Flat foliations of spherically symmetric geometries

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We examine the solution of the constraints in spherically symmetric general relativity when spacetime has a flat spatial hypersurface. It is demonstrated explicitly that, given one flat slice, a foliation by flat slices can be consistently evolved. We show that when the sources are finite these slices do not admit singularities and we provide an explicit bound on the maximum value assumed by the extrinsic curvature. If the dominant energy condition is satisfied, the projection of the extrinsic curvature orthogonal to the radial direction possesses a definite sign. We provide both necessary and sufficient conditions for the formation of apparent horizons in this gauge which are qualitatively identical to those established earlier for extrinsic time foliations of spacetime, [J. Guven and N. Ó Murchadha, Phys. Rev. D **56** 7658 (1997); **56**, 7666 (1997)], which suggests that these conditions possess a gauge invariant validity. [S0556-2821(99)02420-0]

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INTRODUCTION

Despite the impressive array of techniques we currently possess for constructing initial data in general relativity, an unsatisfying feature persists: the sensitivity of the solution to the way we foliate spacetime. Indeed, this often overshadows entirely the physics we are attempting to understand. It is clear that the problem is even worse at the dynamical level.

In order to solve the constraints it is traditional to appeal to an extrinsic time foliation of spacetime. This involves some restriction on the extrinsic curvature so as to define the slicing. One such foliation, in particular, has a long history: in asymptotically flat spacetimes, the mean extrinsic curvature is set to zero. In the simplified context of spherically symmetric general relativity it is possible to examine explicitly how sensitively the solution of the constraints depends on this choice of gauge. This is easy in this case because the extrinsic curvature is completely specified by two scalar functions and the mean curvature is a linear combination of these two scalars. One can then treat any extrinsic time foliation as a specification of the ratio of these scalars. This has been the approach adopted in [1–5] in our examination of the constraints.

In [3–5] we restricted our attention to extrinsic curvatures which, at every point on the slice, are timelike vectors in superspace. In these articles we were able to demonstrate that the solution to the constraints is largely independent of the particular extrinsic time foliation we choose. Remarkably, we appear to have a robust characterization of the physical landmarks of the spatial geometry; specifically, it is possible to provide necessary and sufficient conditions for the formation of apparent horizons which are not overwhelmed by the gauge.

In order to be really satisfied that we have control over the gauge, we propose in this paper to place our prejudice in favor of extrinsic time foliations to the test, i.e., by abandoning such foliations entirely in favor of a foliation which is specified completely by a condition on the intrinsic geometry of the leaves of the foliation. We will explore the simplest choice: that the spatial geometry be flat. It is well known that such slices are provided in the Schwarzschild spacetime geometry by Lemaître coordinates (see, e.g., [6]). Recently, these coordinates have been exploited to describe the canonical reduction of the theory with a source consisting of a shell of massive dust [7]. Indeed, there has been a revival of interest in the canonical reduction of spherically symmetric general relativity (see, for example, [8,9] and references therein). However, when the sources are not confined to a shell, the canonical procedure in an extrinsic time foliation is not generally tractable. The notable exception is the polar slicing [10,11]. But simplicity comes at a price: the corresponding chart is pathological if the slice possesses an apparent horizon, as it ultimately will in a classically collapsing geometry. Flat slicings, while remarkably simple, escape this shortcoming.

Historically, flat foliations of spacetime are of interest due to the role they played in the seventies as a provider of useful guidelines for the construction of the proof of the positive mass theorem. Brill and Jang [12] showed that if the spatial geometry is flat, and the extrinsic curvature falls off more rapidly than $r^{-3/2}$ at infinity, the only solution of the vacuum Einstein equations is flat. One might, therefore, be tempted to conclude that such foliations are pathological, inconsistent as they appear to be with the presence of gravitational waves. As we will see, however, the falloff of the extrinsic curvature in this gauge is slow (exactly as $r^{-3/2}$) and thus Brill and Jang's argument is not applicable. Indeed, this slicing possesses the unusual feature that the Arnowitt-Deser-Misner (ADM) mass is encoded completely by the extrinsic curvature. While we do not have gravitational waves to contend with in a spherically symmetric geometry, it is still possible that the gauge exhibit pathologies associated with the

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sources. However, we demonstrate here that if the sources are finite, all solutions of the constraints are non-singular if they are non singular at the origin. Therefore, this gauge steers clear of singularities because the extrinsic curvature (which is now the only measure of the geometry) is always bounded when the sources are. This should be contrasted with the situation in an extrinsic time foliation where the solutions may well be singular if the sources are large but finite [2,3].

