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University College Cork, Ireland
Coláiste na hOllscoile Corcaigh

Large amplitude steady periodic waves for fixed-depth rotational flows

David Henry
Faculty of Mathematics, University of Vienna, Nordbergstraße 15
1090 Vienna, Austria
david.henry@univie.ac.at

Abstract

We consider steady periodic water waves with vorticity which propagate over a flat bed with a specified fixed mean-depth $d > 0$. Following a novel reformulation of the governing equations, we use global bifurcation theory to establish a global continuum of solutions throughout which the mean-depth is a fixed quantity. Furthermore, we establish the limiting behaviour of solutions in this continuum, which include the existence of weak stagnation points, that are characteristic of large-amplitude steady periodic water waves.

Keywords: Global bifurcation, stagnation points, steady periodic waves, vorticity, fixed-depth flows.
AMS Subject Classification (2000): 35Q31, 35J25.

1 Introduction

In the following paper we rigorously prove the existence of large-amplitude steady periodic water waves, which propagate over a flat bed with a specified fixed mean-depth $d > 0$, and which have a general vorticity distribution. The distinguishing feature of this paper is that we maintain the fixedness of the mean-depth d throughout the continuum of solutions. Recently, following the seminal paper of Constantin and Strauss [13], the existence of large amplitude water waves with vorticity was proven to exist for discontinuous vorticity distributions, stratification, and surface tension, cf. [16, 50, 51]. In all of these works the mass-flux p_0, defined in (5) below, is a fixed constant throughout the global bifurcation procedure, and accordingly no *a priori* control or determination of the mean-depth d is possible for the resulting global continuum of solutions which are proven to exist. Indeed, recent numerical investigations in [39, 40] show that the mean-depth d varies throughout the continuum of solutions [13] for fixed mass-flux p_0. However, both intuitively and physically, it is more natural, and indeed desirable, to have solutions for which the mean-depth d is fixed, rather than having the mass-flux p_0 fixed. In the current paper we address this issue, and gain control on the mean-depth d by employing a novel reformulation of the governing equations which originated in [31]. The reformulation is achieved by performing a semi-hodograph change of variable, and introducing a modified-height function— a procedure which, although redolent of the standard Dubreil-Jacotin [20] transform employed in [13, 16, 50, 51], now allows us to control the mean-depth d once we take care of some additional mathematical complications.

The question of establishing the existence of solutions to the water wave problem has occupied the minds of researchers for centuries, presumably since the field of hydrodynamics was rigorously established in the 18^{th} century. Nevertheless, due to the intractability of the governing equations for water waves, there are only a handful of explicit solutions which are known to exist. For deep-water gravity waves, in 1802 Gerstner [24] found an explicit solution in the Lagrangian formulation of the full water wave equations (which was later independently discovered by Rankine). The resulting wave is a periodic travelling wave with a specific vorticity distribution (see [5, 7, 28] for a modern treatment of Gerstner's wave), and although the prescription of the flow is quite specific and rigid, remarkably this flow has been recently adapted to describe a wide-variety of interesting, and physically varied, water waves (cf. [7, 6, 8, 47]).

In spite of the fact that Gerstner's wave, which is one of the few explicit water wave solutions known, is intrinsically rotational, most of the rigorous analytical work concerning the existence of water waves had, at least until recently, focused on the irrotational case (see [52] for a survey of this work). The reason for this, no doubt, is that the presence of vorticity adds significant intrinsic mathematical complications to the problem. While irrotational flows may be regarded as being suitable for modelling waves which enter a body of still water [35, 42], more physically complicated and realistic flows generally possess vorticity, for example flows which model wave-current interactions [36, 48] or flows generated by wind-shear [38]. In 1934 Dubreil-Jacotin [20] used power series to show the existence of small-amplitude waves with vorticity, however a rigorous proof of the existence of large amplitude waves proved elusive until the breakthrough paper of Constantin and Strauss [13] in 2004 (a noteworthy first approach to this question, using numerical simulations, is given in [19]). This breakthrough was followed by a wide body of work on flows with vorticity, establishing such properties as stability of solutions [15], the symmetry of solutions [9, 10, 11], the analyticity of the surface profile and the streamlines for waves with vorticity [12, 30, 29, 32, 33, 44, 45] and extending the proof of the existence of small-amplitude solutions to more general rotational flows, such as flows which experience surface tension, have critical layers or stagnation points, or flows with discontinuous vorticity [4, 14, 16, 17, 22, 34, 43, 49, 51].

The aim of this paper is to rigorously prove the existence of large-amplitude steady periodic water waves, which propagate over a flat bed with a specified fixed mean-depth $d > 0$, and which have a general vorticity distribution which satisfies (16). In doing so, we extend the work of the author in [31], where the existence of small-amplitude waves with fixed-depth and vorticity is proven using local bifurcation theory. We will extend the local bifurcation curve of [31] to a global continuum of solutions using the global bifurcation theory of Rabinowitz [46], which we implement with the generalised Leray-Schauder degree developed by Healey and Simpson [27], which is applicable to operators which are not compact perturbations of the identity, and which have nonlinear boundary conditions. In so doing, we prove the existence of large-amplitude waves, as detailed in the main result of this paper, which we state as follows. Here the functions (u, v, η) represent the horizontal speed, the vertical speed, and the free-surface of the flow respectively, and $C^{k,\alpha}_{per}$ represents the usual Hölder spaces, with the subscript *per* indicating functions which exhibit periodicity and evenness in the $q-$variable.

Theorem 1.1. *Let the wave speed $c > 0$, the wavelength L and the fixed mean-depth $d > 0$ be given. For $\alpha \in (0, 1)$ let the vorticity function $\gamma \in C^{1,\alpha}(-1, 0)$ satisfy* (16). *Then there exists a continuum $\mathcal{S}^0$ of solutions $(u, v, \eta) \in C^{2,\alpha}_{per}(\overline{D}_\eta) \times C^{2,\alpha}_{per}(\overline{D}_\eta) \times C^{3,\alpha}_{per}(\mathbb{R})$ of the water wave equations* (3) *such that*

(i) u, v, η have period L in the $x-$variable;

(ii) within each period the (non-flat) wave profile η has a single maximum and minimum;

(iii) if $x = 0$ is the location of the wave crest, then u, η are symmetric, while v is anti-symmetric about the line $x = 0$, with $v(x, \cdot) > 0$ for $x \in (0, \pi), y > -d$;

(iv) the wave profile is strictly decreasing from crest to trough;

(v) the flow beneath each wave has no stagnation points, that is, $u < c$ throughout the fluid.

Furthermore, the continuum $\mathcal{S}^0$ contains a single laminar flow (where all streamlines, including the wave surface, are flat horizontal lines), and there is a sequence (u_n, v_n, η_n) for which

$$\textit{either} \quad \lim_{n\to\infty} \left\{ \max_{\overline{D_{\eta_n}}} u_n \right\} = c \quad \textit{or} \quad \lim_{n\to\infty} \left\{ \min_{\overline{D_{\eta_n}}} u_n \right\} = -\infty. \tag{1}$$

We remark that if the mass flux is bounded throughout $\mathcal{S}^0$, then in fact the first limiting behaviour in (1) must occur, implying the existence of a sequence of solutions which possess, in the limit, a weak stagnation point. This behaviour is precluded for the small-amplitude waves which were proven to exist in [31], and it is in this sense that the resulting waves are of large-amplitude. In the following section we outline some results, and notation, from the paper [31] which are relevant here, while Theorem 1.1 is then proven in Section 3.

2 Preliminaries

We consider steady periodic travelling surface waves propagating over water of depth $d > 0$, where d is fixed, and where the external restoration force is gravity. If we fix the undisturbed water surface to be located at $y = 0$, then this fixes the location of the flat bed to be $y = -d$, and furthermore if $\eta(x, t)$ represents the free surface of the wave for any fixed time t, then we must have the mean of η equal to zero, that is

$$\int \eta(x, t)dx = 0, \tag{2}$$

where we integrate over an interval the size of a wavelength. The free surface η is a priori undetermined and is therefore an unknown in the problem. We assume that the steady travelling waves move with a constant wavespeed $c > 0$, and so all functions have a x, t relationship of the form $x - ct$, and we transform to a new reference frame moving alongside the wave by using the change of coordinates $(x - ct, y) \mapsto (x, y)$. In this frame the flow is time independent. Let us denote the closure of the fluid domain by $\overline{D_\eta} = \{(x, y) \in \mathbb{R}^2 : -d \leq y \leq \eta(x)\}$. The governing equations for the motion of the perfect (inviscid and incompressible) fluid take the form of Euler's equation, together with boundary conditions, as given by

$$
\begin{aligned}
u_x + v_y &= 0, && \text{in} \quad D_\eta && \text{(3a)}\\
(u-c)u_x + vu_y &= -P_x, && \text{in} \quad D_\eta && \text{(3b)}\\
(u-c)v_x + vv_y &= -P_y - g, && \text{in} \quad D_\eta && \text{(3c)}\\
v &= (u-c)\eta_x && \text{on } y = \eta(x), && \text{(3d)}\\
P &= P_{atm} && \text{on } y = \eta(x), && \text{(3e)}\\
v &= 0 && \text{on } y = -d, && \text{(3f)}
\end{aligned}
$$

where $P(x, y)$ is the pressure distribution function, P_{atm} is the constant atmospheric pressure and g is the gravitational constant of acceleration. For two-dimensional motion, the vorticity is given by

$$\omega = u_y - v_x. \tag{3g}$$

We now make the additional assumption that there are no weak-stagnation points, that is,

