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General spherically symmetric constant mean curvature foliations of the Schwarzschild solution

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We consider a family of spherical three-dimensional spacelike slices embedded in the Schwarzschild solution. The mean curvature is constant on each slice but can change from slice to slice. We give a simple expression for an everywhere positive lapse and thus we show how to construct foliations. There is a barrier preventing the mean curvature from becoming large, and we show how to avoid this so as to construct a foliation where the mean curvature runs all the way from zero to infinity. No foliation exists where the mean curvature goes from minus to plus infinity. There are slicings, however, where each slice passes through the bifurcation sphere $R = 2M$ and the lapse only vanishes at this one point, and is positive everywhere else, while the mean curvature does run from minus to plus infinity. Symmetric foliations of the extended Schwarzschild spacetime degenerate at a critical point, where we show that the lapse function exponentially approaches zero.

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I. INTRODUCTION

Constant mean curvature (CMC) foliations of the Schwarzschild geometry have been constructed by Brill, Cavalho, and Isenberg [1]. These foliations degenerate when the lapse "collapses"; foliations in the vicinity of this critical point have been investigated in [2]. In both of these papers, it is assumed that the trace of the extrinsic curvature is not only a constant on each slice, but retains this constant value from slice to slice. In this paper we analyze more general families of CMC foliations, with the trace K of the extrinsic curvature that can change with time. The collapse of the lapse can again be described by analytic approximations.

The standard way of viewing general relativity as a dynamical system is by considering the 4-manifold as foliated by a sequence of spacelike 3-surfaces [3,4]. Each 3-surface inherits a 3-metric, g_{ij}, and an extrinsic curvature $K_{ij} = (1/2)\mathcal{L}_{\mathbf{n}} g_{ij}$ [5] where $\mathcal{L}_{\mathbf{n}}$ is the Lie derivative along the normal, and thus K_{ij} is a geometric object, the analogue of the time derivative of the metric. Each slicing is equivalent to a choice of time function. A standard choice is to demand that the trace of the extrinsic curvature, $g^{ij}K_{ij}$, usually written as K and known as the mean curvature of the surface, be constant along each slice. Hence these are the constant mean curvature, or CMC, slices. CMC slices are attractive for a number of reasons. The Einstein equations give an evolution equation for K, it is (in vacuum)

$$\nabla^2 N - K^{ij}K_{ij}N = N\mathcal{L}_{\mathbf{n}}K = \frac{\partial K}{\partial t} - N^i\partial_i K, \qquad (1)$$

where N and N^i are the lapse and shift, respectively.

If K is a spatial constant, the shift term drops out and the equation reduces to

$$\nabla^2 N - K^{ij}K_{ij}N = \frac{\partial K}{\partial t}, \qquad (2)$$

and this can be regarded as an equation for the lapse function, N, of a CMC slicing. This is a nice linear elliptic equation that satisfies the maximum principle.

In the standard, conformal, methods of constructing initial data for the gravitational field, choosing the trace of the extrinsic curvature to be a constant simplifies the equations. One gets a single, nonlinear scalar equation for the conformal factor instead of a coupled system of nonlinear equations [6].

By taking the trace of the equation defining K_{ij}, one can show $\mathcal{L}_{\mathbf{n}}\sqrt{g} = \sqrt{g}K$. This tells us that K is just the fractional time rate of change of the volume along the normal, and, in a cosmology, the CMC slices are the "Hubble time" slices, with the instantaneous value of K equalling the Hubble "constant." In Minkowski space, on the other hand, the "mass hyperboloids," $t^2 - r^2 = m^2$, are CMC slices, with $K = 3/m$. Slices in general asymptotically flat spacetimes mimic the behavior of the mass hyperboloids, in that they are everywhere spacelike but become null at null infinity. This feature makes them of value for the analysis of gravitational or any other form of radiation, see e.g. [7].

One major advantage of CMC slicings for numerical relativity is that, if we consider gravitational waves of a fixed wavelength, the number of wave cycles that a CMC slice intercepts is finite. To resolve a wave in a numerical computation, we need the separation between data points to be less than the wavelength. This means that the domain of a code can extend all the way to null infinity and track the waves all the way out with a finite number of grid points. This phenomenon is independent of whether one compactifies or not.

II. EXPLICIT CMC FOLIATIONS

We possess a great deal of understanding about the spherical CMC slices of the Schwarzschild solution [2].

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We can write the 3-metric and extrinsic curvature analytically in terms of the Schwarzschild radius R. They depend only on two parameters, K, the trace of the extrinsic curvature, and C, that quantifies the transverse-traceless part of the extrinsic curvature. The expressions are

$$dS^2 = \frac{dR^2}{1 - \frac{2M}{R} + (\frac{KR}{3} - \frac{C}{R^2})^2} + R^2 d\Omega^2. \tag{3}$$

The extrinsic curvature is diagonal, with the nonzero components being

$$K^R_R = \frac{K}{3} + \frac{2C}{R^3}; \qquad K^\theta_\theta = K^\phi_\phi = \frac{K}{3} - \frac{C}{R^3}. \tag{4}$$