We examine the occurrence of apparent horizons in the intrinsic time gauge. We must abandon the geometrically satisfying but nonetheless gauge dependent identification of an apparent horizon with the formation of a bag with a neck (minimal surface) in the spatial geometry. Because the geometry is flat, there is no bag to speak of, much less the kind of bag envisaged by Wheeler. Now apparent horizons occur due to the action of extrinsic curvature alone. However, it is no longer enough that the extrinsic curvature be large, it also must possess the appropriate sign. This is due to the peculiarity of this gauge that, when the dominant energy condition is satisfied, the tangential projection K_R of the extrinsic curvature tensor has a definite sign.

We next search for necessary and sufficient conditions for the appearance of apparent horizons. Remarkably, once we take into account the obvious obstruction associated with the sign of K_R, we can reproduce the inequalities we derived earlier in the extrinsic time gauges [4,5]. This suggests a validity which transcends their gauge dependent derivation.

It remains to confirm that the flat foliation is consistent with the dynamics. We do this by constructing explicitly both the lapse and the shift necessary to maintain the flatness of the intrinsic geometry and we write down explicitly the exterior form of the spacetime metric.

Schoen and Yau in [19] gave a sufficient condition for the appearance of trapped surfaces in general initial data sets. It turns out that the only place these estimates can be realistically tested is on intrinsically flat manifolds. We compare the Schoen and Yau sufficient condition with ours for the case of a constant density intrinsically flat sphere and find that ours is better by a factor of 3.

Finally, in the same spirit which has prompted us to consider flat foliations in the first place, we examine the flat foliation as a special case of foliations with negative spatial scalar curvature.

FLAT FOLIATIONS

We recall that the constraints are given by (we exploit the notation introduced in [1])

$$K_R[K_R+2K_{\mathcal{L}}]-\frac{1}{R^2}[2(RR')'-R'^2-1]=8\pi\rho, \quad (1)$$

and

$$K_R'+\frac{R'}{R}(K_R-K_{\mathcal{L}})=4\pi J, \quad (2)$$

where the line element of the spherically symmetric spatial geometry is parametrized by

$$ds^2=dl^2+R^2d\Omega^2, \quad (3)$$

and we have expanded the extrinsic curvature (n^a is the outward pointing unit normal to the two-sphere of fixed l) as,

$$K_{ab}=n_an_bK_{\mathcal{L}}+(g_{ab}-n_an_b)K_R. \quad (4)$$

All derivatives are with respect to the proper radius of the spherical geometry, l. The spatial geometries we wish to consider are R^3 ($l\in 0,\infty$) with a single asymptotically flat region with a regular center, $l=0$. The appropriate boundary condition on the metric at $l=0$ is then

$$R(0)=0. \quad (5)$$

We recall that $R'(0)=1$ if the geometry is regular at this point. We assume that both the energy density of matter, ρ, and its radial flow, J,are appropriately bounded functions of l on some compact support.

When we consider general spherically symmetric initial data we start off with six functions, $(g_{rr},g_{\theta\theta},K_R,K_{\mathcal{L}},\rho,J)$, which satisfy the two constraints. Because the dynamics reside in the matter field, so that the geometry is purely kinematical, it seems natural to choose (ρ,J) as the independent variables. This still leaves us with four dependent objects satisfying the two constraints. One of the extra degrees of freedom is obviously the coordinate choice on the three slice. We can fix this more-or-less independently of everything else. One natural choice (which we use) is to set $g_{rr}\equiv 1$; the other standard choices are to arrange that the metric be conformally flat or that the radial coordinate be the areal (Schwarzschild) radius.

This leaves us with one extra variable among the three $(g_{\theta\theta}\equiv R^2,K_R,K_{\mathcal{L}})$ so we choose some relationship between them. Such a condition should fix the slicing, the way that the given slice is embedded into the spacetime. When we solve the constraints completely we determine both the intrinsic geometry and the extrinsic curvature.