$$u < c \tag{3h}$$

throughout the fluid. This is a physically reasonable assumption for water waves, without underlying currents containing strong non-uniformities, and which are not near breaking [42]. The non-stagnation condition is also essential mathematically in our reformulations of the water wave equations, as we see below. We work with periodic waves, and we choose the period to be 2π without loss of generality. For suppose we are dealing with water waves of wavelength L in the governing equations (3a)–(3f), then after performing the following scaling of variables

$$(x, y, t, g, \omega, \eta, u, v, P, c) \mapsto (\kappa x, \kappa y, \kappa t, \kappa^{-1} g, \kappa^{-1}\omega, \kappa\eta, u, v, P, c)$$

where $\kappa = \frac{2\pi}{L}$ is the wavenumber, we end up with a $2\pi-$periodic system in the new variables identical to (3) except g, ω are replaced by $\kappa^{-1}g, \kappa^{-1}\omega$. We define the stream function ψ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v, \tag{4}$$

and we fix the constant by setting $\psi = 0$ on $y = \eta(x)$. Relations (3d) and (3f) tell us that ψ is constant on both boundaries of D_η, and so it follows from integrating (4) and using (3h) that $\psi = -p_0$ on $y = -d$, where

$$p_0 = \int_{-d}^{\eta(x)} (u(x, y) - c)dy < 0 \tag{5}$$

is the relative mass flux. For a given solution (u, v, η) of the water wave problem (3) the mass-flux is constant, since, from (3a) and (3d) we get

$$\frac{dp_0}{dx} = (u(x, \eta(x)) - c)\eta'(x) - v(x, \eta(x)) = 0.$$

However, the value of this constant may vary for different solutions of the system (3), since in this paper we do not force p_0 to be fixed along the continuum of solutions. Regarding how p_0 may vary for different solutions, let $(u_j, v_j, \eta_j), j = 1, 2$ be two solutions of the water wave problem (3), then we have

$$p_{0_1} - p_{0_2} = \int_{-d}^{\eta_1(x)} (u_1(x, y) - u_2(x, y))\, dy - \int_{\eta_1(x)}^{\eta_2(x)} (u_2(x, y) - c)dy \tag{6}$$

and so the values of the mass-flux p_{0_j} will converge if the solutions (u_j, v_j, η_j) converge in the L^∞ norm. The following inequalities will be useful:

$$\inf_{-d \le y \le \eta(0)} (c - u(0, y)) \cdot d \le |p_0| \le \sup_{-d \le y \le \eta(\pi)} (c - u(\pi, y)) \cdot d, \tag{7}$$

since $\eta(\pi) < 0 < \eta(0)$. We can reformulate the governing equations in the moving frame in terms of the stream function [7, 31] as follows:

$$\begin{aligned}
\Delta\psi &= \omega && \text{in} \quad -d < y < \eta(x), && \text{(8a)}\\
|\nabla\psi|^2 + 2g(y + d) &= Q && \text{on} \quad y = \eta(x), && \text{(8b)}\\
\psi &= 0 && \text{on} \quad y = \eta(x), && \text{(8c)}\\
\psi &= -p_0 && \text{on} \quad y = -d. && \text{(8d)}
\end{aligned}$$

Here Q, the hydraulic head, is a constant of motion for each flow, and it will play a significant role in our global bifurcation analysis. The next step in reworking the governing equations is to transform the fluid domain D_η, with the unknown free boundary η, into the fixed semi-infinite rectangular strip $\overline{R} = \mathbb{R} \times [-1, 0]$. We achieve this by applying the semi-Lagrangian hodograph transformation, which was first introduced in [31], defined by

$$(x, y) \mapsto (q, p) := (x, \psi(x, y)/p_0). \tag{9}$$

We can see clearly now that the non-stagnation condition (3h) is vital in order to ensure that the change of variables (9) represents an isomorphism. In the following it will be useful to represent the top and the bottom of the closed rectangle $\overline{R}$ by

$$T = \{(q, p) : q \in [-\pi, \pi], p = 0\}, \quad B = \{(q, p) : q \in [-\pi, \pi], p = p_0\}.$$

We remark that the transformation (9) has a singular advantage over the standard Dubreil-Jacotin transformation, which featured in [13], insofar as it will allow us to reexpress the system (8) in the form (12) below which enables us to fix concretely the value of the depth $d > 0$ when we engage in bifurcation theory in later sections. The reformulation is achieved by the following procedure. Firstly, it can be easily shown [31] that $\omega_q = 0$, and so the vorticity is a function of p alone: $\omega = \gamma(p)$, where γ will be referred to as the vorticity function [31]. Following the transformation (9) we can reformulate the governing equations (8) in terms of the modified-height function,

$$h(q,p) = \frac{y}{d} - p, \tag{10}$$

where $h(q,-1) = 0$ for all $q \in (-\pi,\pi)$, the condition (31a) takes the form

$$\int_{-\pi}^{\pi} h(q,0)dq = 0,$$

and

$$h_p + 1 = \frac{p_0}{d(u-c)}, \quad h_q = \frac{v}{d(u-c)}. \tag{11}$$

The reformulation of (8) takes the form [31]:

$$\left(\frac{1}{d^2} + h_q^2\right) h_{pp} - 2h_q(h_p+1)h_{pq} + (h_p+1)^2 h_{qq} + \frac{\gamma(p)}{p_0}(h_p+1)^3 = 0,$$
$$\text{in } -1 < p < 0, \tag{12a}$$

$$\frac{1}{d^2} + h_q^2 + \frac{(h_p+1)^2}{p_0^2}[2gd(h+1) - Q] = 0, \quad p = 0, \tag{12b}$$

$$h = 0, \quad p = -1, \tag{12c}$$

where the mass-flux $p_0 = p_0^{(h)}$ is constant for each solution h, the depth $d > 0$ is a constant which is fixed throughout, and the non-stagnation condition (3h) is equivalent (by (11)) to

$$h_p + 1 > 0. \tag{12d}$$

It was shown in [31] that the systems (3), (8) and (12) are equivalent to each other, and a solution of (12) is given by $h \in C_{per}^{3,\alpha}(R)$. We note that (6) and (11) imply that the value of the mass-flux $p_0^{(h_j)}$ converges to $p_0^{(h)}$ if the sequence $\{h_j\}$ converges to h in the $C_{per}^1(R)$ norm. In the following, for notational convenience, unless the precise value of the mass-flux has a direct impact on our considerations, we will suppress the superscript and simply write p_0.

2.1 Existence of nonlinear waves of small amplitude

We now outline briefly how the existence of small-amplitude wave solutions to system (12), for general classes of vorticity functions, is rigorously proven in [31] using local bifurcation theory. The trivial laminar flow solutions $H(p)$ of the modified-height system (12) (that is, solutions which have no $q-$dependence and where the streamlines of the resulting flow are horizontal) take the form

$$H(p) = \int_0^p \frac{ds}{\sqrt{\lambda + \Gamma(s)}} + \frac{1}{2gd}\left[Q - \frac{p_0^2}{d^2}\lambda\right] - (p+1), \quad -1 < p \leq 0,$$

where

$$\Gamma(p) = 2\frac{d^2}{p_0}\int_0^p \gamma(s)ds, \quad -1 \leq p \leq 0,$$

and

$$\lambda = \frac{1}{(1+H_p)^2}\bigg|_{p=0} = \frac{d^2(u-c)^2}{p_0^2}\bigg|_{on\ the\ flat\ surface},$$

with

$$\Gamma_{\min} = \min_{p\in[-1,0]} \Gamma(p) \leq 0, \tag{13}$$

and $\lambda > -\Gamma_{\min}$. In [31] we apply the Crandall-Rabinowitz local bifurcation theorem [18] to prove, for a quite general class of vorticity functions, the existence of small amplitude solutions to (12) of the form

$$h(q,p) = H(p;\lambda) + \epsilon m(q,p),$$

where $0 \not\equiv m \in C^{3,\alpha}_{per}(\overline{R})$. These small-amplitude nontrivial solutions to (12) are in the form of a localised curve of non-laminar solutions bifurcating from the curve of trivial laminar solutions. The existence of this bifurcating curve is intrinsically tied to the question of the existence of a function $m(q,p) = m(p)\cos(q)$, where $m(p)$ satisfies the weighted Sturm-Liouville problem

$$(a^3 m_p)_p = d^2 a m, \quad -1 < p < 0, \tag{14a}$$

$$a^3 m_p = \frac{gd^3}{p_0^2} m, \qquad p = 0, \tag{14b}$$

$$m = 0 \qquad p = -1. \tag{14c}$$

We associate to (14) the minimisation problem:

$$\mu(\lambda) = \inf_{\phi\in H^1(-1,0),\ \phi(-1)=0,\ \phi\not\equiv 0} \mathbb{F}(\phi,\lambda), \tag{15}$$

$$\text{with}\ \ \mathbb{F}(\phi,\lambda) = \frac{-gd^3\phi^2(0) + p_0^2\int_{-1}^0 a^3\phi_p^2 dp}{p_0^2 d^2 \int_{-1}^0 a\phi^2 dp},$$

where the Hilbert space $H^1(-1,0)$ is the standard Sobolev space of square summable functions on $[-1,0]$ whose first derivative is also square summable [21]. In [31] we show that a solution $m(p)$ of (14) exists precisely when there is a critical value λ^* for which $\mu(\lambda^*) = -1$ in (15). Furthermore, such a groundstate $m(p)$, which we normalise to have $m(0) = 1$, is unique, for suppose the difference of the two is denoted $M(p)$. Then $M(p)$ satisfies the system (14), with $M(0) = M_p(0) = 0$, and by the uniqueness property of solutions to linear second order differential equations [26] we must have $M(p) \equiv 0$. The following lemma establishes a property of the groundstate $m(p)$ which will be used in later discussions.

