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Financial Modelling with 2-EPT Probability Density Functions

A thesis submitted in fulfilment of the requirements

for the degree of Doctor of Mathematics

National University of Ireland, Cork
May, 2013

Thesis by Hugh Conor Sexton

Under the Supervision of

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Ireland

Contents

Declaration of Work

I, Hugh Conor Sexton, hereby declare that this Ph.D. thesis entitled *Financial Modelling with 2-EPT Probability Density Functions* has been written entirely by me under the supervision of Prof. Bernard Hanzon, Head of Mathematics Department, National University of Ireland, Cork.

This thesis has not previously been submitted for any other award at this University or at any other Institution.

Hugh Conor Sexton _____ , Monday May 13^{th}, 2013

Abstract

The class of all Exponential-Polynomial-Trigonometric (EPT) functions is classical and equal to the Euler-d'Alembert class of solutions of linear differential equations with constant coefficients. The class of non-negative EPT functions defined on $[0, \infty)$ was discussed in Hanzon and Holland (2010) of which EPT probability density functions are an important subclass. EPT functions can be represented as $\mathbf{c}e^{\mathbf{A}x}\mathbf{b}$, where $\mathbf{A}$ is a square matrix, $\mathbf{b}$ a column vector and $\mathbf{c}$ a row vector where the triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is the minimal realization of the EPT function. The minimal triple is only unique up to a basis transformation. Here the class of 2-EPT probability density functions on $\mathbb{R}$ is defined and shown to be closed under a variety of operations. The class is also generalised to include mixtures with the pointmass at zero. This class coincides with the class of probability density functions with rational characteristic functions. It is illustrated that the Variance Gamma density is a 2-EPT density under a parameter restriction.

A discrete 2-EPT process is a process which has stochastically independent 2-EPT random variables as increments. It is shown that the distribution of the minimum and maximum of such a process is an EPT density mixed with a pointmass at zero. This density can be derived exactly using two different approaches provided. The Fourier Transform of these distributions correspond to the discrete time Wiener-Hopf factors of the discrete time 2-EPT process.

A distribution of daily log-returns, observed over the period 1931-2011 from a prominent US index, is approximated with a 2-EPT density function. Without

the non-negativity condition, it is illustrated how this problem is transformed into a discrete time rational approximation problem. The rational approximation software RARL2 is used to carry out this approximation. The non-negativity constraint is then imposed via a convex optimisation procedure after the unconstrained approximation.

Sufficient and necessary conditions are derived to characterise infinitely divisible EPT and 2-EPT functions. Infinitely divisible 2-EPT density functions generate 2-EPT Lévy processes.

An assets log returns can be modelled as a 2-EPT Lévy process. Closed form pricing formulae are then derived for European Options with specific times to maturity. Formulae for discretely monitored Lookback Options and 2-Period Bermudan Options are also provided. Certain Greeks, including Delta and Gamma, of these options are also computed analytically.

MATLAB scripts are provided for calculations involving 2-EPT functions. Numerical option pricing examples illustrate the effectiveness of the 2-EPT approach to financial modelling.

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First and foremost, I thank my parents for their continued support throughout my education, without whom this would not have been possible.

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List of Figures

Glossary

$$
\begin{aligned}
\sigma(\mathbf{A}) &= \text{Spectrum of Square Matrix } \mathbf{A} \\
\mathbb{H}_+ &= \{\omega \mid Re(\omega) > 0\} \\
\mathbb{H}_- &= \{\omega \mid Re(\omega) < 0\} \\
\mathbb{C}_+ &= \{\omega \mid Im(\omega) > 0\} \\
\mathbb{C}_- &= \{\omega \mid Im(\omega) < 0\} \\
\mathbf{I_A} &= \text{Identity matrix with dimension equal to those of } \mathbf{A} \\
\mathbf{I}_n &= n \times n \text{ Identity Matrix} \\
\mathbf{A} \oplus \mathbf{B} &= \mathbf{A} \otimes \mathbf{I_B} \ + \ \mathbf{I_A} \otimes \mathbf{B} \, , \qquad \text{(Kronecker Sum)} \\
\phi(s) &= \int e^{-sx} f(x) dx \ \text{ , Laplace Transform of } f \\
\Phi(is) &= \int e^{isx} f(x) dx \ \text{ , Characteristic Function of } f
\end{aligned}
$$

A rational function p/q has support on the set $\mathbb{S}$ if $q(s) \neq 0$ for all $s \in \mathbb{S}$.

Chapter 1

Introduction

1.1. Introduction

1.1.1. 2-EPT Probability Density Functions

The recent surge in computing power has meant that many previously computationally expensive modelling techniques have now become feasible. These developments have enabled moves away from traditional Gaussian assumptions which were often implemented due to their analytic tractability rather than their goodness of fit. We define the flexible class of two-sided Exponential Polynomial Trigonometric (2-EPT) densities on the whole real line. On $[0,\ \infty)$ as well as $(-\infty,\ 0)$ these probability density functions are non-negative EPT functions. EPT functions were discussed in Hanzon and Holland (2010b) and can be represented with the minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ as $\mathbf{c}e^{\mathbf{A}x}\mathbf{b}$. An EPT function can be interpreted as the impulse response of a continuous time stable system whose Laplace Transform is a rational transfer function. This interpretation and the use of minimal realizations allows us to apply methods from systems theory in a financial modelling context. The class of 2-EPT densities corresponds to those densities with a strictly proper rational characteristic function / transform. The more general class of probability measures on $\mathbb{R}$ with (proper) rational charac-

teristic functions is also considered whose densities correspond to mixtures of the pointmass at zero ("delta distribution") and 2-EPT functions. It will be seen that the class of 2-EPT densities is closed under many operations which makes 2-EPT densities an ideal financial modelling tool.

1.2. Motivation

1.2.1. Black-Scholes Shortcomings

Much of the success of the Black-Scholes model can be attributed to the closed form risk neutral option prices it generates. However, such enviable results and analytic tractability stem from its simplistic modelling assumptions of a pure diffusion process in which log-returns have a Gaussian distribution. Geometric Brownian Motion driven models have laid the foundations for the phenomenal growth seen in financial engineering, both in the literature and in industry, despite the fact that its poor fit has been acknowledged since Mandelbrot (1963). It has been well documented that the assumption of Gaussian returns cannot be justified by empirical studies and this is portrayed in Figures (1a) and (1b) where the peaked mean and heavy tails of the empirical distribution are obvious. These features are more accentuated when the holding periods become shorter as shown in Geman (2002).

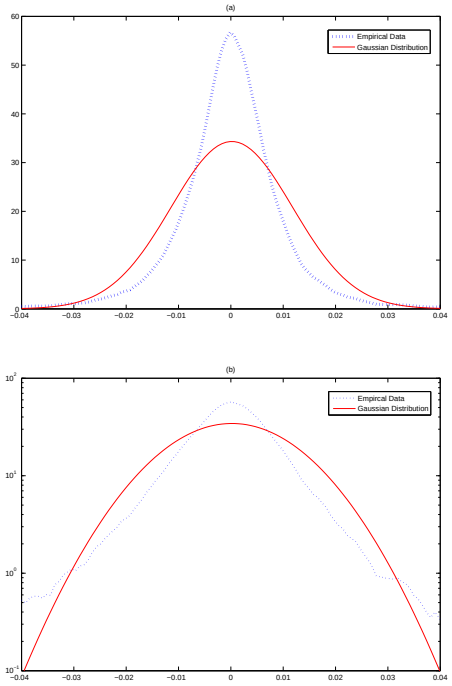

Figure 1.1: *Poor Fit of Gaussian Distribution compared to Empirical Returns (Daily Dow Jones Industrial Average log-returns 1931-2011) (a) illustrates the peaked mean of observed returns while (b) shows a semi-log plot (y-axis uses a log scale) of the heavy tails of empirical returns with respect to the Gaussian tails*

The Black-Scholes model assumes continuous asset paths creating a complete market where delta hedging strategies are used to derive risk neutral option prices. Within this complete markets framework vanilla options are clearly redundant as they can be replicated with a portfolio of risky assets and bonds. However, once jumps are introduced to the price process, and they are clearly identifiable in equity data, the continuous hedging arguments break down but lead to a more realistic, incomplete market. Therefore, without continuous hedging these options are no longer futile and become completing assets, hedging against jump risk.

The most obvious implication of using the mis-specified Black-Scholes model is the underestimation of the probability of extreme events due to the light tails of the Gaussian distribution. Immediately, this means that deep out of-the-money options are underpriced and short term options also mis-priced. Some

firms have countered this problem by using adjusting the volatility parameter depending on the moneyness of the option, but this approach clearly contradicts the Black-Scholes model. The assumption of Gaussian returns meant financial institutions often held insufficient capital reserves to cope with extreme events. Portfolio insurance follows from the assumption of pure diffusion models allowing derivatives exposure to be delta hedged while neglecting gap risk. Large intra-day prices moves, as seen on October 19^{th} 1987 for instance, can result in heavy losses for firms engaging in such practices. A classic example of the consequences of the Black-Scholes shortcomings was demonstrated by the collapse of the hedge fund Long Term Capital Management in 1998, whose principals included the Nobel laureates Myron Scholes and Robert Merton. The fund was dedicated to exploiting mis-pricings identified by their famous model.

1.2.2. Modelling Alternatives

Significant research has been carried out seeking more realistic models which better capture the stylized characteristics of asset returns including excess kurtosis, skewness and jumps. Mandelbrot (1963) advocated modelling returns using α-stable distributions and while a realistic fit was achieved on many timescales relatively little attention has been paid to such models. The primary reason behind their decline is that the probability density function of an α-stable law is not known in closed form excluding the Gaussian and the Cauchy densities, both deemed unsuitable for financial modelling purposes. The Cauchy density is considered inappropriate as it does not possess finite moments. Another disadvantage of the using α-stable distributions to model log-returns is that (excluding the Gaussian) such distributions do not tend to Gaussian over time, a property financial returns exhibit.

The Variance Gamma density is described in Section 2.3. A Variance Gamma process has independent increments which have a Variance Gamma distribution. The Variance Gamma price process is a shifted exponential Variance

Gamma process and has emerged as a popular modelling alternative to Geometric Brownian Motion since been introduced in Carr, Madan, Chang (1998). The Variance Gamma process is a pure jump process while the density is flexible with tails decaying exponentially and asymmetrically if desired. The distribution even allows one to control the excess kurtosis directly. However, a major drawback of the Variance Gamma distribution is the complicated structure of its density, which includes a modified spherical Bessel function, making even simple calculations difficult. For this reason very few Variance Gamma related results are derived via the density function. This is in stark contrast with the Black-Scholes model where closed form option pricing formulae exist and option price sensitivities can be computed analytically. Numerical integration is required to make such calculations within the Variance Gamma framework. Figure 2 illustrates the difference between the Variance Gamma Price Process, a pure jump process, and Geometric Brownian Motion, a pure diffusion process.

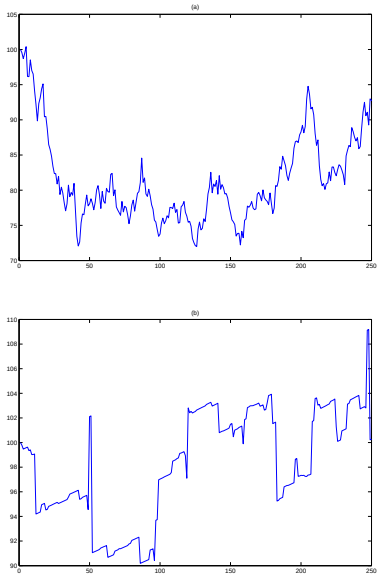

Figure 1.2: *(a) Geometric Brownian Motion (Pure Diffusion Process) compared against (b) an Exponential Variance Gamma Process (Pure Jump Process)*

The Variance Gamma density takes three positive parameters, namely (C, G, M), as inputs. The characteristic function of the Variance Gamma density can be

raised to any positive power by adjusting the value of the parameter C. It can be seen that the characteristic function is infinitely divisible. Hence, the density is infinitely divisible and it follows that the Variance Gamma process is a Lévy process. Therefore, the Variance Gamma price process is an exponential Lévy process. The fact that the underlying process is a Lévy process has undoubtedly contributed to the growth of the model as much research has been conducted into the properties of such processes as we will see in Chapter 5.

The relationship between Lévy processes and infinitely divisible distributions is very close as there exists an infinitely divisible distribution for every Lévy process. A classical result, also given in Kyprianou and Loeffen (2005), proves the converse, that for every infinitely divisible distribution there exists a unique Lévy process. Lévy processes are characterized by their Lévy triple, comprising of a drift component, a diffusion component and a Lévy measure. The Lévy measure specifies the intensity of the jumps and their sizes in process. These processes are ideally suited to financial modelling due to their flexibility and ability to generate realistic asset returns. Modelling with Lévy processes preserves the stationarity and independence of increments while allowing for jumps, distributional asymmetry and heavier tails than the Gaussian distribution.

A downside associated with Lévy processes is that the formulae for the densities generating such processes often contain special functions or may not even have a closed form. A classic example is the generalized hyperbolic distribution whose special cases include the Variance Gamma and Inverse-Gaussian distributions, both containing special functions. The Meixner distribution generates Lévy processes popular in financial modelling but again its density is not suitable for analytic computations. For this reason, calculations involving these density functions often require numerical techniques. Option pricing formulae, based on models using the aforementioned Lévy processes, are not available in closed form to date. Typically these prices are obtained using the Fast Fourier Transform of Carr and Madan (1998).

As already mentioned, the Variance Gamma distribution takes three paramet-

ers (C, G, M) as inputs and if C is restricted to be an integer, a strictly proper rational characteristic function is observed implying the existence of a 2-EPT density. The 2-EPT realization of the density can be recovered and once in this form can be manipulated easily. Assuming an asset has a Variance Gamma price process (exponential Lévy process) with C integer, we show how closed form option pricing formulae can be obtained. It is also possible to derive certain Greeks of the option, including delta and gamma, analytically.

1.2.3. 2-EPT Lévy Processes

A necessary and sufficient condition is derived in Section 4.3 to characterise an infinitely divisible EPT functions. This result is extended in Section 4.6 to provide a similar necessary and condition to identify infinitely divisible 2-EPT probability density functions. Infinitely divisible 2-EPT distributions generate 2-EPT Lévy processes. The increments of 2-EPT Lévy processes have 2-EPT distributions over certain timeframes. The Lévy triple of such processes can be derived and it is seen that 2-EPT Lévy processes are pure jump processes of finite variation. Both Madan (1999) and Geman (2002) propose modelling the log-returns of assets using processes of finite variation. The path of an assets log-returns observed in practice are of finite variation, compared with Brownian Motion models which assume paths of infinite variation. It is obvious that we can benefit from the vast probabilistic theory on Lévy processes. It will be seen later in Section 6.4.4 that 2-EPT densities can approximate the Gaussian quite well. If this approximating density is infinitely divisible then its associated Lévy process will approximate Brownian Motion.

The 2-EPT price process is defined in Section 6.2 as an exponential 2-EPT Lévy process. Assuming that an assets log-return over a period τ have an infinitely divisible 2-EPT distribution, then it is illustrated that the risk neutral closed form pricing formulae exist for European options with integer multiples of τ to maturity. Similarly, closed form pricing formulae can also be derived for

discretely monitored Lookback and 2-Period Bermudan Options for assets with such a 2-EPT price process. The greeks of these options can also be computed analytically.

1.2.4. 2-EPT Operational Advantages

The arguments above make a logical and compelling case for adopting a 2-EPT approach to financial modelling. The claim is strengthened when the machinery available for 2-EPT calculations is unveiled. The range of operations under which the class of 2-EPT functions is closed allows for a broad spectrum of problems to be tackled under a single framework. We will consider some of the operations here to illustrate the advantages of 2-EPT densities. Many of the operations rely on the ability to additively decompose the rational Laplace Transform of the 2-EPT density into the sum of two proper rational functions with poles located in the open right and open left half planes.

The first operation of note to be considered is the sum of 2-EPT densities. It is seen that the class of 2-EPT densities are closed under summation and the computation is straightforward. Summation in this context is equivalent to mixtures of densities, themselves quite important in financial modelling. It is also seen that the class of 2-EPT functions is closed under multiplication.

It is shown that all moments of 2-EPT densities exist and analytic expressions are available for such moments about the origin. These moments may be computed from the derivatives of rational characteristic function which results in standard formulae for the moments. Alternatively the k^{th} moment may be calculated by integrating $x^k f(x)$ over the relevant domain where $f(x)$ represents the 2-EPT probability density function.

The convolution of two 2-EPT functions is another 2-EPT function. Given the 2-EPT modelling assumption, that log-returns over a fixed period τ have a 2-EPT distribution, it is clear that the sum of independent log-returns, over periods of length τ or integer multiplies of τ, again have a 2-EPT distribution.

The distribution of the maxima and minima of a set of independent 2-EPT random variables is shown to be 2-EPT. In conjunction with the convolution formula it is possible to derive the probability density function of the maximum or minimum of a 2-EPT discrete time process. The density of these extrema of such processes is shown to be an EPT density mixed with a pointmass at zero. This property is central to obtaining analytic pricing formula for discretely monitored Lookback options.

The class of 2-EPT functions is closed under scaling while translation appears to be the only notable operation under which the class is not closed. A translated 2-EPT random variable does not have a 2-EPT probability density function.

1.2.5. 2-EPT Computations in Practice

From a theoretical perspective it seems that modelling with 2-EPT densities can be justified. We examine whether such techniques are feasible to implement in practice. We consider the problem of approximating an empirically observed distribution of daily log-returns, from the Dow Jones Industrial Average (DJIA) index 1931-2011, with a 2-EPT probability density function. The non-negative log-returns on $[0, \infty)$ are approximated first with a EPT function using the approximation software RARL2. This procedure is then repeated to fit a EPT function to the data on $(-\infty, 0]$ to give a 2-EPT function. Paramount to these approximations is the assurance of non-negativity and this is imposed via a convex optimisation algorithm. Using this non-negative 2-EPT approximation of the distribution of daily log returns, the density of the annual log-return is computed using the convolution formula. By modelling an assets log-returns using this 2-EPT distribution, prices for European Call Options with one year to maturity and daily monitored Lookback Options are then calculated and plotted over a range of initial asset prices. The Delta and Gamma of these options are are also calculated and plotted. These computations involve large matrices but computation times remain small.

MATLAB is an ideal computing environment for 2-EPT modelling as it is designed for easy matrix manipulation. All numerical examples and results presented in this text have been generated using MATLAB. The Control Systems Toolbox contains many inbuilt functions to implement systems theory results such as deriving minimal realizations. The website "www.2-ept.com" was designed in conjunction with this research as a source for 2-EPT related literature and software. There are various algorithms available to download from the site to assist with 2-EPT calculations most notably the Budan-Fourier algorithm of Hanzon and Holland (2010) and the convex optimisation procedure to approximate negative EPT functions with non-negative EPT functions. Both these algorithms are described in Chapters 3 and 7. Conveniently the approximation software RARL2 used to approximate empirical distributions with 2-EPT functions is also implemented in MATLAB.

1.3. Thesis Structure and Publications

1.3.1. Structure

This introductory chapter concludes with a literature review of the techniques employed throughout the thesis. The review provides a description of previous work carried out in each area and motivation for its inclusion here.

Chapter 2 builds the foundation for the thesis by formulating EPT and 2-EPT probability density functions mathematically. The generalization of mixtures of 2-EPT functions with the Dirac distribution is also given and correspond to generalised 2-EPT distributions. The chapter continues by demonstrating how numerous operations can be carried out within the 2-EPT framework. It is then shown how, under a parameter restriction, the Variance Gamma density is a 2-EPT density.

Chapter 3 shows how an empirical distribution of log-returns can be approximated with a 2-EPT probability density function. The unconstrained approxim-

ation is shown to be a discrete time rational approximation problem which is solved using the rational approximation software RARL2. However, the software does not implement the non-negativity constraint by default. We prove that when the spectrum of $\mathbf{A}$ contains a unique dominant real eigenvalue, the Budan-Fourier technique of Hanzon and Holland (2010) can be used to test for non-negativity on the EPT function on $[0, \infty)$. If the 2-EPT approximation returned does assume negative values, then non-negativity is imposed by adjusting the realization using a convex optimisation algorithm. The chapter culminates with a comparison of the error of the 2-EPT approximation of the empirical distribution versus the errors of Gaussian and Variance Gamma approximations.

2-EPT Lévy Processes are then examined in Chapter 4. It is proven how to characterise an infinitely divisible 2-EPT probability density function. If the distribution is infinitely divisible, the Lévy triple associated with the 2-EPT probability density function can be derived. An example involving the Variance Gamma density is also provided.

Chapter 5 begins by deriving an easy to implement recursive algorithm to compute the EPT probability density function (mixed with a pointmass at zero) of the maximum or minimum of a discrete time 2-EPT process of known length. A discrete time 2-EPT process with a geometrically distributed stopping time is then defined. It is proven (using Wiener-Hopf factorization) that the Laplace Transforms of the distributions of the extrema of the process can be obtained. It is then possible to isolate the Laplace Transform of the distribution of the extrema of 2-EPT process of known length.

Chapter 6 illustrates the benefits of adopting a 2-EPT approach to financial modelling from an option pricing perspective. Firstly, the risk neutral 2-EPT asset price process is derived such that the discounted price process is a martingale. Closed form expressions are then derived for European Call Options and their Greeks. An optimisation algorithm is described to show how the risk neutral 2-EPT price process can be obtained by calibrating the model option prices to a set of empirically observed option prices. It is then shown how to derive

closed form pricing formulae for other European Options such as Put Options, Binary Options and their Greeks. Closed form formulae are then derived for some more exotic options, including a two-period Bermudan option, equivalent to a compound option (i.e. an option on an option). Discretely Monitored Look-back options, with fixed and floating strikes, are also shown to have closed form pricing formulae. Straightforward computations are demonstrated to calculate Value-at-Risk levels and the Expected-Shortfall, for assets with a 2-EPT price process.

Chapter 7 describes some relevant MATLAB functionality available to users for assistance with EPT/2-EPT calculations. The chapter begins by introducing some inbuilt MATLAB commands. A short description is also provided to several beta-version scripts made available by the author free for download on "www.2-ept.com". Chapter 8 concludes with a brief synopsis of what has been achieved to date and some ideas for further research.

1.3.2. Papers and Presentations

The mathematical formulation of EPT and 2-EPT densities with techniques for performing many of the operations was presented at the ERNSI conference in Nice in September, 2011. A significant proportion of Chpater 3, regarding the approximation of empirical data with a non-negative EPT function was presented at the tri-annual SYSID conference in Brussels, 2012. The associated paper, co-authored with Prof. Bernard Hanzon and Dr. Martine Olivi[1], entitled *Rational Approximation of Transfer Functions for Non-Negative EPT Densities* was published in the SYSID 2012 conference proceedings. The results of Chapter 5 on infinitely divisible 2-EPT distributions were presented at the Bachelier Finance Conference in Sydney, in June 2012. This work culminated in a paper written with Prof. Bernard Hanzon and Prof. Finbarr Holland[2] which has been submitted to the Bernoulli Journal for Mathematical Statist-

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ics and Probability under the title *2-EPT Lévy Processes*. A follow on paper, *State Space Calculations for two-sided EPT Densities with Financial Modelling Applications* co-authored with Prof. Bernard Hanzon has been submitted to the same journal. The contents of the latter paper included the mathematical formulation of 2-EPT probability density functions with illustrations of some 2-EPT operations and the paper concluded with first half of Chapter 6 deriving option prices and their greeks for assets with a 2-EPT price process. Draft versions of all papers and presentations are available on "www.2-ept.com".

1.4. Literature Review

1.4.1. Phase Type and Matrix Exponential Distributions

Phase type distributions were introduced in Neuts (1975) and have been used in a wide range of stochastic modelling applications including telecommunications, queuing theory, survival analysis, etc.. Phase Type distributions have provided the basis for many models as they are algorithmically tractable and constitute a versatile class of distributions defined on the non negative half line. A phase-type distribution can be described as the *distribution of the time to absorption in a finite Markov chain of dimension* $n+1$*, where one state is absorbing and the remaining states are transient.* Such a phase-type distribution of dimension n is represented by row vector $\mathbf{s}$ of length n and a square $n \times n$ matrix $\mathbf{P}$. The distribution is given by

$$f(x) \;=\; \mathbf{s}e^{\mathbf{P}x}\mathbf{P}\mathbf{e}$$

where $\mathbf{e} = (1, 1, ..., 1, 1)^R$.

The vector $\mathbf{s}$ can be interpreted as the initial probability vector among the n transient states while the matrix $\mathbf{P}$ can be thought of as the one-step transition probabilities. This results in restrictions on the entries of $\mathbf{P}$, where $p_{i,j}$ denotes the entry in the i^{th} and j^{th} column (i.e. the probability of moving from the

i^{th} to the j^{th} state). Therefore $\mathbf{P}$ can only contain non-negative entries such that $0 \leq p_{i,j} \leq 1$ for all $i, j \in \{1, 2, ..., n\}$. The transition probabilities from each state i must sum to 1 i.e. $\sum_{j=1}^{n} p_{i,j} = 1$. Hence from this definition it is clear that phase-type distributions have a probabilistic interpretation. This is another facet which contributed to their growth. The algorithmically tractable models stem from properties that, like EPT densities, Phase-Type distributions have rational characteristic functions and are closed under a variety of operations. Phase type distributions are dense in the class of all distributions, see Asmussen (2000). This means, that any distribution on $(0, \infty)$ can be approximated arbitrarily close by a Phase-Type distribution. However, for a given rational characteristic function a phase-type distribution may not exist.

This restrictive nature of phase-type distributions lead to the emergence of the matrix exponential distributions as an alternative whose density could be found immediately from their Laplace Transform. Much of the literature on matrix exponentials is based on methods described in Asmussen and O'Cinneide (1998). A matrix-exponential distribution h is described by the triple $(\mathbf{a}, \mathbf{T}, \mathbf{t})$ where

$$h(x) \; = \; \mathbf{a} e^{\mathbf{T}x} \mathbf{t}.$$

The Laplace Transform of the distribution is the rational function $H(u)$ given by

$$H(s) \; = \; \frac{a_p s^{p-1} + a_{p-1} s^{p-2} + ... + a_1}{s^p + b_p s^{p-1} + b_{p-1} s^{p-2} + ... + b_1} + \alpha_0 \; = \; \frac{a(s)}{b(s)}, \qquad (1.1)$$

where $p \geq 1$, $a_1, a_2, ..., a_p, b_1, b_2, ..., b_p$ are all real and $0 \leq \alpha_0 \leq 1$. According to Asmussen and Bladt (1997), the entries of the triple $(\mathbf{a}, \mathbf{T}, \mathbf{t})$ can be chosen to be the coefficients of the Laplace Transform as follows

$$\mathbf{a} \; = \; (a_1, a_2, ..., a_p) \qquad , \qquad \mathbf{t} = (0, 0, 0, ..., 0, 1)^T$$

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 & 1 \\ -b_1 & -b_2 & -b_3 & ... & -b_{p-1} & -b_p \end{pmatrix}$$

If $a(s)$ and $b(s)$ are co-prime polynomials then $(\mathbf{a}, \mathbf{T}, \mathbf{t})$ is a minimal realization.

It is seen above that the realization is easily derived from the coefficients of the polynomials in the numerator and denominator of rational transfer function. The realization of a matrix-exponential distribution is not required to be minimal. For this reason much of the literature can not be directly applied to minimal realizations where simpler calculations often exist. The class of matrix exponential functions defined on the halfline $[0, \infty)$ is equal to the class of EPT functions (mixed with a pointmass) on the same halfine. Asmussen (2000) does give an account of how some operations are carried out in the matrix exponential case but again limited to the non-negative half-line (i.e. matrix-exponential distributions are only defined on the half-line). Hence we see that the 2-EPT case as proposed here utilising minimal realizations is a more general class than either phase-type or matrix exponential while providing greater computational efficiency in the process.

1.4.2. Minimal Realization

A state space model is minimal if it has the minimal number of states among all state space models with the same transfer function. Then the state space model is minimal if and only if it is both controllable and observable. Consider a minimal state space model of order n. If the rational transfer function of this model is represented as the quotient of two co-prime polynomials, the denominator will be a polynomial of degree n.

The most obvious advantage of using minimal realizations is the size of the matrices will be minimalised, saving computation time. Algorithms for deriving minimal realizations can be found in Kailath (1980). Minimal realizations also allow us to immediately determine the stability of the **A** matrix by checking the locations of its eigenvalues. It is the use of minimal realizations which primarily differentiates this EPT approach from the matrix exponential techniques.

1.4.3. Positive Realization

Positive realizations were introduced by Luenberger (1979) to ensure positive responses from a state space model. From Farina (1996), a continuous time positive system is described by its triple $(\mathbf{A}_+, \mathbf{b}_+, \mathbf{c}_+)$ where all off-diagonal entries of $\mathbf{A}_+$ are non-negative while $\mathbf{b}_+$ and $\mathbf{c}_+$ also contain only non-negative elements. The realization of a discrete-time positive linear system $(\tilde{\mathbf{A}}_+, \tilde{\mathbf{b}}_+, \tilde{\mathbf{c}}_+)$ contains all non-negative real elements. There are some strict conditions to ensure the existence of such a realization which are described in Farina (1996). If a positive realization exists for a transfer function of McMillan degree n, which is not always the case, then the positive realization of lowest order may be of order greater than n. Another drawback of positive realizations is that the corresponding density will be strictly positive or identically zero for all $x \geq 0$. This contrasts with minimal realizations where zero responses are allowed so the EPT function can have zeros on $[0, \infty)$. Farina (2000) provides a comprehensive treatment of positive systems.

1.4.4. EPT Literature

The class of functions which are solutions of homogenous linear differential equations with constant real coefficients were first denoted as EPT functions in Hanzon and Holland (2010). It was noted that EPT functions could be used to model forward interest rates such as Nelson-Siegel curves while in systems theory

they appear as impulse response functions from a linear system. In many applications, non-negative EPT functions play a crucial role. Hanzon and Holland (2010) developed the Budan-Fourier algorithm to examine the non-negativity of an EPT function on a finite interval. Hanzon and Holland (2010b) elucidated to the necessity of a dominant real pole in the system of a non-negative function. Similarly Hanzon and Holland (2012) present some necessary conditions and a sufficient condition to ensure non-negativity of an EPT function on the half line $[0, \infty)$.

Hanzon and Ober (2001) considered the class of proper rational probability density functions which could be represented in state space form. The density function can be interpreted as the transfer function of a linear system. The operations examined required the additive decomposition of the rational density function. Using state space techniques the operations considered are similar to those we treat in Chapter 2 including the scaling and translation of the associated random variables and the product and convolution of the rational densities. Hanzon and Ober (2001) and Hanzon and Scherrer (2008) investigate applications of filtering involving proper rational probability density functions by making use of the state space methods mentioned above.

Discrete densities defined on the set of non-negative integers with proper rational generating functions are considered in Hanzon and Ober (2002). The operations of scaling, translation, convolution, mixing and product are carried out using state space systems. Examples of such densities include the geometric distribution and also the truncated Poisson.

1.4.5. Wiener-Hopf Factors for Discrete Processes

We define a discrete time process as a finite sum of independent and identically distributed random variables. This process can be thought of as a random walk. The characteristic functions of the distributions of the minimum and the maximum of discrete time processes are referred to as the Wiener-Hopf factors of the

process. Relationships proven in Spitzer (1956) show that the two Wiener-Hopf factors with support in the upper and lower half planes can be obtained for a discrete time process whose length is geometrically distributed. These factors are crucial for pricing options which depend on the extrema of these discrete time processes such as Lookback Options. However the results of Spitzer (1956) require a non-trivial Laplace inversion to obtain these factors. Petrella and Kou (2004) provides an algorithm to numerically compute the factors and in doing so provides option prices for the aforementioned path-dependent options. The same algorithm can be used to compute the greeks of such options. This contrasts with the 2-EPT approach where the distribution of the discretely monitored extrema can be computed exactly.

In the Black-Scholes model, there are closed form pricing formulae available for Lookback options with both fixed and floating strike prices and a collection of such formulae can be found in Haug (2007). Excluding the already mentioned shortcomings of the Black-Scholes model, a further drawback of these formulae is that it is assumed that the asset prices are monitored continuously which is not observed in practice. Based on practical and regulatory arguments, Broadie, Glasserman, and Kou (1997) advocate the pricing of Lookback options using discretely monitored asset price processes as opposed to continuously monitored processes. These arguments favour a 2-EPT approach as the values of Lookback Options can be calculated analytically in terms of the EPT density of the maximum or minimum of the discretely monitored process.

