EXACT EQUATORIAL WATER WAVES IN THE $f-$PLANE

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Abstract. We present an exact solution for geophysical ocean waves in the Equatorial region which is three-dimensional, nonlinear, explicit in the Lagrangian formulation, and which incorporates a meridional current that is transverse Equatorial.

1. Introduction

In this paper we present an exact solution for geophysical ocean waves in the Equatorial region. The solution we obtain is three-dimensional, nonlinear, explicit in the Lagrangian formulation, and it incorporates a meridional current which is transverse Equatorial. To construct our solution we employ the $f-$plane approximation of the geophysical governing equations which is applicable for oceanic flows within a restricted meridional range of approximately 2° latitude from the Equator [12,14,31]. The Equatorial region experiences a vast range of fascinating oceanographic phenomena which remain largely mysterious. Large-scale currents, such as the vast Equatorial Undercurrent (EUC), and wave-current interactions play a major role in the geophysical dynamics of the Equatorial region [12, 15, 32, 33], and are thought to play a major role in the formation of the El Niño and La Niña phenomena [27].

Allied to this, there are a number of compelling mathematical reasons why the study of ocean waves in the Equatorial region generates great interest. Along the Equator the Coriolis forces become less pronounced in the governing equations, and accordingly we may naturally employ more tractable (although still nonlinear) approximations to the full governing equations, the so-called $\beta-$plane and $f-$plane approximations respectively. Additionally, the Equator acts as a natural waveguide and equatorial waves are predominantly zonal [14,31]. Recently, the existence of a variety of fluid motions in the Equatorial region was rigorously proven in [6,9,23,24] using various mathematical analytic techniques. However, perhaps of greater interest with respect to potentially gaining insight into the dynamics of the Equatorial region was the derivation of an exact and explicit solution to the $\beta-$plane governing equations derived by Constantin in [7].

The solution presented in [7] represents a significant generalisation of the celebrated Gerstner's wave to the geophysical setting, in the sense that it defines a inherently three-dimensional eastward-propagating geophysical wave which is Equatorially-trapped and

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whose dispersion relation is dependant on the Coriolis parameter, while simply ignoring the Coriolis terms recovers Gerstner's wave solution. Gerstner's wave is a two-dimensional wave propagating over a fluid domain of infinite depth (cf. [3, 5, 19]), and perhaps the primary importance of Gerstner's wave is the fact that it represents the only known explicit and exact solution of the nonlinear periodic gravity wave problem with a non-flat free-surface. Interestingly Gerstner's solution may also be modified to describe edge-waves propagating over a sloping bed [2, 34]. Subsequent to [7], a vast and wide-ranging variety of Gerstner-type exact and explicit solutions, which model a number of different physical and geophysical scenarios, were derived and analysed in [8, 10, 11, 17, 20–22, 26, 28, 29].

In this paper we present a three-dimensional exact and explicit (in the Lagrangian formulation) solution of the $f-$plane governing equations which may travel either westwards or eastwards in the zonal direction, and which also has an meridional velocity that is independent of the latitude but which may otherwise be prescribed arbitrarily (this meridional velocity term represents a transverse Equatorial current). Artefacts of the $f-$plane governing equations, in contrast to the $\beta-$plane setting (cf. [7, 20]), are that exact solutions to these equations are not Equatorially-trapped and additionally they may travel both westwards and eastwards. Admitting westward propagating waves is interesting from the viewpoint of the geophysical dynamics in the Equatorial region [7, 15].

Quite aside from the dearth of explicit examples, a benefit which may be reaped from finding exact finite-amplitude solutions to a water wave problem is the potential to regard more complex flows as perturbations of these solutions [30]. Consequently, detailed information about the more complex flows may be gleaned from knowledge of the exact solution by controlling the perturbations. Therefore, while allowing for a transverse Equatorial current is a mathematical curiosity and an interesting extension of the $f-$plane solution presented in [25] in its own right, it is hoped that the additional physical complexity in this solution may be beneficial with respect to potential generalisations in this direction.