Lemma 2.1. *Let $m(p)$ be the unique solution of* (14)*, which has $m(0) = 1$, then $m(p) > 0$ for $p \in (-1,0)$.*

Proof. Suppose $m(p^*) = 0$ for some $p^* \in (-1,0)$. Then multiplying (14a) by m and integrating on $[p^*,0]$ we get

$$\frac{gd^3}{p_0^2} m^2(0) = \int_{p^*}^{0} a^3 m_p^2 dp + \int_{p^*}^{0} d^2 a m^2 dp.$$

Therefore we can see that

$$m_1 = \begin{cases} m(p) & p \in [p^*,0], \\ 0 & p \in [-1,p^*], \end{cases}$$

satisfies (15), and so also represents a groundstate of (14). By the uniqueness properties of the groundstate [31] this implies that $m(p) \equiv m_1(p)$, and in particular $m(-1) = m_p(-1) = 0$. However, by uniqueness properties of solutions [26] to the linear differential equation (14), this implies that $m(p) \equiv 0$, which is a contradiction. The boundary condition at $p = 0$ then implies that $m(p) > 0$ for $p \in (-1,0)$. □

In the course of proving the existence of small-amplitude water waves in [31] we obtain the following result, which provides us with a condition which is sufficient for local bifurcation to occur.

Proposition 2.2 ([31])**.** *Suppose that*

$$\frac{\sqrt{2}}{3}\gamma_\infty^{\frac{3}{2}}|p_0|^{\frac{1}{2}}|p_1|^{\frac{1}{2}} + \frac{2\sqrt{2}}{5}\gamma_\infty^{\frac{1}{2}}|p_0|^{\frac{3}{2}}|p_1|^{\frac{3}{2}} < g, \tag{16}$$

where $\gamma_\infty = \|\gamma\|_{C[-1,0]}$ and $p_1 = \min\{p \in [-1,0] : \Gamma(p) = \Gamma_{min}\}$, where $\Gamma_{\min}$ is defined in (13)*. Then there exist non-trivial solutions to the linearised problem* (14)*.*

Remark We note that $p_1 = 0$ for $\gamma \geq 0$ and so in this case we easily see that (16) holds.

For vorticity functions γ which satisfy condition (16), Proposition (2.2) proves the existence of critical values λ^* such that $\mu(\lambda^*) = -1$, thereby proving the existence of solutions to the linearisation (14) of the water wave problem (12). In turn, in [31], using the local bifurcation theory of Crandall-Rabinowitz we prove that there exists some $\epsilon_0 > 0$ such that there is a local bifurcation curve $\mathcal{C} = \{(\lambda(s), h^s) \in \mathbb{R} \times X : |s| < \epsilon_0\}$ of nontrivial solutions to the full water wave system (12). We remark that the considerations of [31] prove also that this critical value λ^* is, in fact, unique.

For local bifucation λ is the appropriate bifurcation parameter, however in the global bifurcation setting it is more fruitful to take Q as our parameter. This is a valid choice since

$$Q(\lambda) = 2gd \int_{-1}^{0} \frac{ds}{\sqrt{\lambda + \Gamma(s)}} + \frac{p_0^2}{d^2}\lambda > 0, \tag{17}$$

it follows that Q is a positive, convex function of λ, with minimum occurring at the unique value $\lambda_0 > 0$ where

$$\frac{p_0^2}{gd^3} = \int_{-1}^{0} \frac{ds}{(\lambda_0 + \Gamma(s))^{\frac{3}{2}}}.$$

Therefore $Q(\lambda)$ is monotonically decreasing for $-\Gamma_{\min} < \lambda < \lambda_0$ and monotonically increasing for $\lambda > \lambda_0$, and therefore there is a suitable bijective correspondence between values of λ and $Q(\lambda)$ near the critcal point λ^*. If we define the Banach spaces

$$X = \left\{ h \in C^{3,\alpha}_{per}(\overline{R}) : \int_T h(q,0)dq = 0, h = 0 \text{ on } B \right\}, \quad Y = C^{1,\alpha}_{per}(\overline{R}) \times C^{2,\alpha}_{per}(T),$$

then, restricting the bifurcating curve if necessary, local bifurcation theory [31] ensures that there is a local curve of solutions $\mathcal{C}_{loc} = \{(Q(s), h^s) \in \mathbb{R} \times X : |s| < \epsilon_0\}$ with $(Q(0), h^0)$ being the laminar flow solution (Q^*, H^*). In the following we denote

$$\mathcal{C}^+_{loc} = \{(Q(s), h^s) \in \mathbb{R} \times X : 0 \leq s < \epsilon_0\}, \tag{18a}$$
$$\mathcal{C}^-_{loc} = \{(Q(s), h^s) \in \mathbb{R} \times X : -\epsilon_0 < s \leq 0\}. \tag{18b}$$

The small-amplitude wave solutions of (12) take the form [31]

$$h^s(q,p) = H(p; Q(s)) + sm(p)\cos(q), \tag{19}$$

and so we see that the solutions of (12) which are located on $\mathcal{C}^+_{loc}$ correspond to waves with their crest located at $x = 0$, while those located on $\mathcal{C}^-_{loc}$ have their troughs located at $x = 0$.

3 Global bifurcation theory

The water wave system (12) can be represented in terms of the operator $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) : \mathbb{R} \times X \to Y = Y_1 \times Y_2$, where

$$\mathcal{G}_1(Q,h) = \left(\frac{1}{d^2} + h_q^2\right) h_{pp} - 2h_q(h_p+1)h_{pq} + (h_p+1)^2 h_{qq} + \frac{\gamma(p)}{p_0}(h_p+1)^3, \tag{20a}$$

$$\mathcal{G}_2(Q,h) = \frac{1}{d^2} + h_q^2 + \frac{(h_p+1)^2}{p_0^2}[2gd(h+1) - Q]. \tag{20b}$$

For $0 < \delta < 1$ define the set $\mathcal{O}_\delta \subset \mathbb{R} \times X$ by

$$\mathcal{O}_\delta = \left\{ (Q,h) \in \mathbb{R} \times X : h_p + 1 > \delta \text{ in } \overline{R}, h + 1 < \frac{Q-\delta}{2gd} \text{ on } T, \delta < |p_0| < \frac{1}{\delta} \right\}.$$

By its definition, and following from (12d), the operator $h \mapsto \mathcal{G}_1(Q,h)$ is uniformly elliptic on $\mathcal{O}_\delta$ and the boundary operator $\mathcal{G}_2(Q,h)$ remains uniformly oblique (that is, it has a non-tangential component on T). Define $\mathcal{S}_\delta^0$ to be the connected component of $\overline{\{(Q,h) \in \mathcal{O}_\delta : \mathcal{G}(Q,h) = 0, h_q \not\equiv 0\}} \subset \mathbb{R} \times X$ which contains (Q^*, H^*). Hence $\mathcal{S}_\delta^0$ contains the local bifurcation curve $\mathcal{C}_{loc}$ for $\delta > 0$ small enough. Our first goal in this section is to prove the following.

Theorem 3.1 (Global Bifurcation Theorem). *Let $\delta > 0$ be small enough. Then one of the following alternatives holds:*

1. *$\mathcal{S}_\delta^0$ is unbounded in $\mathbb{R} \times X$.*
2. *$\mathcal{S}_\delta^0$ contains another trivial point $(Q(\lambda), H(\lambda))$ with $\lambda \neq \lambda^*$.*
3. *$\mathcal{S}_\delta^0$ contains a point $(Q,h) \in \partial\mathcal{O}_\delta$.*

Furthermore, there exists a continuous curve $\mathcal{K}_\delta$ in $\mathbb{R} \times X$ for which $\mathcal{C}_{loc}^+ \subset \mathcal{K}_\delta \subset \mathcal{S}_\delta^0$ and at each point $\mathcal{K}_\delta$ has a locally analytic reparametrization. Similar to the alternatives above, either $\mathcal{K}_\delta$ is unbounded in $\mathbb{R} \times X$, or it contains a point $(Q,h) \in \partial\mathcal{O}_\delta$, or else $\mathcal{K}_\delta$ is a closed loop.

The proof of the above theorem follows exactly along the lines of the global bifurcation theorem of Rabinowitz [46]. However Rabinowitz's theory, which utilises the Leray-Schauder degree, applies only to operators which are compact perturbations of the identity, and the water wave problem formulation (20) does not belong to this category of operators. We must instead employ a variant of this degree which is applicable to certain nonlinear operators that we say are "admissible". This generalised Leray-Schauder degree was initially developed by Kielhöfer [37] for linear boundary conditions, and then further extended by Healey and Simpson [27] to allow for nonlinear boundary conditions. The first implementation of this generalised degree in the analysis of water waves with vorticity was in the breakthrough paper of Constantin and Strauss [13] where they proved the existence of large-amplitude water waves with vorticity which have a fixed mass-flux. In short, a nonlinear operator is an admissible mapping if it satisfies the criteria of Lemmas 3.2, 3.3 and 3.4 in the next section. Given an admissible operator, it can be shown, exactly as in Rabinowitz's global bifurcation theory except using the generalised degree of Healey-Simpson, the first two alternatives in Theorem 3.1 must hold. The third alternative in Theorem 3.1 arises from the possibility that the continuum of solutions $\mathcal{S}_\delta^0$ touches the boundary of $\mathcal{O}$, in which case either the uniform ellipticity of the operator (20a), or the uniform obliqueness of the boundary operator (20b), break down, or the mass-flux vanishes or blows-up. The existence of the real-analytic curve $\mathcal{K}_\delta$ follows from real-analytic global bifurcation theory [3].