In this paper we propose to foliate spacetime using the intrinsic geometry to mark time. The simplest possible choice is the one we will adopt. Let us suppose that

$$R(l)=l \quad (6)$$

everywhere so that the spatial geometry is flat R^3 everywhere. This condition is, *a priori*, no more restrictive than any of the other slicing conditions we have considered. The scalar curvature now vanishes with the result that the Hamiltonian constraint reduces to the algebraic condition on K_{ab} in terms of ρ,

$$K_R[K_R+2K_{\mathcal{L}}]=8\pi\rho. \quad (7)$$

The momentum constraint, apparently at least, is only modified in a trivial way: R'/R is replaced by $1/l$.

To solve the constraints we eliminate the extrinsic curvature scalar $K_{\mathcal{L}}$ from Eq. (7) in favor of K_R and ρ and substitute into (2). We get

$$(l^{3/2}K_R)' = 4\pi l^{3/2}\left(\frac{\rho}{lK_R} + J\right). \tag{8}$$

The equation is no longer linear in K_R. This is not too surprising. The constraints in general relativity are non-linear to start with and the physics is now completely encoded in the extrinsic geometry.

Lemaître slicing of the Schwarzschild geometry

As mentioned above, one example of such a flat slicing is given by the Lemaître coordinatization of the Schwarzschild solution. We start off with the Schwarzschild solution in standard coordinates

$$ds^2 = -\left(1-\frac{2m}{R}\right)dt^2 + \left(1-\frac{2m}{R}\right)^{-1}dR^2 + R^2 d\Omega^2, \tag{9}$$

and change the time coordinate via

$$t = \tau + 4m\sqrt{\frac{R}{2m}} + 2m\ln\left|\frac{\sqrt{R}-\sqrt{2m}}{\sqrt{R}+\sqrt{2m}}\right|. \tag{10}$$

Differentiating Eq. (10), we get

$$dt = d\tau + \left(1-\frac{2m}{R}\right)^{-1}\sqrt{\frac{2m}{R}}dR, \tag{11}$$

and when this is substituted into Eq. (9) we obtain

$$ds^2 = -\left(1-\frac{2m}{R}\right)d\tau^2 - 2\sqrt{\frac{2m}{R}}dR\,d\tau + dR^2 + R^2 d\Omega^2. \tag{12}$$

It is clear from Eq. (12) that the spatial geometry of the τ equal constant slices is flat [14]. In addition, Lemaître coordinates are non-singular on the horizon at $R=2m$. The price one pays, however, is that the form of the spacetime metric is no longer diagonal in R and τ. Leimaître coordinates cover half of the maximal extension of the Schwarzschild geometry. For our choice, this is the region above the principal diagonal on the Kruskal diagram. Note that the spatial geometry is regular at the Schwarzschild singularity at $R=0$. Each spatial slice intersects this singularity.

It is simple to construct the extrinsic curvature tensor. The Wald ([13]) definition of the extrinsic curvature gives

$$\partial_0 g_{ab} = 2NK_{ab} + N_{a;b} + N_{b;a}. \tag{13}$$

In the coordinate choice given by Eq. (12) we have $\partial_0 g_{ab} = 0$, $N\equiv 1$, and $N_i = (-\sqrt{2m/R},0,0)$ which gives

$$K_{RR} = -\frac{1}{2}\sqrt{\frac{2m}{R^3}}, \quad K_{\theta\theta} = +\sqrt{2mR},$$

$$K_{\phi\phi} = +\sqrt{2mR}\sin^2\theta. \tag{14}$$

From the definition (4) we get

$$K_{\mathcal{L}} = -\frac{1}{2}\sqrt{\frac{2m}{R^3}}, \quad K_R = +\sqrt{\frac{2m}{R^3}}. \tag{15}$$

Clearly these satisfy Eqs. (7) and (8) with vanishing sources. Both $K_{\mathcal{L}}$ and K_R diverge at $R=0$. The singularity in the spacetime curvature is encoded completely in the extrinsic curvature.

Momentarily static solutions

A peculiarity of the flat foliation is that K_R responds directly to ρ. Unlike its behavior in extrinsic time foliations, K_R does not vanish if J is identically zero unless ρ also is identically zero. Clearly, if there are any sources acting, this foliation does not admit any moment of time symmetry solutions with $K_{ab}=0$. In this respect, the simple constant density ''star'' solutions, with ρ constant on some compact support and $J=0$, should not be confused with genuine constant density solutions occurring in a momentarily static configuration.