The sixth-order polynomial

$$k^2 = 1 - \frac{2M}{R} + \left(\frac{KR}{3} - \frac{C}{R^2}\right)^2 \tag{5}$$

plays a key role. It has several meanings, $k = dR/dL$, where L is the proper distance along the slice, therefore $2k/R$ is the 2-mean curvature of the round 2-spheres as embedded in the 3-slice. Also $k = N_K$, where N_K is the Killing lapse, the dot product of the ''timelike'' Killing vector with the unit normal to the slice. For small values of K and C, k^2 is positive for both small and large R, with only two roots. We must have positive k^2 for the metric to make physical sense. Let us label the larger of these two roots R_t. One can show easily that $R_t < 2M$. The zone where k^2 is positive from $R = R_t$ to $R = \infty$ corresponds to a slice that starts out at one future null infinity (if $K > 0$), reaches a minimum surface at $R = R_t$ (hence t' for ''throat''), and continues on to the other future null infinity. If $C < 8M^3K/3$ it crosses below the bifurcation point, if $C > 8M^3K/3$ it crosses through the upper quadrant. As C becomes larger, while holding K fixed, the polynomial k^2 rises up, the two roots approach each other until they finally touch at a value of R which we call $R_\star$. The value of K and C at that point equal

$$K_\star = \frac{2R_\star - 3M}{\sqrt{2MR^3_\star - R^4_\star}}; \qquad C_\star = \frac{3MR^3_\star - R^4_\star}{3\sqrt{2MR^3_\star - R^4_\star}}. \tag{6}$$

This is a critical point of the slicing. As (C, K) change so as to approach this point one gets the ''collapse of the lapse'' and ''slice stretching'' that one is used to in maximal slicing [2,8]. If we increase C even further, k^2 is positive over the entire range $R = (0, \infty)$. When this is true, the CMC slice starts at null infinity and plunges into the singularity. We have previously analyzed the CMC slices where we hold K fixed and just allowed C to change [2]. Obviously there exists a much richer class of CMC slicings, where both C and K change. One really only needs to give some relationship between C and K. However, it is easier to introduce some parameter time t and write $C(t)$ and $K(t)$. One can always reparametrize, which will change the form of both $C(t)$ and $K(t)$. However, the ratio, $(dK/dt)/(dC/dt)$, is unchanged.

The $G_{RR} = 0$ Einstein equation can be written as

$$\partial_t (R K^\theta_\theta)|_{R=\text{const}} = k^3 \partial_R \frac{N}{k}. \tag{7}$$

This equation is solved by the lapse function N, where

$$N \equiv \beta k + k \int_R^\infty dr \frac{\dot{C} - \frac{r^3}{3}\dot{K}}{r^2 k^3}. \tag{8}$$

Here β is a (time-dependent) constant. One can verify that the lapse equation (2), is solved by N as given in Eq. (8).

It is clear that Eq. (2) is a linear elliptic inhomogeneous equation, so therefore the solution can be written as a linear combination of a ''particular'' solution of the inhomogeneous equation combined with a solution of the homogeneous equation $(\nabla^2 - K^{ij}K_{ij})\psi = 0$. Since k is the Killing lapse, we know that

$$(\nabla^2 - K^{ij}K_{ij})k = 0. \tag{9}$$

We also know that we can change C without changing K. This also gives us a lapse that satisfies the homogeneous equation, call it $\phi(\dot{C})$. It can be chosen to depend linearly on $\dot{C}$ (one may have to subtract off some multiple of k). Finally, the inhomogeneous solution can be can be adjusted so as to linearly depend on $\dot{K}$ (again, one may have to subtract off multiples of k and $\dot{C}$). This is why we get the three constants β, $\dot{C}$, and $\dot{K}$ appearing explicitly in Eq. (8).

Let us now restrict ourselves to the case where C and K are small enough that k has a zero at $R = R_t$. Let us also assume $K > 0$. This means that the slice comes in from one future null infinity, has a minimal surface at $R = R_t$, and goes out to the other future null infinity. We know that the Killing lapse k, the solution to Eq. (9), is antisymmetric, passing through zero at $R = R_t$. By adding an appropriate multiple of k we can find the solution that is proportional to $\dot{K}$ and the solution proportional to $\dot{C}$ that are symmetric around the throat.

Let us have a spherical CMC slicing of the Schwarzschild solution. This can be described as a curve in (C, K) space. There is still some freedom. The term in the lapse proportional to k can be thought of as ''pure gauge.'' It slides the slice along the Killing vector without changing either K or C. Since we are interested in the effects of changing K and C we would like to eliminate this freedom. The obvious way is to adjust the lapse by adding an appropriate amount of k so that the lapse is symmetric around the throat. This gives us a Neumann boundary condition $dN/dL = dR/dL dN/dR = k dN/dR = 0$ at $R = R_t$ where L is the proper distance along the slice. Since $k = 0$ at the throat, this looks like a trivial equation. It is not, because dN/dR generically blows up at the throat like $1/k$. This has the effect of reducing the number of constants in the solution from three

to two. The second freedom is the choice of time parametrization of the slicing. It appears that the "natural" choice is to set $\beta = 1$ in Eq. (8). This allows the time translation vector of the slicing to coincide with the timelike Killing vector at null infinity. It turns out that we can satisfy both conditions, i.e., $\beta = 1$ and the symmetry at the throat, simultaneously.

It is interesting to look at expression Eq. (8) in the limit of large R. In that limit $k \approx KR/3$. Therefore the integrand is dominated by $-9\dot{K}/K^3r^2$. This integrates to $9\dot{K}/K^3r$, so the integral equals $-9\dot{K}/K^3R + O(1/R^2)$. The k outside the integral becomes $KR/3$ so we finally get

$$N \approx \beta k - \frac{3\dot{K}}{K^2} + O(1/R). \tag{10}$$

Therefore $N/k \to \beta$ for large R. Since k is the Killing lapse, if $\beta = 1$ we can have the time translation of the slicing equal the Killing vector. With this choice the time parameter is the retarded time.