The layout of the remainder of this paper will be structured as follows. In the next section we prove that the operator (20) is admissible in the sense of Healey and Simpson, from which Theorem (3.1) follows. Following this, we will establish nodal properties which are intrinsic to all solutions in the continuum $\mathcal{S}_\delta^0$ and the curve $\mathcal{K}_\delta$, and rule out the second alternative in Theorem 3.1. The nodal properties inherited by the solutions lead to the proof of statements (*iii*) and (*iv*) of Theorem 1.1. In the final section we prove that the remaining alternatives in Theorem 3.1 lead to the limiting behaviour of the solutions in $\mathcal{S}_\delta^0$ and $\mathcal{K}_\delta$ as outlined in Theorem 1.1.

3.1 On the admissibility of $\mathcal{G}$

In the following three lemmas we establish the admissibility of the operator (20), in the sense of [27], thereby proving Theorem 3.1.

Lemma 3.2 (Proper map)**.** *If $K \subset Y$ is compact and D is a closed bounded set in $\overline{\mathcal{O}_\delta}$, then $\mathcal{G}^{-1}(K) \cup D$ is compact in $\mathbb{R} \times X$.*

Proof. Let $\{(f_j, g_j)\}$ be a convergent sequence in $Y = Y_1 \times Y_2$ with $\mathcal{G}(Q_j, h_j) = (f_j, g_j)$, that is, $\mathcal{G}_1(Q_j, h_j) = f_j, \mathcal{G}_2 = (Q_j, h_j) = g_j$, for $(Q_j, h_j) \in \overline{\mathcal{O}_\delta}$ such that $\{h_j\}$ is bounded in X and $\{Q_j\}$ is bounded in $\mathbb{R}$. We want to prove that there exists a convergent subsequence of $\{(Q_j, h_j)\}$ in $\mathbb{R} \times X$. For $\theta_j = \partial_q h_j$

we have

$$\left(\frac{1}{d^2}+(\partial_q h_j)^2\right)\partial_p^2\theta_j-2(\partial_q h_j)(\partial_p h_j+1)\partial_{pq}^2\theta_j+((\partial_p h_j)+1)^2\partial_q^2\theta_j$$

$$= F_j+\partial_q f_j \quad \text{in } R, \tag{21a}$$

$$(\partial_q h_j)\partial_q\theta_j+[2gd(h_j+1)-Q]\frac{((\partial_p h_j)+1)}{p_0^2}\partial_p\theta_j = G_j+\frac{1}{2}\partial_q g_j \quad \text{on } p=0, \tag{21b}$$

$$\theta_j = 0 \quad \text{on } p=-1. \tag{21c}$$

where $F_j(\partial_q h,\partial_p h,\partial_q^2 h,\partial_{qp}^2 h,\partial_p^2 h)$ is a cubic polynomial and $G_j(\partial_q h,\partial_p h) = -((\partial_p h_j)+1)^2 gd(\partial_q h_j)/p_0^2$. Now, the sequence $\{F_j(\partial_q h,\partial_p h,\partial_q^2 h,\partial_{qp}^2 h,\partial_p^2 h)\}$ is uniformly bounded in $C^{1,\alpha}_{per}(\overline{R})$, while the sequence $\{f_j\}$ converges in $C^{0,\alpha}_{per}(\overline{R})$, therefore the right-hand side of (21a) is compact in $C^{0,\alpha}_{per}(\overline{R})$. The right-hand side of (21b) is similarly compact in $C^{1,\alpha}_{per}(T)$. By choosing subsequences and relabelling, let us assume that these sequences are convergent in $C^{0,\alpha}_{per}(\overline{R}), C^{1,\alpha}_{per}(T)$ respectively. Taking differences we get from (21a)–(21c) that

$$\left(\frac{1}{d^2}+(\partial_q h_j)^2\right)\partial_p^2(\theta_j-\theta_k)-2(\partial_q h_j)(\partial_p h_j+1)\partial_{pq}^2(\theta_j-\theta_k)$$

$$+((\partial_p h_j)+1)^2\partial_q^2(\theta_j-\theta_k) = F_{jk} \text{ in } \ R,$$

$$(\partial_q h_j)\partial_q(\theta_j-\theta_k)+[2gd(h_j+1)-Q]\frac{((\partial_p h_j)+1)}{p_0^2}\partial_p(\theta_j-\theta_k) = G_{jk} \text{ on } \ T,$$

$$\theta_j-\theta_k = 0 \text{ on } \ B,$$

where

$$F_{jk} := F_j-F_k+f_j-f_k-[((\partial_p h_j)+1)^2-((\partial_p h_k)+1)^2]\partial_q^2\theta_k$$
$$+2[(\partial_q h_j)(\partial_p h_j+1)-(\partial_q h_k)(\partial_p h_k+1)]\partial_{pq}^2\theta_k-\left((\partial_q h_j)^2-(\partial_q h_k)^2\right)\partial_p^2\theta_k,$$
$$G_{jk} := G_j-G_k+\frac{1}{2}\partial_q g_j-\frac{1}{2}\partial_q g_k+(\partial_q h_j-\partial_q h_k)\partial_q\theta_k$$
$$-\left([2gd(h_j+1)-Q]\frac{((\partial_p h_j)+1)}{p_0^2}-[2gd(h_k+1)-Q]\frac{((\partial_p h_k)+1)}{p_0^2}\right)\partial_p\theta_k,$$

and it follows from the compactness considerations discussed above that $\|F_{jk}\|_{C^{0,\alpha}_{per}(\overline{R})}, \|G_{jk}\|_{C^{1,\alpha}_{per}(T)} \to 0$ as $j,k\to\infty$.

Since the system is periodic in the q variable, this allows us to bypass the apparent nonsmoothness of the boundary of R. We now apply Schauder estimates for $\theta_j - \theta_k$, applying the estimates for homogeneous Dirchlet boundary conditions [25, Thm. 6.6] and oblique boundary conditions [25, Thm. 6.30] (this approach is valid since these estimates are local, and the different boundary conditions occur on separated parts of the boundary). It follows that

$$\|\theta_j - \theta_k\|_{C^{2,\alpha}_{per}(\overline{R})} \leq C\left(\|\theta_j - \theta_k\|_{C^0_{per}(\overline{R})} + \|F_{jk}\|_{C^{0,\alpha}_{per}(\overline{R})} + \|G_{jk}\|_{C^{1,\alpha}_{per}(T)}\right), \quad (22)$$

with $C > 0$ independent of j, k. The above estimate implies the convergence

$$\|\theta_j - \theta_k\|_{C^{2,\alpha}_{per}(\overline{R})} \to 0 \quad \text{as} \quad j, k \to \infty.$$

Therefore all third order derivatives of h_j (except perhaps $\partial_p^3 h_j$) form Cauchy sequences in $C^\alpha_{per}(\overline{R})$. However, since we may express $\partial_p^3 h_j$ in terms of the other derivatives of h_j, up to third order (by differentiating the equation $\mathcal{G}_1(Q_j, h_j) = f_j$ with respect to p) we may infer that $\{\partial_p^3 h_j\}$ is also a Cauchy sequence in $C^\alpha_{per}(\overline{R})$. We have thus shown that $\{h_j\}$ has a subsequence which converges in X, and since $\mathcal{G}_2(Q_j, h_j) = g_j$ the corresponding subsequence $\{Q_j\}$ must converge in $\mathbb{R}$ also. □

Lemma 3.3 (Fredholm map)**.** *For each $(Q, h) \in \mathcal{O}_\delta$ the linearised operator $\mathcal{G}_h(Q, h)$ is a Fredholm operator of index 0 from X to Y.*

Proof. We take the Fréchet derivative of (20) to get

$$\begin{aligned}\mathcal{G}_{1h}(Q, h) = \left(\frac{1}{d^2} + h_q^2\right)\partial^2_{pp} - 2h_q(h_p + 1)\partial^2_{pq} + (h_p + 1)^2\partial^2_{qq} + 2h_q h_{pp}\partial_q \\ - 2(h_p + 1)h_{pq}\partial_q - 2h_q h_{pq}\partial_p + 2(h_p + 1)h_{qq}\partial_p + \frac{\gamma(p)}{p_0}3(h_p + 1)^2\partial_p,\end{aligned} \quad (23a)$$

$$\mathcal{G}_{2h}(Q, h) = 2h_q\partial_q + 2\frac{(h_p + 1)}{p_0^2}[2gd(h + 1) - Q]\partial_p + \frac{(h_p + 1)^2}{p_0^2}2gd. \quad (23b)$$

