - e^{2\chi}} \right).$$

We note that although the velocity field (8) for the solution (4) is two-dimensional, the vorticity induced by (4) is (weakly) three-dimensional away from the equator, with the first and third components depending on the (small) geophysical parameter β and the latitude s; the vorticity becomes two-dimensional at the equator, and or in the absence of Coriolis forces. Furthermore, although the underlying current c_0 does not feature directly in the expression for $\boldsymbol{\omega}$ (since it is constant), yet it plays an implicit role as determined by the dispersion relation (19) for the wave phasespeed c below.

The next stage in proving that the solution (4) satisfies the governing equations (3) is to show that the Euler equation (3a) is satisfied by (8) and

(9); as shown above this is equivalent to constructing a pressure function $P(q,s,r)$ for which (11) holds. Adopting the candidate

$$\tilde{P} = \rho\frac{kc^2+2\Omega c}{2k}e^{2\xi} - \rho\mathfrak{g}r + \frac{\rho c_0\beta}{2}s^2 + \rho\frac{kc^2+2\Omega c-\mathfrak{g}}{k}e^{\xi}\cos\theta - \frac{\rho g}{2R}s^2 + P_{atm} \tag{13}$$

we have

$$\begin{aligned}
\tilde{P}_q &= -\rho(kc^2+2\Omega c-\mathfrak{g})e^{\xi}\sin\theta\\
\tilde{P}_s &= -\rho(kc^2+2\Omega c)f_s e^{2\xi} - \rho(kc^2+2\Omega c-\mathfrak{g})f_s e^{\xi}\cos\theta + \rho\beta s c_0 - \rho g s/R\\
\tilde{P}_r &= \rho(kc^2+2\Omega c)e^{2\xi} - \rho\mathfrak{g} + \rho(kc^2+2\Omega c-\mathfrak{g})e^{\xi}\cos\theta.
\end{aligned} \tag{14}$$

A match between (14) and (11) is achieved through imposing a constraint on the physical parameters in the form of the dispersion relation

$$kc^2 + 2\Omega c - \mathfrak{g} = 0. \tag{15}$$

A secondary effect of relation (15) is that it renders the candidate pressure function time-independent on the surface, a condition which is necessary for (3e) to hold. It follows from (13),(14),(15) that the pressure function

$$P(q,s,r) = \rho\mathfrak{g}\left(\frac{e^{2\xi(r,s)}}{2k} - r + \frac{f(s)}{c}\left(c_0 - \frac{g}{2\Omega}\right)\right) + P_{atm} - \rho\mathfrak{g}\left(\frac{e^{2kr_0}}{2k} - r_0\right) \tag{16}$$

satisfies (11), and hence the solution (4) satisfies (3a).

3.1 The free-surface interface

The formulation in (16) has been chosen with a view to establishing that the remaining free-surface boundary conditions (3d) and (3e) hold for the flow prescribed by (4), thereby completing the proof of our Main Result. This will be achieved upon proving that for each fixed latitude s there exists a unique solution $r(s) \le r_0 < 0$ such that $P(s, r(s)) = P_{atm}$ in (16), which is equivalent to

$$\mathcal{P}(s, r(s)) = \frac{e^{2kr_0}}{2k} + r_0, \tag{17}$$

where

$$\mathcal{P}(s,r) := \frac{e^{2k[r - \frac{c\beta}{2\mathfrak{g}}s^2]}}{2k} - r - \frac{(g-2\Omega c_0)}{2\mathfrak{g}R}s^2.$$

At the equator, for $s = 0$, the choice $r(0) = r_0$ works in (17). For $|s| > 0$, we infer that the last term on the right-hand side above is negative for physically plausible values c_0 such that (5) holds, bearing in mind (7), and so $\mathcal{P}(s,r)$ decreases as $|s|$ increases. Since $\lim_{r\to-\infty}\mathcal{P}(s,r) = \infty$, and the relation $\mathcal{P}_r(s,r) = e^{2k[r-\frac{c\beta}{2\mathfrak{g}}s^2]}-1 < 0$ implies that $\mathcal{P}(s,r)$ is a monotonically decreasing function of r, we may infer that, for each fixed $s \neq 0$, there exists a unique $r(s)$ such that the equilibrium (17) holds. Differentiating (17) with respect to s we get

$$r'(s) = \frac{\beta s}{\mathfrak{g}} \cdot \frac{c_0 - \frac{g}{2\Omega} - ce^{2k[r-\frac{c\beta}{2\mathfrak{g}}s^2]}}{1 - e^{2k[r-\frac{c\beta}{2\mathfrak{g}}s^2]}} < 0,$$

with negativity following by way of (5), and so the even function $s \mapsto r(s)$ is decreasing whenever condition (5) holds.

