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# Voting Rules from Random Relations

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**Abstract.** We consider a way of generating voting rules based on a random relation, the winners being alternatives that have the highest probability of being supported. We define different notions of support, such as whether an alternative dominates the other alternatives, or whether an alternative is undominated, and we consider structural assumptions on the form of the random relation, such as being acyclic, asymmetric, connex or transitive. We give sufficient conditions on the supporting function for the associated voting rule to satisfy various properties such as Pareto and monotonicity. The random generation scheme involves a parameter $p$ between zero and one. Further voting rules are obtained by tending $p$ to zero, and by tending $p$ to one, and these limiting rules satisfy a homogeneity property, and, in certain cases, Condorcet consistency. We define a language of supporting functions based on eight natural properties, and categorise the different rules that can be generated for the limiting $p$ cases.

## 1 INTRODUCTION

Aggregation of agent preferences is an important issue in multi-agent systems [26], as well as for human agents; in particular, given a finite set of alternatives $A$ and some representation of agent preferences, one may need to choose an alternative that fits best with the preferences. The use of voting rules has been suggested for this task e.g., [20, 5]. Here we consider as input a weighted binary relation $v$ that expresses a non-negative degree of preference $v(x, y)$ of alternative $x$ over alternative $y$. The output is one or more winner, i.e., best alternative. For instance, in a standard voting scenario in which each agent totally orders the alternatives, the value $v(x, y)$ can be defined as the number $n(x, y)$ of agents that prefer $x$ to $y$ (or it could be some strictly monotonic function of $n(x, y)$). The form of input information $v$ is very flexible: in a multi-agent setting, it is not always desirable to assume that each agent expresses a total order on alternatives [17]; this may not fit with the natural preference representation for the agent; in addition the agent may not want to reveal so much information about their preferences; also, $v$ could be generated by a profile of other forms of relation, such as partial orders, or total preorders; or with a weighted relation being expressed by each agent; furthermore, the representation $v$ also allows the possibility of some agents being assigned a greater degree of importance than others.

We develop in this paper a framework for aggregating multi-agent preferences, including many interesting instances (i.e., different aggregation methods), based on a novel probabilistic model; we sketch the idea in the next few paragraphs. Our approach is based on using the weighted relation $v$ to pick a random binary preference relation between alternatives. The numerical support for an alternative $x$ is the chance that the randomly picked relation $R$ (logically) supports $x$, i.e., $\Pr(R \in \mathrm{Sp}_x)$, where $\mathrm{Sp}_x$ is defined to be the set of relations that support $x$. The output is the set of winners, i.e., the set of alternatives with maximal numerical support. Many different notions of logical support are possible, leading to different definitions of $\mathrm{Sp}_x$, and thus of winner.

In the paper we show that we can achieve desirable properties of the social choice rule by assuming particular properties of the supporting function. We say that *$x$ is dominating in relation $R$* if $R$ contains the set $\mathrm{O}_x = \{(x, y) : y \neq x\}$, so that $x$ is preferred to every other alternative with respect to preference relation $R$. We say that the supporting function Sp satisfies the property Opt if $\mathrm{Sp}_x$ only contains relations $R$ in which $x$ is dominating. In this case, $R$ supports $x$ only if $x$ is dominating in relation $R$. We say that Sp satisfies property Ud if $R \in \mathrm{Sp}_x$ implies $x$ is undominated in $R$, i.e., $R \cap \mathrm{D}_x = \emptyset$, where $\mathrm{D}_x = \{(y, x) : y \neq x\}$, so no alternative dominates $x$. We consider sufficient conditions for desirable properties on the voting rule. In particular, if Sp satisfies both Opt and Ud then we show that the voting rule satisfies natural monotonicity and Pareto properties. This therefore gives a method for generating a large family of voting (and other aggregation) rules that have some good properties.

The random generation method for relation $R$ involves an input parameter $p$ between 0 and 1. If $v(x, y) = 1$ then the chance that random relation $R$ does not contain the pair $(x, y)$ is $1 - p$. More generally, the chance that $(x, y) \notin R$ is equal to $(1 - p)^{v(x,y)}$. If $v(x, y) = K$ then one can imagine $K$ independent Boolean random variables each with chance $p$ of being true; the chance that $R$ contains the pair $(x, y)$ is the chance that any of $K$ random variables is true; thus there are $K = v(x, y)$ independent chances to ensure that $(x, y)$ is in the random relation, each with a probability $p$ of succeeding.

A way to ensure a homogeneity property (in which a linear rescaling of the input $v$ makes no difference) is to consider the result of tending $p$ to either 1 or 0. We show that the set of winners is still always non-empty and that we obtain somewhat simpler structures determining the voting rules.

As well as properties Opt and Ud we consider a weaker form TOpt of property Opt, (relating to whether $x$ is dominating in the transitive closure of $R$) and OOpt, which means that, for $R \in \mathrm{Sp}_x$, $R$ only contains elements of the form $(x, z)$, i.e., $R \subseteq \mathrm{O}_x$. We also consider structural properties that restrict the form of the relation: asymmetry, acyclicity, connex, and transitivity properties. We consider a simple language $\mathfrak{L}$ of logical support, based on these eight properties, with a supporting function being generated by a subset of the eight properties. We completely characterise the voting rules for the language, for the $p \to 0$ case, and for the $p \to 1$ case in which $v$ is non-zero, leading to seven different voting rules in each case: see Theorem 3 and Theorem 4. We show, in particular, that the $p \to 1$ cases lead to a number of well-known voting rules: Borda, the Kemeny rule, Tideman's rule, and maximin.

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Section 2 defines the framework, with Section 3 giving some general properties of the generated social choice rules. Section 4 considers the limit cases as $p$ tends to 0 or 1. Sections 5 and 6 include the classification theorems of the rules generated by the language $\mathfrak{L}$ for the $p \to 0$ and $p \to 1$ cases, respectively. Section 7 discusses related work, with Section 8 concluding.

A short version of this paper appeared as an extended abstract [24].

## 2 RULES FROM RANDOM RELATIONS

In this section we define the formalism that takes the input weighted relation $v$ and generates a set of winners. The other parameters are a number $p \in (0, 1)$, which is used in the picking of the random relation, and, for each alternative $x$, a set of relations $\mathrm{Sp}_x$ that support $x$. The winners are the alternatives with highest chance of being supported. We define some natural properties that can be used to generate $\mathrm{Sp}_x$, and we show that for certain very special cases of supporting sets $\mathrm{Sp}_x$, the value of $p$ does not affect the winners.

### 2.1 Random relations on alternatives

The set $A$ represents a finite set of alternatives. Define $\Delta = A \times A \setminus I = \{(x, y) : x, y \in A, x \neq y\}$, where $(x, y) \in I \iff x = y$. Thus, a subset of $\Delta$ is an irreflexive binary relation on alternatives. We define $\mathcal{V}$ to be the set of all functions $v$ from $\Delta$ to the non-negative reals. An element $v$ of $\mathcal{V}$ is intended to represent some degree of preference for alternative $x$ over alternative $y$. For instance, in a voting scenario, it could represent the number of voters preferring $x$ to $y$, in which case the $v$ will (as well as having certain other properties) be *balanced* i.e., for all $(x, y), (w, z) \in \Delta$, $v(x, y) + v(y, x) = v(w, z) + v(z, w)$.

We generate a random irreflexive binary relation $R$, based on parameter $p \in (0, 1)$, as follows. For each $(x, y) \in \Delta$ we (independently) omit $(x, y)$ from $R$ with chance $(1 - p)^{v(x,y)}$, so the probability that $R$ contains $(x, y)$ equals $1 - (1 - p)^{v(x,y)}$. Based on this, the chance $Pr_p^v(\{R\})$ that the randomly chosen relation is equal to a particular $R$ ($\subseteq \Delta$) is defined as follows:

$$Pr_p^v(\{R\}) = \prod_{(x,y)\in R} (1 - (1 - p)^{v(x,y)}) \times \prod_{(x,y)\in \Delta\setminus R} (1 - p)^{v(x,y)}.$$

Thus if $v(x, y)$ means the number of voters preferring $x$ to $y$, then each such vote gives (independently) a chance $p$ of ensuring that $R$ contains $(x, y)$. Let $\Omega^v$ be the set of pairs $(x, y)$ in $\Delta$ for which $v(x, y)$ is non-zero, i.e., $\{(x, y) : x, y \in A, x \neq y, v(x, y) \neq 0\}$. If $R \not\subseteq \Omega^v$ then $Pr_p^v(\{R\}) = 0$: as one would expect, if $v(x, y) = 0$ then there is zero probability of picking a random $R$ containing $(x, y)$.

