A Constraint and Position Identification (CPI) Approach for the Synthesis of Decoupled Spatial Translational Compliant Parallel Manipulators

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This paper introduces a screw theory based method termed constraint and position identification (CPI) approach to synthesize decoupled spatial translational compliant parallel manipulators (XYZ CPMs) with consideration of actuation isolation. The proposed approach is based on a systematic arrangement of rigid stages and compliant modules in a three-legged XYZ CPM system using the constraint spaces and the position spaces of the compliant modules. The constraint spaces and the position spaces are firstly derived based on the screw theory instead of using the rigid-body mechanism design experience. Additionally, the constraint spaces are classified into different constraint combinations, with typical position spaces depicted via geometric entities. Furthermore, the systematic synthesis process based on the constraint combinations and the geometric entities is demonstrated via several examples. Finally, several novel decoupled XYZ CPMs with monolithic configurations are created and verified by finite elements analysis. The present CPI approach enables experts and beginners to synthesize a variety of decoupled XYZ CPMs with consideration of actuation isolation by selecting an appropriate constraint and an optimal position for each of the compliant modules according to a specific application.

Keywords: *compliant parallel manipulator; conceptual design; position space; constraint space; screw theory*

1. Introduction

The last decade has witnessed the rapid development of compliant parallel manipulators (CPMs). CPMs are used to transmit/transform displacements, forces and energy via the deflection of their flexible parts, rather than employing traditional sliding or rolling interfaces [1]. The application of CPMs has been experiencing a rapid increase due to their advantages such as high precision, simplified manufacture, low part count, and ability to miniaturize [2-6]. Planar translational CPMs [7] play a very important role in a variety of fields such as the atomic force microscope [8, 9], micro-assembly [10], data storage [11], and MEMS process [12]. Recent years, XYZ CPMs are gaining more and more attentions in a wide range of potential applications, especially in micro/nanomanipulation, adjustable mounting, precision alignment and actuation instruments, MEMS sensors and actuators, energy harvesting, as well as consumer products [13-15].

Desired XYZ CPMs should provide highly decoupled translations along three orthogonal axes (X, Y and Z), while the rotational stiffness about the three axes should be much higher than the translational stiffness along the three axes [14, 15]. An XYZ CPM has three non-redundant parallel legs where each leg should assist to actuate one of the translations between the motion stage (MS) and the base stages (BSs). The legs should have the following characteristics [15, 16]: (a) each leg should have an actuated stage (AS) that is permitted to translate in the actuation direction only for actuation isolation (only widely used linear translational actuators such as voice coils, linear motors, and piezoelectric stacks are considered, which cannot tolerate off-axis loads or displacements); (b) the translational motion of an AS should be transmitted to the MS by the leg without influencing the other motions of the MS; and (c) undesired rotational motions of the MS should be constrained by the three legs. Considering the above desired characteristics, the synthesis of XYZ CPMs is still a challenging issue [15].

Most existing XYZ CPMs were designed based on qualitative arguments and rationale [15]. Recent years, many XYZ CPMs were synthesized based on pseudo-rigid-body model (PRBM) substitution approach including the direct substitution and the building block based substitution [14, 17-19]. This method often begins with a rigid-body manipulator which can provide the same motions as the desired XYZ CPM. And then the rigid-body manipulator can be converted to an XYZ CPM by replacing the rigid kinematic joints/chains with equivalent compliant modules/building blocks [3]. It is an efficient approach to synthesize XYZ CPMs for the designers who have a lot of knowledge about the type of rigid-body parallel manipulators. However, this PRBM substitution method only leads designers to the XYZ CPMs which are characterized by the associated rigid-body manipulators [3]. Fortunately, a constraint map for the XYZ CPM synthesis was proposed by Awtar et al in [15].

This constraint map decomposes an XYZ CPM into rigid stages and compliant modules, and each compliant module is allocated specific constraints. In [15], all compliant modules were designed based on the allocated constraints, and an XYZ CPM was presented via assembling the rigid stages and the compliant modules together. As known, different compliant modules can be designed on the basis of the specific constraints, thus new XYZ CPMs can be obtained through replacing the compliant modules of the XYZ CPM designed in [15] with other compliant modules with the same constraints. However, the constraint map is only suitable for certain types of XYZ CPMs, as the constraints of the compliant modules in the XYZ CPMs as shown in [17-19] are beyond the range of the constraint map. Overall, the existing methods of synthesizing XYZ CPMs are limited to some types of XYZ CPMs; hence a more efficient approach of synthesizing XYZ CPMs with consideration of the actuation isolation is needed.

The existing approaches to design compliant modules/joints such as the constraint-based method [20], the screw-theory-based method [21-26], and the freedom and constraint topology (FACT) approach [27-31] have been adopted widely. The constraint-based method is based on the concepts: (a) the motions of a given module is determined by the locations and orientations of the constraint members, wire beams and/or sheets, applied on it; and (b) one non-redundant constraint removes one degree of freedom (DOF) from the given module [32-34]. The constraint-based method is widely used to synthesize compliant modules via selecting appropriate constraints and identifying their locations and orientations [20]. The screw-theory based method and the FACT approach provide mathematical expressions and geometric shapes, respectively, to efficiently help designers identify the location and orientation of each constraint member [21-31]. Besides, the screw-theory-based method and the FACT approach were also employed to design decoupled XYZ CPMs [24-26, 31]; however, the actuation isolation [15] was not taken into consideration. Hopkins et al [35] recently reported several planar CPMs that can decouple displacement-based actuators using the FACT approach. But the FACT method still has the difficulty in synthesizing XYZ CPMs since there is no a three-DOF translational DOF space for a parallel module.

Based on the statements above, an ideal XYZ CPM design process can rely on the following steps: (a) decomposing an XYZ CPM into rigid stages and compliant modules; (b) designing the rigid stages; (c) designing the compliant modules individually using the existing methods, and (d) assembling the rigid stages and compliant modules together. In this design process, the main work is to identify the constraints and positions (locations and orientations) of the compliant modules, which are used to design and assemble the compliant modules. This paper will introduce a screw theory based approach termed constraint and position identification (CPI) approach to identify the constraints and positions of the compliant modules followed by a systematic XYZ CPM design procedure. The constraints of the compliant modules identified by the CPI approach can produce much more types of XYZ CPMs compared with the constraints obtained using the constraint map in [15].

The CPI approach focuses on the conceptual design of decoupled XYZ CPMs with consideration of actuation isolation to meet the kinematic requirements. In early-stage conceptual design, all wire beams are regarded as ideal constraints, which means that the stiffness along the constraint line is infinite large, but those along other directions are infinite small. In addition, this paper assumes that all the compliant modules produced by the decomposition of an XYZ CPM are independent of each other. This means each of the compliant modules can constrain a DOF of the connecting rigid stage totally depending on its own structure rather than using the help of other compliant modules.

The remainder of this paper is organized as follows. Sections 2 and 3 consider the fundamental theories to be used in the CPI approach and the XYZ CPM decomposition, respectively. In Sections 4 and 5, constraint spaces and position spaces of the compliant modules are studied. The design procedure based on the CPI approach is summarized in Section 6, followed by several detailed example demonstrations in Section 7. Finally, the conclusions are drawn in Section 8.

2. Background knowledge

2.1. Twist and wrench

It is well known that motions and constraints can be represented by screw vectors, twists and wrenches, respectively [36]. Twists and wrenches can be represented as twist lines and wrench lines, with locations, orientations and pitches. The pitch describes the translational displacement per rotation in the twist, and it also refers to the coupling between the translational force and the rotational force in the wrench. In practice, a translational constraint can restrict the translational motion along the wrench line with two opposite directions. Similarly, a rotational constraint can restrict the rotational motion about the wrench line with two opposite directions.

Therefore, a constraint can be represented as a wrench with two possible opposite directions, which can be written as Eq. (1) and illustrated in Fig. 1(a).

$$\zeta=\begin{bmatrix}\boldsymbol{f}j\\ \boldsymbol{\tau}j\end{bmatrix}=\begin{cases}\begin{bmatrix}\boldsymbol{f}j & \boldsymbol{r}\times\boldsymbol{f}j+q\boldsymbol{f}j\end{bmatrix}^{\mathrm{T}} & \text{force and moment}\\ \begin{bmatrix}\boldsymbol{f}j & \boldsymbol{r}\times\boldsymbol{f}j\end{bmatrix}^{\mathrm{T}} & q=0,\text{ pure force}\\ \begin{bmatrix}\boldsymbol{0} & \boldsymbol{\tau}j\end{bmatrix}^{\mathrm{T}} & q\to\infty,\text{pure moment}\end{cases} \tag{1}$$

where $\boldsymbol{\zeta}$ is a wrench. $\boldsymbol{f}$ and $\boldsymbol{\tau}$ are two three-dimensional vectors which represent translational and rotational loads, respectively. $\boldsymbol{r}$ is a 3×1 location vector which points from the origin of the coordinate system to a point on the wrench line. The pitch is defined by $q=(\boldsymbol{f}\cdot\boldsymbol{\tau})/(\boldsymbol{f}\cdot\boldsymbol{f})$. Here, $j=\pm1$ which describes the two possible opposite directions of the wrench.

The motion that can be restricted by a constraint can be represented as a twist with two possible opposite directions, which can be written as Eq. (2) and illustrated in Fig. 1(b).

$$\xi=\begin{bmatrix}\boldsymbol{w}i\\ \boldsymbol{v}i\end{bmatrix}=\begin{cases}\begin{bmatrix}\boldsymbol{w}i & \boldsymbol{c}\times\boldsymbol{w}i+p\boldsymbol{w}i\end{bmatrix}^{\mathrm{T}} & \text{screw motion}\\ \begin{bmatrix}\boldsymbol{w}i & \boldsymbol{c}\times\boldsymbol{w}i\end{bmatrix}^{\mathrm{T}} & p=0,\text{ pure rotation}\\ \begin{bmatrix}\boldsymbol{0} & \boldsymbol{v}i\end{bmatrix}^{\mathrm{T}} & p\to\infty,\text{ pure translation}\end{cases} \tag{2}$$

where $\boldsymbol{\xi}$ is a screw vector associated with the motion, $\boldsymbol{w}$ and $\boldsymbol{v}$ are two three-dimensional vectors which represent rotational and translational motions, respectively. $\boldsymbol{c}$ is a 3×1 location vector which points from the origin of the coordinate system to a point on the twist line. The pitch is defined by $p=(\boldsymbol{w}\cdot\boldsymbol{v})/(\boldsymbol{w}\cdot\boldsymbol{w})$. And $i=\pm1$ which describes the two possible opposite directions of the twist.

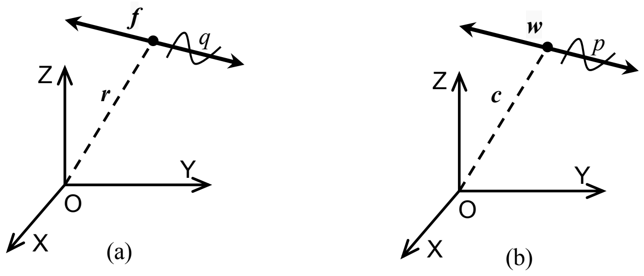


Fig. 1 Illustration of a wrench and a twist: (a) a wrench $\boldsymbol{\zeta}$ with a location vector $\boldsymbol{r}$, an orientation vector $\boldsymbol{f}$, and a scalar value of pitch q; and (b) a twist $\boldsymbol{\xi}$ with a location vector $\boldsymbol{c}$, an orientation vector $\boldsymbol{w}$, and a scalar value of pitch p.

In a coordinate system O-XYZ, the twists and wrenches along and about the three axes are defined as principal twists as shown in Fig. 2(a) and principal wrenches as shown in Fig. 2(b), respectively [23]. Based on Eqs. (1) and (2), the principal twists and the principal wrenches can be written as

$$\left.\begin{aligned}\xi_{rx}&=[i_{rx}\quad 0\quad 0\quad 0\quad 0\quad 0]^{T}\\ \xi_{ry}&=[0\quad i_{ry}\quad 0\quad 0\quad 0\quad 0]^{T}\\ \xi_{rz}&=[0\quad 0\quad i_{rz}\quad 0\quad 0\quad 0]^{T}\\ \xi_{tx}&=[0\quad 0\quad 0\quad i_{tx}\quad 0\quad 0]^{T}\\ \xi_{ty}&=[0\quad 0\quad 0\quad 0\quad i_{ty}\quad 0]^{T}\\ \xi_{tz}&=[0\quad 0\quad 0\quad 0\quad 0\quad i_{tz}]^{T}\end{aligned}\right\}\ \text{and}\ \left.\begin{aligned}\zeta_{tx}&=[j_{tx}\quad 0\quad 0\quad 0\quad 0\quad 0]^{T}\\ \zeta_{ty}&=[0\quad j_{ty}\quad 0\quad 0\quad 0\quad 0]^{T}\\ \zeta_{tz}&=[0\quad 0\quad j_{tz}\quad 0\quad 0\quad 0]^{T}\\ \zeta_{rx}&=[0\quad 0\quad 0\quad j_{rx}\quad 0\quad 0]^{T}\\ \zeta_{ry}&=[0\quad 0\quad 0\quad 0\quad j_{ry}\quad 0]^{T}\\ \zeta_{rz}&=[0\quad 0\quad 0\quad 0\quad 0\quad j_{rz}]^{T}\end{aligned}\right\} \tag{3}$$

where the subscripts tx, ty and tz indicate the translations along X-, Y- and Z-axes, and the subscripts rx, ry and rz indicate the rotations about X-, Y- and Z-axes, respectively. All non-zero elements in the screw vectors equal to ±1, which represent the two possible opposite directions of the screw vector. Obviously, any one twist or wrench can be described as the linear combination of the principal twists or principal wrenches.

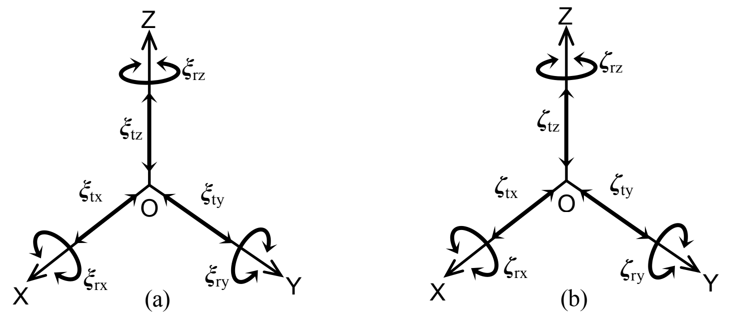


Fig. 2 Illustration of principal twists and principal wrenches: (a) principal twists, and (b) principal wrenches.

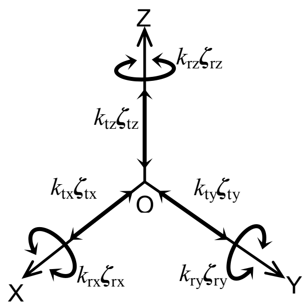


Fig. 3 Representation of the linear combination of the principal wrenches.

In an O-XYZ coordinate system, the constraint of a compliant module can be described as the linear combination of the principal wrenches as shown in Fig. 3, denoted by

$$\zeta=k_{tx}\zeta_{tx}+k_{ty}\zeta_{ty}+k_{tz}\zeta_{tz}+k_{rx}\zeta_{rx}+k_{ry}\zeta_{ry}+k_{rz}\zeta_{rz} \tag{4}$$

where k_{tx}, k_{ty}, k_{tz}, k_{rx}, k_{ry} and k_{rz} are termed stiffness coefficients as the value of each stiffness coefficient is directly proportional to the stiffness of the compliant module in the same direction. For a compliant module, if the stiffness in a direction is infinitely large compared with others, the compliant module has a degree of constraint (DOC) in this direction; else if the stiffness in a direction is infinitely small compared with others, the compliant module has a DOF in this direction. For convenience, this paper specifies that the stiffness coefficient equals 1 if the associated stiffness is infinitely large, while the stiffness coefficient equals 0 if the associated stiffness is infinitely small. In other words, if a stiffness coefficient equals to 1, the direction associated with the stiffness coefficient is the direction of a DOC, otherwise it is the direction of a DOF.

The principal twist or the principal wrench has two possible opposite directions, and the instantaneous direction of a principal wrench is always in the opposite instantaneous direction of the counterpart principal twist. Take $\boldsymbol{\zeta}_{\mathrm{tx}}$ and $\boldsymbol{\xi}_{\mathrm{tx}}$ as shown in Fig. 4(a) for instance, the wrench $\boldsymbol{\zeta}_{\mathrm{tx}}$ always resists the motion denoted by the twist $\boldsymbol{\xi}_{\mathrm{tx}}$, hence the instantaneous directions of the wrench and the twist are always opposite, which are represented in Figs. 4(b) and 4(c).

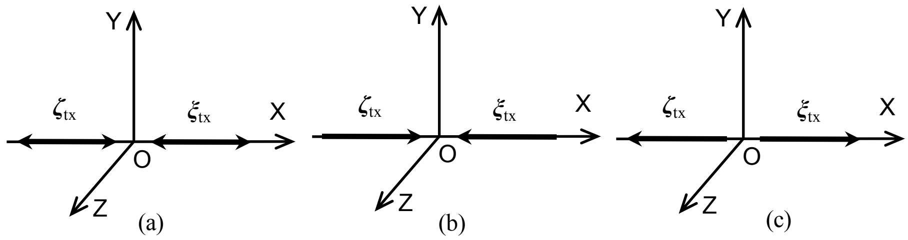


Fig. 4 The instantaneous direction of a principal wrench: (a) ζ_{tx} and ξ_{tx} are bidirectional screw vectors, (b) and (c) ζ_{tx} is always in the opposite direction of ξ_{tx}.

2.2 DOF identification based on the relationship of twists and wrenches

This section briefly reviews how to identify the DOF of a rigid stage based on the screw theory. The *dot product* of a wrench and a twist in a coordinate system is introduced in Eq. (5) [36, 37].

$$\boldsymbol{\xi}\circ\boldsymbol{\zeta}=\begin{bmatrix}\boldsymbol{v}i & \boldsymbol{w}i\end{bmatrix}\begin{bmatrix}\boldsymbol{f}j\\ \boldsymbol{\tau}j\end{bmatrix}=\boldsymbol{v}i\cdot\boldsymbol{f}j+\boldsymbol{w}i\cdot\boldsymbol{\tau}j \tag{5}$$

The operator '∘' means the *dot product* of a twist and a wrench. If the result of the *dot product* equals to zero, the motion associated with the twist will not produce work under the action of the wrench. In other words, if the *dot product* equals to zero, the wrench will not resist the motion associated with the twist, i.e., the wrench is reciprocal to the twist. Therefore, if a twist of a rigid stage is reciprocal to all wrenches applied to the rigid stage, the motion associated with the twist will not be constrained, and the twist is a DOF of the rigid stage. However, if the *dot product* is not equal to zero, the motion associated with the twist will be constrained, i.e., it is a DOC of the rigid stage.