1.4.6. Infinite Divisibility and Lévy Processes

The history of infinitely divisible distributions is awash with famous mathematical names from the twentieth century. The pioneer of research into such distributions was the Italian de Finetti who studied the distributions with respect to stochastic processes with independent increments. Although he did not use the term “infinitely divisible” which was later coined in 1936 by G.M Bawly, he

posed *de Finetti's problem* around 1929 which was to find a general formula for the characteristic function of an infinitely divisible distribution. Kolmogorov addressed *de Finetti's Problem*, published in English in Kolmogorov (1990), yielding his canonical representation of an infinitely divisible characteristic function. However, independently from Kolmogorov, Lévy also described a general solution to *de Finetti's problem* which is given in Lévy (1937). It was Khintchine who gave the first formal definition of an infinitely divisible distribution which read *"a distribution of a random variable which for any positive integer n can be represented as a sum of n identically distributed independent random variables is called an infinitely divisible distribution"*. Khintchine (1937) then showed that Lévy's representation could be derived from the work of Kolmogorov which gave rise to the well known Lévy-Khintchine formula for infinitely divisible characteristic functions. This formula forms the basis for describing infinitely divisible distributions and we make use of it in Chapter 5 to determine the Lévy triple of an infinitely divisible 2-EPT probability density function.

Two other classical texts often cited in relation to infinitely divisible distributions include those of Feller (1971) and Widder (1941). The former states that a function is the Laplace Transform of an infinitely divisible probability distribution on $[0, \infty)$ if and only if it is the exponential of an analytic function on the same domain whose derivative is completely monotonic. There is also a requirement relating to the scaling of the Laplace transform at the origin. Bernsteins Theorem from Widder (1941) gives a necessary and sufficient condition that a function is completely monotonic.

Lukacs (1970) gives many of the elementary properties of infinitely divisible distributions. He also investigates analytic characteristic functions providing results on the location of the zeros for such functions to be infinitely divisible. We arrive at the same result in Section 4.3 where the zeros of an infinitely divisible Laplace transform are dominated by its poles. The necessity of a dominant real pole in the transfer function is also elucidated to which is crucial in the pursuit of non-negative EPT functions.

As already stated, Kyprianou and Loeffen (2005) proves a well known result that there exists a unique Lévy process for every infinitely divisible distribution. The works of Bertoin (1998) and Sato (1999) both provide comprehensive accounts of Lévy processes. A basic introduction to modelling and pricing with Lévy processes is given by Schoutens (2003). Cont and Tankov (2003) provides a more technical examination of the applications of Lévy processes in finance and this text is cited extensively in Chapter 5 as it provides a thorough analysis of the properties of pure jump processes. A useful collection of articles concerned with pricing exotic derivatives in the presence of Lévy processes is Kyprianou et al (2005).

Chapter 2

2-EPT Probability Density Functions

2.1. Mathematical Formulation

2.1.1. 2-EPT Probability Density Functions

We begin by defining the class of real EPT functions $f : [0, \infty) \mapsto \mathbb{R}$ given by

$$f(x) = Re\Big(\sum_{k=1}^{K} p_k(x)\, e^{\mu_k x}\Big) \tag{2.1}$$

where $Re(z)$ denotes the real part of a complex number z, $p_k(x) \in \mathbb{C}[x]$ a polynomial with complex coefficients for each $k = 1, 2, ...K$ and $\mu_k \in \mathbb{C}$ for each $k = 1, 2, ..., K$. The class can be characterized as the class of real continuous functions on $[0, \infty)$ whose Laplace Transform exists and is a strictly proper rational function. A more detailed description of the class and its derivation can be found in Hanzon and Holland (2010).

This class contains the real polynomials, real exponential functions and scaled real trigonometric polynomials (such as $sin(\nu x), cos(\nu x)$, $\nu \in \mathbb{R}$ but not

$tan(\nu x)$) and all products and sums of such functions as the set of functions is a ring over $\mathbb{R}$.

An EPT function defined on the positive real half line can be represented in the form

$$f(x) \;=\; \mathbf{c}\, e^{\mathbf{A}\, x}\, \mathbf{b} \qquad\qquad , \qquad\qquad x \;\geq\; 0 \qquad\qquad (2.2)$$

where $\mathbf{A}$ is a real $n \times n$ matrix, $\mathbf{c}$ a real $1 \times n$ row vector and $\mathbf{b}$ a real $n \times 1$ column vector. The triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is the minimal realization of the function. We now give some examples of EPT functions and their minimal realizations. Firstly, consider the polynomial $f(x) = 3x^2 + 2x - 5$ which can be written as an EPT function using the realization $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ where

$$\mathbf{A} \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad , \qquad \mathbf{b} = (1,0,0)^T \qquad , \; \mathbf{c} = (-5,2,6)$$

Therefore

$$f(x) \;=\; \mathbf{c}e^{\mathbf{A}x}\mathbf{b} \;=\; 3x^2 + 2x - 5$$

Secondly, the EPT function $g(x) = \cos(x) = \mathbf{c}_c e^{\mathbf{A}_c x}\mathbf{b}_c$ has a minimal realization given by

$$\mathbf{A}_c \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad , \qquad \mathbf{b}_c = (1,0)^T \;, \qquad \mathbf{c}_c = (1,0)$$

It should be noted that minimal realizations are only unique up to a basis transformation.

We consider EPT probability density functions where the triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ is such that $f(x) \geq 0 \; \forall \; x \geq 0$. We elucidate later to necessary conditions and sufficient conditions on the triple to ensure non-negative responses. However if $f(x)$ is non-negative then a Perron-Frobenius type result given in Hanzon and Holland (2010b) states that $\mathbf{A}$ must contain a dominant real eigenvalue denoted λ_M such

that $\lambda_M = \max_{\lambda \in \sigma(\mathbf{A})} Re(\lambda)$ where $\sigma(\mathbf{A})$ is the spectrum of $\mathbf{A}$. The Perron-Frobenius type result from Hanzon and Holland (2010b) also requires minimal realizations.

Apart from non-negativity, integrability of the function is a further requirement. The function must be normalized to ensure it integrates to unity. Integrability is most easily characterised in the context of minimal realizations (see Section 1.3). If the triple $(\mathbf{A},\ \mathbf{b},\ \mathbf{c})$ is a minimal realization then $\mathbf{c}e^{\mathbf{A}x}\mathbf{b}$ is integrable if and only if the eigenvalues of $\mathbf{A}$ are all located in the open left half plane, i.e. $\sigma(\mathbf{A}) \subset \mathbb{H}_-$. From this it follows immediately that $\mathbf{A}$ is invertible. If all eigenvalues of $\mathbf{A}$ are in the open left half plane then we have the normalisation constant

$$\int_0^\infty \mathbf{c}e^{\mathbf{A}x}\mathbf{b}dx \ = \ -\mathbf{c}\mathbf{A}^{-1}\mathbf{b} \tag{2.3}$$

The triple $(\mathbf{A}, \mathbf{b}, \tilde{\mathbf{c}})$ now represents a probability density function defined on the positive half real line where $\tilde{\mathbf{c}} \ = \ -\frac{\mathbf{c}}{\mathbf{c}\mathbf{A}^{-1}\mathbf{b}}$.

We will investigate probability density functions which can be written as two separate EPT functions depending on the sign of x

$$f(x) = \begin{cases} \mathbf{c}_N \, e^{\mathbf{A}_N \, x} \, \mathbf{b}_N & \text{if } \ \mathrm{x} \leq 0 \\ \\ \mathbf{c}_P \, e^{\mathbf{A}_P \, x} \, \mathbf{b}_P & \text{if } \ \mathrm{x} > 0 \end{cases} \tag{2.4}$$

where the triples $(\mathbf{A}_P,\ \mathbf{b}_P,\ \mathbf{c}_P)$ and $(\mathbf{A}_N,\ \mathbf{b}_N,\ \mathbf{c}_N)$ have dimensions $n_P \times n_P$, $n_P \times 1$, $1 \times n_P$ and $n_N \times n_N$, $n_N \times 1$, $1 \times n_N$ respectively. Given $f(x) \ \geq 0$ on $\mathbb{R}$ we can examine the integrability of the function. A finite normalization constant requires that, when using minimal realizations, $\mathbf{A}_P$ is stable with $\sigma(\mathbf{A}_P) \subset \mathbb{H}_-$ and $\mathbf{A}_N$ is anti-stable. $\mathbf{A}_N$ is anti-stable implying $\sigma(\mathbf{A}_N) \subset \mathbb{H}_+$, i.e. stable on the negative half line if all its eigenvalues are located in the open right half plane. Once these conditions are met, f as defined in Eq. (2.69) can be classed as a 2-EPT density. The normalisation constant is then given by $\mathbf{c}_N\mathbf{A}_N^{-1}\mathbf{b}_N - \mathbf{c}_P\mathbf{A}_P^{-1}\mathbf{b}_N$. Again the Perron-Frobenius type result implies that if

$f(x)$ is non-negative, both $\mathbf{A}_N$ and $\mathbf{A}_P$ must contain dominant real eigenvalues denoted λ_{M_+} and λ_{M_-} respectively. It is clear that $\lambda_{M_+} = \min_{\lambda \in \sigma(\mathbf{A}_N)} Re(\lambda)$ while $\lambda_{M_-} = \max_{\lambda \in \sigma(\mathbf{A}_P)} Re(\lambda)$.

We can also see that a 2-EPT probability density function has a strictly proper Laplace Transform, well defined when $Re(s) \in (\lambda_{M_-}, \lambda_{M_+})$. The Laplace Transform of the 2-EPT random variable X is given by

$$\begin{aligned}
\phi(s) &= \phi_N(s) + \phi_P(s) \\
&= \mathbb{E}(e^{-sX}) \\
&= \int_{-\infty}^{0} \mathbf{c}_N \, e^{\mathbf{A}_N x} \, \mathbf{b}_N \, e^{-s x} \, dx + \int_{0}^{\infty} \mathbf{c}_P \, e^{\mathbf{A}_P x} \, \mathbf{b}_P \, e^{-s x} \, dx \\
&= -\mathbf{c}_N \, (\mathbf{I}s - \mathbf{A}_N)^{-1} \, \mathbf{b}_N + \mathbf{c}_P \, (\mathbf{I}s - \mathbf{A}_P)^{-1} \, \mathbf{b}_P \\
&= \frac{p(s)}{q(s)} \qquad (2.5)
\end{aligned}$$

where $\mathbf{I}$ is a suitably sized identity matrix with $p(s)$ and $q(s)$ co-prime polynomials. Here $p(s)$ is a polynomial of degree m while $q(s)$ is a polynomial of degree n with $n > m$ implying $\phi(s)$ is a strictly proper rational function. For $\phi(s)$ to correspond to the Laplace Transform of a probability density function it must hold that ϕ is scaled such that $\phi(0) = 1$. Proposition 1.1 proves that the class of 2-EPT probability density functions corresponds to the class of probability density functions which are continuous on $(-\infty, \infty) \backslash \{0\}$ with a strictly proper rational Laplace Transform.

Note that it is possible to split the rational Laplace Transform into two distinct rational functions, ϕ_N and ϕ_P, whose poles are located in the open right half plane and open left half plane respectively.

Such a rational function can be represented in state space form as

$$\phi = \pi \left(\begin{array}{cc|c} \mathbf{A}_P & 0 & \mathbf{b}_P \\ 0 & \mathbf{A}_N & \mathbf{b}_N \\ \hline \mathbf{c}_P & -\mathbf{c}_N & \mathbf{d} \end{array} \right) \qquad (2.6)$$

where π maps any

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c} & \mathbf{d} \end{array}\right) \tag{2.7}$$

to ϕ and $\phi(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d}$. For our purposes $0 \leq \mathbf{d} < 1$ but so far we have assumed $\mathbf{d} = 0$.

Numerous operations can then be carried out on the characteristic function using some of the state space techniques of Hanzon and Ober (2001) and Hanzon and Scherrer (2008).

Proposition 1.1 *The class of 2-EPT probability density functions corresponds to the class of probability density functions which are continuous on* $(-\infty, \infty)\backslash\{0\}$ *with a strictly proper rational Laplace Transform.*

Consider the probability density function f defined on $(-\infty, \infty)$, continuous on $(-\infty, \infty)\backslash\{0\}$ with a strictly proper Laplace Transform ϕ.

$$\phi(s) = \int_{-\infty}^{\infty} e^{-sx} f(x)dx = \frac{p(s)}{q(s)}, \tag{2.8}$$

where p and q are co-prime polynomials of order m and n respectively where $m < n$. It allows holds that $\phi(0) = 1$ since f is a probability density function. Hence, ϕ has no poles at $s = 0$. The Perron-Frobenius type result for rational Laplace Transforms implies that the singularities of ϕ, in $\mathbb{H}_+$ and $\mathbb{H}_-$, nearest the origin are located on the real axis where $\mathbb{H}_-$ and $\mathbb{H}_+$ are the half-planes defined below. This result can also be found in Lukacs and Szazs (1970). Therefore it follows that there is no pole of ϕ located on the imaginary axis.

Let $\Lambda(r(s)) = \{s \in \mathbb{C} | r(s) = 0\}$, be the set of zeros of the polynomial $r(s)$. Also, let $\mathbb{H}_- = \{s \in \mathbb{C} | Re(s) < 0\}$ and $\mathbb{H}_+ = \{s \in \mathbb{C} | Re(s) > 0\}$.

The rational function ϕ can be additively decomposed as follows

$$\frac{p(s)}{q(s)} = \frac{p_1(s)}{q_1(s)} + \frac{p_2(s)}{q_2(s)}, \tag{2.9}$$

such that $\Lambda(q_1(s)) \subset \mathbb{H}_-$ and $\Lambda(q_2(s)) \subset \mathbb{H}_+$. Again, p_1 and q_1 are co-prime polynomials of orders m_1 and n_1 respectively. p_2 and q_2 are co-prime polynomials of orders m_2 and n_2 respectively. $n_1 + n_2 = n$. Since ϕ is a strictly proper rational function, it follows that $m_1 < n_1$ and $m_2 < n_2$ implying both p_1/q_1 and p_2/q_2 are both strictly proper rational functions.

The density function f can be written as follows

$$f(x) = \begin{cases} f_2(x) & \text{if } \mathrm{x} \leq 0 \\ \\ f_1(x) & \text{if } \mathrm{x} > 0 \end{cases} \tag{2.10}$$

where f_2 and f_1 are real non-negative functions on $(-\infty, 0]$ and $(0, \infty)$ respectively. We also have that $\int_0^\infty f_1(x)dx \leq 1$ and $\int_{-\infty}^0 f_2(x)dx \leq 1$.

$$k(s) = \int_0^\infty e^{-sx} f_1(x)dx \tag{2.11}$$

$$h(s) = \int_{-\infty}^0 e^{-sx} f_2(x)dx \tag{2.12}$$

It follows that $k(s)$ has support in $\mathbb{H}_+$ while $h(s)$ has support in $\mathbb{H}_-$. Similarly we have that

$$\frac{p(s)}{q(s)} = k(s) + h(s) \tag{2.13}$$

$$= \frac{p_1(s)}{q_1(s)} + \frac{p_2(s)}{q_2(s)} \tag{2.14}$$

By uniqueness of the additive decomposition it follows that

$$\frac{p_1(s)}{q_1(s)} = k(s) \tag{2.15}$$

$$\frac{p_2(s)}{q_2(s)} = h(s) \tag{2.16}$$

It follows that f_1 and f_2 are EPT functions defined on $(0, \infty)$ and $(-\infty, 0]$ respectively and therefore f is a 2-EPT probability density function■.

2.1.1.1. Characteristic Function of 2-EPT Function

The characteristic function of a 2-EPT density function is also rational. If we consider a 2-EPT random variable X whose minimal realization is given by $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$, then the characteristic function of X is given by

$$\begin{aligned}
\Phi(is) &= \mathbb{E}(e^{isX}) \\
&= \int_{-\infty}^{0} \mathbf{c}_N \, e^{\mathbf{A}_N \, x} \, \mathbf{b}_N \, e^{is \, x} \, dx \; + \; \int_{0}^{\infty} \mathbf{c}_P \, e^{\mathbf{A}_P \, x} \, \mathbf{b}_P \, e^{is \, x} \, dx \\
&= \mathbf{c}_N \, (\mathbf{I}iu \; + \; \mathbf{A}_N)^{-1} \, \mathbf{b}_N \; - \; \mathbf{c}_P \, (\mathbf{I}is \; + \; \mathbf{A}_P)^{-1} \, \mathbf{b}_P \\
&= \Phi_N(is) + \Phi_P(is) \\
&= \frac{p(is)}{q(is)}. \qquad (2.17)
\end{aligned}$$

Like the Laplace Transform, the rational characteristic function can be decomposed into the sum of two rational functions Φ_N and Φ_P which have support on $\mathbb{C}_-$ and $\mathbb{C}_+$ respectively. As can be seen from comparing Eq. (2.17) with Eq. (2.20), the characteristic function is the Laplace Transform evaluated at $-is$ instead of s.

2.1.2. Mixture with Delta Distribution

In Proposition 1.1 we saw that the class of 2-EPT density functions is the class of density functions on $\mathbb{R}$ with strictly proper rational Laplace Transforms. However, if one considers the broader question of all probability distributions on $\mathbb{R}$ with rational Laplace Transforms then the class turns out to be larger than the 2-EPT class considered so far. For example the point mass at zero ("delta distribution") is in this class although it does not have an EPT probability density function. Corollary 1.1 shows that the class of all probability density functions, continuous on $(-\infty, \infty)\backslash\{0\}$, containing a pointmass at zero with a proper rational Laplace Transform is equal to the class of 2-EPT probability density functions mixed with a pointmass at zero. Consider the mixture of the

delta distribution, $\delta(x)$, and 2-EPT probability densities:

$$\mathbf{d}\delta(x) \; + \; (1 \; - \; \mathbf{d})f(x) \tag{2.18}$$

where $f(x)$ is a 2-EPT probability density function and $\mathbf{d}$ a scalar such that $0 \leq \mathbf{d} \leq 1$. For some operations examined, it is more natural to consider this larger class of probability distributions while for other operations it is not.

The state space representation of the rational characteristic function is given by

$$\phi \; = \; \pi \left(\begin{array}{cc|c} \mathbf{A}_P & 0 & \mathbf{b}_P \\ 0 & \mathbf{A}_N & \mathbf{b}_N \\ \hline \mathbf{c}_P & -\mathbf{c}_N & \mathbf{d} \end{array} \right) \qquad = \qquad \pi \left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c} & \mathbf{d} \end{array} \right) \qquad , \qquad 0 \leq \mathbf{d} \leq 1 \tag{2.19}$$

The Laplace Transform of the this probability density function is the proper rational function $\phi(s)$

$$\phi(s) \; = \; -\mathbf{c}_N \left(\mathbf{I}s \; - \; \mathbf{A}_N\right)^{-1} \mathbf{b}_N \; + \; \mathbf{c}_P \left(\mathbf{I}s \; - \; \mathbf{A}_P\right)^{-1} \mathbf{b}_P + \mathbf{d} = \frac{p(s)}{q(s)} \tag{2.20}$$

where p and q are co-prime polynomials of order n and $0 \leq \mathbf{d} \leq 1$.

For the remainder of the thesis we will work with the 2-EPT probability densities unless otherwise stated where we work with the larger class which includes this mixture with the delta distribution. A 2-EPT/EPT function mixed with a Dirac function (pointmass at zero) will be referred to as a generalised 2-EPT/EPT function.

Corollary 1.1 *The class of all probability density functions, continuous on* $(-\infty, \infty)\backslash\{0\}$*, containing a pointmass at zero with a proper rational Laplace Transform is equal to the class of generalised 2-EPT probability density functions.*

Let f be a probability density function defined on $(-\infty, \infty)$, continuous on $(-\infty, \infty)\backslash\{0\}$ with a pointmass at zero and a proper rational Laplace Trans-

form ϕ given by

$$\phi(s) \;=\; \int_{-\infty}^{\infty} e^{-sx} f(x)dx \;=\; \frac{p(s)}{q(s)}, \tag{2.21}$$

where p and q are co-prime polynomials, both of order n. Using similar arguments to Proposition 1.1 the rational function $\phi(s)$ can be decomposed into the sum of two strictly proper rational functions p_1/q_1 and p_2/q_2 and the constant $\mathbf{d}$ as follows

$$\frac{p(s)}{q(s)} \;=\; \frac{p_1(s)}{q_1(s)} + \frac{p_2(s)}{q_2(s)} + \mathbf{d}, \tag{2.22}$$

such that $\Lambda(q_1(s)) \subset \mathbb{H}_-$ and $\Lambda(q_2(s)) \subset \mathbb{H}_+$. The density function f can be written as

$$f(x) = \begin{cases} f_2(x) & \text{if } \mathrm{x} < 0 \\ \\ 0 & \text{with probability } \mathbf{d} \\ \\ f_1(x) & \text{if } \mathrm{x} > 0 \end{cases} \tag{2.23}$$

where $0 < \mathbf{d} < 1$. We can define the probability density function $\hat{f}$ as

$$\hat{f}(x) = \begin{cases} \frac{f_2(x)}{1-\mathbf{d}} & \text{if } \mathrm{x} < 0 \\ \\ \frac{f_1(x)}{1-\mathbf{d}} & \text{if } \mathrm{x} > 0 \end{cases} \tag{2.24}$$

which has a strictly proper rational Laplace Transform

$$\hat{\phi}(s) = \frac{\phi(s) - \mathbf{d}}{1 - \mathbf{d}} \;=\; \frac{p_1(s)}{q_1(s)(1-\mathbf{d})} + \frac{p_2(s)}{q_2(s)(1-\mathbf{d})} \tag{2.25}$$

By proposition 1.1 $\hat{f}$ must be a 2-EPT probability density function implying f is a generalised 2-EPT probability density function.■

2.2. Operations

We now demonstrate how a series of operations can be carried out in the 2-EPT framework. The realizations resulting from these formulae may not always be

minimal. Algorithms from Kailath (1980) may be used to derive the associated minimal realization. In Chapter 7 we elucidate to the MATLAB function `minreal` which can be used to find the minimal realization from a non-minimal realization. In Section 2.2.0.1 we give an example of how a minimal realization can be computed from a non-minimal realization.

2.2.0.1. Minimal Realization

Consider a realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ of order n of a linear time invariant system. The observability $(\mathcal{O}_n)$ and controlability $(\mathcal{C}_n)$ matrices of the system can be defined as follows

$$\mathcal{O}_n(\mathbf{c}, \mathbf{A}) = \begin{pmatrix} \mathbf{c} \\ \mathbf{cA} \\ \vdots \\ \mathbf{cA}^{n-1} \end{pmatrix} \tag{2.26}$$

$$\mathcal{C}_n(\mathbf{b}, \mathbf{A}) = (\mathbf{b}\ \mathbf{Ab}\ \ldots \mathbf{A}^{n-1}\mathbf{b}) \tag{2.27}$$

The realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is called observable is $\mathcal{O}_n(\mathbf{c}, \mathbf{A})$ has full rank. Similarly the realization is controllable if $\mathcal{C}_n(\mathbf{b}, \mathbf{A})$ is of full rank. A theorem from Kalman proves that *"A realization* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ *is minimal if and only if it is controllable and observable"*.

Furthermore minimal state space realizations are only unique up to a transformation. If $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ and $(\tilde{\mathbf{A}}, \tilde{\mathbf{b}}, \tilde{\mathbf{c}}, \tilde{\mathbf{d}})$ are two minimal realizations of a given system, then there exists a unique invertible matrix $\mathbf{T}$ such that

$$\tilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{AT}\ ,\ \tilde{\mathbf{b}} = \mathbf{T}^{-1}\mathbf{b}\ \ ,\ \tilde{\mathbf{c}} = \mathbf{cT}\ ,\ \tilde{\mathbf{d}} = \mathbf{d} \tag{2.28}$$

Let $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ be an N-dimensional realization. Suppose $(\mathbf{c}, \mathbf{A})$ is observable but $(\mathbf{b}, \mathbf{A})$ is not controllable then there exists a matrix $\mathbf{P}$ such that

$$\mathbf{PAP}^T = \begin{pmatrix} \mathbf{A}_{nc} & 0 \\ \mathbf{A}_{21} & \mathbf{A}_c \end{pmatrix} \ , \ \mathbf{Pb} = (0 \ \mathbf{b}_c)^T \ , \ \mathbf{cP}^T = (\mathbf{c}_{nc} \ \mathbf{c}_c) \tag{2.29}$$

where $(\mathbf{b}_c, \ \mathbf{A}_c)$ is controllable. Then the realization $(\mathbf{A}_c, \mathbf{b}_c, \mathbf{c}_c, \mathbf{d})$ is minimal.

Similarly if $(\mathbf{b}, \mathbf{A})$ was controllable but $(\mathbf{c}, \ \mathbf{A})$ was not observable then there exists a matrix $\mathbf{Q}$ such that

$$\mathbf{QAQ}^T = \begin{pmatrix} \mathbf{A}_{no} & \mathbf{A}_{12} \\ 0 & \mathbf{A}_o \end{pmatrix} \ , \ \mathbf{Qb} = (\mathbf{b}_{no} \ \mathbf{b}_o)^T \ , \ \mathbf{cQ}^T = (0 \ \mathbf{c}_o) \tag{2.30}$$

where $(\mathbf{c}_o, \ \mathbf{A}_o)$ is observable implying the realization $(\mathbf{A}_o, \mathbf{b}_o, \mathbf{c}_o, \mathbf{d})$ is minimal.

If a realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is neither controllable or observable then a controllable realization $(\mathbf{A}_c, \mathbf{b}_c, \mathbf{c}_c, \mathbf{d})$ can be derived using the techniques described above. If $(\mathbf{A}_c, \mathbf{b}_c, \mathbf{c}_c, \mathbf{d})$ is not observable, the minimal realization can be computed using the method described above. If $(\mathbf{A}_c, \mathbf{b}_c, \mathbf{c}_c, \mathbf{d})$ is observable then it is also minimal.

2.2.0.2. Computing Minimal Realizations

Consider the transfer function H given by

$$H(s) \ = \ \frac{s^2 - 3s + 2}{s^3 - 8s^2 + 21s - 18} \ = \ \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \tag{2.31}$$

which can be represented by the state space realization $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 18 & -21 & 8 \end{pmatrix} \ , \ \mathbf{b} = (0, 0, 1)^T \ , \ \mathbf{c} = (2, -3, 1) \tag{2.32}$$

The controllability matrix can be computed and is seen to be

$$\mathcal{C}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 8 \\ 1 & 8 & 43 \end{pmatrix} \tag{2.33}$$

where rank($\mathcal{C}_3$)=3 implying $(\mathbf{b}, \mathbf{A})$ is controllable. The observability matrix is

$$\mathcal{O}_3 = \begin{pmatrix} 2 & -3 & 1 \\ 18 & -19 & 5 \\ 90 & -87 & 21 \end{pmatrix} \tag{2.34}$$

but rank($\mathcal{O}_3$) $= 2 < 3$ which means $(\mathbf{c}, \mathbf{A})$ is not observable. From Eq. (2.30) we know that there exists a matrix $\mathbf{Q}$ which can be used to derive the observable realization from the un-observable realization via a similarity transformation. The staircase algorithm of Rosenbrock (1970) can be implemented to find the appropriate transformation matrix $\mathbf{Q}$. It follows that

$$\mathbf{Q} = \begin{pmatrix} 0.2182 & 0.4364 & 0.8728 \\ 0.8164 & 0.4082 & -0.4082 \\ -0.5345 & 0.8017 & -0.2672 \end{pmatrix} \tag{2.35}$$

where

$$\mathbf{QAQ}^T = \begin{pmatrix} 2.0000 & 2.4054 & -24.9031 \\ 0.0000 & -1.0000 & 12.2202 \\ -0.0000 & -1.30930 & 7.0000 \end{pmatrix} \tag{2.36}$$

and

$$\mathbf{Qb} = (0.8728 \ -0.4082 \ -0.2673)^T \ , \ \mathbf{cQ}^T = (0.0000 \, 0.0000 \ -3.7416)$$

Hence the observable realization is $(\mathbf{A}_o, \mathbf{b}_o, \mathbf{c}_o)$ where

$$\mathbf{A}_o = \begin{pmatrix} -1.0000 & 12.2202 \\ -1.30930 & 7.0000 \end{pmatrix}, \ \mathbf{b}_o = (-0.4082 \ -0.2673)^T, \ \mathbf{c}_o = (0.0000 \ -3.7416)$$

We must now investigate if the realization $(\mathbf{A}_o, \mathbf{b}_o, \mathbf{c}_o)$ is controllable. The corresponding controllability matrix is

$$\mathcal{C}_2(\mathbf{b}_o, \mathbf{A}_o) = \begin{pmatrix} -0.4082 & -2.8577 \\ -0.2672 & -1.3363 \end{pmatrix} \tag{2.37}$$

and we note that $\text{rank}(\mathcal{C}_2(\mathbf{b}_o, \mathbf{A}_o)) = 2$ meaning $\mathcal{C}_2(\mathbf{b}_o, \mathbf{A}_o)$ is controllable and therefore $(\mathbf{A}_o, \mathbf{b}_o, \mathbf{c}_o)$ must be a minimal realization. It can be seen that

$$H_2(s) \ = \ \mathbf{c}_o(s\mathbf{I} - \mathbf{A}_o)^{-1}\mathbf{b}_o \ = \ \frac{s-1}{s^2 - 6s + 9} \ = \ H \tag{2.38}$$

where $H_2 = H$ from Eq. (2.31) confirming that $(\mathbf{A}_o, \mathbf{b}_o, \mathbf{c}_o)$ is indeed a minimal realization of $(\mathbf{A}, \mathbf{b}, \mathbf{c})$.

2.2.1. Translation

Let X be a random variable with a 2-EPT density function, f, given by the triple $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ on $[0, \infty)$ and $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ on $(-\infty, 0]$ as given in (2.69) above and let $x_0 \ \in \mathbb{R}^1$. Then the random variable $X + x_0$ has a density function $g(x)$ such that $g(x) \ = \ f(x - x_0)$. We will derive the probability density function of g in terms of the minimal realization of f is given by

$$g(x) \ = \ f(x - x_0) \ = \ \begin{cases} \mathbf{c}_N \ e^{\mathbf{A}_N \ x} \ e^{-\mathbf{A}x_0} b_N & \text{if } \mathrm{x} \leq x_0 \\ \\ \mathbf{c}_P \ e^{\mathbf{A}_P \ x} \ e^{-\mathbf{A}x_0} b_P & \text{if } \mathrm{x} > x_0 \end{cases} \tag{2.39}$$

The function $g(x)$ is 2-EPT if and only if its Laplace Transform is a proper rational function. The Laplace transform of $g(x)$ is given by

$$\begin{aligned}\phi(s) &= \int_{-\infty}^{\infty} f(x)\, e^{-s\,x}\, dx \qquad (2.40)\\ &= \int_{-\infty}^{x_0} \mathbf{c}_N\, e^{\mathbf{A}_N\, x}\, e^{-\mathbf{A}_N x_0}\, e^{-s\,x}\, \mathbf{b}_N\, dx + \int_{x_0}^{\infty} \mathbf{c}_P\, e^{\mathbf{A}_P\, x}\, e^{-\mathbf{A}_P x_0}\, e^{-s\,x}\, \mathbf{b}_P\, dx\\ &= -\mathbf{c}_N\, (s\mathbf{I} - \mathbf{A}_N)^{-1} e^{-s\mathbf{I}x_0}\mathbf{b}_N + \mathbf{c}_P\, (s\mathbf{I} - \mathbf{A}_P)^{-1} e^{-s\mathbf{I}x_0}\mathbf{b}_P\end{aligned}$$

which is not rational. Hence a translated 2-EPT function is not an 2-EPT function.

2.2.2. Scaling

Let X be a random variable with a 2-EPT density function given in (2.69) and cumulative distribution function given by F. Now let $\alpha \in \mathbb{R}^1$ such that $\alpha > 0$ and $Y = \alpha X$. Y has a density function given by g with cumulative distribution G. We obtain that

$$\begin{aligned}G(y_0) &= \mathbb{P}(Y \leq y_0) \qquad (2.41)\\ &= \mathbb{P}(\alpha\, X \leq y_0)\\ &= \mathbb{P}(X \leq \frac{y_0}{\alpha})\\ &= F(\frac{y_0}{\alpha})\end{aligned}$$

Differentiating we then get that

$$g(y_0) = f(\frac{y_0}{\alpha})\frac{1}{\alpha} \qquad (2.42)$$

and leaving $x_0 = \frac{y_0}{\alpha}$ we obtain that

$$\alpha\, g(\alpha\, x_0) = f(x_0) \qquad (2.43)$$

Hence if our original triples for X were $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ and $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ the associated triples for Y are given by $(\alpha\ \mathbf{A}_P, \mathbf{b}_P, \alpha\ \mathbf{c}_P)$ and $(\alpha\ \mathbf{A}_N, \mathbf{b}_N, \alpha\ \mathbf{c}_N)$. This feature of treating the triples separately depending on the sign of x will be seen throughout our work.

Some care has to be taken when $\alpha < 0$. If we consider the triples as outlined in Eq. (2.69), for $\alpha < 0$ we get

$$g(y) = \begin{cases} |\alpha|\ \mathbf{c}_N\ e^{\alpha\ \mathbf{A}_N\ y}\ \mathbf{b}_N & \text{if } \mathrm{y} > 0 \\ \\ |\alpha|\ \mathbf{c}_P\ e^{\alpha\ \mathbf{A}_P\ y}\ \mathbf{b}_P & \text{if } \mathrm{y} \leq 0 \end{cases} \tag{2.44}$$

where the density is flipped and scaled around the origin.

