CORA Cork Open Research Archive
Cartlann Taighde Oscailte Chorcaí

Title	Instability of equatorial water waves with an underlying current
Authors	Genoud, Francois;Henry, David
Publication date	2014-04-01
Original Citation	Genoud, F. and Henry, D. (2014) 'Instability of Equatorial Water Waves with an Underlying Current', Journal of Mathematical Fluid Mechanics, 16 (4), pp. 661-667. doi: 10.1007/s00021-014-0175-4
Type of publication	Article (peer-reviewed)
Link to publisher's version	https://link.springer.com/article/10.1007/s00021-014-0175 - 10.1007/s00021-014-0175-4
Rights	© 2014 Springer Basel. This is a post-peer-review, pre-copyedit version of an article published in Journal of Mathematical Fluid Mechanics. The final authenticated version is available online at: https://doi.org/10.1007/s00021-014-0175-4
Download date	2026-10-31 04:58:10
Item downloaded from	https://hdl.handle.net/10468/12155



UCC
University College Cork, Ireland
Coláiste na hOllscoile Corcaigh

INSTABILITY OF EQUATORIAL WATER WAVES WITH AN UNDERLYING CURRENT

FRANÇOIS GENOUD AND DAVID HENRY

Abstract. In this paper we use the short-wavelength instability approach to derive an instability threshold for exact trapped equatorial waves propagating eastwards in the presence of an underlying current.

1. Introduction

In this paper we consider the hydrodynamical stability of a three-dimensional exact solution of the governing equations for equatorial geophysical water waves which was recently presented in [26]. This solution is explicit in the Lagrangian formulation, and it prescribes trapped equatorial waves which propagate eastwards in the presence of a constant underlying current. Physically, the equator acts as a natural waveguide, leading to trapped zonal waves which decay exponentially away from the equator, cf. the discussions in [7, 18]. Large-scale currents and wave-current interactions play a major role in the geophysical dynamics of the equatorial region, and it has been proposed that the interplay between equatorial currents in the ocean and atmosphere is one of the major generating mechanisms for El Niño and La Niña phenomena, cf. [30] and the discussions in [8, 10, 27]. In [34], Mollo-Christensen introduced a current-like term into Gerstner's solution for gravity waves in order to describe billows between two fluids, and this idea was extended to the geophysical setting in [26].

Hydrodynamical stability is an important area of fluid mechanics which has been addressed systematically since the nineteenth century, following the seminal work of Kelvin, Helmholtz, Rayleigh and Reynolds, cf. [16, 17, 21] for an overview of the mathematical approach to hydrodynamic stability. From a mathematical viewpoint, establishing the hydrodynamical stability or instability of a flow is difficult, given that

1991 *Mathematics Subject Classification.* Primary: 76B15; Secondary: 76B47, 35B50, 26E05.

Key words and phrases. Geophysical flows, instability, currents.

the fully nonlinear governing equations for fluid motion are highly intractable, and accordingly there exist only a handful of explicit exact solutions. Among these is the famous Gerstner's wave [3, 23, 25], an exact solution that is explicit in the Lagrangian formulation. Recently, Gerstner-type solutions have been derived and adapted to model a number of different physical and geophysical scenarios [4, 7, 9, 11, 33, 35] with explicit exact solutions in a Lagrangian framework [2].

The notion of hydrodynamical stability runs as follows. We suppose that a given fluid motion is perturbed by an infinitesimal disturbance, and the question of interest is as to how that infinitesimal perturbation evolves as time progresses. Physically, the question of hydrodynamic stability is important for numerous reasons. For instance, unstable flows cannot be observed in practice since they are rapidly destroyed by any minor perturbations or disturbances. In this paper we employ the short-wavelength instability method to prove that if the wave steepness exceeds a certain value, then the equatorial water waves presented in [26] are unstable under short wavelength perturbations. The short-wavelength instability method, which was independently developed by the authors of [1, 19, 32], examines how a localised and rapidly-varying infinitesimal perturbation will evolve by way of a system of ODEs. For certain solutions which have an explicit Lagrangian formulation, it transpires that the short wavelength instability analysis is remarkably elegant, and the criteria for instability (4.4)–(4.5) takes on a tangible and explicit formulation in terms of the wave steepness. This was first established in [31] for Gerstner's solution to the gravity water wave problem, and recently for geophysical flows in [13] and for edgewaves in [29]. The wave-steepness instability criteria which applies to the equatorially trapped wave with an underlying constant current is presented in Proposition 4.1 below.