2.3 Coordinate transformation of wrenches

This section depicts review how to transform a wrench from a coordinate system 'A' to another coordinate system 'B'. Equation (6) [37] shows that a coordinate transformation matrix **T**, which can be used to perform the transformation of a wrench from one coordinate system to another.

$$\mathbf{T}=\begin{bmatrix} \mathbf{R}_{\mathrm{xyz}} & \mathbf{0} \\ \mathbf{D}\times\mathbf{R}_{\mathrm{xyz}} & \mathbf{R}_{\mathrm{xyz}} \end{bmatrix} \text{ where } \mathbf{D}=\begin{bmatrix} 0 & -d_{\mathrm{z}} & d_{\mathrm{y}} \\ d_z & 0 & -d_{\mathrm{x}} \\ -d_{\mathrm{y}} & d_{\mathrm{x}} & 0 \end{bmatrix} \tag{6}$$

where the sub-matrix $\mathbf{R}_{\mathbf{xyz}}$ is a 3×3 rotation matrix and the sub-matrix **D** is a 3×3 location skew-symmetric matrix. Based on this transformation matrix, a wrench in a coordinate system 'A' can be represented in another coordinate system 'B' via pre-multiplying the coordinate transformation matrix **T**. The entries d_{x}, d_{y} and d_{z} in the sub-matrix **D** are the coordinates of the origin of the coordinate system 'A' in the coordinate system 'B'.

3. XYZ CPM decomposition

As mentioned in Section 1, a desired XYZ CPM has three non-redundant parallel legs, each leg of which should assist to actuate one of the translations between the MS and the BSs, and should include an AS. Therefore, the rigid stages in an XYZ CPM are MS, ASs (AS-X, AS-Y and AS-Z are associated with the motions of the MS along the X-, Y- and Z-axes, respectively) and BSs (BS-X, BS-Y and BS-Z are associated with the motions of the MS along the X-, Y- and Z-axes, respectively). The compliant module between the MS and the AS is defined as the passive module (PM) and the compliant module connecting the AS to the BS is termed as the active module (AM). Hence, the compliant modules in an XYZ CPM are the PMs (PM-X, PM-Y and PM-Z are associated with the motions of the MS along the X-, Y- and Z-axes, respectively), and the AMs (AM-X, AM-Y and AM-Z are associated with the motions of the MS along the X-, Y- and Z-axes, respectively). Note that the compliant modules, PMs and AMs, can be of the parallel, serial or hybrid type. The rigid stages and compliant modules of an XYZ CPM are represented in Fig. 5(a).

Note that this paper assumes that the compliant modules, PMs and AMs, are independent of others as mentioned earlier. Due to this assumption, a small number of *decoupled* XYZ CPMs such as the one shown in Appendix A won't be synthesized using the CPI approach, which, however, can be obtained using the appropriate modifications on the decoupled XYZ CPMs proposed in this paper.

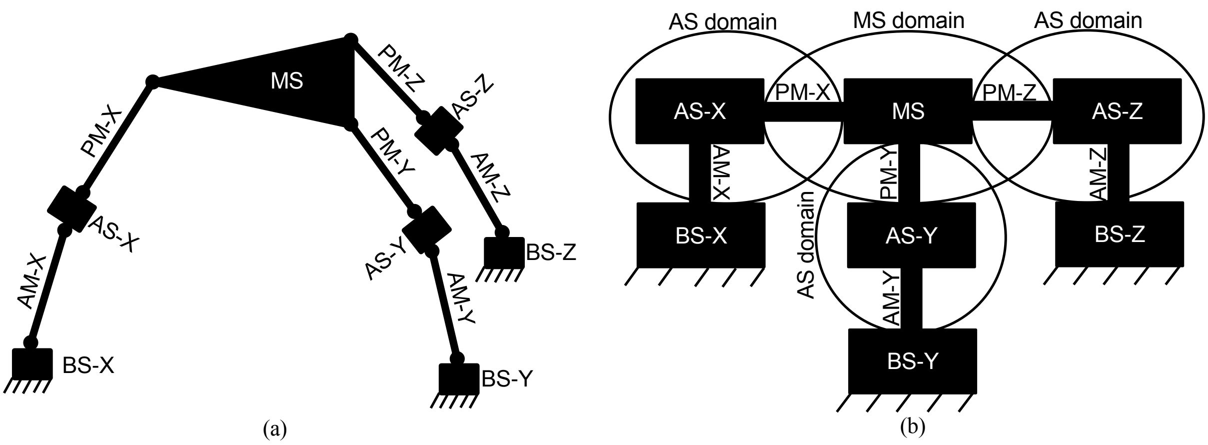


Fig. 5 Illustration of the rigid stages and compliant modules of an XYZ CPM: (a) the rigid stages and compliant modules in an XYZ CPM, and (b) the rigid stages and compliant modules in different analysis domains, MS domain and AS domains.

Based on the above decomposition, the main work of synthesizing an XYZ CPM is to design the compliant modules, which includes: (a) identifying the constraints of the compliant modules to synthesize the compliant modules using the existing design approaches; and (b) identifying the positions (locations and orientations) of the compliant modules in an XYZ CPM. For the XYZ CPM shown in Fig. 5(a), the permitted motions of the MS and the ASs are specified. More specifically, the MS is permitted to translate in the three orthogonal directions, and each AS is constrained to translate in the direction of the force of the actuator. The motions of the MS or the ASs are controlled by the compliant modules connecting with them. Therefore, the constraints of the compliant modules can be identified according to the permitted motions of the MS and ASs. In order to identify the constraints of the compliant modules easily, the rigid stages and the compliant modules are assigned to different analysis domains as shown in Fig. 5(b). The constraints of the compliant modules in a domain are subject to the permitted motions of the rigid stage in the domain. That is to say that the constraints of the MS arise from the three connected PMs in parallel, and the constraints of each AS arise from the adjacent PM and AM in parallel. Therefore, the DOC of the MS equals to the total contribution of the DOC of the three PMs. Similarly, the DOC of each AS equals to the total contribution of the DOC of the PM and the AM in the same leg. This paper decomposes compliant modules based on DOC (or constraints) only since the DOC relationship between a rigid stage and the associated compliant modules can be represented by simple addition expression only. In addition, Due to decomposing compliant mechanisms based on constraints only, the AM does not need to be a translational joint so that some of the AM's constraints can be transmitted to the PM to constrain the AS motion.

The constraint space of a compliant module is the combination of all the permitted constraints of the compliant module. As shown in Fig. 5(b), the three PMs are all included in the MS domain, and there are intersection fields between the MS domain and the AS domains, thus the constraint spaces of the PMs and those of the AMs interrelate with each other. For this reason, this paper names both the constraint spaces of the PMs and those of the AMs as the constraint spaces of compliant modules. The position space of a compliant module is the combination of all permitted positions in an XYZ CPM system where the constraint of this compliant module in the XYZ CPM system remains unchanged if the position of the compliant module changes within the position space. The position space of a compliant module can obtained according to the constraints of the compliant module.

4. Constraint spaces

4.1 Actuation constraints and coordinate system definition

In an XYZ CPM system, the three translational motions of the MS are actuated by the three linear translational actuators, and the forces of the actuators can be regarded as wrenches. A linear translational actuator can provide a translational force along the actuation axis, but cannot tolerate transverse forces/displacements [15]. Therefore, the force of a linear translational actuator can be represented by a pure force wrench. This is also the reason why an AS is allowed to translate in one direction only. The pure force wrenches of the three actuators in an XYZ CPM system can be written as Eq. (7a) based on Eq. (1).

$$\boldsymbol{\zeta}_{\mathrm{x}}=[\boldsymbol{f}_{\mathrm{x}};\ \boldsymbol{r}_{\mathrm{x}}\times\boldsymbol{f}_{\mathrm{x}};]^{\mathrm{T}},\ \boldsymbol{\zeta}_{\mathrm{y}}=[\boldsymbol{f}_{\mathrm{y}};\ \boldsymbol{r}_{\mathrm{y}}\times\boldsymbol{f}_{\mathrm{y}};]^{\mathrm{T}},\ \text{and}\ \boldsymbol{\zeta}_{\mathrm{z}}=[\boldsymbol{f}_{\mathrm{z}};\ \boldsymbol{r}_{\mathrm{z}}\times\boldsymbol{f}_{\mathrm{z}};]^{\mathrm{T}} \tag{7a}$$

where $\boldsymbol{\zeta}_{\mathrm{x}}$, $\boldsymbol{\zeta}_{\mathrm{y}}$ and $\boldsymbol{\zeta}_{\mathrm{z}}$ are the three pure force wrenches in a global coordinate system $\mathrm{O_m}$-$\mathrm{X_mY_mZ_m}$. $\boldsymbol{f}_{\mathrm{x}}$, $\boldsymbol{f}_{\mathrm{y}}$ and $\boldsymbol{f}_{\mathrm{z}}$ are the directions of the three pure force wrenches, and $\boldsymbol{r}_{\mathrm{x}}$, $\boldsymbol{r}_{\mathrm{y}}$ and $\boldsymbol{r}_{\mathrm{z}}$ are the three location vectors. In the global coordinate system, the MS is permitted to translate along the $\mathrm{X_m}$-, $\mathrm{Y_m}$- and $\mathrm{Z_m}$-axes, which can be described as three translational twists $\boldsymbol{\xi}_{\mathrm{m\text{-}tx}}$, $\boldsymbol{\xi}_{\mathrm{m\text{-}ty}}$ and $\boldsymbol{\xi}_{\mathrm{m\text{-}tz}}$ based on Eq. (2). As required by the decoupling characteristic, each actuator drives one of the three motions without influencing other two motions, thus the following equations can be obtained based on Eq. (5).

$$\boldsymbol{\xi}_{\mathrm{m\text{-}tx}}\circ\boldsymbol{\zeta}_{\mathrm{x}}\neq0,\boldsymbol{\xi}_{\mathrm{m\text{-}tx}}\circ\boldsymbol{\zeta}_{\mathrm{y}}=0,\boldsymbol{\xi}_{\mathrm{m\text{-}tx}}\circ\boldsymbol{\zeta}_{\mathrm{z}}=0,\boldsymbol{\xi}_{\mathrm{m\text{-}ty}}\circ\boldsymbol{\zeta}_{\mathrm{x}}=0,\boldsymbol{\xi}_{\mathrm{m\text{-}ty}}\circ\boldsymbol{\zeta}_{\mathrm{y}}\neq0,\boldsymbol{\xi}_{\mathrm{m\text{-}ty}}\circ\boldsymbol{\zeta}_{\mathrm{z}}=0,\boldsymbol{\xi}_{\mathrm{m\text{-}tz}}\circ\boldsymbol{\zeta}_{\mathrm{x}}=0,\boldsymbol{\xi}_{\mathrm{m\text{-}tz}}\circ\boldsymbol{\zeta}_{\mathrm{y}}=0,\ \text{and}\ \boldsymbol{\xi}_{\mathrm{m\text{-}tz}}\circ\boldsymbol{\zeta}_{\mathrm{z}}\neq0 \tag{7b}$$

It can be concluded from Eq. (7b) that: (a) $\boldsymbol{f}_{\mathrm{x}}$, $\boldsymbol{f}_{\mathrm{y}}$ and $\boldsymbol{f}_{\mathrm{z}}$ should be parallel to the $\mathrm{X_m}$-, $\mathrm{Y_m}$- and $\mathrm{Z_m}$-axes, respectively; and (b) $\boldsymbol{r}_{\mathrm{x}}$, $\boldsymbol{r}_{\mathrm{y}}$ and $\boldsymbol{r}_{\mathrm{z}}$ can be any vectors. In other words, the three actuation forces only need to parallel to the $\mathrm{X_m}$-, $\mathrm{Y_m}$- and $\mathrm{Z_m}$-axes, respectively.

A PM is used to transmit the force of an actuator from the AS to the MS without influencing the forces transmitted by the other two PMs, so the PM cannot be compressed and elongated along the direction of the actuation force transmitted by the PM, i.e. the PM has a pure force wrench along the actuation direction. A PM connects the MS and one of the ASs using its two ends, so the force of the actuator is input from the end connecting the AS and output from the end connecting the MS. Both the input force and the output force of a PM should be parallel to the axis of the global coordinate system, but the input force and the output force are not always collinear. According to the description above, three PM local coordinate systems (O_{px}-$X_{px}Y_{px}Z_{px}$, O_{py}-$X_{py}Y_{py}Z_{py}$ and O_{pz}-$X_{pz}Y_{pz}Z_{pz}$) and three AM local coordinate systems (O_{ax}-$X_{ax}Y_{ax}Z_{ax}$, O_{ay}-$X_{ay}Y_{ay}Z_{ay}$ and O_{az}-$X_{az}Y_{az}Z_{az}$) are set up as shown in Fig. 6.

Figure 6 shows that: (a) the MS is permitted to move in the X_m-, Y_m- and Z_m-axes of the global coordinate system; (b) the PM local coordinate systems are located at the center points of the interfaces between the MS and the three PMs, and are fixed to the MS; (c) each PM is represented as two parallel lines, a parallel line connector, and two small virtual rigid stages which are used to connect the PM to the MS and the AS. The two parallel lines indicate the input direction and the output direction of the actuation force transmitted by the PM; (d) each PM has a pure force wrench along the X-axis of the PM local coordinate system; (e) each AS is constrained to translate only along the X-axis of the PM coordinate system; (f) the AM local coordinate systems are located at the center points of the interfaces between the ASs and the AMs, and fixed to the ASs; (g) each AM is expressed by one straight line and two virtual rigid stages which are used to connect the AM to the AS and the BS; and (h) each BS is bound to the ground, whose six DOF are completely constrained by the ground. The constraint spaces and the position spaces will be identified based on the established coordinate systems (Fig. 6) in the next two sections.

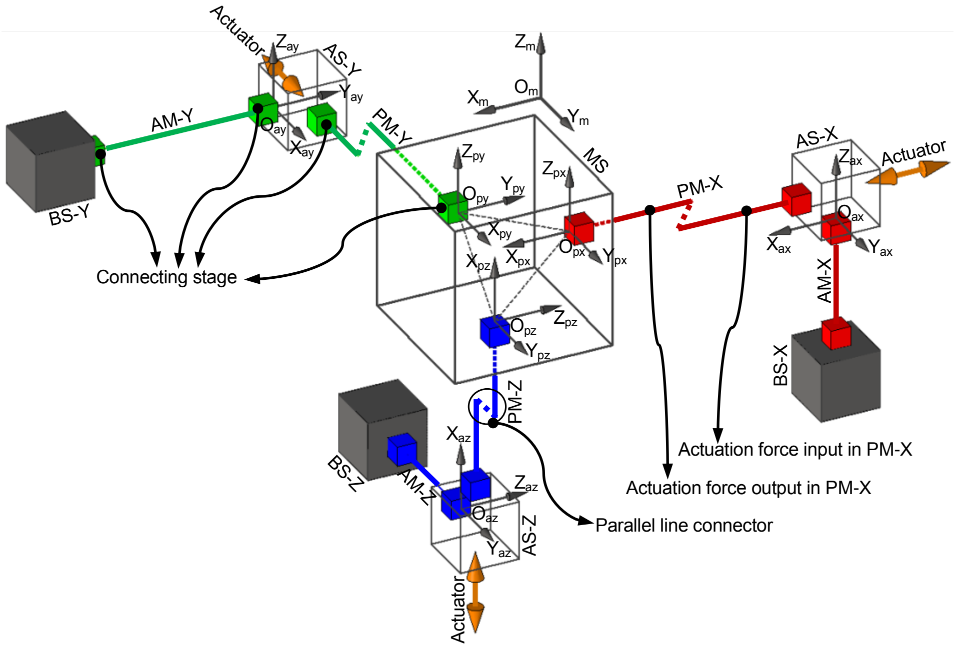


Fig. 6 Illustration of the rigid stages, compliant modules and actuators in an XYZ CPM system and representation of the coordinate systems (global coordinate system O_m-$X_mY_mZ_m$; PM local coordinate systems O_{px}-$X_{px}Y_{px}Z_{px}$, O_{py}-$X_{py}Y_{py}Z_{py}$ and O_{pz}-$X_{pz}Y_{pz}Z_{pz}$; and AM local coordinate systems O_{ax}-$X_{ax}Y_{ax}Z_{ax}$, O_{ay}-$X_{ay}Y_{ay}Z_{ay}$ and O_{az}-$X_{az}Y_{az}Z_{az}$). (Online version in color.)

4.2 Constraint space identification

The motions of the rigid stages and the constraints of the compliant modules are defined as shown in Table 1. In Table 1, $\xi_{*\text{-tx}}$, $\xi_{*\text{-ty}}$, $\xi_{*\text{-tz}}$, $\xi_{*\text{-rx}}$, $\xi_{*\text{-ry}}$, and $\xi_{*\text{-rz}}$ ('$*$' denotes m, ax, ay, az, bx, by or bz) are the three translational DOF along X-, Y- and Z-axes and three rotational DOF about X-, Y- and Z-axes of the global or local coordinate systems. ζ_{px}, ζ_{py} and ζ_{pz} are the constraints of the three PMs, and ζ_{ax}, ζ_{ay} and ζ_{az} are the constraints of the three AMs. $\zeta_{*\text{-tx}}$, $\zeta_{*\text{-ty}}$, $\zeta_{*\text{-tz}}$, $\zeta_{*\text{-rx}}$, $\zeta_{*\text{-ry}}$ and $\zeta_{*\text{-rz}}$ ('$*$' denotes px, py, pz, ax, ay, or az) are the principal wrenches in the local coordinate systems, and $k_{*\text{-tx}}$, $k_{*\text{-ty}}$, $k_{*\text{-tz}}$, $k_{*\text{-rx}}$, $k_{*\text{-ry}}$ and $k_{*\text{-rz}}$ ('$*$' can be px, py, pz, ax, ay, or az) are the stiffness coefficients in the local coordinate systems.

For an XYZ CPM, the motions associated with $\xi_{m\text{-tx}}$, $\xi_{m\text{-ty}}$, $\xi_{m\text{-tz}}$, $\xi_{ax\text{-tx}}$, $\xi_{ay\text{-tx}}$, and $\xi_{az\text{-tx}}$ are permitted, while the others should be constrained. All the motions of the BSs are restricted by the ground. Moreover, the permitted constraints of the compliant modules can be identified in the different domains based on Eq. (5). The twist and the wrench in Eq. (5) should be in the same coordinate system, so the wrenches in Table 1 should be transformed to the appropriate coordinate systems. More specifically, the ζ_{px}, ζ_{py} and ζ_{pz} should be transformed to the global coordinate system, because these wrenches are used to identify whether the twists $\xi_{m\text{-tx}}$, $\xi_{m\text{-ty}}$, $\xi_{m\text{-tz}}$, $\xi_{m\text{-rx}}$, $\xi_{m\text{-ry}}$ and $\xi_{m\text{-rz}}$ are constrained or not. Similarly, the ζ_{ax}, ζ_{ay} and ζ_{az} should be transformed to the coordinate system O_{px}-$X_{px}Y_{px}Z_{px}$, O_{py}-$X_{py}Y_{py}Z_{py}$ and O_{pz}-$X_{pz}Y_{pz}Z_{pz}$, respectively. These coordinate transformation matrices can be derived as shown in Eqs. (8) and (9) based on Eq. (6).

Table 1 The definitions of the motions of the rigid stages and the constraints of the compliant modules in an XYZ CPM.