While expression Eq. (8) is easy to understand near infinity, it is difficult to see from it what happens near the throat. To find the cleanest expression, we have to manipulate Eq. (8). It is easy to see that

$$\frac{d}{dR}\left(\frac{1}{k}\right) = -\frac{1}{k^3}\left(\frac{M}{R^2} + \frac{K^2R}{9} + \frac{KC}{3R^2} - \frac{2C^2}{R^5}\right). \tag{11}$$

Therefore we have

$$\frac{1}{r^2k^3} = -\left(\frac{1}{M + \frac{KC}{3} + \frac{K^2r^3}{9} - \frac{2C^2}{r^3}}\right)\frac{d}{dr}\left(\frac{1}{k}\right), \tag{12}$$

and the expression for the lapse can be rewritten as

$$N = \beta k - k\int_R^\infty \left(\frac{\dot{C} - \frac{\dot{K}r^3}{3}}{M + \frac{KC}{3} + \frac{K^2r^3}{9} - \frac{2C^2}{r^3}}\right)\frac{d}{dr}\left(\frac{1}{k}\right)dr. \tag{13}$$

Now integrate by parts to get

$$N = \beta k + k\int_R^\infty \frac{1}{k}\frac{d}{dr}\left(\frac{\dot{C} - \frac{\dot{K}r^3}{3}}{M + \frac{KC}{3} + \frac{Kr^3}{9} - \frac{2C^2}{r^3}}\right)dr - k\left(\frac{1}{k}\frac{\dot{C} - \frac{\dot{K}r^3}{3}}{M + \frac{KC}{3} + \frac{K^2r^3}{9} - \frac{2C^2}{r^3}}\right)\Bigg|_R^\infty. \tag{14}$$

If we change the range of integration from the throat to R, instead of from R to ∞ we get

$$N = \beta k + k\int_{R_t}^\infty \frac{1}{k}\frac{d}{dr}\left(\frac{\dot{C} - \frac{\dot{K}r^3}{3}}{M + \frac{KC}{3} + \frac{Kr^3}{9} - \frac{2C^2}{r^3}}\right)dr - k\int_{R_t}^R \frac{1}{k}\frac{d}{dr}\left(\frac{\dot{C} - \frac{\dot{K}^2r^3}{3}}{M + \frac{KC}{3} + \frac{Kr^3}{9} - \frac{2C^2}{r^3}}\right)dr + \frac{\dot{C} - \frac{\dot{K}R^3}{3}}{M + \frac{KC}{3} + \frac{K^2R^3}{9} - \frac{2C^2}{R^3}}. \tag{15}$$

It turns out that if we choose

$$\beta = -\int_{R_t}^\infty \frac{1}{k}\frac{d}{dr}\left(\frac{\dot{C} - \frac{\dot{K}r^3}{3}}{M + \frac{KC}{3} + \frac{Kr^3}{9} - \frac{2C^2}{r^3}}\right)dr, \tag{16}$$

i.e., eliminate the term that is of the form constant $\times\, k$ we get the symmetric lapse function that satisfies $dN/dL = 0$ at the throat. This is

$$N = -k\int_{R_t}^R \frac{1}{k}\frac{d}{dr}\left(\frac{\dot{C} - \frac{\dot{K}r^3}{3}}{M + \frac{KC}{3} + \frac{Kr^3}{9} - \frac{2C^2}{r^3}}\right)dr + \frac{\dot{C} - \frac{\dot{K}R^3}{3}}{M + \frac{KC}{3} + \frac{K^2R^3}{9} - \frac{2C^2}{R^3}}. \tag{17}$$

We obviously assume that we are in the subcritical regime so that R_t exists. It is straightforward, if tedious, to show that this expression satisfies the CMC equation, Eq. (2).

As stated earlier, we can simultaneously set $dN/dL = 0$ at the throat (just by choosing the lapse as given by Eq. (17)) and simultaneously set $\beta = 1$. We start off with a curve in (C, K) space on which we put some coordinate time label. This defines $\dot{C}$ and $\dot{K}$. Now compute

$$\gamma = \int_{R_t}^\infty \frac{1}{k}\frac{d}{dr}\left(\frac{-\dot{C} + \frac{\dot{K}r^3}{3}}{M + \frac{KC}{3} + \frac{Kr^3}{9} - \frac{2C^2}{r^3}}\right)dr, \tag{18}$$

and scale the time by γ, i.e.,

$$d\hat{t} = \gamma dt. \tag{19}$$

This rescales $\dot{C}$ and $\dot{K}$ by γ. Now the value of β for which the two lapse functions, Eq. (8) and (17), are equal is $\beta = 1$!.

The quantity $M + K^2r^3/9 + KC/3 - 2C^2/r^3$ that appears twice in the expression for N is nothing but ($r^2/2$ times) the first derivative of k^2 and this is positive on the entire interval $[R_t, \infty)$. It only goes to zero as we approach the critical point. It is the "going to zero" of this function that gives the classical "collapse of the lapse."