2. Governing equations

In a reference frame with the origin located at a point on earth's surface and rotating with the earth, we take the $x-$axis to be the longitudinal direction (horizontally due east), the $y-$axis to be the latitudinal direction (horizontally due north) and the $z-$axis to be vertically upwards. We take the earth to be a perfect sphere of radius $R = 6378km$, which has a constant rotational speed of $\Omega = 73 \cdot 10^{-6} rad/s$, and $g = 9.8ms^{-2}$ is the standard gravitational acceleration at the earth's surface. The governing equations for geophysical ocean waves [15, 22]

are given by

$$u_t + uu_x + vu_y + wu_z + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\frac{1}{\rho}P_x, \tag{2.1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u\sin\phi = -\frac{1}{\rho}P_y, \tag{2.1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u\cos\phi = -\frac{1}{\rho}P_z - g, \tag{2.1c}$$

together with the equation of incompressibility

$$\nabla\cdot\mathbf{U} = 0. \tag{2.2a}$$

Here $\mathbf{U} = (u, v, w)$ is the velocity field of the fluid, the variable ϕ represents the latitude, ρ is the density of the fluid (which we take to be constant), and P is the pressure of the fluid. For latitudes within 5° of the equator we operate within the framework of the $\beta-$plane approximation [15] of the governing equations (2.1)

$$u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta yv = -\frac{1}{\rho}P_x, \tag{2.2ba}$$

$$v_t + uv_x + vv_y + wv_z + \beta yu = -\frac{1}{\rho}P_y, \tag{2.2bb}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g, \tag{2.2bc}$$

where $\beta = 2\Omega/R = 2.28\cdot 10^{-11}m^{-1}s^{-1}$. The boundary conditions for the fluid on the free-surface η are given by

$$w = \eta_t + u\eta_x + v\eta_y, \tag{2.2c}$$

$$P = P_0 \qquad \text{on } y = \eta(x, y, t), \tag{2.2d}$$

where P_0 is the constant atmospheric pressure. The kinematic boundary condition on the surface simply states that all surface particles remain confined to the surface. For trapped equatorial waves the wave surface profile decays in the latitudinal direction away from the equator, and we assume the water to be infinitely deep. Since we model waves which propagate over a flow with a constant underlying current, the flow converges rapidly with increasing depth to a constant uniform current

$$(u, v) \to (-c_0, 0) \quad \text{as} \quad y \to -\infty. \tag{2.2e}$$

The direction of the current has a bearing on the dynamics of the exact solution that we outline below, and additionally it plays a role in either increasing or decreasing the likelihood of instability of the flow with respect to short wavelength perturbations.

3. Exact solution of (2.2b)

Recently, in [26], an exact solution to the $\beta-$plane equations (2.2b) with a constant underlying current was presented. The solution represents steady waves travelling in the longitudinal direction due east, which have a constant speed of propagation $c > 0$, in the presence of a constant underlying current of strength c_0. The Eulerian coordinates of fluid particles (x, y, z) are expressed as functions of the time t and the Lagrangian labelling variables (q, r, s), for $q \in \mathbb{R}$, $r \in (-\infty, r_0]$ where $r_0 < 0$, and s in an interval which depends on the direction and size of c_0, as follows:

$$x = q - c_0 t - \frac{1}{k} e^{k[r-f(s)]} \sin\left[k(q-ct)\right], \tag{3.1a}$$

$$y = s, \tag{3.1b}$$

$$z = r + \frac{1}{k} e^{k[r-f(s)]} \cos\left[k(q-ct)\right]. \tag{3.1c}$$

Here k is the wavenumber and the function $f(s)$ takes the form

$$f(s) = \frac{c\beta}{2\gamma} s^2,$$

where

$$\gamma = 2\Omega c_0 + g > 0$$

for all physically plausible values of the current c_0. The function $f(s)$ ensures a latitudinal decay in the fluid motion away from the equator when $c > 0$. Defining

$$\chi = k\left(r - f(s)\right), \ \theta = k(q - ct),$$

the determinant of the Jacobian of the transformation (3.1) is $1 - e^{2\chi}$, which is time independent. Thus it follows that the flow defined by (3.1) must be volume preserving, ensuring that (2.2a) holds in the Eulerian setting [2]. In order for the transformation (3.1) to be well-defined, and to ensure that our flow has the appropriate decay properties (in both the vertical and the latitudinal directions), it is necessary that