Item	Motion of rigid stage	Constraint of compliant module	Coordinate system
MS	$\xi_{m\text{-}tx}$, $\xi_{m\text{-}ty}$, $\xi_{m\text{-}tz}$, $\xi_{m\text{-}rx}$, $\xi_{m\text{-}ry}$, $\xi_{m\text{-}rz}$	/	O_m-$X_mY_mZ_m$
PM-X	/	$\zeta_{px}= k_{px\text{-}tx}\zeta_{px\text{-}tx}+ k_{px\text{-}ty}\zeta_{px\text{-}ty}+ k_{px\text{-}tz}\zeta_{px\text{-}tz}+ k_{px\text{-}rx}\zeta_{px\text{-}rx} + k_{px\text{-}ry}\zeta_{px\text{-}ry}+ k_{px\text{-}rz}\zeta_{px\text{-}rz}$	O_{px}-$X_{px}Y_{px}Z_{px}$
PM-Y	/	$\zeta_{py}= k_{py\text{-}tx}\zeta_{py\text{-}tx}+ k_{py\text{-}ty}\zeta_{py\text{-}ty}+ k_{py\text{-}tz}\zeta_{py\text{-}tz}+ k_{py\text{-}rx}\zeta_{py\text{-}rx} + k_{py\text{-}ry}\zeta_{py\text{-}ry}+ k_{py\text{-}rz}\zeta_{py\text{-}rz}$	O_{py}-$X_{py}Y_{py}Z_{py}$
PM-Z	/	$\zeta_{pz}= k_{pz\text{-}tx}\zeta_{pz\text{-}tx}+ k_{pz\text{-}ty}\zeta_{pz\text{-}ty}+ k_{pz\text{-}tz}\zeta_{pz\text{-}tz}+ k_{pz\text{-}rx}\zeta_{pz\text{-}rx} + k_{pz\text{-}ry}\zeta_{pz\text{-}ry}+ k_{pz\text{-}rz}\zeta_{pz\text{-}rz}$	O_{pz}-$X_{pz}Y_{pz}Z_{pz}$
AS-X	$\xi_{ax\text{-}tx}$, $\xi_{ax\text{-}ty}$, $\xi_{ax\text{-}tz}$, $\xi_{ax\text{-}rx}$, $\xi_{ax\text{-}ry}$, $\xi_{ax\text{-}rz}$	/	O_{px}-$X_{px}Y_{px}Z_{px}$
AS-Y	$\xi_{ay\text{-}tx}$, $\xi_{ay\text{-}ty}$, $\xi_{ay\text{-}tz}$, $\xi_{ay\text{-}rx}$, $\xi_{ay\text{-}ry}$, $\xi_{ay\text{-}rz}$	/	O_{py}-$X_{py}Y_{py}Z_{py}$
AS-Z	$\xi_{az\text{-}tx}$, $\xi_{az\text{-}ty}$, $\xi_{az\text{-}tz}$, $\xi_{az\text{-}rx}$, $\xi_{az\text{-}ry}$, $\xi_{az\text{-}rz}$	/	O_{pz}-$X_{pz}Y_{pz}Z_{pz}$
AM-X	/	$\zeta_{ax}= k_{ax\text{-}tx}\zeta_{ax\text{-}tx}+ k_{ax\text{-}ty}\zeta_{ax\text{-}ty}+ k_{ax\text{-}tz}\zeta_{ax\text{-}tz}+ k_{ax\text{-}rx}\zeta_{ax\text{-}rx} + k_{ax\text{-}ry}\zeta_{ax\text{-}ry}+ k_{ax\text{-}rz}\zeta_{ax\text{-}rz}$	O_{ax}-$X_{ax}Y_{ax}Z_{ax}$
AM-Y	/	$\zeta_{ay}= k_{ay\text{-}tx}\zeta_{ay\text{-}tx}+ k_{ay\text{-}ty}\zeta_{ay\text{-}ty}+ k_{ay\text{-}tz}\zeta_{ay\text{-}tz}+ k_{ay\text{-}rx}\zeta_{ay\text{-}rx} + k_{ay\text{-}ry}\zeta_{ay\text{-}ry}+ k_{ay\text{-}rz}\zeta_{ay\text{-}rz}$	O_{ay}-$X_{ay}Y_{ay}Z_{ay}$
AM-Z	/	$\zeta_{az}= k_{az\text{-}tx}\zeta_{az\text{-}tx}+ k_{az\text{-}ty}\zeta_{az\text{-}ty}+ k_{az\text{-}tz}\zeta_{az\text{-}tz}+ k_{az\text{-}rx}\zeta_{az\text{-}rx} + k_{az\text{-}ry}\zeta_{az\text{-}ry}+ k_{az\text{-}rz}\zeta_{az\text{-}rz}$	O_{az}-$X_{az}Y_{az}Z_{az}$
BS-X	$\xi_{bx\text{-}tx}$, $\xi_{bx\text{-}ty}$, $\xi_{bx\text{-}tz}$, $\xi_{bx\text{-}rx}$, $\xi_{bx\text{-}ry}$, $\xi_{bx\text{-}rz}$	/	O_m-$X_mY_mZ_m$
BS-Y	$\xi_{by\text{-}tx}$, $\xi_{by\text{-}ty}$, $\xi_{by\text{-}tz}$, $\xi_{by\text{-}rx}$, $\xi_{by\text{-}ry}$, $\xi_{by\text{-}rz}$	/	O_m-$X_mY_mZ_m$
BS-Z	$\xi_{bz\text{-}tx}$, $\xi_{bz\text{-}ty}$, $\xi_{bz\text{-}tz}$, $\xi_{bz\text{-}rx}$, $\xi_{bz\text{-}ry}$, $\xi_{bz\text{-}rz}$	/	O_m-$X_mY_mZ_m$

$$\mathbf{T}_{px}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -z_{px} & y_{px} & 1 & 0 & 0 \\ z_{px} & 0 & -x_{px} & 0 & 1 & 0 \\ -y_{px} & x_{px} & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{T}_{py}=\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -z_{py} & 0 & y_{py} & 0 & -1 & 0 \\ 0 & -z_{py} & -x_{py} & 1 & 0 & 0 \\ x_{py} & y_{py} & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{T}_{pz}=\begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ y_{pz} & -z_{pz} & 0 & 0 & 0 & -1 \\ -x_{pz} & 0 & -z_{pz} & 0 & 1 & 0 \\ 0 & x_{pz} & y_{pz} & 1 & 0 & 0 \end{bmatrix} \tag{8}$$

where $\mathbf{T}_{px}$, $\mathbf{T}_{py}$ and $\mathbf{T}_{pz}$ are the coordinate transformation matrices from the PM local coordinate systems to the global coordinate system. Points (x_{px}, y_{px}, z_{px}), (x_{py}, y_{py}, z_{py}) and (x_{pz}, y_{pz}, z_{pz}) are the coordinates of the origins of the PM local coordinate systems in the global coordinate system.

$$\mathbf{T}_{ax}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -z_{ax} & y_{ax} & 1 & 0 & 0 \\ z_{ax} & 0 & -x_{ax} & 0 & 1 & 0 \\ -y_{ax} & x_{ax} & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{T}_{ay}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -z_{ay} & y_{ay} & 1 & 0 & 0 \\ z_{ay} & 0 & -x_{ay} & 0 & 1 & 0 \\ -y_{ay} & x_{ay} & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{T}_{az}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -z_{az} & y_{az} & 1 & 0 & 0 \\ z_{az} & 0 & -x_{az} & 0 & 1 & 0 \\ -y_{az} & x_{az} & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

where $\mathbf{T}_{ax}$, $\mathbf{T}_{ay}$ and $\mathbf{T}_{az}$ are the coordinate transformation matrices from the AM local coordinate systems O_{ax}-$X_{ax}Y_{ax}Z_{ax}$, O_{ay}-$X_{ay}Y_{ay}Z_{ay}$ and O_{az}-$X_{az}Y_{az}Z_{az}$ to the PM local coordinate systems O_{px}-$X_{px}Y_{px}Z_{px}$, O_{py}-$X_{py}Y_{py}Z_{py}$ and O_{pz}-$X_{pz}Y_{pz}Z_{pz}$, respectively. Points (x_{ax}, y_{ax}, z_{ax}), (x_{ay}, y_{ay}, z_{ay}) and (x_{az}, y_{az}, z_{az}) are the coordinates of the origins of the AM local coordinate systems in the PM local coordinate systems.

Based on Eqs. (6), (8) and (9), the $\boldsymbol{\zeta}_{\mathrm{px}}$, $\boldsymbol{\zeta}_{\mathrm{py}}$ and $\boldsymbol{\zeta}_{\mathrm{pz}}$ can be transformed to the global coordinate system via pre-multiplying $\mathbf{T}_{\mathrm{px}}$, $\mathbf{T}_{\mathrm{py}}$ and $\mathbf{T}_{\mathrm{pz}}$, respectively, which are represented in Eqs. (10)-(12). And the $\boldsymbol{\zeta}_{\mathrm{ax}}$, $\boldsymbol{\zeta}_{\mathrm{ay}}$ and $\boldsymbol{\zeta}_{\mathrm{az}}$ can be transformed to the PM local coordinate systems via pre-multiplying $\mathbf{T}_{\mathrm{ax}}$, $\mathbf{T}_{\mathrm{ay}}$ and $\mathbf{T}_{\mathrm{az}}$, respectively, which are shown in Eqs. (13)-(15).

$$\zeta_{\mathrm{px}}'=\mathbf{T}_{\mathrm{px}}\zeta_{\mathrm{px}}=\left[k_{\mathrm{px\text{-}tx}}j_{\mathrm{px\text{-}tx}},k_{\mathrm{px\text{-}ty}}j_{\mathrm{px\text{-}ty}},k_{\mathrm{px\text{-}tz}}j_{\mathrm{px\text{-}tz}},k_{\mathrm{px\text{-}rx}}j_{\mathrm{px\text{-}rx}},k_{\mathrm{px\text{-}ry}}j_{\mathrm{px\text{-}ry}}+z_{\mathrm{px}}j_{\mathrm{px\text{-}tx}},k_{\mathrm{px\text{-}rz}}j_{\mathrm{px\text{-}rz}}-y_{\mathrm{px}}j_{\mathrm{px\text{-}tx}}\right]^{\mathrm{T}} \tag{10}$$

$$\zeta_{\mathrm{py}}'=\mathbf{T}_{\mathrm{py}}\zeta_{\mathrm{py}}=\left[k_{\mathrm{py\text{-}ty}}j_{\mathrm{py\text{-}ty}},k_{\mathrm{py\text{-}tx}}j_{\mathrm{py\text{-}tx}},k_{\mathrm{py\text{-}tz}}j_{\mathrm{py\text{-}tz}},-k_{\mathrm{py\text{-}ry}}j_{\mathrm{py\text{-}ry}}-z_{\mathrm{py}}j_{\mathrm{py\text{-}tx}},k_{\mathrm{py\text{-}rx}}j_{\mathrm{py\text{-}rx}},k_{\mathrm{py\text{-}rz}}j_{\mathrm{py\text{-}rz}}+x_{\mathrm{py}}j_{\mathrm{py\text{-}tx}}\right]^{\mathrm{T}} \tag{11}$$

$$\zeta_{\mathrm{pz}}'=\mathbf{T}_{\mathrm{pz}}\zeta_{\mathrm{pz}}=\left[k_{\mathrm{pz\text{-}tz}}j_{\mathrm{pz\text{-}tz}},k_{\mathrm{pz\text{-}ty}}j_{\mathrm{pz\text{-}ty}},k_{\mathrm{pz\text{-}tx}}j_{\mathrm{pz\text{-}tx}},y_{\mathrm{pz}}j_{\mathrm{pz\text{-}tx}}-k_{\mathrm{pz\text{-}rz}}j_{\mathrm{pz\text{-}rz}},k_{\mathrm{pz\text{-}ry}}j_{\mathrm{pz\text{-}ry}}-x_{\mathrm{pz}}j_{\mathrm{pz\text{-}tx}},k_{\mathrm{pz\text{-}rx}}j_{\mathrm{pz\text{-}rx}}\right]^{\mathrm{T}} \tag{12}$$

$$\zeta_{\mathrm{ax}}'=\mathbf{T}_{\mathrm{ax}}\zeta_{\mathrm{ax}}=\left[k_{\mathrm{ax\text{-}tx}}j_{\mathrm{ax\text{-}tx}},k_{\mathrm{ax\text{-}ty}}j_{\mathrm{ax\text{-}ty}},k_{\mathrm{ax\text{-}tz}}j_{\mathrm{ax\text{-}tz}},k_{\mathrm{ax\text{-}rx}}j_{\mathrm{ax\text{-}rx}},k_{\mathrm{ax\text{-}ry}}j_{\mathrm{ax\text{-}ry}}+z_{\mathrm{ax}}j_{\mathrm{ax\text{-}tx}},k_{\mathrm{ax\text{-}rz}}j_{\mathrm{ax\text{-}rz}}-y_{\mathrm{ax}}j_{\mathrm{ax\text{-}tx}}\right]^{\mathrm{T}} \tag{13}$$

$$\zeta_{\mathrm{ay}}'=\mathbf{T}_{\mathrm{ay}}\zeta_{\mathrm{ay}}=\left[k_{\mathrm{ay\text{-}tx}}j_{\mathrm{ay\text{-}tx}},k_{\mathrm{ay\text{-}ty}}j_{\mathrm{ay\text{-}ty}},k_{\mathrm{ay\text{-}tz}}j_{\mathrm{ay\text{-}tz}},k_{\mathrm{ay\text{-}rx}}j_{\mathrm{ay\text{-}rx}},k_{\mathrm{ay\text{-}ry}}j_{\mathrm{ay\text{-}ry}}+z_{\mathrm{ay}}j_{\mathrm{ay\text{-}tx}},k_{\mathrm{ay\text{-}rz}}j_{\mathrm{ay\text{-}rz}}-y_{\mathrm{ay}}j_{\mathrm{ay\text{-}tx}}\right]^{\mathrm{T}} \tag{14}$$

$$\zeta_{\mathrm{az}}'=\mathbf{T}_{\mathrm{az}}\zeta_{\mathrm{az}}=\left[k_{\mathrm{az\text{-}tx}}j_{\mathrm{az\text{-}tx}},k_{\mathrm{az\text{-}ty}}j_{\mathrm{az\text{-}ty}},k_{\mathrm{az\text{-}tz}}j_{\mathrm{az\text{-}tz}},k_{\mathrm{az\text{-}rx}}j_{\mathrm{az\text{-}rx}},k_{\mathrm{az\text{-}ry}}j_{\mathrm{az\text{-}ry}}+z_{\mathrm{az}}j_{\mathrm{az\text{-}tx}},k_{\mathrm{az\text{-}rz}}j_{\mathrm{az\text{-}rz}}-y_{\mathrm{az}}j_{\mathrm{az\text{-}tx}}\right]^{\mathrm{T}} \tag{15}$$

In Eqs. (10)-(15), the $j_{*\text{-tx}}$, $j_{*\text{-ty}}$, $j_{*\text{-tz}}$, $j_{*\text{-rx}}$, $j_{*\text{-ry}}$ and $j_{*\text{-rz}}$ ('$*$' denotes px, py, pz, ax, ay, or az) equal to ±1. As discussed in Section 2.1, the instantaneous direction of a principal wrench is always in the opposite instantaneous direction of the counterpart principal twist. Therefore, the elements in an entry of the screw vectors as shown in Eqs. (10)-(15) should have the same sign. Take Eq. (10) for example, the elements $k_{\mathrm{px\text{-}rz}}j_{\mathrm{px\text{-}rz}}$ and $-yy_{\mathrm{px}}j_{\mathrm{px\text{-}tx}}$ in the entry $k_{\mathrm{px\text{-}rz}}j_{\mathrm{px\text{-}rz}}-y_{\mathrm{px}}j_{\mathrm{px\text{-}tx}}$ are always with the same sign. So $k_{\mathrm{px\text{-}rz}}j_{\mathrm{px\text{-}rz}}-y_{\mathrm{px}}j_{\mathrm{px\text{-}tx}}$ can be rewritten as ±1 multiplied by the sum of the absolute values of $k_{\mathrm{px\text{-}rz}}$ and y_{px}, i.e. $\pm(|k_{\mathrm{px\text{-}rz}}|+|y_{\mathrm{px}}|)$. As a result, Eqs. (10)-(15) can be rewritten as

$$\zeta_{\mathrm{px}}'=\mathbf{T}_{\mathrm{px}}\zeta_{\mathrm{px}}=\left[k_{\mathrm{px\text{-}tx}}j_{\mathrm{px\text{-}tx}},k_{\mathrm{px\text{-}ty}}j_{\mathrm{px\text{-}ty}},k_{\mathrm{px\text{-}tz}}j_{\mathrm{px\text{-}tz}},k_{\mathrm{px\text{-}rx}}j_{\mathrm{px\text{-}rx}},\pm\left(\left|k_{\mathrm{px\text{-}ry}}\right|+\left|z_{\mathrm{px}}\right|\right),\pm\left(\left|k_{\mathrm{px\text{-}rz}}\right|+\left|y_{\mathrm{px}}\right|\right)\right]^{\mathrm{T}} \tag{16}$$

$$\zeta_{\mathrm{py}}'=\mathbf{T}_{\mathrm{py}}\zeta_{\mathrm{py}}=\left[k_{\mathrm{py\text{-}ty}}j_{\mathrm{py\text{-}ty}},k_{\mathrm{py\text{-}tx}}j_{\mathrm{py\text{-}tx}},k_{\mathrm{py\text{-}tz}}j_{\mathrm{py\text{-}tz}},\pm\left(\left|k_{\mathrm{py\text{-}ry}}\right|+\left|z_{\mathrm{py}}\right|\right),k_{\mathrm{py\text{-}rx}}j_{\mathrm{py\text{-}rx}},\pm\left(\left|k_{\mathrm{py\text{-}rz}}\right|+\left|x_{\mathrm{py}}\right|\right)\right]^{\mathrm{T}} \tag{17}$$

$$\zeta_{\mathrm{pz}}'=\mathbf{T}_{\mathrm{pz}}\zeta_{\mathrm{pz}}=\left[k_{\mathrm{pz\text{-}tz}}j_{\mathrm{pz\text{-}tz}},k_{\mathrm{pz\text{-}ty}}j_{\mathrm{pz\text{-}ty}},k_{\mathrm{pz\text{-}tx}}j_{\mathrm{pz\text{-}tx}},\pm\left(\left|y_{\mathrm{pz}}\right|+\left|k_{\mathrm{pz\text{-}rz}}\right|\right),\pm\left(\left|k_{\mathrm{pz\text{-}ry}}\right|+\left|x_{\mathrm{pz}}\right|\right),k_{\mathrm{pz\text{-}rx}}j_{\mathrm{pz\text{-}rx}}\right]^{\mathrm{T}} \tag{18}$$

$$\zeta_{\mathrm{ax}}'=\mathbf{T}_{\mathrm{ax}}\zeta_{\mathrm{ax}}=\left[k_{\mathrm{ax\text{-}tx}}j_{\mathrm{ax\text{-}tx}},k_{\mathrm{ax\text{-}ty}}j_{\mathrm{ax\text{-}ty}},k_{\mathrm{ax\text{-}tz}}j_{\mathrm{ax\text{-}tz}},k_{\mathrm{ax\text{-}rx}}j_{\mathrm{ax\text{-}rx}},\pm\left(\left|k_{\mathrm{ax\text{-}ry}}\right|+\left|z_{\mathrm{ax}}\right|\right),\pm\left(\left|k_{\mathrm{ax\text{-}rz}}\right|+\left|y_{\mathrm{ax}}\right|\right)\right]^{\mathrm{T}} \tag{19}$$

$$\zeta_{\mathrm{ay}}'=\mathbf{T}_{\mathrm{ay}}\zeta_{\mathrm{ay}}=\left[k_{\mathrm{ay\text{-}tx}}j_{\mathrm{ay\text{-}tx}},k_{\mathrm{ay\text{-}ty}}j_{\mathrm{ay\text{-}ty}},k_{\mathrm{ay\text{-}tz}}j_{\mathrm{ay\text{-}tz}},k_{\mathrm{ay\text{-}rx}}j_{\mathrm{ay\text{-}rx}},\pm\left(\left|k_{\mathrm{ay\text{-}ry}}\right|+\left|z_{\mathrm{ay}}\right|\right),\pm\left(\left|k_{\mathrm{ay\text{-}rz}}\right|+\left|y_{\mathrm{ay}}\right|\right)\right]^{\mathrm{T}} \tag{20}$$

$$\zeta_{\mathrm{az}}'=\mathbf{T}_{\mathrm{az}}\zeta_{\mathrm{az}}=\left[k_{\mathrm{az\text{-}tx}}j_{\mathrm{az\text{-}tx}},k_{\mathrm{az\text{-}ty}}j_{\mathrm{az\text{-}ty}},k_{\mathrm{az\text{-}tz}}j_{\mathrm{az\text{-}tz}},k_{\mathrm{az\text{-}rx}}j_{\mathrm{az\text{-}rx}},\pm\left(\left|k_{\mathrm{az\text{-}ry}}\right|+\left|z_{\mathrm{az}}\right|\right),\pm\left(\left|k_{\mathrm{az\text{-}rz}}\right|+\left|y_{\mathrm{az}}\right|\right)\right]^{\mathrm{T}} \tag{21}$$