At fixed-latitudes $y = s$ the free-surface $z = \eta(x,s,t)$ is implicitly prescribed by setting $r = r(s)$ in (4c) for the unique value $r(s) < r_0$ which solves (17), and it follows directly that condition (3e) is fulfilled. By its very design, this method of prescription of the free-surface $z = \eta(x,y,t)$ ensures that the kinematic boundary condition (3d) holds: all particles originating on the wave surface will remain at the surface for all time. Furthermore, at each fixed-latitude $y = s$ in a coordinate system moving with the mean flow (which we take to be fixed if $c_0 = 0$), the free-surface is an inverted trochoid (since (12) holds) and particle trajectories are given by closed circles. In the limiting case $r_0 \to 0$ in (12) the free-surface approaches a cycloid, with singular cusps at the crests [2], at the equator ($s = 0$). It is worth noting that, as opposed to the typical Eulerian approach [1], the Lagrangian labelling variables in (4) do not represent the initial position of the particle they define, but rather the centre of the circle described by the particle motion. We further note that closed particle trajectories are rarely encountered beneath irrotational periodic travelling surface gravity water waves, cf. [2], rather this feature is indicative of the flows with vorticity. If $c_0 \neq 0$, the particles move in trochoidal orbits with respect to a fixed-coordinate system.

4 Global validity of (4)

We have thus far proven that the image of the mapping (4) from Lagrangian to Eulerian coordinates is compatible with the governing equations (3). We now provide a rigorous mathematical justification that the prescribed flow is dynamically possible, which follows if we can show that the mapping (4) is a global diffeomorphism between $\mathcal{D}$ to the fluid domain. This ensures that it is possible to have a three-dimensional, nonlinear motion of the entire fluid body described by (4), characterising wave-current interactions, whereby fluid particles never collide, and furthermore they encompass the entire infinite fluid region beneath the free-surface interface.

Remark 1 In order to prove that the mapping (4) is a global isomorphism we can simplify matters by setting of $t = 0$: the general case can be recovered by changing variables $(q, s, r) \mapsto (q + ct, s, r)$, coupled with translating the $x-$variable by $(c_0 + c)t$. Bearing this in mind we define the operator

$$\begin{aligned}\mathcal{F}(q, s, r) &= (x(0; q, s, r), y(0; q, s, r), z(0; q, s, r)) \\ &= \left(q - \frac{1}{k}e^{k[r-f(s)]}\sin kq, s, r + \frac{1}{k}e^{k[r-f(s)]}\cos kq\right),\end{aligned}$$

and since $G(q, s, r) = \mathcal{F}(q, s, r) - (q, 0, 0)$ is $q-$periodic, with period $2\pi/k$, we can focus on the truncated domain $\tilde{\mathcal{D}} = (0, 2\pi/k) \times (-s_0, s_0) \times (-\infty, r_0)$.

Lemma 4.1. *The map $\mathcal{F}$ is a local diffeomorphism from $\tilde{\mathcal{D}}$ into its range, and it is globally injective on $\tilde{\mathcal{D}}$.*