**Example 1** *Consider the set of alternatives $A = \{a, b, c, d\}$ and $v \in \mathcal{V}$ represented by the following table, with e.g., $v(a, b) = 5$; $v$ may, for example, arise from a profile with eight voters: two voters with preference order $a > b > c > d$, and three voters with each of $a > b > d > c$ and $c > b > d > a$, so that then $v(x, y)$ is the number of voters preferring $x$ to $y$, for different alternatives $x$ and $y$.*

| $v(x,y)$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | – | 5 | 5 | 5 |
| $b$ | 3 | – | 5 | 8 |
| $c$ | 3 | 3 | – | 5 |
| $d$ | 3 | 0 | 3 | – |

*Let $R$ equal $\mathrm{O}_b$, which is defined to be $\{(b, a), (b, c), (b, d)\}$. Let $q = 1 - p$ and let $r$ equal $\sum_{(x,y)\in\Delta\setminus \mathrm{O}_b} v(x, y) = 32$. Then $Pr_p^v(\{R\})$ equals $(1 - q^{v(b,a)})(1 - q^{v(b,c)})(1 - q^{v(b,d)})q^r$, that is, $(1 - q^3)(1 - q^5)(1 - q^8)q^{32}$.* ∎

The following simple technical result implies that increasing $p$ corresponds to a scaling up of $v$ (since $\lambda > 1 \iff p' > p$):

**Proposition 1** *Let $p, p' \in (0, 1)$, let $v \in \mathcal{V}$, and let $v' = \lambda v$ for some real $\lambda > 0$. If $(1 - p') = (1 - p)^\lambda$ then for all $R \subseteq \Delta$, $Pr_p^{v'}(\{R\}) = Pr_{p'}^v(\{R\})$.*

### 2.2 Supporting functions

We assume that for each alternative $x \in A$, we have a rule for determining whether or not relation $R$ supports $x$, and we define $\mathrm{Sp}_x$ to be the set of all relations $R$ that support $x$. Thus, for each $x \in A$, $\mathrm{Sp}_x$ is a set of subsets of $\Delta$. We define $\mathcal{SP}$ to be the set of supporting functions [over $A$], i.e., the set of functions Sp that associate with each alternative $x$ a non-empty set $\mathrm{Sp}_x$ of irreflexive binary relations on $A$ (so that $\emptyset \neq \mathrm{Sp}_x \subseteq 2^\Delta$). There are lots of different ways of defining supporting functions. We give some basic instances below.

We first define for alternative $x \in A$:

- $\mathrm{D}_x = \{(y, x) : y \in A \setminus \{x\}\}$, the set of pairs in which $x$ is dominated; and
- $\mathrm{O}_x = \{(x, y) : y \in A \setminus \{x\}\}$, the set of pairs in which $x$ is dominating.

**Basic supporting functions** Ud**,** Opt**,** TOpt **and** OOpt**:** For alternative $x \in A$, we define:

- $\mathrm{Ud}_x = \{R \subseteq \Delta : R \cap \mathrm{D}_x = \emptyset\}$, i.e., the set of all irreflexive relations $R$ in which $x$ is undominated.
- $\mathrm{Opt}_x = \{R \subseteq \Delta : R \supseteq \mathrm{O}_x\}$, the set of $R$ in which $x$ directly dominates all other alternatives.
- $\mathrm{TOpt}_x = \{R \subseteq \Delta : \mathrm{Tr}(R) \supseteq \mathrm{O}_x\}$, which contains all $R$ whose transitive closure contains $\mathrm{O}_x$. Thus, $R \in \mathrm{TOpt}_x$ if and only if every other alternative is reachable from $x$ w.r.t. $R$, viewing $R$ as a directed graph on alternatives.
- $\mathrm{OOpt}_x = \{R \subseteq \Delta : R \subseteq \mathrm{O}_x\}$, which only contains subsets of $\mathrm{O}_x$.

### 2.3 Defining winners $W_p^{\mathrm{Sp}}(v)$, given $p \in (0, 1)$

Given $v \in \mathcal{V}$, a supporting function $\mathrm{Sp} \in \mathcal{SP}$, and a value $p \in (0, 1)$, we consider, for each alternative $x$, the probability $\Pr_p^v(\mathrm{Sp}_x)$ of $\mathrm{Sp}_x$, i.e., $\sum_{R\in \mathrm{Sp}_x} Pr_p^v(\{R\})$. This generates a social choice rule in the obvious way: we define the associated set of winners, $W_p^{\mathrm{Sp}}(v)$, to be the set of alternatives $x$ that maximise $\Pr_p^v(\mathrm{Sp}_x)$, so that $x \in W_p^{\mathrm{Sp}}(v)$ if and only if for all alternatives $y$, $\Pr_p^v(\mathrm{Sp}_x) \geq \Pr_p^v(\mathrm{Sp}_y)$.

**Example 1 continued:** Suppose we define $\mathrm{Sp}_x$ to be $\mathrm{Ud}_x$ for $x \in A$; with this definition, relation $R$ supports $b$ if and only if $b$ is not dominated in $R$, i.e., there exists no pair of the form $(x, b)$ in $R$. In other words, $R \in \mathrm{Sp}_b$ if and only if $R \subseteq \Delta \setminus \mathrm{D}_b$, where $\mathrm{D}_b = \{(a, b), (c, b), (d, b)\}$. Thus, $Pr_p^v(\mathrm{Ud}_b) = \sum_{R\subseteq\Delta\setminus \mathrm{D}_b} Pr_p^v(\{R\})$, which can be shown to be equal to $\prod_{(x,y)\in \mathrm{D}_b} q^{v(x,y)} = q^s$ where $s = \sum_{(x,y)\in \mathrm{D}_b} v(x, y) = 8$. Similarly, $Pr_p^v(\mathrm{Ud}_a) = q^9$, $Pr_p^v(\mathrm{Ud}_c) = q^{13}$, and $Pr_p^v(\mathrm{Ud}_d) = q^{18}$. This shows that whatever value is chosen for $p \in (0,1)$, $b$ is the unique winner, since it has the highest probability of being supported: $Pr_p^v(\mathrm{Ud}_b) > Pr_p^v(\mathrm{Ud}_x)$ for $x \neq b$.

Suppose we instead define $\mathrm{Sp}_x$ to be $\mathrm{Ud}_x \cap \mathrm{Opt}_x$ for all $x \in A$. Now $\mathrm{Sp}_x$ contains all relations $R$ in which (i) $x$ is undominated and (ii) $x$ dominates the other alternatives, i.e., all $R$ such that $\mathrm{O}_x \subseteq R \subseteq \Delta \setminus \mathrm{D}_x$. It can be shown that $Pr_p^v(\mathrm{Sp}_b) = (1-q^3)(1-q^5)(1-q^8)q^8$ and $Pr_p^v(\mathrm{Sp}_a) = (1-q^5)^3 q^9$. With e.g., $p = 0.5$, this makes $b$ the unique winner; actually, $b$ is the unique winner unless $p$ is very small (less than around 0.0693), when $a$ becomes the winner.

In fact, Proposition 2 below implies that defining $\mathrm{Sp}_x = \mathrm{Ud}_x$ leads to the Borda voting rule for any value of $p$. We will see later (see the discussion of case (v) of Theorem 4) that for $p$ close to 1, defining $\mathrm{Sp}_x = \mathrm{Ud}_x \cap \mathrm{Opt}_x$ still generates the Borda voting rule. Varying the definition of $\mathrm{Sp}_x$ leads to other voting rules; with $p$ tending to 1, $\mathrm{Sp}_x = \mathrm{Opt}_x$ generates the maximin rule; if we add the condition that $R$ is asymmetric we obtain the Tideman rule, and adding either acyclicity or transitivity conditions leads to the Kemeny rule (see Section 6). ■

## 2.4 Generating supporting functions

**Basic structural properties** As**,** Ac**,** C **and** T**:** Along with the basic supporting functions Ud, Opt, TOpt and OOpt defined above, we can also consider properties that restrict the form of the relations $R$. A *structural property* is then just a set of subsets of $\Delta$. We define the structural properties As (asymmetry), Ac (acyclicity), C (connex) and T (transitivity) as follows, where $R$ is an arbitrary subset of $\Delta$ (i.e., an arbitrary irreflexive relation on $A$).

- $R \in \mathrm{As} \iff [(x,y) \in R \Rightarrow (y,x) \notin R]$.
- $R \in \mathrm{Ac} \iff R$ is acyclic.
- $R \in \mathrm{C} \iff \forall (x,y) \in \Delta$, $(x,y) \in R$ or $(y,x) \in R$.
- $R \in \mathrm{T} \iff (x,y),(y,z) \in R$ and $x \neq z \Rightarrow (x,z) \in R$.

**Generation of** Sp **from a set of properties:** The basic supporting functions can be treated as properties of a supporting function: for $\mathrm{Sp} \in \mathcal{SP}$, we say that Sp *satisfies supporting function* $Z$ if for all $x \in A$, $\mathrm{Sp}_x \subseteq Z_x$. Similarly, Sp *satisfies structural property* $Z$ if for all $x \in A$, $\mathrm{Sp}_x \subseteq Z$.

Let $\Gamma$ be the union of a non-empty set of supporting functions $\Gamma_1$ and a set of structural properties $\Gamma_2$. We say that supporting function Sp is *generated by* $\Gamma$ ($\mathrm{Sp} = \mathrm{Sp}(\Gamma)$) if for all $x \in A$, $\mathrm{Sp}_x = \bigcap_{Z \in \Gamma_1} Z_x \cap \bigcap_{Z \in \Gamma_2} Z$. The definition implies that $\mathrm{Sp}(\Gamma)$ satisfies each element of $\Gamma$. For $x \in A$ we abbreviate $(\mathrm{Sp}(\Gamma))_x$ to $\mathrm{Sp}_x(\Gamma)$. For example, consider Sp generated by $\{\mathrm{Opt}, \mathrm{C}, \mathrm{Ac}\}$, i.e., by the supporting function Opt and the pair of structural properties C and Ac; then $\mathrm{Sp}_x = \mathrm{Sp}_x(\{\mathrm{Opt}, \mathrm{C}, \mathrm{Ac}\})$ is the set of strict total orders on $A$ in which $x$ is the top element, since relation $R$ is in $\mathrm{Sp}_x(\{\mathrm{Opt}, \mathrm{C}, \mathrm{Ac}\})$ if and only if it is in $\mathrm{Opt}_x \cap \mathrm{C} \cap \mathrm{Ac}$, where $R \in \mathrm{C} \cap \mathrm{Ac}$ means that it is a strict total order, and $R \in \mathrm{Opt}_x$ implies that $x$ dominates the other alternatives. Below, especially in Sections 5 and 6, we analyse rules generated by the basic supporting functions (from Section 2.2) and by the basic structural properties defined above.