According to the permitted motions of the MS, Eq. (22) can be derived based on Eq. (5) in the MS domain. And the values of the stiffness coefficients as shown in Eq. (23) can be identified based on Eqs. (16)-(18) and (22).

$$\left.\begin{array}{l}\boldsymbol{\xi}_{\text{m-tx}}\circ\boldsymbol{\zeta}_{\text{px}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{m-tx}}\circ\boldsymbol{\zeta}_{\text{py}}'=0 \text{ and } \boldsymbol{\xi}_{\text{m-tx}}\circ\boldsymbol{\zeta}_{\text{pz}}'=0\\ \boldsymbol{\xi}_{\text{m-ty}}\circ\boldsymbol{\zeta}_{\text{px}}'=0 \text{ and } \boldsymbol{\xi}_{\text{m-ty}}\circ\boldsymbol{\zeta}_{\text{py}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{m-ty}}\circ\boldsymbol{\zeta}_{\text{pz}}'=0\\ \boldsymbol{\xi}_{\text{m-tz}}\circ\boldsymbol{\zeta}_{\text{px}}'=0 \text{ and } \boldsymbol{\xi}_{\text{m-tz}}\circ\boldsymbol{\zeta}_{\text{py}}'=0 \text{ and } \boldsymbol{\xi}_{\text{m-tz}}\circ\boldsymbol{\zeta}_{\text{pz}}'\neq 0\\ \boldsymbol{\xi}_{\text{m-rx}}\circ\boldsymbol{\zeta}_{\text{px}}'\neq 0 \text{ and/or } \boldsymbol{\xi}_{\text{m-rx}}\circ\boldsymbol{\zeta}_{\text{py}}'\neq 0 \text{ and/or } \boldsymbol{\xi}_{\text{m-rx}}\circ\boldsymbol{\zeta}_{\text{pz}}'\neq 0\\ \boldsymbol{\xi}_{\text{m-ry}}\circ\boldsymbol{\zeta}_{\text{px}}'\neq 0 \text{ and/or } \boldsymbol{\xi}_{\text{m-ry}}\circ\boldsymbol{\zeta}_{\text{py}}'\neq 0 \text{ and/or } \boldsymbol{\xi}_{\text{m-ry}}\circ\boldsymbol{\zeta}_{\text{pz}}'\neq 0\\ \boldsymbol{\xi}_{\text{m-rz}}\circ\boldsymbol{\zeta}_{\text{px}}'\neq 0 \text{ and/or } \boldsymbol{\xi}_{\text{m-rz}}\circ\boldsymbol{\zeta}_{\text{py}}'\neq 0 \text{ and/or } \boldsymbol{\xi}_{\text{m-rz}}\circ\boldsymbol{\zeta}_{\text{pz}}'\neq 0\end{array}\right\} \tag{22}$$

$$\left.\begin{array}{l}k_{\text{px-tx}}=1 \text{ and } k_{\text{px-ty}}=0 \text{ and } k_{\text{px-tz}}=0\\ k_{\text{py-tx}}=1 \text{ and } k_{\text{py-ty}}=0 \text{ and } k_{\text{py-tz}}=0\\ k_{\text{pz-tx}}=1 \text{ and } k_{\text{pz-ty}}=0 \text{ and } k_{\text{pz-tz}}=0\\ k_{\text{px-rx}}=1 \text{ or/and } k_{\text{py-ry}}=1 \text{ or/and } k_{\text{pz-rz}}=1\\ k_{\text{px-ry}}=1 \text{ or/and } k_{\text{py-rx}}=1 \text{ or/and } k_{\text{pz-ry}}=1\\ k_{\text{px-rz}}=1 \text{ or/and } k_{\text{py-rz}}=1 \text{ or/and } k_{\text{pz-rx}}=1\end{array}\right\} \tag{23}$$

It should be noted that Eq. (23) shows the completed solutions for the PM constraints to the MS including the exact constraints and redundant constraints. However, only independent constraints within each PM are considered. It should be elaborated that the process of deriving the values of the stiffness coefficients associated with the rotational DOF of the MS. Take $\boldsymbol{\xi}_{\text{m-rx}}\circ\boldsymbol{\zeta}_{\text{px}}'\neq 0$, $\boldsymbol{\xi}_{\text{m-rx}}\circ\boldsymbol{\zeta}_{\text{py}}'\neq 0$, and $\boldsymbol{\xi}_{\text{m-rx}}\circ\boldsymbol{\zeta}_{\text{pz}}'\neq 0$ in Eq. (22) for example, based on Eqs. (16)-(18), the following equations can be obtained: $k_{\text{px-rx}}\neq 0$ or/and $|k_{\text{py-ry}}|+|z_{\text{py}}|\neq 0$ or/and $|k_{\text{pz-rz}}|+|y_{\text{pz}}|\neq 0$. Suppose that $k_{\text{px-rx}}=0$ $k_{\text{py-ry}}=0$ and $k_{\text{pz-rz}}=0$, the motions associated $\boldsymbol{\xi}_{\text{m-rx}}$ cannot be constrained if the global coordinate system is moved to the locations where $z_{\text{py}}= y_{\text{pz}}=0$. In other words, if $k_{\text{px-rx}}=0$ $k_{\text{py-ry}}=0$ and $k_{\text{pz-rz}}=0$, there is still a line about which the MS can rotate. Therefore, at least one of the stiffness coefficients $k_{\text{px-rx}}$, $k_{\text{py-ry}}$ and $k_{\text{pz-rz}}$ should not be zero as shown in the fourth row in Eq. (23). Other stiffness coefficients in Eq. (23) associated with the rotational DOF of the MS can also be obtained based on the process described as above. It should be pointed out that the results as shown in Eq. (23) are independent of the positions of the PM local coordinate systems (or the PMs).

In the same way, according to the permitted motions of the ASs in the AS domains, Eq. (24) can be derived based on Eq. (5) if the constraints of the PMs to the ASs are not considered. And the values of the stiffness coefficients can be identified as shown in Eq. (25) based on Eqs. (19)-(21) and (24). The values of the stiffness coefficients in Eq. (25) are not subject to the positions of the AM local coordinate systems (or the AMs) neither.

$$\left.\begin{array}{l}\boldsymbol{\xi}_{\text{ax-tx}}\circ\boldsymbol{\zeta}_{\text{ax}}'=0 \text{ and } \boldsymbol{\xi}_{\text{ax-ty}}\circ\boldsymbol{\zeta}_{\text{ax}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{ax-tz}}\circ\boldsymbol{\zeta}_{\text{ax}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{ax-rx}}\circ\boldsymbol{\zeta}_{\text{ax}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{ax-ry}}\circ\boldsymbol{\zeta}_{\text{ax}}'\neq 0 \text{and } \boldsymbol{\xi}_{\text{ax-rz}}\circ\boldsymbol{\zeta}_{\text{ax}}'\neq 0\\ \boldsymbol{\xi}_{\text{ay-tx}}\circ\boldsymbol{\zeta}_{\text{ay}}'=0 \text{ and } \boldsymbol{\xi}_{\text{ay-ty}}\circ\boldsymbol{\zeta}_{\text{ay}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{ay-tz}}\circ\boldsymbol{\zeta}_{\text{ay}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{ay-rx}}\circ\boldsymbol{\zeta}_{\text{ay}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{ay-ry}}\circ\boldsymbol{\zeta}_{\text{ay}}'\neq 0 \text{and } \boldsymbol{\xi}_{\text{ay-rz}}\circ\boldsymbol{\zeta}_{\text{ay}}'\neq 0\\ \boldsymbol{\xi}_{\text{az-tx}}\circ\boldsymbol{\zeta}_{\text{az}}'=0 \text{ and } \boldsymbol{\xi}_{\text{az-ty}}\circ\boldsymbol{\zeta}_{\text{az}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{az-tz}}\circ\boldsymbol{\zeta}_{\text{az}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{az-rx}}\circ\boldsymbol{\zeta}_{\text{az}}'\neq 0 \text{ and } \boldsymbol{\xi}_{\text{az-ry}}\circ\boldsymbol{\zeta}_{\text{az}}'\neq 0 \text{and } \boldsymbol{\xi}_{\text{az-rz}}\circ\boldsymbol{\zeta}_{\text{az}}'\neq 0\end{array}\right\} \tag{24}$$

$$\left.\begin{array}{l}k_{\text{ax-tx}}=0 \text{ and } k_{\text{ax-ty}}=1 \text{ and } k_{\text{ax-tz}}=1 \text{ and } k_{\text{ax-rx}}=1 \text{ and } k_{\text{ax-ry}}=1 \text{ and } k_{\text{ax-ry}}=1\\ k_{\text{ay-tx}}=0 \text{ and } k_{\text{ay-ty}}=1 \text{ and } k_{\text{ay-tz}}=1 \text{ and } k_{\text{ay-rx}}=1 \text{ and } k_{\text{ay-ry}}=1 \text{ and } k_{\text{ay-ry}}=1\\ k_{\text{az-tx}}=0 \text{ and } k_{\text{az-ty}}=1 \text{ and } k_{\text{az-tz}}=1 \text{ and } k_{\text{az-rx}}=1 \text{ and } k_{\text{az-ry}}=1 \text{ and } k_{\text{az-ry}}=1\end{array}\right\} \tag{25}$$

Equation (23) shows that there are 27 (3×3×3=27) permitted constraint combinations for the PM-X, PM-Y and PM-Z if the redundant constraint from the three PMs to the MS is not considered, i.e. exactly-constrained design. It can be seen from Eq. (25) that each AM has five constraints in the AM local coordinate system if the constraints of the PM to the AS are not considered.

At this stage, ASs are completely constrained by the AMs (each AM has five constraints in the AM local coordinate system) with the 27 permitted exact-constraint combinations for the PMs, which are shown in Table B.1 in Appendix. B. The constraint combinations in Table B.1 form the basic constraint space of the compliant modules (i.e. B-constraint space), which is a subspace of the constraint spaces of the compliant modules.

As shown in Fig. 5(b), each PM is included not only in the MS domain but also in the AS domain. Therefore, each AS can be constrained by the PM besides the AM. The following two conditions should be met if a PM can constrain an extra rotational DOF of the AS connecting to the PM: (a) the PM in one leg should have the constraint associated with the extra rotational DOF of the AS; and (b) this extra rotational DOF of the AS should be restricted by the other leg(s). Based on the two conditions specified above, another subspace of the constraint space termed as T-constraint space can be gained via transmitting some of the rotational constraints (each PM has one translational constraint which is used to actuate the MS, so the translational constraints of the PMs cannot be transmitted to the AMs) from the AMs to the PMs in the B-constraint space. One or more redundant constraints can be added for the PMs and AMs in the B-constraint space and the T-constraint space, and therefore the constraint space with redundant constraints is defined as S-constraint space. Overall, the complete constraint space of the compliant modules consists of the B-constraint space, the T-constraint space and the S-constraint space, the relationships of which are represented in Fig. 7.

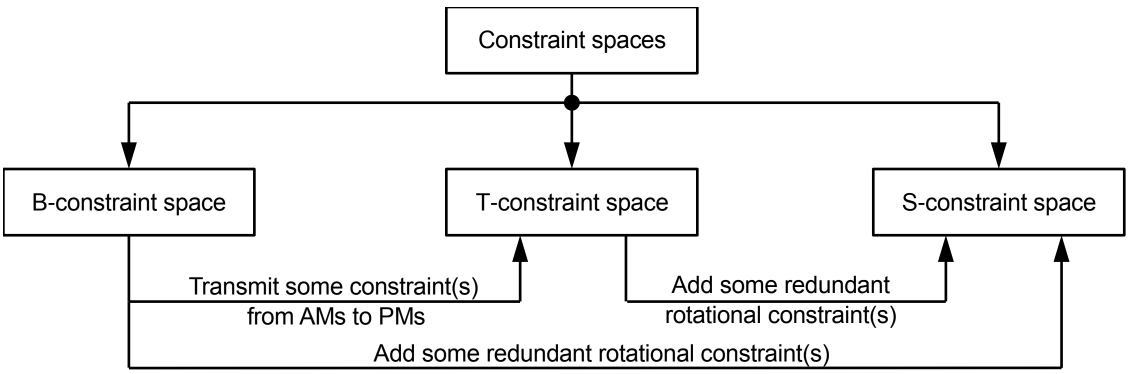


Fig. 7 Relationships among the B-constraint space, T-constraint space and S-constraint space.

5. Position space

5.1 PM position space

From the discussion in Section 4.2, the results of the stiffness coefficients shown in Eq. (23) are independent of the positions of the PMs. Therefore, PMs can translate freely, which cannot affect the constraints of the PMs to the XYZ CPM system. Moreover, each PM cannot rotate about the Y- and Z-axes of the PM local coordinate system, because the direction of the pure force wrench of the PM should be parallel to the local X-axis. If using three straight lines to represent the three PMs and ignoring considering the rotations of the three PMs about the three X-axes of the three PM local coordinate systems, the permitted positions of the three PMs can be shown in Fig. 8.

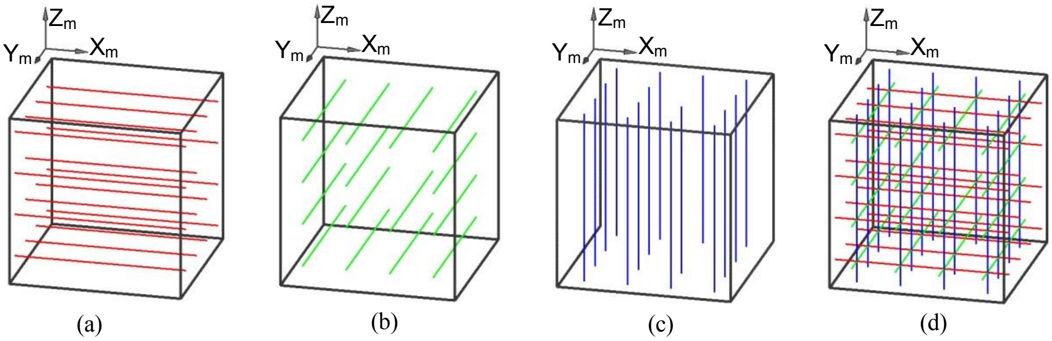


Fig.8 Permitted positions of the three PMs excluding considering the rotations about the X-axes of the three PM local coordinate systems: (a) permitted positions of the PM-X (red lines), (b) permitted positions of the PM-Y (green lines), (c) permitted positions of the PM-Z (blue lines), and (d) permitted position combination of the three PMs. (Online version in color.)

The following work is to identify if a PM can rotate about the X-axis of the PM local coordinate system via taking PM-X for example. Based on Table 1 and Eq. (23), the wrench of the PM-X can be written as

$$\zeta_{\text{px}} = \zeta_{\text{px-tx}} + k_{\text{px-rx}}\zeta_{\text{px-rx}} + k_{\text{px-ry}}\zeta_{\text{px-ry}} + k_{\text{px-rz}}\zeta_{\text{px-rz}} \tag{26a}$$

The coordinate transformation matrix in terms of the rotation about the X-axis is shown in Eq. (26b) based on Eq. (6).

$$\mathbf{T}_{\text{px-rx}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 & 0 & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & 0 & 0 & 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{26b}$$

where α is the angle of the rotation about the X-axis of the PM local coordinate system O_{px}-$X_{px}Y_{px}Z_{px}$. After the rotation, the wrench of the PM-X in the PM local coordinate system can be represented as $\boldsymbol{\zeta}_{px\text{-}r}$, which can be written in Eq. (27) based on Eqs. (26a) and (26b).

$$\zeta_{px\text{-}r}=\mathbf{T}_{px\text{-}rx}\zeta_{px}=\begin{bmatrix} k_{px\text{-}tx}\,j_{px\text{-}tx} \\ 0 \\ 0 \\ k_{px\text{-}rx}\,j_{px\text{-}rx} \\ k_{px\text{-}ry}\cos(\alpha)\,j_{px\text{-}ry}-k_{px\text{-}rz}\sin(\alpha)\,j_{px\text{-}rz} \\ k_{px\text{-}ry}\sin(\alpha)\,j_{px\text{-}ry}+k_{px\text{-}rz}\cos(\alpha)\,j_{px\text{-}rz} \end{bmatrix}=\begin{bmatrix} k_{px\text{-}tx}\,j_{px\text{-}tx} \\ 0 \\ 0 \\ k_{px\text{-}rx}\,j_{px\text{-}rx} \\ \pm\left(\left|k_{px\text{-}ry}\cos(\alpha)\right|+\left|k_{px\text{-}rz}\sin(\alpha)\right|\right) \\ \pm\left(\left|k_{px\text{-}ry}\sin(\alpha)\right|+\left|k_{px\text{-}rz}\cos(\alpha)\right|\right) \end{bmatrix} \tag{27}$$

where $k_{px\text{-}ry}\cos(\alpha)j_{px\text{-}ry}-k_{px\text{-}rz}\sin(\alpha)j_{px\text{-}rz}$ is equivalent to $\pm(|k_{px\text{-}ry}\cos(\alpha)|+|k_{px\text{-}rz}\sin(\alpha)|)$ because $k_{px\text{-}ry}\cos(\alpha)j_{px\text{-}ry}$ and $-k_{px\text{-}rz}\sin(\alpha)j_{px\text{-}rz}$ always have the same sign. Similarly, $k_{px\text{-}ry}\sin(\alpha)j_{px\text{-}ry}+k_{px\text{-}rz}\cos(\alpha)j_{px\text{-}rz}$ is equivalent to $\pm(|k_{px\text{-}ry}\sin(\alpha)|+|k_{px\text{-}rz}\cos(\alpha)|)$.