By the definition of $\mathcal{O}_\delta$, and from (12d), we can see that the linear operator $\mathcal{G}_{1h}$ is uniformly elliptic (in both the $(q, p)-$variables in $\overline{R}$ as well as for $(Q, h) \in \mathcal{O}_\delta$) while the linear operator $\mathcal{G}_{2h}$ is strictly oblique. For fixed $(Q, h) \in \mathcal{O}_\delta$, let $\mathcal{L}$ denote the Fréchet derivative operator $\mathcal{G}_h(Q, h)$:

$$\mathcal{L} : X_0 \to Y_0,$$

where

$$X_0 = \left\{ h \in C^{2,\alpha}_{per}(\overline{R}) : \int_T h(q,0)dq = 0, h = 0 \text{ on } B \right\},\ Y_0 = C^{0,\alpha}_{per}(\overline{R}) \times C^{1,\alpha}_{per}(T),$$

are Banach spaces. Since the outward normal $\nu = (0,1)$ on T, and $\sup_{q\in[-\pi,\pi]} 2gd(h(q,0)+1) - Q < 0$, if we choose

$$\sigma > \sup_{q\in[-\pi,\pi]} \frac{(h_p(q,0)+1)^2}{p_0^2} 2gd,$$

then the operator $\mathcal{L}_\sigma : X_0 \to Y_0$, defined by

$$\mathcal{L}_\sigma w := \mathcal{L}w - \sigma(w, w|_T),$$

satisfies the following existence theorem [25, Thm. 6.31]: for any $F = (f,g) \in Y_0$, $\mathcal{L}_\mu w = F$ has a unique solution $w_F \in X_0$. Furthermore, similar to (22), we have the Schauder estimate

$$\|w\|_{C^{2,\alpha}_{per}(\overline{R})} \le C_0 \left(\|w\|_{C^0_{per}(\overline{R})} + \|\mathcal{G}_{1h}(Q,h)w\|_{C^{0,\alpha}_{per}(\overline{R})} + \|\mathcal{G}_{2h}(Q,h)w\|_{C^{1,\alpha}_{per}(T)} \right), \tag{24}$$

where the constant $C_0 > 0$ depends only on $\|h\|_{C^{2,\alpha}_{per}(\overline{R})}$. Therefore the inverse mapping $\mathcal{L}_\mu^{-1} : F \mapsto w_F$ is bounded from Y_0 to X_0, and since $C^{2,\alpha}_{per}(\overline{R})$ is compactly embedded in both $C^{0,\alpha}_{per}(\overline{R})$ and $C^{1,\alpha}_{per}(T)$, we may regard $\mathcal{L}_\mu^{-1}$ as a compact map from Y_0 to Y_0. Given $F \in Y_0$, a function $w \in X_0$ solves $\mathcal{L}w = F$ if and only if

$$\frac{1}{\sigma} w + (\mathcal{L}_\mu)^{-1} w = \frac{1}{\sigma} (\mathcal{L}_\mu)^{-1} F.$$

Furthermore, the operator defined by $w \mapsto \frac{1}{\sigma} w + (\mathcal{L}_\mu)^{-1} w$ from Y_0 to Y_0 is Fredholm, since it is the compact perturbation of the Fredholm operator $w \mapsto \frac{1}{\sigma} w$. As a consequence, $\ker(\mathcal{L}) \subset X_0$ must be finite-dimensional and $\operatorname{ran}(\mathcal{L}) \subset Y_0$ is closed with finite co-dimension, implying that $\mathcal{G}_h(Q,h) : X_0 \to Y_0$ is a Fredholm operator.

We now show that $\mathcal{G}_h(Q,h) : X \to Y$ is a Fredholm operator of index zero, noting that the considerations of the previous paragraph imply that $\mathcal{G}_h(Q,h)$ has a finite dimensional kernel in X, and furthermore the range of $\mathcal{G}_h(Q,h)$ must have finite co-dimension in Y. In order to prove that the range is closed in Y we work as follows. Given $w \in X$, then taking the $q-$derivatives of $\mathcal{G}_h(Q,h)w$ and using the classical Schauder estimates [25, Thm. 6.30], there is a constant $C_1 > 0$, independent of w, such that

$$\begin{aligned} &\|w_q\|_{C^{2,\alpha}_{per}(\overline{R})} \\ &\le C_1 \left(\|w_q\|_{C^0_{per}(\overline{R})} + \|\partial_q \mathcal{G}_{1h}(Q,h)w\|_{C^{0,\alpha}_{per}(\overline{R})} + \|\partial_q \mathcal{G}_{2h}(Q,h)w\|_{C^{1,\alpha}_{per}(T)} \right). \end{aligned}$$

From (23a) we get the relation

$$\partial_p^3 w = d^2 \partial_p \left(\frac{\mathcal{G}_{1h}(Q,h)w + F(w_q, w_p, w_{qp}, w_{qq})}{\left(1 + d^2 h_q^2\right)} \right)$$

where F is a cubic polynomial expression, together with the Schauder estimate above, to get an estimate for $\|w_{ppp}\|_{C^{0,\alpha}_{per}(\overline{R})}$, with the end effect that we have

$$\|w\|_{C^{3,\alpha}_{per}(\overline{R})} \leq C \left(\|w\|_{C^1_{per}(\overline{R})} + \|\mathcal{G}_{1h}(Q,h)w\|_{Y_1} + \|\mathcal{G}_{2h}(Q,h)w\|_{Y_2} \right) \tag{25}$$

for all $w \in X$, and the constant C depends solely on $\|h\|_X$. The fact that the range of $\mathcal{G}_h(Q,h)$ is closed now follows, for if $\{w_n\}$ is a sequence such that $\mathcal{G}_h(Q,h)w_n = F_n \to F$ in Y, then (24) and (25) imply that $\{w_n\}$ form a Cauchy sequence in X, and clearly its limit w satisfies $\mathcal{G}_h(Q,h)w = F$. Therefore, $\mathcal{G}_h(Q,h)$ is a Fredholm operator, and since the index of a Fredholm operator is a continuous function, it is constant on the open and connected set $\mathcal{O}_\delta$, and the value must be zero since the index of $\mathcal{G}_h(Q,h)$ is zero at the local bifurcation point. $\square$

Lemma 3.4 (Spectral properties)**.**

(i) For every $M > 0$ there exist constants $c_1, c_2 > 0$ such that for all $(Q,h) \in \mathcal{O}_\delta$ with $|Q| + \|h\|_X \leq M$, we have

$$c_1 \|\psi\|_X \leq \mu^{\frac{\alpha}{2}} \|(A - \mu)\psi\|_{Y_1} + \mu^{\frac{1+\alpha}{2}} \|B\psi\|_{Y_2}$$

for all $\psi \in X$ and for all real $\mu \geq c_2$, where $A = A(Q,h) = \mathcal{G}_{1h}(Q,h)$ and $B = B(Q,h) = \mathcal{G}_{2h}(Q,h)$.

(ii) Define the spectrum

$$\Sigma(Q,h) = \{\mu \in \mathbb{C} : A - \mu \text{ not isomorphic from } X_0 = \{\psi \in X : B\psi = 0\}, \\ \text{endowed with the norm } \|\psi\|_{Y_1} + \|A\psi\|_{Y_1}, \text{ onto } Y_1\}$$

Then $\Sigma(Q,h)$ consists entirely of eigenvalues of finite multiplicity with no finite accumulation points. Furthermore, there is a neighbourhood $\mathcal{N}$ of $[0,\infty)$ in the complex plane such that $\Sigma(Q,h) \cup \mathcal{N}$ is a finite set.

(iii) For all $(Q,h) \in \mathcal{O}_\delta$, the boundary operator $\mathcal{G}_{2h}(Q,h) : X \to Y_2$ is onto.

Proof. The proof of (i) follows what the standard argument of Agmon [2] (see also [7, 27]). We introduce the independent variable $t \in (-2,2)$ to the problem, and consider the elliptic operator $A + \partial_t^2$ over the cylindrical domain $\Omega^t = R \times (-2,2) \subset \mathbb{R}^3$, with a homogeneous Dirichlet condition on the additional boundary corresponding to $t = \pm 2$. The new operator is uniformly elliptic and furthermore satisfies the complementing condition of [1] on the boundary of Ω^t (see [1, 7, 27] for a description of the complementing condition). Therefore, from [1] it follows that Schauder-type estimates similar to (24) apply to this operator on the augmented domain Ξ with top boundary $T \times [-2,2]$. Using a standard trick [7, 13, 27] we insert the function $w(q,p,t) = e^{i\sqrt{\mu}t}\phi(t)\psi(q,p)$ into the modified version of estimate (24) (where $\psi \in X$, and ϕ is a smooth cut-off function, with support in $(-2,2)$ and $\phi(t) = 1$ for $t \in [-1,1]$), giving us

$$\|w\|_{C^{2,\alpha}_{per}(\overline{\Omega}^t)} \leq C\left(\|w\|_{C^0_{per}(\overline{\Omega}^t)} + \|(A+\partial_t^2)w\|_{C^{0,\alpha}_{per}(\overline{\Omega}^t)} + \|Bw\|_{C^{1,\alpha}_{per}(T\times[-2,2])}\right). \tag{26}$$

We wish to derive Schauder-type estimates, in terms of the original domain R and top boundary T, from the above estimate concerning the augmented cylindrical domain. To this end, we note that

$$(A+\partial_t^2)w = e^{i\sqrt{\mu}t}\left\{\phi(t)(a-\mu)\psi + [2i\sqrt{\mu}\phi'(t) + \phi''(t)]\psi\right\},$$

and straightforward, if laborious, calculations show that there exist constants $C_1, C_2, C_3, C_4 > 0$ which are independent of $\mu > 1$ and $\psi \in X$ such that

$$\begin{aligned}
\|e^{i\sqrt{\mu}t}\phi(t)\psi\|_{C^{2,\alpha}_{per}(\overline{\Omega}^t)} &\geq C_0\left(\|\psi\|_{C^{2,\alpha}_{per}(\overline{R})} + \mu\|\psi\|_{C^{0,\alpha}_{per}(\overline{R})}\right),\\
\|e^{i\sqrt{\mu}t}\phi(t)\psi\|_{C^{0,\alpha}_{per}(\overline{\Omega}^t)} &\leq C_1\mu^{\alpha/2}\|\psi\|_{C^{0,\alpha}_{per}(\overline{\Omega}^t)},\\
\|e^{i\sqrt{\mu}t}[2i\sqrt{\mu}\phi'(t)+\phi''(t)]\psi\|_{C^{0,\alpha}_{per}(\overline{R})} &\leq C_2\mu^{(\alpha+1)/2}\|\psi\|_{C^{0,\alpha}_{per}(\overline{\Omega}^t)},\\
\|(A-\mu)e^{i\sqrt{\mu}t}\phi(t)\psi\|_{C^{0,\alpha}_{per}(\overline{R})} &\leq C_3\mu^{\alpha/2}\|(A-\mu)\psi\|_{C^{0,\alpha}_{per}(\overline{R})},\\
\|Be^{i\sqrt{\mu}t}\phi(t)\psi\|_{C^{1,\alpha}_{per}(T\times[-2,2])} &\leq C_4\mu^{(\alpha+1)/2}\|B\psi\|_{C^1_{per}(T)}.
\end{aligned}$$