This lapse, while maintaining the CMCness of the slices, will not keep the metric in the form Eq. (3), because the normal to the slice is not along the $R =$ constant direction. It is easy to work out what the appropriate shift is since the Killing vector is along this direction. We know that the Killing lapse is $\alpha_K = k$ and we also know that $\alpha_K^2 - \beta_K^2 = 1 - 2M/R$. Therefore $\beta_K^2 = (KR/3 - C/R^2)^2 \Rightarrow \beta_K^R = k(-KR/3 + C/R^2)$. In turn we get $N^R = N(-KR/3 + C/R^2)$. Since $g_{RR} = 1/k^2$ we get $N_R = N(-KR/3 + C/R^2)$. This argument works for both choice of lapse, Eq. (8) and (17). In the upper half plane, if $N > 0$, we want $N^R < 0$ for large R because the normal to the CMC slice is leaning over more towards null infinity than the Killing vector is. This shift does not vanish at the throat for the symmetric N as distinct from the Killing shift. It will be positive (if N is positive) there and we seek a symmetric solution. This means that we have a discontinuity in the shift across the throat. This is to be expected, because, as we move forward in time, the coordinate range of R

expands. The opposite will occur in the lower half plane. From the formula for the symmetric N, Eq. (17), we can easily read off the value of the lapse at the throat as

$$N(R=R_t)=\frac{\frac{dC}{dt}-\frac{dK}{dt}\frac{R_t^3}{3}}{M+\frac{K^2R_t^3}{9}+\frac{KC}{3}-\frac{2C^2}{R_t^3}}. \tag{20}$$

This is obviously positive if $dC/dt - dK/dt(R_t^3/3) > 0$. If we have a symmetric slicing and if $N = 0$ at any one point then it must be zero on an entire closed 2-sphere. If dK/dt is positive, we can immediately see from applying the maximum principle to the CMC equation, Eq. (2), that if the lapse is positive in the center it is positive everywhere. One way to show this is to assume the opposite. Let us assume that we have a region on the 3-surface where $N > 0$ in the interior, and $N = 0$ on the boundary. Therefore $dN/dR \leq 0$ on the boundary. Multiply the lapse equation, Eq. (2), by N and integrate over the region in which $N \geq 0$. One gets an immediate contradiction. Therefore we have a foliation instead of just a slicing. This kind of argument was first introduced by [8]. This expression for the central lapse in Eq. (17) gives us an immediate consistency check. Let us consider the situation where the variations in C and K are such that the radius of the throat does not change. Since $R = R_t$ is the zero of k^2, by inspecting the formula for k^2, Eq. (5), the condition is that

$$\frac{\delta K R_t}{3}-\frac{\delta C}{R_t^2}=0. \tag{21}$$

But, of course, if the radius does not change the lapse at the throat must vanish. This is exactly what we get from Eq. (20).

III. TWO SPACETIME METRICS

Let us start with a CMC slice whose 3-metric is given by 3, and let us drag this along the timelike Killing vector, i.e., we set $\dot{K} = 0$, $\dot{C} = 0$. We can now write down the 4-metric associated with this slicing (it is not a foliation because the lapse vanishes at the throat). It is

$$\bar{g}_{\mu\nu}=\begin{pmatrix} -\left(1-\frac{2M}{R}\right), & \frac{1}{k}\left(\frac{C}{R^2}-\frac{KR}{3}\right) & 0 & 0 \\ \frac{1}{k}\left(\frac{C}{R^2}-\frac{KR}{3}\right), & \frac{1}{k^2} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{pmatrix},$$

$$\bar{g}^{\mu\nu}=\begin{pmatrix} -\frac{1}{k^2}, & \frac{1}{k}\left(\frac{C}{R^2}-\frac{KR}{3}\right) & 0 & 0 \\ \frac{1}{k}\left(\frac{C}{R^2}-\frac{KR}{3}\right), & \left(1-\frac{2M}{R}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}.$$

Another spacetime metric can be constructed using the same 3-metric but choosing a lapse function that allows C and K to change with time, i.e., either Eq. (8) or Eq. (17). This has the form

$$\bar{g}_{\mu\nu}=\begin{pmatrix} -\frac{N^2}{k^2}\left(1-\frac{2M}{R}\right), & \frac{N}{k^2}\left(\frac{C}{R^2}-\frac{KR}{3}\right) & 0 & 0 \\ \frac{N}{k^2}\left(\frac{C}{R^2}-\frac{KR}{3}\right), & \frac{1}{k^2} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{pmatrix},$$

$$\bar{g}^{\mu\nu}=\begin{pmatrix} -\frac{1}{N^2}, & \frac{1}{N}\left(\frac{C}{R^2}-\frac{KR}{3}\right) & 0 & 0 \\ \frac{1}{N}\left(\frac{C}{R^2}-\frac{KR}{3}\right), & k^2 & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}.$$

IV. MATCHING CMC TO MAXIMAL SLICING

Using the explicit form of the metric(s) in the previous section, we can show that one can smoothly match a spherical CMC slicing (with either a constant or time-dependent K) to a maximal slicing in an extended Schwarzschild solution. The matching is performed along a timelike surface with fixed Schwarzschild radius $R = R_j > 2M$. We can handle the case where the CMC slices are on the "inside" and the maximal slices are "outside," and the converse.