The PM-X is able to rotate about the X-axis of the PM local coordinate system if the wrenches $\boldsymbol{\zeta}_{px}$ and $\boldsymbol{\zeta}_{px\text{-}r}$ maintain equivalent. It can be concluded from Eq. (27) that: (a) the PM-X can rotate about the X-axis if $k_{px\text{-}ry}=k_{px\text{-}rz}=0$ or 1; and (b) the PM-X cannot rotate about the X-axis if $k_{px\text{-}ry}\neq k_{px\text{-}rz}$.

Overall, the position space of a PM is summarized as follows:

(a) The PM can translate freely, but cannot rotate about the Y- and Z-axes of the PM local coordinate system, which is represented in Fig. 8;

(b) The PM can rotate about the X-axis of the PM local coordinate system if the PM has the rotational constraints about the Y- and Z-axes in the PM local coordinate system, else if the PM has no the two rotational constraints.

(c) The PM cannot rotate about the X-axis of the PM local coordinate system if the PM can provide one of the two rotational constraints about the Y- and Z-axes in the PM local coordinate system.

Note that in this section only the position space for the PM as a whole is considered. Actually, each PM can be comprised of several compliant joints in series, so the position spaces for the individual compliant joints exist within the PM [38], which is out of the scope of this paper.

5.2 AM position space

According to Section 4.2, the results of the stiffness coefficients as shown in Eq. (25) are independent of the positions of the AMs. Therefore, the AM can translate freely, which cannot affect the constraints of the AMs to the XYZ CPM system. Moreover, each AM cannot rotate about the Y- and Z-axes of the AM local coordinate system so that the DOF direction of the AM keeps the same as the direction of the force of the actuator.

The following work is to identify whether an AM can rotate about the X-axis of the AM local coordinate system via taking AM-X for example. Based on Table 1 and the AM constraint space discussed in Section 4.2, the general wrench of the AM-X can be written as

$$\zeta_{ax}= \zeta_{ax\text{-}ty}+\zeta_{ax\text{-}tz}+ k_{ax\text{-}rx}\zeta_{ax\text{-}rx}+ k_{ax\text{-}ry}\zeta_{ax\text{-}ry}+ k_{ax\text{-}rz}\zeta_{ax\text{-}rz}.$$

The coordinate transformation matrix in terms of the rotation about the X-axis is shown in Eq. (28) based on Eq. (6).

$$\mathbf{T}_{ax\text{-}rx}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) & 0 & 0 & 0 \\ 0 & \sin(\beta) & \cos(\beta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\beta) & -\sin(\beta) \\ 0 & 0 & 0 & 0 & \sin(\beta) & \cos(\beta) \end{bmatrix} \tag{28}$$

where β is the angle of the rotation about the X-axis of the AM local coordinate system. ζ_{ax} can be transformed to $\zeta_{ax\text{-}r}$ as shown in Eq. (29) after the rotation.

$$\zeta_{ax\text{-}r}=\mathbf{T}_{ax\text{-}rx}\zeta_{ax}=\begin{bmatrix} 0 \\ k_{ax\text{-}ty}\cos(\beta)\, j_{ax\text{-}ty} - k_{ax\text{-}tz}\sin(\beta)\, j_{ax\text{-}tz} \\ k_{ax\text{-}ty}\sin(\beta)\, j_{ax\text{-}ty} + k_{ax\text{-}tz}\cos(\beta)\, j_{ax\text{-}tz} \\ k_{ax\text{-}rx}\, j_{ax\text{-}rx} \\ k_{ax\text{-}ry}\cos(\beta)\, j_{ax\text{-}ry} - k_{ax\text{-}rz}\sin(\beta)\, j_{ax\text{-}rz} \\ k_{ax\text{-}ry}\sin(\beta)\, j_{ax\text{-}ry} + k_{ax\text{-}rz}\cos(\beta)\, j_{ax\text{-}rz} \end{bmatrix}=\begin{bmatrix} 0 \\ \pm\left(|\cos(\beta)|+|\sin(\beta)|\right) \\ \pm\left(|\sin(\beta)|+|\cos(\beta)|\right) \\ k_{ax\text{-}rx}\, j_{ax\text{-}rx} \\ \pm\left(|k_{ax\text{-}ry}\cos(\beta)|+|k_{ax\text{-}rz}\sin(\beta)|\right) \\ \pm\left(|k_{ax\text{-}ry}\sin(\beta)|+|k_{ax\text{-}rz}\cos(\beta)|\right) \end{bmatrix} \tag{29}$$

On the basis of Eq. (29), one can obtain that: (a) the AM-X can rotate about the X-axis if $k_{ax\text{-}ry}=k_{ax\text{-}rz}=0$ or 1; and (b) the AM-X cannot rotate about the X-axis if $k_{ax\text{-}ry}\neq k_{ax\text{-}rz}$.

Overall, the position space of an AM is summarized as follows:

(a) The AM can translate freely, but cannot rotate about the Y- and Z-axes of the AM local coordinate system;

(b) The AM can rotate about the X-axis of the AM local coordinate system if the AM has the rotational constraints about the Y- and Z-axes in the AM local coordinate system, else if the AM has no the two rotational constraints. The permitted rotations of the three AMs are illustrated in Fig. 9.

(c) The AM cannot rotate about the X-axis of the AM local coordinate system if the AM can provide one of the two rotational constraints about the Y- and Z-axes in the AM local coordinate system.

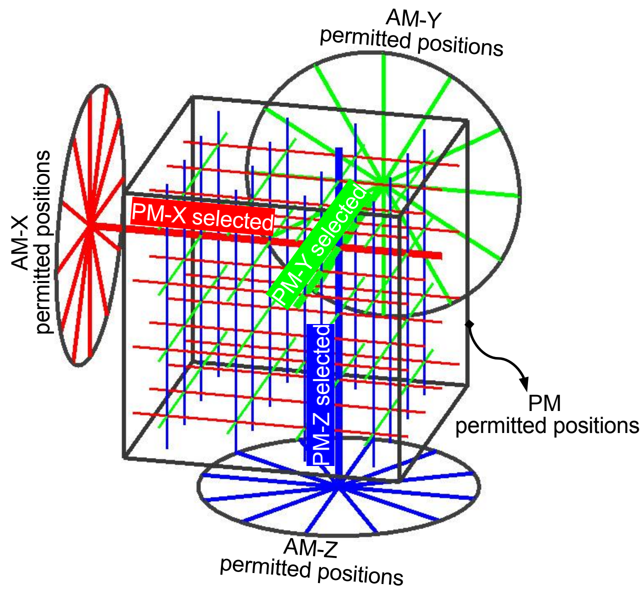


Fig. 9 Permitted rotations of the AMs about the X-axes in the three AM local coordinate systems (PM-X and AM-X are regarded as red lines; PM-Y and AM-Y are regarded as green lines; and PM-Z and AM-Z are regarded as blue lines). (Online version in color.)

Similar to the position space discussion for the PM in Section 5.1, if one AM is comprised of several joints in series, the position spaces for the individual compliant joints within the AM is also out of the scope of this paper.

6. CPI approach based synthesis procedure

The CPI approach is summarized as follows: once the constraint spaces and the position spaces are identified using the screw theory at first, the compliant modules in an XYZ CPM can be synthesized based on the constraints selected from the constraint spaces, and then the compliant modules are combined with the rigid stages based on the positions selected from the position spaces. The selections differ from case to case depending on the design requirements. In practice, the final XYZ CPM is chosen from several candidate XYZ CPMs designed based on different selections. In addition, further modification is needed to make the XYZ CPM have good characteristics such as compact configuration and easy fabrication. It should be emphasized that the CPI approach is based on a systematic arrangement of rigid stages and compliant modules using the constraint spaces and position spaces.

The CPI approach based synthesis procedure is described in Fig. 10.

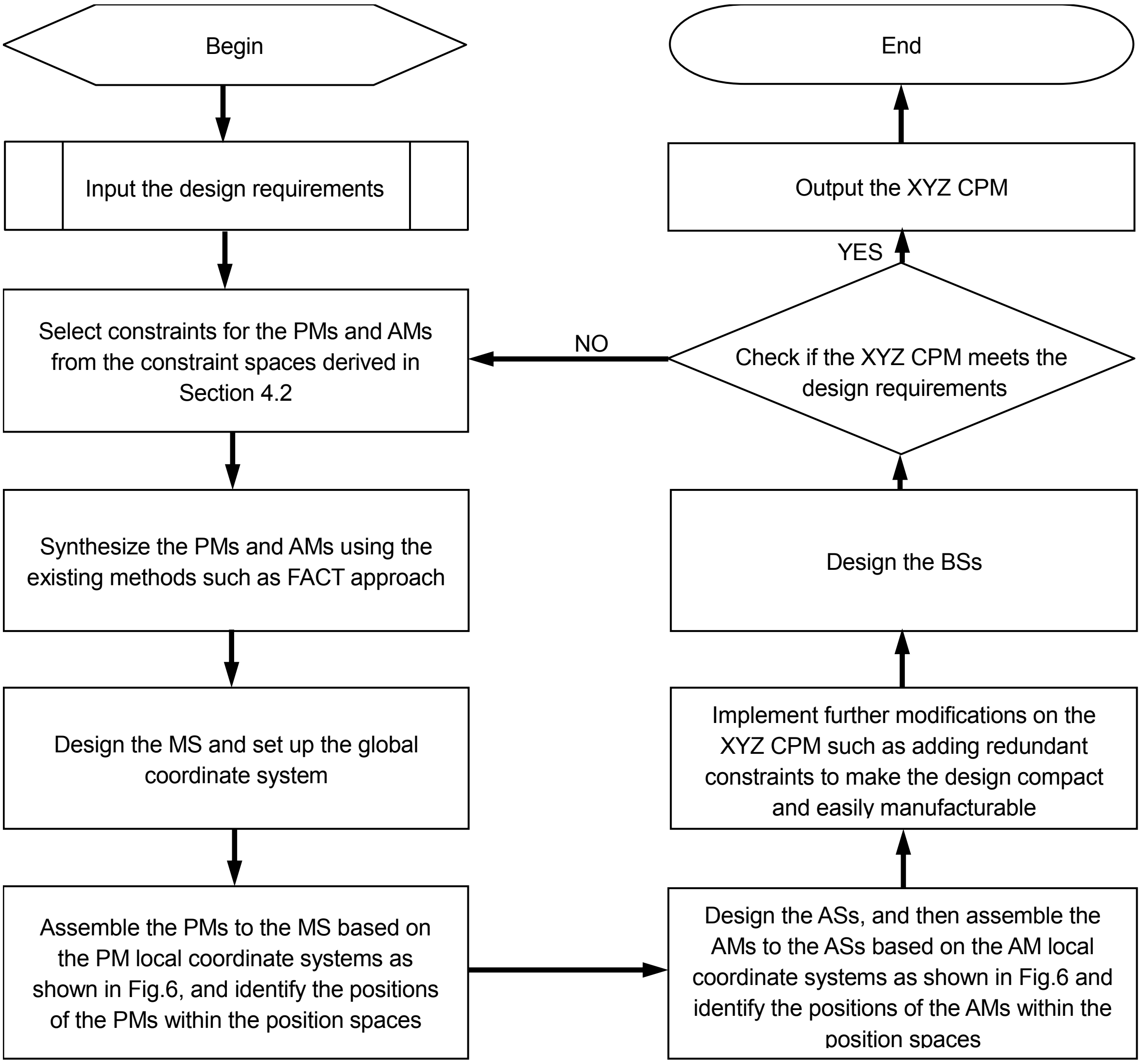


Fig. 10 Flow chart for the CPI approach design procedure.

7. Synthesis of XYZ CPMs using the CPI approach

This section will use design examples to demonstrate how to synthesize XYZ CPMs using the present CPI approach. Suppose that the objective is to design XYZ CPMs with monolithic configuration such as the ones proposed in [15, 19].

Step 1: Select constraints for the PMs and AMs from the constraint spaces shown in Appendix B. According to the design requirement, it is better to select the combination in which the PM-X, PM-Y and PM-Z have the same constraints and the AM-X, AM-Y and AM-Z also have the same constraints, i.e., isotropic legs. The reason is that compliant modules can be designed with the same structure if the compliant modules have the same constraints. Thus the combination 16 in the B-constraint space (Appendix B) is selected as the constraints of the compliant modules. In order to obtain more monolithic XYZ CPMs, other three cases are derived based on the combination 16, which are shown in Table 2. In Table 2, the case 1 is the combination 16 in Appendix B. The case 2 is derived via selecting three redundant rotational constraints for the PMs in the case 1 so that the case 2 belongs to the S-constraint space. The case 3 is determined through transmitting three rotational constraints (the underlined rotational constraints in the case 1) from the AMs to the PMs in the case 1 and then adding three redundant rotational constraints for the PMs, so the case 3 still belongs to the S-constraint space. If all redundant rotational constraints are selected for the PMs and AMs, the case 4 is generated, which is a case within the S-constraint space.

Table 2 Four constraint combination cases for the PMs and AMs.

Item	**Module**	**Constraints of the compliant modules in the PM and AM local coordinate systems**		
		X	**Y**	**Z**
Case 1	PMs	$T_{px\text{-}tx}R_{px\text{-}rz}$	$T_{py\text{-}tx}R_{py\text{-}ry}$	$T_{pz\text{-}tx}R_{pz\text{-}ry}$
	AMs	$T_{ax\text{-}ty}T_{ax\text{-}tz}R_{ax\text{-}rx}\underline{R_{ax\text{-}ry}}R_{ax\text{-}rz}$	$T_{ay\text{-}ty}T_{ay\text{-}tz}R_{ay\text{-}rx}R_{ay\text{-}ry}\underline{R_{ay\text{-}rz}}$	$T_{az\text{-}ty}T_{az\text{-}tz}R_{az\text{-}rx}R_{az\text{-}ry}\underline{R_{az\text{-}rz}}$
Case 2	PMs	$T_{px\text{-}tx}R_{px\text{-}ry}R_{px\text{-}rz}$	$T_{py\text{-}tx}R_{py\text{-}ry}R_{py\text{-}rz}$	$T_{pz\text{-}tx}R_{pz\text{-}ry}R_{pz\text{-}rz}$
	AMs	$T_{ax\text{-}ty}T_{ax\text{-}tz}R_{ax\text{-}rx}R_{ax\text{-}ry}R_{ax\text{-}rz}$	$T_{ay\text{-}ty}T_{ay\text{-}tz}R_{ay\text{-}rx}R_{ay\text{-}ry}R_{ay\text{-}rz}$	$T_{az\text{-}ty}T_{az\text{-}tz}R_{az\text{-}rx}R_{az\text{-}ry}R_{az\text{-}rz}$
Case 3	PMs	$T_{px\text{-}tx}R_{px\text{-}rx}R_{px\text{-}ry}R_{px\text{-}rz}$	$T_{py\text{-}tx}R_{py\text{-}rx}R_{py\text{-}ry}R_{py\text{-}rz}$	$T_{pz\text{-}tx}R_{pz\text{-}rx}R_{pz\text{-}ry}R_{pz\text{-}rz}$
	AMs	$T_{ax\text{-}ty}T_{ax\text{-}tz}R_{ax\text{-}rx}R_{ax\text{-}rz}$	$T_{ay\text{-}ty}T_{ay\text{-}tz}R_{ay\text{-}rx}R_{ay\text{-}ry}$	$T_{az\text{-}ty}T_{az\text{-}tz}R_{az\text{-}rx}R_{az\text{-}ry}$
Case 4	PMs	$T_{px\text{-}tx}R_{px\text{-}rx}R_{px\text{-}ry}R_{px\text{-}rz}$	$T_{py\text{-}tx}R_{py\text{-}rx}R_{py\text{-}ry}R_{py\text{-}rz}$	$T_{pz\text{-}tx}R_{pz\text{-}rx}R_{pz\text{-}ry}R_{pz\text{-}rz}$
	AMs	$T_{ax\text{-}ty}T_{ax\text{-}tz}R_{ax\text{-}rx}R_{ax\text{-}ry}R_{ax\text{-}rz}$	$T_{ay\text{-}ty}T_{ay\text{-}tz}R_{ay\text{-}rx}R_{ay\text{-}ry}R_{ay\text{-}rz}$	$T_{az\text{-}ty}T_{az\text{-}tz}R_{az\text{-}rx}R_{az\text{-}ry}R_{az\text{-}rz}$

Step 2: Synthesize the PMs and AMs based on the constraints selected in Step 1. In this example, several parallel compliant modules are designed using the FACT method [27] as shown in Fig. 11. The compliant module in Fig. 12(a), a 4-DOC parallel module, is designed via deleting two of the beams of the compliant module shown in Fig. 11(c). The compliant module in Fig. 12(b)/(c), a 4-DOC serial module, is conceived by stacking two compliant modules in Fig. 11(d) together. These compliant modules include all required compliant modules to be used in the 4 cases in step 1; however, other compliant modules can be designed based on the same constraints if necessary.

Step 3: Choose cubes as the MSs (the dimensions can be ignored in this early-stage design) for these cases, and set up the global coordinate system as shown in Fig. 6.

Step 4: Assemble the PMs based on the PM local coordinate systems as shown in Fig. 6 and the position spaces of the PMs as illustrated in Fig. 8(d). Here, the rotations of the PMs about the X-axes of the PM local coordinate systems should be identified based on the specific constraints of the PMs. The 2-DOC module in Fig. 11(a) is selected as the PM for the case 1. The positions of the three PMs are identified as shown in Fig. 13(a), and the orientations of the PMs should be subject to the PM local coordinate systems as illustrated in Fig. 13(b). The three PMs cannot rotate about the three local X-axes due to their constraints based on the results in Section 5.1. Similarly, the PMs and their selected positions for the cases 2-4 are shown in Figs. 14-17, of which both Figs. 16 and 17 are for the case 4.

Step 5: Design ASs for the four cases. There are no specific requirements about the ASs.

Step 6: Identify the positions of the AMs in terms of the position spaces. For the case 1, the 5-DOC-1 module, shown in Fig. 11(c), is selected as the AM. The permitted positions and selected positions of the AMs are represented in Fig. 13(c) based on Fig. 9 where the translations of the AMs are not considered in these cases. The positions for the three AMs as shown in Fig. 13(c) are selected to make the XYZ CPM compact. Similarly, the positions of the AMs for the cases 2-4 are shown in Figs. 14-17, of which both Figs. 16 and 17 are for the case 4.

Step 7: Make further modifications for the four cases. For example, an inactive module is added to the XYZ CPM in the case 1.

Step 8: Design BSs for the four cases. It should be noticed that the intermediate stages of the PMs of the XYZ CPMs as shown in Figs. 15 and 16 are selected as the equivalent BSs, because the intermediate stages can also provide the equivalent constraints to the ASs.

Step 9: Check if the final XYZ CPMs in the four cases meet the design requirements. The three novel XYZ CPMs, 4-4-XYZ CPM, 4-5-XYZ CPM-1 and 4-5-XYZ CPM-2 as shown in Figs. 15 and 16, are created in this paper. The FEA results shown in Figs. 18-20 validate their decoupled translational motions. The 2-5-XYZ CPM and 3-5-XYZ CPM as shown in Figs. 13 and 14 were also proposed by Hao in [18] and [19]. A prototype for the 3-5-XYZ CPM is also demonstrated in Fig. 21. The 4-5-XYZ CPM-3 illustrated in Fig. 17 was already reported by Awtar et al in [15]. Apparently, these resulting XYZ CPMs can be manufactured easily via cutting in the three orthogonal directions.