2. Governing equations

The governing equations for geophysical ocean waves, in a reference frame with the origin located at a point fixed on Earth's surface and rotating with the Earth, are given by [14, 31]

$$u_t + uu_x + vu_y + wu_z + 2\Omega w \cos\phi - 2\Omega v \sin\phi = -\frac{1}{\rho} P_x, \tag{2.1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u \sin\phi = -\frac{1}{\rho} P_y, \tag{2.1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u \cos\phi = -\frac{1}{\rho} P_z - g, \tag{2.1c}$$

together with the equation of incompressibility

$$\nabla \cdot \mathbf{U} = 0. \tag{2.2a}$$

Here the $x-$axis is in the longitudinal direction (horizontally due east), the $y-$axis along the latitudinal direction (horizontally due north) and the $z-$axis is perpendicular to the Earth's surface pointing vertically upwards. Then $\mathbf{U} = (u, v, w)$ is the velocity field of the fluid, the variable ϕ represents the latitude, P is the pressure of the fluid and ρ is the constant density of the fluid. The Earth is taken to be a perfect sphere of radius $R = 6378km$ with constant rotational speed of $\Omega = 73 \cdot 10^{-6} rad/s$, and $g = 9.8ms^{-2}$ is the gravitational acceleration at the surface of the Earth. In the Equatorial region, where the latitude ϕ is relatively small, the full governing equations for geophysical water waves (2.1) may be rendered more tractable by approximating the Coriolis terms. When ϕ is non-constant, but small, the $\beta-$plane approximation [14],

$$\sin\phi \approx \phi, \cos\phi \approx 1,$$

reduces the governing equations to the following form:

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v &= -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + \beta y u &= -\frac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u &= -\frac{1}{\rho}P_z - g, \end{aligned}$$

where $\beta = 2\Omega/R = 2.28 \cdot 10^{-11} m^{-1}s^{-1}$. Effectively, the Coriolis terms for the curved Earth's surface which appear in (2.1) are approximated by a planar model. If we further restrict our focus to purely Equatorial waves (and so we work essentially at a constant latitude) we get the $f-$plane approximation [12, 14]

$$u_t + uu_x + vu_y + wu_z + 2\Omega w = -\frac{1}{\rho}P_x, \tag{2.2ba}$$

$$v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho}P_y, \tag{2.2bb}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g, \tag{2.2bc}$$

The boundary conditions for the fluid on the free-surface η are given by

$$w = \eta_t + u\eta_x + v\eta_y, \tag{2.2c}$$

$$P = P_0 \qquad \text{on } y = \eta(x, y, t), \tag{2.2d}$$

where P_0 is the constant atmospheric pressure. The kinematic boundary condition on the surface simply states that all surface particles remain confined to the surface. We assume that the fluid domain is infinitely deep, with the fluid motion vanishing rapidly with increasing depth,

$$(u, v) \to (0, 0) \ \text{ as } \ y \to -\infty. \tag{2.2e}$$

3. Exact solution of (2.2b)

We may re-express the $f-$plane equations of motion (2.2b) in the form

$$\frac{Du}{Dt} + 2\Omega w = -\frac{1}{\rho}P_x, \tag{3.1a}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}P_y, \tag{3.1b}$$

$$\frac{Dw}{Dt} - 2\Omega u = -\frac{1}{\rho}P_z - g, \tag{3.1c}$$

where D/Dt is the material or convective derivative. We now claim that the system

$$x = q - \frac{1}{k}e^{kr}\sin[k(q-ct)], \tag{3.2a}$$

$$y = s + \psi(q,r)t, \tag{3.2b}$$

$$z = r + \frac{1}{k}e^{kr}\cos[k(q-ct)], \tag{3.2c}$$

represents an exact solution of (3.1). Here the Eulerian coordinates (x,y,z) of fluid particles are expressed as functions of the time t, the Lagrangian labelling variables $(q,r,s)\in(\mathbb{R},(-\infty,r_0),\mathbb{R})$, where $r_0<0$, and the arbitrary function ψ. For notational convenience let us choose

