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General relativistic versus Newtonian: A universality in spherically symmetric radiation hydrodynamics for quasistatic transonic accretion flows

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We compare Newtonian and general relativistic descriptions of the stationary accretion of self-gravitating fluids onto compact bodies. Spherical symmetry and thin gas approximation are assumed. Luminosity depends, among other factors, on the temperature and the contribution of gas to the total mass, in both—general relativistic (L_{GR}) and Newtonian (L_N)—models. We discover a remarkable universal behavior for transonic flows: the ratio of respective luminosities L_{GR}/L_N is independent of the fractional mass of the gas and depends on asymptotic temperature. It is close to 1 in the regime of low asymptotic temperatures and can grow several times at high temperatures. These conclusions are valid for a wide range of polytropic equations of state.

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I. INTRODUCTION

The classical accretion model of Bondi [1] has been generalized to radiation hydrodynamics in the 1970s. Shakura [2] discussed steady radiating flows attracted by a pointlike Newtonian mass. Thorne and his coworkers formulated models of stationary accretion in the Schwarzschild spacetime [3,4]. The self-gravity of infalling radiating gases has been included into the general relativistic model by Park and Miller [5] and Rezzolla and Miller [6] in the 1990s. The steady accretion of Newtonian self-gravitating flows has been recently investigated [7,8].

In this paper we compare the general relativistic and Newtonian models for transonic flows. We make a number of simplifying assumptions—spherical symmetry, a polytropic equation of state $p = K\rho_0^\Gamma$, and the thin gas approximation in the transport equation [9]. It is assumed that the accretion is quasistationary.

To begin with, there are two kinds of possible general relativistic effects. The first is related to the backreaction—and that includes self-gravity and the dependence on the fraction of total mass in the accreting gas (defined later as $1 - y$)—and the other is related to the asymptotic speed of sound a_∞. (We shall occasionally use the term "asymptotic temperature"; both quantities are related, for ideal polytropic gases $a^2 = kT/\Gamma$. Here k is the Boltzmann constant.) The same boundary conditions are assumed in both models.

It is already known that the Newtonian description is not adequate for the accretion of hot transonic test fluids [10,11] in pure hydrodynamics. Thus it is not surprising to discover—we do—that the Newtonian description can fail in radiation hydrodynamics. But it is surprising that there emerges a particular universality that is unknown in the existing literature. To rephrase it, let us point out that each of the two luminosities, L_{GR} (general relativistic) and L_N (Newtonian), taken separately depends on the gas mass fraction $1 - y$ and the asymptotic temperature T_∞. Yet we find that their ratio L_{GR}/L_N depends only on the asymptotic temperature. This universality means that the general relativistic enhancement of luminosity, the ratio L_{GR}/L_N, can be found by solving the accretion of test fluids in hydrodynamics without radiation. This is a much simpler algebraic problem than the original one, investigated in [10,11]. These facts are valid for polytropic equations of state with $1 < \Gamma < 5/3$.

The order of the rest of this paper is as follows. Section II discusses notation and equations. Section III shows how the validity of the thin gas approximation constrains the choice of boundary data. We describe the form of boundary data for the accretion problem. Section IV discusses the notion of sonic points and transonic flows in radiation hydrodynamics. In the first part of Sec. V we formulate a low-radiation condition and prove the universality of L_{GR}/L_N. The luminosity of hot accreting gases in a general relativistic model can be much higher than that given by the related Newtonian counterpart. In the second part of Sec. V we find a sector of luminous transonic flows which again gives a universal ratio, but now the asymptotic temperature is low and $L_{\mathrm{GR}}/L_N \approx 1$. Section VI describes the numerical scheme that is used in this work, gives the equation of state, and details numerical values for most of the boundary data. Section VII reviews numerical results. The last section summarizes the main conclusions.

II. FORMALISM AND EQUATIONS

A. General relativistic accretion

The metric

$$ds^2 = -N^2dt^2 + \hat{a}dr^2 + R^2(d\theta^2 + \sin^2(\theta)d\phi^2) \quad (1)$$

uses comoving coordinates t, r, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$: time, coordinate radius, and two angle variables, respectively. R denotes the areal radius and N is the lapse. The radial velocity of gas is given by $U = \frac{1}{N}\frac{dR}{dt}$. We assume relativistic units with $G = c = 1$.

The energy-momentum tensor reads $T_{\mu\nu} = T^B_{\mu\nu} + T^E_{\mu\nu}$, where the baryonic part is given by $T^B_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$ with the timelike normalized four-velocity U_μ, $U_\mu U^\mu = -1$. The radiation part $T^E_{\mu\nu}$ possesses only four nonzero components, $T^{0E}_0 \equiv -\rho^E = -T^{rE}_r$ and $T^E_{r0} = T^E_{0r}$. A comoving observer would measure local mass densities, the material density $\rho = T^{B\mu\nu}U_\mu U_\nu$, and the radiation density ρ^E, respectively. The baryonic current reads $j^\mu \equiv \rho_0 U^\mu$, where ρ_0 is the baryonic mass density. Its conservation is expressed by the equation

$$\nabla_\mu j^\mu = 0. \tag{2}$$

Let n_μ be the unit normal to a coordinate sphere lying in the hypersurface $t = \text{const}$ and let k be the related mean curvature scalar, $k = \frac{R}{2}\nabla_i n^i = (1/\sqrt{\hat{a}})\partial_r R$. The quantity $j = U_\mu n^\nu N T^{\mu E}_\nu/\sqrt{\hat{a}} = N T^{0E}_r/\sqrt{\hat{a}}$ is interpreted as the comoving radiation flux density. We assume the polytropic equation of state $p = K\rho_0^\Gamma$, with constants K and Γ. The internal energy density h and the rest and baryonic mass densities are related by $\rho = \rho_0 + h$, where $h = p/(\Gamma - 1)$.

There are four conservation equations that originate from the contracted Bianchi identities, $\nabla_\mu T^{\mu B}_\nu = -\nabla_\mu T^{\mu E}_\nu = F_\nu$ (here $\nu = 0,\ r$). The radiation force density F_ν describes the interaction between baryons and radiation. This formulation of general relativistic radiation hydrodynamics agrees with that of Park and Miller [5], Rezzolla and Miller [6], and (on the fixed, Schwarzschildean, background) Thorne, Flammang, and Żytkow [4].