2.2.3. Sums of 2-EPT Densities

We consider weighted sums of 2-EPT functions $f_i(x)$, each with minimal realization $(\mathbf{A}_{P,i}, \mathbf{b}_{P,i}, \mathbf{c}_{P,i})$ for $x > 0$ and $(\mathbf{A}_{N,i}, \mathbf{b}_{N,i}, \mathbf{c}_{N,i})$ for $x \leq 0$. The addition of 2-EPT densities gives rise directly to mixture densities. A mixture density is a convex combination of probability density functions such that the properties of non-negativity and integration to unity are preserved. Hence we obtain an expression of the form

$$g(x) = \sum_{i=1}^{N} w_i f_i(x) \qquad , \qquad \sum_{i=1}^{N} w_i = 1 \tag{2.45}$$

where we restrict $w_i > 0\ \forall\ i$. The mixture density $g(x)$ on the whole real line in the simple case of $N = 2$ and weight $w_1 = w$ and $w_2 = 1 - w$ is given by

$$g(x) = \begin{cases} w\ \mathbf{c}_{N,1}\ e^{\mathbf{A}_{N,1}\ x}\ \mathbf{b}_{N,1} + (1-w)\ \mathbf{c}_{N,2}\ e^{\mathbf{A}_{N,2}\ x}\ \mathbf{b}_{N,2} = \mathbf{c}_N\ e^{\mathbf{A}_N\ x}\ \mathbf{b}_N & \text{if } \mathrm{x} \leq 0 \\ \\ w\ \mathbf{c}_{P,1}\ e^{\mathbf{A}_{P,1}\ x}\ \mathbf{b}_{P,1} + (1-w)\ \mathbf{c}_{P,2}\ e^{\mathbf{A}_{P,2}\ x}\ \mathbf{b}_{P,2} = \mathbf{c}_P\ e^{\mathbf{A}_P\ x}\ \mathbf{b}_P & \text{if } \mathrm{x} > 0 \end{cases} \tag{2.46}$$

for $0 < w < 1$, where for $x > 0$ we have $\mathbf{c}_P = (w\mathbf{c}_{P,1},\ (1-w)\mathbf{c}_{P,2})$, $\mathbf{b}_P = (\mathbf{b}_{P,1},\ \mathbf{b}_{P,2})^T$ and similarly for $\mathbf{c}_N$ and $\mathbf{b}_N$ when $x \leq 0$. We split the

A matrices into a block diagonal structure as follows

$$\mathbf{A}_N = \begin{pmatrix} \mathbf{A}_{N,1} & 0 \\ 0 & \mathbf{A}_{N,2} \end{pmatrix} \quad , \quad \mathbf{A}_P = \begin{pmatrix} \mathbf{A}_{P,1} & 0 \\ 0 & \mathbf{A}_{P,2} \end{pmatrix} \tag{2.47}$$

This approach provides an ideal framework for working with mixture densities and determining their properties in straightforward manner. The approach can easily be extended for a general $N > 2$ provided $w_i > 0$. If $w_i < 0$ for any i we remain in the 2-EPT class but non-negativity of the density may be violated. To test for non-negativity, the reader is referred to Hanzon and Holland (2010) and Hanzon and Holland (2012).

2.2.4. Product of 2-EPT Densities

Suppose we wish to calculate the product of two 2-EPT probability density functions $f_1(x)$ and $f_2(x)$ defined over the whole real line in both instances. We define $f_i(x)$ as

$$f_i(x) = \begin{cases} \mathbf{c}_{N,i}\, e^{\mathbf{A}_{N,i}\, x}\, \mathbf{b}_{N,i} & \text{if } x \leq 0 \\ \\ \mathbf{c}_{P,i}\, e^{\mathbf{A}_{P,i}\, x}\, \mathbf{b}_{P,i} & \text{if } x > 0 \end{cases} \tag{2.48}$$

where $\mathbf{A}_{N,i}$ and $\mathbf{A}_{P,i}$ are square matrices of size $n_{N,i}$ and $n_{P,i}$ respectively. As seen earlier we have to treat the EPT functions on the positive and negative domains separately, as follows

$$g(x) = f_1(x)\, f_2(x) = \begin{cases} \mathbf{c}_{N,1}\, e^{\mathbf{A}_{N,1}\, x}\, \mathbf{b}_{N,1}\, \mathbf{c}_{N,2}\, e^{\mathbf{A}_{N,2}\, x}\, \mathbf{b}_{N,2} = \mathbf{c}_N\, e^{\mathbf{A}_N\, x}\, \mathbf{b}_N & \text{if } x \leq 0 \\ \\ \mathbf{c}_{P,1}\, e^{\mathbf{A}_{P,1}\, x}\, \mathbf{b}_{P,1}\, \mathbf{c}_{P,2}\, e^{\mathbf{A}_{P,2}\, x}\, \mathbf{b}_{P,2} = \mathbf{c}_P\, e^{\mathbf{A}_P\, x}\, \mathbf{b}_P & \text{if } x > 0 \end{cases} \tag{2.49}$$

For $x > 0$ we obtain the triple $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$

$$\begin{aligned} \mathbf{A}_P &= \mathbf{A}_{P,1} \otimes \mathbf{I}_{n_{P,2}} + \mathbf{I}_{n_{P,1}} \otimes A_{P,2} \\ \mathbf{b}_P &= \mathbf{b}_{P,1} \otimes \mathbf{b}_{P,2} \\ \mathbf{c}_P &= \mathbf{c}_{P,1} \otimes \mathbf{c}_{P,2} \end{aligned} \tag{2.50}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Similarly the triple can be obtained for $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$. However the product of two probability densities is not guaranteed to be a probability density function in the sense that it does not necessarily integrate to unity and may have to be normalized. Hence for $g(x)$ to be a probability density function we require

$$\tilde{\mathbf{c}}_P = \frac{\mathbf{c}_P}{\mathbf{c}_N \mathbf{A}_N^{-1} \mathbf{b}_N - \mathbf{c}_P \mathbf{A}_P^{-1} \mathbf{b}_P} \tag{2.51}$$

$$\tilde{\mathbf{c}}_N = \frac{\mathbf{c}_N}{\mathbf{c}_N \mathbf{A}_N^{-1} \mathbf{b}_N - \mathbf{c}_P \mathbf{A}_P^{-1} \mathbf{b}_P} \tag{2.52}$$

and the realization $(\mathbf{A}_N, \mathbf{A}_P, \mathbf{b}_N, \mathbf{b}_P, \tilde{\mathbf{c}}_N, \tilde{\mathbf{c}}_P)$ then represents the 2-EPT probability density function.

2.2.5. Moments

It will be seen that all moments of 2-EPT densities exist. This can be seen by noticing that all higher order derivatives of the rational characteristic function Φ are again rational functions with the same pole locations as Φ. We first examine a closed form expression for moments of a 2-EPT random variable using derivatives of the characteristic function Φ and then examine how to calculate the k^{th} moments by integrating x^k multiplied by the density over the domain.

2.2.5.1. Calculating Moments using Characteristic Function

We have already seen that the characteristic function of an 2-EPT density can be written in terms of the sum of two rational functions. We also know that the k^{th} moment is given by the k^{th} derivative of the characteristic function evaluated at zero. The characteristic function is described in Eq. 2.17. The k^{th} moment of a random variable X is then given by

$$\mathbb{E}(X^k) = (-i)^k \Phi^k(0),$$

where Φ^k represents the k^{th} derivative of Φ. If X is a 2-EPT random variable with minimal realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ then it follows that

$$\mathbb{E}(X^k) \;=\; -k!\, \mathbf{c}_N\, (-\mathbf{A}_N)^{-(k+1)}\, \mathbf{b}_N \;+\; k!\, \mathbf{c}_P\, (-\mathbf{A}_P)^{-(k+1)}\, \mathbf{b}_P. \tag{2.53}$$

Note that $\mathbf{A}_N$ and $\mathbf{A}_P$ are invertible as we are using minimal realizations.

Also if the realization for the 2-EPT density function is given by the minimal $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ as in Eq. (2.7) then the k^{th} moment is

$$\mathbb{E}(X^k) \;=\; k!\, \mathbf{c}(-\mathbf{A})^{-(k+1)}\mathbf{b} \tag{2.54}$$

2.2.5.2. Calculating Moments Using Direct Integration

Now we consider how to find the k^{th} moment of a 2-EPT density $f(x)$ by exploiting its 2-EPT form and integrating $x^k f(x)$ over the relevant domain. x^k can be represented as a polynomial of order k which has the form $p(x) \;=\; x^k \;=\; \mathbf{c}_k\, e^{\mathbf{a}_k\, x}\, \mathbf{b}_k$ where $\mathbf{a}_k$ is a square $(k+1) \times (k+1)$ nilpotent matrix.

Hence if we are looking for the truncated k^{th} moment over the interval $[-d,\, d]$ where $d \;\in\; \mathbb{R}$, $d \;>\; 0$ and $f(x)$ is as given in (2.69) we obtain

$$\begin{aligned}
\int_{-d}^{d} x^k\, f(x)\, dx \;&=\; \int_{-d}^{d} \mathbf{c}_k\, e^{\mathbf{a}_k\, x}\, \mathbf{b}_k\, f(x)\, dx \qquad (2.55)\\
&=\; \int_{-d}^{0} \mathbf{c}_k\, e^{\mathbf{a}_k\, x}\, \mathbf{b}_k\, \mathbf{c}_N\, e^{\mathbf{A}_N\, x} \mathbf{b}_N\, dx + \int_{0}^{d} \mathbf{c}_k\, e^{\mathbf{a}_k\, x}\, \mathbf{b}_k\, \mathbf{c}_P\, e^{\mathbf{A}_P\, x} \mathbf{b}_P\, dx\\
&=\; \int_{-d}^{0} \tilde{\mathbf{c}}_N\, e^{\tilde{\mathbf{A}}_N\, x} \tilde{\mathbf{b}}_N\, dx + \int_{0}^{d} \tilde{\mathbf{c}}_P\, e^{\tilde{\mathbf{A}}_P\, x} \tilde{\mathbf{b}}_P\, dx\\
&=\; \tilde{\mathbf{c}}_N\, \tilde{\mathbf{A}}_N^{-1}\, \tilde{\mathbf{b}}_N - \tilde{\mathbf{c}}_N\, \tilde{\mathbf{A}}_N^{-1}\, e^{-\tilde{\mathbf{A}}_N\, d}\, \tilde{\mathbf{b}}_N + \tilde{\mathbf{c}}_P\, \tilde{\mathbf{A}}_P^{-1}\, e^{\tilde{\mathbf{A}}_P\, d}\, \tilde{\mathbf{b}}_P - \tilde{\mathbf{A}}_P\, \tilde{\mathbf{A}}_P^{-1}\, \tilde{\mathbf{b}}_P
\end{aligned}$$

where the triples $(\tilde{\mathbf{A}_P},\, \tilde{\mathbf{b}_P},\, \tilde{\mathbf{c}_P})$ and $(\tilde{\mathbf{A}_N},\, \tilde{\mathbf{b}_N},\, \tilde{\mathbf{c}_N})$ can be determined using the product formulae in Section 2.2.4. In the limit $d \;\to\; \infty$ we obtain

$$\int_{-\infty}^{\infty} x^k\, \mathbf{c}\, e^{\mathbf{A}\, x}\mathbf{b}\, dx \;\;=\;\; \tilde{\mathbf{c}}_N\, \tilde{\mathbf{A}}_N^{-1}\, \tilde{\mathbf{b}}_N - \tilde{\mathbf{c}}_P\, \tilde{\mathbf{A}}_P^{-1}\, \tilde{\mathbf{b}}_P \tag{2.56}$$

2.2.6. Minimum of 2-EPT Random Variables

In financial as well as probabilistic applications it is often quite important to find certain properties of the minimum of a number of independent random variables. Here we propose a straightforward approach to finding the density of the minimum of two 2-EPT random variables. We consider the case where we have two independent random variables X, Y with 2-EPT density functions f_1 and f_2 respectively such that

$$f_i(x) = \begin{cases} \mathbf{c}_{N,i}\, e^{\mathbf{A}_{N,i}\, x}\, \mathbf{b}_{N,i} & \text{if } \mathrm{x} \leq 0 \\ \\ \mathbf{c}_{P,i}\, e^{\mathbf{A}_{P,i}\, x}\, \mathbf{b}_{P,i} & \text{if } \mathrm{x} > 0 \end{cases} \tag{2.57}$$

for $i = 1, 2$. Hence it can be seen that the associated cumulative distribution functions are again 2-EPT and given by

$$F_i(x) = \begin{cases} \mathbf{c}_{N,i}\, \mathbf{A}_{N,i}^{-1}\, e^{\mathbf{A}_{N,i}\, x}\, \mathbf{b}_{N,i} & \text{if } \mathrm{x} \leq 0 \\ \\ \mathbf{c}_{P,i}\, \mathbf{A}_{P,i}^{-1}\, e^{\mathbf{A}_{P,i}\, x}\, \mathbf{b}_{P,i} \underbrace{-\mathbf{c}_{P,i}\, \mathbf{A}_{P,i}^{-1}\, \mathbf{b}_{P,i} + \mathbf{c}_{N,i}\, \mathbf{A}_{N,i}^{-1}\, \mathbf{b}_{N,i}}_{=1} & \text{if } \mathrm{x} > 0 \end{cases} \tag{2.58}$$

It is straightforward to see that in the limit $x \to \infty$ we must have $\lim_{x\to\infty} F_i(x) = 1$. This implies that the cumulative distribution functions can be written as

$$F_i(x) = \begin{cases} \mathbf{c}_{N,i}\, \mathbf{A}_{N,i}^{-1}\, e^{\mathbf{A}_{N,i}\, x}\, \mathbf{b}_{N,i} & \text{if } \mathrm{x} \leq 0 \\ \\ \mathbf{c}_{P,i}\, \mathbf{A}_{P,i}^{-1}\, e^{\mathbf{A}_{P,i}\, x}\, \mathbf{b}_{P,i} + 1 & \text{if } \mathrm{x} > 0 \end{cases} \tag{2.59}$$

We now let $W = \min(X, Y)$ and note that

$$\begin{aligned} \mathbb{P}[W \leq w] &= 1 - \mathbb{P}[X > w]\mathbb{P}[Y > w] \qquad \ldots \qquad \textit{by independence} \\ &= 1 - (1 - \mathbb{P}[X \leq w])(1 - \mathbb{P}[Y \leq w]) \\ &= 1 - (1 - F_1(w))(1 - F_2(w)) \\ &= F_1(w) + F_2(w) - F_1(w)F_2(w) \end{aligned} \tag{2.60}$$

As before we have to treat the cases where $w > 0$ and $w \leq 0$ separately. It can be seen that for $w \leq 0$ we obtain

$$\begin{aligned}
\mathbb{P}[W \leq w] &= \mathbf{c}_{N,1}\, \mathbf{A}_{N,1}^{-1}\, e^{\mathbf{A}_{N,1}\, w}\, \mathbf{b}_{N,1} + \mathbf{c}_{N,2}\, \mathbf{A}_{N,2}^{-1}\, e^{\mathbf{A}_{N,2}\, w}\, \mathbf{b}_{N,2} \quad (2.61)\\
&\quad - \mathbf{c}_{N,1}\, \mathbf{A}_{N,1}^{-1}\, e^{\mathbf{A}_{N,1}\, w}\, \mathbf{b}_{N,1}\, \mathbf{c}_{N,2}\, \mathbf{A}_{N,2}^{-1}\, e^{\mathbf{A}_{N,2}\, w}\, \mathbf{b}_{N,2}\\
&= \mathbf{c}_{N,1}\, \mathbf{A}_{N,1}^{-1}\, e^{\mathbf{A}_{N,1}\, w}\, \mathbf{b}_{N,1} + \mathbf{c}_{N,2}\, \mathbf{A}_{N,2}^{-1}\, e^{\mathbf{A}_{N,2}\, w}\, \mathbf{b}_{N,2} \quad (2.62)\\
&\quad -\mathbf{c}_N\, \tilde{\mathbf{A}}_{N,1,N,2}^{-1}\, e^{\mathbf{A}_{N,1,N,2}\, w}\, \mathbf{b}_N \qquad \forall\ w \leq 0
\end{aligned}$$

where $\mathbf{c}_N = \mathbf{c}_{N,1} \otimes \mathbf{c}_{N,2}$, $\mathbf{b}_N = \mathbf{b}_{N,1} \otimes \mathbf{b}_{N,2}$, $\tilde{\mathbf{A}}_{N,1,N,2} = (\mathbf{A}_{N,1} \otimes \mathbf{A}_{N,2})$ and $\mathbf{A}_{N,1,N,2} = \mathbf{A}_{N,1} \otimes \mathbf{I}_{N,2} + \mathbf{I}_{N,1} \otimes \mathbf{A}_{N,2}$.

We consider the case where $w > 0$ and see that

$$\begin{aligned}
\mathbb{P}[W \leq w] &= 1 - (1 - F_1(w))(1 - F_2(w)) \quad (2.63)\\
&= 1 - \mathbf{c}_{P,1}\, \mathbf{A}_{P,1}^{-1}\, e^{\mathbf{A}_{P,1}\, w}\, \mathbf{b}_{P,1}\, \mathbf{c}_{P,2}\, \mathbf{A}_{P,2}^{-1}\, e^{\mathbf{A}_{P,2}\, w}\, \mathbf{b}_{P,2}\\
&= 1 - \mathbf{c}_P\, \tilde{\mathbf{A}}_{P,1,P,2}^{-1}\, e^{\mathbf{A}_{P,1,P,2}\, w}\, \mathbf{b}_P
\end{aligned}$$

where $\mathbf{c}_P = \mathbf{c}_{P,1} \otimes \mathbf{c}_{P,2}$, $\mathbf{b}_P = \mathbf{b}_{P,1} \otimes \mathbf{b}_{P,2}$, $\tilde{\mathbf{A}}_{P,1,P,2} = (\mathbf{A}_{P,1} \otimes \mathbf{A}_{P,2})$ and $\mathbf{A}_{P,1,P,2} = \mathbf{A}_{P,1} \otimes \mathbf{I}_{P,2} + \mathbf{I}_{P,1} \otimes \mathbf{A}_{P,2}$.

By differentiating we can find the corresponding density function $h(w)$

$$h(w) = \begin{cases} \mathbf{c}_{N,1}\, e^{\mathbf{A}_{N,1}\, w}\, \mathbf{b}_{N,1} + \mathbf{c}_{N,2}\, e^{\mathbf{A}_{N,2}\, w}\, \mathbf{b}_{N,2} - \mathbf{c}_N\, \tilde{\mathbf{A}}_{N,1,N,2}^{-1}\, \mathbf{A}_{N,1,N,2}\, e^{\tilde{\mathbf{A}}_{N,1,N,2}\, w}\, \mathbf{b}_N & \text{if } w \leq 0 \\ -\mathbf{c}_P\, \tilde{\mathbf{A}}_{P,1,P,2}^{-1}\, \mathbf{A}_{P,1,P,2}\, e^{\tilde{\mathbf{A}}_{P,1,P,2}\, w}\, \mathbf{b}_P & \text{if } w > 0 \end{cases} \quad (2.64)$$

It should be noted that h is again 2-EPT as it is composed of sums of EPT functions.

2.2.6.1. Minimum of 2-EPT Random Variable and Zero

The operation $\min(X, 0)$, where X is a 2-EPT random variable, is of special importance in Chapter 4 so the operation will be examined in detail here. Let $W = \min(X, 0)$ where X has a minimal realization as described above and we

note that

$$\mathbb{P}[X > 0] \;=\; \int_0^\infty \mathbf{c}_{P,1} e^{\mathbf{A}_{P,1} x} \mathbf{b}_{P,1} dx \;=\; -\mathbf{c}_{P,1} \mathbf{A}_{P,1}^{-1} \mathbf{b}_{P,1}.$$

It follows that the generalized EPT distribution h of W must contain a point-mass at zero of $-\mathbf{c}_{P,1}\mathbf{A}_{P,1}^{-1}\mathbf{b}_{P,1}$ and is given by

$$h(w) = \begin{cases} \mathbf{c}_{N,1}\, e^{\mathbf{A}_{N,1}\, w}\, \mathbf{b}_{N,1} & \forall \;\; \mathrm{w} < 0 \\ 0 & \text{with probability } -\mathbf{c}_{P,1}\mathbf{A}_{P,1}^{-1}\mathbf{b}_{P,1} \\ 0 & \forall \;\; \mathrm{w} > 0 \end{cases} \tag{2.65}$$

2.2.7. Maximum of 2-EPT Random Variables

As above we define two random variables X, Y each with 2-EPT densities f_1 and f_2 respectively given by Eq. (2.57) whose cumulative distributions are given by Eq. (2.59).

We let $W = Max(X, Y)$ where X, Y are independent random variables. For $w \leq 0$ we obtain

$$\begin{aligned} \mathbb{P}[W \leq w] &= \mathbb{P}[X \leq w,\, Y \leq w] \\ &= \mathbb{P}[X \leq w]\mathbb{P}[Y \leq w] \\ &= \mathbf{c}_{N,1}\, \mathbf{A}_{N,1}^{-1}\, e^{\mathbf{A}_{N,1}\, x}\, \mathbf{b}_{N,1}\, \mathbf{c}_{N,2}\, \mathbf{A}_{N,2}^{-1}\, e^{\mathbf{A}_{N,2}\, y}\, \mathbf{b}_{N,2} \\ &= \mathbf{c}_N\, \tilde{\mathbf{A}}_{N,1,N,2}^{-1}\, e^{\mathbf{A}_{N,1,N,2}\, w}\, \mathbf{b}_N \end{aligned} \tag{2.66}$$

We now consider the case for $w > 0$ and see that

$$\begin{aligned} \mathbb{P}[W \leq w] &= \mathbb{P}[X \leq w,\, Y \leq w] \\ &= (1 + \mathbf{c}_{P,1}\, \mathbf{A}_{P,1}^{-1}\, e^{\mathbf{A}_{P,1}\, w}\, \mathbf{b}_{P,1})(1 + \mathbf{c}_{P,2}\, \mathbf{A}_{P,2}^{-1}\, e^{\mathbf{A}_{P,2}\, w}\, \mathbf{b}_{P,2}) \\ &= 1 + \mathbf{c}_{P,1}\, \mathbf{A}_{P,1}^{-1}\, e^{\mathbf{A}_{P,1}\, w}\, \mathbf{b}_{P,1} + \mathbf{c}_{P,2}\, \mathbf{A}_{P,2}^{-1}\, e^{\mathbf{A}_{P,2}\, w}\, \mathbf{b}_{P,2} \\ &+ \mathbf{c}_{P,1}\, \mathbf{A}_{P,1}^{-1}\, e^{\mathbf{A}_{P,1}\, w}\, \mathbf{b}_{P,1}\, \mathbf{c}_{P,2}\, \mathbf{A}_{P,2}^{-1}\, e^{\mathbf{A}_{P,2}\, w}\, \mathbf{b}_{P,2} \\ &= 1 + \mathbf{c}_{P,1}\, \mathbf{A}_{P,1}^{-1}\, e^{\mathbf{A}_{P,1}\, w}\, \mathbf{b}_{P,1} + \mathbf{c}_{P,2}\, \mathbf{A}_{P,2}^{-1}\, e^{\mathbf{A}_{P,2}\, w}\, \mathbf{b}_{P,2} + \mathbf{c}_P\, \tilde{\mathbf{A}}_{P,1,P,2}^{-1}\, e^{\mathbf{A}_{P,1,P,2}\, w}\, \mathbf{b}_P \end{aligned} \tag{2.67}$$

By differentiating we can find the corresponding density function for the maximum $h(w)$

$$h(w) = \begin{cases} \mathbf{c}_N \ \tilde{\mathbf{A}}^{-1}_{N,1,N,2} \ \mathbf{A}_{N,1,N,2} \ e^{\mathbf{A}_{N,1,N,2} \ w} \ \mathbf{b}_N & \text{if } \mathrm{w} \leq 0 \\ \mathbf{c}_{P,1} \ e^{\mathbf{A}_{P,1} \ w} \ \mathbf{b}_{P,1} + \mathbf{c}_{P,2} \ e^{\mathbf{A}_{P,2} \ w} \ \mathbf{b}_{P,2} + \mathbf{c}_P \ \tilde{\mathbf{A}}^{-1}_{P,1,P,2} \ \mathbf{A}_{P,1,P,2} \ e^{\mathbf{A}_{P,1,P,2} \ w} \ \mathbf{b}_P & \text{if } \mathrm{w} > 0 \end{cases} \tag{2.68}$$

Again h is 2-EPT.

Using a similar argument to in Section 2.2.6.1, the generalized EPT distribution of $W = \max(X, 0)$ is given by

$$h(w) = \begin{cases} 0 & \forall \ \mathrm{w} < 0 \\ 0 & \text{with probability } -\mathbf{c}_{N,1}\mathbf{A}^{-1}_{N,1}\mathbf{b}_{N,1} \\ \mathbf{c}_{P,1} e^{\mathbf{A}_{P,1} w} \mathbf{b}_{P,1} & \forall \ \mathrm{w} > 0 \end{cases} \tag{2.69}$$

2.2.8. Convolution of 2-EPT Probability Densities

Given two stochastically independent random variables X, Y, the characteristic function for the sum of the random variables is given by the product of their characteristic functions. It has been shown that the 2-EPT density functions have rational characteristic functions which here we denote Φ_X and Φ_Y respectively. We denote the triple for Φ_X by $(\mathbf{A}_1, \ \mathbf{b}_1, \ \mathbf{c}_1)$ and for Φ_Y by $(\mathbf{A}_2, \ \mathbf{b}_2, \ \mathbf{c}_2)$ as we have done in Eq. (2.6) and (2.7). We implement a state space calculus approach to calculate the product of Φ_X and Φ_Y. The state space realizations can be written as

$$\left(\begin{array}{c|c} \mathbf{A}_i & \mathbf{b}_i \\ \hline \mathbf{c}_i & 0 \end{array} \right) \tag{2.70}$$

$\mathbf{A}_i$ an $(n_i \times n_i)$ matrix, $\mathbf{b}_i$ a $(n_i \times 1)$ column vector and $\mathbf{c}_i$ a $(1 \times n_i)$ row vector. The $\mathbf{A}_i$ matrix comprises of $\mathbf{A}_{P,i}$ and $\mathbf{A}_{N,i}$ block diagonalised. The characteristic function for the sum of $X + Y$ is the product of their characteristic

functions and given by

$$\Phi_{X+Y} = \pi \left(\begin{array}{cc|c} \mathbf{A}_1 & \mathbf{b}_1\mathbf{c}_2 & 0 \\ 0 & \mathbf{A}_2 & \mathbf{b}_2 \\ \hline \mathbf{c}_1 & 0 & 0 \end{array} \right) = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c} & 0 \end{array} \right) \tag{2.71}$$

according to the well known formula which is also given in Hanzon and Ober (2001). It is clear that some basis transformations will have to be performed to ensure the $\mathbf{A}$ matrix is block diagonal with eigenvalues split depending on the sign of their real parts. We know that this splitting is possible as the rational characteristic function can be decomposed into the sum of two rational functions with poles in either half plane. Once $\mathbf{A}$ is block diagonal with a minimal realization we will be in a position to carry out all other operations cited. Similar transformations will have to be completed on the vectors $\mathbf{b}$ and $\mathbf{c}$ as illustrated below.

2.2.8.1. Generalised Convolution Formula

We will now consider the above convolution problem when the densities involved are mixtures of the pointmass at zero and 2-EPT functions, as described in Section 2.1.2. Let X, Y be random variables with proper rational characteristic functions, Φ_X and Φ_Y, which have minimal realizations given by $(\mathbf{A}_1,\ \mathbf{b}_1,\ \mathbf{c}_1,\ \mathbf{d}_1)$ and $(\mathbf{A}_2,\ \mathbf{b}_2,\ \mathbf{c}_2,\ \mathbf{d}_2)$ respectively. Assume $\mathbf{d}_1 > 0$ and $\mathbf{d}_2 > 0$. The state space realization for the characteristic function of the random variable $Z = X + Y$ is given by

$$\Phi_Z \ = \ \Phi_X\Phi_Y = \pi \left(\begin{array}{cc|c} \mathbf{A}_1 & \mathbf{b}_1\mathbf{c}_2 & \mathbf{d}_2\mathbf{b}_1 \\ 0 & \mathbf{A}_2 & \mathbf{b}_2 \\ \hline \mathbf{c}_1 & \mathbf{d}_1\mathbf{c}_2 & \mathbf{d}_1\mathbf{d}_2 \end{array} \right) \tag{2.72}$$

2.2.8.2. Additive Decomposition of Rational Function in State Space Form

A simple Matlab function is available to perform the calculation in one step and is described in 7.5.1. However for completeness we illustrate how to transform 'by hand' the **A** matrix to block diagonal form while separating the eigenvalues according to the sign of their real part. We first perform a state space basis transformation to split the eigenvalues so those with positive real part are all in the upper block and those with negative real part in the lower block or vice-versa. This can be done in a step by step method as is shown later in Section 2.3.3. It is crucial to achieving a unique solution that the **A** matrix is brought back to upper diagonal form after each operation. Assuming the sequence of operations can be expressed by the square matrix **Q**, the state space representation will now be given by

$$\phi_Z = \pi \left(\begin{array}{c|c} \mathbf{QAQ}^{-1} & \mathbf{Qb} \\ \hline \mathbf{cQ}^{-1} & 0 \end{array} \right) \tag{2.73}$$

We note that such a transformation does not alter the density as we examine the Taylor series expansion of the original density

$$\mathbf{c}\, e^{\mathbf{A}x}\, \mathbf{b} \;=\; \sum_{k=0}^{\infty} \frac{\mathbf{c}\,(\mathbf{A}x)^k\,\mathbf{b}}{k!} \;=\; \mathbf{cb} \;+\; \mathbf{c}\,\mathbf{A}x\,\mathbf{b} \;+\; \frac{\mathbf{c}\,(\mathbf{A}x)^2\,\mathbf{b}}{2!} \;+\; \ldots \tag{2.74}$$

while after the transformation we obtain

$$\begin{aligned} \mathbf{cQ}^{-1}\, e^{\mathbf{QA}x\mathbf{Q}^{-1}}\, \mathbf{Qb} \quad &= \quad \mathbf{cQ}^{-1}\mathbf{Qb} \;+\; \mathbf{cQ}^{-1}\,\mathbf{QA}x\mathbf{Q}^{-1}\,\mathbf{Qb} \;+\; \frac{\mathbf{cQ}^{-1}\,(\mathbf{QA}x\mathbf{Q}^{-1})^2\,\mathbf{Qb}}{2!} \\ &= \quad \mathbf{cb} \;+\; \mathbf{c}\,\mathbf{A}x\,\mathbf{b} \;+\; \frac{\mathbf{c}\,(\mathbf{A}x)^2\,\mathbf{b}}{2!} \;+\; \ldots \end{aligned} \tag{2.75}$$

The $\tilde{\mathbf{A}}$ Matrix is now of the form

$$\mathbf{QAQ}^{-1} \;=\; \tilde{\mathbf{A}} = \left(\begin{array}{cc} \mathbf{A}_P & \mathcal{V} \\ 0 & \mathbf{A}_N \end{array} \right) \tag{2.76}$$

where the eigenvalues of $\mathbf{A}_P$ have all negative real parts and the eigenvalues of $\mathbf{A}_N$ have positive real parts. However we must now eliminate the upper right block matrix $\mathcal{V}$. This again is done via a state space basis transformation by letting

$$\mathbf{T} = \begin{pmatrix} I & \mathcal{T} \\ 0 & I \end{pmatrix} \tag{2.77}$$

where $\mathbf{T}$ has the same dimensions as $\tilde{\mathbf{A}}$ above, $\mathbf{I}$ is the identity matrix and $\mathcal{T}$ is a square sub-matrix the same size as $\mathcal{V}$.

$$\mathbf{TQAQ}^{-1}\mathbf{T}^{-1} = \mathbf{T}\tilde{\mathbf{A}}\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{A}_P & \mathcal{T}\mathbf{A}_N - \mathbf{A}_P\mathcal{T} + \mathcal{V} \\ 0 & \mathbf{A}_N \end{pmatrix} \tag{2.78}$$

We choose $\mathcal{T}$ to be the unique solution to the sylvester equation

$$\mathcal{T}\mathbf{A}_N \; - \; \mathbf{A}_P\mathcal{T} \; + \; \mathcal{V} \; = \; 0 \tag{2.79}$$

The transformed $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ triple is then given by $(\mathbf{TQAQ}^{-1}\mathbf{T}^{-1},\ \mathbf{TQb},\ \mathbf{cQ}^{-1}\mathbf{T}^{-1})$.