Two solutions for each (ρ, J)

One can show using Eq. (8) that there are two solutions for each specification of (ρ, J). We write Eq. (8) as

$$K_R' + \frac{3}{2}l^{-1}K_R = 4\pi l^{-1}\frac{\rho}{K_R} + 4\pi J. \tag{16}$$

Suppose that K_R is finite at the origin, $l=0$. Then

$$K_R(0) = \pm\sqrt{\frac{8\pi\rho(0)}{3}}, \tag{17}$$

which is independent of J. Having fixed $K_R(0)$, Eq. (16) allows us to integrate out to find $K_R(l)$.

The momentum constraint and the quasi-potential local mass

An alternative casting of the momentum constraint is as

$$(l^3K_R^2)' = 8\pi l^2(\rho + JlK_R). \tag{18}$$

We recall that in this gauge the spherical quasi-local mass m (the Misner-Sharp, Hawking, *et al.* mass) [15,16] (see also [17] and [1] for references), defined by

$$m(l) = \frac{1}{2}R(1-R'^2) + R^3K_R^2, \tag{19}$$

is completely determined by extrinsic curvature:

$$m(l) = \frac{1}{2}l^3K_R^2, \tag{20}$$

and our rewriting of the constraint, Eq. (18), can be identified with the integrability condition on m,

$$m' = 4\pi R^2(\rho R' + JRK_R). \tag{21}$$

Thus we can solve Eq. (18) as

$$K_R = \pm \frac{1}{l^{3/2}} \sqrt{2m(l)}, \tag{22}$$

where $m(l)$ is the quasi-local mass. Therefore, a necessary condition for the existence of a flat slice through a given spherical spacetime is that the mass function be positive. m is manifestly positive in this gauge.

The geometry is necessarily singular if the constant of integration $m(0)$ is nonvanishing. Indeed, in vacuum, $m(l) = m(0)$ and we reproduce a Lemaître slice of the Schwarzschild geometry.

Outside the support of matter,

$$K_R^2 = \frac{2m(l_0)}{l^3}. \tag{23}$$

The ADM mass is encoded completely in $|K_R|$. We note also that K_R has the same asymptotic falloff as the gauge, $K_{\mathcal{L}} + 0.5K_R = 0$. In this gauge, the Hamiltonian constraint assumes the intrinsic geometric form,

$$^3\mathcal{R} = 16\pi\rho, \tag{24}$$

where $^3\mathcal{R}$ is the scalar curvature. It is equally legitimate, however, to treat Eq. (24) itself as a source dependent intrinsic time gauge condition. The Hamiltonian constraint then reduces to the algebraic condition, $K_{\mathcal{L}} + 0.5K_R = 0$. This way, the extrinsic time gauge can be viewed as entirely equivalent to an intrinsic time gauge, albeit an unconventional one depending explicitly on the source.

$J=0$ is exactly solvable

When $J=0$, for any given ρ, Eq. (18) is exactly solvable. Suppose that the weak energy condition, $\rho \geqslant 0$, holds. Then

$$K_R = \pm \frac{1}{l^{3/2}} \sqrt{2M(l)}, \tag{25}$$

where [18]

$$M(l) = 4\pi \int_0^l dl l^2 \rho, \tag{26}$$

is the bare material mass and equals the mass function. It must be positive. One solution is minus the other and K_R has a definite sign. Indeed, when the dominant energy condition holds, this property continues to hold.

K_R has a definite sign

If $\rho \geqslant c|J|$ for any constant c, then K_R possess a definite sign. In particular, this is true if the dominant energy condition, $c=1$, is satisfied. To prove this, let us suppose that K_R is positive at $l=0$. Suppose that it falls to zero on the support of ρ. Where $K_R \sim 0+$, the combination $\rho/lK_R + J$ appearing on the right-hand side (RHS) of Eq. (8) is positive so that the LHS, $(l^{3/2}K_R)'$, is also positive. K_R is therefore increasing and cannot fall through zero. An identical argument applies for negative K_R. In this gauge, dominant energy places a very strong constraint on K_R. Note, however, that the radial extrinsic curvature, $K_{\mathcal{L}}$, which is now determined by Eq. (7), does not possess a definite sign.

The effect of introducing a current source, J, on a solution with $J=0$ is easily deduced from Eq. (18). If K_R is positive, then a positive current increases K_R, whereas a negative one decreases it. In particular, if J has a definite sign (positive say), the solution has everywhere greater K_R than the corresponding solution with $J=0$.