Proof. The inequality (12) ensures that the Jacobian matrix (10) has a non-zero determinant throughout $\tilde{\mathcal{D}}$, and since $\mathcal{F}$ has continuous partial derivatives an application of the Inverse Function Theorem proves that $\mathcal{F}$ is a local diffeomorphism into its range. If we assume that $\mathcal{F}(q_1, s_1, r_1) = \mathcal{F}(q_2, s_2, r_2)$, where (q_1, s_1, r_1) and (q_2, s_2, r_2) are arbitrary points of $\mathcal{D}$, then by definition $s_1 = s_2$: accordingly we fix s in subsequent considerations, and examine injectivity with respect to q and $r < r(s)$, where $r(s)$ is the solution of (17). We can re-express $\mathcal{F}(q, r) = (q, r) + h(q, r)$ where

$$h(q, r) = \frac{1}{k}e^{k[r-f(s)]}\left(-\sin kq, \cos kq\right),$$

and the Mean-Value Theorem ensures that

$$|h(q_1, r_1) - h(q_2, r_2)| \le \max_{\sigma\in[0,1]} \|Dh_{\sigma(q_1,r_1)+(1-\sigma)(q_2,r_2)}\| \cdot |(q_1, r_1) - (q_2, r_2)|, \quad (18)$$

where $\|\cdot\|$ is the usual matrix operator norm, and $|\cdot|$ is the Euclidean norm. Direct computation yields that $\|Dh_{(q,r)}\| = e^{k[r-f(s)]}$, hence (18) yields

$$\begin{aligned}|\mathcal{F}(q_1, r_1) - \mathcal{F}(q_2, r_2)| &\ge |(q_1, r_1) - (q_2, r_2)| - e^{k[\max\{r_1,r_2\}-f(s)]} \cdot |(q_1, r_1) - (q_2, r_2)| \\ &\ge \left(1 - e^{k[r(s)-f(s)]}\right) \cdot |(q_1, r_1) - (q_2, r_2)|\end{aligned}$$

and injectivity follows since (12) holds throughout $\tilde{\mathcal{D}}$, ensuring that the term in parenthesis is non-zero. □

Remark 2 We note that in the limiting case $r_0 \to 0$ all considerations above apply except at the cusps when $s = 0$, in which case the map $\mathcal{F}$ is merely continuous. In this setting a straightforward adaptation of Lemma 4.1 proves that global injectivity holds for $\mathcal{F}$ throughout the domain $\mathcal{D}$.

To prove our final theorem we make use of the following degree-theoretical result, the *Invariance of Domain* Theorem [24, 27], which we state as:

Theorem 4.2. *If $U \subset \mathbb{R}^n$ is open and $F : \overline{U} \to \mathbb{R}^n$ is a continuous one-to-one mapping, then $F : U \to F(U)$ is a homeomorphism. Furthermore, we have $F\left(\overline{\partial U}\right) = \partial F(\overline{U})$.*

The proof that the mapping (4) is a global diffeomorphism between $\mathcal{D}$ and the infinite fluid domain bounded above by the free-surface interface follows from the next result.

Theorem 4.3. *The mapping* (4) *is a global diffeomorphism between $\mathcal{D}$ and the fluid domain bounded above by the free-surface interface $z = \eta(x, y, t)$. For $r_0 < 0$ the free surface has a smooth profile, and in the limiting case $r_0 = 0$ the surface is smooth except when $s = 0$, in which case it is piecewise smooth with upward cusps.*

Proof. The previous result, Lemma 4.1, ensures that $\mathcal{F}$ is a local diffeomorphism from $\tilde{\mathcal{D}}$ into its image, which is also globally injective. Furthermore, the Invariance of Domain Theorem 4.2 implies that the mapping $\mathcal{F}$ is a homemorphism. Accordingly it follows that $\mathcal{F}$ is a global diffeomorphism. A further application of the Invariance of Domain Theorem ensures that the mapping $\mathcal{F}$ maps the boundaries of $\tilde{\mathcal{D}}$ into the boundaries of its image. The considerations of Remarks 1 and 2 enable us to extend these observations to the full domain $\mathcal{D}$, thereby completing the proof of our theorem. $\square$

5 Fluid characteristics

5.1 Geoid prescribed by (3a)

When the fluid is at rest the free-surface is a geoid and the pressure is constant there; accordingly in the absence of motion ($u = v = w = 0$) the modified $\beta-$plane governing equations (3a) requires a very specific pressure distribution. If the free surface is a surface of constant atmospheric pressure ($P_{atm} = 1$ atm $= 1.01325$ bar), then

$$P(x, y, z, t) = P_{atm} - \frac{\rho g}{2R}\, y^2 - \rho g\, z$$

throughout the fluid, so that the free surface is given by

$$z = \frac{P_{atm}}{\rho g} - \frac{1}{2R}y^2.$$

The above distortion from a constant value of z is consistent with, and indeed a consequence of, the β-plane approximation [8, 10, 16]. It corresponds to a free surface geoid following the curvature of Earth away from the equator, as the curved surface of the Earth drops below the tangent plane at the Equator.