2.2.9. Composition Formulae

2.2.9.1. Composition of Rational Functions

We restate Proposition 3.2 from Hanzon and Scherrer (2008) for completeness as the result holds when the rational functions being composed have poles in both half planes. Letting g_1 and g_2 each be mixtures of 2-EPT probability density functions with pointmasses at zero, as shown in Section 2.1.2, and state space realizations $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{d}_1)$ and $(\mathbf{A}_2, \mathbf{b}_2, \mathbf{c}_2, \mathbf{d}_2)$ respectively. Then the Laplace Transforms of g_1 and g_2 are the proper rational functions G_1 and G_2 respectively. Proposition 3.2 from Hanzon and Scherrer (2008) states that the the composition, $G = G_1 \circ G_2$, is again a proper rational function with the

realization of G given by $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ where

$$\begin{aligned} \mathbf{A} &= \mathbf{I} \otimes \mathbf{A}_2 + (\mathbf{A}_1 - \mathbf{d}_2\mathbf{I})^{-1} \otimes b_2 c_2 \qquad (2.80) \\ \mathbf{b} &= -(\mathbf{A}_1 - \mathbf{d}_2\mathbf{I})^{-1}\mathbf{b}_1 \otimes \mathbf{b}_2 \\ \mathbf{c} &= \mathbf{c}_1(\mathbf{A}_1 - \mathbf{d}_2\mathbf{I})^{-1} \otimes \mathbf{c}_2 \\ \mathbf{d} &= \mathbf{d}_1 - \mathbf{c}_1(\mathbf{A}_1 - \mathbf{d}_2\mathbf{I})^{-1}\mathbf{b}_1 \qquad (2.81) \end{aligned}$$

A condition which must be satisfied is that $\mathbf{d}_2$ is not an eigenvalue of $\mathbf{A}_1$, $\mathbf{d}_2 \notin \sigma(\mathbf{A}_1)$.

2.2.9.2. Application of Composition Formula with Matrix Geometric Generating Function

In probability theory one can define a new random variable Y as the sum of N independent copies of a random variable X where N itself is a random variable with a probability distribution on the non-negative integers. If X is a 2-EPT random variable and N a random variable with a matrix geometric distribution then Y is also a 2-EPT random variable. Let the density function of X be f. Here we derive the 2-EPT realization for Y based on the realizations for X and N. An in-depth description of realizations for matrix geometric distributions can be found in Hanzon and Ober (2002).

Using the formulae above we consider the composition of two rational functions, G and ϕ, where G is the rational generating function of a matrix geometric distribution and ϕ, the Laplace Transform of a 2-EPT probability density function mixed with a pointmass at zero. The discrete matrix geometric distribution has a minimal realization $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{d}_1)$ where the probabilities are given by

$$\begin{aligned} p(0) &= \mathbf{d}_1 \\ p(k) &= \mathbf{c}_1\, \mathbf{A}_1^{k-1}\mathbf{b}_1 \qquad\qquad \forall\, k \in (1, 2, 3, ...) \end{aligned}$$

The realization of ϕ is $(\mathbf{A}_2, \mathbf{b}_2, \mathbf{c}_2, \mathbf{d}_2)$ where we assume $\mathbf{d}_2 > 0$ although we relax this assumption later.

We have that

$$\begin{aligned} G(z) &= \sum_{k=0}^{\infty} z^k p(k) \\ &= \mathbf{d}_1 + \sum_{k=1}^{\infty} z^k \, \mathbf{c}_1 \, \mathbf{A}_1^{k-1} \mathbf{b}_1 \\ &= \mathbf{d}_1 + \mathbf{c}_1 (\frac{\mathbf{I}}{z} - \mathbf{A}_1)^{-1} \mathbf{b}_1 \end{aligned} \tag{2.82}$$

is the rational generating function of p.

The random variable Y is defined as

$$Y = \sum_{i=0}^{N} X_i. \tag{2.83}$$

It follows that the distribution of Y is given by

$$g(y) = p(0) + p(1) f^1(y) + p(2) f^2(y) + ... + p(k) f^k(y) + ... \tag{2.84}$$

where $f^k(y)$ is the distribution of $\sum_{i=1}^{k} X_i$ i.e. $f^k(y)$ is the k-fold convolution of f with itself. The Laplace Transform of f^k is $(\phi)^k$. Hence, the Laplace transform of Y is

$$\sum_{k=0}^{\infty} \phi^k(s) p(k). \tag{2.85}$$

Therefore, we examine

$$\begin{aligned} G(\phi(s)) &= \sum_{k=0}^{\infty} \phi^k(s) p(k) \\ &= \mathbf{d}_1 + \mathbf{c}_1 \Big(\frac{\mathbf{I}}{\phi(s)} - \mathbf{A}_1 \Big)^{-1} \mathbf{b}_1 \end{aligned} \tag{2.86}$$

It should be clear that

$$\phi(s) = \mathbf{c}_2 (s\mathbf{I} - \mathbf{A}_2)^{-1} \mathbf{b}_2 + \mathbf{d}_2 \tag{2.87}$$

To find the realization of the composition in Eq. (2.86) we must first exploit a well known identity for the inverse of a transfer function. We wish to represent $(\phi^{-1}(s))$ by the 4-tuple $(\tilde{\mathbf{A}_2}, \tilde{\mathbf{b}_2}, \tilde{\mathbf{c}_2}, \tilde{\mathbf{d}_2})$. The identity gives us that

$$\phi^{-1}(s) \;=\; \mathbf{d}_2^{-1} \;-\; \mathbf{d}_2^{-1}\mathbf{c}_2(s\mathbf{I} \;-\; \mathbf{A}_2 \;+\; \mathbf{b}_2\,\mathbf{d}_2^{-1}\mathbf{c}_2)^{-1}\mathbf{b}_2\mathbf{d}_2^{-1} \tag{2.88}$$

Hence we obtain that

$$\begin{aligned}
\tilde{\mathbf{A}_2} &= \mathbf{A}_2 \;-\; \mathbf{b}_2\,\mathbf{d}_2^{-1}\mathbf{c}_2 \\
\tilde{\mathbf{b}_2} &= \mathbf{b}_2\mathbf{d}_2^{-1} \\
\tilde{\mathbf{c}_2} &= -\mathbf{d}_2^{-1}\mathbf{c}_2 \\
\tilde{\mathbf{d}_2} &= \mathbf{d}_2^{-1}
\end{aligned} \tag{2.89}$$

Implementing Proposition (3.2) from Hanzon and Scherrer (2008) we have that $G(\phi(u))$ is a rational function with realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ given by

$$\begin{aligned}
\mathbf{A} &= \mathbf{I} \otimes \tilde{\mathbf{A}_2} \;+\; (\mathbf{A}_1 \;-\; \tilde{\mathbf{d}_2}I)^{-1} \otimes \tilde{\mathbf{b}_2}\tilde{\mathbf{c}_2} \\
\mathbf{b} &= -(\mathbf{A}_1 \;-\; \tilde{\mathbf{d}_2}\mathbf{I})^{-1}\mathbf{b}_1 \otimes \tilde{\mathbf{b}_2} \\
\mathbf{c} &= \mathbf{c}_1(\mathbf{A}_1 \;-\; \tilde{\mathbf{d}_2}\mathbf{I})^{-1} \otimes \tilde{\mathbf{c}_2} \\
\mathbf{d} &= \mathbf{d}_1 \;-\; \mathbf{c}_1(\mathbf{A}_1 \;-\; \tilde{\mathbf{d}_2}I)^{-1}\mathbf{b}_1
\end{aligned} \tag{2.90}$$

Again the condition which must be satisfied from Hanzon and Scherrer (2008) states that $\tilde{\mathbf{d}_2}$ is not an eigenvalue of $\mathbf{A}_1$.

We can examine the limit of equations (2.90) where $\mathbf{d}_2 \to 0$, equivalent to $\tilde{\mathbf{d}_2} \to \infty$, and it can be shown that when we take the limit $\mathbf{d}_2 \to 0$ the function

$G(\phi(u))$ has a realization given by

$$\begin{aligned} \mathbf{A} &= \mathbf{I} \otimes \mathbf{A}_2 + \mathbf{A}_1 \otimes \mathbf{b}_2\mathbf{c}_2 \\ \mathbf{b} &= \mathbf{b}_1 \otimes \mathbf{b}_2 \\ \mathbf{c} &= \mathbf{c}_1 \otimes \mathbf{c}_2 \\ \mathbf{d} &= \mathbf{d}_1 \end{aligned} \tag{2.91}$$

and thus has no restrictions with respect to the eigenvalues of $\mathbf{A}_1$.

2.2.9.3. Example of Composition Formula with Matrix Geometric Generating Function

We now consider an actuarial example to demonstrate the usefulness of the approach in Section 2.2.9.2. A practical example is where the number of insurance claims is distributed according to a matrix geometric distribution and claim sizes have an EPT distribution. We compute the EPT distribution for the total claims liability.

Let the realization of the matrix geometric distribution be $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{d}_1)$ and consider the scalar case where

$$\begin{aligned} p(0) &= \mathbf{d}_1 && 0 \leq \mathbf{d}_1 \leq 1 \\ p(k) &= (1-\mathbf{d}_1)(1-\lambda)\,\lambda^{k-1} \;=\; \mathbf{c}_1\mathbf{A}_1^{k-1}\mathbf{b}_1 && \forall\, k \;>\; 0 \end{aligned}$$

Hence we have that $\mathbf{A}_1 \;=\; \lambda$ where $0 \;<\; \lambda \;<\; 1$, $\mathbf{b}_1 \;=\; 1$ and

$$\mathbf{c}_1 \;=\; (1-\mathbf{d}_1)(1-\lambda) \tag{2.92}$$

We then assume that claim sizes have an EPT density with minimal realization of order n, $(\mathbf{A}_2, \mathbf{b}_2, \mathbf{c}_2, 0)$, i.e. $\mathbf{d}_2 \;=\; 0$ so the EPT density does not have a pointmass at zero. We assume that claim sizes are non-negative and therefore

$\sigma(\mathbf{A}_2) \subset \mathbb{H}_-$. Allowing for negative claim sizes could easily be included although it does not make sense in this example. The total claims liability then has an EPT density mixed with a pointmass at zero and realization given by $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ obtained from Eq. (2.91) where

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_2 \ + \ \lambda \mathbf{b}_2 \mathbf{c}_2 \\ \mathbf{b} &= \mathbf{b}_2 \\ \mathbf{c} &= (1 - \mathbf{d}_1)(1 - \lambda) \ \mathbf{c}_2 \\ \mathbf{d} &= \mathbf{d}_1 \end{aligned} \tag{2.93}$$

It is clear that the transfer function given by the minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ has a maximum McMillan degree of n.

2.3. Variance Gamma and Implementation Example

2.3.1. Introduction to Variance Gamma Densities in 2-EPT Form

The Variance Gamma (VG) process is a popular flexible asset price process in Financial Modelling and has been the subject of much research since its introduction in Carr, Madan, Chang (1998). The density itself is extremely flexible as one can easily control its mean, skewness and kurtosis through its three parameters (C,G,M). The parameters C, G, M are positive real values. The density in closed form is

$$f_{VG}(x; C, G, M) = \frac{(GM)^C}{\sqrt{\pi}\,\Gamma(C)} \, exp\Big(\frac{(G-M)x}{2}\Big)\Big(\frac{|x|}{G+M}\Big)^{C-1/2} K_{C-1/2}\Big(\frac{(G+M)|x|}{2}\Big) \tag{2.94}$$

where K_ν denotes a Modified Spherical Bessel function of the third kind. The corresponding characteristic function has a much simpler expression given by

$$\Phi(is; C, G, M) \ = \ \Big(\frac{G\ M}{G\ M + (M\ -\ G)is + s^2}\Big)^C \tag{2.95}$$

It is clear that the characteristic function is rational for integer C. Using Abramowitz and Stegun (1965) pg. 443 we see that for integer C Eq. (2.94) can be written as

$$f_{VG}(x;C,G,M) = exp\Big(\frac{(G-M)x-(G+M)|x|}{2}\Big)\underbrace{\frac{(MG)^C}{(C-1)!}\sum_{k=0}^{C-1}\frac{(C-1+k)!(G+M)^{-C-k}|x|^{C-1-k}}{(C-1-k)!\;k!}}_{p(x)} \tag{2.96}$$

We now see that for integer C Eq. (2.96) is in 2-EPT form with exponential and polynomial components. If we split the density around the origin we get

$$f_{VG}(x;C,G,M) = \begin{cases} exp(G\;x)\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!\;(C-1-s)!} & \text{if } x \leq 0 \\ exp(-M\;x)\frac{(MG)^C}{(C-1)!}\sum_{s=0}^{C-1}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}|x|^s}{s!\;(C-1-s)!} & \text{if } x > 0 \end{cases} \tag{2.97}$$

We note the polynomial parts of (2.97) are identical for all x implying $\mathbf{c}_P = \mathbf{c}_N$ and $\mathbf{b}_P = \mathbf{b}_N$. As we have seen already a polynomial can represented in EPT format, $p(x) = \mathbf{c}\; e^{\mathbf{a}\;x}\; \mathbf{b}$. We must now find the coefficients of the polynomial such that they are entries of the $\mathbf{c}$ row vector and they can be seen to be

$$c_s = \frac{(MG)^C}{(C-1)!}\frac{(2(C-1)-s)!(G+M)^{-2C+1+s}}{(C-1-s)!} \qquad ... \qquad s \in \{0,1,...,C-1\} \tag{2.98}$$

The $s!$ factor from Eq. (2.97) is omitted in Eq. (2.98) as it will be included in the taylor expansion of e^x. Similarly $\mathbf{b} = (1,\;0,\ldots,\;0)^T$, a $(C \times 1)$ column vector. We then have $\mathbf{a}$, a square $C \times C$ nilpotent matrix given by

$$\mathbf{a} = \begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & 1 & 0 \end{pmatrix} \tag{2.99}$$

This methodology works well for $x > 0$ since the polynomial is given in terms of the absolute value of x so care must be taken when $x \leq 0$ giving us $p(x) = \mathbf{c}\ e^{-\mathbf{a}\,x}\ \mathbf{b}$. Hence we obtain the following representations for the density

$$f_{VG}(x;C,G,M) = \begin{cases} \mathbf{c}\ e^{G\,x}\ e^{-\mathbf{a}\,x}\ \mathbf{b} & \text{if } x \leq 0 \\ \\ \mathbf{c}\ e^{-M\,x}\ e^{\mathbf{a}\,x}\ \mathbf{b} & \text{if } x > 0 \end{cases} \tag{2.100}$$

$$f_{VG}(x;C,G,M) = \begin{cases} \mathbf{c}\ e^{(G\mathbf{I}-\mathbf{a})\,x}\ \mathbf{b} & \text{if } x \leq 0 \\ \\ \mathbf{c}\ e^{(-M\mathbf{I}+\mathbf{a})\,x}\ \mathbf{b} & \text{if } x > 0 \end{cases} \tag{2.101}$$

where $\mathbf{I}$ is the $C \times C$ identity matrix. Finally we let $\mathbf{A}_N = G\mathbf{I} - \mathbf{a}$ and $\mathbf{A}_P = -M\mathbf{I} + \mathbf{a}$ implying

$$f_{VG}(x;C,G,M) = \begin{cases} \mathbf{c}\ e^{\mathbf{A}_N\,x}\ \mathbf{b} & \text{if } x \leq 0 \\ \\ \mathbf{c}\ e^{\mathbf{A}_P\,x}\ \mathbf{b} & \text{if } x > 0 \end{cases} \tag{2.102}$$

It is clear that all eigenvalues of $\mathbf{A}_P$ are negative, all identically $-M$ while all eigenvalues of $\mathbf{A}_N$ are positive G. Thus $\mathbf{A}_P$ is stable and $\mathbf{A}_N$ anti-stable and both are non-singular.

2.3.2. State Space Representation of Variance Gamma Characteristic Function

Suppose $X \sim VG(C,G,M)$, we have already seen in Eq. (2.95) that the characteristic function is rational when C is integer. Consider the most basic case when $C = 1$ we get

$$\begin{aligned} \phi(si;1,G,M) &= \Big(\frac{G\,M}{G\,M + (M - G)is + u^2}\Big) \\ &= \frac{M\,G}{(M + G)(si + G)} + \frac{-M\,G}{(M + G)(si - M)} \\ &= \phi_P(si) + \phi_N(si) \end{aligned} \tag{2.103}$$

It is obvious that the rational characteristic function can be decomposed into the sum of two rational functions, $\phi_N(si)$ and $\phi_P(si)$, with poles located in the open upper half plane, $\mathbb{C}_+$, and open lower half plane, $\mathbb{C}_-$, respectively. The state space representation is then given by

$$\left(\begin{array}{cc|c} -M & 0 & 1 \\ 0 & G & 1 \\ \hline -\frac{MG}{(M+G)} & \frac{MG}{(M+G)} & 0 \end{array}\right) \tag{2.104}$$

Consider the case where $C = 2$ which is equivalent to the product of the two identical characteristic functions with $C = 1$ which can be found using the convolution formula from Section 2.2.8 or alternatively we could find the state space representation using the formulations given in Eq. (2.97) to Eq. (2.102)

$$\left(\begin{array}{cccc|c} G & 0 & 0 & 0 & 1 \\ -1 & G & 0 & 0 & 0 \\ 0 & 0 & -M & 0 & 1 \\ 0 & 0 & 1 & -M & 0 \\ \hline \frac{2(MG)^2}{(M+G)^3} & (\frac{MG}{M+G})^2 & -\frac{2(MG)^2}{(M+G)^3} & -(\frac{MG}{M+G})^2 & 0 \end{array}\right) \tag{2.105}$$

Both methods give equivalent realizations as it was already noted that minimal realizations are unique only up to a state space transformations.

2.3.3. Variance Gamma Additive Decomposition Example

If we are required to carry out certain calculations using the characteristic function or attempting to recover the density from the characteristic function it is necessary to have the eigenvalues of **A** split according to their positive and negative real part as stated in Section (2.2.8.2) already. The **A** matrix must also be block diagonalised. We will consider such a case say where the characteristic function of Z is the product of the characteristic functions of X and Y which both have Variance Gamma distributions $VG(1, G_X, M_X)$ and $VG(1, G_Y, M_Y)$

respectively. Hence we find the state space representation for Z using the convolution formula given in Section 2.2.8

$$\phi(Z) = \left(\begin{array}{cccc|c} G_X & 0 & \alpha_X & -\alpha_X & 0 \\ 0 & -M_X & \alpha_X & -\alpha_X & 0 \\ 0 & 0 & G_Y & 0 & 1 \\ 0 & 0 & 0 & -M_Y & 1 \\ \hline \alpha_Y & -\alpha_Y & 0 & 0 & 0 \end{array}\right) \tag{2.106}$$

where

$$\alpha_X = \frac{M_X G_X}{M_X + G_X} \qquad \alpha_Y = \frac{M_Y G_Y}{M_Y + G_Y} \tag{2.107}$$

To split the eigenvalues we must perform some row operations while ensuring $\mathbf{A}$ maintains its upper diagonal structure. To achieve this we pre multiply by $\mathbf{Q}_1$ and post multiply by $\mathbf{Q}_1^{-1}$

$$\begin{aligned} \mathbf{Q}_1\ \mathbf{A}\ \mathbf{Q}_1^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} G_X & 0 & \alpha_X & -\alpha_X \\ 0 & -M_X & \alpha_X & -\alpha_X \\ 0 & 0 & G_Y & 0 \\ 0 & 0 & 0 & -M_Y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} G_X & \alpha_X & 0 & -\alpha_X \\ 0 & G_Y & 0 & 0 \\ 0 & \alpha_X & -M_X & -\alpha_X \\ 0 & 0 & 0 & -M_Y \end{pmatrix} \end{aligned}$$

We can see the eigenvalues are split depending on the sign of their real part. However we must ensure $\mathbf{Q}_1\ \mathbf{A}\ \mathbf{Q}_1^{-1}$ remains upper diagonal. Hence, we pre multiply by $\mathbf{Q}_2$ and post multiply by $\mathbf{Q}_2^{-1}$

$$\mathbf{Q}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.108}$$

and q is a scalar giving us

$$\begin{aligned}\tilde{\mathbf{A}} = \mathbf{Q}_2(\mathbf{Q}_1\ \mathbf{A}\ \mathbf{Q}_1^{-1})\mathbf{Q}_2^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} G_X & \alpha_X & 0 & -\alpha_X \\ 0 & G_Y & 0 & 0 \\ 0 & \alpha_X & -M_X & -\alpha_X \\ 0 & 0 & 0 & -M_Y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} G_X & \alpha_X & 0 & -\alpha_X \\ 0 & G_Y & 0 & 0 \\ 0 & f(q) & -M_X & -\alpha_X \\ 0 & 0 & 0 & -M_Y \end{pmatrix} \end{aligned} \tag{2.109}$$

where $f(q)$ is a linear function of q and can be easily solved such that $\tilde{\mathbf{A}} = \mathbf{Q}_2(\mathbf{Q}_1\ \mathbf{A}\ \mathbf{Q}_1^{-1})\mathbf{Q}_2^{-1}$ remains upper diagonal. In this instance we find that

$$q = \frac{-\alpha_X}{M_X + G_Y} \tag{2.110}$$

We now transform $\tilde{\mathbf{A}}$ such that it is block diagonal. As was outlined earlier in Section (2.2.8.2) we introduce a matrix $\mathbf{T}$ where here $\mathcal{T}$ is a 2×2 submatrix and let

$$\mathbf{A}_N = \begin{pmatrix} G_X & \alpha_X \\ 0 & G_Y \end{pmatrix}, \qquad \mathbf{A}_P = \begin{pmatrix} -M_X & -\alpha_X \\ 0 & -M_Y \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} 0 & -\alpha_X \\ 0 & 0 \end{pmatrix} \tag{2.111}$$

giving

$$\begin{aligned}\mathbf{T}\tilde{A}\mathbf{T}^{-1} &= \begin{pmatrix} \mathbf{I}_{2x2} & \mathcal{T} \\ 0 & \mathbf{I}_{2x2} \end{pmatrix} \begin{pmatrix} \mathbf{A}_N & \mathcal{V} \\ 0 & \mathbf{A}_P \end{pmatrix} \begin{pmatrix} \mathbf{I}_{2x2} & -\mathcal{T} \\ 0 & \mathbf{I}_{2x2} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A}_N & \mathcal{T}\mathbf{A}_P - \mathbf{A}_N\mathcal{T} + \mathcal{V} \\ 0 & \mathbf{A}_P \end{pmatrix} \end{aligned}$$

We then solve the sylvester equation for $\mathcal{T}$ above which in this case we find that

$$\mathcal{T} = \begin{pmatrix} 0 & \frac{-\alpha_X}{M_Y + G_X} \\ 0 & 0 \end{pmatrix} \tag{2.112}$$

The final "**A**" matrix is

$$\mathbf{TQ_2Q_1AQ_1^{-1}Q_2^{-1}}T^{-1} = \begin{pmatrix} G_X & \alpha_X & 0 & 0 \\ 0 & G_Y & 0 & 0 \\ 0 & 0 & -M_X & -\alpha_X \\ 0 & 0 & 0 & -M_Y \end{pmatrix} \tag{2.113}$$

The triple for the state space representation is then given by

$(\mathbf{TQ_2Q_1AQ_1^{-1}Q_2^{-1}T^{-1}},\ \mathbf{TQ_2Q_1b},\ \mathbf{cQ_1^{-1}Q_2^{-1}T^{-1}})$

Chapter 3

Fitting EPT Densities

3.1. Introduction

The problem of approximating an empirical probability distribution with a non-negative EPT function is considered here. Despite the fact that the observed data may be non-negative it is still possible that the approximating EPT function is negative on certain intervals. This problem has been observed in practice and is also encountered in model reduction when approximating a high order non-negative system with a lower order system.

It has been elucidated to in Section 1.4.3 that positive realizations do ensure non-negativity but it was stated that such realizations may not exist. Also positive realizations may not be minimal.

We demonstrate how to fit a minimal non-negative EPT density on a particular set of data. The dataset to be approximated initially is the positive daily Dow Jones Industrial Average (DJIA) daily log returns over 80 years which are shown in Figure 3.1. (Fitting another EPT function on $(-\infty, 0]$ to the negative log returns can be carried out by the same procedure). The adjusted closing prices are used which are freely available on *finance.yahoo.com* using the ticker symbol DJI.

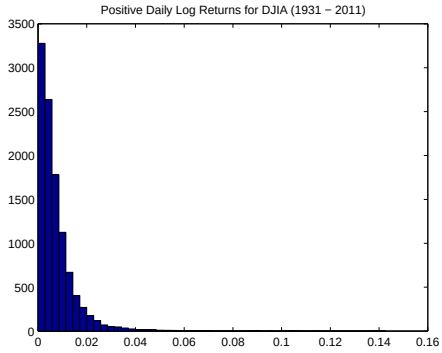

Figure 3.1: Histogram of Daily Log returns for DJIA for (1931 - 2011)

Hence the goal of the chapter is to describe a procedure to find a triple $(\mathbf{A}, \mathbf{b}, \mathbf{c}) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n}$ of McMillan degree n minimizing the criterion

$$||h(x) - \mathbf{c}e^{\mathbf{A}x}\mathbf{b}||_2^2 \tag{3.1}$$

where $h(x)$ is a density function that represents the data ($h(x)$ is directly related to a histogram of the data; a precise description of h is given in Section 3.2). Here the norm is the L^2 norm over the half-line $[0, \infty)$. Section 3.2 shows how the tail of the function can be estimated. The necessity of a dominant real pole in the spectrum of $\mathbf{A}$ is also discussed. In section 3.3 a brief comparison of using Maximum Likelihood Estimation versus an RARL2 approximation for the EPT density is provided.

Section 3.4 illustrates that our L^2 fitting problem is equivalent to a discrete time rational approximation problem in H^2. The unconstrained problem, in which the non-negativity constraint is not imposed, is tackled using the rational approximation software RARL2 to find an observable pair $(\mathbf{A}, \mathbf{c})$ and an optimal $\mathbf{b}$. In a separate section, 3.5, and in Section 3.5.1, the Fourier coefficients of the transform of $h(x)$, which form the inputs to RARL2, are derived directly as inner products of $h(x)$ with a set of orthogonal basis functions on $[0, \infty)$. Results from unconstrained RARL2 approximations are also provided.

The non-negativity problem is described in Section 3.6. By assuming the pres-

ence of a dominant real pole and solving the unconstrained problem with RARL2 we show that non-negativity can be imposed by choosing an optimal $\mathbf{b}_C^*$ for a given pair $(\mathbf{A}, \mathbf{c})$ such that the EPT function with realization $(\mathbf{A}, \mathbf{b}_C^*, \mathbf{c})$ is non-negative on $[0, \infty)$. Choosing $\mathbf{b}_C^*$ is shown to be a convex optimisation problem which is solved in Section 3.8. Central to the convex optimisation algorithm is the Budan-Fourier method of Hanzon and Holland (2010) which examines non-negativity of an EPT function on a finite interval $[0, T]$. Section 3.7 proves that a finite interval can be constructed on which to apply the Budan-Fourier technique to obtain a necessary and sufficient test for non-negativity of the EPT function on $[0, \infty)$. This is possible due to the assumed presence of a strictly dominant real pole in the spectrum of $\mathbf{A}$. Section 3.9 gives an example of the convex optimisation procedure.

Section 3.11 considers the problem of ensuring a continuous (at the origin) 2-EPT probability density function. In Section 3.12 two Variance Gamma densities are fitted to the data and a Gaussian density. The quality of fit of these densities are compared to the 2-EPT approximation. Lastly Section 3.12.1 compares the first four moments of the data to the corresponding moments of the 2-EPT density approximation.

Much of the work in this chapter was completed during a research visit to INRIA, Sophia-Antipolis in May 2011 under the additional guidance of Martine Olivi. A paper Hanzon, Olivi and Sexton (2012) containing many of these results was presented at the SYSID 2012 conference and published in the proceedings.

3.2. Extrapolating the Tail

We take a histogram of daily log returns for DJIA for the period 1931-2011 as our starting point (see Figure 3.1). Firstly we scale the data such that all data points are located in the interval $[0, 1]$. We denote the resulting function by $r(x)$. This is a step function. As its support is only on the finite interval $[0, 1]$

we extrapolate the tail of the empirical density to the whole positive real line with an exponential function.

There is a vast literature on extreme value theory discussing techniques to estimate the tail behaviour of a distribution of empirical asset returns. Much of the research available focuses on using more sophisticated functions to approximate this behaviour. Such functions include the Pareto, Gumbel, Weibull and Frechet distributions. Unfortunately, none of these functions can be be used to approximate the tail behaviour in the our case as we are restricted to using EPT functions with a unique dominant real pole.

Assume we have N_{obs} observations in the sample which are ordered such that $X_1 \leq X_2 \leq ... \leq X_{N_{obs}}$. One (regression) technique of identifying the strictly dominant real pole and its coefficient is to select all observations greater than or equal to $X_{K_{obs}}$. The parameters $\mu_M > 0$ and $\lambda_M < 0$ are then chosen to minimise

$$\min_{\mu_M, \lambda_M} \sum_{i=N-K_{bin}}^{N} (\log(r(x_i)) - \log(\mu_M e^{\lambda_M x_i}))^2 \tag{3.2}$$

where $x_{i+1} - x_i = \delta x$ for all $i = 0, 1, ..., N-1$ and $\delta x = 1/N$. K_{bin} is then chosen as the interval $[x_i, x_{i+1}]$ containing the observation $X_{K_{obs}}$. $X_{K_{obs}}$ must be chosen to capture tail behaviour only, so should not be chosen too big, but also not too small, in order to get a good estimate of μ_M and λ_M. The literature is also unclear on how to determine where the tail of the distribution begins. In our case we chose the upper 1% of returns for this estimation procedure which included over 100 observations. The 99^{th} percentile of empirical returns begin at $X_{K_{obs}} = 0.2769$ (after scaling). In this situation we let $N = 500$ so $K_{bin} = 138$.

We then define the function $h(x)$ on $[0, \infty)$ as

$$h(x) = \begin{cases} r(x) & \text{if } x \leq 1 \\ \\ \mu_M e^{\lambda_M x} & \text{if } x > 1 \end{cases} \tag{3.3}$$

In Figure 3.2 we can see $h(x)$ on $[0, 0.3]$.

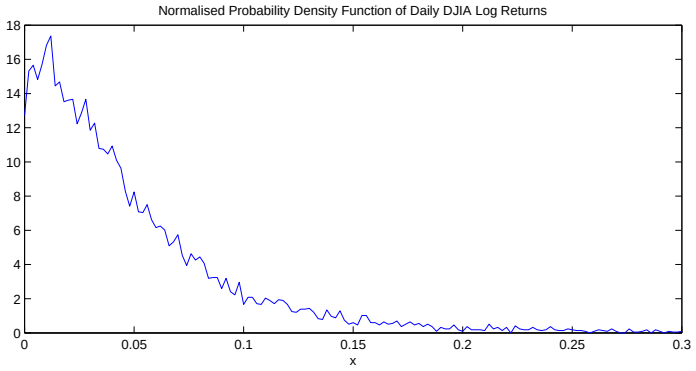

Figure 3.2: Normalised Density, $h(x)$, for observed daily log returns over $[0, 0.3]$

As we proceed to examining non-negative EPT approximations we elucidate to the importance of the Perron-Frobenius type result which states *that if* $f(x) = \boldsymbol{c}e^{\boldsymbol{A}x}\boldsymbol{b}$ *is non-negative* $\forall\ x \geq 0$ *then* $\lambda_M = \max_{\lambda \in \sigma(A)} Re(\lambda)$ *is an element of the spectrum of* $\boldsymbol{A}$. This result puts a constraint on the eigenvalues of the approximating $\mathbf{A}$ matrix in that it must contain a dominant real eigenvalue (equivalently the rational characteristic function must contain a dominant real pole). This condition is far from sufficient and as the analysis in Hanzon and Holland (2012) shows the non-negativity constraint is a complicated one in general.

We do not impose this condition on $\sigma(\mathbf{A})$ in Sections 3.4 or 3.5, instead allowing RARL2 to recover the dominant real pole during the approximation procedure.

3.3. RARL2 vs. Maximum Likelihood Estimation

Maximum Likelihood Estimation is a well documented standard procedure for estimating the parameters of a probability density function. In the EPT case, an EPT function of order 3 has 15 parameters which would need to be estimated. There are also restrictions on these parameters such that the resulting EPT function is stable, integrates to unity and is non-negative, although we

will see later that the condition of non-negativity can be relaxed. The literature on MLE for Matrix-Exponential (or Matrix-Analytic) distribution is relatively sparse. Fackrell (2003) provides an in-depth description on MLE procedures for fitting a Matrix-Exponential distribution to a set of data. We will introduce Fackrells technique but it will be seen to be quite cumbersome in comparison to the RARL2 method implemented later.