## 2.5 Two rules that are independent of $p$

For most supporting functions Sp, the choice of $p$ affects the set of winners, often very considerably.[2] However, here we show that choosing Sp to be either Ud or OOpt leads to rules that are independent of $p$, both being extensions of the Borda voting rule [7, 28]. They are both instances of cases in which a set $\mathrm{Sp}_x$ consists of all relations $R$ that are subsets of some set $S_x$. If $\mathrm{Sp} = \mathrm{Ud}$ then $S_x = \Delta \setminus \mathrm{D}_x$ and the winners are those $x$ that minimise $v^+(\mathrm{D}_x) = \sum_{z \neq x} v(z,x)$, which is the sum of votes against $x$. If we define Sp as OOpt then $S_x = \mathrm{O}_x$ and the winners are those elements maximising $v^+(\mathrm{O}_z) = \sum_{z \neq x} v(x,z)$, the sum of votes for $x$. (For balanced $v$, the two rules are equivalent.)

Note that in such cases the best alternatives $x$ are those for which there exists $R \in \mathrm{Sp}_x$ that maximises $v^+(R)$ among $R \in \bigcup_{z \in A} \mathrm{Sp}_z$, where $v^+(R) = \sum_{(x,y) \in R} v(x,y)$. This holds more generally for rules based on tending $p$ to one, explored in Sections 4 and 6.[3]

**Proposition 2** *Suppose that for each $x \in A$, $\mathrm{Sp}_x$ is of the form $\{R : R \subseteq S_x\}$ for some $S_x \subseteq \Delta$. Then, $Pr_p^v(\mathrm{Sp}_x) = (1-p)^{v^+(\Delta)} \times (1-p)^{-v^+(S_x)}$, and $x \in W_p^{\mathrm{Sp}}(v)$ if and only if $x \in argmax_z v^+(S_z)$. In particular, we have $x \in W_p^{\mathrm{Ud}}(v)$ if and only if $x \in argmin_z v^+(\mathrm{D}_z)$, and $x \in W_p^{\mathrm{OOpt}}(v)$ if and only if $x \in argmax_z v^+(\mathrm{O}_z)$.*

## 2.6 $\mathcal{V}$-rules and voting rules

Define a $\mathcal{V}$-rule to be a function $W$ from $\mathcal{V}$ to $2^A \setminus \{\emptyset\}$. Thus, for any $\mathrm{Sp} \in \mathcal{SP}$ and $p \in (0,1)$, $W_p^{\mathrm{Sp}}$ (i.e., the function $v \mapsto W_p^{\mathrm{Sp}}(v)$) is a $\mathcal{V}$-rule.

We can generate a voting rule from a $\mathcal{V}$-rule (and in particular, from $W_p^{\mathrm{Sp}}$) in different ways. Let us define a *voting rule* (over set of alternatives $A$) to be a function from the set $\mathcal{P}$ of profiles over $A$ to $2^A \setminus \{\emptyset\}$, where a profile over $A$ is finite sequence of total orders over $A$. For profile $\pi \in \mathcal{P}$, define $\pi_* \in \mathcal{V}$ by $\pi_*(x,y)$ equalling the number of voters who prefer $x$ to $y$. Given a $\mathcal{V}$-rule $W$, the function $\pi \mapsto W(\pi_*)$ is a voting rule. We say that $v \in \mathcal{V}$ is *profile-based* if there exists a profile $\pi \in \mathcal{P}$ such that $v = \pi_*$. Clearly if $v$ is profile-based then it is balanced.

Other transformations from profiles to $\mathcal{V}$ lead to other voting rules. In particular, for real $\epsilon > 0$, define $\pi_\epsilon$ to be $\pi_* + \epsilon$, so that $\forall (x,y) \in \Delta$, $\pi_\epsilon(x,y) = \pi_*(x,y) + \epsilon$. More generally, for strictly monotonic $f$ on the non-negative reals we define $\pi_f \in \mathcal{V}$ by $\pi_f(x,y) = f(\pi_*(x,y))$. Of particular interest is the case in which $f(0) > 0$, since then $\pi_f$, like $\pi_\epsilon$, is a non-zero element of $\mathcal{V}$, i.e., $\pi_f(x,y) > 0$ for all $(x,y) \in \Delta$.

# 3 GENERAL PROPERTIES OF $\mathcal{V}$-RULE $W_p^{\mathrm{Sp}}$

Various properties of voting rules have been studied, as ways of judging how intuitive a particular voting rule is, e.g., [28, 19, 15, 21]. We consider versions of these properties for $\mathcal{V}$-rules. We discuss neutrality, homogeneity, two versions of Pareto properties, and a monotonicity property. (Anonymity is clearly not an issue here, since $\mathcal{V}$ makes no reference to individual voters.)

To address neutrality, we consider the function $\tau_{x,y}$ that switches labels $x$ and $y$, where $x$ and $y$ are two different elements of $A$.

For $v \in \mathcal{V}$, and $x \neq y \in A$, we define $v^{x,y}$ to be $v$ with the labels of $x$ and $y$ exchanged, i.e., $\tau_{x,y}$ on pairs followed by $v$. We say that $\mathcal{V}$-rule $W$ is *neutral* if for all $v \in \mathcal{V}$ and $(x,y) \in \Delta$, $W(v^{x,y})$ is equal to $\tau_{x,y}(W(v))$, i.e., $W(v)$ with the labels of $x$ and $y$ exchanged.

[2] This is illustrated by the contrasting results in Sections 5 and 6.

[3] For the proofs, making use of many auxiliary results, see [25].

We say that supporting function $\mathrm{Sp} \in \mathcal{SP}$ is *neutral* if for all different $x, y \in A$, we have $R \in \mathrm{Sp}_y \iff \tau_{x,y}(R) \in \mathrm{Sp}_x$. Being neutral is a very natural property for Sp; in particular, Sp is neutral if it is generated by any of the basic supporting functions and structural properties (see Sections 2.2 and 2.4); in particular, any element of the language $\mathfrak{L}$ considered later is neutral.

**Proposition 3 (Neutrality)** *Let* Sp *be a neutral element of* $\mathcal{SP}$. *Then for any* $p \in (0,1)$, $W_p^{\mathrm{Sp}}$ *is a neutral* $\mathcal{V}$*-rule.*

**Homogeneity:** We say that $\mathcal{V}$-rule $W$ is *homogeneous* if for any real $\lambda > 0$ and any $v \in \mathcal{V}$, $W(\lambda v) = W(v)$. Proposition 1 implies that $W_p^{\mathrm{Sp}}$ being homogeneous is equivalent to the condition that for all $p' \in (0,1)$ and all $v \in \mathcal{V}$, $W_{p'}^{\mathrm{Sp}}(v) = W_p^{\mathrm{Sp}}(v)$, i.e., independence of the rule with respect to $p$ (which also means that $W_p^{\mathrm{Sp}}$ is homogeneous for one value of $p$ in $(0,1)$ if and only if it holds for all values of $p$). Proposition 2 hence gives examples of homogeneous rules, since it relates to rules that are independent of the value of $p$.

We now consider properties that relate to the Pareto Principle for voting rules, stating that a Pareto dominated alternative is not a winner (where $x$ Pareto dominates $y$ if every agent agrees that $x$ is better than $y$).

Firstly, it is convenient to define a notion of null elements of $\mathcal{V}$, in which no alternative is supported.

**Definition of $\mathrm{Sp}_x^v$, and null elements of $\mathcal{V}$:** We abbreviate $\mathrm{Sp}_x \cap 2^{\Omega^v}$ to $\mathrm{Sp}_x^v$. Given $\mathrm{Sp} \in \mathcal{SP}$, we say that $v$ is *null for* Sp if for all $x \in A$, $\mathrm{Sp}_x^v = \emptyset$, i.e., if $2^{\Omega^v} \cap \bigcup_{x \in A} \mathrm{Sp}_x = \emptyset$; otherwise, we say that $v$ is *non-null for* Sp. If $v$ is null for Sp then each $\mathrm{Sp}_x$ has zero probability ($\mathrm{Pr}_p^v(\mathrm{Sp}_x) = 0$ for all alternatives $x$, since $Pr_p^v$ is zero outside $2^{\Omega^v}$) so trivially every alternative is a winner, i.e., $W_p^{\mathrm{Sp}}(v) = A$ (for any $p \in (0,1)$).

The constraints chosen on the $\mathcal{V}$-rule in Propositions 4 and 5 relate to the fact that if $x$ Pareto-dominates $y$ according to profile $\pi$ then $\pi_*(x,y) > \pi_*(y,x) = 0$, and for all $z \in A \setminus \{x,y\}$, $\pi_*(x,z) \geq \pi_*(y,z)$ and $\pi_*(z,x) \leq \pi_*(z,y)$ (where $\pi_*$ is the $\mathcal{V}$-rule defined in Section 2.6, with $\pi_*(x,y)$ being the number of voters who prefer $x$ to $y$).