It can be concluded that all the obtained XYZ CPMs meet the early-stage design requirement, and they are compact and can be fabricated easily. Further comparisons can be made based on specific working conditions, non-linear kinematostatic analysis, dynamic analysis, etc. In addition, two non-monolithic designs showing the PM rotations about the X-axes of the PM local coordinate systems can be seen in Appendix C.

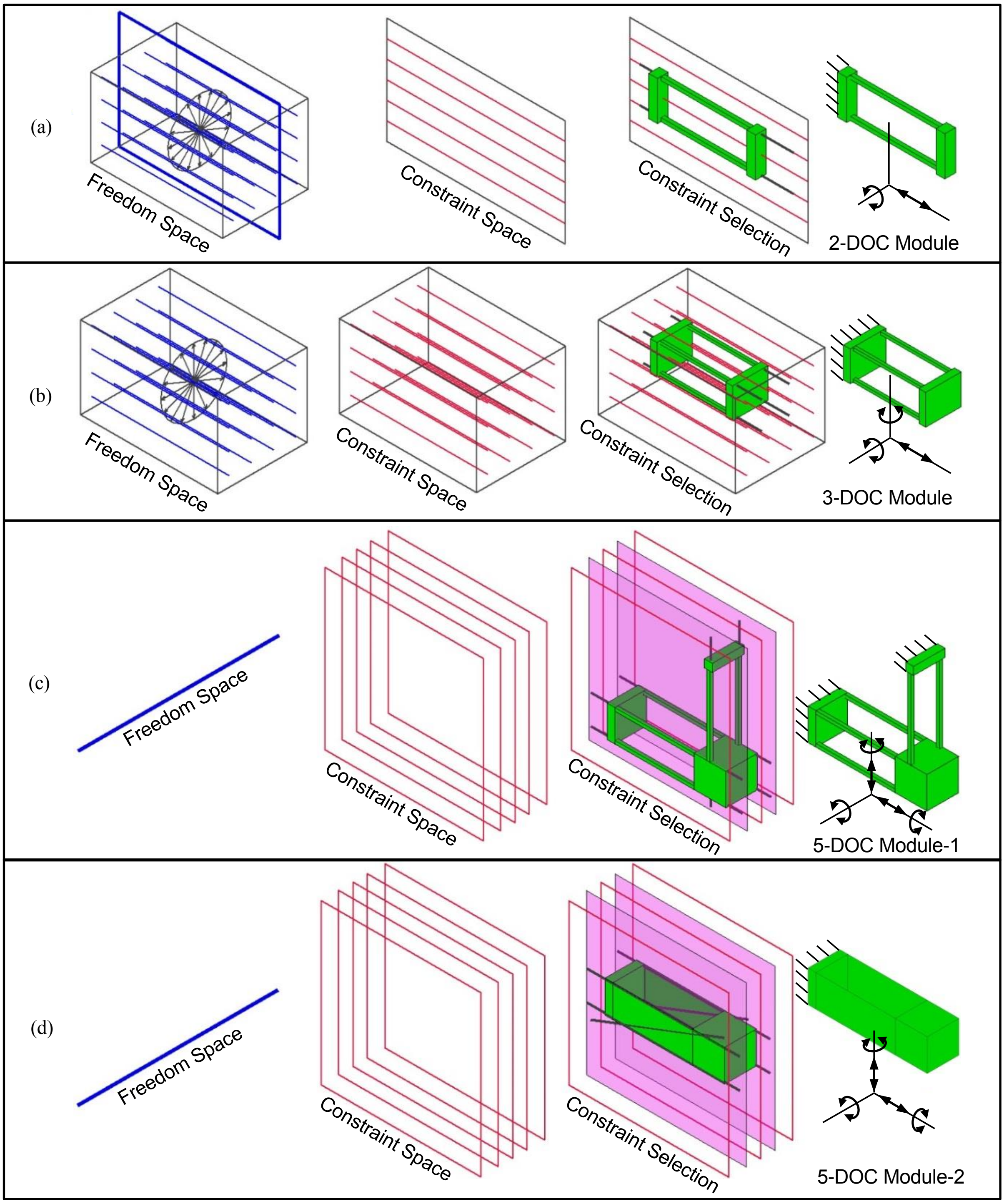


Fig. 11 Parallel compliant modules with different DOC (or constraints) designed using the FACT method: (a) a 2-DOC module design, (b) a 3-DOC module design, (c) a 5-DOC module termed 5-DOC-1 module design, and (d) a 5-DOC module termed 5-DOC-2 module design. (Online version in color.)

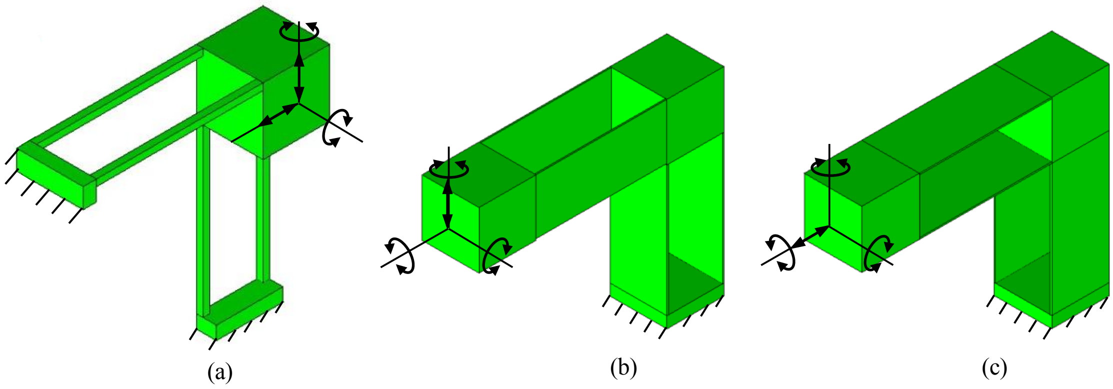


Fig. 12 Compliant modules with different DOC (or constraints): (a) a 4-DOC module termed 4-DOC-1 module, (b) a 4-DOC module termed 4-DOC-2 module, and (c) a 4-DOC module termed 4-DOC-3 module. (Online version in color.)

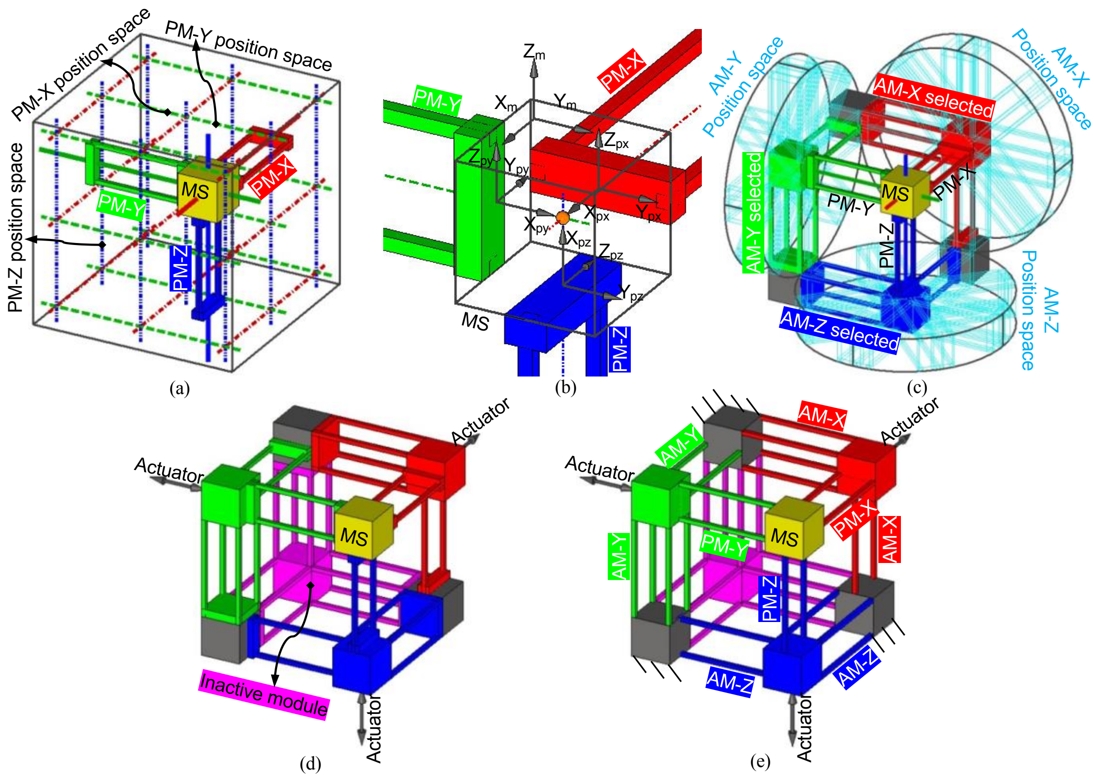


Fig. 13 An XYZ CPM designed based on the constraints in the case 1: (a) determining the PM positions, (b) the orientations of the PMs in the PM local coordinate systems, (c) selecting the AM positions, (d) adding an inactive module, and (e) the final XYZ CPM termed 2-5-XYZ CPM (i.e. XYZ CPM with 2-DOC PM and 5-DOC AM). (Online version in color.)

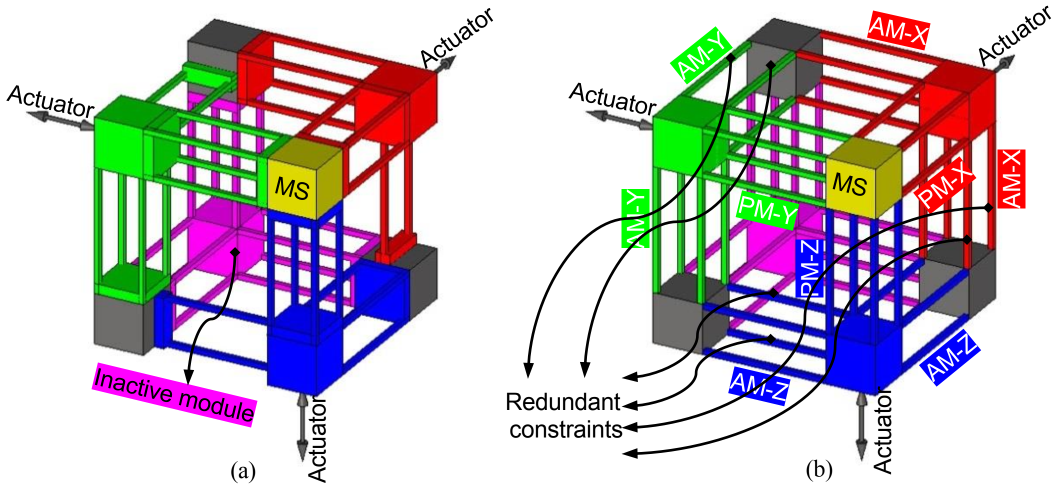


Fig. 14 An XYZ CPM designed based on the constraints in the case 2: (a) replacing the PMs of the XYZ CPM shown in Fig. 13(d) with the 3-DOC module in Fig. 11(b), and (b) the final XYZ CPM termed 3-5-XYZ CPM (i.e. XYZ CPM with 3-DOC PM and 5-DOC AM) obtained by adding redundant constraints on the AMs of the XYZ CPM shown in Fig. 14(a). (Online version in color.)

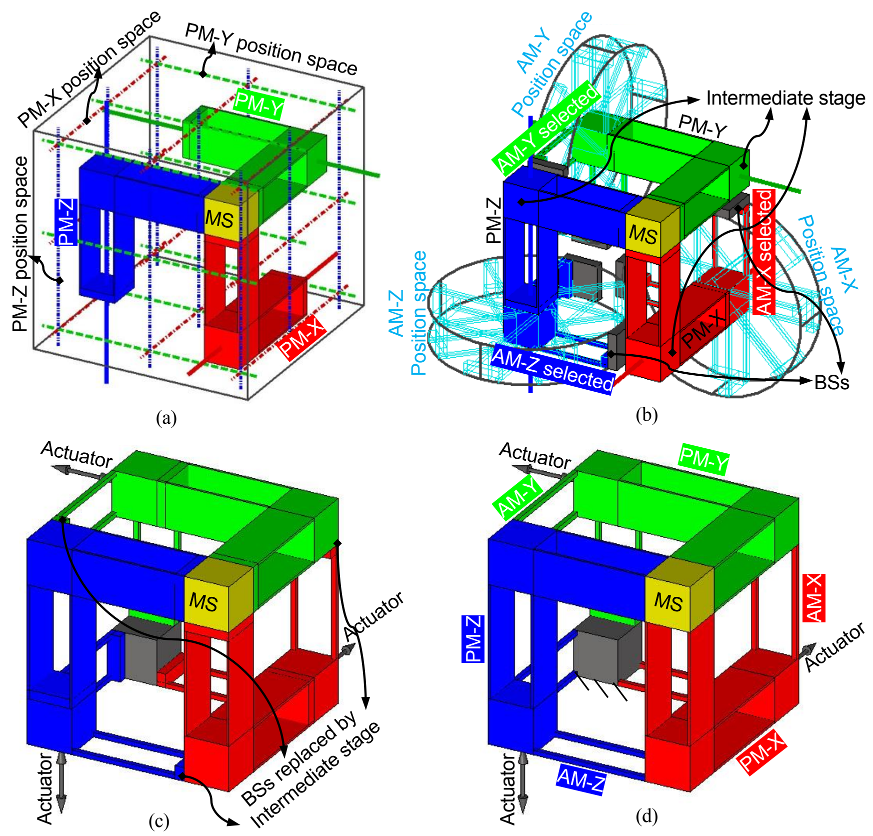


Fig. 15 An XYZ CPM designed based on the constraints in the case 3: (a) determining the PM positions, (b) selecting the AM positions, (c) selecting the intermediate stages as the BSs (because the intermediate stages can provide the constraints which the BSs can offer), and (d) the final XYZ CPM termed 4-4-XYZ CPM (i.e. XYZ CPM with 4-DOC-2 PM and 4-DOC-1 AM). (Online version in color.)

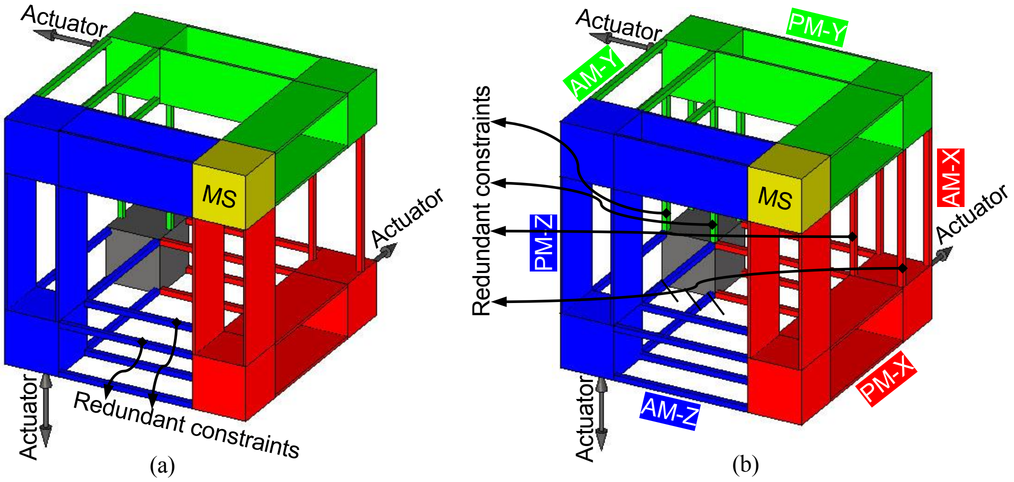


Fig. 16 An XYZ CPM designed via replacing the AMs of the 4-4-XYZ CPM with the 5-DOC-1 module based on the constraints in the case 4 and adding redundant constraints: (a) the final XYZ CPM termed 4-5-XYZ CPM-1 (i.e. XYZ CPM-1 with 4-DOC-2 PM and 5-DOC AM) via adding two wire beams to the AM-Z, and (b) the final XYZ CPM termed 4-5-XYZ CPM-2 (i.e. XYZ CPM-2 with 4-DOC-2 PM and 5-DOC AM) via adding other four wire beams to the AMs of the 4-5-XYZ CPM-1. (Online version in color.)

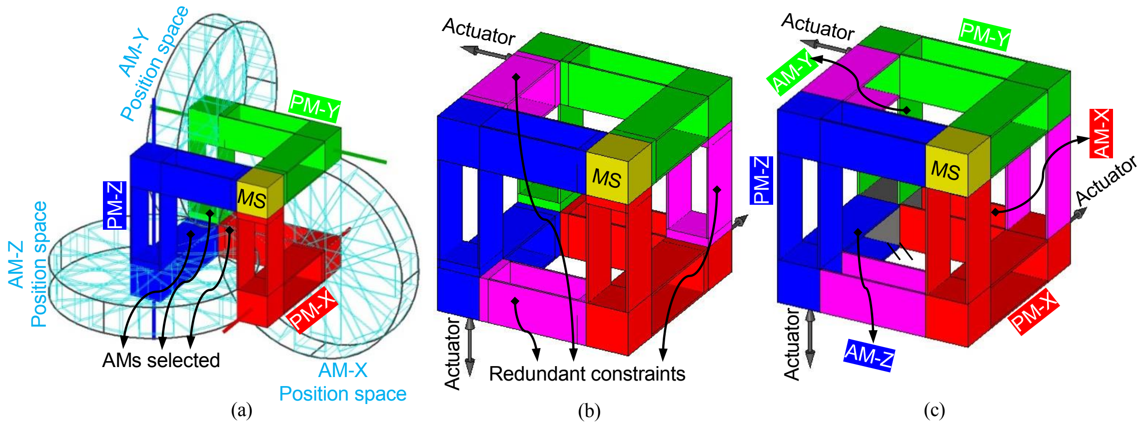
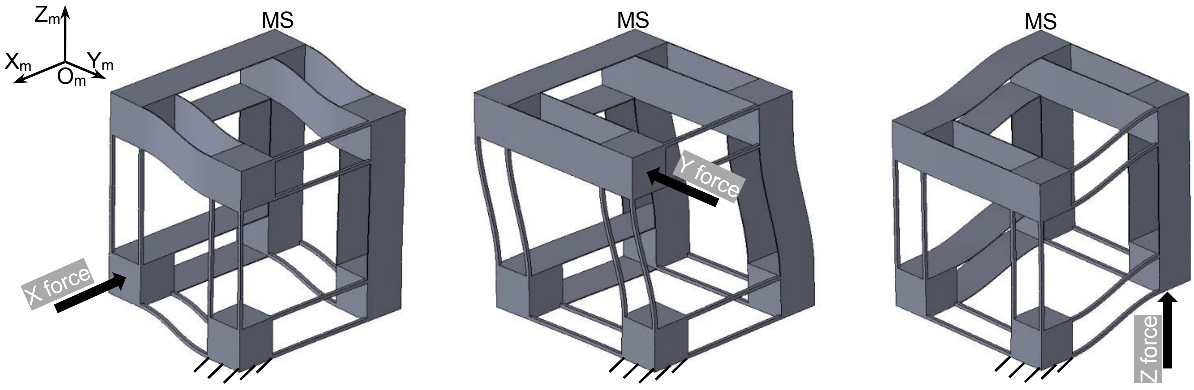
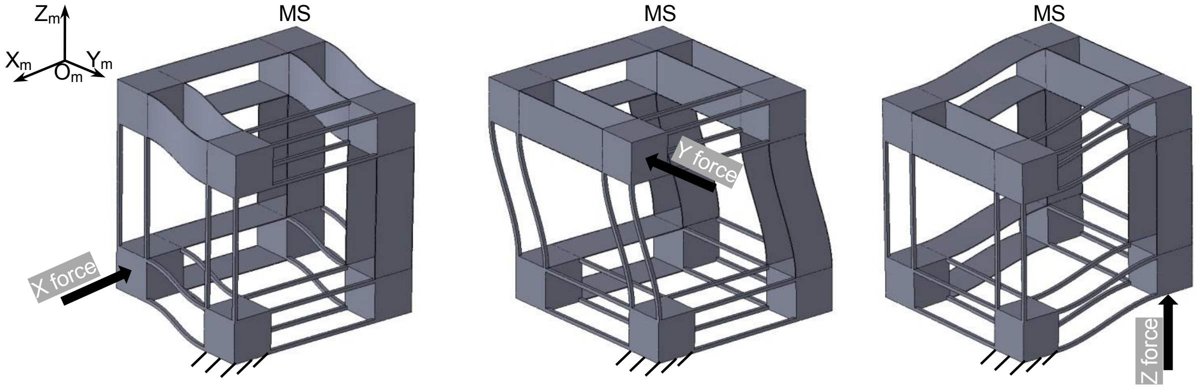


Fig. 17 An XYZ CPM designed via replacing the AMs of the 4-4-XYZ CPM with the 5-DOC-2 module based on the constraints in the case 4: (a) determining the AM positions, (b) adding redundant constraints, and (c) the final XYZ CPM termed 4-5-XYZ CPM-3 (i.e. XYZ CPM-3 with 4-DOC-2 PM and 5-DOC AM). (Online version in color.)