It was seen in Eq. (1.1) that the Laplace Transform of a Matrix-Exponential distribution ($\mathbf{a}e^{\mathbf{T}x}\mathbf{t}$) is a rational polynomial $H = a/b$. The roots of the polynomial b correspond to the eigenvalues of $\mathbf{T}$. By re-writing the Matrix-Exponential function, Fackrell (2003) first estimates the eigenvalues of $\mathbf{T}$ using the multidimensional non-linear optimisation Nelder-Mead algorithm. However this algorithm does not have the ability to change a real eigenvalue into a complex eigenvalue. Therefore, before the algorithm is run the number of real zeros must be specified. Hence, an order 5 approximation would require three iterations of the algorithm, firstly with five real zeros, secondly with three real zeros and a complex pair and finally an approximation with one real zero and two complex pairs. The coefficients of the polynomial b can then found from eigenvalues. The coefficients of the polynomial a can then be derived to maximise the likelihood function for each case depending on the number of real poles in H. The final choice of coefficients for the polynomials a/b will those which maximise the likelihood function. It is also possible that a zero of the polynomial a could cancel out a zero of the polynomial in the denominator b meaning a minimal realization is not guaranteed.

Due to the aforementioned issues surrounding the maximum likelihood estimation procedure, we examined alternative techniques. The rational approximation method known as RARL2 was an obvious candidate. The procedure computes a pre-specified order n stable rational L^2-approximation to a given transfer function of a discrete-time system. The discrete-time transfer function to be approximated can be passed as either a finite number of Fourier coefficients, a high order realization or pointwise values on the unit circle. RARL2

uses a parametrization of stable all-pass systems represented by balanced output realizations. There are a number of advantages to using such parametrizations as stability is guaranteed while differentiability and well-conditioning are also assured. RARL2 was developed at INRIA and a more comprehensive description with further references can be found in Olivi (2010). This approximation method has been implemented in MATLAB and is available from INRIA. Although the algorithm can not enforce the non-negativity constraint, it is possible to ensure integration to unity. We now illustrate how this discrete time rational approximation software can be used to solve our approximation problem given in Eq. (3.1).

3.4. Discrete Time Rational Approximation

As is well-known, minimizing Eq. (3.1) is equivalent to minimizing the $L_2(i\mathbf{R})$ distance between the characteristic functions (Fourier transforms) of $h(x)$ and $f(x)$, where $f(x) = \mathbf{c}e^{\mathbf{A}x}\mathbf{b}$. The characteristic function of $\mathbf{c}e^{\mathbf{A}x}\mathbf{b}$ is a rational function with poles in the open left half plane and fixed McMillan degree. In this way the problem of minimizing Eq. (3.1) can be translated into a rational approximation problem under a "least squares" norm. In systems theory this problem is known as the H_2 model approximation problem for continuous time systems. The software RARL2 is devised to handle similiar problems for discrete time models. We use an isometric transformation to map the continuous time system into a discrete time system and then apply the software.

RARL2 is ideally suited to this rational approximation problem as it uses a parameterization of the observable pairs $(\mathbf{A}, \mathbf{c})$ which is convenient for enforcing the stability constraint and for optimisation due to their differentiability properties. Without the non-negativity constraint the criterion function to be minimised is Eq. (3.1) over the set of minimal triples $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ where $\mathbf{A}$ is stable and of fixed degree n.

The continuous time transfer function is found via the transform

$$\mathbb{E}[e^{-sx}] \;=\; F(s) \;=\; \int_0^\infty \mathbf{c}e^{\mathbf{A}x}\mathbf{b}e^{-sx}dx \;=\; \mathbf{c}(s\mathbf{I} \;-\; \mathbf{A})^{-1}\mathbf{b} \tag{3.4}$$

which is a strictly proper rational function. The transform of $h(x)$ is given by

$$H(s) \;=\; \frac{\mu_M e^{\lambda_M - s}}{s-\lambda_M} + \sum_{n=0}^{N-1} r(x_n)\Big(-\frac{1}{s}e^{-sx_{n+1}} + \frac{1}{s}e^{-sx_n}\Big) \tag{3.5}$$

To transform the continuous time rational function into discrete time we use a well known isometry. In discrete time the stable $\mathbf{A}$ matrix will have all its eigenvalues located inside the unit disk.

Marmorat et al (2012) give a map from continuous to discrete time which we implement here. The map is

$$F(s) \mapsto \tilde{F}(z) \;=\; \frac{\sqrt{2}}{z-1}F\big(\frac{z+1}{z-1}\big) \tag{3.6}$$

achieved using the Möbius transform

$$z \mapsto s \;=\; \frac{z+1}{z-1} \tag{3.7}$$

These formulae allow us to derive a discrete time realization, ($\tilde{\mathbf{A}},\ \tilde{\mathbf{b}},\ \tilde{\mathbf{c}}$), from the continuous time realization ($\mathbf{A},\ \mathbf{b},\ \mathbf{c}$). The state space formulae of the transformation are given by

$$\begin{aligned}
\tilde{\mathbf{A}} &= -(\mathbf{I}-\mathbf{A})^{-1}(\mathbf{I}+\mathbf{A}) \\
\tilde{\mathbf{b}} &= \sqrt{2}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{b} \\
\tilde{\mathbf{c}} &= \mathbf{c}
\end{aligned} \tag{3.8}$$

Hence the discrete time rational function is given by

$$\tilde{F}(z) = \tilde{\mathbf{c}}(z\mathbf{I} \;-\; \tilde{\mathbf{A}})^{-1}\tilde{\mathbf{b}} \tag{3.9}$$

Equations (3.6) and (3.7) can be used to transform $H(s)$ into its discrete time counterpart $\tilde{H}(z)$. We let

$$i\omega \;=\; \frac{e^{i\theta}+1}{e^{i\theta}-1},\ \theta \in [-\pi,0)\cup(0,\pi] \tag{3.10}$$

giving us

$$\begin{aligned}\tilde{H}(e^{i\theta}) \;&=\; \frac{\sqrt{2}}{e^{i\theta}-1}H(i\omega) = \Big(\frac{\sqrt{2}}{e^{i\theta}-1}\Big)\frac{\mu_M e^{\lambda_M - i\omega}}{i\omega-\lambda_M} \\ &+\; \frac{\sqrt{2}}{e^{i\theta}-1}\Big(\sum_{n=0}^{N-1}\hat{r}(x_n)\Big(\frac{1}{i\omega}e^{-i\omega x_n}-\frac{1}{i\omega}e^{-i\omega x_{n+1}}\Big)\Big)\end{aligned} \tag{3.11}$$

where $\hat{r}(x) = r(x) - \mu_M e^{\lambda_M x}$.

RARL2 seeks to minimise the L_2 norm

$$\min_{\tilde{\mathbf{A}},\,\tilde{\mathbf{b}},\,\tilde{\mathbf{c}}} ||\tilde{H} \;-\; \tilde{F}||_2^2 \tag{3.12}$$

which is equivalent to Eq. (3.1).

3.5. RARL2 Approximation from Fourier Coefficients

Let $\{g_n(x)\}_{n=-1,-2,-3,\ldots}$ denote the sequence of functions for $n<0$ given by

$$g_n(x) \;=\; \sqrt{2}e^{-x}\sum_{j=0}^{|n|-1}\binom{-n-1}{j}\frac{(-2x)^j}{j!}\ ,\ x\in[0,\infty) \tag{3.13}$$

which form an orthonormal basis for the Hilbert space $L^2[0,\infty)$ of real square integrable functions on $[0,\infty)$. The function $h(x)$ can then be represented by a weighted series of orthonormal functions

$$h(x) \;=\; \sum_{n<0} c_n g_n(x) \tag{3.14}$$

The coefficients are given by

$$c_n \;=< g_n, h >= \int_0^\infty g_n(x)h(x)dx \tag{3.15}$$

which can be obtained by computing the integral in Eq. (3.15), numerically on $[0, 1]$ and analytically on $[1, \infty)$. It is proven in Section 3.5.1 that the coefficients c_n are the Fourier coefficients of the discrete time function $\tilde{H}$. It is clear that g_n is an Exponential-Polynomial (EP) function and subsequently the approximation of $h(x)$ derived from a truncated sum $\sum_{n=-1,-2,\ldots,-M} c_n g_n(x)$ is also EP.

The RARL2 software then uses these Fourier coefficients to determine a high order rational approximant and to calculate a first approximation of this in the form of the discrete time minimal system of degree n. This is then used as a starting point for the RARL2 approximation which uses a gradient search algorithm to locate a local minimum, in the form of a minimal realization of degree n, of the criterion in Eq. (3.12). The algorithm will report the smallest local minimum found.

By calculating the Fourier coefficients of $h(x)$ as described in Eqs. (3.13) to (3.15) we evaluate the performance of two RARL2 approximations, one of order 3 and another of order 6. Figure 3.3 shows the discrete time the approximation where the order 3 and order 6 approximation are almost indistinguishable.

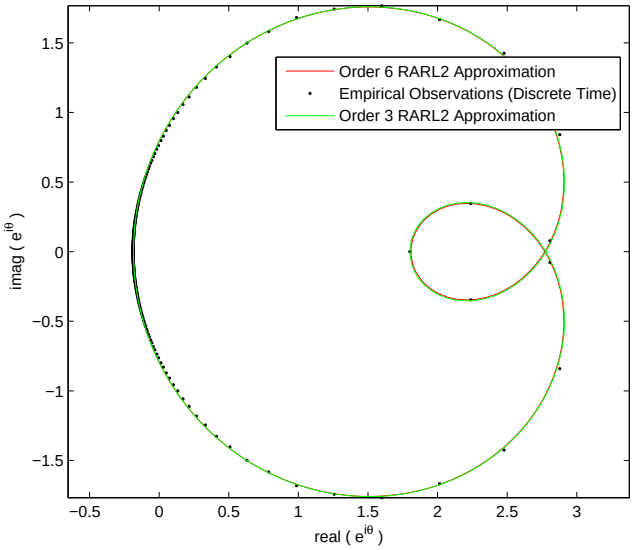

Figure 3.3: Discrete Time Approximations of order (3) (green) and of Order (6) (red)

Figure 3.4 illustrates the normalised empirical data, $h(x)$, (integrates to unity) and the RARL2 approximations in continuous time.

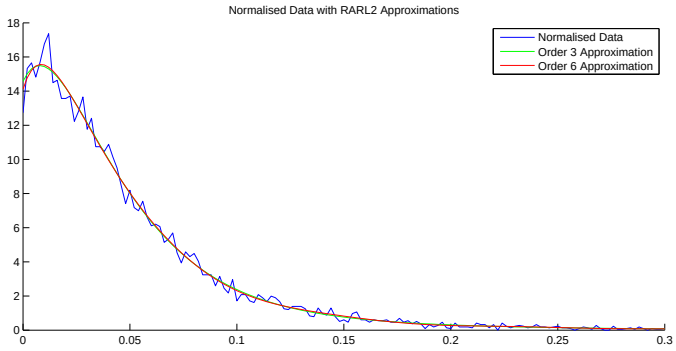

Figure 3.4: The Density Function $h(x)$ is in blue. RARL2 Approximations of Order 3 (green) and of Order 6 (red)

The approximation of order 3 has a square error of 0.0501 compared with the order 6 approximation which has a square error of 0.0441, demonstrating the improvement of fit using higher order approximations. The poles of the rational transfer function of the order 3 approximation are $(-119.5601,\ -283.2162 \pm$

$78.1844i$) while the zeros are $(-1.5595, -610.2373)$. The order 3 approximation contains a dominant real pole of -119.5601. Using techniques described in Section 7 we conclude that this approximation is non-negative for all $x > 0$. The poles of the order 6 rational transfer function are $(-269.626 \pm 107.333i, -18.275 \pm 195.347i, -37.492, -119.560)$ and the zeros are $(-18.194 \pm 195.435i, -38.207, -160.835, -569.021)$. However the dominant pole of the order 6 approximation is complex and the resulting EPT is negative on certain intervals.

From Section 3.2 the dominant real pole was estimated as $\lambda_M = -119.5601$ with its coefficient $\mu_M = 31.587$.

For the above EPT approximations, the empirical density was sampled at 500 equally spaced points on $[0, 1]$. The first 170 Fourier Coefficients were then calculated using this data. The output was unchanged when these input parameters (170,500) were perturbed slightly e.g. $\pm 10\%$. It should also be noted that very similar results were achieved when the inputs were not the Fourier Coefficients but pointwise values on the unit circle i.e. Eq. 3.11 evaluated for $\theta \in [-\pi, \pi]$.

3.5.1. Derivation of Fourier Coefficients

We begin by defining the Hardy space of the disk $L^2(\mathbb{T})$ which can be split into two orthogonal subspaces

$$L^2(\mathbb{T}) \;=\; H^2 \oplus H^2_\perp \tag{3.16}$$

where $H^2_\perp$ is the subspace of the Hardy space of the exterior of the disk consisting of strictly proper functions whose Fourier coefficients of non-negative index are zero.

We can represent the discrete time rational transfer function $\tilde{H}(z)$ defined on

$H_\perp^2$ in terms of its Fourier coefficients as

$$\begin{aligned} \tilde{H}(z) &= \sum_{n=-\infty}^{0} c_n z^n \\ \tilde{H}(e^{it}) &= \sum_{n=-\infty}^{0} c_n e^{int} \end{aligned} \tag{3.17}$$

where c_n are the Fourier coefficients given by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \tilde{H}(e^{it})e^{-int}dt \; = < \tilde{H},\; e^{int} >_{L^2(\mathbb{T})}$$

The problem can be transformed into continuous time using the isometry from Eq. (3.6) and the Möbius transforms from Eq. (3.7)

$$\begin{aligned} H(s) &= \frac{\sqrt{2}}{s-1}\tilde{H}\Big(\frac{s+1}{s-1}\Big) \\ &= \frac{\sqrt{2}}{s-1}\sum_{n=-\infty}^{\infty} c_n\Big(\frac{s+1}{s-1}\Big)^n \end{aligned} \tag{3.18}$$

This is an isometry from $L^2(\mathbb{T})$ to $L^2(i\mathbb{R})$ as it sends the Hardy space $H_\perp^2$ onto $H^2(\mathbb{C}^+)$ where $\mathbb{C}^+$ denotes the open right half plane.
The Fourier coefficients are transformed as

$$c_n \;=\; \frac{1}{2\pi} < H,\; G_n >_{H^2(\mathbb{C}^+)} \tag{3.19}$$

where

$$G_n(s) \;=\; \sqrt{2}\frac{(s+1)^n}{(s-1)^{n+1}} \tag{3.20}$$

Using Plancherel's Theorem we can transform the Fourier coefficients from $H^2(\mathbb{C}^+)$ to $L^2(0,\infty)$ such that

$$c_n \; = < h,\; g_n >_{L^2(0,\infty)} \tag{3.21}$$

As we only consider the Fourier coefficients with $n < 0$ we see that g_n is the Fourier inverse of G_n given by

$$g_n(t) = \sqrt{2}e^{-t}\Big(\sum_{j=0}^{-n-1} \binom{-n-1}{j} \frac{(-2t)^j}{j!} \Big) \tag{3.22}$$

The reader is referred to Partington (1997), especially Corollary 1.4.2, for more information on such orthonormal sequences of functions, and Marmorat et al (2012) for a more in depth discussion on the transformation from discrete to continuous time in a similar setting.

3.6. Toward a Non-Negative EPT Approximation

If applying the unconstrained method as described in Sections 3.4 and 3.5 using RARL2 produces an EPT function with negative values for any $x \geq 0$ then we must alter our approach to ensure non-negativity. An important tool is the generalized Budan-Fourier algorithm of Hanzon and Holland (2010) can be used to test for non-negativity of an EPT function on a finite interval $[0, T]$. For each real EPT function on a finite interval $[0, T] \subset \mathbb{R}$ a generalized Budan-Fourier sequence with associated sets of boundary points can be constructed. The construction is given directly in terms of the parameters of the EPT function. The generalized Budan-Fourier sequence with associated sets of boundary points, allows one to find the minimum of the EPT function on [0, T] by repeatedly using bisection techniques. This allows one to characterize non-negative EPT functions on a finite interval.

Again $h(x)$ is the true function to be approximated with the EPT function $f(x) = \mathbf{c}e^{\mathbf{A}x}\mathbf{b}$ of McMillan degree n, the minimization problem with the non-negativity constraint we wish to solve is given by Eq. (3.1).

Recalling the Perron-Frobenius result from Section 3.2 we will now assume the presence of a unique strictly dominant real eigenvalue, $\lambda_M < 0$, of multiplicity one in the spectrum of $\mathbf{A}$ (so all other eigenvalues in the spectrum have real

part strictly less than λ_M). This assumption may be slightly restrictive as the Perron-Frobenius result allows for multiple dominant eigenvalues. The strictly dominant eigenvalue and its coefficient, μ_M, are estimated as shown in Section 3.2.

The dominant eigenvalue and coefficient can then be handled separately from the other parameters in the approximation by transforming $h(x)$ and $f(x)$ as follows

$$\begin{aligned} \hat{h}(x) &= h(x) - \mu_M e^{\lambda_M x} \\ \hat{f}(x) &= f(x) - \mu_M e^{\lambda_M x} = \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}} \end{aligned} \tag{3.23}$$

where $\hat{f}(x) = \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}$ is an EPT function of McMillan degree $(n-1)$.

To ensure λ_M remains the dominant pole of $f(x)$ the eigenvalues of $\hat{\mathbf{A}}$ are constrained in the approximation such that $\sigma(\hat{\mathbf{A}}) \subset \{s | Re(s) < \lambda_M\}$. This restriction on $\sigma(\hat{\mathbf{A}})$ is similar to the stability condition and can be enforced in RARL2.

As noted already, the RARL2 algorithm so far does not take the non-negativity constraint into consideration. Implementing RARL2 after the above transformations yields the criterion

$$||\hat{h}(x) - \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}||_2^2 \tag{3.24}$$

for which a local minimum is found using the gradient algorithms described in Marmorat and Olivi (2007) where the pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ range over the set of output normal observable pairs as described in Hanzon et al (2006). For a given pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ it is a linear problem to solve for the optimal $\hat{\mathbf{b}}^*$. The Fourier coefficients of $\hat{h}(x)$ may be computed by evaluating the integral in Eq. (3.15) on $[0, 1]$ as $\hat{h}(x) = 0$ for all $x > 1$.

The non-empty convex set $\mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ can then be defined for any pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ such that if $\hat{\mathbf{b}}^* \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ then non-negativity is guaranteed by $f(x) \geq 0$ or

equivalently

$$\hat{f}(x) \;=\; \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}^{*} \geq -\mu_M e^{\lambda_M x} \qquad \forall\, x \;\geq\; 0 \tag{3.25}$$

However if $\hat{\mathbf{b}}^{*} \;\notin\; \mathcal{B}(\hat{\mathbf{A}},\; \hat{\mathbf{c}})$ then using the convex optimisation algorithm, described in Section 3.8, it is possible to find the constrained optimal $\hat{\mathbf{b}}^{*}_{C} \;\in\; \mathcal{B}(\hat{\mathbf{A}},\; \hat{\mathbf{c}})$ s.t. $\hat{\mathbf{b}}^{*}_{C}$ is the solution for $\hat{\mathbf{b}}$ to the minimization problem

$$\min_{\{\hat{\mathbf{b}} \in\, \mathcal{B}(\hat{\mathbf{A}},\, \hat{\mathbf{c}})\}} ||\hat{h}(x) - \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}||_2^2 \tag{3.26}$$

The final triple for the non-negative EPT function f can represented in state space form by

$$\left(\begin{array}{cc|c} \lambda_M & 0 & 1 \\ 0 & \hat{\mathbf{A}} & \hat{\mathbf{b}}^{*}_{C} \\ \hline \mu_M & \hat{\mathbf{c}} & 0 \end{array}\right)$$

3.7. Finite Interval to Test for Non-Negativity

Given an EPT function with the following minimal realization with strictly dominant real pole $\lambda_M < 0$ and coefficient $\mu_M > 0$,

$$\left(\begin{array}{cc|c} \lambda_M & 0 & 1 \\ 0 & \hat{\mathbf{A}} & \hat{\mathbf{b}} \\ \hline \mu_M & \hat{\mathbf{c}} & 0 \end{array}\right) \tag{3.27}$$

If $\hat{\mathbf{b}} = \hat{\mathbf{b}}^{*}$ then the triple $(\hat{\mathbf{A}}, \hat{\mathbf{b}}^{*}, \hat{\mathbf{c}})$ minimizes the criterion in Eq. (3.24) for a given pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$.

Define the non-empty set $\mathcal{B}_1(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ for a pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ as follows:

$$\mathcal{B}_1(\hat{\mathbf{A}}, \hat{\mathbf{c}}) \;=\; \left\{ \hat{\mathbf{b}} \;\middle|\; ||\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}||_2 \;\leq\; 2||\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}^*||_2 \right\} \tag{3.28}$$

Note that the minimising point $\hat{\mathbf{b}}_C^*$ will necessarily lie in the bounded set $\mathcal{B}_1(\hat{\mathbf{A}}, \hat{\mathbf{c}})$.

Proof

The zero vector lies in the convex set $\mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ and we also have $\hat{\mathbf{b}}_C^* \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$. We let $q(x) = \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}^*$ and $p(x) = \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}_C^*$ and using the triangular inequality we can show that

$$||p(x) - q(x)||_2 \leq ||q(x)||_2 \Longrightarrow ||p(x)||_2 \leq 2||q(x)||_2 \tag{3.29}$$

Proposition 3.1

Consider the EPT function with minimal realization given by

$$\left(\begin{array}{cc|c} \lambda_M & 0 & 1 \\ 0 & \hat{\mathbf{A}} & \hat{\mathbf{b}}^* \\ \hline \mu_M & \hat{\mathbf{c}} & 0 \end{array}\right) \tag{3.30}$$

where $\sigma(\hat{\mathbf{A}}) \;\subset\; \{s | Re(s) < \lambda_M\}$. There exists a $T_0 > 0$ such that for any $\hat{\mathbf{b}} \in \mathcal{B}_1(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ it holds that $\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}} \geq -\mu_M e^{\lambda_M x}$ for all $x \in [0, T_0]$ if and only if $\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}} \geq -\mu_M e^{\lambda_M x}$ for all $x \geq 0$.

Proof

Since $\sigma(\hat{\mathbf{A}}) \;\subset\; \{s | Re(s) < \lambda_M\}$, the EPT function above can be scaled by a factor $e^{-\lambda_M x}$ which transforms the non-negativity condition to be considered to

$$\hat{\mathbf{c}} e^{\tilde{\mathbf{A}}x} \hat{\mathbf{b}} \geq -\mu_M \qquad , \qquad \forall\, x \;\geq\; 0 \tag{3.31}$$

where $\tilde{\mathbf{A}} = (\hat{\mathbf{A}} - \mathbf{I}\lambda_M)$ such that $\sigma(\tilde{\mathbf{A}}) \subset \mathbb{H}_-$.

For any $\hat{\mathbf{b}} \in \mathcal{B}_1(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ we have

$$||\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}||_2 \leq 2||\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}^*||_2 \tag{3.32}$$

$\mathbf{Q}$ is defined as the positive definite observability grammian

$$\mathbf{Q} = \int_0^\infty e^{\hat{\mathbf{A}}^T y} \, \hat{\mathbf{c}}^T \, \hat{\mathbf{c}} \, e^{\hat{\mathbf{A}}y} dy \tag{3.33}$$

The L_2 norm is then given by

$$\hat{\mathbf{b}}^T \, \mathbf{Q} \, \hat{\mathbf{b}} = ||\hat{\mathbf{c}} \, e^{\hat{\mathbf{A}}x} \, \hat{\mathbf{b}}||_2^2 \tag{3.34}$$

By letting $\lambda_{\min} = \min\{\lambda \in \sigma(\mathbf{Q})\}$ such that

$$\hat{\mathbf{b}}^T \, \mathbf{Q} \, \hat{\mathbf{b}} \geq \hat{\mathbf{b}}^T \, \hat{\mathbf{b}} \, \lambda_{\min} = ||\hat{\mathbf{b}}||^2 \lambda_{\min} \tag{3.35}$$

it can be seen that

$$||\hat{\mathbf{b}}||^2 \lambda_{\min} \leq \hat{\mathbf{b}}^T \mathbf{Q} \, \hat{\mathbf{b}} \leq 4D^2 \tag{3.36}$$

where $D^2 = \hat{\mathbf{b}}^{*T} \mathbf{Q} \, \hat{\mathbf{b}}^*$. Hence there is an obvious bound on $\hat{\mathbf{b}}$

$$||\hat{\mathbf{b}}||^2 \leq \frac{4D^2}{\lambda_{\min}} \tag{3.37}$$

As the pair $(\tilde{\mathbf{A}}, \hat{\mathbf{c}})$ is observable and $\tilde{\mathbf{A}}$ an asymptotically stable matrix it must hold

$$\lim_{x \to \infty} \hat{\mathbf{c}} \, e^{\tilde{\mathbf{A}}x} \to 0 \tag{3.38}$$

implying that $\forall \, \epsilon > 0 \, \exists \, T_0 > 0$ s.t. $\forall \, x > T_0$, $||\hat{\mathbf{c}} e^{\tilde{\mathbf{A}}x}||^2 < \epsilon$.

To find such a T_0 we define the Lyapunov function

$$V(x) = \hat{\mathbf{c}} \, e^{\tilde{\mathbf{A}}x} \, \mathbf{M} \, e^{\tilde{\mathbf{A}}^T x} \, \hat{\mathbf{c}}^T \tag{3.39}$$

where $\mathbf{M}$ is the positive definite solution to the Lypanunov equation

$$\tilde{\mathbf{A}}\,\mathbf{M} \;+\; \mathbf{M}\,\tilde{\mathbf{A}}^{T} \;=\; -\mathbf{I} \tag{3.40}$$

We see that $V(x)$ is monotonically decreasing

$$V'(x) \;=\; \hat{\mathbf{c}}\, e^{\tilde{\mathbf{A}}x}\,(\tilde{\mathbf{A}}\,\mathbf{M} \;+\; \mathbf{M}\,\tilde{\mathbf{A}}^{T})\, e^{\tilde{\mathbf{A}}^{T}x}\,\hat{\mathbf{c}}^{T} \;<\; 0 \tag{3.41}$$

$\forall\, x \;>\; 0$. Letting $\tilde{\lambda}_{\min} \;=\; \min\{\lambda \;\in\; \sigma(\mathbf{M})\}$ it is clear

$$\begin{aligned} \tilde{\lambda}_{\min}||\hat{\mathbf{c}}e^{\tilde{\mathbf{A}}x}||^2 \quad &\leq \quad \hat{\mathbf{c}}\, e^{\tilde{\mathbf{A}}x}\,\mathbf{M}\, e^{\tilde{\mathbf{A}}^{T}x}\,\hat{\mathbf{c}}^{T} \\ ||\hat{\mathbf{c}}e^{\tilde{\mathbf{A}}x}||^2 \quad &\leq \quad \frac{V(x)}{\tilde{\lambda}_{\min}} \end{aligned} \tag{3.42}$$

We let

$$K \;=\; \frac{4D^2}{\tilde{\lambda}_{\min}\lambda_{\min}} \tag{3.43}$$

For all $\hat{\mathbf{b}} \in \mathcal{B}_1(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ we must have that

$$KV(x) \geq \hat{\mathbf{c}}\, e^{\tilde{\mathbf{A}}x}\hat{\mathbf{b}}\hat{\mathbf{b}}^{T} e^{\tilde{\mathbf{A}}^{T}x}\,\hat{\mathbf{c}}^{T} \tag{3.44}$$

We can then solve for T_0 such that

$$V(T_0) \;=\; \frac{\mu_M^2}{K} \;=\; \frac{\mu_M^2\tilde{\lambda}_{\min}\lambda_{\min}}{4D^2} \tag{3.45}$$

proving Proposition 3.1. ■.

It should be noted that T_0 can be computed independently of $\hat{\mathbf{b}}$ as Eq. (3.45) does not depend on $\hat{\mathbf{b}}$.

3.8. Convex Optimisation with Budan Fourier Algorithm

Consider the EPT function

$$f(x) \;=\; \mu_M e^{\lambda_M x} + \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}} \tag{3.46}$$

as described in Section 3.6. If $\hat{\mathbf{b}} = \hat{\mathbf{b}}^* \notin \mathcal{B}(\hat{\mathbf{A}},\ \hat{\mathbf{c}})$ then the function $f(x)$ will have negative values for some $x \geq 0$.

We intend to apply a convex optimisation algorithm to solve the minimization problem

$$\min_{\{\hat{\mathbf{b}} \in \mathcal{B}(\hat{\mathbf{A}},\ \hat{\mathbf{c}})\}} ||\hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} (\hat{\mathbf{b}}^* - \hat{\mathbf{b}})||_2^2 \tag{3.47}$$

We see that Eq. (3.26) can be re-written as

$$\begin{aligned} & ||\hat{h}(x) - \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}^* + \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}^* - \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}||_2^2 \\ = \quad & ||\hat{h}(x) - \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}^*||_2^2 + ||\hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} (\hat{\mathbf{b}}^* - \hat{\mathbf{b}})||_2^2 \end{aligned} \tag{3.48}$$

which is possible due to the orthogonality of $(\hat{h}(x) - \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}^*)$ and $\hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} (\hat{\mathbf{b}}^* - \hat{\mathbf{b}})$ as the function $q(x) = \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}}^*$ is the projection of $\hat{h}$ onto the linear space of functions $\{p(x) = \hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \beta | \beta \in \mathbb{R}^{n-1}\}$. Hence it is clear that minimising the criterion in Eq. (3.26) is equivalent to minimising the criterion in Eq. (3.47).

Section 3.7 derives a T_0 such that if the EPT function is non-negative on $[0, T_0]$ then non-negativity is assured on $[0, \infty)$. This is possible due to the presence of the real strictly dominant pole and a restriction on the norm of $\hat{\mathbf{b}}$. Then the generalized Budan-Fourier algorithm can determine whether non-negativity on the positive half line holds by using just the interval $[0, T_0]$.

As a starting guess for the Convex Optimisation algorithm an initial $\hat{\mathbf{b}}_0 \in \{\hat{\mathcal{B}}(\hat{\mathbf{A}},\ \hat{\mathbf{c}}) \bigcap \hat{\mathcal{B}}_1(\hat{\mathbf{A}},\ \hat{\mathbf{c}})\}$ is always available since the zero vector itself lies in this set. Using the Budan-Fourier algorithm it is possible to choose α, $0 \leq \alpha \leq 1$, such that $\hat{\mathbf{b}}_1 = \alpha \hat{\mathbf{b}}^*$ lies on the boundary of $\mathcal{B}(\hat{\mathbf{A}},\ \hat{\mathbf{c}})$ and it follows that $\hat{\mathbf{b}}_1 \in \hat{\mathcal{B}}_1(\hat{\mathbf{A}},\ \hat{\mathbf{c}})$. This may be a good initial guess if no other obvious can-

didates are available.

The implementation of the algorithm in MATLAB is described in Section 7.4. The inbuilt convex optimisation function `fmincon` is the focal point of this algorithm and is responsible for steps 1 and 4 below. Assuming $\hat{\mathbf{b}}_i \in \{\hat{\mathcal{B}}(\mathbf{A}, \mathbf{c}) \bigcap \hat{\mathcal{B}}_1(\hat{\mathbf{A}},\ \hat{\mathbf{c}})\}$, the criterion function to be minimised is

$$\mathcal{C}_i = ||\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x}(\hat{\mathbf{b}}_i - \hat{\mathbf{b}}^*)||_2^2 \tag{3.49}$$

Firstly T_0 is computed using the method described in Section 3.7 and we set the penalty value $P \in \mathbb{R}$ as an extremely large constant. The algorithm then proceeds as follows:

1. Compute $\hat{\mathbf{b}}_{i+1}$ by perturbing $\hat{\mathbf{b}}_i$ in a direction to decrease the criterion function $\mathcal{C}_i$.

2. Check if $\hat{\mathbf{b}}_{i+1} \in \hat{\mathcal{B}}_1(\hat{\mathbf{A}},\ \hat{\mathbf{c}})$. If true proceed to next step, otherwise penalise $\mathcal{C}_i$ by setting $\mathcal{C}_{i+1} = P > \mathcal{C}_i$ and goto step 4.

3. Examine if $\hat{\mathbf{b}}_{i+1} \in \hat{\mathcal{B}}(\hat{\mathbf{A}},\ \hat{\mathbf{c}})$ by testing if $\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}_{i+1}} \geq -\mu_M e^{\lambda_M x}$ for all $x \in [0, T_0]$. If true, compute $\mathcal{C}_{i+1}$ using Eq (3.49), otherwise set $\mathcal{C}_{i+1} = P > \mathcal{C}_i$.

4. If $\mathcal{C}_{i+1} \leq \mathcal{C}_i$, compute $\hat{\mathbf{b}}_{i+2}$ by perturbing $\hat{\mathbf{b}}_{i+1}$, otherwise revert back to $\hat{\mathbf{b}}_i$ and compute a new $\hat{\mathbf{b}}_{i+1}$ by perturbing $\hat{\mathbf{b}}_i$ in a different direction to Step (1).