**Proposition 4 (Pareto-1)** *Suppose that* Sp *satisfies* Opt *(i.e., for all* $x \in A$, $\mathrm{Sp}_x \subseteq \mathrm{Opt}_x$*) and consider any* $y \in A$ *and* $v \in \mathcal{V}$. *Assume that for some* $x \in A \setminus \{y\}$, $v(y,x) = 0$. *Then* $\mathrm{Sp}_y^v$ *is empty, and so,* $\mathrm{Pr}_p^v(\mathrm{Sp}_y) = 0$. *Hence, if* $v$ *is non-null for* Sp *then* $y \notin W_p^{\mathrm{Sp}}(v)$.

**Proposition 5 (Pareto-2)** *Suppose that* Sp *is neutral and satisfies* Opt *and* Ud, *and let* $x \neq y$ *be elements of* $A$. *Assume that* $v$ *satisfies the following properties:* $v(x,y) > v(y,x)$, *and for all* $z \in A \setminus \{x,y\}$, $v(x,z) \geq v(y,z)$ *and* $v(z,x) \leq v(z,y)$. *Then, either* $\mathrm{Sp}_y^v$ *is empty or for any* $p \in (0,1)$, $\mathrm{Pr}_p^v(\mathrm{Sp}_x) > \mathrm{Pr}_p^v(\mathrm{Sp}_y)$. *Hence, if* $v$ *is non-null for* Sp *then* $y \notin W_p^{\mathrm{Sp}}(v)$, *and thus,* $y$ *is not a winner.*

We give simple sufficient conditions for a natural monotonicity property: if Sp satisfies Opt and Ud and $x$ is a winner, and we only increase votes for $x$ and only decrease votes against $x$, then $x$ remains a winner:

**Proposition 6 (Monotonicity)** *Let* $x \in A$ *and assume that* Sp *satisfies* Opt *and* Ud, *and that* $v$ *and* $v'$ *are such that for all* $z \neq x$, $v'(x,z) \geq v(x,z)$ *and* $v'(z,x) \leq v(z,x)$, *and* $v$ *and* $v'$ *are equal on all other elements of* $\Delta$. *Let* $y \in A \setminus \{x\}$. *If* $\mathrm{Pr}_p^v(\mathrm{Sp}_x) > \mathrm{Pr}_p^v(\mathrm{Sp}_y)$ *then* $\mathrm{Pr}_p^{v'}(\mathrm{Sp}_x) > \mathrm{Pr}_p^{v'}(\mathrm{Sp}_y)$*; and, if* $\mathrm{Pr}_p^v(\mathrm{Sp}_x) \geq \mathrm{Pr}_p^v(\mathrm{Sp}_y)$ *then* $\mathrm{Pr}_p^{v'}(\mathrm{Sp}_x) \geq \mathrm{Pr}_p^{v'}(\mathrm{Sp}_y)$. *Assume that* $x \in W_p^{\mathrm{Sp}}(v)$. *Then* $x \in W_p^{\mathrm{Sp}}(v')$, *and if* $v'$ *is not null for* Sp, *we have* $W_p^{\mathrm{Sp}}(v') \subseteq W_p^{\mathrm{Sp}}(v)$.

Let us say that voting rule $U$ (on $A$) satisfies the *Pareto property* if $x \notin U(\pi)$ (i.e., $x$ is not a winner) whenever profile $\pi$ ($\in \mathcal{P}$) and alternative $x$ ($\in A$) are such that there exists $y$ with $\pi_*(x,y) = 0$ (all voters prefer $y$ to $x$). We say that $U$ satisfies *monotonicity* if $x \in U(\pi)$ implies $x \in U(\pi') \subseteq U(\pi)$ whenever $\pi, \pi' \in \mathcal{P}$ and $x \in A$ are such that $\pi'$ is equal to $\pi$ on all voters except one in which the position of $x$ is improved without changing the relative positions of other alternatives.

The following result shows that we can obtain, with our framework, voting rules that satisfy monotonicity and the Pareto property, in a very wide range of different ways: by choosing any value $p \in (0,1)$, and any supporting function Sp that satisfies the two basic properties Opt and Ud, and by choosing any strictly monotonic function $f$ in the generation of the $\mathcal{V}$-rule from the profile $\pi$ (see Section 2.6). Condition $f(0) > 0$, ensuring that $\pi_f$ is always non-null, avoids exceptions to do with null cases (cf. Propositions 4, 5 and 6).

**Proposition 7** *Consider any* $p \in (0,1)$, *and any strictly monotonic function* $f$ *on the non-negative reals with* $f(0) > 0$, *and assume that* $\mathrm{Sp} \in \mathcal{SP}$ *satisfies* Opt *and* Ud. *The voting rule* $\pi \mapsto W_p^{\mathrm{Sp}}(\pi_f)$ *satisfies monotonicity, and, if* Sp *is neutral, it satisfies the Pareto property.*

## 4 LIMIT CASES WHEN $p \to 1$ OR $p \to 0$

As well as considering fixed value of $p \in (0,1)$, we can also consider the effect of tending $p$ to zero or one. An advantage of this is that it leads to rules that are homogeneous, in that multiplying $v$ by a positive scalar will not affect the result (this can be seen as a consequence of the property expressed by Proposition 1). There are different ways of generating such limiting functions. A first idea might be to consider, for alternative $x$, the limit of $\mathrm{Pr}_p^v(\mathrm{Sp}_x)$ as $p$ tends to 1 (or 0). However, it can easily happen that e.g., for all $x \in A$, $\lim_{p\to 1} \mathrm{Pr}_p^v(\mathrm{Sp}_x) = 0$, which will lead to a trivial social choice rule which excludes no alternatives. An alternative is to consider if $\lim_{p\to 1} \frac{\mathrm{Pr}_p^v(\mathrm{Sp}_x)}{\mathrm{Pr}_p^v(\mathrm{Sp}_y)} \geq 1$ for all alternatives $y$. Although this is often reasonable, there are cases where it can be less decisive than one would like. Instead, to compare alternatives $x$ and $y$ we use $Q_{\mathrm{Sp}}^{v,p}(x,y)$, defined below. We define, for $x \neq y$,

$$Q_{\mathrm{Sp}}^{v,p}(x,y) = \frac{Pr_p^v(\mathrm{Sp}_x \setminus \mathrm{Sp}_y)}{Pr_p^v(\mathrm{Sp}_y \setminus \mathrm{Sp}_x)} = \frac{\sum_{R \in \mathrm{Sp}_x \setminus \mathrm{Sp}_y} Pr_p^v(R)}{\sum_{S \in \mathrm{Sp}_y \setminus \mathrm{Sp}_x} Pr_p^v(S)}.$$

We have that $Q_{\mathrm{Sp}}^{v,p}(x,y) = 1/Q_{\mathrm{Sp}}^{v,p}(y,x)$. The ratio is defined to be $\infty$ if the denominator is zero and the numerator is non-zero; if both numerator and denominator are zero, then $Q_{\mathrm{Sp}}^{v,p}(x,y)$ is defined to be 1. This happens when $Pr_p^v(\mathrm{Sp}_x \setminus \mathrm{Sp}_y) = Pr_p^v(\mathrm{Sp}_y \setminus \mathrm{Sp}_x) = 0$.

The winners set $W_p^{\mathrm{Sp}}$ (see Section 2.3) can also be expressed in terms of the function $Q_{\mathrm{Sp}}^{v,p}(x,y)$: for any $p \in (0,1)$, $x \in W_p^{\mathrm{Sp}}(v)$ if and only if for all alternatives $y$, $Q_{\mathrm{Sp}}^{v,p}(x,y) \geq 1$.

### 4.1 Winners When $p \to 1$ and $p \to 0$

Consider any given supporting function Sp and any weighted relation $v \in \mathcal{V}$. We define $\mathcal{V}$-rules $W_{\to 1}^{\mathrm{Sp}}(v)$, $\overline{W}_{\to 1}^{\mathrm{Sp}}(v)$ and $W_{\to 0}^{\mathrm{Sp}}(v)$ as follows, where Sp is an arbitrary supporting function, and $v$ is a weighted relation; $x \in A$ is an alternative.

- $x \in W_{\to 1}^{\mathrm{Sp}}(v)$ if and only if for all $y \in A \setminus \{x\}$, $\lim_{p \to 1} Q_{\mathrm{Sp}}^{v,p}(x,y) \geq 1$. Alternative $x$ is then said to be a *strong* $(p \to 1)$*-winner* [given Sp and $v$].

- $x \in \overline{W}^{\mathrm{Sp}}_{\to 1}(v)$ if and only if for all $y \in A \setminus \{x\}$, $\lim_{p \to 1} Q^{v,p}_{\mathrm{Sp}}(x,y) > 0$. Then $x$ is a *weak $(p \to 1)$-winner*. $\overline{W}^{\mathrm{Sp}}_{\to 1}(v)$ is a somewhat less decisive social choice rule than $W^{\mathrm{Sp}}_{\to 1}(v)$; clearly, we always have $W^{\mathrm{Sp}}_{\to 1}(v) \subseteq \overline{W}^{\mathrm{Sp}}_{\to 1}(v)$.
- $x \in W^{\mathrm{Sp}}_{\to 0}(v)$ if and only if for all $y \in A \setminus \{x\}$, $\lim_{p \to 0} Q^{v,p}_{\mathrm{Sp}}(x,y) \geq 1$, and $x$ is said to be a *$(p \to 0)$-winner*.[4]

Naturally, the set $W^{\mathrm{Sp}}_{\to 1}(v)$ of strong $(p \to 1)$-winners is a subset of $\overline{W}^{\mathrm{Sp}}_{\to 1}(v)$, the set of weak $(p \to 1)$-winners. Effectively, the strong winners are generated by a tie-breaking over the weak winners. For examples of this, see the discussion of parts (ii) and (iii) of Theorem 4 below.