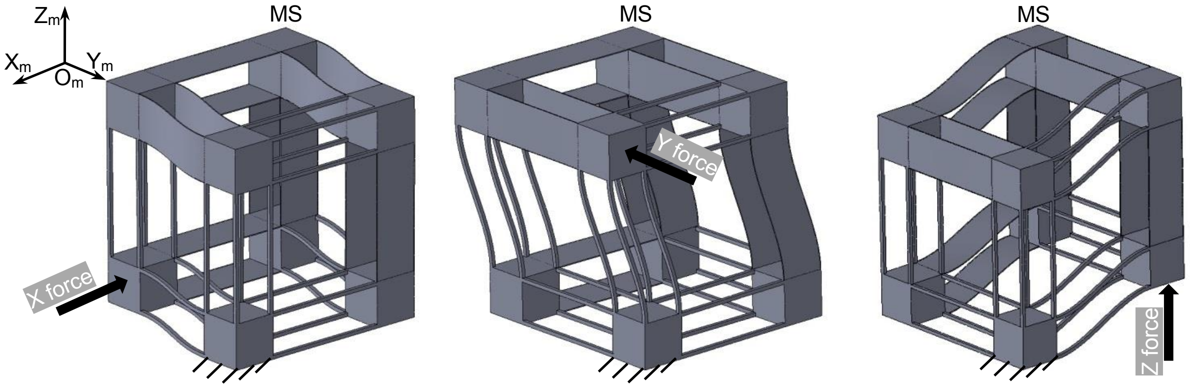
(a) (b) (c)

Fig. 18 The FEA results of the 4-4-XYZ CPM: (a) X motion only, (b) Y motion only, and (c) Z motion only.



(a) (b) (c)

Fig. 19 The FEA results of the 4-5-XYZ CPM-1: (a) X motion only, (b) Y motion only, and (c) Z motion only.



(a) (b) (c)

Fig. 20 The FEA results of the 4-5-XYZ CPM-2: (a) X motion only, (b) Y motion only, and (c) Z motion only.

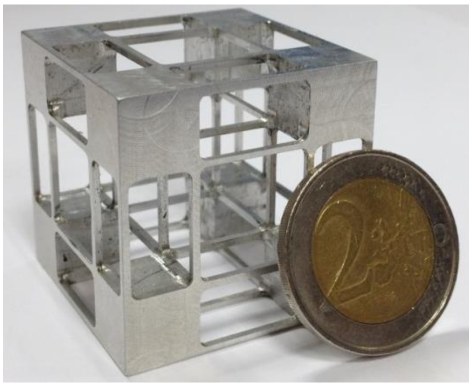


Fig. 21 A prototype of the 3-5-XYZ CPM.

8. Conclusion

A novel CPI approach for synthesizing decoupled XYZ CPMs with consideration of actuation isolation has been proposed in this paper. The XYZ CPMs designed using the CPI approach have the following characteristics:

(a) Each XYZ CPM has three non-redundant parallel legs between the MS and the BSs. Note that redundant legs can be added in the further modification design step for symmetrical arrangement etc.

(b) Each leg has an AS that is permitted to translate in one actuation direction only.

(c) The translational motion of each AS is transmitted to the MS without influencing the other two translations of the MS.

(d) Non-desired rotational motions of MSs are constrained by the three legs.

(e) The structures and positions of the legs can be adjusted under the constraint spaces and the position spaces to meet a variety of design requirements and applications.

The CPI approach is a systematic arrangement approach for the rigid stages and the compliant modules in an XYZ CPM system according to the constraint spaces and the position spaces. The constraint spaces and the position spaces have been derived based on screw theory rather than design experience. The constraint spaces are classified into three different types (B-constraint space, T-constraint space, and S-constraint space), and the mainly-used positions in the position spaces are illustrated by geometric shapes. Therefore, the CPI approach is an efficient method not only for experts but also for beginners. The synthesis process has been demonstrated step by step via several monolithic XYZ CPMs.

The proposed constraint spaces contain a number of constraint combinations. Moreover, a number of XYZ CPMs can be designed based on only one of the combinations, because: (a) each compliant module in an XYZ CPM system has many permitted positions to select; and (b) each compliant module can be designed with different structures such as parallel structure, serial structure, hybrid structure. Therefore, a variety of XYZ CPMs can be synthesized using the CPI approach based on the constraint spaces and position spaces.

It is noted that the present CPI approach focuses on the early-stage conceptual design. The nonlinear characteristics such as parasitic motion over the large range of motion are left for future work. Additionally, coupled XYZ CPMs can be synthesized through further modifications of the decoupled XYZ CPMs presented using the CPI approach, which is detailed in Appendix D.

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Appendix A: Monolithic decoupled XYZ CPM design with PM constraints coupled

The decomposition method used in this paper is an effective way to analyze the compliant modules of an XYZ CPM, which is suitable for designing most decoupled XYZ CPMs with considering actuation isolation. However, there is a limitation resulting from the decomposition. As the aforementioned assumption in Section 1, the PMs and AMs are regarded as independent compliant modules. Take an AS which is constrained by one PM and one AM for example, if the PM and the AM are independent to each other, any one DOF of the AS can be constrained by the PM and/or the AM, but is not constrained by the combination of the PM and the AM without redundant constraints. Due to this assumption, one kind of decoupled XYZ CPMs cannot be synthesized using this CPI approach. This kind of decoupled XYZ CPMs have the following characteristics: (a) the PMs in different legs are combined together to constraint a DOF of the MS, but any one of the PMs cannot resist the DOF. and/or (b) the AM and the PM in a leg are combined together to constrain a DOF of the AS, but the AM or the PM cannot resists the DOF of the AS separately. One specific example is shown below.

If two redundant constraint beams are added for each AM of the XYZ CPM as shown in Fig. 13(c), the XYZ CPM-1 as demonstrated in Fig. A. 1(a) can be obtained by the CPI approach. The XYZ CPM-2 as illustrated in Fig. A. 1(b) can be obtained through rotating the three PMs of the XYZ CPM-1 at the same 45 degrees. The decoupled translations of the XYZ CPM-2 can be seen in Figs. A. 1(c)-(e).

Each of the PM in the XYZ CPM-2 system cannot constrain any one of the rotations of the MS of the XYZ CPM-2 about the X-, Y- and Z-axes with regard to the coordinate system O-XYZ, but the three rotations can be resisted by the three PMs of the XYZ CPM-2 together. Therefore, the constraints of the PMs in XYZ CPM-2 system are not independent to each other. The coupled constraints of PMs can be explained by understanding constraint devices such as spheres in vees [33].

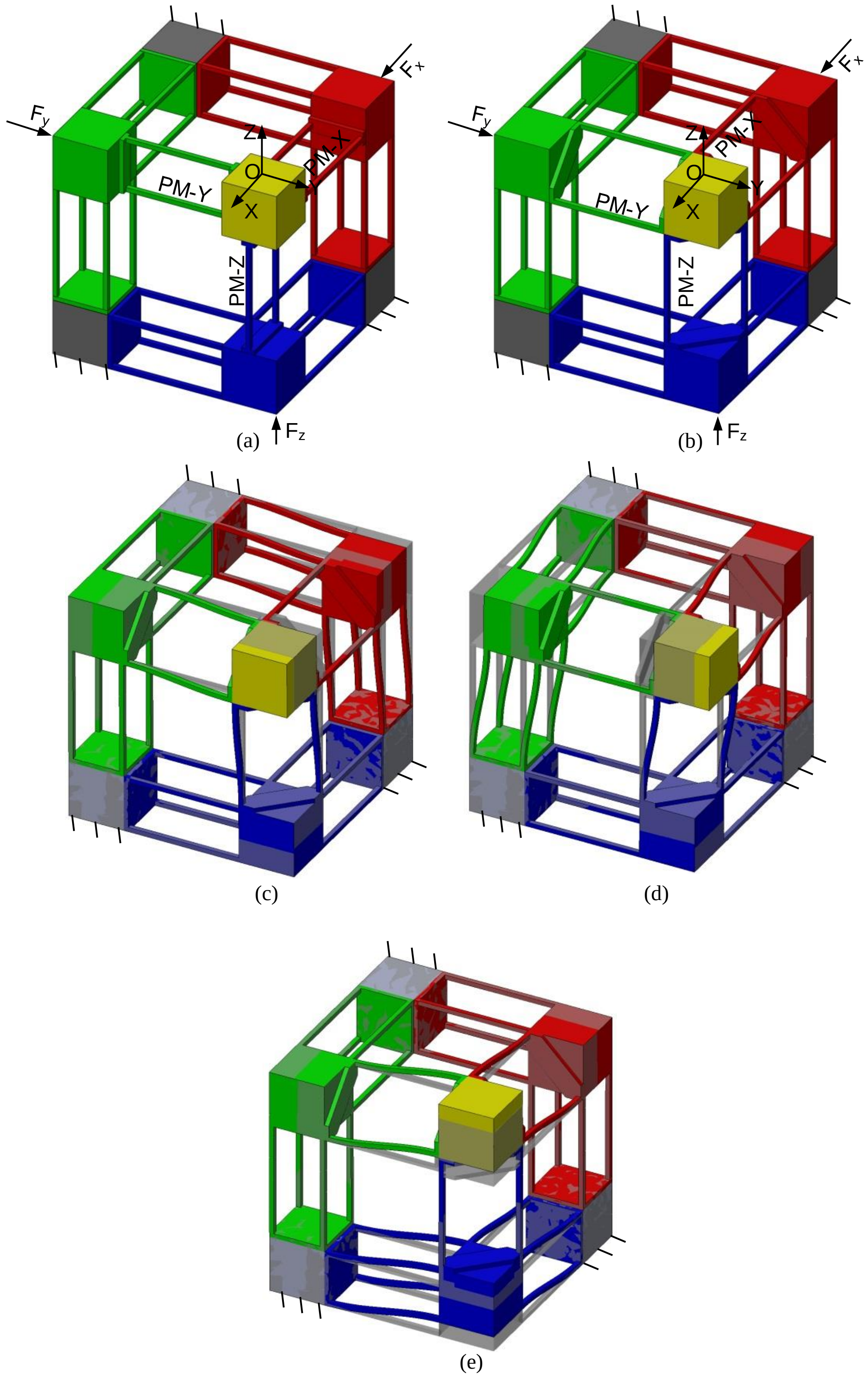


Fig. A.1 Decoupled XYZ CPM designs: (a) decoupled XYZ CPM-1, (b) decoupled XYZ CPM-2, (c) X direction motion of the decoupled XYZ CPM-2, (d) Y direction motion of the decoupled XYZ CPM-2, and (e) Z direction motion of the decoupled XYZ CPM-2. (Online version in color)

Appendix B: Constraint spaces of PMs and AMs

B-constraint space is represented in Table B.1, where T and R mean translational constraint and rotational constraint, respectively. The subscripts px, py and pz indicate the three PM local coordinate systems, and the subscripts ax, ay and az indicate the three AM local coordinate systems. The subscripts -tx, -ty and -tz represent the translational constraints along the three axes of each local coordinate system, and -rx, -ry and -rz indicate the rotational constraints about the three axes of each local coordinate system. The underlined rotational constraints can be transmitted from the AMs to the PMs.

The T-constraint space can be derived via transmitting some of the underlined rotational constraints from the AMs to PMs. The S-constraint space can be obtained through adding redundant rotational constraints to the AMs and PMs in the B-constraint space and T-constraint space. The B-constraint space, T-constraint space and S-constraint space compose the constraint spaces of the compliant modules.

Table B.1 B-constraint space.

Combination	Module	DOC of the modules in the three legs in the local coordinate systems		
		X	**Y**	**Z**
Combination 1	PMs	$T_{px-tx}R_{px-rx}R_{px-ry}R_{px-rz}$	T_{py-tx}	T_{pz-tx}
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}R_{ax-ry}R_{ax-rz}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}\underline{R_{ay-ry}}\underline{R_{ay-rz}}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}\underline{R_{az-ry}}\underline{R_{az-rz}}$
Combination 2	PMs	$T_{px-tx}R_{px-rx}R_{px-ry}$	$T_{py-tx}R_{py-rz}$	T_{pz-tx}
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}R_{ax-ry}\underline{R_{ax-rz}}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}\underline{R_{ay-ry}}R_{ay-rz}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}\underline{R_{az-ry}}\underline{R_{az-rz}}$
Combination 3	PMs	$T_{px-tx}R_{px-rx}R_{px-ry}$	T_{py-tx}	$T_{pz-tx}R_{pz-rx}$
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}R_{ax-ry}\underline{R_{ax-rz}}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}\underline{R_{ay-ry}}\underline{R_{ay-rz}}$	$T_{az-ty}T_{az-tz}R_{az-rx}\underline{R_{az-ry}}\underline{R_{az-rz}}$
Combination 4	PMs	$T_{px-tx}R_{px-rx}R_{px-rz}$	$T_{py-tx}R_{py-rx}$	T_{pz-tx}
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}\underline{R_{ax-ry}}R_{ax-rz}$	$T_{ay-ty}T_{ay-tz}R_{ay-rx}\underline{R_{ay-ry}}\underline{R_{ay-rz}}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}\underline{R_{az-ry}}\underline{R_{az-rz}}$
Combination 5	PMs	$T_{px-tx}R_{px-rx}$	$T_{py-tx}R_{py-rx}R_{py-rz}$	T_{pz-tx}
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}\underline{R_{ax-ry}}\underline{R_{ax-rz}}$	$T_{ay-ty}T_{ay-tz}R_{ay-rx}\underline{R_{ay-ry}}R_{ay-rz}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}\underline{R_{az-ry}}\underline{R_{az-rz}}$
Combination 6	PMs	$T_{px-tx}R_{px-rx}$	$T_{py-tx}R_{py-rx}$	$T_{pz-tx}R_{pz-rx}$
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}\underline{R_{ax-ry}}\underline{R_{ax-rz}}$	$T_{ay-ty}T_{ay-tz}R_{ay-rx}\underline{R_{ay-ry}}\underline{R_{ay-rz}}$	$T_{az-ty}T_{az-tz}R_{az-rx}\underline{R_{az-ry}}\underline{R_{az-rz}}$
Combination 7	PMs	$T_{px-tx}R_{px-rx}R_{px-rz}$	T_{py-tx}	$T_{pz-tx}R_{pz-ry}$
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}\underline{R_{ax-ry}}R_{ax-rz}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}\underline{R_{ay-ry}}\underline{R_{ay-rz}}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}R_{az-ry}\underline{R_{az-rz}}$
Combination 8	PMs	$T_{px-tx}R_{px-rx}$	$T_{py-tx}R_{py-rz}$	$T_{pz-tx}R_{pz-ry}$
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}\underline{R_{ax-ry}}\underline{R_{ax-rz}}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}\underline{R_{ay-ry}}R_{ay-rz}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}R_{az-ry}\underline{R_{az-rz}}$
Combination 9	PMs	$T_{px-tx}R_{px-rx}$	T_{py-tx}	$T_{pz-tx}R_{pz-rx}R_{pz-ry}$
	AMs	$T_{ax-ty}T_{ax-tz}R_{ax-rx}\underline{R_{ax-ry}}\underline{R_{ax-rz}}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}\underline{R_{ay-ry}}\underline{R_{ay-rz}}$	$T_{az-ty}T_{az-tz}R_{az-rx}R_{az-ry}\underline{R_{az-rz}}$
Combination 10	PMs	$T_{px-tx}R_{px-ry}R_{px-rz}$	$T_{py-tx}R_{py-ry}$	T_{pz-tx}
	AMs	$T_{ax-ty}T_{ax-tz}\underline{R_{ax-rx}}R_{ax-ry}R_{ax-rz}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}R_{ay-ry}\underline{R_{ay-rz}}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}\underline{R_{az-ry}}\underline{R_{az-rz}}$
Combination 11	PMs	$T_{px-tx}R_{px-ry}$	$T_{py-tx}R_{py-ry}R_{py-rz}$	T_{pz-tx}
	AMs	$T_{ax-ty}T_{ax-tz}\underline{R_{ax-rx}}R_{ax-ry}\underline{R_{ax-rz}}$	$T_{ay-ty}T_{ay-tz}\underline{R_{ay-rx}}R_{ay-ry}R_{ay-rz}$	$T_{az-ty}T_{az-tz}\underline{R_{az-rx}}\underline{R_{az-ry}}\underline{R_{az-rz}}$