This procedure is repeated until no further improvement is possible (i.e. $|\mathcal{C}_n - \mathcal{C}_{n-1}| < \epsilon$ for a large number of consecutive iterations) implying the global minimum has been located at $\hat{\mathbf{b}}_C^*$. The essential result is that at each point, obtained in the algorithm, that is not equal to the global minimizing point, a direction exists in which the criterion function is improved. To see this consider the piece of straight line connecting a point obtained in the algorithm with the globally minimizing point. Due to convexity, the piece of straight line will lie wholly in the set $\hat{\mathcal{B}}(\hat{\mathbf{A}},\ \hat{\mathbf{c}})$, while the criterion will be strictly decreasing along

this straight line (until the global minimum is be reached). The convex optimization algorithm used tries to find a direction in which the criterion function is improved. The result states that such a direction can always be found except if we would have arrived at the global optimum. The algorithm is illustrated in Figure 3.5. Note, the steps in the Fig 3.5 do not correspond to steps in the algorithm outlined above.

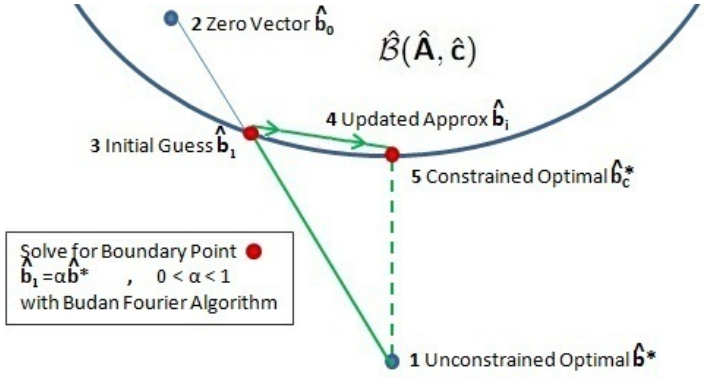

Figure 3.5: Convex Optimisation Algorithm for Non-Negative EPT Density Function using Budan Fourier Technique

3.9. Non-Negative Convex Optimisation Algorithm Example

We now provide an example to illustrate the performance of the non-negative convex optimisation algorithm described in Sections 3.8. The function we will approximate is

$$h(x) \;=\; 16e^{-\frac{x}{2}} - 30e^{-x} + 15e^{-2x} \;=\; \mu_M e^{\lambda_M x} + \hat{\mathbf{c}}e^{\hat{\mathbf{A}}x}\hat{\mathbf{b}}^* \tag{3.50}$$

The dominant pole $\lambda_M = -\frac{1}{2}$ and its coefficient $\mu_M = 16$ are easily identifiable. The minimal realization of $h(x)$ in state space form is given by

$$\left(\begin{array}{ccc|c} -0.5 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ \hline 16 & -30 & 15 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} \lambda_M & 0 & & 1 \\ 0 & \hat{\mathbf{A}} & & \hat{\mathbf{b}}^* \\ \hline \mu_M & \hat{\mathbf{c}} & & 0 \end{array}\right) \tag{3.51}$$

It is clear from Figure 3.6 that $h(x)$ is negative on the interval $[0.1897, 0.6575]$. We will approximate $h(x)$ with the non-negative EPT function $f(x)$

$$f(x) \;=\; \mu_M e^{\lambda_M x} + \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}_C^* \tag{3.52}$$

For the pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ we will use the algorithm described in Section 3.8 to derive $\hat{\mathbf{b}}_C^*$ minimising the L_2 norm

$$\min_{\{\hat{\mathbf{b}} \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})\}} ||\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} (\hat{\mathbf{b}}^* - \hat{\mathbf{b}})||_2^2 \tag{3.53}$$

where the convex set $\mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ is as defined in Section 3.6.

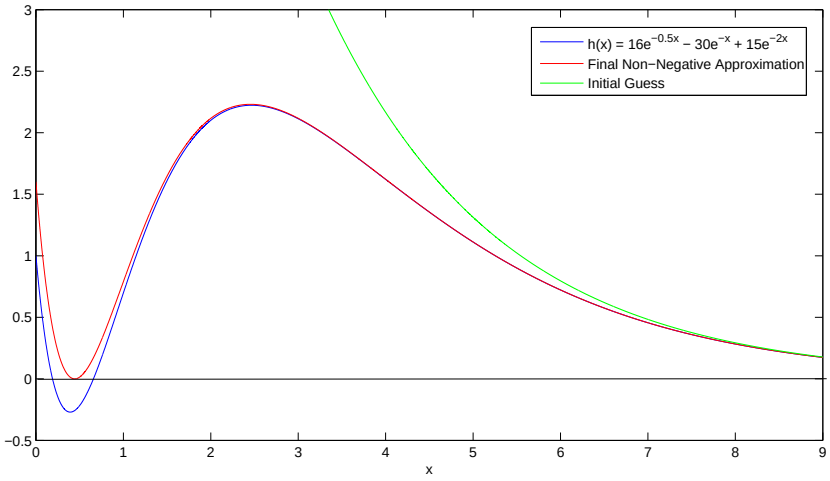

Figure 3.6: Non-Negative Approximation

Using the technique from Section 3.7 we can construct the finite interval on which it is necessary to check for non-negativity. In this instance the interval

is $[0,\ 9.2156]$. Hence if $f(x)$ is non-negative on $[0,\ 9.2156]$ then we know that $f(x)$ is non-negative on $[0, \infty)$.

We begin the convex optimisation with the initial guess, $\hat{\mathbf{b}}_1 = (0.9797,\ 0.9797)^T$ which is a convex combination of the zero vector and $\hat{\mathbf{b}}^*$. Running the algorithm we find the $\hat{\mathbf{b}}_C^*$ minimising Eq. (3.53) is $\hat{\mathbf{b}}_C^* = (0.9984, 1.0366)^T$ which yields an L_2 norm of 0.1296. The non-negative approximating EPT function f has a state space realization

$$\left(\begin{array}{ccc|c} -0.5 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0.9982 \\ 0 & 0 & -2 & 1.0359 \\ \hline 16 & -30 & 15 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \lambda_M & 0 & 1 \\ 0 & \hat{\mathbf{A}} & \hat{\mathbf{b}}_C^* \\ \hline \mu_M & \hat{\mathbf{c}} & 0 \end{array}\right) \tag{3.54}$$

We can calculate the minimum of $f(x)$ which is a small positive number of the order 10^{-15} indicating that $\hat{\mathbf{b}}_C^*$ does indeed lie on the boundary of the set $\mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$. The optimisation procedure took 65 seconds to compute.

3.10. Additional Remarks

If we have $\hat{\mathbf{b}}^* \notin \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ then we can find $\hat{\mathbf{b}}_C^* \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ for a given pair $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ but it is still possible to adapt the $(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ and derive a new $\hat{\mathbf{b}}_C^* \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ to try and improve the non-negative approximation. Also further iterations can be used to determine the strictly dominant pole and the triple $(\hat{\mathbf{A}}, \hat{\mathbf{b}}_C^*, \hat{\mathbf{c}})$ jointly. (We can use $h - \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}_C^*$ to perform a log-linear regression or an L^2 optimal order one EPT approximation!).

3.11. 2-EPT Approximation

To fit an EPT function with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ to the data on $(-\infty, 0]$ such that $\sigma(\mathbf{A}_N) \subset \mathbb{H}_+$ is identical to the method described above. The 2-EPT function with realization $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P, \mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ approximates the data on the whole real line. This 2-EPT function is then scaled to ensure integration to

unity.

Using RARL2 it is possible to impose continuity at the origin restricting

$$\mathbf{c}_P\mathbf{b}_P \;=\; C_0 \;=\; \mathbf{c}_N\mathbf{b}_N \tag{3.55}$$

An iterative procedure is required to derive the optimal C_0 which minimizes the square error between the 2-EPT function and the empirical data. Although such a condition will reduce the quality of fit of the 2-EPT approximation, it seems like a sensible restriction when we continue on to option pricing within the 2-EPT framework. If the underlying 2-EPT density has a discontinuity at the origin, then a similar discontinuity will be observed in the associated vanilla option prices. In chapter 6 we will see that the vanilla option pricing formulae are indirectly split about the origin so the discontinuity persists. Such a discontinuity in option prices does not violate any no-arbitrage requirements but from a practitioners perspective it should be avoided.

A further restriction which could be considered would be to ensure the derivative of the density function is continuous at zero although this would require significant changes in the approximation procedure to implement. Such a condition would ensure the continuity of the Greeks of the option prices.

The point mass at zero of the 2-EPT density could be estimated as the proportion of zeros in the dataset. A scaling factor would then be used in the RARL2 approximation procedure to ensure integration to unity.

3.12. 2-EPT Vs. Variance Gamma Vs. Gaussian Approximation

We now evaluate the quality of the 2-EPT approximation against the Variance Gamma and benchmark Gaussian distributions. We will impose the condition of continuity at the origin, as given in Section 3.11 for the 2-EPT approximation. We consider two Variance Gamma approximations, the first with no constraint on the input parameters while the second approximation, restricts the input

parameter "C" to be an integer such that it can be represented as a 2-EPT distribution. The empirical density being approximated is the set of DJIA log returns (1931 - 2011) and will be approximated on the interval $[-0.2563,\ 0.1427]$ which includes all returns over the period in question. Letting $h(x)$ be the normalised density of log returns observable at $N+1$ equally spaced points on $[-0.2563,\ 0.1427]$ and assuming $f(x)$ represents the approximating density we estimate the squared distance between f and h as the sum

$$\sum_{i=0}^{N} (h(x_i) \ - \ f(x_i))^2 \ \delta x \tag{3.56}$$

where $x_i \ - \ x_{i-1} \ = \ \delta x \ \forall \ i \ = \ \{1,\ 2,\ ...,\ N\}$ given that

$$\delta x \ = \ \frac{0.2563 + 0.1427}{N} \tag{3.57}$$

We will use the squared distance between the empirical and approximating densities as the measurement to evaluate the fit.

The parameters estimated for the Variance Gamma and Gaussian distributions were chosen to minimise Eq. (3.56). The goodness of fit is illustrated in Figure 3.7.

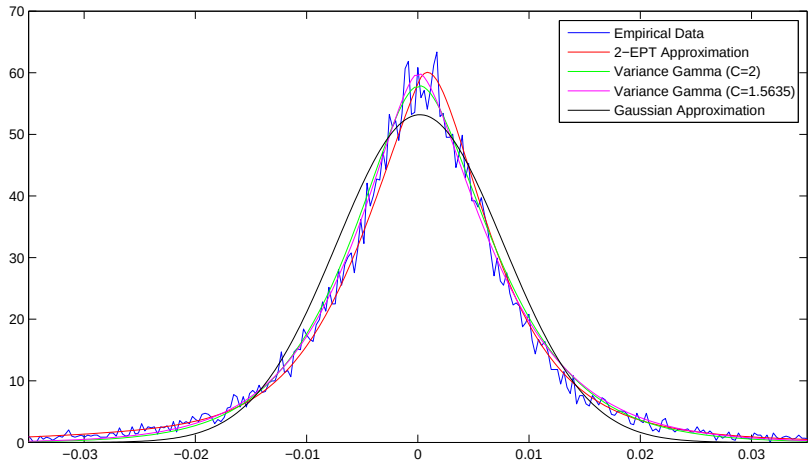

Figure 3.7: 2-EPT, Variance Gamma and Gaussian Approximations of Empirical Log Returns. N = 400

From Figure 3.7 it is difficult to distinguish between the quality of fit for the Variance Gamma and 2-EPT distributions. However it is clear that the Gaussian density provides a poor fit with tails decaying too quickly and not sufficiently peaked at the mean.

Table 3.1 shows the error (Squared Distance calculated via Eq. (3.56)) for each of the approximations versus the empirical density.

	Error	Parameters
2-EPT (Order 3)	0.2498	$(\mathbf{A}_N,\ \mathbf{b}_N,\ \mathbf{c}_N,\ \mathbf{A}_P,\ \mathbf{b}_P,\ \mathbf{c}_P)$
Variance Gamma	0.2341	C = 1.5635, G = 202.5676, M = 189.1945
Variance Gamma(C=2)	0.2686	C = 2, G = 239.7714, M = 225.9878
Gaussian	0.7747	$\mu = 0.000191$, $\sigma = 0.00751$

Table 3.1: Errors Associated with Density Approximations and respective parameters.

We now perform a Chi-Square test to evaluate the goodness of fit further. The null hypothesis of the test is $H_0 : f_0 = g$ where f_0 represents the distri-

bution of the empirical data while g is distribution being tested (namely the 2-EPT, Variance Gamma, Variance Gamma (C=2) and Gaussian distributions) as described in Table 3.1. The alternative hypothesis is $H_A : f_0 \neq g$. We also calculate the average log likelihood ($\hat{l}$) for each distribution. We have that

$$\hat{l} = \frac{1}{N_{obs}} \sum_{i=1}^{N_{obs}} \log(g(x_i)) \tag{3.58}$$

Higher values of $\hat{l}$ imply a better fit. The average log-likelihood is just a scaled version of the log-likelihood as $N_{obs} = 20,734$ for all comparisons.

	Chi-Square Stat	P-Value	Average Log-Likelihood
2-EPT (Order 3)	0.0224	1	3.2148
Variance Gamma	0.0751	0.9993	3.1818
Variance Gamma(C=2)	0.1419	0.9982	3.1626
Gaussian	401.6704	1.32e-25	2.7759

Table 3.2: Chi-Square Test Statistics and associated p-values. Average Log-Likelihood also provided.

From the in Table 3.2 it is clear that the Gaussian distribution is not a suitable approximation for the data (p-value 1.32e-25) so we reject the null hypothesis. However in the other 3 cases (the 2-EPT and both Variance Gamma distributions) we cannot reject the null hypothesis and therefore we assume that these distributions are good approximations for the data. It is also obvious that the 2-EPT distribution produces the lowest Chi-Square Statistic and highest p-value. The 2-EPT distribution also produces the highest average log-likelihood. Therefore, both sets of analysis indicate that the 2-EPT approximation provides the best fit for the data.

It was necessary to check certain assumptions before we performed the Chi-Square test. Firstly the data observations must be independent which was confirmed by examining the autocorrelation plot of the data. The data was

binned into 400 equally sized intervals on $(-0.0450, 0.0411)$. These parameters were chosen to meet the requirement for a chi-square test that 80% of the intervals have an expected count of greater than 5. The full set of data was used to perform the likelihood analysis.

The state space realization of the 2-EPT probability density function is given by

$$\left(\begin{array}{c|c} \mathbf{A}_N & \mathbf{b}_N \\ \hline \mathbf{c}_N & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 99.5597 & 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & 61.4545 & -97.6067 & 3.5058 \\ 0.0000 & 370.1559 & 302.1875 & 7.7742 \\ \hline 29.6202 & 5.52403 & 1.1851 & 0 \end{array}\right) \tag{3.59}$$

and

$$\left(\begin{array}{c|c} \mathbf{A}_P & \mathbf{b}_P \\ \hline \mathbf{c}_P & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -119.5601 & 0.0000 & 0.0000 & 1.0000 \\ 0.0000 & -33.0987 & 160.9465 & 2.5729 \\ 0.0000 & -426.6732 & -533.3337 & 10.3280 \\ \hline 31.5888 & 7.9015 & 0.6081 & 0 \end{array}\right) \tag{3.60}$$

where

$$\mathbf{c}_{N_2}\mathbf{b}_{N_2} \;=\; \mathbf{c}_{P_2}\mathbf{b}_{P_2} \tag{3.61}$$

It should also be pointed out that $\sigma(\mathbf{A}_N) = (99.5597, 181.8210 \pm 147.1108i)$ and $\sigma(\mathbf{A}_P) = (-119.5601,\ -283.2162 \pm 78.1844i)$. It is clear from the dominant poles of both spectrums $\sigma(\mathbf{A}_N)$ and $\sigma(\mathbf{A}_P)$ that the tail of the EPT function defined on $(-\infty, 0]$ decays slower than the tail of the EPT function defined on $[0, \infty)$. This is expected as large negative returns occur more frequently than corresponding positive returns.

The tail behaviour of the 2-EPT density differs substantially from the tail behaviour of both Variance Gamma Densities. Here we will analyse the tails of the distributions on $[0, \infty)$ but a similar argument can be made for the same analysis on $(-\infty, 0]$. This difference can be seen by the fact that the positive tails of the Variance Gamma densities have dominant eigenvalues of -189.1945 and -225.9878 compared with the 2-EPT density which has a dominant eigenvalue of -119.5601. The reason there is such a discrepancy in the decay is that the

tail behaviour of the Variance Gamma densities was not estimated explicitly like in the 2-EPT case. The positive tail behaviour of the Variance Gamma density is governed by the input parameter M. However, in the analysis above, the parameters C, G, M were chosen to minimise the overall square distance error, as given by Eq. (3.56), rather than capture the tail behaviour. The fact that the tail of the 2-EPT density can be estimated independently from the centre of the distribution illustrates another advantage of using 2-EPT densities to approximate empirical densities.

It should be noted that the 2-EPT density function is non-negative and integrates to unity. It is clear from Table 3.1 that the Variance Gamma distribution with $C = 1.5635$ provides the best approximation with a square error of 0.2341. However this distribution can not be represented with a finite state space realization. The 2-EPT approximation of order (3) has a square error of 0.2498 which is better than the constrained Variance Gamma approximation with "C" integer which has an error of 0.2686. Both these approximations are significantly better than the Gaussian approximation which has a square error of 0.7747 where μ and σ were chosen to minimise the square error.

3.12.1. 2-EPT Moments

Using the formulae to calculate the moments of a 2-EPT random variable given in Section 2.2.5, the moments of the fitted 2-EPT density given in Section 3.12 are computed around the mean.

	2-EPT	Empirical
Mean	1.2079×10^{-4}	1.9101×10^{-4}
Volatility	0.01057	0.01106
Skewness	-0.6231	-0.5946
Kurtosis	8.4205	27.7125

Table 3.3: 2-EPT Density and Empirical Moments about the mean

From Table 3.3 we see that the moment approximations are quite reasonable. The annual drift of the 2-EPT model is 3.04% compared with a realized annual drift of 4.81%. Assuming there are 252 trading days in the year the daily volatility estimate equates to an annualized volatility of 16.813%. The annual volatility was calculated using the "N-Fold Convolution" function which is described in Section 7.5.3 and simply derives the convolution of a 2-EPT density with itself N times. The realized annual volatility calculated directly from the data was 15.27%. The negative skewness of the 2-EPT is a good approximation for the empirical density. The kurtosis of the 2-EPT density underestimates the true kurtosis exhibited implying that the tails of the 2-EPT density may still be too light for this purpose.

Chapter 4

2-EPT Lévy Processes

4.1. Introduction

The importance of infinitely divisible distributions has been elucidated to in Chapter 1 where their connection with Lévy processes was noted. It was stated that each infinitely divisible distribution gives rise to a unique Lévy process and the converse is also true. In this Chapter we give a method for characterising infinitely divisible EPT random variables. It is also proven how to characterise infinitely divisible mixtures of an EPT densities with Dirac distributions.

It is then shown how the Laplace Transform of an infinitely divisible 2-EPT probability density function can be factored into the Laplace Transforms of two infinitely divisible EPT functions. A necessary and sufficient condition is then derived to characterise infinitely divisible 2-EPT probability density functions. The Lévy triple of an infinitely divisible 2-EPT probability density function is also derived. The chapter concludes with an example illustrating that the Variance Gamma probability density function is indeed infinitely divisible.

The contents of this chapter were presented at the Bachelier Finance Conference in Sydney in June 2012. The associated paper, Hanzon, Holland and Sexton (2012), contains many of the results of this chapter and has been submitted to

the Bernoulli Journal of Mathematical Statistics and Probability for publication.

4.2. Mathematical Formulation

We begin with the definition of a Levy process from Cont and Tankov (2003)

Lévy Process *A càdlàg stochastic process (Paths are right-continuous with left limits everywhere, with probability one)* $(X_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}$ *such that* $X_0 = 0$ *is called a Lévy process if it possesses the following properties*

- *Independent increments: for every increasing sequence of times* $t_0, t_1, ..., t_n$, *the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ *are independent.*
- *Stationary increments: the law of* $X_{t+h} - X_t$ *does not depend on* t.
- *Stochastic continuity:* $\forall\, \epsilon > 0,\ \lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0.$

The Lévy-Khintchine formula is used to characterise infinitely divisible distributions in terms of its Lévy triple. There are a number of representations of the formula but we present the most common form within the Financial Mathematics literature which is taken from Kyprianou and Loeffen (2005)

Lévy-Khintchine Formula *A probability law* f *of a real valued random variable is infinitely divisible with characteristic exponent* Ψ,

$$\int_{\mathbb{R}} e^{-isx} f(x)dx \;=\; e^{-\Psi(s)}, \qquad s \;\in\; \mathbb{R}, \tag{4.1}$$

if and only if there exists a triple (a, σ, ν) *where* $a \in \mathbb{R}$, $\sigma \geq 0$ *and* ν *a measure concentrated on* $\mathbb{R}\backslash\{0\}$ *satisfying* $\int_{\mathbb{R}} \min\{1, x^2\} d\nu(x) \;<\; \infty$, *such that*

$$\Psi(s) \;=\; ias + \frac{1}{2}\sigma^2 s^2 + \int_{\mathbb{R}} (1 - e^{isx} + isx\mathbb{I}_{\{|x|<1\}}) d\nu(x), \tag{4.2}$$

for all s *in* $\mathbb{R}$.

The measure ν is the unique Lévy measure. The characteristic exponent from

Eq. (4.2) can be decomposed as

$$\Psi(s) \;=\; ias + \frac{1}{2}\sigma^2 s^2 \;+\; \int_0^\infty (1 - e^{isx} + isx\mathbb{I}_{\{x<1\}})d\nu_P(x) + \int_{-\infty}^0 (1 - e^{isx} + isx\mathbb{I}_{\{x>-1\}})d\nu_N(x), \quad (4.3)$$

where $\nu_P(x) = 0$ for all $x < 0$ and $\nu_N(x) = 0$ for all $x > 0$.

Proposition 3.9 from Cont and Tankov (2003) provides a formulation for Lévy processes of finite variation.

Finite Variation Lévy Processes *A Lévy process is of finite variation if and only if its characteristic triplet* (a, σ, ν) *satisfies*

$$\sigma = 0 \qquad \textit{and} \qquad \int_{|x|\leq 1} |x| d\nu(x) \;<\; \infty \quad (4.4)$$

4.3. Infinitely Divisible EPT Distributions

We begin by examining an EPT probability density function f defined on $[0, \infty)$ by the minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c})$

$$f(x) \;=\; \mathbf{c}e^{\mathbf{A}x}\mathbf{b}, \qquad x \geq 0, \quad (4.5)$$

such that $\sigma(\mathbf{A}) \subset \mathbb{H}_-$. As f is non-negative on the half line $[0, \infty)$ it stated in Section 2.1.1 that a Perron-Frobenius type result implies that the spectrum of $\mathbf{A}$ contains a negative dominant real eigenvalue λ_M such that $\lambda_M = \max_{\lambda\in\sigma(\mathbf{A})} Re(\lambda)$.

The Laplace transform of f for $s > 0$ is given by

$$F(s) \;=\; \int_0^\infty e^{-sx} f(x)dx \;=\; \int_0^\infty e^{-sx}\mathbf{c}e^{\mathbf{A}x}\mathbf{b}dx \;=\; \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \;=\; \frac{p(s)}{q(s)}. \quad (4.6)$$

F is a strictly proper rational function as p and q are co-prime polynomials of order m and n respectively where $n > m$.

We let $\Lambda(\mathbf{A}, \mathbf{b}, \mathbf{c}) = \{s|\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} = 0\}$ be the set of zeros of the function $\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$. A result from Lukacs (1970) states that an infinitely divisible analytic function cannot have any zero inside its strip of convergence. If a rational function $\mathbf{c}(s\mathbf{I} - A)^{-1}\mathbf{b}$ is infinitely divisible it must hold that $\max_{\lambda \in \Lambda(\mathbf{A},\mathbf{b},\mathbf{c})} Re(\lambda) \leq \lambda_M$. Only triples $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ satisfying this necessary condition will be considered, otherwise the resulting EPT probability density function is not infinitely divisible. As $\lambda_M < 0$ it holds that $\Lambda(\mathbf{A}, \mathbf{b}, \mathbf{c}) \subset \mathbb{H}_-$. In systems theory a transfer function with all poles and zeros located in the half plane is referred to as "minimum phase".

It is known, Feller (1971), that a function g is the Laplace transform of an infinitely divisible probability distribution on $[0, \infty)$, if and only if $g = e^{-w}$ where the derivative of w is completely monotonic on $[0, \infty)$ and $w(0+) = 0$.

Completely Monotonic Function *A function J defined on $(0, \infty)$ is said to be completely monotonic if it possesses derivatives $J^n(x)$ for all $n = 0, 1, 2, ...$ such that*

$$(-1)^n J^{(n)}(x) \geq 0, \tag{4.7}$$

for all $x > 0$

It is known from Bernsteins Theorem, given in Widder (1941) (p. 160), that a necessary and sufficient condition that a function J is completely monotonic on $[0, \infty)$ is

$$J(s) = \int_0^\infty e^{-sx} d\nu(x), \tag{4.8}$$

for $0 \leq s < \infty$ where ν is a finite, non-negative measure on $[0, \infty)$ and the integral converges for $0 \leq s < \infty$. We conclude that a non-identically zero completely monotonic function $J(s)$ cannot vanish for any positive s.

From the results of Feller (1971) above, if F is the Laplace transform of the infinitely divisible distribution f on $[0, \infty)$ then $F = e^{-h}$ and the derivative of

h given by

$$h' \;=\; -\frac{F'}{F}, \tag{4.9}$$

must be completely monotonic on $[0,\infty)$. By construction it holds that $h(0) = -\log(F(0)) = 0$ as $F(0) = 1$ since f is a probability density function. It is clear that if h' is completely monotonic on $[0,\infty)$ then there exists some positive measure ν such that

$$h'(s) \;=\; \int_0^\infty e^{-sx} d\nu(x), \qquad s > 0. \tag{4.10}$$

Since $F = e^{-h}$ and $h(0) = 0$, by Fubini

$$h(s) \;=\; \int_0^s h'(t)dt \;=\; \int_0^\infty \frac{1-e^{-sx}}{x} d\nu(x), \qquad s > 0. \tag{4.11}$$

In particular we have that

$$\int_0^\infty \frac{1-e^{-x}}{x} d\nu(x) \;=\; h(1) \;<\; \infty.$$

However,

$$\frac{1-e^{-x}}{x} \;>\; \frac{1}{1+x}, \qquad x \;>\; 0,$$

implying that

$$\int_0^\infty \frac{1}{x+1} d\nu(x) \;<\; \infty.$$

It follows from this that

$$\int_1^\infty \Big|\frac{1-e^{-sx}}{x}\Big| d\nu(x) \;\le\; 2\int_1^\infty \frac{1}{x} d\nu(x) \;<\; \infty, \qquad s \in \mathbb{H}_+,$$

whence

$$\int_0^\infty \frac{1-e^{-sx}}{x} d\nu(x) \;=\; \int_0^1 \frac{1-e^{-sx}}{x} d\nu(x) + \int_1^\infty \frac{1-e^{-sx}}{x} d\nu(x),$$

exists everywhere on $\mathbb{H}_+$ and defines a function that is analytic there. In other words,

$$h(s) \;=\; \int_0^\infty \frac{1-e^{-sx}}{x} d\nu(x)\;,\; \forall\; s \;\in\; \mathbb{H}_+. \tag{4.12}$$

A limiting argument assures us that

$$h(i\omega) \;=\; \int_0^\infty \frac{1-e^{-i\omega x}}{x} d\nu(x), \qquad -\infty < \omega < \infty\;.$$

$\nu(x)$ defined on $(0,\infty)$ must be determined such that $h'(s)$ is the Laplace transform of $\nu(x)$. It is clear that

$$h'(s) \;=\; -\frac{F'(s)}{F(s)} \;=\; \frac{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-2}\mathbf{b}}{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}}.$$

The strictly proper rational function F can be written as

$$F(s) \;=\; \frac{\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b}}{\det(s\mathbf{I}-\mathbf{A})}, \tag{4.13}$$

where the zeros of the polynomial $\mathbf{c}(s\mathbf{I}-\mathbf{A})^{*}\mathbf{b}$ are the zeros of $F(s)$. $\Big[\mathbf{M}^*$ denotes the adjoint matrix of the square matrix $\mathbf{M}$, such that $\mathbf{M}^*\mathbf{M} = \mathbf{M}\mathbf{M}^* = \det(\mathbf{M})\,\mathbf{I}\Big]$.

Lemma 4.1 *Suppose q is the characteristic polynomial of a square matrix $\boldsymbol{Q}$ where $\sigma(\boldsymbol{Q}) \subset \mathbb{H}_-$. Then, for $s > 0$, $q'(s)/q(s)$ is the Laplace transform of the trace of* $\exp(\boldsymbol{Q}x)$

Proof

$$Tr(e^{\mathbf{Q}x}) \;=\; \sum_{n=0}^{\infty} \frac{x^n Tr(\mathbf{Q}^n)}{n!},$$

but

$$Tr(\mathbf{Q}^n) \;=\; \sum_{\lambda\in\sigma(\mathbf{Q})} \lambda^n.$$

Hence for $s > 0$

$$\begin{aligned}\int_0^\infty e^{-sx} Tr(e^{\mathbf{Q}x})dx &= \sum_{\lambda\in\sigma(\mathbf{Q})} \int_0^\infty \Big(\sum_{n=0}^\infty \frac{x^n\lambda^n}{n!}\Big)e^{-sx}dx \\ &= \sum_{\lambda\in\sigma(\mathbf{Q})} \int_0^\infty e^{\lambda x}e^{-sx}dx \\ &= \sum_{\lambda\in\sigma(\mathbf{Q})} \frac{1}{s-\lambda} \\ &= \frac{q'(s)}{q(s)}. \qquad \blacksquare\end{aligned}$$

Lemma 4.2 *Let F be a minimum phase strictly proper rational function and $F = e^{-h}$. Then $h' = -F'/F$, is the Laplace transform of $Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}x})$ where $\boldsymbol{B}$ is the companion matrix of $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}$ if $\deg(\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}) > 0$. If $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}$ is constant then h' is the Laplace Transform of $Tr(e^{\boldsymbol{A}x})$.*

Proof

If $\deg(\mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b}) > 0$ let $F = p/q$ where $p(s) = \mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b}$ and $q(s) = \det(s\mathbf{I}-\mathbf{A})$. Using the quotient rule

$$-\frac{F'(s)}{F(s)} = \frac{q'(s)}{q(s)} - \frac{p'(s)}{p(s)}.$$

As F is of minimum phase, $Tr(e^{\mathbf{A}x})$ and $Tr(e^{\mathbf{B}x})$ are integrable on $(0,\infty)$ and by Lemma 4.1

$$\begin{aligned}-\frac{F'(s)}{F(s)} &= \int_0^\infty e^{-sx}Tr(e^{\mathbf{A}x})dx - \int_0^\infty e^{-sx}Tr(e^{\mathbf{B}x})dx \\ &= \int_0^\infty e^{-sx}\Big(Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})\Big)dx.\end{aligned}$$

If $\mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b} = K$ where $K \in \mathbb{R}$ then $F(s) = K/q(s)$ and $q(s) = \det(s\mathbf{I}-\mathbf{A})$. It can be seen that

$$-\frac{F'(s)}{F(s)} = \frac{q'(s)}{q(s)}, \tag{4.14}$$

and similarly

$$-\frac{F'(s)}{F(s)} \;=\; \int_0^\infty e^{-sx} Tr(e^{\mathbf{A}x})dx. \qquad \blacksquare \tag{4.15}$$

Theorem 4.1 *Given a minimal triple* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ *that defines a not identically zero EPT probability density function whose Laplace Transform is of minimum phase. If* $\deg(\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}) > 0$, $\boldsymbol{B}$ *is the companion matrix of* $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}$ *and* $F(s)$ *is the Laplace transform of* $f(x) = \boldsymbol{c}e^{\boldsymbol{A}x}\boldsymbol{b}$*, then*

$$F(s) \;=\; \exp\Big(-\int_0^\infty (1-e^{-sx})\frac{Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}x})}{x}dx\Big), \tag{4.16}$$

Thus F *is infinitely divisible iff* $Tr(e^{\boldsymbol{A}x}) \geq Tr(e^{\boldsymbol{B}x})$ *for all* $x \geq 0$

If $\boldsymbol{c}(s\boldsymbol{I}-\boldsymbol{A})^*\boldsymbol{b}$ *is constant then the Laplace transform of* $f(x) = \boldsymbol{c}e^{\boldsymbol{A}x}\boldsymbol{b}$ *is given by* $F(s)$

$$F(s) \;=\; \exp\Big(-\int_0^\infty (1-e^{-sx})\frac{Tr(e^{\boldsymbol{A}x})}{x}dx\Big), \tag{4.17}$$

where $F(s)$ *is infinitely divisible iff* $Tr(e^{\boldsymbol{A}x}) \geq 0$ *for all* $x \geq 0$.