**$v^+$ and $v^\times$:** We will define some notation that allows the expression of characterisations of the different kinds of winners. For $R \subseteq \Delta$ recall that $v^+(R)$ equals $\sum_{(x,y) \in R} v(x,y)$. Informally, we will sometimes refer to $v^+(R)$ as the sum of votes for $R$. Similarly, we define $v^\times(R)$ to be the number $\prod_{(x,y) \in R} v(x,y)$.

We write $\mathrm{Sp}^v_{x,y}$ for $\mathrm{Sp}^v_x \setminus \mathrm{Sp}^v_y$, where $\mathrm{Sp}^v_x = \mathrm{Sp}_x \cap 2^{\Omega^v}$.

For the $p \to 1$ case, we define $g^v_{x,y}(\mathrm{Sp})$, abbreviated to $g_{x,y}$, to be $\max\{v^+(R) \,:\, R \in \mathrm{Sp}^v_x \setminus \mathrm{Sp}^v_y\}$, where the max of an empty set is here defined to zero. Define $h^v_{x,y}(\mathrm{Sp})$ (abbreviated to $h_{x,y}$) to be $|\{R \in \mathrm{Sp}^v_{x,y} \,:\, v^+(R) = g_{x,y}\}|$.

Relating to the $p \to 0$ case, for $\Omega \subseteq \Delta$ we define $N^v_{x,y}(\mathrm{Sp})$ (usually abbreviated to $N_{x,y}$) to be $\min\{|R| \,:\, R \in \mathrm{Sp}^v_{x,y}\}$, where the min over an empty set is here defined to be $\infty$. We define $E^v_{x,y}(\mathrm{Sp}) = \sum_{R \in \mathrm{Sp}^v_{x,y},\, |R| = N_{x,y}} v^\times(R)$.

The strong $(p \to 1)$-winners, and the $(p \to 0)$-winners, can be shown to be the undominated elements in $A$ with respect to the irreflexive relations $\succ^{v,\mathrm{Sp}}_{\to 1}$ and $\succ^{v,\mathrm{Sp}}_{\to 0}$ (respectively), defined as follows. For different $x, y \in A$,

$x \succ^{v,\mathrm{Sp}}_{\to 1} y \iff \lim_{p \to 1} Q^{v,p}_{\mathrm{Sp}}(x,y) > 1.$

$x \succ^{v,\mathrm{Sp}}_{\to 0} y \iff \lim_{p \to 0} Q^{v,p}_{\mathrm{Sp}}(x,y) > 1.$

The two theorems below characterise the winners, in terms of $g_{x,y}$ and $h_{x,y}$ for the $p \to 1$ cases, and in terms of $N_{x,y}$ and $E_{x,y}$ for the $p \to 0$ cases. Roughly speaking, the weak $(p \to 1)$-winners $x$ are those such that $\mathrm{Sp}^v_x$ contains a relation $R$ maximising $v^+(R)$; and (very roughly speaking) the $(p \to 0)$-winners $x$ are those such that $\mathrm{Sp}^v_x$ contains a relation $R$ minimising cardinality, and then maximising the sum over minimal cardinality sets $R$ of $v^\times(R)$.

**Theorem 1** *For any $v \in \mathcal{V}$ and* Sp, *relation $\succ^{v,\mathrm{Sp}}_{\to 1}$ is transitive, and sets $W^{\mathrm{Sp}}_{\to 1}(v)$ and $\overline{W}^{\mathrm{Sp}}_{\to 1}(v)$ are non-empty. $W^{\mathrm{Sp}}_{\to 1}(v)$ equals the set of alternatives that are undominated with respect to relation $\succ^{v,\mathrm{Sp}}_{\to 1}$, i.e., $x \in W^{\mathrm{Sp}}_{\to 1}(v)$ if and only if $x \in A$ and there does not exist $y \in A$ with $y \succ^{v,\mathrm{Sp}}_{\to 1} x$. For $x, y \in A$, we have $x \succ^{v,\mathrm{Sp}}_{\to 1} y \iff g_{x,y} > g_{y,x}$ or [$g_{x,y} = g_{y,x}$ and $h_{x,y} > h_{y,x}$].*

**Theorem 2** *For any $v \in \mathcal{V}$ and* Sp, *relation $\succ^{v,\mathrm{Sp}}_{\to 0}$ is transitive, and $W^{\mathrm{Sp}}_{\to 0}(v)$ is non-empty. $W^{\mathrm{Sp}}_{\to 0}(v)$ equals the set of alternatives that are undominated with respect to relation $\succ^{v,\mathrm{Sp}}_{\to 0}$. For $x, y \in A$, we have $x \succ^{v,\mathrm{Sp}}_{\to 0} y \iff N_{x,y} < N_{y,x}$ or [$N_{x,y} = N_{y,x}$ and $E_{x,y} > E_{y,x}$].*

[4] One could also define $x$ to be a weak $(p \to 0)$-winner, if for all $y \in A \setminus \{x\}$, $\lim_{p \to 0} Q^{v,p}_{\mathrm{Sp}}(x,y) > 0$. However, it turns out that this is a less interesting definition, because then the winners, although dependent on $\Omega^v$, do not otherwise depend on $v$, so that, e.g., changing a non-zero value $v(y,z)$ to another non-zero value will not change the set of winners.

**Unary supporting functions:** characterising winners is simpler if the supporting function Sp is *unary*, i.e., for any different alternatives $x$ and $y$, the best relations in $\mathrm{Sp}_x$ are not in $\mathrm{Sp}_y$. For the $p \to 1$ case, the best relations are those maximising $v^+$. For the $p \to 0$ case, similar remarks apply, but with the best relations having minimal cardinality.

## 4.2 The language $\mathfrak{L}$

As shown in Section 2.4, a set $\Gamma$ of basic properties generates a supporting function $\mathrm{Sp}(\Gamma)$, thus leading to a $\mathcal{V}$-rule. We define $\mathfrak{L}$ to be the set of all subsets $\Gamma$ of the set of properties $\{\mathrm{OOpt}, \mathrm{Opt}, \mathrm{TOpt}, \mathrm{Ud}, \mathrm{As}, \mathrm{Ac}, \mathrm{T}, \mathrm{C}\}$ that contain at least one of $\{\mathrm{OOpt}, \mathrm{Opt}, \mathrm{TOpt}, \mathrm{Ud}\}$ (else $\mathrm{Sp}_x(\Gamma)$ is independent of $x$) and such that $\Gamma$ does not include both OOpt and C (the latter pair being incompatible, which would make each $\mathrm{Sp}_x(\Gamma)$ empty). In Sections 5 and 6 we analyse this language for the limit cases.

## 4.3 Some properties for limit cases

The neutrality, Pareto and monotonicity properties for $p \in (0,1)$ (Propositions 3, 4, 5 and 6) in Section 3 extend to the $p \to 1$ and $p \to 0$ cases.

Proposition 1 implies the following:

**Proposition 8** *For any* $\mathrm{Sp} \in \mathcal{SP}$, *$\mathcal{V}$-rules $W^{\mathrm{Sp}}_{\to 1}$, $\overline{W}^{\mathrm{Sp}}_{\to 1}$ and $W^{\mathrm{Sp}}_{\to 0}$ are homogeneous.*

In addition we have a sufficient condition for the Condorcet property for the $p \to 1$ case. First, define transformation $\omega_x$, by $\omega_x(R) = (R \setminus \mathrm{D}_x) \cup \mathrm{O}_x$. Thus, $\omega_x$ turns $R$ into a relation $\omega_x(R)$ in which $x$ is undominated and dominates all other alternatives. We say that $\omega$ *respects* Sp, if for all $y \neq x$ and for all $R \in \mathrm{Sp}_y$ we have $\omega_x(R) \in \mathrm{Sp}_x$. Thus, for $\mathcal{X} \in \{\mathrm{OOpt}, \mathrm{Opt}, \mathrm{TOpt}, \mathrm{Ud}\}$, $\omega$ respects $\mathcal{X}$ if $R \in \mathcal{X}_x \Rightarrow \omega_x(R) \in \mathcal{X}_x$. For structural property $\mathcal{X}$, such as $\mathcal{X} \in \{\mathrm{Ac}, \mathrm{As}, \mathrm{T}, \mathrm{C}\}$, we say that $\omega$ *respects* $\mathcal{X}$ if $R \in \mathcal{X} \Rightarrow \omega_x(R) \in \mathcal{X}$. For many natural Sp, we have $\omega$ respects Sp; in fact $\omega$ respects $\mathcal{X}$ for each $\mathcal{X} \in \{\mathrm{Opt}, \mathrm{TOpt}, \mathrm{Ud}, \mathrm{Ac}, \mathrm{As}, \mathrm{T}, \mathrm{C}\}$, and any Sp generated by a subset of these.