We wish to match a patch of spacetime with metric

$$g_{\mu\nu}=\begin{pmatrix} -\frac{N^2}{k^2}\left(1-\frac{2M}{R}\right), & \frac{N}{k^2}\left(\frac{C}{R^2}-\frac{KR}{3}\right) & 0 & 0 \\ \frac{N}{k^2}\left(\frac{C}{R^2}-\frac{KR}{3}\right), & \frac{1}{k^2} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{pmatrix}$$

with

$$\frac{N}{k}\equiv\beta_K(t)+\int_R^\infty dr\frac{\dot{C}-\frac{r^3}{3}\dot{K}}{r^2k^3} \tag{22}$$

along some cylinder of constant Schwarzschild radius $R = R_j$ to a patch of spacetime with metric

$$\tilde{g}_{\mu\nu}=\begin{pmatrix} -\frac{\tilde{N}^2}{\tilde{k}^2}\left(1-\frac{2M}{R}\right), & \frac{\tilde{N}}{\tilde{k}^2}\frac{\tilde{C}}{R^2} & 0 & 0 \\ \frac{\tilde{N}}{\tilde{k}^2}\frac{\tilde{C}}{R^2}, & \frac{1}{\tilde{k}^2} & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta \end{pmatrix}$$

with

$$\frac{\tilde{N}}{\tilde{k}}\equiv\beta_{\tilde{C}}(t)+\int_R^\infty dr\frac{\dot{\tilde{C}}}{r^2\tilde{k}^3}, \tag{23}$$

with $\tilde{k}^2 = 1 - \frac{2M}{R} + \frac{\tilde{C}^2}{R^4}$.

We need to check that the Israel-Darmois junction conditions [9] are satisfied. Note that we allow β to be two different time-dependent functions, one in each patch. Really, we only need $\beta_K(t)$ to change with time, it makes sense to keep $\beta_{\tilde{C}} = 1$. This guarantees that the time in the "maximal" zone is the proper time at infinity. The Israel-Darmois junction conditions are that intrinsic metric and the extrinsic curvature of the three-dimensional matching surface be continuous. We can actually arrange that the whole 4-metric be continuous.

The key condition is that we choose $\tilde{C}$, K, and C to satisfy $\tilde{C} = C - \frac{KR_j^3}{3}$. This has to be satisfied on every time slice. Therefore we also want $\dot{\tilde{C}} = \dot{C} - \frac{\dot{K}R_j^3}{3}$. This guarantees that $k = \tilde{k}$ along the matching surface. We also can use the free parameter $\beta_K(t)$ to maintain $\tilde{N} = N$. This guarantees that the 4-metrics to the left and right of the surface $R = R_j$ are the same.

Now we need to look at the extrinsic curvature. The condition $\tilde{C} = C - \frac{KR_j^3}{3}$ guarantees that $\tilde{K}_{\theta\theta} = K_{\theta\theta}$ and $\tilde{K}_{\phi\phi} = K_{\phi\phi}$. This means that we only need to check K_{00}. The normals to the surface $R = R_j$ are respectively $\tilde{n}_\mu = (0, \frac{1}{\tilde{k}} \cdot 0, 0)$ and $n_\mu = (0, \frac{1}{k} \cdot 0, 0)$. Hence $\tilde{n}_\mu = n_\mu$. We have $K_{00} = n_{0;0} = -\Gamma^R_{00} n_R = -\frac{1}{2} g^{R\mu}(2g_{0\mu,0} - g_{00,\mu}) n_R$. Since the metrics match on the surface, we have $g_{0\mu,0} = \tilde{g}_{0\mu,0}$. Therefore the problem reduces to comparing $(\tilde{N}/\tilde{k})_{,R}$ to $(N/k)_{,R}$. Looking at Eqs. (22) and (23). This reduces to comparing $\dot{\tilde{C}}$ to $\dot{C} - \frac{\dot{K}R_j^3}{3}$. These are obviously equal. Therefore we can match both the intrinsic metric and the extrinsic curvature along the 3-surface defined by $R = R_j$.

The metric along the matching surface is C^∞ while the matching of the extrinsic curvatures guarantees that the metric perpendicular to the surface is C^1. Further, consider the Hamiltonian and momentum constraints along the matching surface. The behavior of the metric and extrinsic curvature means that there is no jump discontinuity in ${}^{(3)}R - K^{ab}K_{ab} - K^2$ or $\nabla_a(K^{ab} - g^{ab}K)$ across the surface. This means that there are no stresses along the matching surface.

Consider the Hamiltonian and momentum constraints on a spacelike CMC slice. These are ${}^{(3)}R - K^{ab}K_{ab} + K^2 = 0$ and $\nabla_a(K^{ab} - g^{ab}K) = 0$. If we write $K^{ab} = K_t^{ab} + 1/3 K g^{ab}$ we can rewrite the constraints as ${}^{(3)}R - K_t^{ab}K_{ab}^t = -(2/3)K^2$ and $\nabla_a K_t^{ab} = 0$. Therefore the initial data can be regarded as maximal data coupled to a constant negative-density "matter" field at rest with $16\pi\rho = -(2/3)K^2$. Therefore we can regard this as a negative cosmological constant. We can even naturally deal with the situation where the cosmological constant is time dependent since we are free to allow K to change from slice to slice.

Thus we can consider this matching as gluing a sphere in a maximally sliced anti-de Sitter spacetime to a maximal Schwarzschild exterior. This would involve holding K fixed in time. Alternatively, we could have the interior maximal and the exterior be CMC. Obviously, we are only considering the Schwarzschild spacetime in a spherically symmetric slicing. This has no gravitational radiation. Nevertheless, the community of people who numerically analyze binary black hole collisions might find such slicings appealing. The black holes could be put in the maximal zone while the radiation could be analyzed in the asymptotically null CMC zone.

V. FOLIATIONS VERSUS SLICINGS

If dK/dt is positive and if $dC/dt - dK/dt(R_t^3/3) > 0$, then the lapse function is positive everywhere and we have a foliation. Why is this? How restrictive a condition is it?