One can find the mean curvature k from the Einstein constraint equations $G_{\mu 0} = 8\pi T_{\mu 0}$ ([10,12]),

$$k = \sqrt{1 - \frac{2m(R)}{R} + U^2}, \tag{3}$$

where $m(R)$ is the quasilocal mass,

$$m(R) = M - 4\pi\int_R^{R_\infty} dr r^2\left(\rho + \rho^E + \frac{Uj}{k}\right). \tag{4}$$

The integration in (4) extends from R to the outer boundary R_∞ of the ball of gas. Its external boundary is connected to the Schwarzschild vacuum spacetime by a transient zone of a negligible mass. Thus the asymptotic mass M of the Schwarzschild spacetime is approximately equal to $m(R_\infty)$.

In the polar gauge foliation one has a new time $t_S(t, r)$ with $\partial_{t_S} = \partial_t - NU\partial_R$. The quantity $4\pi NkR^2(j(1 + (\frac{U}{k})^2) + 2U\rho^E/k)$ is the radiation flux measured by an observer located at R in coordinates (t_S, R). One can show that

$$\begin{aligned}\partial_{t_S} m(R) &= (\partial_t - NU\partial_R)m(R)\\ &= 4\pi\left(NkR^2 j\left(1 + \left(\frac{U}{k}\right)^2\right) + 2N\rho_E U\right)_R^{R_\infty}\\ &\quad + 4\pi(NUR^2(\rho + p))_R^{R_\infty} + A_\infty,\end{aligned} \tag{5}$$

where A_∞ is the value of $-4\pi NUR^2(\rho + \rho_E + \frac{Uj}{k})$ at $R = R_\infty$. The mass contained in the annulus (R, R_∞) changes if the fluxes on the right-hand side, one directed outward and the other inward, do not cancel. The local baryonic flux reads $\dot{M} = -4\pi UR^2\rho_0$, it is not constant $(\partial_R\dot{M} \neq 0)$, and its boundary value reads $\dot{M}_\infty$.

The accretion process is said to be stationary (or quasistationary) if all relevant physically observables, which are measured at a fixed areal radius R, remain approximately constant during time intervals much smaller than the runaway instability time scale $T = M/\dot{M}_\infty$. That means that $\partial_{t_S} X \equiv (\partial_t - NU\partial_R)X = 0$ for $X = \rho_0, \rho, j, U \ldots$.

The above assumptions imply that in the thin gas approximation $F_0 = 0$ and the radiation force density has only one nonzero component $F_r = \kappa k\sqrt{\hat{a}}\rho_0 j$ [8]. Baryons and radiation interact through the elastic Thomson scattering. κ is a material constant, in standard units $\kappa = \sigma/(m_p c)$ and c, σ, and m_p are, respectively, the speed of light, the Thomson cross section, and the proton mass.

The full system of equations in a form suitable for numerics has been obtained in [8]. It consists of

(i) The total energy conservation

$$\begin{aligned}&\dot{M}N\frac{\Gamma - 1}{\Gamma - 1 - a^2} + 2\dot{M}N\frac{\rho^E}{\rho_0}\\ &\quad = 4\pi R^2 jNk\left(1 + \frac{U^2}{k^2}\right) + C;\end{aligned} \tag{6}$$

the constant C is the asymptotic energy flux inflowing through the sphere of a radius R_∞.

(ii) The local radiation energy conservation

$$\begin{aligned}&\left(1 - \frac{2m(R)}{R}\right)\frac{N}{R^2}\frac{d}{dR}(R^2\rho^E)\\ &\quad = -\kappa k^2 Nj\rho_0 + 2N(U\rho^E - kj)\frac{dU}{dR}\\ &\qquad + 2k(jU - k\rho^E)\frac{dN}{dR} + 8\pi NR\left(j^2 - j\rho^E\frac{U}{k}\right).\end{aligned} \tag{7}$$

(iii) The equation related to the relativistic Euler equation (below $a = \sqrt{dp/d\rho}$ is the speed of sound)

$$\frac{d}{dR}\ln a^2 = -\frac{\Gamma - 1 - a^2}{a^2 - \frac{U^2}{k^2}}\left[\frac{1}{k^2 R}\left(\frac{m(R)}{R} - 2U^2 + 4\pi R^2\left(\rho_E + p + j\frac{U}{k}\right)\right) - \kappa j\left(1 - \frac{a^2}{\Gamma - 1}\right)\right]. \tag{8}$$

(iv) The baryonic mass conservation

$$\frac{dU}{dR} = -\frac{U}{\Gamma - 1 - a^2}\frac{d}{dR}\ln a^2 - \frac{2U}{R} + \frac{4\pi R j}{k}. \tag{9}$$

(v) The equation for the lapse

$$\frac{dN}{dR} = N\left(\kappa j\frac{\Gamma - 1 - a^2}{\Gamma - 1} + \frac{d}{dR}\ln(\Gamma - 1 - a^2)\right). \tag{10}$$

Equations (3), (4), and (6)–(10) give the complete model used in numerical calculations.

The asymptotic data for the accretion must satisfy several physical conditions. We assume the inequalities $a_\infty^2 \gg M/R_\infty \gg U_\infty^2$ ensuring, as demonstrated by Karkowski, Malec, and Roszkowski [7] and Mach *et al.* [13], that the assumption of stationary accretion is reasonably well satisfied. These inequalities are probably needed to ensure stability (see a discussion in [7] and studies of stability of accreting flows in Newtonian hydrodynamics [13]). In the asymptotic region $j_\infty \approx \rho_\infty^E$ and the total luminosity is well approximated by $L_0 = 4\pi R_\infty^2 j_\infty$. The total luminosity is related to the asymptotic accretion rate $\dot{M}_\infty$ by [8]

$$L_0 = \alpha\dot{M}_\infty \equiv \left(1 - \frac{N(R_0)}{k(R_0)}\sqrt{1 - \frac{2m(R_0)}{R_0}}\right)\dot{M}_\infty. \tag{11}$$

Here R_0 is the size of the compact core and the quantity α can be interpreted as a binding energy per unit mass.