$$r - f(s) \leq r_0 < 0.$$

We note that this relation implies that $c > 0$ for our flow, and so the waves must propagate in an easterly direction. As we are modelling trapped equatorial waves, we take $v \equiv 0$ throughout the fluid, and we

can express (2.2b) as

$$\frac{Du}{Dt} + 2\Omega w = -\frac{1}{\rho}P_x, \tag{3.2a}$$

$$\frac{Dv}{Dt} + \beta y u = -\frac{1}{\rho}P_y, \tag{3.2b}$$

$$\frac{Dw}{Dt} - 2\Omega u = -\frac{1}{\rho}P_z - g, \tag{3.2c}$$

where D/Dt is the material derivative. The velocity gradient tensor

$$\nabla\mathbf{U} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} = \frac{cke^{\chi}}{1-e^{2\chi}} \begin{pmatrix} -\sin\theta & 0 & \cos\theta + e^{\chi} \\ f_s(e^{\chi} - \cos\theta) & 0 & -f_s \sin\theta \\ -e^{\chi} + \cos\theta & 0 & \sin\theta \end{pmatrix}, \tag{3.3}$$

and so the vorticity is given by $\omega = (w_y - v_z, u_z - w_x, v_x - u_y)$

$$= \left(-s\frac{kc^2\beta}{g}\frac{e^{\chi}\sin\theta}{1-e^{2\chi}}, -\frac{2kce^{2\chi}}{1-e^{2\chi}}, s\frac{kc^2\beta}{g}\frac{e^{\chi}\cos\theta - e^{2\chi}}{1-e^{2\chi}} \right).$$

The motion prescribed by (3.1) satisfies (3.2) when the pressure function is

$$P = \rho\gamma\left(\frac{e^{2\chi}}{2k} - r + \frac{c_0}{c}f(s)\right) + P_0 - \rho g\left(\frac{e^{2kr_0}}{2k} - r_0\right),$$

and when the dispersion relation holds:

$$c = \frac{\sqrt{\Omega^2 + k(2\Omega c_0 + g)} - \Omega}{k}, \quad c_0 \neq c. \tag{3.4}$$

The flow determined by (3.1) satisfies the governing equations (2.2b), and the free-surface $z = \eta(x - ct, y)$ is defined parametrically at fixed latitudes $y = s$ by setting $r = r(s) \leq r_0$, where $r(s)$ is the unique solution of the equation

$$\frac{e^{2k[r - \frac{c\beta}{2\gamma}s^2]}}{2k} - r + \frac{c_0\beta}{2\gamma}s^2 - \frac{e^{2kr_0}}{2k} + r_0 = 0.$$

When $c_0 \leq 0$, which corresponds to a following current, the above equation has a unique solution for all $s \in \mathbb{R}$, and the solution (3.1) defines a travelling trapped equatorial wave. For $c_0 > 0$, an adverse current, the solution exists only for s in a restricted region, cf. [26]. The surface wave prescribed by (3.1), for fixed values of s and t, is an upside down trochoid, cf. [6, 7, 25]. Also, the steepness of the wave-profile, defined to be half the amplitude multiplied by the wavenumber, is given by

$$\tau(s) = e^{\chi},$$

which is maximum $\tau_0 = e^{kr_0}$ at the equator. In the absence of a current, or in a reference frame moving with uniform speed c_0, we can see from (3.1) that the particle trajectories for the underlying flow are closed circles in a fixed latitudinal plane. The effect of the current in (3.1) is to transport the particles in the zonal direction with uniform speed c_0. The existence of closed particle paths is typical of Gerstner-type waves, and it is a phenomenon which does not apply to most irrotational water waves. In the setting of both finite [5,11] and infinite [14] depth Stokes waves, and for linear irrotational waves [28], the particle trajectories are in fact not closed.

4. Instability analysis

The main result of this paper may be stated as follows:

Proposition 4.1. *The equatorial waves propagating eastward over a constant underlying current, as prescribed by (3.1), are unstable to short wavelength perturbations if the steepness of the wave*

$$e^{kr_0} > \frac{3\Omega + \sqrt{\Omega^2 + k(2\Omega c_0 + g)}}{\Omega + 3\sqrt{\Omega^2 + k(2\Omega c_0 + g)}} \gtrapprox \frac{1}{3}. \tag{4.1}$$