The extrinsic geometry is non-singular everywhere

When ρ and J are finite, and the center is non-singular, K_R is finite and bounded away from zero everywhere. However, the only way the extrinsic geometry can become singular is by K_R diverging or $K_{\mathcal{L}}$ diverging (for which it is necessary that $K_R = 0$) so that the flat slice is singularity avoiding.

We can place an explicit bound on K_R as follows. We integrate Eq. (18)

$$L^3 K_R^2 = 8\pi \int_0^L dl l^2 (\rho + JlK_R). \tag{27}$$

We then have

$$L^2 (K_R^2)_{\mathrm{Max}} \leqslant 8\pi \left(\frac{1}{3} L^2 \rho_{\mathrm{Max}} + \frac{1}{4} L^3 J_{\mathrm{Max}} (K_R)_{\mathrm{Max}} \right), \tag{28}$$

so that (by solving the quadratic)

$$L(K_R)_{\mathrm{Max}} \leqslant \pi J_{\mathrm{Max}} L^2 + \left[\pi^2 (J_{\mathrm{Max}} L^2)^2 + \frac{8\pi}{3} \rho_{\mathrm{Max}} L^2 \right]^{1/2}. \tag{29}$$

K_R is clearly finite when ρ and J are. We note that the bound is more sensitively dependent on J_{Max} than it is on ρ_{Max}. Finally, we have seen that K_R cannot vanish on the support of ρ when the (generalized) dominant energy condition is satisfied. Thus $K_{\mathcal{L}}$, which is determined by Eq. (7), is also bounded.

APPARENT HORIZONS

We define the two optical scalars,

$$\omega_{\pm} = 2(R' \pm RK_R). \tag{30}$$

A future (past) apparent horizon forms when $\omega_{\pm} = 0$. The surface is future (past) trapped when $\omega_{\pm} \leqslant 0$. In this gauge, the appearance of an apparent horizon is entirely due to the action of extrinsic curvature.

We have already seen that solutions fall into two categories when the dominant energy condition is satisfied: those with $K_R > 0$ and those with $K_R < 0$. On solutions with K_R positive (negative), future (past) trapped surfaces are impos-

sible. Let us therefore focus on the occurrence of future (past) trapped surfaces in solutions with $K_R<0$ ($K_R>0$).

Suppose that $K_R<0$ and that the solution is free of future apparent horizons so that $lK_R>-1$. It is then clear from Eq. (8) that $(l^{3/2}K_R)'<0$ so that $l^{3/2}K_R$ decreases monotonically from zero at the origin (it saturates at the value $-\sqrt{2m}$ outside the source). Thus, if $l^{3/2}K_R$ is not monotonic, the solution must possess an apparent horizon. Of course the converse of this statement is false: monotonic $l^{3/2}K_R$ does not necessarily imply that the geometry is free of an apparent horizon [recall that any solution with $J=0$ given by Eq. (25) is monotonic].

A sufficient condition for the formation of trapped surfaces can be obtained as follows:

We integrate Eq. (8) from $l=0$ up to $l=L$:

$$L^{1/2}(\omega_+-1)_{l=L}=4\pi\int_0^L dl l^{3/2}\left(\frac{\rho}{lK_R}+J\right). \tag{31}$$

Let us suppose that the surface at $l=L$ is not future trapped, so that $\omega_{\pm}\geqslant 0$, nor does any trapped surface exist in the interior ($lK_R\geqslant -1$). Then

$$L^{1/2}\geqslant 4\pi\int_0^L dl l^{3/2}\left(\frac{\rho}{-lK_R}-J\right). \tag{32}$$

If $lK_R>0$, the inequality is vacuous as we would expect. If $lK_R<0$, then $1/(-lK_R)>1$, so that

$$L^{1/2}\geqslant 4\pi\int_0^L dl l^{3/2}(\rho-J). \tag{33}$$

In addition,

$$\int_0^L l^2 f(l)dl\leqslant L^{1/2}\int_0^L l^{3/2}f(l)dl, \tag{34}$$

for any positive function, $f(l)$, so that, with dominant energy, we obtain

$$L\geqslant 4\pi\int_0^L dl l^2(\rho-J)=M-P, \tag{35}$$

where

$$P=4\pi\int_0^L dl l^2 J. \tag{36}$$

It is clear that Eq. (34) is sharp as it is saturated with f peaked sharply about L.