B. Newtonian approximation

The notation is as in the preceding part of this section. The mass accretion flux is now R-independent, in contrast to the general relativistic case, $\partial_R \dot{M} = 0$, and the baryonic mass density ρ_0 coincides with ρ. $\phi(R)$ is the Newtonian gravitational potential,

$$\phi(R) = -\frac{M(R)}{R} - 4\pi\int_R^{R_\infty} r\rho(r)dr; \tag{12}$$

$M(R) \equiv M - 4\pi\int_R^{R_\infty} r^2\rho(r)dr$ is the mass contained within the sphere R.

The Newtonian model can be described by two basic equations [7]:

(i) The energy conservation equation

$$L_0 - L(R) = \dot{M}\left(\frac{a_\infty^2}{\Gamma - 1} + \frac{U_\infty^2}{2} + \phi(\infty) - \frac{a^2}{\Gamma - 1} - \frac{U^2}{2} - \phi(R)\right). \tag{13}$$

(ii) The luminosity equation

$$L = L_0 \exp\left(\frac{-\kappa\dot{M}}{4\pi R}\right) = L_0\exp\left(\frac{-L_0\tilde{R}_0}{L_E R}\right). \tag{14}$$

Notice that the luminosity has the same form as in the case of test fluids [2]. Here we introduced the Eddington luminosity $L_E = 4\pi M/\kappa$ while $\tilde{R}_0 \equiv GM/|\phi(R_0)|$ is a kind of modified size measure of the compact body. In the case of test fluids $\tilde{R}_0 = R_0$. We assume $L_0 = |\phi(R_0)|\dot{M}$; notice, however, that for small α this relation [with $\alpha = |\phi(R_0)|$] appears as the Newtonian limit of Eq. (11).

We shall give a sketch of the proof that Eqs. (13) and (14) constitute a limit of the general relativistic model, assuming the thin gas approximation. Let us assume obvious nonrelativistic conditions (we return to our convention $G = c = 1$) that (i) velocities U^2, $a^2 \ll 1$ and (ii) mass concentrations $2M(R)/R \ll 1$ (for any R bigger than the size R_0 of the compact core) are small. We shall require also that most of the mass rests in the core, (iii) $M_g < M/2$. Here $M_g \equiv 4\pi\int_R^{R_\infty} drr^2\frac{\rho_0}{k}$ represents the mass M_g of gas in the annulus (R, R_∞). This assumption is made only for the sake of simplicity; we believe that it can be eliminated. Finally, we need that (iv) the comoving radiation energy density ρ^E and the comoving radiation flux density j are of the same order of magnitude. We shall comment on that. We already assumed that at the boundary R_∞ both quantities are equal. It follows by the volume integration of both sides of Eq. (7) that, under conditions (i)–(iii),

$$(R^2\rho^E)|_R^{R_\infty} \approx -\kappa\int_R^{R_\infty} drr^2\rho_0 j. \tag{15}$$

The right-hand side of (15) can estimated from above by $\tau\sup(|j|R^2)$ and from below by $\tau\inf(|j|R^2)$. Here τ is the optical thickness (see the next section for the definition). If $\tau \ll 1$, then in fact $\rho^E \approx j$.[1] It is clear that (iv) is not generically valid if the gas is opaque, $\tau \gg 1$.

[1]It might well be that by a careful and tedious calculation one can eliminate the condition (iv), assuming a weaker condition that $\tau \ll 1$ (or even $\tau < 1$).

Define an auxiliary quantity

$$\hat{L} \equiv -2\dot{M}N\frac{\rho^E}{\rho_0} + 4\pi R^2 jNk\left(1 + \frac{U^2}{k^2}\right). \qquad (16)$$

$\hat{L}$ represents local luminosity as measured by an observer stationary at R [8]. We already have shown that conditions (i)–(iv) imply $\rho^E \approx j$. In the nonrelativistic regime $|U| \ll 1$ and careful investigation of (16) leads to the equality $\hat{L} \approx 4\pi R^2 jN$. It follows from Eq. (6) that at the boundary of the accretion cloud $\hat{L}(R_\infty) = L_{\mathrm{GR}}$; $\hat{L}(R_\infty)$ is the total luminosity. It is convenient to replace $\dot{M}$ by $\dot{M} \equiv \dot{M}_\infty + 16\pi^2 \int_R^{R_\infty} drjr^3 \frac{\rho_0}{k}$ [8]; the new quantity $\dot{M}_\infty$ is constant and coincides with $\dot{M}$ at the boundary R_∞. We have $16\pi^2 \int_R^{R_\infty} drjr^3 \frac{\rho_0}{k} \leq 16\pi^2 \sup(jr^2) \int_R^{R_\infty} drr\frac{\rho_0}{k} \leq 4\pi \sup(jr^2)\frac{4\pi}{R} \int_R^{R_\infty} drr^2 \frac{\rho_0}{k}$. We conclude that

$$16\pi^2 \int_R^{R_\infty} drjr^3 \frac{\rho_0}{k} \ll 4\pi \sup(jr^2), \qquad (17)$$

since we assumed $M_g < M/2$ and in the Newtonian limit holds $2M/R \ll 1$. The term $4\pi \sup(jr^2)$ is equal to $\sup\hat{L} = \hat{L}(R_\infty)$; this in turn is equal to $\alpha M_\infty \ll M_\infty$. The last inequality again exploits the fact that in the Newtonian limit $\alpha = |\phi(R_0)| \ll 1$. Thus we can conclude that $\dot{M} \approx \dot{M}_\infty$ and the energy conservation equation becomes

$$\dot{M}_\infty N \frac{\Gamma - 1}{\Gamma - 1 - a^2} = \hat{L} + C. \qquad (18)$$

By differentiating Eq. (6) and employing Eq. (10) one can easily derive the following equation:

$$\frac{d}{dR}\hat{L} = \dot{M}_\infty N\kappa j. \qquad (19)$$

Now, the assumptions (i)–(iv) imply that the metric functions

$$k \approx 1 - \frac{M(R)}{R} + \frac{U^2}{2} \qquad (20)$$

and the lapse

$$N \approx 1 + \phi(R) + \frac{U^2}{2} \qquad (21)$$

are close to unity.