The proof of Proposition 4.1 will be presented in the remainder of this section. We employ the short-wavelength instability method, which examines the evolution of a localised and rapidly-varying infinitesimal perturbation represented at time t by the wave packet

$$\mathbf{u}(\mathbf{X}, t) = \varepsilon \mathbf{b}(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, t) e^{i\Phi(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, t)/\delta}. \tag{4.2}$$

Here $\mathbf{X} = (x, y, z)$, Φ is a scalar function, and at $t = 0$ we have

$$\Phi(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, 0) = \mathbf{X} \cdot \xi_{\mathbf{0}}, \quad \mathbf{b}(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, 0) = \mathbf{b_0}(\mathbf{X}, \xi_{\mathbf{0}}).$$

The normalised wave vector $\xi_{\mathbf{0}}$ is subject to the transversality condition $\xi_{\mathbf{0}} \cdot \mathbf{b_0} = 0$, and $\mathbf{b_0}$ is the normalised amplitude of the short-wavelength perturbation of the flow which has the velocity field $\mathbf{U}(\mathbf{X}) \equiv (u\, v\, w)^T(x, y, z)$. Then the evolution in time of $\mathbf{X}$, of the perturbation amplitude $\mathbf{b}$, and of the wave vector $\xi = \nabla\Phi$, is governed at the leading order in the small parameters ϵ and δ by the system of ODEs

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{U}(\mathbf{X}, t), \\ \dot{\xi} = -(\nabla \mathbf{U})^T \xi, \\ \dot{\mathbf{b}} = -L\mathbf{b} - (\nabla \mathbf{U})\mathbf{b} + ([L\mathbf{b} + 2(\nabla \mathbf{U})\mathbf{b}] \cdot \xi) \dfrac{\xi}{|\xi|^2}, \end{cases} \tag{4.3}$$

with initial conditions

$$\mathbf{X}(0) = \mathbf{X_0}, \ \xi(0) = \xi_{\mathbf{0}}, \ \mathbf{b}(0) = \mathbf{b_0}.$$

Here $(\nabla \mathbf{U})^T$ is the transpose of the velocity gradient tensor (3.3) and, for the system defined by (3.1), $L = L(\mathbf{X})$ is given by

$$L = \begin{pmatrix} 0 & -\beta y & 2\Omega \\ \beta y & 0 & 0 \\ -2\Omega & 0 & 0 \end{pmatrix},$$

cf. [13] for details.

The system of ODEs (4.3) describing the evolution of a rapidly-varying perturbation was independently derived in [1, 19, 32]. The instability criterion for Lagrangian flows for which $\mathbf{X}(0) = \mathbf{X_0}$ is determined by the exponent

$$\Lambda(\mathbf{X_0}) = \limsup_{t\to\infty} \frac{1}{t} \ln \left(\sup_{|\xi_0|=|\mathbf{b_0}|=1, \xi_0 \cdot \mathbf{b_0}=0} \{|\mathbf{b}(\mathbf{X}, \xi_0, \mathbf{b_0}, t)|\} \right). \tag{4.4}$$

If

$$\Lambda(\mathbf{X_0}) > 0 \tag{4.5}$$

for a given fluid trajectory then particles become separated at an exponential rate, and accordingly the flow is unstable [20]. Therefore, establishing (4.5) provides us with a criterion for instability of a flow. For certain solutions [13, 31] which have an explicit Lagrangian formulation, it transpires that the short wavelength instability analysis is remarkably elegant, and the criterion for instability (4.5) takes on a tangible and explicit formulation in terms of the wave steepness. We now derive this instability criterion for the flow determined by (3.1).

We choose the latitudinal wave vector $\xi_0 = (0\ 1\ 0)^T$, and from (3.3) we then have $\xi(t) = (0\ 1\ 0)^T$ for all $t \geq 0$. It follows that the evolution of $\mathbf{b} = (b_1, b_2, b_3)$ is governed by

$$\begin{cases} \dot{b_1} = \beta s b_2 - 2\Omega b_3 + \dfrac{kce^\chi \sin\theta}{1-e^{2\chi}} b_1 - \dfrac{kc^2 \beta s e^\chi (e^\chi - \cos\theta)}{(2\Omega c_0 + g)(1-e^{2\chi})} b_2 + \dfrac{kce^\chi (e^\chi - \cos\theta)}{1-e^{2\chi}} b_3, \\ \dot{b_2} = 0, \\ \dot{b_3} = 2\Omega b_1 - \dfrac{kce^\chi (e^\chi + \cos\theta)}{1-e^{2\chi}} b_1 + \dfrac{kc^2 \beta s e^\chi \sin\theta}{(2\Omega c_0 + g)(1-e^{2\chi})} b_2 - \dfrac{kce^\chi \sin\theta}{1-e^{2\chi}} b_3. \end{cases} \tag{4.6}$$