If we hold K fixed and increase C we automatically get a foliation [2]. The lapse is everywhere positive and the slices move forward in time. However, if we hold C fixed and increase K, the slices move forward near both infinities and backward in the middle. One way of seeing this is to look at the (implicit) formula for R_t,

$$k^2 = 1 - \frac{2M}{R_t} + \left(\frac{KR_t}{3} - \frac{C}{R_t^2}\right)^2 = 0. \tag{24}$$

Now compute dR_t/dK. We get

$$\left(\frac{M}{R^2} + \frac{K^2 R}{9} + \frac{KC}{3R^2} - \frac{2C^2}{R^5}\right) dR_t + \left(\frac{KR_t}{3} - \frac{C}{R_t^2}\right)\frac{dKR_t}{3} = 0. \tag{25}$$

This gives

$$\frac{dR_t}{dK} = \frac{3(\frac{M}{R^2} + \frac{K^2 R}{9} + \frac{KC}{3R^2} - \frac{2C^2}{R^5})}{R_t(\frac{C}{R_t^2} - \frac{KR_t}{3})} > 0. \tag{26}$$

This shows that R_t increases as K increases. However, in the upper quadrant of the Schwarzschild solution the Schwarzschild radius decreases as one moves forward in time from the bifurcation sphere (at $R = 2M$) to the singularity at $R = 0$.

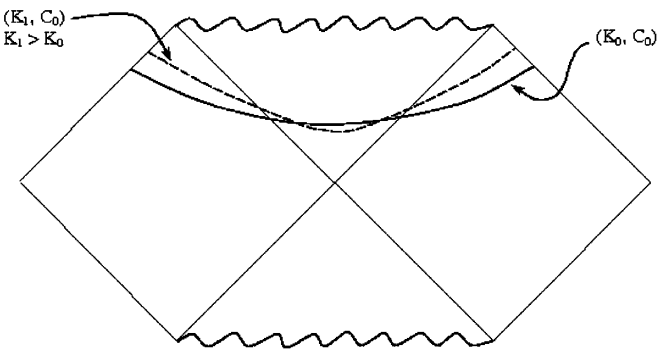


FIG. 1. K changes but C is fixed; the slices can intersect.

Similarly, we have

$$\frac{dR_t}{dC} = -\frac{R_t^2(\frac{M}{R^2} + \frac{K^2R}{9} + \frac{KC}{3R^2} - \frac{2C^2}{R^5})}{R_t(\frac{C}{R_t^2} - \frac{KR_t}{3})} < 0. \tag{27}$$

Therefore R_t decreases with increasing C and so the slice moves forward in time.

Therefore to have the slices moving forward everywhere, when K changes, one needs to simultaneously increase C as well as K. This behavior is illustrated in Figs. 1 and 2.

One way of finding CMC slices in Schwarzschild is by means of a height function approach, see [2,8]. This involves considering a slice defined as $t = 0$ where $t = T - h(R)$, and where T and R are the standard Schwarzschild coordinates. The condition that the slice be CMC reduces to a second order differential equation for h. This can be explicitly integrated once to give [2]

$$\frac{dh}{dR} = \frac{\frac{KR}{3} - \frac{C}{R^2}}{(1 - \frac{2M}{R})k}. \tag{28}$$

In retrospect, the fact that the lapse function can be written in such a compact form should not have been too surprising. The lapse function can be viewed as the derivative of the height function with respect to K and C. Therefore one might expect to write the first derivative of the lapse as a function, and the lapse itself as a simple integral, just as we do.

We can get an interesting slicing that is ''almost'' a foliation by setting $C = 8M^3K/3$. All these slices have their throats at the bifurcation sphere, $R = 2M$. These slices all touch at $R = 2M$, and so the lapse is zero there. Otherwise, as K increases, the lapse is positive everywhere else. This slicing allows K to run all the way from $-\infty$ to $+\infty$. If we use K as our label time, the lapse function of this slicing is

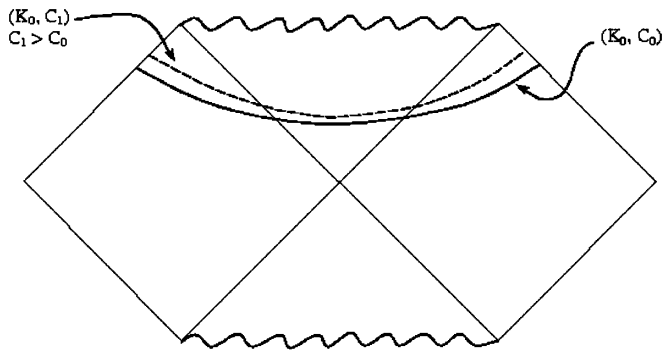


FIG. 2. The trace K is constant and C changes; the slices form a foliation.

$$N = \frac{1}{3}\frac{8M^3 - R^3}{M + \frac{K^2R^3}{9} + \frac{8M^3K^2}{9} - \frac{128M^6K^2}{9R^3}} + \frac{k}{3}\int_{2M}^{R}\frac{dr}{k}\frac{d}{dr} \times \left(\frac{8M^3 - r^3}{M + \frac{K^2r^3}{9} + \frac{8M^3K^2}{9} - \frac{128M^6K^2}{9r^6}}\right). \tag{29}$$

This slicing covers all of the left and right quadrants, but never enters the upper or lower quadrants, the ''black'' or ''white'' hole zones. This shows us that, while we can get a foliation that covers the range in K of $[0, \infty)$, we cannot find a foliation that allows K to run the whole range $(-\infty, +\infty)$ because the Schwarzschild solution is time symmetric and a foliation that gets to $K = +\infty$ must break this. To some degree, this is a word game. If you prespecify the range to be covered, i.e., one seeks a foliation that goes from $K = -D/M$ to $K = +D/M$, where D is a large number, one can do this. Start off with a moment of time symmetry slice, i.e., $(K = 0, C = 0)$ and choose a curve in (K, C) space so that dC/dK is only infinitesimally bigger than $R_t^3/3$. This will reach any desired value of K, in particular $K = D/M$. Now add the time reversal of this, and we have the desired object. Of course, all such foliations eventually run into the critical curve at some finite value of K. This, however, would be bigger than the specified D/M.