The condition $v(x,y) > v(y,x)$ expresses a direct preference for $x$ over $y$. The result below roughly states that if $x$ is directly preferred to every other alternative and $\omega$ respects Sp, and that Sp satisfies the asymmetry property then $x$ is the unique weak $(p \to 1)$-winner.

**Proposition 9 (Condorcet property for $(p \to 1)$)** *Assume that $\omega$ respects* Sp *and that* Sp *satisfies* As. *Suppose $x \in A$, and for all other alternatives $y \in A \setminus \{x\}$, $v(x,y) > v(y,x)$. Then $x \in \overline{W}^{\mathrm{Sp}}_{\to 1}(v)$, i.e., $x$ is a weak $(p \to 1)$-winner. If, in addition, $v$ is non-null and for all $y \in A \setminus \{x\}$, $\mathrm{Sp}_y \cap \mathrm{Opt}_x \cap \mathrm{Ud}_x = \emptyset$ then $W^{\mathrm{Sp}}_{\to 1}(v) = \{x\}$.*

*Suppose that $\Gamma \in \mathfrak{L}$, and that $\Gamma \not\ni$ OOpt and either $\Gamma \ni$ As or $\Gamma \ni$ Ac. Then $x$ is the unique $(p \to 1)$-winner for $\Gamma$ and non-null $v$.*

The extra condition that $\mathrm{Sp}_y \cap \mathrm{Opt}_x \cap \mathrm{Ud}_x = \emptyset$ is a very weak one, just saying that $R$ does not support $y$ if $x$ is undominated in $R$ and dominates $y$ and the other alternatives.

We say that voting rule $U$ satisfies the *Condorcet property* if $U(\pi) = \{x\}$ whenever profile $\pi$ and $x \in A$ are such that for all $y \in A \setminus \{x\}$, $\pi_*(x,y) > \pi_*(y,x)$. Assume that $\omega$ respects Sp and that Sp satisfies As and for all $y \in A \setminus \{x\}$, $\mathrm{Sp}_y \cap \mathrm{Opt}_x \cap \mathrm{Ud}_x = \emptyset$. The above result implies that the voting rule given by $\pi \mapsto W^{\mathrm{Sp}}_{\to 1}(\pi_f)$ satisfies the Condorcet property, where $f$ is any strictly monotonic function on the non-negative reals with $f(0) > 0$.

### 4.4 Equivalence of sets $\Gamma$

We will see that for many different $\Gamma, \Gamma' \in \mathfrak{L}$, $\Gamma$ and $\Gamma'$ are equivalent in that they generate the same winners. We formalise this as follows.

We say that sets $\Gamma$ and $\Gamma'$ in $\mathfrak{L}$ are *$(p \to 0)$-equivalent*, abbreviated to $\Gamma \equiv_0 \Gamma'$, if for all $v \in \mathcal{V}$, $W^{\mathrm{Sp}(\Gamma)}_{\to 0}(v) = W^{\mathrm{Sp}(\Gamma')}_{\to 0}(v)$, so that $\Gamma$ and $\Gamma'$ generate the same set of $(p \to 0)$-winners.

Sets $\Gamma$ and $\Gamma'$ in $\mathfrak{L}$ are *$(p \to 1)$-$\Delta$-equivalent*, abbreviated to $\Gamma \equiv^+_1 \Gamma'$, if for all non-zero $v$ (i.e., with $\Omega^v = \Delta$), $W^{\mathrm{Sp}(\Gamma)}_{\to 1}(v) = W^{\mathrm{Sp}(\Gamma)}_{\to 1}(v)$ and $\overline{W}^{\mathrm{Sp}(\Gamma)}_{\to 1}(v) = \overline{W}^{\mathrm{Sp}(\Gamma')}_{\to 1}(v)$, so the weak $(p \to 1)$-winners are the same, as are the strong $(p \to 1)$-winners.

It can be shown that $\{\mathrm{Ud}, \mathrm{Ac}\} \equiv^+_1 \{\mathrm{Opt}, \mathrm{Ud}, \mathrm{Ac}, \mathrm{C}\}$, for example; this is essentially because, with either definition of $\Gamma$, the relations $R$ in $\mathrm{Sp}_x(\Gamma)$ maximising $v^+$ are the total orders with top element $x$, and so both definitions of $\Gamma$ give rise to the same strong and weak winners.

We say that sets $\Gamma$ and $\Gamma'$ in $\mathfrak{L}$ are *simply equivalent* if for all $x \in A$, we have $\mathrm{Sp}_x(\Gamma) = \mathrm{Sp}_x(\Gamma')$. Clearly, if $\Gamma$ and $\Gamma'$ are simply equivalent then they are $(p \to 0)$-equivalent and $(p \to 1)$-$\Delta$-equivalent.

For example, $\{\mathrm{Opt}, \mathrm{Ac}\}$ and $\{\mathrm{Opt}, \mathrm{Ud}, \mathrm{Ac}\}$ are simply equivalent, because for any $x \in A$, $\mathrm{Opt}_x \cap \mathrm{Ac} \subseteq \mathrm{Ud}_x$ (if $x$ is dominating w.r.t. an acyclic relation then it is undominated), and so $\mathrm{Opt}_x \cap \mathrm{Ac} = \mathrm{Opt}_x \cap \mathrm{Ud}_x \cap \mathrm{Ac}$.

## 5 THE $p \to 0$ RULES GENERATED BY $\mathfrak{L}$

In this section we consider all the rules with $p$ tending to 0 generated by the language $\mathfrak{L}$, that is, all $\mathcal{V}$-rules $W^{\mathrm{Sp}}_{\to 0}$ for Sp being $\mathrm{Sp}(\Gamma)$ for some member $\Gamma$ of $\mathfrak{L}$. The language $\mathfrak{L}$ can be shown to have 176 different subsets. However, there are many logical connections between the eight properties, implying instances of simple equivalence (see Section 4.4). For instance, $\mathrm{Ud}_x \cap \mathrm{C} \subseteq \mathrm{Opt}_x$, and $\mathrm{As} \cap \mathrm{T} \subseteq \mathrm{Ac}$, and $\mathrm{Ac} \subseteq \mathrm{As}$. Also, if $R \in \mathrm{Opt}_x \cap \mathrm{As}$ then $R \in \mathrm{Ud}_x$, i.e., $\mathrm{Opt}_x \cap \mathrm{As} \subseteq \mathrm{Ud}_x$, which implies that $\{\mathrm{Opt}, \mathrm{As}\}$ and $\{\mathrm{Opt}, \mathrm{As}, \mathrm{Ud}\}$ are simply equivalent. This means that $\mathrm{Sp}(\{\mathrm{Opt}, \mathrm{As}\})$ is the same function as $\mathrm{Sp}(\{\mathrm{Opt}, \mathrm{As}, \mathrm{Ud}\})$ and so leads to the same $\mathcal{V}$-rules. However, there are more subtle connections that imply equivalence of two elements of $\mathfrak{L}$ in terms of the $\mathcal{V}$-rules they generate for the $p \to 0$ case, because of the characterisation of $W^{\mathrm{Sp}}_{\to 0}$ given in Theorem 2. For instance, the minimal cardinality elements[5] of $\mathrm{Sp}_x(\{\mathrm{Opt}, \mathrm{C}\})$ are all asymmetric and so in As, and it can then be shown that $\mathrm{Sp}(\{\mathrm{Opt}, \mathrm{C}\})$ and $\mathrm{Sp}(\{\mathrm{Opt}, \mathrm{C}, \mathrm{As}\})$ lead to the same $\mathcal{V}$-rule $W^{\mathrm{Sp}}_{\to 0}$, and so are $(p \to 0)$-equivalent. The two types of equivalences reduce the number of different $p \to 0$ rules generated by $\mathfrak{L}$ to just seven, as stated by Theorem 3.

Parts (i) and (ii) relate to the two cases that are independent of $p$ discussed in Proposition 2. Part (i) implies, for instance, that the $p \to 0$ $\mathcal{V}$-rule based on $\mathrm{Sp}(\{\mathrm{Ud}, \mathrm{T}, \mathrm{Ac}, \mathrm{As}\})$ gives the same rule as Sp defined by $\mathrm{Sp}_x = \mathrm{Ud}_x$, with the winners $W^{\mathrm{Sp}}_{\to 0}(v)$ being the alternatives $x$ minimising $v^+(\mathrm{D}_x)$. The OOpt $p \to 0$ $\mathcal{V}$-rule in (ii) returns $x$ maximising $v^+(\mathrm{O}_x)$.

The Opt rule (iii) is somewhat similar to (ii): it returns $x$ maximising $v^\times(\mathrm{O}_x)$. Thus, if one pre-processes by an exponential, for example with $v \mapsto W^{\mathrm{Sp}}_{\to 0}(2^v)$, then the winners are $x$ maximising $\prod_{y \neq x} 2^{v(x,y)} = 2^{v^+(\mathrm{O}_x)}$, i.e., maximising $v^+(\mathrm{O}_x)$ as in case (ii).

[5] For the $p \to 1$ case in Section 6 we have a similar type of equivalence based instead on the subset-maximal elements of $\mathrm{Sp}_x(\Gamma)$; see the supplementary document for the details.