Combination	Module	DOC of the modules in the three legs in the local coordinate systems		
		X	**Y**	**Z**
Combination 12	PMs	$T_{\text{px-tx}}R_{\text{px-ry}}$	$T_{\text{py-tx}}R_{\text{py-ry}}$	$T_{\text{pz-tx}}R_{\text{pz-rx}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}R_{\text{ax-ry}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}R_{\text{ay-ry}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}R_{\text{az-rx}}\underline{R_{\text{az-ry}}}\underline{R_{\text{az-rz}}}$
Combination 13	PMs	$T_{\text{px-tx}}R_{\text{px-rz}}$	$T_{\text{py-tx}}R_{\text{py-rx}}R_{\text{py-ry}}$	$T_{\text{pz-tx}}$
	AMs	$T_{\text{x-ty}}T_{\text{x-tz}}\underline{R_{\text{x-rx}}}\underline{R_{\text{x-ry}}}R_{\text{x-rz}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}R_{\text{ay-rx}}R_{\text{ay-ry}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}\underline{R_{\text{az-ry}}}\underline{R_{\text{az-rz}}}$
Combination 14	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}R_{\text{py-rx}}R_{\text{py-ry}}R_{\text{py-rz}}$	$T_{\text{pz-tx}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}R_{\text{ay-rx}}R_{\text{ay-ry}}R_{\text{ay-rz}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}\underline{R_{\text{az-ry}}}\underline{R_{\text{az-rz}}}$
Combination 15	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}R_{\text{py-rx}}R_{\text{py-ry}}$	$T_{\text{pz-tx}}R_{\text{pz-rx}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}R_{\text{ay-rx}}R_{\text{ay-ry}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}R_{\text{az-rx}}\underline{R_{\text{az-ry}}}\underline{R_{\text{az-rz}}}$
Combination 16	PMs	$T_{\text{px-tx}}R_{\text{px-rz}}$	$T_{\text{py-tx}}R_{\text{py-ry}}$	$T_{\text{pz-tx}}R_{\text{pz-ry}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}R_{\text{ax-rz}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}R_{\text{ay-ry}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}R_{\text{az-ry}}\underline{R_{\text{az-rz}}}$
Combination 17	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}R_{\text{py-ry}}R_{\text{py-rz}}$	$T_{\text{pz-tx}}R_{\text{pz-ry}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}R_{\text{ay-ry}}R_{\text{ay-rz}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}R_{\text{az-ry}}\underline{R_{\text{az-rz}}}$
Combination 18	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}R_{\text{py-ry}}$	$T_{\text{pz-tx}}R_{\text{pz-rx}}R_{\text{pz-ry}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}R_{\text{ay-ry}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}R_{\text{az-rx}}R_{\text{az-ry}}\underline{R_{\text{az-rz}}}$
Combination 19	PMs	$T_{\text{px-tx}}R_{\text{px-ry}}R_{\text{px-rz}}$	$T_{\text{py-tx}}$	$T_{\text{pz-tx}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}R_{\text{ax-ry}}R_{\text{ax-rz}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}\underline{R_{\text{ay-ry}}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}\underline{R_{\text{az-ry}}}R_{\text{az-rz}}$
Combination 20	PMs	$T_{\text{px-tx}}R_{\text{px-ry}}$	$T_{\text{py-tx}}R_{\text{py-rz}}$	$T_{\text{pz-tx}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}R_{\text{ax-ry}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}\underline{R_{\text{ay-ry}}}R_{\text{ay-rz}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}\underline{R_{\text{az-ry}}}R_{\text{az-rz}}$
Combination 21	PMs	$T_{\text{px-tx}}R_{\text{px-ry}}$	$T_{\text{py-tx}}$	$T_{\text{pz-tx}}R_{\text{pz-rx}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}R_{\text{ax-ry}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}\underline{R_{\text{ay-ry}}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}R_{\text{az-rx}}\underline{R_{\text{az-ry}}}R_{\text{az-rz}}$
Combination 22	PMs	$T_{\text{px-tx}}R_{\text{px-rz}}$	$T_{\text{py-tx}}R_{\text{py-rx}}$	$T_{\text{pz-tx}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}R_{\text{ax-rz}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}R_{\text{ay-rx}}\underline{R_{\text{ay-ry}}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}\underline{R_{\text{az-ry}}}R_{\text{az-rz}}$
Combination 23	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}R_{\text{py-rx}}R_{\text{py-rz}}$	$T_{\text{pz-tx}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}R_{\text{ay-rx}}\underline{R_{\text{ay-ry}}}R_{\text{ay-rz}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}\underline{R_{\text{az-ry}}}R_{\text{az-rz}}$
Combination 24	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}R_{\text{py-rx}}$	$T_{\text{pz-tx}}R_{\text{pz-rx}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}R_{\text{ay-rx}}\underline{R_{\text{ay-ry}}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}R_{\text{az-rx}}\underline{R_{\text{az-ry}}}R_{\text{az-rz}}$
Combination 25	PMs	$T_{\text{px-tx}}R_{\text{px-rz}}$	$T_{\text{py-tx}}$	$T_{\text{pz-tx}}R_{\text{pz-ry}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}R_{\text{ax-rz}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}\underline{R_{\text{ay-ry}}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}R_{\text{az-ry}}R_{\text{az-rz}}$
Combination 26	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}R_{\text{py-rz}}$	$T_{\text{pz-tx}}R_{\text{pz-ry}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}\underline{R_{\text{ay-ry}}}R_{\text{ay-rz}}$	$T_{\text{az-ty}}T_{\text{az-tz}}\underline{R_{\text{az-rx}}}R_{\text{az-ry}}R_{\text{az-rz}}$
Combination 27	PMs	$T_{\text{px-tx}}$	$T_{\text{py-tx}}$	$T_{\text{pz-tx}}R_{\text{pz-rx}}R_{\text{pz-ry}}R_{\text{pz-rz}}$
	AMs	$T_{\text{ax-ty}}T_{\text{ax-tz}}\underline{R_{\text{ax-rx}}}\underline{R_{\text{ax-ry}}}\underline{R_{\text{ax-rz}}}$	$T_{\text{ay-ty}}T_{\text{ay-tz}}\underline{R_{\text{ay-rx}}}\underline{R_{\text{ay-ry}}}\underline{R_{\text{ay-rz}}}$	$T_{\text{az-ty}}T_{\text{az-tz}}R_{\text{az-rx}}R_{\text{az-ry}}R_{\text{az-rz}}$

Appendix C: Non-monolithic decoupled designs

Based on the position space concept as studied in Section 5, each of the PMs in the XYZ CPM as shown in Fig. 17(a) can rotate as a whole about the X-axis of the PM local coordinate system. Figure C. 1(a) shows an XYZ CPM obtained by rotating the PMs of the XYZ CPM shown in Fig. 17(a) at 45 degrees about the X-axes of the PM local coordinate systems. And the decoupled translational motions of the XYZ CPM are derived using FEA method and demonstrated in Figs. C. 1(b)-(d). Another XYZ CPM illustrated in Fig. B. 2 can be designed by rotating the PM-X of the XYZ CPM shown in Fig. 17(a) at 90 degrees about the X-axis of the PM local coordinate system. The two novel XYZ CPMs are proposed firstly in this paper, which can be used in some specific applications.

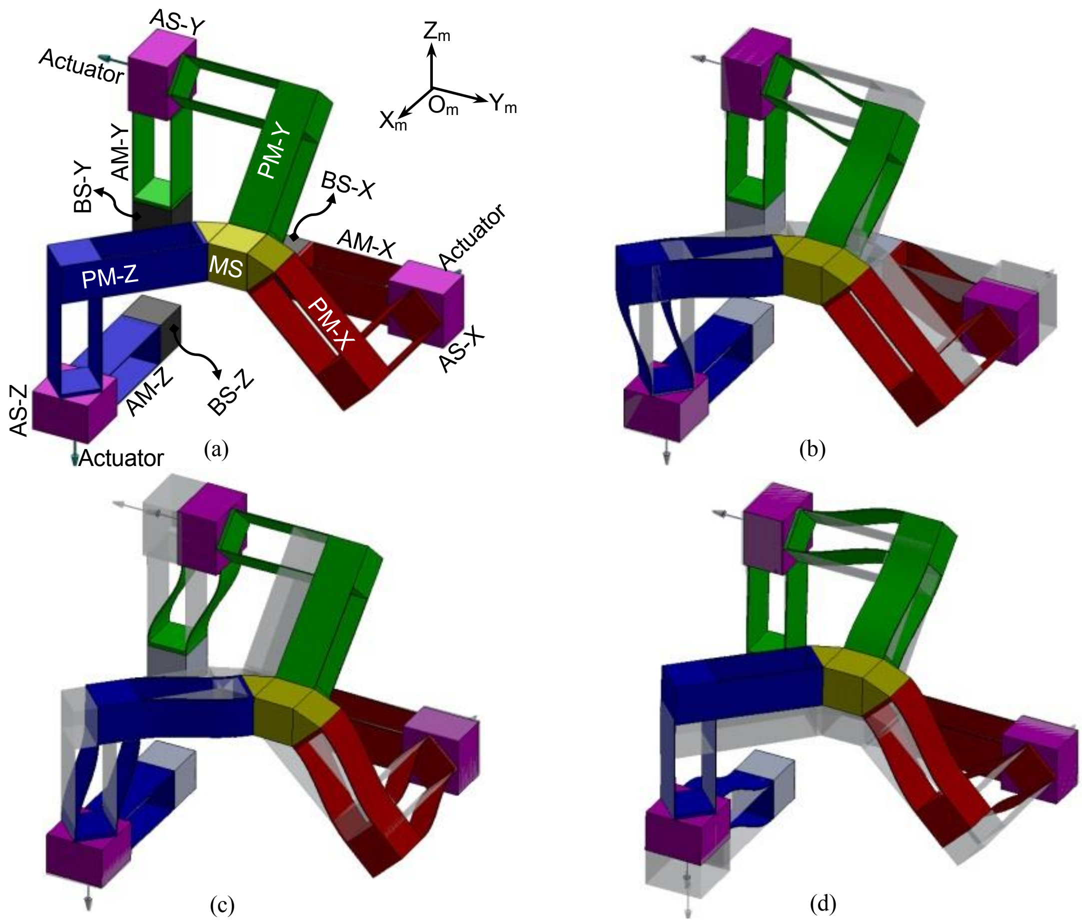


Fig. C.1 An XYZ CPM with PMs of the XYZ CPM shown in Fig. 17(a) rotating at 45 degrees about the X-axes of the PM local coordinate systems: (a) XYZ CPM without motion, (b) X motion only, (c) Y motion only, and (d) Z motion only. (Online version in color.)

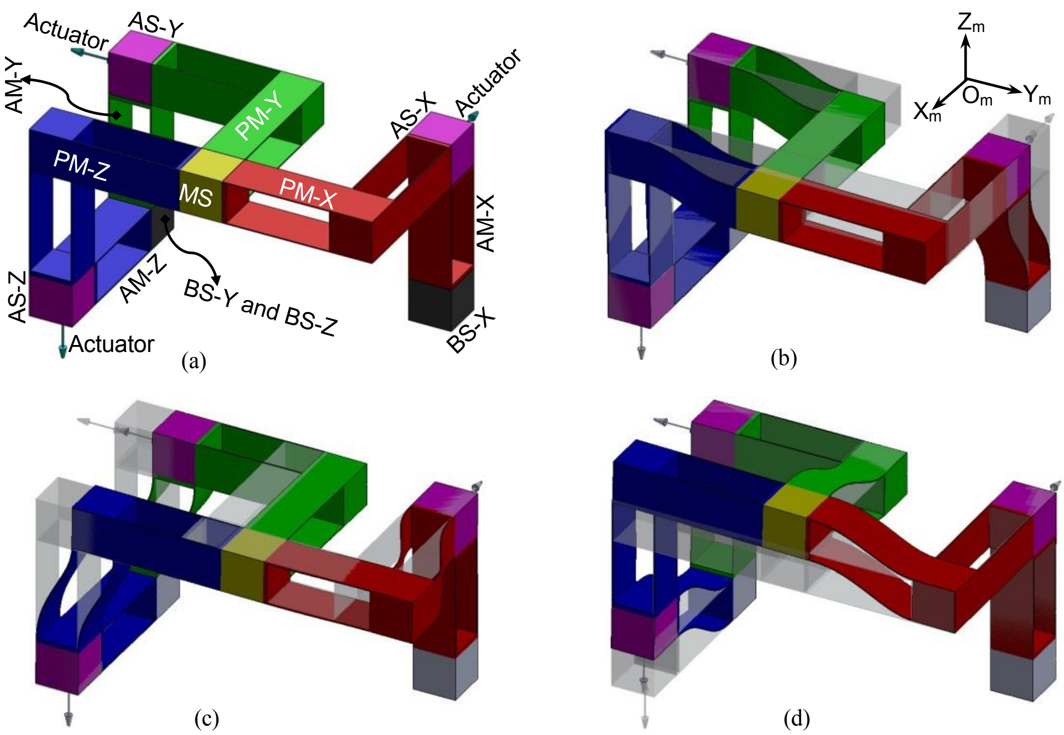


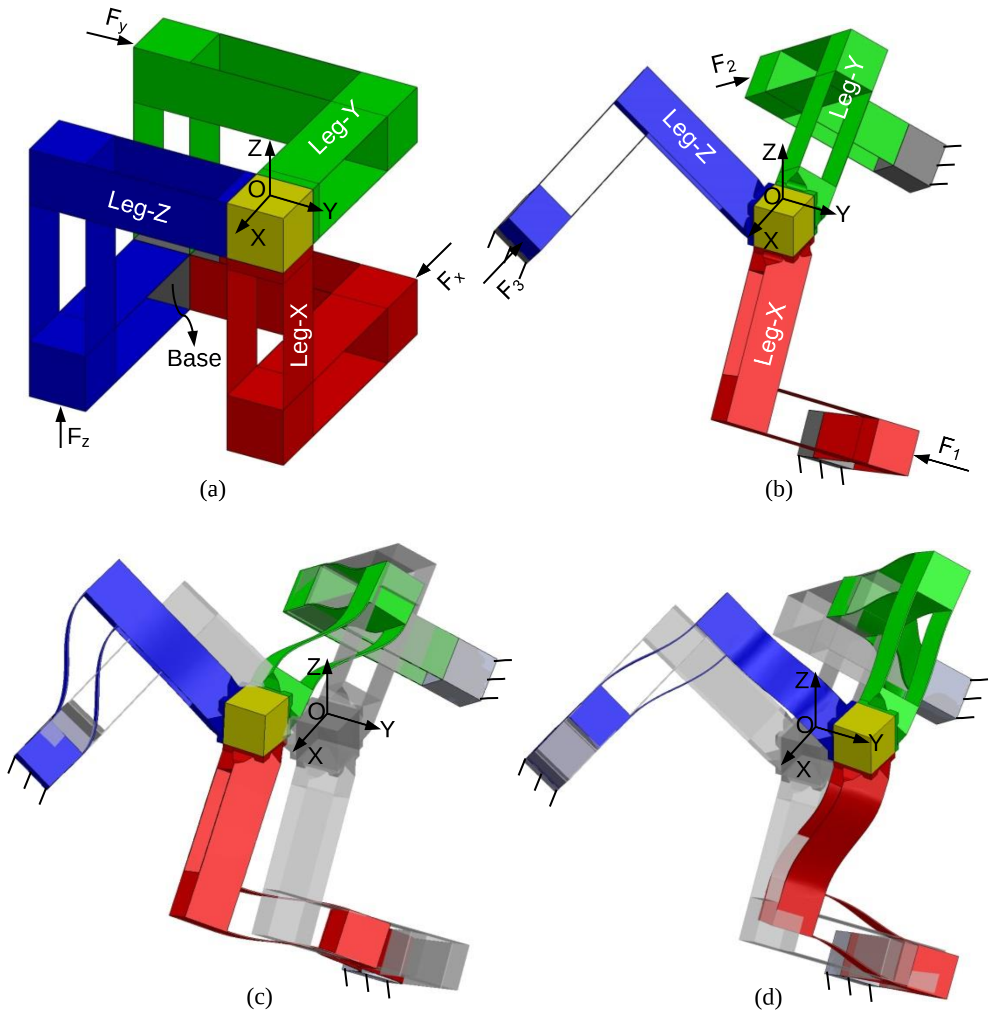
Fig. C.2 An XYZ CPM with PM-X of the XYZ CPM shown in Fig. 17(a) rotating at 90 degrees about the X-axis of the PM local coordinate system: (a) XYZ CPM without motion, (b) X motion only, (c) Y motion only, and (d) Z motion only. (Online version in color.)

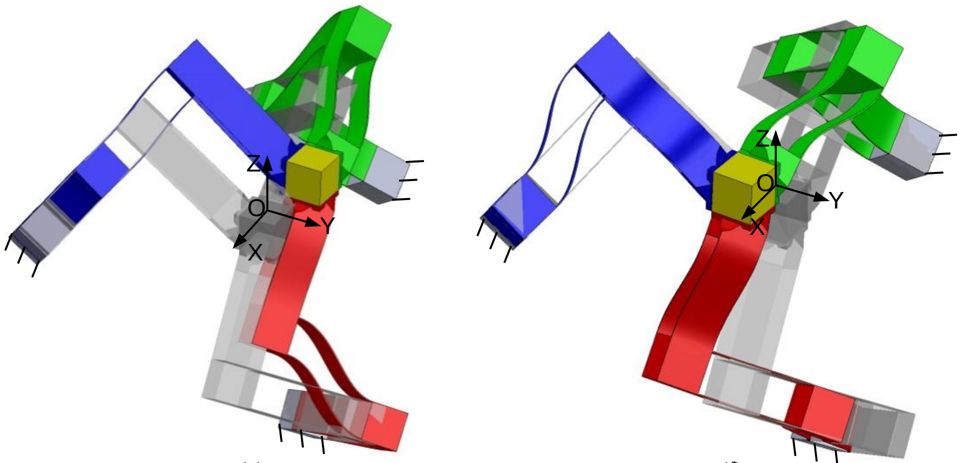
Appendix D: Coupled XYZ CPM design

The CPI approach does not focus on designing coupled XYZ CPMs, but a number of coupled XYZ CPMs can be obtained through making appropriate modifications on the decoupled XYZ CPMs obtained using the CPI approach. The modification is usually based on the position space concept with one typical example demonstrated as below.

Figure D. 1(a) shows a decoupled XYZ CPM, which is also illustrated in Fig. 17(a). If the decoupled XYZ CPM is decomposed into one motion stage (MS) and three legs, each of the legs has three translational DOF along the X-, Y- and Z-axes of the coordinate system O-XYZ. Therefore, each of the legs can constrain the three rotational DOF of the MS. Based on the method of identifying position space of a compliant module studied in Section 3, each of the legs can freely rotate and translate along and about the X-, Y- and Z-axes without affecting the MS three-axis translations. However, in order to make the MS controllable by the three actuation forces to have a spatial motion, the directions of any two of the three actuation forces cannot be parallel or collinear. Since when any two actuation forces are parallel or collinear, there is at least one leg to be redundant.

As a result, the coupled XYZ CPM as shown in Fig. D. 1(b) is gained via rotating the three legs about the specific axes through the following steps. The Leg-X rotates about the Z-axis at minus 45 degrees and then rotates about the Y-axis at minus 45 degrees; the Leg-Y rotates about the X-axis at 45 degrees and then rotates about the Y-axis at 45 degrees; and the Leg-Z rotates about the Y-axis at minus 45 degrees and then rotates about the X-axis at minus 45 degrees. These rotations result in the coupled actuation forces for the MS's motion. When only one of the three forces is applied, the MS will translate along the X-, Y- and Z-axes at the same time as represented in Figs. D. 1(c)-(e). When F_2 and F_3 forces are applied, the motions of the MS can be seen in Fig. D. 1(f). In other words, if only one of the three motions along the X-, Y- and Z-axes is needed, the three forces should be exerted simultaneously.





(e) (f)

Fig. D.1 A coupled XYZ CPM designed via some appropriate modification on the decoupled XYZ CPM as shown in Fig. 17(a): (a) the decoupled XYZ CPM also shown in Fig. 17(a), (b) the coupled XYZ CPM, (c) F_1 force applied only, (d) F_2 force applied only, (c) F_3 force applied only, and (d) F_2 and F_3 forces applied. (Online version in color.)