Noting that the choice $b_2(0) = 0$ implies $b_2(t) = 0$ for all $t \geq 0$, and accordingly $\xi(t) \cdot \mathbf{b}(t) = 0$, the system (4.6) reduces to

$$\dot{B} = \begin{pmatrix} \frac{kce^\chi \sin\theta}{1-e^{2\chi}} & -2\Omega + \frac{kce^\chi (e^\chi - \cos\theta)}{1-e^{2\chi}} \\ 2\Omega - \frac{kce^\chi (e^\chi + \cos\theta)}{1-e^{2\chi}} & -\frac{kce^\chi \sin\theta}{1-e^{2\chi}} \end{pmatrix} B, \tag{4.7}$$

where $B = \begin{pmatrix} b_1 \\ b_3 \end{pmatrix}$. This system is nonautonomous, however the change of variables induced by the matrix

$$P = \begin{pmatrix} \cos(kct/2) & \sin(kct/2) \\ -\sin(kct/2) & \cos(kct/2) \end{pmatrix}$$

transforms the planar system (4.7) to an autonomous system for $Q = P^{-1}B$,

$$\frac{d}{dt}Q(t) = DQ(t),$$

where

$$D = \begin{pmatrix} \frac{kce^{\chi}}{1-e^{2\chi}}\sin(kq) & -\frac{kce^{\chi}}{1-e^{2\chi}}\cos(kq) - 2\Omega + \frac{kce^{\chi}}{1-e^{2\chi}} - \frac{kc}{2} \\ -\frac{kce^{\chi}}{1-e^{2\chi}}\cos(kq) + 2\Omega - \frac{kce^{\chi}}{1-e^{2\chi}} + \frac{kc}{2} & -\frac{kce^{\chi}}{1-e^{2\chi}}\sin(kq) \end{pmatrix}.$$

Since $B = PQ$, and P is periodic with time, we deduce that the short-wavelength rapidly-varying perturbation $\mathbf{u}$, defined in (4.2), grows exponentially with time if D has a positive eigenvalue, which occurs if

$$e^{\chi} > \frac{4\Omega + kc}{4\Omega + 3kc}.$$

Using the dispersion relation (3.4), this gives us condition (4.1). Therefore, if the steepness of the wave is sufficiently large at the equator, the localised small perturbation (4.2) of grows at an exponential rate and the flow is consequently unstable. We note that since $\Omega \ll 1$ the left-hand side of (4.1) is of the form

$$\frac{3\tilde{\epsilon} + 1}{\tilde{\epsilon} + 3} \gtrapprox \frac{1}{3},$$

where

$$\tilde{\epsilon} = \frac{\Omega}{\sqrt{\Omega^2 + k(2\Omega c_0 + g)}} \ll 1.$$

In particular, we deduce that an adverse current with $c_0 > 0$ favours instability in the sense that the threshold on the steepness for the wave to be unstable is decreased compared to the case without current. Conversely, this threshold is increased by a following current with $c_0 < 0$.

Acknowledgment. The authors would like to acknowledge the support of the ERC Advanced Grant “Nonlinear studies of water flows with vorticity”.

References

[1] B. J. Bayly, Three-dimensional instabilities in quasi-two dimensional inviscid flows, in *Nonlinear Wave Interactions in Fluids*, edited by R. W. Miksad et al., pp. 71–77, ASME, New York, 1987.

[2] A. Bennett, *Lagrangian fluid dynamics*, Cambridge University Press, Cambridge, 2006.

[3] A. Constantin, On the deep water wave motion, *J. Phys. A* 34 (2001), 1405–1417.

[4] A. Constantin, Edge waves along a sloping beach, *J. Phys. A* 34 (2001), 9723–9731.

[5] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.* 166 (2006), 523–535.

[6] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Conference Series in Applied Mathematics, Vol. 81, SIAM, Philadelphia, 2011.

[7] A. Constantin, An exact solution for equatorially trapped waves, *J. Geophys. Res. Oceans* 117 (2012), C05029.

[8] A. Constantin, On the modelling of Equatorial waves, *Geophys. Res. Lett.*, 39 L05602 (2012).

[9] A. Constantin, Some three-dimensional nonlinear Equatorial flows, *J. Phys. Oceanogr.* 43 (2013), 165–175.