$$\xi = kr,\ \ \theta = k(q-ct),$$

and we calculate directly from (3.2) that

$$u = \frac{Dx}{Dt} = ce^{\xi}\cos\theta, \tag{3.3a}$$

$$v = \frac{Dy}{Dt} = \psi(q,r), \tag{3.3b}$$

$$w = \frac{Dz}{Dt} = ce^{\xi}\sin\theta. \tag{3.3c}$$

Hence the solution (3.2) above comprises a wave-like term in the zonal direction, with an additional latitudinally–independent meridional current term $\psi(q,r)$ which is constant with respect to time. The wavelike term is a travelling wave moving with constant wave phasespeed c (which is determined by the dispersion relations (3.11) below) and $k=2\pi/L$ is the wavenumber, where the wavelength L is a positive constant. The Jacobian matrix of the transformation (3.2) is given by

$$\begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} 1-e^{\xi}\cos\theta & \psi_q(q,r)t & -e^{\xi}\sin\theta \\ 0 & 1 & 0 \\ -e^{\xi}\sin\theta & \psi_r(q,r)t & 1+e^{\xi}\cos\theta \end{pmatrix}. \tag{3.4}$$

The Jacobian has a time independent determinant $1-e^{2kr}$ (which is non-zero since $r_0<0$) thus it follows that the flow defined by (3.2) must be volume preserving, ensuring that

(2.2a) holds in the Eulerian setting [1]. It is clear from (3.3) that the wave motion is inherently three-dimensional if $\psi \not\equiv 0$. We can explicitly show that the vorticity $\boldsymbol{\omega}$ of the ensuing flow is also three-dimensional if the current term ψ is non-constant. To calculate the vorticity we find the inverse of the Jacobian (3.4),

$$\begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial r}{\partial z} \end{pmatrix} = \frac{1}{1-e^{2\xi}} \begin{pmatrix} 1+e^{\xi}\cos\theta & -t[\psi_q(1+e^{\xi}\cos\theta)+\psi_r e^{\xi}\sin\theta] & e^{\xi}\sin\theta \\ 0 & 1-e^{2\xi} & 0 \\ e^{\xi}\sin\theta & -t[\psi_r(1-e^{\xi}\cos\theta)+\psi_q e^{\xi}\sin\theta] & 1-e^{\xi}\cos\theta \end{pmatrix}. \tag{3.5}$$

Then we have

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial r}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial q} & \frac{\partial v}{\partial q} & \frac{\partial w}{\partial q} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} & \frac{\partial w}{\partial s} \\ \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial w}{\partial r} \end{pmatrix} = \frac{1}{1-e^{2\xi}} \times$$

$$\begin{pmatrix} 1+e^{\xi}\cos\theta & -t[\psi_q(1+e^{\xi}\cos\theta)+\psi_r e^{\xi}\sin\theta] & e^{\xi}\sin\theta \\ 0 & 1-e^{2\xi} & 0 \\ e^{\xi}\sin\theta & -t[\psi_r(1-e^{\xi}\cos\theta)+\psi_q e^{\xi}\sin\theta] & 1-e^{\xi}\cos\theta \end{pmatrix} \begin{pmatrix} -kce^{\xi}\sin\theta & \psi_q & kce^{\xi}\cos\theta \\ 0 & 0 & 0 \\ kce^{\xi}\cos\theta & \psi_r & kce^{\xi}\sin\theta \end{pmatrix}$$

$$= \frac{1}{1-e^{2\xi}} \begin{pmatrix} -cke^{\xi}\sin\theta & \psi_q(1+e^{\xi}\cos\theta)+\psi_r e^{\xi}\sin\theta & cke^{\xi}(\cos\theta+e^{\xi}) \\ 0 & 0 & 0 \\ cke^{\xi}(\cos\theta-e^{\xi}) & \psi_q e^{\xi}\sin\theta+\psi_r(1-e^{\xi}\cos\theta) & cke^{\xi}\sin\theta \end{pmatrix},$$

and so the vorticity, defined by $\boldsymbol{\omega} = \nabla \times \mathbf{U} = (w_y - v_z, u_z - w_x, v_x - u_y)$, takes the form

$$\boldsymbol{\omega} = \left(-\psi_q e^{kr}\sin\theta - \psi_r(1-e^{kr}\cos\theta), -\frac{2kce^{2kr}}{1-e^{2kr}}, \psi_q(1+e^{kr}\cos\theta)+\psi_r e^{kr}\sin\theta\right).$$