VI. CRITICAL FOLIATIONS

Henceforth we assume the subcritical regime—a minimal surface at R_t and the Neumann boundary condition $dN/dL = 0$ where L is the proper distance along the slice. This guarantees that the slices are symmetric about the throat. The line element, expressed in terms of comoving time t and the areal radius R, takes the following form:

$$ds^2 = -dt^2\left(1 - \frac{2m}{R}\right)\lambda^2 - 2\lambda\frac{\frac{C}{R^2} - \frac{KR}{3}}{k}dtdR + \frac{dR^2}{k^2} + R^2d\Omega^2. \tag{30}$$

Here $\lambda = 1 + \int_R^\infty dr\frac{\dot{C} - \frac{r^3}{3}\dot{K}}{r^2k^3}$. Let us define the following integrals:

$$X = -\int_R^\infty \frac{\frac{2}{3}K^2r^2 + \frac{12C^2}{r^4}}{k(2M + \frac{2Kr^3}{9} + \frac{2KC}{3} - \frac{4C^2}{r^3})^2}dr, \qquad Y = \int_R^\infty \frac{1}{k}\frac{r^2(6m + 2KC - \frac{24C^2}{r^3})}{(2M + \frac{2Kr^3}{9} + \frac{2KC}{3} - \frac{4C^2}{r^3})^2}dr. \tag{31}$$

The formula for the lapse function, $N = k\lambda$, can be rewritten as

$$N = k + 2\frac{-\frac{R^3}{3}\frac{dK}{dt} + \frac{dC}{dt}}{M + \frac{K^2R^3}{9} + \frac{KC}{3} - \frac{2C^2}{R^3}} - \frac{2}{3}\frac{dK}{dt}kY + 2\frac{dC}{dt}kX. \tag{32}$$

This representation of the lapse N is convenient when there

exist minimal surfaces. From (32) one clearly sees that if the minimal surface exists at R_t, $k(R_t) = 0$, then

$$N = 2\frac{-\frac{R_t^3}{3}\dot{K} + \dot{C}}{M + \frac{K^2R_t^3}{9} + \frac{KC}{3} - \frac{2C^2}{R_t^3}}. \tag{33}$$

One finds from (32) that at the throat $dN/dL = \frac{1}{2} \times \frac{d}{dR}k^2(1 + 2\dot{C}X - \frac{2}{3}\dot{K}Y)$. Therefore the condition $dN/dL = 0$ yields the differential equation

$$\dot{C} = \frac{-1}{2X} + \frac{\dot{K}Y}{3X}; \tag{34}$$

this is a highly nonlinear relation, since both integrals X and Y depend in a convoluted way on C and K. Nevertheless, it is possible to give a compact analytic description of this foliation near the critical point C_*, K_*, R_* of the foliation. At this point vanish both k^2 and the first derivative $\frac{dk^2}{dR}$. The integrals X and Y become divergent at the critical point, but the right-hand side of (34) is finite everywhere and vanishes at the critical point. Define

$$\Delta = \frac{1}{3}K^2R_t^3 - \frac{3C^2}{R_t^3} + R_t, \quad \kappa = \frac{2}{3}K^2R_t^2 + 12\frac{C^2}{R_t^4}. \tag{35}$$

Now assume that K, C, R_t are close to critical values K_*, C_*, R_*. One can show after a lengthy analysis and a number of careful estimates, that

$$\begin{aligned} X &\approx -\sqrt{2\kappa}\frac{R_t}{\Delta^2}, \\ Y &\approx -\sqrt{2\kappa}\frac{R_t^4}{\Delta^2} + 3\sqrt{\frac{2}{\kappa}}\frac{R_t^3}{\Delta} + \left(\frac{\pi}{2} - 1\right)\frac{72C^2}{R_t\Delta\kappa^{3/2}}. \end{aligned} \tag{36}$$

The calculation is completely analogous to that of Section VII in [2] and it is sketched in the appendix. Define $\delta \equiv R_t - R_*$ and $\epsilon \equiv C_* - C$. Moreover, let $\delta K \equiv K_t - K_*$ and $\frac{\delta K}{\delta} \to 0$ as R_t tends to R_*. Then (up to terms of lower order) $\Delta \approx \delta\kappa$ and $\epsilon \approx -\frac{B}{A}\delta^2$. Here $A = \frac{R_*^2}{2}\kappa_*$, $B = -2C_* + \frac{2}{3}K_*R_*^3$ and κ_* is the value of κ at the critical point. The insertion of the above information into Eqs. (34) and (36) yields the differential equation

$$\frac{d}{dt}\left(\epsilon - \frac{1}{3}\delta K R_*^3\right) \approx -\frac{|B|\kappa^{1/2}}{\sqrt{2}R_*^3}\left(\epsilon - \frac{1}{3}\delta K R_*^3\right). \tag{37}$$

Define $\Gamma = \frac{|B|\kappa^{1/2}}{\sqrt{2}R_*^3}$. The asymptotic behavior of the lapse function near the critical point follows from (33), the analysis of the decaying of k^2 near R_* and (37). One obtains

$$N = N_0 e^{-t\Gamma/2}. \tag{38}$$

In the particular case of the trace K being independent of time, the estimate (38) coincides with the result derived earlier in [2].