Then the function $\hat{L}$ satisfies with good accuracy the differential equation

$$\frac{d}{dR}\hat{L} \approx \dot{M}_\infty \kappa \frac{\hat{L}}{4\pi R^2}, \qquad (22)$$

which is solved by $\hat{L} = L_0 \exp(-\kappa \frac{\dot{M}_\infty}{4\pi R})$. Thus we obtain the same form of a solution as in the Newtonian equation (14). Furthermore, one can approximate Eq. (18) by a suitable Newtonian model in the region (R_*, R_∞). Expanding the lapse N [keeping only the first order terms in $M(R)/R$ and U^2, as in (21)] and replacing $\frac{\Gamma-1}{\Gamma-1-a^2}$ by $1 + \frac{a^2}{\Gamma-1}$, we arrive at the Newtonian equation (13).

III. THIN GAS APPROXIMATION AND BOUNDARY DATA

The thin gas approximation demands that the optical thickness [9] of the cloud is smaller than 1, i.e.,

$$\tau = \int_{R_0}^{R_\infty} n(r)\sigma dr < 1, \qquad (23)$$

where n is the baryonic number density. Notice that $n = \frac{\rho_0}{m_p}$ if we assume the monoatomic hydrogenic gas. Assuming that n decreases to the asymptotic value n_∞, we arrive at $1 > R_\infty n_\infty \sigma$. Thus the rough condition for the validity of the thin gas approximation is that the radiation free path $l \equiv 1/(n_\infty \sigma)$ is not shorter than the size of the cloud, $l > R_\infty$. This implies $\rho_0 < \frac{m_p}{R_\infty \sigma}$ and (taking into account that $\rho_0 \approx \rho$) estimates the mass of gas, $M_g < \frac{4\pi}{3\kappa c} R_\infty^2$. Denote the solar mass by $M_\odot$ and define $10^s \equiv \frac{R_\infty}{M}$. One obtains an estimate consistent with the thin gas approximation

$$\frac{M_g}{M} < 10^{-21} \times 10^{2s} \times \frac{M}{M_\odot}. \qquad (24)$$

We choose s and M that give the right-hand side of (24) of the order of unity. In such a case a significant part of the total mass M would be contributed by the gas itself. That could allow for the strong impact of backreaction and self-gravitation onto accretion. It is clear that there is a scaling freedom—one can trade the size (represented by the exponent s) for the total mass without changing the bound in (24).

The boundary data set is the same for the Newtonian and general relativistic models. Thus we specify in both cases the same values of asymptotic masses M, masses of the core, the binding energy per unit mass $\alpha = |\phi(R_0)|$, the asymptotic speed of sound a_∞, and the size R_∞. The total luminosity L_0 is not a free data, but it results from equations. We assume identical equations of state in the two models.

IV. SONIC POINTS AND LUMINOSITY

We shall study transonic flows. For these flows there exists a radius R_* such that $a_* = |\vec{U}_*|$; the speed of sound is equal to the length of the spatial part of the velocity vector. Henceforth all quantities denoted by asterisk will refer to a sonic point.

It is clear from the inspection of equations that the regularity of solutions demands a particular relation for the fraction m_*/R_*; here m_* is the mass within the sonic sphere. In the Newtonian model the three characteristics, a_{*N}, U_{*N}, and m_{*N}/R_{*N}, are related as below [7]:

$$a_{*N}^2 = U_{*N}^2 = \frac{m_{*N}}{2R_{*N}}\left(1 - \frac{L_{*N}\kappa}{4\pi m_{*N}}\right) = \frac{m_{*N}}{2R_{*N}}\left(1 - \frac{L_{*N}M}{L_E m_{*N}}\right). \tag{25}$$

In the last equation appears the Eddington luminosity L_E. It is clear that the necessary condition for the critical Newtonian flow—that is, possessing a sonic point—reads

$$\frac{L_{*N}M}{L_E m_{*N}} < 1. \tag{26}$$

Define $x \equiv L_0/L_E$ and $y \equiv m_*/M$. Since $L_* \leq L_0$, the inequality $x < y$ becomes the necessary condition for a sonic point. In the general relativistic model, at the sonic point $a^2 = \frac{U^2}{k^2}$; the denominator of the right-hand side of Eq. (8) vanishes and that implies the vanishing of the numerator. One obtains

$$\frac{1}{k^2R}\left(\frac{m}{R} - 2U^2 + 4\pi R^2\left(\rho_E + p + j\frac{U}{k}\right)\right) = \kappa j\left(1 - \frac{a^2}{\Gamma - 1}\right). \tag{27}$$

Let us remark, that in the Newtonian limit $a^2 \ll 1$ and $4\pi R_{*\mathrm{GR}}^2(\rho_{*E} + p_* + j\frac{U_*}{k_*}) \ll \frac{m_{*\mathrm{GR}}}{R_{*\mathrm{GR}}}$. Therefore, in this limit Eq. (27) coincides with Eq. (25). It is obvious that radiation pushes the sonic point inward; if the size of a compact object is bigger than the value of R_* predicted by (26) and (27), then the flow becomes subsonic.

V. UNIVERSALITY IN L_{GR}/L_N

We assume in this section the polytropic equation of state $p = Kn^\Gamma$ with $\Gamma < 5/3 - \epsilon$, for some small $\epsilon > 0$. This restriction is due to the peculiar character of the equation of state corresponding to $\Gamma = 5/3$ The constancy of L_{GR}/L_N is valid for all Γ's, although the specific value of this ratio depends on the equation of state.

A. Low luminosities

The natural reference quantity for radiating systems is the Eddington luminosity L_E. It can be roughly described as the luminosity at which the infall of gas is prevented. Thus one might define weakly radiating systems as radiating with a luminosity L_0 (herein $L_0 = L_{\mathrm{GR}}$ or $L_0 = L_N$) that is much smaller than the Eddington luminosity, $L_0 \ll L_E$. We will adopt a different definition, for reasons that will become clear.

The (XY) *condition.* We will say that an accretion system satisfies the (XY) condition if $x \ll y$.