It is interesting to note that if ψ is non-constant then the vorticity $\boldsymbol{\omega}$ is three-dimensional and it also has a steady periodic time-dependence owing to the presence of the θ term. In all circumstances vorticity is present in the flow. This fact is in stark contrast with Stokes waves, in which case the assumption of irrotational flow prevents closed particle paths (see the discussion in [10,13,18]). In our setting, for $\psi \equiv 0$ in (3.2), we permit circular particle paths as is the case for Gerstner's wave— see [16]. To prove that (3.2) defines an exact solution of (3.1), we calculate

$$\frac{Du}{Dt} = kc^2 e^{\xi}\sin\theta, \tag{3.6a}$$
$$\frac{Dv}{Dt} = 0, \tag{3.6b}$$
$$\frac{Dw}{Dt} = -kc^2 e^{\xi}\cos\theta, \tag{3.6c}$$

and inserting the terms from (3.3) and (3.6) into (3.1) gives us

$$P_x = -\rho(kc^2 e^{\xi}\sin\theta + 2\Omega c e^{\xi}\sin\theta), \tag{3.7a}$$
$$P_y = 0, \tag{3.7b}$$
$$P_z = -\rho(-kc^2 e^{\xi}\cos\theta - 2\Omega c e^{\xi}\cos\theta + g). \tag{3.7c}$$

Multiplying both sides of (3.7) by the Jacobian matrix (3.4) we derive the following expression for the pressure gradient in terms of the Lagrangian variables

$$\begin{pmatrix} P_q \\ P_s \\ P_r \end{pmatrix} = -\rho \begin{pmatrix} (kc^2 + 2\Omega c - g)e^{\xi}\sin\theta \\ 0 \\ -(kc^2+2\Omega c)e^{2\xi} - (kc^2 + 2\Omega c - g)e^{\xi}\cos\theta + g \end{pmatrix}. \tag{3.8}$$

If we can prescribe a suitable expression for the pressure function P such that (3.8) holds it will the follow that (3.2) is indeed an exact solution of the governing equations (2.2). Choosing the expression

$$\tilde{P} = \rho\frac{kc^2 + 2\Omega c}{2k}e^{2\xi} - \rho g r + \rho\frac{kc^2 + 2\Omega c - g}{k}e^{\xi}\cos\theta + P_0 \tag{3.9}$$

we have

$$\begin{aligned}
\tilde{P}_q &= -\rho(kc^2 + 2\Omega c - g)e^{\xi}\sin\theta \\
\tilde{P}_s &= 0 \\
\tilde{P}_r &= \rho(kc^2 + 2\Omega c)e^{2\xi} - \rho g + \rho(kc^2 + 2\Omega c - g)e^{\xi}\cos\theta,
\end{aligned}$$

which matches the right hand side of (3.8). The pressure must be time independent on the surface due to (2.2d) so we need to eliminate terms containing θ in (3.9). Therefore

$$kc^2 + 2\Omega c - g = 0, \tag{3.10}$$

which we solve to get the dispersion relation

$$c = \frac{\pm\sqrt{\Omega^2 + kg} - \Omega}{k}. \tag{3.11}$$

We note that choosing the plus sign above gives $c > 0$ for which the wavelike term is eastward propagating, whereas choosing the minus sign gives $c < 0$ and so the wavelike term is then propagating westwards. This freedom in direction is an artefact of the $f-$plane governing equations, such a situation is excluded in the broader $\beta-$plane setting whereby waves are exclusively eastward propagating [7,20]. Piecing everything together, expression (3.9) reduces to

$$P = \rho g\left(\frac{e^{2kr}}{2k} - r\right) + P_0 - \rho g\left(\frac{e^{2kr_0}}{2k} - r_0\right), \tag{3.12}$$

which is a suitable expression for the pressure distribution in the fluid. The above considerations prove that the flow determined by (3.2) satisfies the full set of governing equations (2.2).

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