The inequality Eq. (35) depends only on the physical measures of the initial data, i.e., ρ, and J, and the size of the region, L. It assumes the same form as the inequality we obtained in [4]. Indeed, in this gauge, we equal the best constant we obtained in [4].

Note that if the dominant energy condition is violated, the inequality Eq. (34) is not valid. No inequality analogous to Eq. (35) appears to hold.

We can exploit the universal bound on K_R, Eq. (29) to obtain a corresponding necessary condition.

Let the first apparent horizon occur at $l=L$. Then

$$L=8\pi\int_0^L dl l^2(\rho+JlK_R), \tag{37}$$

so that

$$L\leqslant\frac{8\pi L^3}{3}\rho_{\rm Max}+2\pi L^4 J_{\rm Max}(K_R)_{\rm Max}. \tag{38}$$

We now exploit the bound Eq. (29) for $L(K_R)_{\rm Max}$ to obtain

$$1\leqslant\frac{8\pi}{3}\rho_{\rm Max}L^2+2\pi J_{\rm Max}L^2 \times\left[\pi J_{\rm Max}L^2+\left(\pi^2(J_{\rm Max}L^2)^2+\frac{8\pi}{3}\rho_{\rm Max}L^2\right)^{1/2}\right]. \tag{39}$$

On rearrangement, Eq. (39) can be cast

$$1\leqslant 4\pi^2(J_{\rm Max}L^2)^2+\frac{8\pi}{3}\rho_{\rm Max}L^2. \tag{40}$$

If, in addition, we exploit dominant energy, we can replace $J_{\rm Max}$ by $\rho_{\rm Max}$ and solve the quadratic to get

$$\frac{1}{3\pi}\left(\frac{\sqrt{13}}{2}-1\right)\leqslant\rho_{\rm Max}L^2. \tag{41}$$

Thus if $\rho_{\rm Max}L^2<(\sqrt{13}/2-1)/3\pi$, the region cannot contain an apparent horizon. The constant of proportionality is comparable to that appearing in Eq. (57) of [5] with $\alpha=1$. The derivation is, however, considerably simpler. In the extrinsic time foliation, the derivation depended in an essential way on the application of a weighted Poincaré inequality. This improvement is clearly related to the singularity avoidance of the intrinsic time gauge we have exploited here.

CONSISTENCY OF FLAT FOLIATION WITH THE EVOLUTION

The condition that the flat foliation be preserved under evolution, $\partial_0\gamma_{ab}=0$, implies that the extrinsic curvature is proportional to a Killing form:

$$2NK_{ab}=-\nabla_a N_b-\nabla_b N_a. \tag{42}$$

In the spherically symmetric geometry we are considering this reduces to the set of conditions

$$N_l'=NK_{\mathcal{L}}$$

$$N_l=lNK_R. \tag{43}$$

Here N and N_l are respectively the lapse and the radial shift. Eliminating N, we obtain

$$N_l' = \frac{1}{l}\frac{K_{\mathcal{L}}}{K_R} N_l, \tag{44}$$

with the solution,

$$N_l = \exp\left(- \int dl\, \alpha / l \right)$$

$$N = \exp\left(- \int dl\, \alpha / l \right) / l K_R, \tag{45}$$

where $\alpha = -K_{\mathcal{L}}/K_R$ is the ratio of extrinsic curvature scalars introduced in [3]. Thus flat slicing is consistent with evolution. Indeed, both the lapse and shift are completely determined without appealing to the dynamical Einstein equations for $K_{\mathcal{L}}$ and K_R.

In the exterior region we have $\alpha = 1/2$, so that $N_l = N_l(L)(L/l)^{1/2}$ and $N = N_l(L) L^{1/2} (2m_0)^{-1/2}$ where $N_l(L)$ is the boundary shift. The lapse is constant. The boundary condition $N \to 1$ at infinity therefore fixes $N_l(L)$. Let τ be the time coordinate defined by this foliation. We then have the exterior spacetime metric

$$ds^2 = -\left(1 - \frac{2m_0}{l}\right) d\tau^2 \pm 2\sqrt{\frac{2m_0}{l}}\, d\tau dl + dl^2 + l^2 d\Omega^2. \tag{46}$$

This is the Schwarzschild metric expressed in Lemaître coordinates as we have already seen in Eq. (12). The spacetime is completely characterized by the shift.