For case (iv) and e.g., $\Gamma = \{\mathrm{TOpt}\}$, the rule can be characterised as follows. For alternative $x \in A$, consider any function $g$ from $A \setminus \{x\}$ to $A$, and let $R_g = \{(g(y), y) : y \in A \setminus \{x\}\}$. Let $F_x$ be the set of such functions $g$ with $\mathrm{Tr}(R_g) \supseteq \mathrm{O}_x$, i.e., such that every other alternative is reachable from $x$, when viewing $R_g$ as a directed graph. The winners $W^{\mathrm{Sp}}_{\to 0}(v)$ are $x \in A$ that maximise $\sum_{g \in F_x} v^\times(R_g)$.

For case (v) with e.g., $\mathrm{Sp} = \mathrm{Sp}(\{\mathrm{Ud}, \mathrm{C}, \mathrm{As}\})$ it can be shown that the winners $W^{\mathrm{Sp}}_{\to 0}(v)$ are $x$ maximising $\prod_{y \neq x} \frac{v(x,y)}{v(x,y)+v(y,x)}$. For balanced $v$, the winners are thus $x$ maximising $\prod_{y \neq x} v(x,y) = v^\times(\mathrm{O}_x)$, therefore giving the same results as (iii) for balanced $v$, which includes profile-based $v$. (vi) seems an especially complicated rule. The winners in (vii) are $x$ maximising the sum of $v^\times(R)$ over all strict total orders with $x$ top.

**Theorem 3** *Consider any* $\Gamma \in \mathfrak{L}$.

*(i)* *If* $\Gamma \subseteq \{\mathrm{Ud}, \mathrm{T}, \mathrm{Ac}, \mathrm{As}\}$ *and* $\Gamma \ni \mathrm{Ud}$ *then* $\Gamma \equiv_0 \{\mathrm{Ud}\}$. *[argmin$_x v^+(\mathrm{D}_x)$ rule]*
*(ii)* *If* $\Gamma \subseteq \{\mathrm{OOpt}, \mathrm{Ud}, \mathrm{T}, \mathrm{Ac}, \mathrm{As}\}$ *and* $\Gamma \ni \mathrm{OOpt}$ *then* $\Gamma \equiv_0 \{\mathrm{OOpt}\}$. *[argmax$_x v^+(\mathrm{O}_x)$ rule]*
*(iii)* *If* $\Gamma \not\ni \mathrm{C}$ *and either* $\Gamma \ni \mathrm{Opt}$ *or* $\Gamma \supseteq \{\mathrm{TOpt}, \mathrm{T}\}$ *or* $\Gamma \supseteq \{\mathrm{TOpt}, \mathrm{OOpt}\}$ *then* $\Gamma \equiv_0 \{\mathrm{Opt}\}$. *[argmax$_x v^\times(\mathrm{O}_x)$ rule]*
*(iv)* *If* $\Gamma \subseteq \{\mathrm{TOpt}, \mathrm{Ud}, \mathrm{Ac}, \mathrm{As}\}$ *and* $\Gamma \ni \mathrm{TOpt}$ *then* $\Gamma \equiv_0 \{\mathrm{TOpt}\}$.
*(v)* *If* $\Gamma \subseteq \{\mathrm{Ud}, \mathrm{Opt}, \mathrm{TOpt}, \mathrm{C}, \mathrm{As}\}$ *and* $\Gamma \ni \mathrm{C}$ *and either* $\Gamma \ni \mathrm{Ud}$ *or* $\Gamma \ni \mathrm{Opt}$ *then* $\Gamma \equiv_0 \{\mathrm{Ud}, \mathrm{C}\}$.
*(vi)* $\{\mathrm{TOpt}, \mathrm{As}, \mathrm{C}\} \equiv_0 \{\mathrm{TOpt}, \mathrm{C}\}$.
*(vii)* *If* $\Gamma \ni \mathrm{C}$ *(and so* $\Gamma \not\ni \mathrm{OOpt}$*) and either* $\Gamma \ni \mathrm{Ac}$ *or* $\Gamma \ni \mathrm{T}$ *then* $\Gamma \equiv_0 \{\mathrm{Opt}, \mathrm{C}, \mathrm{Ac}\}$.

*Furthermore, cases (i)–(vii) are mutually exclusive and cover every element of* $\mathfrak{L}$.

## 6 $p \to 1$ CASES GENERATED BY $\mathfrak{L}$ WHEN $\Omega = \Delta$

In this section we consider $\mathcal{V}$-rules generated by the language $\mathfrak{L}$ with $p \to 1$. There are a much larger number of different such rules than for the $p \to 0$ case; for instance, including C or T in $\Gamma$ can make a (usually) small change to the $\mathcal{V}$-rule. Because of this, we only consider inputs $v$ that are non-zero, i.e., with $\Omega^v = \Delta$, and classify the sets in $\mathfrak{L}$ on this subset of $\mathcal{V}$.

**Theorem 4** *Consider any* $\Gamma \in \mathfrak{L}$.

*(i)* *If* $\Gamma \ni \mathrm{OOpt}$ *then* $\Gamma \equiv^+_1 \{\mathrm{OOpt}\}$. *[argmax$_x v^+(\mathrm{O}_x)$ rule]*
*(ii)* *If* $\Gamma \not\ni \mathrm{OOpt}$ *and either* $\Gamma \ni \mathrm{Ac}$ *or* $\Gamma \supseteq \{\mathrm{As}, \mathrm{T}\}$ *then* $\Gamma \equiv^+_1 \{\mathrm{Ud}, \mathrm{Ac}\}$.
*(iii)* *If* $\Gamma \subseteq \{\mathrm{Opt}, \mathrm{TOpt}, \mathrm{Ud}, \mathrm{As}, \mathrm{C}\}$ *and* $\Gamma \ni \mathrm{As}$ *and [either* $\Gamma \ni \mathrm{Ud}$ *or* $\Gamma \ni \mathrm{Opt}$*] then* $\Gamma \equiv^+_1 \{\mathrm{Ud}, \mathrm{As}\}$. *[argmin$_x \sum_{y \neq x} \max(v(y,x) - v(x,y), 0)$ rule]*
*(iv)* $\{\mathrm{TOpt}, \mathrm{As}, \mathrm{C}\} \equiv^+_1 \{\mathrm{TOpt}, \mathrm{As}\}$.
*(v)* *If* $\Gamma \subseteq \{\mathrm{Opt}, \mathrm{TOpt}, \mathrm{Ud}, \mathrm{T}, \mathrm{C}\}$ *and* $\Gamma \ni \mathrm{Ud}$ *then* $\Gamma \equiv^+_1 \{\mathrm{Ud}\}$. *[argmin$_x v^+(\mathrm{D}_x)$ rule]*
*(vi)* *If* $\Gamma \subseteq \{\mathrm{Opt}, \mathrm{TOpt}, \mathrm{C}\}$ *and* $\Gamma \ni \mathrm{Opt}$ *then* $\Gamma \equiv^+_1 \{\mathrm{Opt}\}$. *// [argmax$_x \min_{y \neq x} v(x,y)$ rule]*
*(vii)* *If* $\Gamma \subseteq \{\mathrm{Opt}, \mathrm{TOpt}, \mathrm{C}, \mathrm{T}\}$ *and either* $\Gamma \supseteq \{\mathrm{Opt}, \mathrm{T}\}$ *or [*$\Gamma \ni \mathrm{TOpt}$ *and* $\Gamma \not\ni \mathrm{Opt}$*] then* $\Gamma \equiv^+_1 \{\mathrm{Opt}, \mathrm{T}\}$.

*Furthermore, cases (i)–(vii) are mutually exclusive and cover every element of* $\mathfrak{L}$.

Case (i) generated by $\mathrm{Sp} = \mathrm{OOpt}$ selects alternatives $x$ that maximise $v^+(\mathrm{O}_x)$, with $\overline{W}^{\mathrm{Sp}}_{\to 1}(v) = W^{\mathrm{Sp}}_{\to 1}(v)$, and thus, for profiles, is the Borda rule. Recall that this Sp is one of the cases that is independent of $p$ (see Proposition 2). It is unary, and the unique relation $R$ in $\mathrm{Sp}_x$ maximising $v^+(R)$ is $\mathrm{O}_x$.

(ii): This rule is generated (for instance) by $\{\mathrm{Opt}, \mathrm{Ac}\}$. It is unary, and the elements $R \in \mathrm{Sp}_x$ maximising $v^+(R)$ are all the total orders which have $x$ top. The weak winners $x \in \overline{W}^{\mathrm{Sp}}_{\to 1}(v)$ are those which are top elements for total orders with maximal sum of votes, so the voting rule $\pi \mapsto \overline{W}^{\mathrm{Sp}}_{\to 1}(\pi_*)$ agrees with the Kemeny rule [18, 13]. For the strong winners, ties are (partially) broken by counting the number of total orders $R$ with $x$ top that maximise $v^+(R)$.

(iii) is generated e.g., by $\{\mathrm{Opt}, \mathrm{As}\}$, and is unary. Weak winners are $x$ that minimise $\sum_{y \neq x} \max(v(y,x) - v(x,y), 0)$, with natural partial tie breaking for the strong winners, by minimising, among the weak winners $x$, $|\{y \in A \setminus \{x\} : v(y,x) = v(x,y)\}|$. For profiles $\pi \in \mathcal{P}$, the voting rule $\pi \mapsto \overline{W}^{\mathrm{Sp}}_{\to 1}(\pi_*)$ agrees with Tideman's rule [22, 23, 3].