By comparing the exponent of $F(s)$ in Eq. (4.16) to the characteristic exponent of the Lévy-Khintchine formula in Eq. (4.2) the Lévy triple (a, σ, ν) of the EPT probability density function with realization $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ can be determined. The drift component a must be chosen such that

$$a \;=\; -\int_0^1 x\, d\nu(x) \tag{4.18}$$

while the diffusion term $\sigma = 0$. If $\deg(\mathbf{c}(s\mathbf{I}-\mathbf{A})^*\mathbf{b}) > 0$ the Radon-Nikodym derivative of the Lévy measure of the EPT density

$$\nu'(x) \;=\; \frac{Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})}{x}, \qquad x > 0, \tag{4.19}$$

and $\nu'(x) = 0$ for all $x < 0$. If $\deg(\mathbf{c}(s\mathbf{I} - \mathbf{A})^*\mathbf{b} = 0$, then

$$\nu'(x) \;=\; \frac{Tr(e^{\mathbf{A}x})}{x}, \qquad x > 0. \tag{4.20}$$

Again we must have $\nu'(x) = 0$ for all $x < 0$.

According to the definition in Eq. (4.4) the Lévy process is of finite variation if a from Eq. (4.18) is finite. Consider the case where $\deg(\mathbf{c}(s\mathbf{I} - \mathbf{A})^*\mathbf{b} > 0$ and note that

$$\begin{aligned} a &= -\int_0^1 \Big(Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})\Big)dx \\ &= -\Big[Tr(\mathbf{A}^{-1}e^{\mathbf{A}x}) - Tr(\mathbf{B}^{-1}e^{\mathbf{B}x})\Big]_0^1 \\ &= -Tr(\mathbf{A}^{-1}e^{\mathbf{A}}) + Tr(\mathbf{B}^{-1}e^{\mathbf{B}}) + Tr(\mathbf{A}^{-1}) - Tr(\mathbf{B}^{-1}) < \infty \end{aligned} \tag{4.21}$$

which holds as $\{\sigma(\mathbf{A}) \cup \sigma(\mathbf{B})\} \subset \mathbb{H}_-$. When $\mathbf{c}(s\mathbf{I} - \mathbf{A})^*\mathbf{b}$ is constant it can be seen that

$$a \;=\; -Tr(\mathbf{A}^{-1}e^{\mathbf{A}}) + Tr(\mathbf{A}^{-1}) \;<\; \infty \tag{4.22}$$

Therefore Lévy processes generated from infinitely divisible EPT distributions are of finite variation.

Consider the case where $\deg(\mathbf{c}(s\mathbf{I} - \mathbf{A})^*\mathbf{b}) > 0$, then $x\nu'(x)$ is an Exponential-Trigonometric (ET) function given by

$$x\nu'(x) \;=\; Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x}) \tag{4.23}$$

It is clear that F defined in Eq. (4.16) is infinitely divisible if and only if $x\nu'(x)$ is non-negative on $[0, \infty)$.

It has been stated that $\sigma(\mathbf{A})$ contains a dominant real eigenvalue $\lambda_M < 0$. Denote $\tilde{\lambda}_M = \max_{\lambda \in \sigma(\mathbf{B})} Re(\lambda)$ as the dominant eigenvalue in $\mathbf{B}$. It is clear

that if $x\nu'(x) \geq 0$, as in Eq. (4.23) for all $x \geq 0$ then it must hold that $\lambda_M \geq \tilde{\lambda}_M$. If $\lambda_M < \tilde{\lambda}_M$ then there would exist a $K > 0$ such that

$$Tr(e^{\mathbf{A}K}) < Tr(e^{\mathbf{B}K}). \tag{4.24}$$

This coincides with the result from Lukacs (1970) such that an analytic characteristic function of an infinitely divisible distribution can have no zeros located in its strip of convergence. This property can be used as a quick check to confirm that the Laplace Transform $F(s)$ is not infinitely divisible.

If F as given in Theorem 4.1 is infinitely divisible then it is clear that

$$\begin{aligned}
\nu'(x) = \quad & \frac{Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x})}{x} \quad \geq 0, \qquad \forall x > 0 \\
\iff \quad & Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x}) \quad \geq 0, \qquad \forall x > 0 \\
\implies \quad & \lim_{x \downarrow 0} Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}x}) \geq 0 \\
= \quad & Tr(\mathbf{I_A}) - Tr(\mathbf{I_B}) \qquad \geq 0
\end{aligned} \tag{4.25}$$

where $\mathbf{I_A}$ is an identity matrix with the same dimensions as the square matrix $\mathbf{A}$. It follows that $\dim(\mathbf{A}) \geq \dim(\mathbf{B})$ must hold if F is the Laplace Transform of an infinitely divisible EPT distribution implying that F must not contain more zeros than poles. This result is given in Corollary 4.1.

The Budan-Fourier Algorithm of Hanzon and Holland (2010) can be used to test for non-negativity of an EPT function on a finite interval. If the ET function, $x\nu'(x)$, contains a unique dominant real pole, which must be λ_M, then the aforementioned Budan-Fourier technique can be used to test for non-negativity of the ET function on the whole half line $[0, \infty)$ by examining a finite interval $[0, T]$. This result follows from Proposition 3.1 in Section 3.7 where it is proven how to construct $T > 0$ when an EPT function contains a unique dominant real eigenvalue.

A necessary condition that the function f is infinitely divisible is that f is strictly positive on $[0, \infty)$ and this can be seen in Lemma 4.3.

Lemma 4.3 *Suppose the minimal triple* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ *defines a non-negative EPT function denoted by* f*, that is not identically zero, which in turn defines a finite measure* $f(x)dx$*. If* f *is infinitely divisible it is necessary that* f *is strictly positive on* $[0, \infty)$*.*

Proof

Consider the Laplace Transform of f given by

$$F(s) \;=\; \int_0^\infty e^{-sx} f(x)dx \;=\; \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}, \tag{4.26}$$

for $s \in \mathbb{H}_+$. Then

$$-\frac{F'(s)}{F(s)} \;=\; \frac{\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-2}\mathbf{b}}{\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}}, \tag{4.27}$$

is a rational function. Hence, $-F'/F$ is the Laplace Transform of some continuous function k such that for all $s \in \mathbb{H}_+$

$$-\frac{F'(s)}{F(s)} \;=\; \int_0^\infty e^{-sx} k(x)dx. \tag{4.28}$$

Similarly

$$\int_0^\infty e^{-sx} x f(x)dx \;=\; \int_0^\infty e^{-sx} f(x)dx \int_0^\infty e^{-sx} k(x)dx, \tag{4.29}$$

which is the convolution of f and k. By uniqueness of the transform,

$$xf(x) \;=\; \int_0^\infty f(x-t)k(t)dt,\; 0 < x < \infty. \tag{4.30}$$

By assumption $f \geq 0$. If F is infinitely divisible then as noted earlier $-F'/F$ is completely monotonic. Hence $-F'/F$ is the Laplace Transform of some non-negative measure. Therefore $k \geq 0$.

Finally, if $f(a) = 0$ for some $a > 0$, then

$$0 \;=\; af(a) \;=\; \int_0^a f(a-t)k(t)dt, \tag{4.31}$$

whence, by continuity, $f(a-t)k(t) \equiv 0,\ \forall\ t\ \in\ [0,a]$, which is impossible and the result of Lemma 4.3 follows.■

4.4. Infinitely Divisible EPT Distributions Mixed with Dirac Function

We consider a generalised EPT density function (EPT density mixed with a pointmass at zero) such that the probability density function f defined on $[0,\infty)$ has a minimal realization $(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{d})$ of McMillan degree n and $0 < \mathbf{d} < 1$.

$$f(x) = \begin{cases} 0 & \text{with probability } \mathbf{d} \\ \\ \mathbf{c}e^{\mathbf{A}x}\mathbf{b} & \text{if } x\ >\ 0 \end{cases} \tag{4.32}$$

A Perron-Frobenius type from Hanzon and Holland (2010b) result implies that $\sigma(\mathbf{A})$ contains a dominant real eigenvalue $\lambda_M < 0$ such that $\lambda_M = \max_{\lambda\in\sigma(\mathbf{A})} Re(\lambda)$. The Laplace transform of f is the proper rational function F

$$F(s)\ =\ \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}+\mathbf{d}\ =\ \frac{p(s)}{q(s)}, \tag{4.33}$$

where p and q are co-prime polynomials of degree n. As already mentioned, a result from Lukacs (1970) states that an analytic Laplace Transform of an infinitely divisible probability density function cannot contain any zeros inside its strip of convergence. Letting $\Lambda(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{d}) = \{s|\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}+\mathbf{d}\}$, then for an infinitely divisible probability density function with minimal realization $(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{d})$ it must hold that $\max_{\lambda\in\Lambda(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{d})} Re(\lambda) \leq \lambda_M$. It follows since $\lambda_M < 0$ that $\Lambda(\mathbf{A},\mathbf{b},\mathbf{c},\mathbf{d}) \subset \mathbb{H}_-$ and therefore the F again is of minimum phase.

As in Section 4.3, define $F = e^{-h}$ and it follows that $h' = -F'/F$. Hence, by Feller (1971), F is the Laplace transform of an infinitely divisible distribution if and only if h' is completely monotonic on $[0,\infty)$ and $h(0) = 0$. It holds by construction that $h(0) = \log(F(0)) = 0$. Since F is a proper rational function

and $\mathbf{d} > 0$, a well known identity gives

$$F^{-1}(s) \;=\; \mathbf{d}^{-1} - \mathbf{c}\mathbf{d}^{-1}(s\mathbf{I} - (\mathbf{A} - \mathbf{b}\mathbf{d}^{-1}\mathbf{c}))^{-1}\mathbf{b}\mathbf{d}^{-1}. \tag{4.34}$$

The zeros of F^{-1} correspond to the eigenvalues of $\mathbf{A}$ and the poles of $F^{-1}(s)$ are the eigenvalues of $\mathbf{A} - \mathbf{b}\mathbf{d}^{-1}\mathbf{c}$. Following a similar technique from Section 4.3, Lemma 4.4 can be proven.

Lemma 4.4 *Let F be the minimum phase rational function $F(s) = \boldsymbol{c}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b} + \boldsymbol{d}$ and $F = e^{-h}$. Then $h' = -F'/F$ is the Laplace transform of $Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}_d x})$ where $\boldsymbol{B}_d = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{d}^{-1}\boldsymbol{c}$*

Proof

Let $F = p/q$ where $p(s) = \mathbf{d}\det(s\mathbf{I} - \mathbf{B}_d)$ and $q(s) = \det(s\mathbf{I} - \mathbf{A})$. The remainder of the proof follows as in Lemma 4.2 ■.

Hence we provide a necessary and sufficient condition to characterise an infinitely divisible generalised EPT probability density function.

Theorem 4.2 *Given a minimal realization $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ that defines a not identically zero probability density function f defined as the mixture of an EPT function with a pointmass at zero. Assume the Laplace Transform F of f is of minimum phase. F is given by*

$$F(s) \;=\; \exp\Big(-\int_0^\infty (1 - e^{-sx})\frac{Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}_d x})}{x}dx\Big). \tag{4.35}$$

Then F is infinitely divisible iff $Tr(e^{\boldsymbol{A}x}) - Tr(e^{\boldsymbol{B}_d x}) \geq 0$ for all $x \geq 0$ where $\boldsymbol{B}_d = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{d}^{-1}\boldsymbol{c}$

Comparing Eq. (4.35) to the Lévy-Khintchine formula in Eq. (4.2) we see that the corresponding Lévy triple for the density function with minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ has $\sigma = 0$ indicating a pure jump process with Lévy measure defined by

$$\nu'(x) \;=\; \frac{Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}_d x})}{x} \qquad , \qquad x \geq 0 \tag{4.36}$$

Finally the drift component is

$$\begin{aligned} a &= -\int_0^1 x\, d\nu(x) \\ &= -\int_0^1 \Big(Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}_d x})\Big)dx \\ &= -Tr(\mathbf{A}^{-1}e^{\mathbf{A}}) + Tr(\mathbf{B}_d^{-1}e^{\mathbf{B}_d}) + Tr(\mathbf{A}^{-1}) - Tr(\mathbf{B}_d^{-1}) < \infty \quad (4.37) \end{aligned}$$

which is finite as $\{\sigma(\mathbf{A}) \cup \sigma(\mathbf{B}_d)\} \subset \mathbb{H}_-$. We see from Eq. (4.37) that the associated Lévy process is of finite variation which follows from Eq. (4.4).

The Budan-Fourier method of Hanzon and Holland (2010) can be used to test for non-negativity of the ET function $x\nu'(x) = Tr(e^{\mathbf{A}x}) - Tr(e^{\mathbf{B}_d x})$ on a finite interval. As stated at the end of Section 4.3, results from Proposition 3.1 proves that if $\{\sigma(\mathbf{A}) \cup \sigma(\mathbf{b}_d)\}$ contains a unique dominant real element λ_M, then the Budan-Fourier technique can be used to locate all sign-changing zeros on the half line $[0, \infty)$.

Similar to Section 4.3 it is clear that if F is the Laplace Transform of an infinitely divisible EPT function and $\tilde{\lambda}_M = \max_{\lambda \in \sigma(\mathbf{B}_d)} Re(\lambda)$ then it must hold that $\tilde{\lambda}_M \leq \lambda_M$. This condition is equivalent to the result from Lukacs (1970) regarding the location of the poles and zeros of a rational Laplace Transform of an infinitely divisible distribution.

An identical argument to that given at the end of Section 4.3 in Eq. (4.25) implies the Laplace Transform of an infinitely divisible generalised EPT distribution can not contain more zeros than poles.

4.5. Infinitely Divisible Result for Rational Laplace Transforms

In Sections 4.3 and 4.4 we made use of the fact that EPT and generalised EPT probability density functions, f, could be represented with the minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ where $0 \leq \mathbf{d} < 1$. The case is now considered where the

Laplace Transform of f is the rational function F given by

$$F(s) \;=\; \int_0^\infty e^{-sx} f(x)dx \;=\; \frac{p(s)}{q(s)},\; s > 0,$$

and $p,\ q$ are co-prime polynomials of orders m and n respectively where $m \leq n$. We denote $\Lambda(r(s)) = \{s | r(s) = 0\}$ as the zeros of the polynomial $r(s)$. As noted already, due to the Perron-Frobenius type result, F must contain a dominant real pole implying $\lambda_M \in \Lambda(q(s))$ where $\lambda_M = \max_{\lambda \in \Lambda(q(s))} Re(\lambda)$. Similarly, by the results of Lukacs (1970) a necessary condition for F to be infinitely divisible requires $\max_{\lambda \in \Lambda(p(s))} Re(\lambda) \leq \lambda_M$. Hence, only probability density functions whose rational Laplace Transforms satisfy these conditions are considered.

Following the same method as in Section 4.3, by letting $F = e^{-h}$ it is known that F is infinitely divisible iff there exists a positive measure with Laplace transform $h' = -F'/F$. If $\Lambda(r(s)) \subset \mathbb{H}_-$ then from Lemma 4.1 for $s > 0$

$$\sum_{\lambda \in \Lambda(r(s))} \int_0^\infty e^{\lambda x} e^{-sx} dx \;=\; \frac{r'(s)}{r(s)}, \tag{4.38}$$

and using Lemma 4.2 we obtain the following Theorem.

Theorem 4.3 *Given a probability density function f with rational Laplace Transform $F = p/q$ where p and q are co-prime polynomials of orders m and n respectively such that $m \leq n$. If $\deg(p(s)) > 0$ and $\{\Lambda(p(s)) \cup \Lambda(q(s))\} \subset \mathbb{H}_-$. Then F is the Laplace transform of f given by*

$$F(s) \;=\; \exp\Big(- \int_0^\infty (1 - e^{-sx}) \frac{\sum_{\lambda \in \Lambda(q(s))} e^{\lambda x} - \sum_{\mu \in \Lambda(p(s))} e^{\mu x}}{x} dx \Big). \tag{4.39}$$

F is infinitely divisible iff $\sum_{\lambda \in \Lambda(q(s))} e^{\lambda x} \geq \sum_{\mu \in \Lambda(p(s))} e^{\mu x}$ for all $x \geq 0$

If $\deg(p(s)) = 0$ and $\Lambda(q(s)) \subset \mathbb{H}_-$, then F is the Laplace transform of f given by

$$F(s) \;=\; \exp\Big(- \int_0^\infty (1 - e^{-sx}) \frac{\sum_{\lambda \in \Lambda(q(s))} e^{\lambda x}}{x} dx \Big). \tag{4.40}$$

F is infinitely divisible iff $\sum_{\lambda\in\Lambda(q(s))} e^{\lambda x} \geq 0$ for all $x \geq 0$

The result in Theorem 4.3 coincide with the results obtained in Steutel (1967) for mixtures of exponential functions.

Corollary 4.1 *Given an infinitely divisible probability density function f defined on the half-line $[0,\infty)$ with rational Laplace Transform $F = p/q$ where p and q are co-prime polynomials such that $\{\Lambda(p(s))\cup\Lambda(q(s))\} \subset \mathbb{H}_-$. Then, $\#(\Lambda(p(s))) \leq \#(\Lambda(q(s)))$ where $\#(A)$ denotes the number of elements, including multiplicities, in the set A.*

Proof

If $\deg(p(s)) > 0$ and f is infinitely divisible, then from Theorem 4.3

$$\sum_{\lambda\in\Lambda(q(s))} e^{\lambda x} - \sum_{\mu\in\Lambda(p(s))} e^{\mu x} \;\geq\; 0 \tag{4.41}$$

It follows that

$$\lim_{x\to 0}\Big(\sum_{\lambda\in\Lambda(q(s))} e^{\lambda x} - \sum_{\mu\in\Lambda(p(s))} e^{\mu x}\Big) = \#(\Lambda(q(s))) - \#(\Lambda(p(s))) \geq 0 \tag{4.42}$$

A similar result follows when F is the rational Laplace Transform of an infinitely divisible distribution defined on the half line $(-\infty, 0]$.■

4.6. Infinitely Divisible 2-EPT Distributions

A necessary and sufficient condition to characterise an infinitely divisible 2-EPT probability density function is derived here. Consider the generalised 2-EPT probability density function f, with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P, \mathbf{d})$. As f represents a probability density function, the Perron-Frobenius type result from Hanzon and Holland (2010) implies $\lambda_{M_-} \in \sigma(\mathbf{A}_P)$ and $\lambda_{M_+} \in \sigma(\mathbf{A}_N)$

such that

$$\begin{aligned} \lambda_{M_-} &= \max_{\lambda \in \sigma(\mathbf{A}_P)} Re(\lambda) \\ \lambda_{M_+} &= \min_{\lambda \in \sigma(\mathbf{A}_N)} Re(\lambda). \end{aligned}$$

The Laplace Transform of f, which is analytic for $Re(s) \in (\lambda_{M_-}, \lambda_{M_+})$, is given by

$$F(s) \;=\; -\mathbf{c}_N(s\mathbf{I} - \mathbf{A}_N)^{-1}\mathbf{b}_N + \mathbf{c}_P(s\mathbf{I} - \mathbf{A}_P)^{-1}\mathbf{b}_P + \mathbf{d} \;=\; \frac{p(s)}{q(s)}. \qquad (4.43)$$

where p and q are co-prime polynomials of degree m and n respectively such that $m \leq n$. If $\mathbf{d} > 0$, then $m = n$.

The region $\mathcal{I}$ is defined such that $\mathcal{I} = \{s | Re(s) \in (\lambda_{M_-}, \lambda_{M_+})\}$. Hence, if F is the Laplace Transform of an infinitely divisible distribution, a result from Lukacs (1970) implies

$$\Lambda(p(s)) \cap \mathcal{I} \;=\; \emptyset. \qquad (4.44)$$

Using terminology from Lukacs (1970), $\mathcal{I}$ is referred to as the strip of analyticity (or convergence) for the rational Laplace Transform F.

Therefore if F is infinitely divisible then $F(is) \neq 0$ for all $s \in \mathbb{R}$. The property of no zeros within the strip of analyticity is a necessary condition for infinitely divisible distributions and can be used to confirm that the Laplace Transform F is not infinitely divisible.

If F is infinitely divisible it can be factored as follows

$$\begin{aligned} F(s) &= F_1(s)\, F_2(s) && (4.45) \\ &= \frac{p_1(s)}{q_1(s)}\, \frac{p_2(s)}{q_2(s)}, && (4.46) \end{aligned}$$

such that F_1 has all its zeros and poles located in the open left half plane while F_2 has all its zeros and poles located in the open right half plane. Hence it is clear that $\{\Lambda(p_1(s)) \cup \Lambda(q_1(s))\} \subset \mathbb{H}_-$ and $\{\Lambda(p_2(s)) \cup \Lambda(q_2(s))\} \subset \mathbb{H}_+$. It

follows that $F(0) = 1$ as it is the Laplace Transform of a probability density function. Similarly F_1 and F_2 must be scaled such that $F_1(0) = F_2(0) = 1$.

The polynomials p_1 and p_2 are of degree m_1 and m_2 respectively such that $m_1 + m_2 = m$. Similarly q_1 and q_2 are polynomials of degree n_1 and n_2 respectively where $n_1 + n_2 = n$. If F_1 and F_2 are infinitely divisible then the inequalities $m_1 \leq n_1$ and $m_2 \leq n_2$ must hold based on results from Corollary 4.1 which states that a rational Laplace Transform of an infinitely divisible distribution cannot contain more zeros than poles.

If F is infinitely divisible the necessary condition from Lukacs (1970) implies that $\max_{\lambda \in \Lambda(p_1(s))} Re(\lambda) \leq \lambda_{M_-}$ and $\min_{\lambda \in \Lambda(p_2(s))} Re(\lambda) \geq \lambda_{M_+}$.

Define

$$g_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \\ \sum_{\lambda \in \Lambda(q_1(s))} e^{\lambda x} - \sum_{\mu \in \Lambda(p_1(s))} e^{\mu x} & \text{if } x > 0 \end{cases} \tag{4.47}$$

Using results of Sections 4.3 and 4.5 it is known that the Laplace Transform of g_1 is analytic on the half plane $\{s | Re(s) > \lambda_{M_-}\}$ and therefore so is

$$h_1(s) \;=\; \int_0^\infty \frac{1 - e^{-sx}}{x} g_1(x) dx, \qquad Re(s) > \lambda_{M_-} \tag{4.48}$$

It can be seen that

$$\exp(-h_1(s)) \;=\; \frac{p_1(s)}{q_1(s)} \;=\; F_1(s), \qquad Re(s) > \lambda_{M_-} \tag{4.49}$$

Similarly, define g_2 by

$$g_2(x) = \begin{cases} \sum_{\lambda \in \Lambda(q_2(s))} e^{\lambda x} - \sum_{\mu \in \Lambda(p_2(s))} e^{\mu x} & \text{if } x < 0 \\ \\ 0 & \text{if } x > 0 \end{cases} \tag{4.50}$$

Let

$$h_2(s) \;=\; \int_{-\infty}^{0} \frac{1-e^{-sx}}{x} g_2(x)dx, \qquad Re(s) < \lambda_{M_+} \tag{4.51}$$

A computation shows that

$$\begin{aligned} h_2'(s) &= \sum_{\lambda\in\Lambda(q_2(s))} \frac{1}{\lambda - s} - \sum_{\mu\in\Lambda(p_2(s))} \frac{1}{\mu - s} \\ &= -\frac{q_2'(s)}{q_2(s)} + \frac{p_2'(s)}{p_2(s)}, \;\; Re(s) < \lambda_{M_+} \end{aligned} \tag{4.52}$$

and therefore

$$\exp(h_2(s)) \;=\; \frac{p_2(s)}{q_2(s)} \;=\; F_2(s) \tag{4.53}$$

It follows that

$$F(s) \;=\; F_1(s)F_2(s) \;=\; \exp(-h_1(s) + h_2(s)) \;=\; \exp(-h(s)) \tag{4.54}$$

for all $\lambda_{M_-} < Re(s) < \lambda_{M_+}$ where

$$h(s) \;=\; \int_{-\infty}^{\infty} \frac{1-e^{-sx}}{x} g(x)dx\;, \qquad \lambda_{M_-} < Re(s) < \lambda_{M_+} \tag{4.55}$$

and $g(x) \;=\; g_1(x) - g_2(x)$.

Lemma 4.5 *Using the same notation as above. Suppose the 2-EPT probability density function f is infinitely divisible. Then $xg(x)$ is non-negative for all $x \in (-\infty, \infty)$*

Proof

We use the classical Lévy-Khintchine formula from Feller (1971) which differs slightly from the representation in Eq. (4.2). According to this definition, if F is the characteristic function of an infinitely divisible distribution, then there exists a real constant γ and a bounded non-decreasing function M such that $F(it) = e^{\Psi(t)}$ where

$$\Psi(t) \;=\; i\gamma t \;+\; \int_{-\infty}^{\infty} \Big(e^{itu} - 1 - \frac{itu}{1+u^2}\Big) \frac{(1+u^2)}{u^2} dM(u), \tag{4.56}$$

with the understanding that the integrand is $-t^2/2$ when $u = 0$. Thus, for all real t, $h(it) = -\Psi(t)$, and comparing real parts of the identity we have that

$$\int_{-\infty}^{\infty} \frac{1-\cos(tx)}{x^2} xg(x)dx \;=\; \int_{-\infty}^{\infty} \frac{1-\cos(tu)}{u^2}(1+u^2)dM(u) \tag{4.57}$$

Since all moments of g exist and are finite we may differentiate the LHS as often as we please. Also, by dominated convergence

$$\lim_{t\to 0} \frac{1}{t^2} \int_{-\infty}^{\infty} \frac{1-\cos(tx)}{x^2} xg(x)dx \;=\; \frac{1}{2}\int_{-\infty}^{\infty} xg(x)dx. \tag{4.58}$$

This implies that the limit

$$\lim_{t\to 0} \frac{1}{t^2} \int_{-\infty}^{\infty} \frac{1-\cos(tu)}{u^2}(1+u^2)dM(u) \tag{4.59}$$

exists. But the integrand of the displayed integral is non-negative. Hence, by Fatous lemma

$$\int_{-\infty}^{\infty} (1+u^2)dM(u) \;<\; \infty \tag{4.60}$$

This means that we can differentiate the identity $h(it) = -\Psi(t)$ at least twice. To demonstrate this, note that, for all real δ, t,

$$h(i(t+\delta)) + h(i(t+\delta)) - 2h(it) \;=\; \Psi(i(t+\delta)) - \Psi(i(t+\delta)) + 2\Psi(it) \tag{4.61}$$

For $\delta \neq 0$

$$\int_{-\infty}^{\infty} e^{itx} \frac{1-\cos(\delta x)}{\delta^2 x^2} xg(x)dx \;=\; \int_{-\infty}^{\infty} e^{itu} \frac{1-\cos(\delta u)}{\delta^2 u^2}(1+u^2)dM(u) \tag{4.62}$$

Hence, letting $\delta \to 0$, and invoking the dominated convergence theorem, ($0 \leq 1-\cos(t) \leq t^2/2$) we see that

$$\int_{-\infty}^{\infty} e^{itx} xg(x)dx \;=\; \int_{-\infty}^{\infty} e^{itu}(1+u^2)dM(u),\; -\infty < t < \infty \tag{4.63}$$

By uniqueness of the Fourier Stieltjes transforms, it follows that

$$-xg(-x)dx \;=\; (1+x^2)dM(x) \tag{4.64}$$

Hence, for all real x, $-xg(-x) \geq 0$. Therefore $xg(x) \geq 0$ for all real x which concludes the proof.■

Corollary 4.2 *If $xg(x)$ is non-negative for all $x \in \mathbb{R}$, then both $g_1(x)$ and $g_2(x)$ are also non-negative where g, g_1 and g_2 are as defined above.*

Proof

By definition,

$$xg(x) = \begin{cases} -xg_2(x) & \text{if x} < 0 \\ \\ xg_1(x) & \text{if x} > 0 \end{cases} \tag{4.65}$$

If $xg(x)$ is non-negative for all $x \in \mathbb{R}$ then the result follows.

Theorem 4.4 *Suppose f is an infinitely divisible 2-EPT probability density function. Then f is the convolution of two infinitely divisible generalised EPT functions, denoted f_1 and f_2, defined on $[0,\infty)$ and $(-\infty,0]$ respectively.*

Proof

The rational Laplace Transform F of f can be factored into two rational functions F_1 and F_2 as shown in Eq. (4.45). F_1 is a rational function with $\{\Lambda(p_1(s)) \cup \Lambda(q_1(s))\} \subset \{s|Re(s) \leq \lambda_{M_-}\}$. As F_1 has all its poles and zeros located in the open half plane $\{s|Re(s) \leq \lambda_{M_-}\}$, F_1 can be written as $F_1 = e^{-h_1}$ where h_1 is defined in Eq. (4.48). Since f is infinitely divisible, it follows from Lemma 4.5 and Corollary 4.2 that g_1 as defined in Eqs. (4.47) is non-negative. Therefore, h_1' is completely monotonic and F_1 is infinitely divisible. Hence F_1 is the Laplace Transform of an infinitely divisible generalised EPT probability density function denoted f_1 such that

$$F_1(s) \;=\; \int_0^\infty e^{-sx} f_1(x)dx \tag{4.66}$$

Similarly, F_2 is the Laplace Transform of a generalised EPT probability density function f_2 defined on $(-\infty, 0]$ and for the same reasons as above it follows that F_2 is infinitely divisible. Finally,

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-sx} f(x)dx \;=\; F(s) \quad &= \quad F_1(s)\, F_2(s) \\ &= \quad \int_0^{\infty} e^{-sx} f_1(x)dx \int_{-\infty}^{0} e^{-sx} f_2(x)dx\end{aligned}$$

and so for $-\infty < x < \infty$,

$$\begin{aligned}f(x) \quad &= \quad \int_{-\infty}^{\infty} f_1(x-t) f_2(t)dt \\ &= \quad (f_1 * f_2)(x) \qquad \blacksquare \end{aligned} \tag{4.67}$$

Corollary 4.3 *Suppose Z is an infinitely divisible 2-EPT random variable. Then there exists two generalised EPT random variables X and Y such that $Z = X + Y$. The generalised EPT densities of the random variables X and Y defined on $[0, \infty)$ and $(-\infty, 0]$ respectively are uniquely determined.*

Proof

Theorem 4.4 states that an infinitely divisible 2-EPT probability density function f is the convolution of two generalised EPT density functions f_1 and f_2 defined on $[0, \infty)$ and $(-\infty, 0]$ respectively. Hence, it follows that f_1 and f_2 are the probability density functions for the infinitely divisible random variables X and Y respectively such that $Z = X + Y$, where f is the 2-EPT probability density function of the infinitely divisible random variable Z.■

An infinitely divisible 2-EPT probability density function generates a 2-EPT Lévy process. The Lévy Triple (a, σ, ν) of such a process can be found using the representation of the Laplace Transform, F, of the density function f given in Eq. (4.55) and comparing it to the Lévy-Khintchine formula in Eq. (4.2). It is clear that a 2-EPT Lévy process is a pure jump process as $\sigma = 0$. The

associated Lévy measure ν is defined by

$$\nu'(x) = \begin{cases} \frac{\sum_{\lambda\in\Lambda(q_2(s))} e^{\lambda x} - \sum_{\mu\in\Lambda(p_2(s))} e^{\mu x}}{-x} & \text{if } x < 0 \\ \\ \frac{\sum_{\lambda\in\Lambda(q_1(s))} e^{\lambda x} - \sum_{\mu\in\Lambda(p_1(s))} e^{\mu x}}{x} & \text{if } x > 0 \end{cases} \tag{4.68}$$

The drift component a is chosen as

$$\begin{aligned} a &= -\int_0^1 \Big(\sum_{\lambda\in\Lambda(q_1(s))} e^{\lambda x} - \sum_{\mu\in\Lambda(p_1(s))} e^{\mu x} \Big) dx \\ &- \int_{-1}^0 \Big(\sum_{\lambda\in\Lambda(q_2(s))} e^{\lambda x} - \sum_{\mu\in\Lambda(p_2(s))} e^{\mu x} \Big) dx \\ &< \infty \end{aligned}$$

as $0 \notin \{\Lambda(p(s)) \cup \Lambda(q(s))\}$. Hence we conclude that 2-EPT Lévy processes are of finite variation.