5.2 Wave motion

As mentioned above, the enforced constraint (15) amounts to a dispersion relation for the flow, and solving we get

$$c = \frac{\sqrt{\Omega^2 + k\mathfrak{g}} - \Omega}{k},$$

since we require $c > 0$. Therefore the dispersion relation for the wave phasespeed induced by the flow (4) is given by

$$c = \frac{\sqrt{\Omega^2 + k(2\Omega c_0 + g)} - \Omega}{k}, \tag{19}$$

featuring contributions from the Coriolis force, the centripetal force and the underlying current. We note that (19) holds for $c_0 \neq c$: if $c_0 = c$ then (15) gives us $c = \sqrt{g/k}$, which is the standard dispersion relation for both Gerstner's wave, and gravity waves in deep-water, cf. [2]. This expression is also obtained through ignoring the effects of the Earth's rotation (letting $\Omega \to 0$), in which case the solution (4) effective reduces to Gerstner's two-dimensional gravity water wave. Surface waves with wavelengths of 300 m, propagating at speeds of about 22 m/s, are common in the Pacific – see the discussion in [3]; the corresponding value of the speed predicted by the dispersion relation $c = \sqrt{g/k}$ is therefore quite accurate. For further relevant field data we refer to the discussion in [7]. Finally, we observe that the wave phasespeed $c \to 0$ in the (physically implausible) limit whereby $\mathfrak{g} \to 0$ in (5).

5.3 Stratification

In the absence of an underlying current ($c_0 = 0$) we can admit variable density in our fluid through assuming a steady functional dependence of the form $\rho(x,y,z,t) = \rho(x-ct,y,z)$. The equation of mass conservation (3b) becomes

$$(u-c)\rho_x + w\rho_z = 0, \tag{20}$$

and through direct computation (using (8), (10), and (20)) we find that

$$\rho_q = \rho_x \frac{\partial x}{\partial q} + \rho_y \frac{\partial y}{\partial q} + \rho_z \frac{\partial z}{\partial q} = \rho_x(1 - e^{\xi}\cos\theta) - \rho_z e^{\xi}\sin\theta = 0.$$

Therefore the density ρ is independent of q. Defining the density function by

$$\rho(r,s) = F\left(\frac{e^{2\xi}}{2k} - r + \frac{f(s)}{c}\left(c_0 - \frac{g}{2\Omega}\right)\right),$$

where $F : (0,\infty) \to (0,\infty)$ is a non-decreasing, continuously differentiable function, we may infer that all the considerations of section 3 which apply for a homogeneous fluid may be generalised to the setting of a stratified fluid. The pressure function (16) is adapted by defining, for $\mathcal{F}' = F$ with $\mathcal{F}(0) = 0$, by the function

$$P = g\mathcal{F}\left(\frac{e^{2\xi}}{2k} - r + \frac{f(s)}{c}\left(c_0 - \frac{g}{2\Omega}\right)\right) + P_{atm} - g\mathcal{F}\left(\frac{e^{2kr_0}}{2k} - r_0\right).$$

Acknowledgements

The author wishes to thank the anonymous referees for their constructive comments and observations. The author acknowledges the support of the Science Foundation Ireland (SFI) research grant 13/CDA/2117.

References

[1] A. Bennett, *Lagrangian fluid dynamics*, Cambridge University Press, Cambridge, 2006.

[2] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Conference Series in Applied Mathematics, Vol. 81, SIAM, Philadelphia, 2011.

[3] A. Constantin, An exact solution for equatorially trapped waves, *J. Geophys. Res.* **117** (2012), C05029.

[4] A. Constantin, Some three-dimensional nonlinear Equatorial flows, *J. Phys. Oceanogr.* **43** (2013), 165–175.