Notice the trivial fact that $y < 1$. If (XY) holds, that is, $x \ll y$, then obviously $L_0 \ll L_E$. Thus the (XY) assumption is stronger than just the statement $L_0 \ll L_E$. Another interesting fact is that (XY) guarantees that the characteristics of the sonic point are essentially unchanged by the radiation—see Eqs. (25) and (27). The luminosity is the product of α by the (asymptotic) mass accretion rate, and since the mass accretion rate can be formulated completely in terms of the sonic point parameters, it becomes luminosity independent if $x \ll y$. The general relativistic mass accretion rate $\dot{M}_{\mathrm{GR}}$ within the steadily accreting fluid can be expressed as below [see Eq. (6.1) in [10]]:

$$\dot{M}_{\mathrm{GR}} = \pi m_{*\mathrm{GR}}^2 \rho_{\infty\mathrm{GR}} \frac{R_{*\mathrm{GR}}^2}{m_{*\mathrm{GR}}^2}\left(\frac{a_{*\mathrm{GR}}^2}{a_\infty^2}\right)^{(5-3\Gamma)/2(\Gamma-1)} \times \left(1 + \frac{a_{*\mathrm{GR}}^2}{\Gamma}\right)\frac{1 + 3a_{*\mathrm{GR}}^2}{a_\infty^3}. \tag{28}$$

The corresponding Newtonian expression reads

$$\dot{M}_N = \pi m_{*N}^2 \rho_{\infty N} \frac{R_{*N}^2}{m_{*N}^2}\left(\frac{a_{*N}^2}{a_\infty^2}\right)^{(5-3\Gamma)/2(\Gamma-1)} \frac{1}{a_\infty^3}. \tag{29}$$

Equation (29) has been derived by Kinasiewicz in [14], but it follows also from (28) in the limit of small sound speeds, $a_{*\mathrm{GR}} \ll 1$. The way of writing these two expressions is not accidental. It has been shown in [15] that characteristics of the sonic point—$a_{*\mathrm{GR}}^2$ and $R_{*\mathrm{GR}}^2/m_{*\mathrm{GR}}^2$—do not depend on the fraction of mass carried by the gas. These quantities are dictated just by the asymptotic speed of sound a_∞ in a test fluid model. An analogous result holds in the Newtonian model, as shown in [14]. Therefore $L_{\mathrm{GR}}/L_N = \dot{M}_{\mathrm{GR}}/\dot{M}_N$ is equal to the ratio $F \times m_{*\mathrm{GR}}^2\rho_{\infty\mathrm{GR}}/m_{*N}^2\rho_{\infty N}$, where the coefficient F depends on Γ (and thus on the equation of state) and on the sonic point parameters a_{*N}, $a_{*\mathrm{GR}}$, $R_{*\mathrm{GR}}^2/m_{*\mathrm{GR}}^2$, and R_{*N}^2/m_{*N}^2. Therefore the coefficient F is independent of the mass fraction y. Now the masses are approximately equal, $m_{*\mathrm{GR}} \approx m_{*N}$; this is because the masses within the sonic point are well approximated by the masses of the cores, and the latter are equal by definition. The equality of masses of the cores in both models is one of our boundary conditions. The asymptotic gas densities $\rho_{\infty\mathrm{GR}}$ and $\rho_{\infty N}$ are approximately equal to $(M - m_{*\mathrm{GR}})/V$ ([14,15]); in order to show that one should invoke assumptions concerning boundary conditions $U_\infty^2 \ll \frac{M}{R_\infty} \ll a_\infty^2$. The calculation is long but straightforward. Thus, we finally obtain $L_{\mathrm{GR}}/L_N = F$; the ratio of luminosities is independent of the fraction of mass carried by the gas, in the regime of low luminosities. This means that the appropriate information on the ratio of the relativistic and Newtonian luminosities, L_{GR}/L_N, can be obtained just by the analysis of accreting systems with test gas (and for these see, for instance, results in [10,11]). This is despite the fact that actual values of both luminosities taken separately depend on the contribution of the gas to total mass. We already know that in accretion without radiation the mass accretion rates are maximal when $m_* = 2M/3$ and they tend to zero at both ends: (i) $m_* \to M$ (when the density ρ_∞ tends to zero) and (ii) $m_*/M \to 0$ (when the mass of the core is negligible in comparison to the mass of the fluid) [15]. That implies, for weakly radiating systems, that luminosities behave in a similar way. But still their ratio is constant and independent of the parameter y.

B. High luminosities

The argument of the former subsection does not apply to accreting luminous systems, when the total luminosity L_0 is close to the Eddington limit or more generally when the (XY) condition is broken. That this can happen, is easily illustrated by the Newtonian model. In this model one obtains an equation that relates luminosity x and the mass content y [see Eq. (21) in [7], in units adapted to the convention of this paper]:

$$x = \alpha \frac{\chi_\infty M^2}{4a_\infty^3}(1-y)(y-x)^2\left(\frac{2}{5-3\Gamma}\right)^{(5-3\Gamma)/(2(\Gamma-1))}. \quad (30)$$

Here χ_∞ is a constant. The argument that was used above relied on the fact that the right-hand side of (30) does not depend on x if $x \ll y$. But if x is relatively large, then $\dot{M}$ becomes x-dependent, and (30) yields a relation $x = x(y)$. A similar reasoning can also be applied to the general relativistic model; again $\dot{M}$ depends on x if x is large. In conclusion, for luminous systems the ratio of L_{GR}/L_N can become x- and y-dependent.

Luminous systems are characterized by small values of the asymptotic speed of sound, $a_\infty \ll 1$. We restrict our attention to systems that satisfy the following.

$X(1-Y)$ *condition.* We will say that an accretion system satisfies the $X(1-Y)$ condition if $x \gg 1-y$ and $x < y/2$.

Since $y > x$, the above implies $y > 2/3$. Thus $X(1-Y)$ selects a subclass of luminous accretion systems with moderate contribution of the gas to total mass. Luminous test fluids belong to this category.

The assumptions $a_\infty \ll 1$ and $x < y/2$ imply that in the region extending from the sonic point to R_∞ the infall velocity U is small and the position of the sonic point R_* is large ($R_* \gg M$). Therefore one can approximate Eq. (6) by a suitable Newtonian model in the annular region (R_*, R_∞) (see Sec. II B for the discussion). But Eq. (30) has a unique solution, assuming $x < y < 1$. Therefore the Newtonian limit of the general relativistic model and the Newtonian solution do coincide and the ratio L_{GR}/L_N is not only constant, but it is equal to 1.

VI. NUMERICS

We compare two accreting systems, a Newtonian one and its general relativistic counterpart, that have identical sizes, the same asymptotic masses and identical masses of compact cores, equal asymptotic temperature, and the same binding energy. Thus, it is legitimate to say that the boundary data are ultimately the asymptotic mass, the mass of the core, the binding energy per unit mass $\alpha = |\phi(R_0)|$, the asymptotic speed of sound a_∞, and the size of the system R_∞. The total luminosity L_0 is not part of these data but is the sought result of the two models.