Combining these inequalities with (26) gives

$$\frac{C_0}{C(C_3+C_4)}\|\psi\|_{C^{2,\alpha}_{per}(\overline{R})} \leq \mu^{\alpha/2}\|(A-\mu)\psi\|_{C^{0,\alpha}_{per}(\overline{R})} + \mu^{(\alpha+1)/2}\|B\psi\|_{C^1_{per}(T)}, \quad \psi \in X,$$

provided that $C_0\mu^{(1-\alpha)/2} \geq C(C_1+C_2)$, and a similar inequality may be derived for $\partial_q\psi$. Statement (i) now follows, for $\mu > 1$ sufficiently large, by expressing $\partial_p^2\psi$ in terms of $(A+\partial_t^2)\psi$ and the derivatives of ψ of order less than two.

For statement (ii), following the argumentation of Lemma 3.3, we find that $(A-\mu, B)$ is a Fredholm operator of index zero from X to Y. Part (i)

ensures that it has a trivial kernel, and hence it is one-to-one and onto if μ is sufficiently large. Furthermore, (i) ensures that X_0 is a Banach space and $A : X_0 \to Y_1$ is a bounded linear operator. If $\mu \geq c_2$, then $(A-\mu) : X_0 \to Y_1$ is bijective, with compact inverse due to the compact embedding $X \subset Y_1$. Applying Riesz-Schauder theory we have proven (ii). Finally, the existence of a value of μ such that $(A - \mu, B)$ maps X onto Y ensures that B maps X onto Y_2, proving part (iii). □

We have now established that the operator $\mathcal{G}$ is admissible in the sense of Healey-Simpson [27]. The results of Theorem 3.1 concerning the structure of $\mathcal{S}_\delta^0$ now follows exactly as for the global bifurcation theorem of Rabinowitz [46], but using the generalised degree of Healey-Simpson in place of the Leray-Schauder degree. Additionally, the existence of the real-analytic curve $\mathcal{K}_\delta$ follows from real-analytic global bifurcation theory [3], thereby proving Theorem 3.1.

3.2 Nodal patterns of solutions

In this section we prove that the nodal configuration of all solutions h in the set $\mathcal{S}_\delta^0$, and along the curve $\mathcal{K}_\delta$, is inherited from the linearised eigenfunction at the local bifurcation point (Q^*, H^*). These nodal properties, expressed in (27) below, will be used to eliminate the second alternative in Theorem 3.1, and which additionally rules out $\mathcal{K}_\delta$ being a closed loop. Furthermore, statement (iii) of Theorem 1.1 follows as a consequence of the nodal properties (27), coupled with (3h) and (11).

Let Ω be the open set $(0, \pi) \times (-1, 0)$, and let us denote its sides (excluding the corners) as

$$\begin{aligned} \partial\Omega_t &= \{(q, 0) : q \in (0, \pi)\}, & \partial\Omega_b &= \{(q, -1) : q \in (0, \pi)\}, \\ \partial\Omega_l &= \{(0, p) : p \in (-1, 0)\}, & \partial\Omega_r &= \{(p, \pi) : p \in (-1, 0)\}. \end{aligned}$$

We will prove that all $(Q, h) \in \mathcal{S}_\delta^0$ with $h_q \not\equiv 0$ satisfy either

$$\begin{array}{llll} h_q < 0 \text{ in } \ \Omega \cup \partial\Omega_t, & h_{qp} < 0 \text{ on } \ \partial\Omega_b, & h_{qq} < 0 \text{ on } \ \partial\Omega_l, & h_{qq} > 0 \text{ on } \ \partial\Omega_r, \\ h_{qqp}(0, -1) < 0, & h_{qqp}(\pi, -1) < 0, & h_{qq}(0, 0) < 0, & h_{qq}(\pi, 0) > 0, \end{array} \tag{27}$$

or else the exact opposite inequalities hold. In particular, the inequalities (27) will hold for (Q, h) on $\mathcal{C}_{loc}^+$ close to the bifurcation point (Q^*, H^*), and the opposite inequalities hold for points (Q, h) on $\mathcal{C}_{loc}^-$ close to (Q^*, H^*). By it definition (10), we have $h = 0$ on $\partial\Omega_b$, and since h is even and periodic in q we have $h_q = 0$ on $\partial\Omega_l \cup \partial\Omega_r$.

Lemma 3.5. *The inequalities* (27) *hold in a small neighbourhood of* (Q^*, H^*) *in* $\mathbb{R} \times X$*, along the curve* $\mathcal{C}_{loc}^+ \backslash (Q^*, H^*)$ *coming out of* (Q^*, H^*)*.*

Proof. In the neighbourhood of the bifurcation point (Q^*, H^*), it is easier to work with (λ, w) rather than (Q, h), where $w = h - H(\lambda, p)$, with the laminar solution $H(\lambda, p)$ independent of q. To prove this the lemma we will show that the inequalities (27) hold for w along $\mathcal{C}_{loc}$. We know from local bifurcation theory [31] (see the discussion preceeding (19)) that, for $\epsilon > 0$ small enough, along $\mathcal{C}_{loc}^+$ we have

$$w^\epsilon(q, p) = \epsilon M(p) \cos q + o(\epsilon) \quad \text{in} \quad C^{3,\alpha}(\overline{\Omega}), \tag{28}$$

where $M \in C^{3,\alpha}[-1, 0]$ is the solution to the Sturm-Liouville system (14). Since M is the ground state for the Sturm-Liouville problem, we know from Lemma 2.1 that $M(p) \neq 0$ for $p \in (-1, 0)$, in fact we have

$$M(p) > 0, \quad p \in (-1, 0),$$

and also $M'(-1) > 0$. Hence, from $M(0) = 1, M'(-1)$, and differentiating (28), we get that (27) holds for $\epsilon > 0$ small enough. □

We note that the opposite inequalities hold for points (Q, h) on $\mathcal{C}_{loc}^-$ which are close to (Q^*, H^*)— see the remark following (19).

Lemma 3.6. *For any* $(Q, h) \in \mathcal{S}_\delta^0$ *with* $h_q \not\equiv 0$*, either the inequalities* (27) *or their exact opposites hold.*

Proof. The set $\mathcal{S}_\delta^0$ is connected in $\mathbb{R} \times X$. Furthermore, the set of $(Q, h) \in \overline{\mathcal{O}_\delta}$ which satisfy the inequalities (27) is a nonempty (by Lemma 3.5) open set in $\mathbb{R} \times X$. If the statement were false, then there exists some $(Q, h) \in \mathcal{S}_\delta^0$, with $h_q \not\equiv 0$, for which one of the inequalities in (27) is an equality, even though it is the limit of a sequence of elements for which the inequalities (27) are strict, or the exact opposites of (27) are strict. Without loss of generality, we assume that (Q, h) is a limit of the sequence $\{(Q_n, h_n)\}$ whereby (27) holds for each h_n. If we differentiate the system (12) with respect to q, we get the following system for $\phi = h_q$:

$$\left(\frac{1}{d^2} + h_q^2\right)\phi_{pp} - 2h_q(h_p+1)\phi_{pq} + (h_p+1)^2\phi_{qq} + [2h_q h_{pp} - 2(h_p+1)h_{pq}]\phi_q$$
$$+[3\frac{\gamma(p)}{p_0}(h_p+1)^2 - 2h_q h_{pq} + 2(h_p+1)h_{qq}]\phi_p = 0 \ \text{ in } \Omega, \tag{29a}$$

$$2h_q\phi_q + 2\frac{(h_p+1)}{p_0^2}[2gd(h+1) - Q]\phi_p + \frac{2gd(h_p+1)^2}{p_0^2}\phi = 0, \ p = 0, \tag{29b}$$

$$\phi = 0, \ p = -1. \tag{29c}$$

The above system is uniformly elliptic with an oblique boundary condition on the top boundary. Now, since $\partial_q h_n < 0$ in Ω for each n, we have must have $\phi \le 0$ in $\overline{\Omega}$. Since $h_q \not\equiv 0$, the strong maximum principle implies that $\phi = h_q < 0$ in Ω, and on the boundaries $\partial\Omega_l \cup \partial\Omega_b \cup \partial\Omega_r$ the Hopf inequality [23] holds for ϕ (which attains its maximum, zero, on these boundaries), thereby proving the second, third and fourth inequalities in (27). We now show that $\phi < 0$ on $\partial\Omega_t$. Suppose otherwise, let $\phi = 0$ at some point $(q_0, 0)$, for $q_0 \in (0, \pi)$. Then Hopf's boundary inequality implies that $\phi_p > 0$ at $(q_0, 0)$, which violates the boundary condition (29b) on foot of the definition of $\mathcal{O}_\delta$.