[5] A. Constantin, Some nonlinear, Equatorially trapped, nonhydrostatic internal geophysical waves, *J. Phys. Oceanogr.* **44** (2014), 781–789.

[6] A. Constantin and P. Germain, Instability of some Equatorially trapped waves, *J. Geophys. Res. Oceans* **118** (2013), 2802–2810.

[7] A. Constantin and R. S. Johnson, The dynamics of waves interacting with the Equatorial Undercurrent, *Geophys. Astrophys. Fluid Dyn.* **109** (2015), 311–358.

[8] A. Constantin and R. S. Johnson, An exact, steady, purely azimuthal equatorial flow with a free surface, *J. Phys. Oceanogr.* **46**(2016), 1935–1945.

[9] A. Constantin and R. S. Johnson, Current and future prospects for the application of systematic theoretical methods to the study of problems in physical oceanography, *Phys. Lett. A* **380** (2016), 3007–3012.

[10] A. Constantin and R. S. Johnson, A nonlinear, three-dimensional model for ocean flows, motivated by some observations of the Pacific Equatorial Undercurrent and thermocline, preprint.

[11] B. Cushman-Roisin and J.-M. Beckers, *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects*, Academic, Waltham, Mass., 2011.

[12] A. V. Fedorov and J. N. Brown, Equatorial waves, in *Encyclopedia of Ocean Sciences*, edited by J. Steele, pp. 3679–3695, Academic, San Diego, Calif., 2009.

[13] F. Genoud and D. Henry, Instability of equatorial water waves with an underlying current, *J. Math. Fluid Mech.* **16** (2014), 661–667.

[14] A. Gill, *Atmosphere-ocean dynamics*, Academic Press, New York, 1982.

[15] D. Henry, An exact solution for equatorial geophysical water waves with an underlying current, *Eur. J. Mech. B Fluids* **38** (2013), 18–21.

[16] D. Henry, Equatorially trapped nonlinear water waves in a $\beta-$plane approximation with centripetal forces, *J. Fluid Mech.* **804** (2016), R1.

[17] D. Henry and H.-C. Hsu, Instability of internal equatorial water waves, *J. Diff. Eq.* **258** (2015), 1015–1024.

[18] D. Henry and S. Sastre-Gómez, Mean flow velocities and mass transport for Equatorially-trapped water waves with an underlying current, *J. Math. Fluid Mechanics* **18** (2016), 795–804.

[19] H.-C. Hsu, Some nonlinear internal Equatorial flows, *Nonlinear Analysis: Real World Applications*, **18** (2014), 69-74.

[20] D. Ionescu-Kruse, An exact solution for geophysical edge waves in the $\beta-$plane approximation, *J. Math. Fluid Mech.* **17** (2015), 699–706.

[21] D. Ionescu-Kruse, Instability of equatorially trapped waves in stratified water, *Ann. Mat. Pura Appl.* **195** (2016), 585–599.

[22] T. Izumo, The equatorial current, meridional overturning circulation, and their roles in mass and heat exchanges during the El Niño events in the tropical Pacific Ocean, *Ocean Dyn.*, **55** (2005), 110–123.

[23] G. C. Johnson, M. J. McPhaden and E. Firing, Equatorial Pacific ocean horizontal velocity, divergence, and upwelling, *J. Phys. Oceanogr.* **31** (2001), 839–849.

[24] W. Krawcewicz and J. Wu, *Theory of Degrees with Applications to Bifurcations and Differential Equations*, Wiley-Interscience Publ., Inc., New York, 1997.

[25] A. V. Matioc, An exact solution for geophysical equatorial edge waves over a sloping beach, *J. Phys. A* **45** 365501 (2012).

[26] E. Mollo-Christensen, Gravitational and Geostrophic Billows: Some Exact Solutions, *J. Atmos. Sci.* **35** (1978), 1395–1398.

[27] E. H. Rothe, *Introduction to Various Aspects of Degree Theory in Banach Spaces*, American Mathematical Society, Providence, Rhode Island, 1986.

[28] G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics*, Cambridge University Press, 2006.