[10] A. Constantin, On equatorial wind waves, *Differential and Integral equations*, 26 (2013), 237–252.

[11] A. Constantin, Some nonlinear, equatorially trapped, nonhydrostatic internal geophysical waves, *J. Phys. Oceanogr.* 44 (2014), 781–789.

[12] A. Constantin and J. Escher, Symmetry of steady deep-water waves with vorticity, *Eur. J. Appl. Math.* 15 (2004), 755–768.

[13] A. Constantin and P. Germain, Instability of some equatorially trapped waves, *J. Geophys. Res. Oceans* 118 (2013) 2802–2810.

[14] A. Constantin and W. Strauss, Pressure beneath a Stokes wave, *Comm. Pure Appl. Math.* 63 (2010), 533–557.

[15] B. Cushman-Roisin and J.-M. Beckers, *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects*, Academic, Waltham, Mass., 2011.

[16] P. G. Drazin, *Introduction to hydrodynamic stability*, Cambridge University Press, Cambridge, 2002.

[17] P. G. Drazin and W. H. Reid, *Hydrodynamic stability*, Cambridge University Press, Cambridge, 2004.

[18] A. V. Fedorov and J. N. Brown, Equatorial waves, in *Encyclopedia of Ocean Sciences*, edited by J. Steele, pp. 3679–3695, Academic, San Diego, Calif., 2009.

[19] S. Friedlander and M. M. Vishik, Instability criteria for the flow of an inviscid incompressible fluid, *Phys. Rev. Lett.* 66 (1991), 2204–2206.

[20] S Friedlander and V. Yudovich, Instabilities in fluid motion, *Not. Am. Math. Soc.* 46 (1999), 1358–1367.
[21] S. Friedlander, Lectures on stability and instability of an ideal fluid, in Hyperbolic Equations and Frequency Interactions, 227–304, IAS/Park City Math. Ser. 5, Amer. Math. Soc., Providence, RI, 1999.
[22] I. Gallagher and L. Saint-Raymond, On the influence of the Earth's rotation on geophysical flows, in *Handbook of Mathematical Fluid Mechanics*, vol. 4, edited by S. Friedlander and D. Serre, pp. 201–329, North-Holland, Amsterdam, 2007.
[23] F. Gerstner, Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile, *Ann. Phys.* 2 (1809), 412–445.
[24] D. Henry, On the deep-water Stokes flow, *Int. Math. Res. Not.* (2008), 7 pp..
[25] D. Henry, On Gerstner's water wave, *J. Nonl. Math. Phys.* 15 (2008), 87–95.
[26] D. Henry, An exact solution for equatorial geophysical water waves with an underlying current, *Eur. J. Mech. B Fluids* 38 (2013), 18–21.
[27] D. Henry and A. Matioc, On the existence of equatorial wind waves, *Nonlinear Anal.* 101 (2014), 113–123.
[28] D. Ionescu-Kruse, Particle trajectories in linearized irrotational shallow water flows, *J. Nonlinear Math. Phys.* 15 (2008), 13–27.
[29] D. Ionescu-Kruse, Instability of edge waves along a sloping beach, *J. Diff. Eq.*, DOI: 10.1016/j.jde.2014.03.009.
[30] T. Izumo, The equatorial current, meridional overturning circulation, and their roles in mass and heat exchanges during the El Niño events in the tropical Pacific Ocean, *Ocean Dyn.*, 55 (2005), 110–123.
[31] S. Leblanc, Local stability of Gerstner's waves, *J. Fluid Mech.* 506 (2004), 245–254.
[32] A. Lifschitz and E. Hameiri, Local stability conditions in fluid dynamics, *Phys. Fluids* 3 (1991), 2644–2651.
[33] A. V. Matioc, An exact solution for geophysical equatorial edge waves over a sloping beach, *J. Phys. A* 45 365501 (2012).
[34] E. Mollo-Christensen, Gravitational and Geostrophic Billows: Some Exact Solutions, *J. Atmos. Sci.* 35 (1978), 1395–1398.
[35] R. Stuhlmeier, On edge waves in stratified water along a sloping beach, *J. Nonlinear Math. Phys.* 18 (2011), 127–137.

Faculty of Mathematics, University of Vienna, Vienna, Austria
Email address: francois.genoud@univie.ac.at

School of Mathematical Sciences, University College Cork, Cork, Ireland, Faculty of Mathematics, University of Vienna, Vienna, Austria.
Email address: d.henry@ucc.ie