We now need to prove the four inequalities on the bottom row of (27). At the top right corner, $(\pi, 0)$, we have $h_q = h_{qqq} = h_{qp} = h_{qpp} = 0$, since h is even in q and periodic. We proved above that $h_{qq}(\pi, p) > 0$ for $p \in (-1, 0)$, and so $h_{qq}(\pi, 0) \ge 0$ by continuity. Let us assume that $h_{qq}(\pi, 0) = 0$, then differentiating (29b) with respect to q and evaluating at $(\pi, 0)$ we get $(2(h_p+1)[2gd(h+1) - Q]/p_0^2)\phi_{pq} = 0$, which implies that $\phi_{pq}(\pi, 0) = 0$. Therefore, all first and second derivatives of ϕ vanish as $(\pi, 0)$, which contradicts Serrin's Edge Point Lemma [23]. Hence $h_{qq}(\pi, 0) > 0$. A similar treatment using Serrin's Edge Point Lemma derives the inequalities for the other corners. □

To this point we have proven that, throughout $\mathcal{S}_\delta^0$, the nodal pattern (27) (or its exact opposite) must hold, unless there is a laminar flow solution. We now rule out the possibility of there being a laminar flow solution $(Q(\lambda), H(\lambda))$ apart from (Q^*, H^*), thereby eliminating the second alternative in Theorem 3.1.

Lemma 3.7. *The only solution in $\mathcal{S}_\delta^0$ that is independent of q is the bifurcation point (Q^*, H^*).*

Proof. Let us suppose that $(Q_n, h_n) \in \mathcal{S}_\delta^0$ is a sequence converging to $(Q(\lambda), H(\cdot,\lambda)) \in \mathcal{S}_\delta^0$, such that $\partial_q h_n \not\equiv 0$ and the inequalities (27) hold. Let $\phi_n = \partial_q h_n / \|\partial_q h_n\|_{C^{2,\alpha}_{per}(\overline{R})}$, and take a subsequence $\{\phi_{n_k}\}$ which converges to some $\phi \in C^2_{per}(\overline{R})$ (and which, we infer from regularity theory for the uniformly elliptic oblique boundary problem (29), therefore converges in $C^{2,\alpha}_{per}(\overline{R})$). It follows that the limit ϕ is a function of the form $\partial_q m$, with both m and $\partial_q m$ in $C^{2,\alpha}_{per}(\overline{R})$, and for which $\|\partial_q m\|_{C^{2,\alpha}_{per}(\overline{R})} = 1$. Taking the limit $n_k \to \infty$ in the system (29) we get

$$\frac{1}{d^2}\phi_{pp} + (H_p+1)^2\phi_{qq} + 3\frac{\gamma(p)}{p_0}(H_p+1)^2\phi_p = 0, \qquad \text{in } \Omega, \tag{30a}$$

$$2\frac{(H_p+1)}{p_0^2}[2gd(H+1)-Q]\phi_p + \frac{2gd(H_p+1)^2}{p_0^2}\phi = 0, \qquad p=0, \tag{30b}$$

$$\phi = 0, \qquad p=-1. \tag{30c}$$

Expanding in a sine series, we have

$$\phi(q,p) = \sum_{k=1}^{\infty} \sin(kq) f_k(p),$$

where $f_k \in C^2([-1,0])$ is given by

$$f_k(p) = \frac{2}{\pi}\int_0^{\pi} \sin(kq)\phi(q,p)dq, \quad k \geq 1.$$

Now, $H_p(0)+1 = \frac{1}{\sqrt{\lambda}}$, $\lambda = \frac{d^2}{p_0^2}(Q - 2gd(H(0)+1)$, and multiplying (30) by $\sin(q)$ and integrating over $[-\pi,\pi]$ we get

$$(a^3\partial_p f_1)_p = d^2 a f_1 \ \text{ in } (-1,0), \tag{31a}$$

$$\lambda^{3/2} f_1'(0) = \frac{gd^3}{p_0} f(0), \tag{31b}$$

$$f_1(-1) = 0. \tag{31c}$$

Therefore f_1 is an eigenvalue of (14) with eigenvalue $\mu = -1$. Now, since $\partial_q h_{n_k} < 0$ in Ω (by (27)) and equals zero on $p=-1$, we have $\phi \leq 0$ in Ω, with $\phi(q,-1) = 0$, and so the strong maximum principle implies that $\phi < 0$ in Ω, giving us

$$f_1(p) < 0, \quad p \in (-1,0).$$

This non-vanishing condition implies that f_1 is a groundstate (for otherwise it has nodes where it vanishes), that is, $\mu = -1$ is the lowest eigenvalue. It now follows from the considerations of [31], where we prove the uniqueness of the critical value λ^* giving $\mu(\lambda^*) = -1$ in (15), that $\lambda = \lambda^*$. $\square$

The results of this Section can be summarised as follows:

Proposition 3.8. *The connected set $\mathcal{S}_\delta^0$ is either unbounded in $\mathbb{R} \times X$ or contains a point $(Q, h) \in \partial\mathcal{O}_\delta$ (similarly for the curve $\mathcal{K}_\delta$). Furthermore, any solution h in $\mathcal{S}_\delta^0, \mathcal{K}_\delta$, which is different from the local bifurcation solution (Q^*, H^*), inherits the nodal properties* (27)*.*

3.3 Uniform regularity and bounds in $\mathcal{S}_\delta^0, \mathcal{K}_\delta$

In this section we complete the proof of Theorem 1.1, in particular Lemma 3.11 establishes the limiting behaviour (1) of solutions in the continuum $\mathcal{S}_\delta^0$. For the subsequent considerations we note that, due to the periodicity in q of the functions which we consider, we can ignore the apparent nonsmoothness of the boundary at the lateral sides of the domain R. Also, although we have an oblique boundary condition on the top, and a homogeneous Dirichlet condition on the bottom, due to the local nature of the estimates which we will employ, and the seperatedness of the respective boundaries, we may justifiably combine estimates which apply to the different boundary conditions. Key among these estimates are the Schauder-type estimates which were developed by Liebermann and Trudinger [41], and accordingly we will present their regularity theorem in a little detail. Let us consider the oblique nonlinear elliptic boundary value problem

$$F(h, Dh, D^2h) = 0 \quad \text{in} \quad R, \tag{32a}$$
$$G(h, Dh) = 0 \quad \text{on} \quad p = 0, \tag{32b}$$
$$h = 0 \quad \text{on} \quad p = -1. \tag{32c}$$

Here $F \in C^2(\mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}, \mathbb{R})$ and $G \in C^2(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$, with Dh the gradient, D^2h the Hessian matrix, and $\mathbb{S}$ the space of 2×2 real symmetric matrices. We say that F is elliptic at a point (h, ξ, r) if the matrix $F_r = [\partial F/\partial r_{ij}]_{1\le i,j\le 2}$ is positive definite at this point. If Λ_1, Λ_2 denote the minimum and maximum eigenvalue of F_r, respectively, then F is uniformly elliptic if Λ_2/Λ_1 is bounded. The boundary operator G is oblique at a point $(q, 0)$ if the normal derivative $\chi = G_\xi \cdot (0, -1)$ is positive for all $(h, \xi) \in \mathbb{R} \times \mathbb{R}^2$.

Theorem 3.9 (Liebermann-Trudinger)**.** *Let $h \in C^2(\overline{R})$ be a solution, of period 2π in q, of the boundary value problem* (32)*, with $|h| + |Dh| \le K$ in $\overline{R}$, for some constant $K > 0$. Suppose that for some $M > 0$ the functions*

$F(h,\xi,r)$ and $G(h,\xi)$ satisfy the structure conditions

$$\Lambda_2 \le M\Lambda_1, \tag{33a}$$
$$|F|, |F_{\xi_i}| \le M\Lambda_1, \quad i = 1,2, \tag{33b}$$
$$F_{rr} \le 0, \tag{33c}$$
$$|G|, |G_h|, |G_{\xi_i}|, |G_{hh}|, |G_{h\xi_i}|, |G_{\xi_i\xi_j}| \le M\xi, \quad i,j = 1,2, \tag{33d}$$

for all $(h,\xi,r) \in (\mathbb{R}\times\mathbb{R}^2\times\mathbb{S})$ such that $|h|+|\xi_1|+|\xi_2| \le K$. Then there are positive constant $\mu(M) < 1$ and $C(K,M)$ such that $h \in C^{2,\mu}(\overline{R})$ and

$$\|h\|_{C^{2,\mu}_{per}(\overline{R})} \le C.$$

We are now in a position to prove the following uniform regularity result.