VII. CONCLUSIONS

We consider a family of spherical three-dimensional spacelike slices embedded in the Schwarzschild solution. The mean curvature is constant on each slice but can change from slice to slice. One describes how the slices are stacked by defining the lapse function, that quantifies distance along the normal as one goes from slice to slice. We write down a simple expression for the lapse of any such slicing. This allows us to glue a patch of a Schwarzschild spacetime with a CMC slicing to a patch that is maximally sliced. It is easy to identify those slicings where the lapse is everywhere positive. The slices do not cross so one has a foliation. There is a barrier that prevents the mean curvature from becoming large, and we show how to avoid this so as to construct a foliation where the mean curvature runs all the way from zero to infinity. No foliation exists where the mean curvature goes from minus to plus infinity. However, if we consider the slicing where each slice passes through the bifurcation sphere, the point where $R = 2M$, we almost get a foliation because the lapse only vanishes at this one point and is positive everywhere else, while the mean curvature does run from minus to plus infinity. There exist symmetric foliations of the extended Schwarzschild spacetime. They degenerate at a critical point. We show that the lapse function exponentially approaches zero at this critical point.

ACKNOWLEDGMENTS

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APPENDIX

Below we show how one calculates the integral X defined in the main text. Define the following functions:

$$\begin{aligned} \tilde{F}_1 &\equiv 6\frac{C^2}{r^4} + \frac{K^2r^2}{3}, \\ \frac{\tilde{F}_2}{R_t} &\equiv \frac{K^2R_t^2}{9}\left(\frac{r^2}{R_t^2} + \frac{r}{R_t} - 2\right) - \frac{C^2}{R_t^4}\left(\frac{R_t}{r} + \frac{R_t^2}{r^2} + \frac{R_t^3}{r^3} - 3\right), \\ \tilde{F}_3 &\equiv \frac{2}{9}K^2R_t^3\left(\frac{r^3}{R_t^3} - 1\right) + \frac{4C^2}{R_t^3}\left(1 - \frac{R_t^3}{r^3}\right). \end{aligned} \tag{A1}$$

The function k^2 defined in the main text can be written as follows:

$$k^2 = \left(1 - \frac{R_t}{r}\right)\left(\frac{\Delta}{R_t} + \frac{\tilde{F}_2}{R_t}\right) \tag{A2}$$

and the integral X takes the form

$$X = -2\sqrt{R_t}\int_{R_t}^{\infty}\frac{dr}{\sqrt{1-\frac{R_t}{r}}}\frac{1}{\sqrt{\Delta+\tilde{F}_2}} \times \frac{\tilde{F}_1}{(2m+\frac{2}{3}KC+\frac{2}{9}K^2r^3-\frac{4C^2}{r^3})^2}. \quad \text{(A3)}$$

The function $k(R)$ vanishes at the minimal surface, at R_t. Thus $2m+\frac{2}{3}KC = R_t+\frac{1}{9}K^2R_t^3+\frac{C^2}{R_t^3}$ and the denominator of the third factor of (A3) can be represented as

$$(\Delta+\tilde{F}_3)^2. \quad \text{(A4)}$$

Here appears the function Δ, already defined in the main text. It is now convenient to replace r in the integral X by $y \equiv \sqrt{1-\frac{R_t}{r}}$. Then X reads

$$X = -4\sqrt{R_t}\int_0^1 dy\frac{F_1}{\sqrt{\Delta+y^2F_2}(\Delta+y^2F_3)^2}. \quad \text{(A5)}$$

Here $F_i \equiv \tilde{F}_i/y^2$ for $i=2,\ 3$, and $F_1 \equiv \tilde{F}_1/(1-y^2)^2$. Now define a new variable $z \equiv y/\sqrt{\Delta}$; the integral (A5) becomes

$$X = -4\frac{\sqrt{R_t}}{\Delta^2}\int_0^{1/\sqrt{\Delta}}\frac{dzF_1(z\sqrt{\Delta})}{\sqrt{1+z^2F_2(z\sqrt{\Delta})}(1+z^2F_3(z\sqrt{\Delta}))^2}. \quad \text{(A6)}$$

One can split the integral $\int_0^{1/\sqrt{\kappa\delta}}$ into two parts: $\int_0^{1/\sqrt{\kappa\delta}} = \int_0^{1/\sqrt{10^4\kappa\delta}} + \int_{1/\sqrt{10^4\kappa\delta}}^{1/\sqrt{\kappa\delta}}$. It is easy to check that the contribution coming from the second integral of the integrand of X goes to zero as δ approaches zero. The first integral of the integrand of X in turn is well approximated by the integral

$$X = -4\frac{\sqrt{R_t}}{\Delta^2}\kappa R_t\int_0^{\infty}\frac{dz}{\sqrt{1+\frac{z^2}{2}R_t\kappa}(1+z^2R_t\kappa)^2}. \quad \text{(A7)}$$

This can be explicitly calculated, with the result displayed in the main text. Let us remark that the function $1/\Delta$ explodes to infinity at the critical point of the foliation. Thus in this limit the quantity $X\Delta^2$ becomes equal to $-\sqrt{2\kappa_*}$ and the first of Eqs. (36) appears exact.

The calculation of the other (Y) integral defined in Eq. (31) proceeds in a similar way.

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