It appears convenient in numerical calculations to specify temporarily $\rho_{0\infty}$ and L_0 instead of the mass of the core. Conceptually the computational technique is the same in the two models. For a given $\rho_{0\infty}$ one randomly chooses L_0 (equivalently one could choose an accretion rate, due to relation $\dot{M} = L_0/\alpha$). This choice completely specifies $L(r)$ in the Newtonian model—see formula (14). Asymptotic radiation data for the general relativistic system in turn are given by $j_\infty = \rho_\infty^E = L_0/(4\pi R_\infty^2)$, and the mass accretion rate $\dot{M}_\infty = L_0/\alpha$. During the numerical integration one gets a subsonic solution (if the chosen L_0 is smaller than a critical luminosity) or finds no solution at all (if L_0 is greater than a critical value). Using the bisection method one finds this critical luminosity for which the gas flow becomes transonic. The mass of the core results from computations. Notice that for a given $\rho_{0\infty}$ masses of the core usually differ in the Newtonian and the general relativistic models. The difference is particularly noticeable for high asymptotic sound speeds a_∞. One should change the value of $\rho_{0\infty}$ and repeat the procedure until finally the masses of both cores are the same for both critical flows.

In this way one obtains a boundary of the solution set (in the plane L_0-M_{core}) that consists exclusively of transonic solutions, if the mean free path of photons is larger than the size of the system R_∞.

From a mathematical point of view we have a system of ordinary first order differential equations. The general relativistic problem also includes the integro-algebraic constraint Eq. (6). Numerical calculations start from the values adopted at the outer boundary R_∞ and continue inward until the equality $\alpha = 1 - \frac{N(R)}{k(R)}\sqrt{1 - [(2m(R))/R]}$ (in the GR case) is met at some R; this value of the areal radius is denoted as R_0 and interpreted as the radius of the compact core of the accreting system. For the Newtonian model the calculation continues until the gravitational potential ϕ becomes equal to $-\alpha$. The numerical integration employs the 8th order Runge-Kutta method [16]. The main numerical difficulty is encountered in the vicinity of the sonic point. In the general relativistic case the denominator and the numerator of Eq. (27) vanish for $a^2 = \frac{U^2}{k^2}$. In numerical computation, the division by very small numbers may cause errors and lead to unphysical solutions; therefore a special regularization technique had to be implemented. We omit further discussion of related technicalities, but let us mention that because of this difficulty with the sonic point there appear small numerical errors for $M_{\mathrm{core}} \approx 1$ and $M_{\mathrm{core}} \approx 0.1$ (see Fig. 3). In the Newtonian model one has to deal with the same problem.

We choose specific numerical data, but since the accreting system possesses a simple scaling property—as discussed in one of preceding sections—one can extend the validity of all conclusions to a large family of systems with appropriately scaled masses M and sizes R_∞.

We assume standard gravitational units $G = c = 1$, the size $R_\infty = 10^8 \times M$, and the mass $M = 10^6 M_\odot$, where $M_\odot$ is the solar mass. In the scaling $M = 1$ one gets $\kappa = 3.6258 \times 10^{22}(M_\odot/M)$, that is, $\kappa = 3.6258 \times 10^{16}$. The Eddington luminosity reads $L_E = 3.4658 \times 10^{-22} M/M_\odot = 3.4658 \times 10^{-16}$. These data are arranged to ensure the validity of the thin gas approximation. The optical thickness (23) is always smaller than 1.

VII. RESULTS

In the first part of this section we consider the case with the polytropic index $\Gamma = 3/2$. Figures 1 and 2 show accreting solutions on the luminosity-(mass of the central core) diagram for $\alpha = 0.9$. The squared speed of sound is $a_\infty^2 = 10^{-1}$ and $a_\infty^2 = 10^{-6}$, respectively. The two figures show transonic solution sets for the Newtonian and general relativistic models; they are depicted by dashed and solid lines, respectively. Comparing these figures, we notice that the brightness of a system increases sharply as a_∞^2 decreases. In the case illustrated in the second figure, maximal luminosities go up to one quarter of the Eddington luminosity. In the test gas limit, the interaction between gas and radiation is negligible and the gas accretion can be approximated by the purely hydrodynamic description. Such a case was already analyzed in [10], with the same conclusion as suggested by the comparison of Figs. 1 and 2: the larger the asymptotic speed of sound, the larger the gap between the general relativistic and the Newtonian predictions. The general relativistic model gives significantly larger accretion rates for high asymptotic temperatures. These figures clearly demonstrate that luminosities depend on the fraction of mass deposited in the gas and become maximal when this fraction is not bigger than 1/3. Again, this aspect of the description of the regime of weakly radiating sources agrees with the purely hydrodynamic study of [15]. The position of this maximum depends weakly on the relative luminosity L/L_E and it shifts from $y = 2/3$ in Fig. 1 towards $y = 0.75$ in Fig. 2. It is clear that this effect is due to the influence of the radiation;

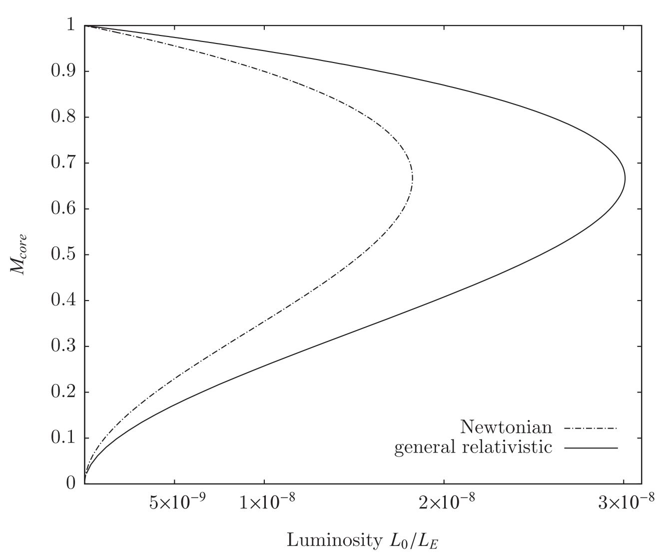

FIG. 1. Luminosity of general relativistic and Newtonian models. $\alpha = 0.9$ and $a_\infty^2 = 10^{-1}$. The abscissa shows the luminosity in terms of the Eddington luminosity L_E and the ordinate shows the mass of the compact core.