Lemma 3.10. *If $\sup_{h\in\mathcal{S}^0_\delta}\{\|h\|_X\} = \infty$, then $\sup_{h\in\mathcal{S}^0_\delta}\{Q + \|h_p\|_{L^\infty(R)}\} = \infty$. Similarly, if $\sup_{h\in\mathcal{K}_\delta}\{\|h\|_X\} = \infty$, then $\sup_{h\in\mathcal{K}_\delta}\{Q + \|h_p\|_{L^\infty(R)}\} = \infty$.*

Proof. Since the proofs are similar, we just show the second statement holds. That is to say, if $\sup_{h\in\mathcal{K}_\delta}\{Q+\|h_p\|_{L^\infty(R)}\}$ is bounded, then so is $\sup_{h\in\mathcal{K}_\delta}\{\|h\|_X\}$. Since $|p_0| = \int_{-d}^{\eta(0)}(c-u)dy \ge \inf_{\overline{D_\eta}}\{c-u\}[\eta(0)+d]$, we get

$$0 \le h(q,p) = \frac{y}{d} - \frac{\psi}{p_0} \le \frac{1}{d}[y+d] \le \frac{1}{d}[\eta(0)+d] \le \frac{|p_0|}{d\inf_{\overline{D_\eta}}\{c-u\}} = \sup_{\overline{R}}\{h_p+1\}.$$

Therefore h is bounded along $\mathcal{K}_\delta$. The bound for h_q follows by considering the system (29), which is uniformly elliptic for $\phi = h_q$. Applying the strong maximum principle [25], we deduce that ϕ must attain its maximum on the boundary of R. So ϕ attains its maximum either on $p=-1$, where $\phi = 0$, or $p = 0$, where

$$p_0^2 h_q^2 = (h_p+1)^2[Q - 2gd(h+1)] - \frac{p_0^2}{d^2} \le (h_p+1)^2 Q - \frac{p_0^2}{d^2}.$$

Since $\inf_{h\in\mathcal{K}_\delta}|p_0| > \delta$, we must have $\sup_{h\in\mathcal{K}_\delta}\{\|h_q\|_{L^\infty(R)}\} < \infty$, and so h in bounded in $C^1_{per}(\overline{R})$ along $\mathcal{K}_\delta$. We now may apply the additional a priori Schauder-type estimates that Theorem 3.9 offers us, as follows. We have

$$F(h,\xi,r) = \left(\frac{1}{d^2} + \xi_1^2\right) r_{22} - 2\xi_1(\xi_2+1)r_{12} + (\xi_2+1)^2 r_{11} + \frac{\gamma(p)}{p_0}(\xi_2+1)^3,$$
$$G(h,\xi) = \frac{1}{d^2} + \xi_1^2 + \frac{(\xi_2+1)^2}{p_0^2}[2gd(h+1) - Q].$$

By using cut-off functions, we may apply Theorem 3.9 to a subset of $C^2_{per}(\overline{R})$ where $h_p + 1 > \delta$ in $\overline{R}$, and $Q - 2gd(h+1) > \delta$ on $p = 0$. Here, we easily see that (33a) and (33b) holds, while $F_{rr} \equiv 0$, giving us (33c). We have

$$\chi = \frac{2(\xi_2 + 1)}{p_0^2}[Q - 2gd(h+1)] = \frac{2\left(\frac{1}{d^2} + \xi_1^2\right)}{1 + \xi_2} \geq \delta_0 > 0,$$

since $\sup_{\overline{R}} h_p < \infty$ for all h and additionally we have $h_p + 1 > \delta$. This implies that (33d) holds, and so the $C^{2,\mu}_{per}(\overline{R})$ norm of the solution $h \in X$ is uniformly bounded along $\mathcal{K}_\delta$.

To prove that the $C^{3,\mu}_{per}(\overline{R})$ norm of the solution $h \in X$ is uniformly bounded along $\mathcal{K}_\delta$, we notice that $\phi = \partial_q h$ satisfies (29), and applying standard Schauder estimates ([25][Theorem 6.30]) for the oblique derivative problem and the $C^{2,\mu}_{per}(\overline{R})$ bounds for $h \in \mathcal{K}_\delta$ give us the uniform boundedness of the $C^{2,\mu}_{per}(\overline{R})$ norm of h_q along $\mathcal{K}_\delta$. This gives us uniform bounds in $C^{0,\mu}_{per}(\overline{R})$ of all the third derivatives of h along $\mathcal{K}_\delta$, except h_{ppp}. To bound this derivative, we can express h_{pp} in terms of the other derivatives of h of order less than or equal to two using equation (12a). It then follows that $\|h_{pp}\|_{C^{1,\mu}_{per}(\overline{R})}$ is uniformly bounded along $\mathcal{K}_\delta$, which in turn implies that $\|h\|_{C^{3,\mu}_{per}(\overline{R})}$, and also $\|h\|_{C^{2,\alpha}_{per}(\overline{R})}$, is uniformly bounded along $\mathcal{K}_\delta$. Finally, to show that $\|h\|_X$ is uniformly bounded along $\mathcal{K}_\delta$, we repeat the procedure of this paragraph, replacing α instead of μ. □

We now set $\mathcal{S}^0 = \bigcup_{\delta > 0} \mathcal{S}^0_\delta$, noting that $\mathcal{S}^0_\delta$ increases in as $\delta > 0$ decreases. All the considerations to this point, from the alternatives of Theorem 3.1, which were refined in Proposition (3.8), coupled with the previous lemma, tell us that for any $\delta > 0$, the following alternatives must hold:

Alt. 1 There exists a sequence $\{(Q_n, h_n)\} \in \mathcal{S}^0_\delta$ with $\lim_{n\to\infty} Q_n = \infty$;

Alt. 2 There exists a sequence $\{(Q_n, h_n)\} \in \mathcal{S}^0_\delta$ with $\lim_{n\to\infty} |\max_{\overline{R}} \partial_p h_n| = \infty$;

Alt. 3 There exists a $(Q, h) \in \mathcal{S}^0_\delta$ with $\partial_p h + 1 = \delta$ somewhere in $\overline{R}$;

Alt. 4 There exists a $(Q, h) \in \mathcal{S}^0_\delta$ with $Q - 2gd(h+1) = \delta$ somewhere on T.

Alt. 5 There exists a $(Q, h) \in \mathcal{S}^0_\delta$ with $\left|p_0^{(h)}\right| = \delta$, or $\left|p_0^{(h)}\right| = 1/\delta$.

The next lemma distils the above alternatives into two separate possibilities:

Lemma 3.11. *We have either* $\sup_{\overline{D_{\eta n}}}\{u_n\} \to c$ *or* $\inf_{\overline{D_{\eta n}}}\{u_n\} \to -\infty$ *along some sequence in* $\mathcal{S}^0$.

Proof. We prove the lemma working with each of the five alternatives above in their turn. Suppose alternative 1 holds for some $\delta > 0$. It follows from Bernouilli's identity, and the fact that $\eta(\pi) < 0$ at the trough, that

$$Q = (c - u(\pi, \eta(\pi)))^2 + 2g(\eta(\pi) + d) \leq (c - u(\pi, \eta(\pi)))^2 + 2gd.$$

Therefore $\inf_{\overline{D_{\eta_n}}}\{u_n\} \to -\infty$. Now suppose alternative 2 holds for some $\delta > 0$, then $\partial_p h_n + 1 = \frac{|p_0{}^{(h_n)}|}{d(c-u_n)} \to \infty$. Since $|p_0{}^{(h_n)}|$ is bounded above by $1/\delta$, then we must have $\sup_{\overline{D_{\eta_n}}}\{u_n\} \to c$. If alternative 3 holds for a sequence of $\delta_n \downarrow 0$, then there exists a corresponding sequence $(Q_n, h_n) \in \mathcal{S}^0$ such that $\inf_{\overline{R}}\{\partial_p h_n + 1\} = \inf_{\overline{D_{\eta_n}}} \frac{|p_0{}^{(h_n)}|}{d(c-u_n)} \to 0$. Since $|p_0{}^{(h_n)}|$ is bounded below by $\delta > 0$, we must have $\inf_{\overline{D_{\eta_n}}}\{u_n\} \to -\infty$. If alternative 4 holds for a sequence of $\delta_n \to 0$, then there exists a corresponding sequence $(Q_n, h_n) \in \mathcal{S}^0$ such that $\inf_T[Q_n - 2gd(h_n + 1)] \to 0$. The nonlinear boundary condition on T gives us

$$\begin{aligned}(u_n - c)^2 = \frac{\left(p_0^{(h_n)}\right)^2}{d^2(\partial_p h_n + 1)^2} &\leq \frac{\left(p_0^{(h_n)}\right)^2 (1 + d^2(\partial_q h_n)^2)}{d^2(\partial_p h_n + 1)^2} \\ &= [Q_n - 2gd(h_n + 1)] \to 0,\end{aligned}$$

and so $\inf_{\overline{D_{\eta_n}}}\{c - u_n\} \to 0$. Finally, we suppose Alternative 5 holds for a sequence of $\delta_n \downarrow 0$, then there exists a corresponding sequence $(Q_n, h_n) \in \mathcal{S}^0$ such that either $\lim_{n\to\infty} |p_0{}^{(h_n)}| = 0$ or $\lim_{n\to\infty} |p_0{}^{(h_n)}| = \infty$. If $\lim_{n\to\infty} |p_0{}^{(h_n)}| = 0$, then since

$$|p_0^{(h_n)}| = \int_{-d}^{\eta_n(0)} (c - u_n(0, y)))\, dy \geq d \cdot \inf_{-d \leq y \leq \eta_n(0)} (c - u_n(0, y))$$

we must have $\inf_{\overline{D_{\eta_n}}}\{c - u_n\} \to 0$. If however $\lim_{n\to\infty} |p_0{}^{(h_n)}| = \infty$ then, since

$$|p_0^{(h_n)}| = \int_{-d}^{\eta_n(\pi)} (c - u_n(\pi, y)))dy \leq \sup_{-d \leq y \leq \eta_n(\pi)} (c - u_n(\pi, y)) \cdot d$$

we must have $\inf_{\overline{D_{\eta_n}}}\{u_n\} \to -\infty$. $\square$

Remark We remark that, if the mass flux is bounded for a sequence of solutions where $\inf_{\overline{D_{\eta_n}}}\{u_n\} \to -\infty$, that is $\lim_{n\to\infty} |p_0^n| < \infty$, then we can show that in fact $\sup_{\overline{D_{\eta_n}}}\{u_n\} \to c$ for this sequence. Therefore, if the mass flux is bounded throughout the global continuum of solutions $\mathcal{S}^0$, then there is sequence of solutions which possess in the limit a weak stagnation point. The boundedness of the mass-flux throughout $\mathcal{S}^0$ appears to be physically reasonable supposition, however it remains to be verified mathematically.

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