NEGATIVE SCALAR CURVATURE

Let us consider instead of the flat slicing, any foliation with negative scalar curvature:

$${}^3\mathcal{R} = -\frac{1}{R^2}[2(RR')' - R'^2 - 1]. \tag{47}$$

It is then simple to demonstrate that $R' \geqslant 1$ everywhere so that $R \geqslant l$. We have, at a critical point of R',

$$R'^2 = 1 - {}^3\mathcal{R} R^2. \tag{48}$$

If the scalar curvature is bounded then so is R'. Such foliations are clearly very different from the extrinsic time foliations we considered in [3–5] with everywhere positive scalar curvature in which $R'^2 \leqslant 1$ and $R \leqslant l$ in regular solutions. With a prescribed value of the scalar curvature, we can solve the constraints exactly as we did in the flat slice. We get

$$(R^{3/2} K_R)' = 4\pi R^{3/2}\left(\frac{R'\tilde{\rho}}{R K_R} + J\right), \tag{49}$$

where we set $\tilde{\rho} = \rho - {}^3\mathcal{R}/16\pi$. We have $\tilde{\rho} \geqslant \rho$ so that whatever energy condition is good with ρ is better with $\tilde{\rho}$. Now, exactly as in a flat slicing, K_R has a definite sign when dominant energy holds.

It is straightforward to demonstrate that a bound can be placed on K_R. Such gauges therefore share with the flat foliation its singularity avoidance. This differs from any foliation with ${}^3\mathcal{R}$ positive where large sources are not always consistent with any singular geometry.

Let us examine the robustness of Eq. (35). It is simple to demonstrate that instead of Eq. (35), we obtain the sharp inequality

$$RR' \geqslant M - P, \tag{50}$$

which does not depend explicitly on ${}^3\mathcal{R}$. Equation (50) coincides with Eq. (35) when $R = l$. We see, however that the more negative the scalar curvature, the greater is the maximum of R', and thus the weaker the inequality.

THE SCHOEN AND YAU CRITERION FOR TRAPPED SURFACES

The first mathematically precise statement of a sufficient condition for the appearance of apparent horizons in a general (i.e., nonsymmetric) initial data set was given by Schoen and Yau in [19]. This condition is difficult to evaluate in general but it is possible to use it in the special case where the spatial geometry is flat, exactly the situation we are discussing in this article.

Schoen and Yau define the size of any compact three dimensional subset (Ω) of a Reimannian manifold as the minor radius of the largest three torus that can be embedded in Ω. They call this $Rad(\Omega)$. In [19] they prove two theorems. Theorem I is a statement that one cannot have a large set with large positive scalar curvature. More precisely, they show that if the scalar curvature of Ω is bounded below by a positive constant, ${}^{(3)}\mathcal{R} \geqslant \mathcal{R}_0 > 0$, then

$$Rad(\Omega) \leqslant \sqrt{\frac{8\pi^2}{3\mathcal{R}_0}}. \tag{51}$$

Theorem II deals with the situation where one has a solution to the initial value constraints (including positive matter) on a set Ω. If the matter satisfies $\rho - |J| \geqslant \lambda > 0$ and if Ω is large in the sense that

$$Rad(\Omega) \geqslant \sqrt{\frac{3\pi^2}{16\pi\lambda}}, \tag{52}$$

then the initial data must have a trapped surface.

If we have a maximal slice, we have that ${}^{(3)}\mathcal{R} = K^{ij}K_{ij} + 16\pi\rho \geqslant 16\pi\lambda$ and so we can use $16\pi\lambda$ in place of $\mathcal{R}_0$ in Theorem I. However, since $8/3 < 3$, we have from Theorem I that we can never get maximal initial data to satisfy Eq. (52). It is clear that the estimates leading to Theorem II are not sharp and some number smaller than 3 would almost certainly suffice. Unfortunately, the constant in Theorem I is also not sharp (see [20]) and the two constants are linked. Therefore our only hope of finding a nontrivial system in which to use Theorem II of Schoen and Yau is to look at nonmaximal initial data. In this case there is at least the possibility of having large ρ, so as to satisfy the condition in Theorem II, while simultaneously having small ${}^{(3)}\mathcal{R}$ to es-

cape the barrier that Theorem I imposes. When one looks at the way that Theorems I and II are derived, to try and find a configuration that satisfies the condition in Theorem II, it is clear that one wants to have no current, because it works against ρ and to have the mass density as uniform as possible. Further, one wants as little transverse traceless (TT) part in the extrinsic curvature as possible as that adds to the scalar curvature. Finally, the metric should be as simple as possible. This leads one to consider the situation where the intrinsic metric is flat, and thus eliminating any barrier due to Theorem I; the extrinsic curvature is pure trace; and the trace is constant, so that there is no current and one has a constant mass density. The other great advantage of the flat metric is that one can easily evaluate their ''torus'' measure of the size of a set.