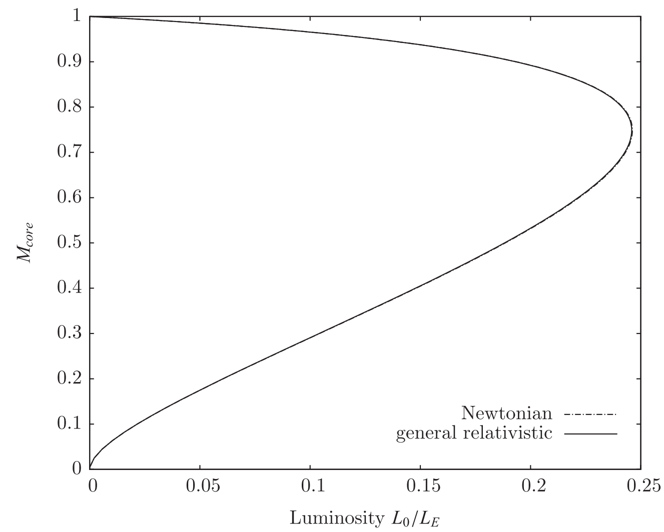

FIG. 2. Luminosity of general relativistic and Newtonian models. $\Gamma = 3/2$, $\alpha = 0.9$, and $a_\infty^2 = 10^{-6}$. The abscissa and ordinate are as in Fig. 1.

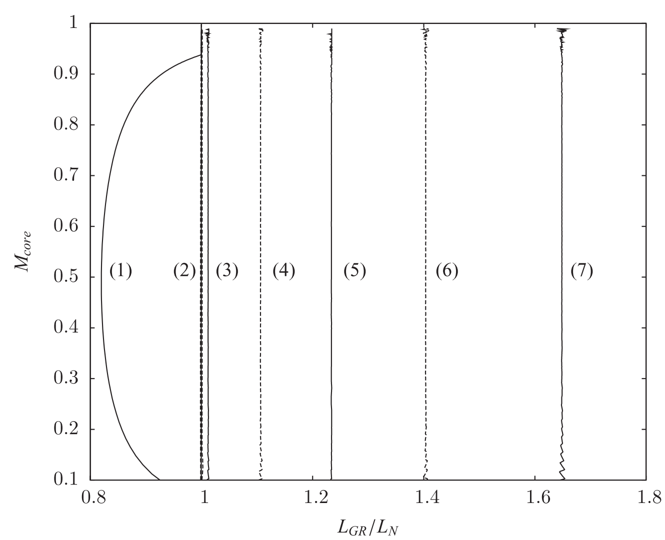

FIG. 3. Binding energy $\alpha = 0.9$, $\Gamma = 3/2$. The values of $L_{\rm GR}/L_N$ are shown on the abscissa. The mass fraction y is put on the ordinate. Asymptotic squared speeds of sound are 10^{-7} (line no. 1); 10^{-6}, 10^{-5}, 10^{-4} (the three close lines are grouped as line 2); 10^{-3} (line 3); 10^{-2} (line 4); 2.5×10^{-2} (line 5); 5×10^{-2} (line 6); 10^{-1} (line 7).

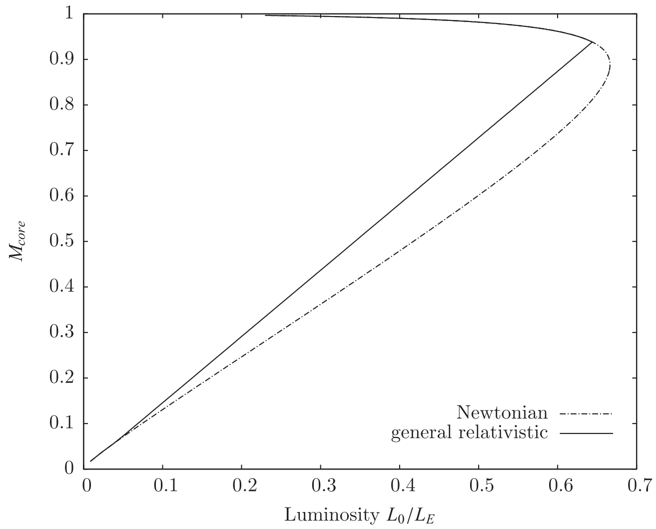

FIG. 4. Binding energy $\alpha = 0.9$, $\Gamma = 3/2$. The luminosities are shown on the abscissa, while the mass fraction y (the core mass) is put on the ordinate. Here $a_\infty^2 = 10^{-7}$.

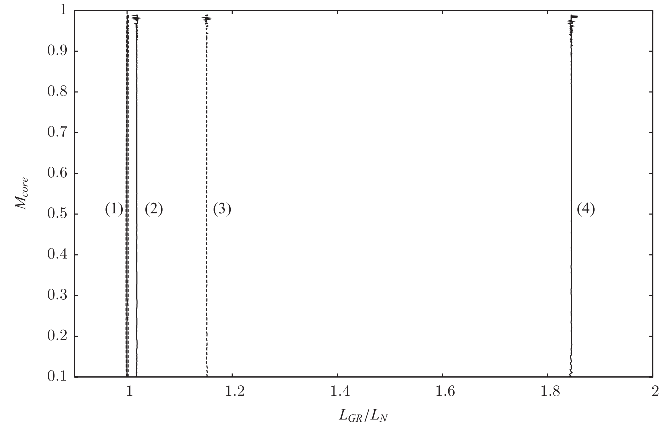

FIG. 6. Binding energy $\alpha = 0.9$, $\Gamma = 14/9$. The values of $L_{\rm GR}/L_N$ are shown on the abscissa. The mass fraction y is put on the ordinate. Asymptotic squared speeds of sound are 10^{-6}, 10^{-5}, 10^{-4} (the three close lines are grouped as line 1); 10^{-3} (line 2); 10^{-2} (line 3), 10^{-1} (line 4).

the higher the luminosity, the larger the mass of the core at the maximum.