(iv) generated by $\{\mathrm{TOpt}, \mathrm{As}\}$ leads to a somewhat complicated $\mathcal{V}$-rule, which however, does satisfy the Cordorcet property, by Proposition 9, as well a Pareto property based on the conditions on $v$ in Proposition 5, and also a natural monotonicity property (see Lemma 25 of the longer version [25]).

Case (v) generated by $\mathrm{Sp} = \mathrm{Ud}$ has $\overline{W}^{\mathrm{Sp}}_{\to 1}(v) = W^{\mathrm{Sp}}_{\to 1}(v)$ equalling $\mathrm{argmin}_x v^+(\mathrm{D}_x)$, with Sp being independent of $p$ (see Proposition 2). For profiles, it is the Borda rule.

The weak winners for rule (vi) generated e.g., by $\mathrm{Sp} = \mathrm{Opt}$, are those $x$ maximising $\min_{z \neq x} v(x,z)$, and the voting rule $\pi \mapsto \overline{W}^{\mathrm{Sp}}_{\to 1}(\pi_*)$ agrees with the maximin rule [27, 13], with natural partial tie-breaking for the strong winners.

Rule (vii) generated by e.g., $\{\mathrm{Opt}, \mathrm{T}\}$ is an interesting non-unary rule with the weak winners being the undominated alternatives, where $x$ dominates $y$ if and only if the cost for $x$ against $y$ is less than the cost for $y$ against $x$, and the cost for $x$ against $y$ is the minimum of $F_B$ over all $B$ such that $x \in B \subseteq A \setminus \{y\}$, where $F_B = \sum_{w \notin B, z \in B} v(w,z)$. This rule satisfies Pareto and monotonicity properties expressed by Propositions 5 and 6, as well as neutrality and homogeneity.

## 7 RELATED WORK

The probabilistic approach described in this paper, which picks one or more winning alternative, contrasts with probabilistic social choice functions [1], which generate (sets of) probability distributions over alternatives. Future work could study the probabilistic social choice functions that result from adapting the definition of winner in Section 2.3 by picking alternative $x$ with chance proportional to $\mathrm{Pr}^v_p(\mathrm{Sp}_x)$.

The input $v \in \mathcal{V}$ of a $\mathcal{V}$-rule can be viewed as a weighted directed graph on alternatives, with non-negative real weights; this suggests the potential of relationships with weighted tournament solutions, C2 functions in the Fishburn's classification [14, 13].

In particular, in certain cases, there is a correspondence between the weak ($p \to 1$)-winners and the winners according to a voting rule generated by median orders [16, 4, 13]. Suppose that $v \in \mathcal{V}$ is based on a (generalised) profile of $m$ relations $R_i \in \mathcal{R}$, where $\mathcal{R}$ is some class of relations that are all tournaments (i.e., $\mathcal{R} \subseteq \mathrm{As} \cap \mathrm{C}$) so that, for different alternatives $x$ and $y$, $(x,y) \in R_i \iff (y,x) \notin R_i$. We can write $v$ as $\sum_{i=1}^m v_i$, where $v_i$ is a $\{0,1\}$-valued element of $\mathcal{V}$ corresponding to $R_i$. Assume also that Sp satisfies As, C and Opt (and hence Ud). Then, Sp is unary, and the weak winners consist of $x \in A$ such that $\mathrm{Sp}_x$ contains an element $R \in \mathcal{M} = \bigcup_{z \in A} \mathrm{Sp}_z$ that maximises $v^+(R)$. For any $i = 1, \ldots, m$, $(v_i)^+(\Delta \setminus R) = |R_i \setminus R| = \frac{1}{2}(|R_i \setminus R| + |R \setminus R_i|)$. Maximising $v^+(R)$ is the same as minimising $v^+(\Delta \setminus R) = v^+(\Delta) - v^+(R)$, and $v^+(\Delta \setminus R) = \sum_{i=1}^m (v_i)^+(\Delta \setminus R) = \sum_{i=1}^m |R_i \setminus R|$, which thus equals half of the *remoteness* between $(R_1, \ldots, R_m)$ and $R$ [16]. This implies that the weak ($p \to 1$)-winners are the winners of the median order (generalised) voting rule based on minimising the $(\mathcal{R}, \mathcal{M})$-score (see page 95 of [13]). This includes, in particular, cases (ii) (Kemeny's rule) and (iii) (Tideman's rule) of Theorem 4.

Theorem 4 also suggests the potential of connections between unary ($p \to 1$) cases and voting rules generated from Maximum Likelihood Estimators (MLE), and consensus-based voting rules [12, 6, 8, 11], since the maximin rule (case (vi)) can be generated, as well as Borda (cases (ii) and (v)) and the rules in Theorem 4(ii) and (iii). In a general sense, our approach with limiting $p$ is reminiscent of the construction of the rules $\mathrm{MLE}^{\infty}_{intr}$, $\mathrm{MLE}^{1}_{intr}$, $\mathrm{MLE}^{\infty}_{tr}$ and $\mathrm{MLE}^{1}_{tr}$ in [12], and they can give similar rules to the $p \to 1$ rules generated with our framework. However, even when they do, the tie-breaking can be very different from the strong winners in our framework, as illustrated on pages 190 and 191 of [12]; however, this may well be because of the different methods for defining the limiting voting rules, with the method used in these publications (see e.g., [11], Section 5) being conceptually elegant, but perhaps mathematically more complex, than our approach in Section 4.1 based on the limiting values of $Q^{v,p}_{\mathrm{Sp}}(x,y)$.

Note that, although the MLE-based models and our framework are both based on probabilistic models, the interpretation of the input profile (or more generally the function $v$) is conceptually completely different: in the MLE framework the input profile is assumed to be the *result* of a random process; whereas in our framework it is the *input* for a random process. In the MLE framework, the input profile is treated as a noisy version of some unknown true profile, based on a probabilistic model for the noise. In contrast, in our framework, the input profile (or function $v$) is treated as a collection of parameters for generating a random relation, which is then used to determine the winners; thus, $v$ is interpreted as a kind of propensity.

## 8 DISCUSSION

We have defined and explored a framework for generating voting rules (and $\mathcal{V}$-rules that allow a more general form of input), based on winners being alternatives that maximise the probability of being supported. We have given some simple sufficient conditions for certain properties of the voting rule. We defined a simple language of supporting functions, and categorised the rules generated for the two limiting cases with $p \to 0/1$.

Our method allows one to generate large (and continuous) families of voting rules that satisfy some good properties. In particular, if we choose some neutral Sp based on (arbitrarily complicated) sets of relations and add the conditions Opt and Ud then we obtain neutral $\mathcal{V}$-rules, and thus also voting rules (using an arbitrary strictly monotonic non-zero function of the positive reals), that satisfy Pareto and monotonicity properties (see Proposition 7). Homogeneity of the voting rule can be enforced by an additional normalisation step. If we additionally consider $p \to 1$ and add the condition As (i.e., intersect each $\mathrm{Sp}_x$ with As) then the voting rule will satisfy the Condorcet property.

The ability to choose between a wide range of voting rules could be useful as a barrier to manipulation, since a different voting rule could be used for each decision (where the voting rule could, for instance, be chosen randomly using a probability distribution which is not known by any of the agents).

For the limiting $p \to 1$ case, it is striking that several well-known voting rules can be generated by choosing different natural choices of the supporting function. In particular, for profile $\pi \in \mathcal{P}$, and any real $\epsilon > 0$, the voting rule $\pi \mapsto \overline{W}^{\mathrm{Sp}}_{\to 1}(\pi_\epsilon)$ is the Borda rule for Sp equalling OOpt or Ud; the maximin rule for $\mathrm{Sp} = \mathrm{Opt}$; it is the Kemeny rule for $\mathrm{Sp} = \mathrm{Sp}(\{\mathrm{Ud}, \mathrm{Ac}\})$ (and for $\mathrm{Sp}(\{\mathrm{Ud}, \mathrm{Ac}, \mathrm{C}\})$, based on total orders) and Tideman's rule for $\mathrm{Sp} = \mathrm{Sp}(\{\mathrm{Ud}, \mathrm{As}\})$ (and for $\mathrm{Sp}(\{\mathrm{Ud}, \mathrm{As}, \mathrm{C}\})$, based on tournaments). Therefore, as well as generating new voting rules, the approach gives a new perspective on standard rules, and it would be interesting to pursue a view of the framework as a rationalisation of certain classes of voting rules [10, 9, 12].

There are many potential avenues of exploration for future research, some of which we raise below.

- As well as analysing further some of the non-standard rules generated by the fairly simple and natural property language $\mathfrak{L}$ (in particular from Theorems 4 and 3), more complex properties of relations can be used to define $\mathrm{Sp}_x$, including, for instance, cardinality constraints on the relation and number of equivalence classes, and the use of tournament solutions [2]. The framework can also be extended to allow structural properties (on $R \in \mathrm{Sp}_x$) that depend on $\Omega^v$, such as a weakened transitivity: $\mathrm{Tr}(R) \cap \Omega^v = R$.
- The method for generating the probability distribution is a rather simple one: independently choosing which elements to include in $R$, and then (in effect) conditioning on the structural assumptions (such as $R$ being transitive); it would be interesting to explore other natural forms of distribution, and what kinds of voting rules would be produced.
- In addition there is the issue of computation and general complexity analyses; in particular, it would be interesting to develop and test Monte-Carlo algorithms for computing an approximation of the set of winners, based on various supporting functions, for the non-limit cases. Some discussion of this can be found in Section 2.7 of [25].

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