In our notation, this is equivalent to choosing $K_R = K_{\mathcal{L}} = \frac{1}{3}\mathrm{tr}\,K = \text{constant}$. From Eq. (2) this gives $J=0$. From Eq. (7) we get that the mass density is constant and satisfies

$$\rho = \rho_0 = \frac{3K_R^2}{8\pi}. \tag{53}$$

If we have a spherical set of radius L satisfying this solution (one can think of it as part of a flat cosmology), it is clear from Eq. (30) that the horizon appears when $|LK_R|=1$. The sufficient condition we have derived [Eq. (35)] when applied to this special case gives $|LK_R| \geqslant \sqrt{2}$, and the necessary condition [Eq. (41)] gives $|LK_R| \geqslant 0.84$. Happily, these numbers lie on each side of 1.

If we apply the Schoen and Yau condition to an intrinsically flat constant density sphere of radius L, we get that $Rad(\Omega) = L/2$ and the Schoen and Yau sufficient condition [Eq. (52)] becomes $|LK_R| \geqslant \sqrt{2}\pi$. This calculation shows that the Schoen and Yau Theorem II is not vacuous. However, their set is 4 times larger than is required and their sufficiency condition a factor of 3 weaker than ours.

CONCLUSIONS

We have examined the constraints in spherically symmetric general relativity using an intrinsic time to foliate spacetime. Specifically we have foliated spacetime with flat spatial slices. The presentation of the initial data on a flat spatial hypersurface is very different from that on a hypersurface belonging to an extrinsic time foliation of spacetime. The Hamiltonian constraint becomes an algebraic constraint on the extrinsic curvature: when the weak energy condition is satisfied, trajectories in superspace lie completely inside the superspace light-cone—the complement in superspace of the allowed region consistent with any standard extrinsic curvature foliation ($K_R=0$ or $\mathrm{Tr}\,K=0$ or, more generally, any of those considered in [3]).

In this foliation, solutions of the constraints *do* exhibit peculiarities: when the sources are finite, there are no singular geometries satisfying the constraints other than those which contain a singularity at their center; when the dominant energy condition is satisfied, K_R possesses a definite sign; they do not admit minimal surfaces. Despite this, we find that the physical description they provide of apparent horizons is completely consistent with that in an extrinsic time foliation. Not only do the natural measures of the material content for necessary and sufficient conditions (ρ_{Max} and M respectively) coincide with those we found when we considered extrinsic time slices but, in addition, the inequalities assume identical forms.

Analogous gauges are applicable with other topologies. For example, in a closed cosmology with S^3 topology one could choose $R(l) = (l_0/2\pi)\sin(\pi l/l_0)$, where l_0 is the interpolar distance.

It would be interesting to examine the canonical reduction and subsequent quantization of spherically symmetric general relativity in this gauge. The fact that many of the features of extrinsic time foliations which are problematic do not occur suggests that flat foliations could provide a valuable alternative, in particular, for the description of gravitational collapse.

Finally, there appears to be no immediate obstruction to the construction of a foliation of a general asymptotically flat spacetime by a gauge of the form, ${}^3\mathcal{R}=0$.

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$$R = \left[\frac{3}{2}\left(\frac{\varrho}{\sqrt{2m}} + \sqrt{2m}\,\tau\right)\right]^{2/3}.$$

Using ϱ as a spatial coordinate the Schwarzschild metric can be written as

$$ds^2=-d\tau^2+\left[\frac{3}{2}\left(\frac{\varrho}{\sqrt{2m}}+\sqrt{2m}\,\tau\right)\right]^{-2/3}d\varrho^2$$

$$+\left[\frac{3}{2}\left(\frac{\varrho}{\sqrt{2m}}+\sqrt{2m}\,\tau\right)\right]^{4/3}d\Omega^2.$$

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