Figure 3 reveals a feature of spherical accretion that confirms the analytic proof (made in one of the preceding sections) that $L_{\rm GR}/L_N$ should be constant, at least for small luminosities. While each individual quantity $L_{\rm GR}$, L_N depends on the contribution of the gas to the total mass, their ratio is roughly constant at a given asymptotic temperature. In the six sets of transonic flows (lines 2–7 in Fig. 3) the ratio $L_{\rm GR}/L_N$ is independent of the mass of accreting gas. We would like to call the reader's attention to line 2, where the maximal value of $x = 1/4$ is achieved at $y = 0.75$ (see Fig. 2). Thus the maximal value of $x/y \approx 1/3$, $x \approx 1 - y$ and still the fraction $L_{\rm GR}/L_N$ is constant. This agrees with the analytic result shown in the second part of Sec. V, although the proof of this requires $x \gg 1 - y$. That suggests that analytic results can be proven under less stringent conditions than stated in Sec. V.

Figure 4 and line 1 in Fig. 3 display data where the backreaction effect causes $L_{\rm GR}/L_N$ to vary (and, in particular, $L_{\rm GR}/L_N$ can be made significantly smaller than 1). Notice, however, that in the general relativistic model the flows cease to be transonic for $y < 0.94$. In contrast, they are always transonic in the Newtonian model. There is a small segment just below $y = 1$, where the $X(1 - Y)$ condition is met and the ratio of luminosities equals 1.

In the final part of this section we investigate how the results depend on the equation of state. Figures 5–7 correspond to different values of the polytropic index, $\Gamma = 4/3$, $\Gamma = 14/9$, and $\Gamma = 1.66$, respectively. We are

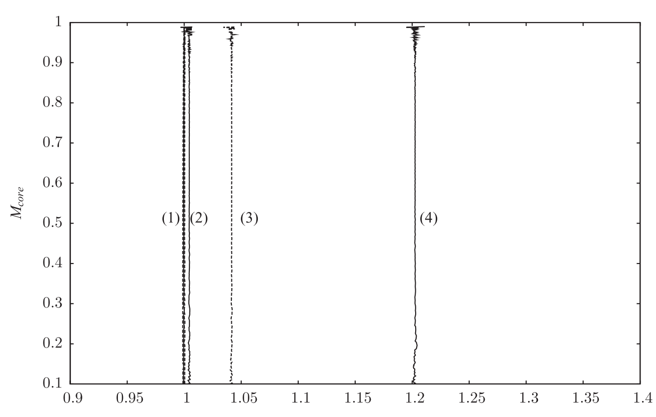

FIG. 5. Binding energy $\alpha = 0.9$, $\Gamma = 4/3$. The values of $L_{\rm GR}/L_N$ are shown on the abscissa. The mass fraction y is put on the ordinate. Asymptotic squared speeds of sound are $a_\infty^2 = 10^{-7}$, $a_\infty^2 = 10^{-6}$, $a_\infty^2 = 10^{-5}$, $a_\infty^2 = 10^{-4}$ (there are four close lines denoted as 1); 10^{-3} (line 2); 10^{-2} (line 3); 10^{-1} (line 4).

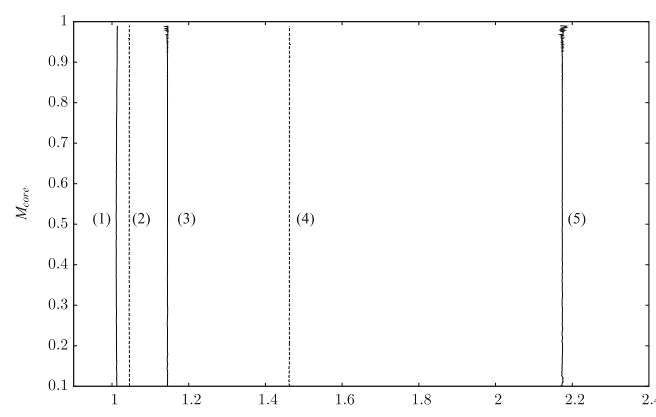

FIG. 7. Binding energy $\alpha = 0.9$, $\Gamma = 1.66$. The values of $L_{\rm GR}/L_N$ are shown on the abscissa. The mass fraction y is put on the ordinate. Asymptotic squared speeds of sound are 10^{-5} (line no. 1), 10^{-4} (line 2), 10^{-3} (line 3), 10^{-2} (line 4), 10^{-1} (line 5).

interested only in the universality aspect of the accretion. It is clear that there emerges, as before, the enhancement of the luminosity due to the relativistic effect. The larger the polytropic index, the larger the enhancement.

VIII. CONCLUSIONS

There are interesting universal properties hidden in generalizations of the classical Bondi accretion model. It is already known that when radiation is absent, transonic flows (that maximize mass accretion rates) correspond to the case when $y \equiv m_*/M = 2/3$, irrespective of the equation of state and the asymptotic speed of sound [14,15,17]. The mass of the core is about $2/3$ of the total mass of an accreting system. This paper deals with radiating accretion flows. We compare luminosities corresponding to transonic solutions of the general relativistic and Newtonian accretion models, assuming the same polytropic equation of state and identical boundary data—asymptotic speed of sound a_∞, size R_∞, total (asymptotic) mass, and fraction $1 - y$ of the total mass contributed by gas. We focus our attention on the investigation of the relation between their relative luminosity (L_{GR}/L_N) and y. When accreting systems are characterized by low luminosity and the condition (XY) of Sec. V holds true (that is, $L_{\mathrm{GR}} \ll L_E \times y$ and $L_N \ll L_E \times y$), then the ratio L_{GR}/L_N is independent of y and can be significantly larger than 1. We have found an example with the largest value of L_{GR}/L_N exceeding 1.6, but in earlier investigation of test fluids with the polytropic index close to $5/3$ the ratio of mass accretion rates $\dot{M}_{\mathrm{GR}}/\dot{M}_N$ exceeded 10 [11], which suggests that L_{GR}/L_N can grow by 1 order of magnitude. On the other hand, when the condition $X(1 - Y)$ of Sec. V is valid (that is, a transonic flow is highly luminous, $x \equiv L_0/L_E \gg 1 - y$ and $x < y/2$, but the contribution of gas to the mass is small), then $L_{\mathrm{GR}}/L_N \approx 1$. These properties of the ratio L_{GR}/L_N have been derived analytically and confirmed (under less stringent conditions) numerically.

ACKNOWLEDGMENTS

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