4.7. Variance Gamma Example

Consider a Variance Gamma probability density function f which has input parameters (C, G, M) where "C" is integer and $G, M > 0$. The Laplace transform F of the density function is a strictly proper rational and analytic for $Re(s) \in (-G, M)$

$$\begin{aligned} F(s) &= \Big(\frac{MG}{MG + (M-G)s - s^2} \Big)^C \\ &= \Big(\frac{-M}{s-M} \Big)^C \Big(\frac{G}{s+G} \Big)^C \\ &= F_1(s)\, F_2(s) \\ &= \mathbf{c}_N (s\mathbf{I} - \mathbf{A}_N)^{-1} \mathbf{b}_N\, \mathbf{c}_P (s\mathbf{I} - \mathbf{A}_P)^{-1} \mathbf{b}_P \end{aligned} \tag{4.69}$$

$F(is)$ is the strictly proper rational characteristic function of f. We define $\mathbf{c}_P$ and $\mathbf{c}_N$ as $1 \times C$ row vectors

$$\mathbf{c}_N \;=\; (0, 0, ..., 0, -M^C) \qquad , \qquad \mathbf{c}_P \;=\; (0, 0, ..., 0, G^C) \tag{4.70}$$

Likewise $\mathbf{b}_P = \mathbf{b}_N = (1, 0, 0, ..., 0)^T$ while $\mathbf{A}_N$ and $\mathbf{A}_P$ are square $C \times C$ matrices given by

$$\mathbf{A}_N \;=\; \begin{pmatrix} M & 0 & 0 & ... & 0 \\ -1 & M & 0 & ... & 0 \\ 0 & -1 & M & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & -1 & M \end{pmatrix} \qquad , \qquad \mathbf{A}_P \;=\; \begin{pmatrix} -G & 0 & 0 & ... & 0 \\ 1 & -G & 0 & ... & 0 \\ 0 & 1 & -G & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & 1 & -G \end{pmatrix}$$

It is clear from Eq. (4.69) that both F_1 and F_2 are strictly proper rational functions. Hence, the results of Section 4.3 can be used to determine if both F_1 and F_2 are the infinitely divisible implying the F is also infinitely divisible. It should be noted that

$$\deg\Big(\mathbf{c}_N(s\mathbf{I} - \mathbf{A}_N)^*\mathbf{b}_N\Big) \;=\; \deg\Big(\mathbf{c}_P(s\mathbf{I} - \mathbf{A}_P)^*\mathbf{b}_P\Big) \;=\; 0, \tag{4.71}$$

From Theorem 4.1 we can conclude that F_2 is infinitely divisible if and only if

$$Tr(e^{\mathbf{A}_P x}) \geq 0, \tag{4.72}$$

for all $x > 0$. It is clear that

$$Tr(e^{\mathbf{A}_P x}) \;=\; Ce^{-Gx} \geq 0 \tag{4.73}$$

for all $x > 0$ so we conclude that F_2 is indeed the Laplace Transform of an infinitely divisible distribution. We also conclude that F_1 is infinitely divisible as

$$Tr(e^{\mathbf{A}_N x}) \;=\; Ce^{Mx} \geq 0 \tag{4.74}$$

for all $x < 0$. Hence we conclude that the 2-EPT random variable X is infinitely divisible. The Lévy triple associated with the infinitely divisible distribution f is given by (a, σ, ν) where $\sigma = 0$,

$$\begin{aligned} a &= -\int_0^1 Ce^{-Gx}dx - \int_{-1}^0 Ce^{Mx}dx && (4.75) \\ &= \frac{C}{G}\Big(e^{-G} - 1\Big) + \frac{C}{M}\Big(1 - e^{-M}\Big) && (4.76) \end{aligned}$$

and

$$\nu'(x) = \begin{cases} \frac{Ce^{Mx}}{-x} & \text{if x} < 0 \\ \\ \frac{Ce^{-Gx}}{x} & \text{if x} > 0 \end{cases} \qquad (4.77)$$

A more involved example characterising the 2-EPT probability density function derived in Section 3.12 as infinitely divisible is presented in Section 6.4.3.1.

Chapter 5

Distribution of Discrete Time Extrema of 2-EPT Processes

5.1. Introduction

This chapter focuses on deriving the probability density functions for the extremum (minimum or maximum) of a discrete time 2-EPT process of known length. Such distributions are of great importance from a financial modelling perspective as they are essential for pricing Lookback options with fixed and floating strikes.

We begin by defining a discrete time 2-EPT process and its extrema. Using a recursive algorithm it will be seen that the probability density function of the extrema for this discrete time process is an EPT function mixed with a point-mass at zero, referred to as a generalised EPT density. This density can be computed exactly using the algorithm, which is implemented in MATLAB in Section 7.5.4.

A discrete 2-EPT process with a geometrically distributed stopping time is then defined. The Laplace Transform of the distribution of the stopped process can

be derived and is a proper rational function. It is then proven, using a unique Wiener-Hopf factorization, that the Laplace Transforms of the distributions of the maxima and minima of this process are obtained. These Laplace Transforms are also seen to be rational functions. Using these results, it is then illustrated how the Laplace Transforms of the distributions of the extrema of a 2-EPT process of set length can be derived.

5.2. Minimum of Discrete Time 2-EPT Process

A discrete time 2-EPT process is defined as the sum of independent and identically distributed 2-EPT random variables. The discrete 2-EPT process $L(T)$ is defined for each $t = 0, 1, 2, ..., T$ as

$$L(t) \; = \; \sum_{k=0}^{t} X_k \tag{5.1}$$

where X_k for all $k > 0$ are independent and identically distributed with a generalised 2-EPT probability density function and $X_0 = 0$. The probability density function for $L(t)$ for all $t = \{1, 2, ..., T\}$ is a generalised 2-EPT distribution and can be found using the convolution formula given in Section 2.2.8. The minimum of the discrete time 2-EPT process is given by

$$M^{-}(T) \; = \; \min_{t \in \{0,1,2,...,T\}} L(t) \tag{5.2}$$

The probability density function of $M(T)$ can be derived using the recursive formula for $M(T)$

$$M^{-}(T) \; = \; \min\{0 \; , \; X_1 \; + \; \min\{0, \; X_2 \; + \; \min\{0, \; X_3 \; + \; ... \; \}\}\} \tag{5.3}$$

As $X_0 \; = \; 0$, it follows that $M^{-}(0)$ is a pointmass at zero. The class of 2-EPT random variables is closed under convolution and also under the minimum operator as seen in Chapter 2. The recursive formula takes the minimum of a

2-EPT random variable with zero, ensuring that $M^-(T)$ is defined on closed left half line. Hence the probability density function of $M^-(T)$ is an EPT function mixed with a pointmass at zero as shown in Section 2.2.6.1.

We can also define an equality in distribution ($\stackrel{d}{=}$) for $M^-(T+1)$ in terms of $M^-(T)$

$$M^-(T+1) \stackrel{d}{=} \min\{0,\ X_{T+1} + M^-(T)\} \tag{5.4}$$

Similarly for the maximum of such a process we define

$$M^+(T) = \max_{t\in\{0,1,2,\ldots,T\}} L(t) \tag{5.5}$$

5.2.1. The Π_- and Π_+ Operator

Let ϕ_t^- be the Laplace Transform of $M^-(t)$ and ϕ_X the Laplace transform of the generalised 2-EPT probability density function of the random variable X. We have that

$$\phi_t^- = \Pi_-(\phi_X \phi_{t-1}^-) \tag{5.6}$$

where the operator Π_- maps the Laplace Transform of a random variable to the Laplace Transform of the truncated random variable. For example, let Y be a random variable with Laplace Transform ϕ_Y. Then $\Pi_-(\phi_Y)$ is the Laplace Transform of $\min(Y, 0)$. Let ϕ_Y be a proper rational function $\phi_Y = b/a$ with no poles on the imaginary axis and the unique decomposition given by

$$\phi_Y(s) = \frac{b(s)}{a(s)} = \frac{b_1(s)}{a_1(s)} + \frac{b_2(s)}{a_2(s)} + d \tag{5.7}$$

where b_i/a_i are strictly proper for $i = 1, 2$. The rational function $b_1(s)/a_1(s)$ is stable (i.e. all zeros of a_1 located in $\mathbb{H}_-$) and $b_2(s)/a_2(s)$ is anti-stable (i.e. all zeros of a_2 located in $\mathbb{H}_+$). The operator Π_- acts as follows

$$\Pi_-(\phi_Y)(s) = \frac{b_1(0)}{a_1(0)} + \frac{b_2(s)}{a_2(s)} + d \tag{5.8}$$

while

$$\Pi_+(\phi_Y)(s) \;=\; \frac{b_1(s)}{a_1(s)} + \frac{b_2(0)}{a_2(0)} + d \tag{5.9}$$

where $\Pi_+(\phi_Y)$ is the Laplace Transform of $\max(Y, 0)$. The operators Π_+ and Π_- are also linear such that

$$\Pi_+(c_1\phi_Y + c_2\phi_Z) \;=\; c_1\Pi_+(\phi_Y) + c_2\Pi_+(\phi_Z) \tag{5.10}$$

where $c_1, c_2 \in \mathbb{R}$.

5.3. Discrete Time 2-EPT Processes with Geometrically Distributed Length

Consider a discrete time 2-EPT process whose length is geometrically distributed. The stopping time of this process, N_p, is geometrically distributed with parameter p where $0 < p < 1$. The distribution of N_p is

$$\mathbb{P}[N_p \;=\; k] \;=\; pq^k \qquad , \qquad \forall\, k \;=\; 0, 1, 2, ... \tag{5.11}$$

where $q \;=\; 1-p$. The value of the discrete time process $L(t)$ for all $t = 0, 1, 2, ...$ is given by

$$L(t) \;=\; \sum_{k=0}^{t} X_i, \qquad\qquad \forall\, k \;=\; 1, 2, 3, ..., N_p \tag{5.12}$$

where X_k are independent and identically distributed generalised 2-EPT random variables with rational Laplace Transform ϕ_X. The Laplace Transform of $L(t)$ is Φ_X^t. Also $X_0 = 0$. The Laplace Transform of the distribution of $L(N_p)$ is given by

$$\phi_p(s) \;=\; \sum_{k\geq 0} pq^k(\phi_X)^k \;=\; \frac{p}{1 - q\phi_X} \tag{5.13}$$

so

$$\phi_p = p + q\phi_X\phi_p \tag{5.14}$$

Since ϕ_X is a proper rational function then it follows that ϕ_p is also a proper rational function which has no poles or zeros on the imaginary axis as $|\phi(is)| \leq 1$ $\forall\ s\ \in\ \mathbb{R}$. The geometric distribution is a special of the negative binomial distribution which is infinitely divisible. It follows that the distribution of $L(N_p)$ is also infinitely divisible.

Next consider $M^-(N_p)$ where M^- is defined in Eq. (5.2) and N_p is a geometrically distributed non-negative random variable as shown in Eq. (5.11). The Laplace Transform Ψ^- of $M^-(N_p)$ is given by

$$\Psi^-(s)\ =\ \sum_{k\geq 0}\mathbb{E}\Big(e^{-sM^-(T)}|T=k\Big)\mathbb{P}(T=k)\ =\ \sum_{k\geq 0}pq^k\phi_k^-(s) \tag{5.15}$$

Since Π_- is linear and due to the recursive relationship in Eq. (5.6), it follows that

$$\Psi^-\ =\ p\ +\ q\Pi_-(\phi_X\Psi^-) \tag{5.16}$$

Analogously, for the Laplace Transform (Ψ_+) of $M^+(N_p)$

$$\Psi^+(s)\ =\ \sum_{k\geq 0}pq^k\phi_k^+(s) \tag{5.17}$$

with the recursive equation

$$\Psi^+\ =\ p\ +\ q\Pi_+(\phi_X\Psi^+) \tag{5.18}$$

5.3.1. Recursive Solution

Consider an arbitrary solution Ψ of the following equation

$$\Psi\ =\ p + q\Pi_-(\phi_X\Psi) \tag{5.19}$$

If this equation is recursively solved, we obtain

$$\begin{aligned}
\Psi &= p + q\Pi_-(\phi_X\Psi) \\
&= p + pq\Pi_-(\phi_X) + q^2\Pi_-(\phi_X\Pi_-(\phi_X\Psi)) \\
&= p + pq\Pi_-(\phi_X) + pq^2\Pi_-(\phi_X\Pi_-(\phi_X)) + q^3\Pi_-(\phi_X\Pi_-(\phi_X\Pi_-(\phi_X\Psi))) \\
&\vdots \\
&= \sum_{k=0}^{K} pq^k\psi_k \; + \; q^{K+1}\theta_{K+1} \qquad (5.20)
\end{aligned}$$

where $\psi_0 = 1$, $\psi_k = \Pi_-(\phi_X\psi_{k-1})$, $\theta_1 = \Pi_-(\phi_X\Psi)$ and $\theta_{k+1} = \Pi_-(\phi_X\theta_k)$ for all $k \geq 1$. The recursive equation (5.6) shows that $\psi_k = \phi_k^-$, i.e. ψ_k is the Laplace Transform of $M^-(k)$. If Ψ is the Laplace Transform of the probability distribution of a random variable then θ_k is such a Laplace Transform for all k and thus $|\theta_k(is)| \leq 1$ holds for all $s \in \mathbb{R}$. This implies that $q^{K+1}\theta_K$ converges uniformly to zero on $(i\mathbb{R})$ and thus on $(i\mathbb{R})$ we have

$$\Psi \; = \; \sum_{k=0}^{K} pq^k\psi_k \; + \; q^{K+1}\theta_{K+1} \; = \; \sum_{k\geq 0} pq^k\phi_k^- \; = \; \Psi^- \qquad (5.21)$$

Hence we see that these results imply that Eq. (5.19) has a unique solution if we require that the solution is the Laplace Transform of a probability density function.

To solve Eq. (5.19) we examine the Laplace Transform of $L(N_p)$ (see Eqs. (5.11) - (5.13)) and factorize as

$$\phi_p \; = \; \phi_-\phi_+ \qquad (5.22)$$

where ϕ_- and ϕ_+ are two proper rational functions with no poles or zeros on the imaginary axis. These factors can be scaled such that $\phi_-(0) = \phi_+(0) = \phi_p(0) = 1$. It follows that

$$q\phi_X \; = \; 1 - \frac{p}{\phi_+\phi_-} \qquad (5.23)$$

which is equivalent Spitzers relationship for discrete time process from Spitzer (1956). The above expression can be substituted into Eq. (5.19) and it follows

that

$$\Psi \;=\; p + \Pi_-(q\phi_X\Psi) \;=\; p + \Pi_-(\Psi) - p\Pi_-\Big(\frac{\Psi}{\phi_+\phi_-}\Big) \tag{5.24}$$

Therefore $\Psi = \phi_-$ is a solution to Eq. (5.16) if

$$\Pi_-(\phi_-) = \phi_- \;\text{ and }\; \Pi_-(\phi_+^{-1}) = 1 \tag{5.25}$$

The first condition implies that ϕ_- is anti-stable with all poles located in $\mathbb{H}_+$. The second condition means that ϕ_+^{-1} is stable implying all zeros of ϕ_+ must be located in $\mathbb{H}_-$. However these conditions do not uniquely determine ϕ_- and ϕ_+ but we can solve to analogous equation to Eq. (5.19) given by

$$\Psi \;=\; p + \Pi_+(q\phi_X\Psi) \;=\; p + \Pi_+(\Psi) - p\Pi_+\Big(\frac{\Psi}{\phi_+\phi_-}\Big) \tag{5.26}$$

We see that $\Psi = \phi_+$ is a solution to the above equation if

$$\Pi_+(\phi_+) = \phi_+ \;\text{ and }\; \Pi_+(\phi_-^{-1}) = 1 \tag{5.27}$$

implying that ϕ_+ is stable and ϕ_- has all its zeros located in $\mathbb{H}_+$.

If we consider the unique Wiener-Hopf factorization of ϕ_p where ϕ_+, ϕ_+^{-1} are stable and ϕ_-, ϕ_-^{-1} are anti-stable then ϕ_- solves Eq. (5.16) and ϕ_+ solves Eq. (5.18). Furthermore if ϕ_- and ϕ_+ are Laplace Transforms of probability density functions then $\phi_- = \Psi_-$ and $\phi_+ = \Psi_+$ implying that ϕ_- and ϕ_+ would be the Laplace Transforms of $M^-(N_p)$ and $M^+(N_p)$ respectively.

In Section 4.6 it was proven that an infinitely divisible random whose probability distribution had a rational Laplace Transform could be factorized as shown in Eq. (4.45) (which will be seen to be equivalent to Eq (5.22)). It was also shown in Theorem 4.4 that an infinitely divisible generalised 2-EPT distribution is the convolution of two infinitely divisible generalised EPT distributions defined on either half plane. These results and those of Corollary 4.3 imply that ϕ_- and ϕ_+ can be chosen to satisfy Eq. (5.25) and Eq. (5.27) respectively such that

$\phi_p = \phi_-\phi_+$. These results from Chapter 4 also imply that this factorization means that ϕ_- and ϕ_+ are Laplace Transforms of the infinitely divisible probability density functions. Therefore, it follows that ϕ_+ is the rational Laplace Transform of $M^+(N_p)$ and ϕ_- is the rational Laplace Transform of $M^-(N_p)$ where $M^+(N_p)$ and $M^-(N_p)$ are defined in Eqs. (5.2) and (5.5) while N_p is a geometrically distributed random variable.

5.4. Distributions of Extrema for 2-EPT Processes

It is clear from the previous section that the Laplace Transform (ϕ_-) of the distribution of the minimum of a generalised 2-EPT process which has a geometrically distributed stopping time (e.g. Eq. (5.12)) can be derived. It then follows that ϕ_- can be represented as the infinite sum

$$\frac{\phi_-}{p} = 1 + q\phi_1^- + q^2\phi_2^- + ... + q^k\phi_k^- + ... \tag{5.28}$$

where ϕ_k^- is the Laplace Transform of $M^-(k)$ for $k \geq 0$. The coefficient of q^k is the Laplace Transform of $M^-(k)$ which can be found as the k^{th} derivative of Eq. (5.28) evaluated at $q = 0$. Hence

$$\phi_k^- = \frac{d^k}{dq^k}\Big(\frac{\phi_-}{p}\Big)\Big|_{q=0} \tag{5.29}$$

An analogous result can be used to derive the Laplace Transform ϕ_k^+ of $M^+(k)$ and is given by

$$\phi_k^+ = \frac{d^k}{dq^k}\Big(\frac{\phi_+}{p}\Big)\Big|_{q=0} \tag{5.30}$$

Chapter 6

Option Pricing, Greeks and Risk Management for 2-EPT Price Processes

6.1. Introduction

This chapter focuses on risk neutral option pricing when the underlying asset has log-returns which follow a 2-EPT density function. We begin by introducing the 2-EPT risk neutral price process, in Section 6.2, which is a shifted exponential Levy process. The value of a European Call Option on such an asset is then derived in closed form in Section 6.3.1. The greeks Delta and Gamma are also calculated.

In Section 6.4 the risk neutral density for the log-returns is derived by calibrating the 2-EPT realization such that model prices (calculated using the 2-EPT option pricing formula from Section 6.3.1) match observed market option prices. Section 6.4.3 compares the greeks of European Call Options to the corresponding Black-Scholes values. This section also illustrates that the risk neutral density

is indeed infinitely divisible.

The discretely monitored 2-EPT asset price process is then defined in Section 6.5.2 under the real world measure $\mathbb{P}$. Using the recursion formula from Section 5.2, it is shown how discretely monitored Lookback options with both fixed and floating strikes can be priced analytically in terms of the distribution of the extrema. The Delta and Gamma of such options can also be computed analytically. Option prices and their Greeks are then calculated for daily monitored Lookback Options using the 2-EPT distribution for daily log returns derived in Chapter 3.

In Section 6.6, a 2-Period Bermudan Option is then considered which has a single early exercise opportunity. It is shown that the pricing formulae for such an option can be computed analytically. It is also shown how pricing such an option is equivalent to pricing a compound option.

The Chapter concludes by examining the risk management techniques of Value-at-Risk and Expected Shortfall when the underlying asset has a 2-EPT price process.

6.2. Risk Neutral 2-EPT Process

In Chapter 3 the empirical log-returns of the Dow Jones Industrial Average (DJIA) Index were approximated with a 2-EPT probability density function. This density will be referred to as the 2-EPT density function under the real world measure with a realization $(\mathbf{A}_{N_S}, \mathbf{b}_{N_S}, \mathbf{c}_{N_S}, \mathbf{A}_{P_S}, \mathbf{b}_{P_S}, \mathbf{c}_{P_S})$. Hence the the 2-EPT price process for this asset under the real world measure $\mathbb{P}$ is given by

$$S(T) \;=\; S(t)e^{Y_\tau} \tag{6.1}$$

where $\tau = (T-t)$ and for a fixed period τ the log returns Y_τ have a 2-EPT density such that $Y_\tau \sim\; 2-EPT(\mathbf{A}_{N_S}, \mathbf{b}_{N_S}, \mathbf{c}_{N_S}, \mathbf{A}_{P_S}, \mathbf{b}_{P_S}, \mathbf{c}_{P_S})$.

For the purposes of derivatives pricing it is clear that the price process model

must be free of arbitrage and therefore we must work under the risk neutral measure $\mathbb{Q}$. The absence of arbitrage is equivalent to the existence of an equivalent martingale measure. In Delbaen and Schachermayer (1994) it is shown that if there exists a change of measure from $\mathbb{P}$ to the risk neutral measure $\mathbb{Q}$, such that, under $\mathbb{Q}$ the discounted price process is a martingale, then under $\mathbb{P}$ the price process must have been a semi-martingale. It is well known that all Lévy processes are semi-martingales. It is also known that any twice differentiable function of a semi-martingale is again a semi-martingale. Therefore all exponential Lévy processes are semi-martingales. Proposition 9.9 from Cont and Tankov (2003) proves that there exists an equivalent martingale measure if the price process is driven by an underlying Lévy process which has both positive and negative jumps. In this case the underlying Lévy process must be a 2-EPT Lévy process which we denote X_τ for all $\tau \geq 0$. Using techniques from Chapter 5 we can determine if X_τ and Y_τ are Lévy processes implying the risk neutral price process below is arbitrage free.

$$S(T) \;=\; S(t)e^{(r+\omega)\tau+X_\tau} \tag{6.2}$$

where r is the constant risk free rate and ω is a constant which is computed in Eq. (6.4). Let $X_\tau \sim 2-EPT(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ represent the risk neutral density. This density cannot be estimated from historical data but must be implied directly from the prices of traded market instruments to ensure no-arbitrage. Under the risk neutral measure $\mathbb{Q}$, asset prices discounted using a money market account are martingales. Hence, the mean rate of return on an asset under $\mathbb{Q}$ is the continuously compounded risk free rate r. It follows that the discounted asset price must satisfy

$$e^{-r\tau}\mathbb{E}_{\mathbb{Q}}[S(T)] = \mathbb{E}_{\mathbb{Q}}[S(t)e^{\omega\tau+X_\tau}] \;=\; S(t) \tag{6.3}$$

Therefore $\omega\tau$ can be calculated explicitly as

$$\begin{aligned} \omega\tau &= \log\left(\mathbb{E}[e^{-X_\tau}]\right) \\ &= -\log(\mathbf{c}_N(\mathbf{I}+\mathbf{A}_N)^{-1}\mathbf{b}_N - \mathbf{c}_P(\mathbf{I}+\mathbf{A}_P)^{-1}\mathbf{b}_P) \end{aligned} \tag{6.4}$$

6.3. European Options and Greeks

6.3.1. Vanilla Option Pricing

Let $C(S,\tau,r,K)$ denote the price of European Call option at time t which expires at $T > t$ on the underlying S with a strike of K and $\tau = T - t$. As described above, we assume $X_{s>0}$ is a 2-EPT Lévy process such that for fixed τ, $X_\tau \sim 2-EPT(\mathbf{A}_N,\mathbf{b}_N,\mathbf{c}_N,\mathbf{A}_P,\mathbf{b}_P,\mathbf{c}_P)$. The risk neutral 2-EPT price process for the asset is given by Eq. (6.2). Hence the risk neutral price of such a European Call Option is given by

$$\begin{aligned} C(S,\tau,r,K) &= e^{-r\tau}\,\mathbb{E}_{\mathbb{Q}}[S(T)-K]^+ \\ &= e^{-r\tau}\mathbb{E}_{\mathbb{Q}}[S(T)|S(T) > K]\mathbb{Q}[S(T)>K] - K\,e^{-r\tau}\mathbb{Q}[S(T) > K] \end{aligned} \tag{6.5}$$

Firstly, consider the second term (i.e. the probability of the option expiring in the money). Using the asset price dynamics defined above we have that

$$\begin{aligned} \mathbb{Q}[S(T) > K] &= \mathbb{Q}[S(t)e^{(\omega+r)\tau+X_\tau} > K] \qquad (6.6)\\ &= \mathbb{Q}[X_\tau > \log(K/S(t)) - (r+\omega)(\tau)] \\ &= 1-\mathbb{Q}[X_\tau < -(\log(S(t)/K)+(r+\omega)\tau)] \\ &= 1-\phi(-d) \qquad (6.7) \end{aligned}$$

where $d = \log(S(t)/K) + (r+\omega)\tau$. For $d \leq 0$

$$\begin{aligned} 1-\phi(-d) &= \mathbb{Q}[X > -d] && (6.8)\\ &= \int_{-d}^{\infty} \mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx && (6.9)\\ &= -\mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_P d}\mathbf{b}_P \end{aligned}$$

For $d > 0$

$$\begin{aligned} 1-\phi(-d) &= \mathbb{Q}[X > -d] && (6.10)\\ &= \int_{-d}^{0} \mathbf{c}_N\, e^{\mathbf{A}_N\, x}\, \mathbf{b}_N\, dx + \int_{0}^{\infty} \mathbf{c}_P\, e^{\mathbf{A}_P\, x}\, \mathbf{b}_P\, dx \\ &= 1 - \mathbf{c}_N\mathbf{A}_N^{-1}e^{-\mathbf{A}_N d}\mathbf{b}_N \end{aligned}$$

The term $\mathbb{E}_{\mathbb{Q}}[S(T)|S(T) > K]\mathbb{Q}[S(T) > K]$ must be be evaluated

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[S(T)|S(T) > K]\mathbb{Q}[S(T) > K] &= \int_{K}^{\infty} S(T)\, f(S(T))\, dS(T) && (6.11)\\ &= \int_{-d}^{\infty} S(t)e^{(r+\omega)\tau+x}\, f(x)\, dx \end{aligned}$$

where f represents the 2-EPT probability density function with minimal realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. Again the cases when $d \leq 0$ and $d > 0$ must be treated individually. Consider the case where $d > 0$ and note that

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[S(T)|S(T) > K]\mathbb{Q}[S(T) > K] &= \int_{-d}^{0} S(t)e^{(r+\omega)\tau+x}\, \mathbf{c}_N\, e^{\mathbf{A}_N\, x}\, \mathbf{b}_N\, dx \\ &+ \int_{0}^{\infty} S(t)e^{(r+\omega)\tau+x}\, \mathbf{c}_P\, e^{\mathbf{A}_P\, x}\, \mathbf{b}_P\, dx \\ &= S(t)e^{(r+\omega)\tau}\Big(\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}\mathbf{b}_N && (6.12)\\ &- \mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^{-(\mathbf{A}_N+\mathbf{I})d}\mathbf{b}_N - \mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}\mathbf{b}_P\Big) \end{aligned}$$

For $d \leq 0$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[S(T)|S(T) > K]\mathbb{Q}[S(T) > K] &= \int_{-d}^{\infty} S(t)e^{(r+\omega)\tau+x}\, \mathbf{c}_P\, e^{\mathbf{A}_P\, x}\, \mathbf{b}_P\, dx \\ &= -S(t)e^{(r+\omega)\tau}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P && (6.13) \end{aligned}$$

The price for the European call option as outlined above for $d > 0$ is given by

$$\begin{aligned} C(S,\tau,r,K) &= S(t)e^{\omega\tau}\Big(\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}\mathbf{b}_P - \mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^{-(\mathbf{A}_N+\mathbf{I})d}\mathbf{b}_N - \mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}\mathbf{b}_P\Big) \\ &- K\,e^{-r\tau}\,(1-\mathbf{c}_N\mathbf{A}_N^{-1}e^{-\mathbf{A}_N d}\mathbf{b}_N) \end{aligned} \tag{6.14}$$

$$\tag{6.15}$$

A similar expression for the price can be obtained when $d \leq 0$.

$$C(S,\tau,r,K) = -S(t)e^{\omega\tau}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P + K\,e^{-r\tau}\,\mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_P d}\mathbf{b}_P \tag{6.16}$$

6.3.1.1. Option Prices in EPT Form

An interesting point to note is that if we write $S(t)$ in terms of d we get

$$\begin{aligned} S(t) &= e^{d+\log(K)-(r+\omega)\tau} \\ &= Ke^d e^{-(r+\omega)\tau} \end{aligned} \tag{6.17}$$

We denote n_N as the McMillan degree of the EPT function with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$. Similarly n_P denotes the McMillan degree of the EPT function given by the realization $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. Keeping K, r, ω and τ fixed we can write the price of a call option, for $d > 0$ from Eq. (6.14), as an EPT function in terms of d

$$\begin{aligned} C(d,\tau,r,K) &= Ke^{-r\tau}\Big(\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^d\mathbf{b}_N - \mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^{-\mathbf{A}_N d}\mathbf{b}_N - \mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^d\mathbf{b}_P\Big) \\ &- K\,e^{-r\tau}\,(1-\mathbf{c}_N\mathbf{A}_N^{-1}e^{-\mathbf{A}_N d}\mathbf{b}_N) \end{aligned} \tag{6.18}$$

which is EPT, as it is the sum of EPT functions, and has a maximum McMillan degree of : $n_N + 2$
For $d \leq 0$ we get

$$\begin{aligned} C(d,\tau,r,K) &= -Ke^{-r\tau}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-\mathbf{A}_P d}\mathbf{b}_P + K\,e^{-r\tau}\,\mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_P d}\mathbf{b}_P \\ &= Ke^{-r\tau}\mathbf{c}_P\Big(-(\mathbf{A}_P+\mathbf{I})^{-1} + \mathbf{A}_P^{-1}\Big)e^{-\mathbf{A}_P d}\mathbf{b}_P \end{aligned} \tag{6.19}$$

which is clearly an EPT function in terms of d with a McMillan degree n_P

6.3.2. Greeks for European Options

It should be clear to the reader that differentiation of the option value functions of the form in Eq. (6.14) and Eq. (6.16) with respect to parameters not involved in the realization is straightforward. Hence calculating certain Greeks does not pose problems. Here we give an example of delta, $\frac{\partial C}{\partial S}$, for a European Call Option with $d \leq 0$

$$\begin{aligned} \frac{\partial C(S,\tau,r,K)}{\partial S} &= -e^{\omega\tau}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P + e^{\omega\tau}\mathbf{c}_P e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P \\ &- \frac{K}{S}\, e^{-r\tau}\, \mathbf{c}_P e^{-\mathbf{A}_P d}\mathbf{b}_P \end{aligned} \tag{6.20}$$

We can also derive delta for $d > 0$ using Eq. (6.14). Delta could also have been calculated using the EPT expressions in Eq. (6.18) and (6.19).

The second derivative of the option price with respect to the initial asset price, known as gamma, can also be derived. For $d \leq 0$ we use the option price derived in Eq. (6.16) and observe that

$$\begin{aligned} \frac{\partial^2 C(S,\tau,r,K)}{\partial S^2} &= \frac{e^{\omega\tau}}{S}\mathbf{c}_P e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P - \frac{e^{\omega\tau}}{S}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P \\ &+ \frac{K}{S^2}\, e^{-r\tau}\, \mathbf{c}_P e^{-\mathbf{A}_P d}\mathbf{b}_P + \frac{K}{S^2}\, e^{-r\tau}\, \mathbf{c}_P \mathbf{A}_P^{-1} e^{-\mathbf{A}_P d}\mathbf{b}_P \end{aligned} \tag{6.21}$$

The Greeks rho, $\frac{\partial C}{\partial r}$, and theta, $\frac{\partial C}{\partial \tau}$ could also be easily calculated.

6.4. 2-EPT Risk Neutral Calibration

In Section 6.2 the risk neutral 2-EPT price process was defined. It was stated earlier that the risk neutral price process could be derived by calibrating the 2-EPT density of log-returns such that the 2-EPT model option prices equal the observed market option prices. The prices of options on the DJIA were obtained from Bloomberg and are given in Appendix A.1. We consider a set of European Call Options with M days to maturity where $\tau = M/252$ as there are 252

trading days in a year. $P(S, K, M)$ represents the market price of a European Call Option on an asset with value S, strike K and M days to maturity.

The underlying asset is assumed to have a 2-EPT price process as described in Eq. (6.2). The risk neutral daily log-returns of the underlying have a 2-EPT density function f with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. The 2-EPT density for the risk-neutral log-returns of the asset over M days can found as the M fold convolution of f with itself and has a realization $(\mathbf{A}_{N_M}, \mathbf{b}_{N_M}, \mathbf{c}_{N_M}, \mathbf{A}_{P_M}, \mathbf{b}_{P_M}, \mathbf{c}_{P_M})$. The criterion function to be minimised is

$$\sum_j \left(C(S, K_j, \tau, (\mathbf{A}_{N_M}, \mathbf{b}_{N_M}, \mathbf{c}_{N_M}, \mathbf{A}_{P_M}, \mathbf{b}_{P_M}, \mathbf{c}_{P_M})) - P(S, K_i, \tau)\right)^2 \quad (6.22)$$

where j runs over the set of options with M days to maturity. This criterion function is minimised by determining the risk neutral (daily) 2-EPT density f whose M-fold convolution has the realization $(\mathbf{A}_{N_M}, \mathbf{b}_{N_M}, \mathbf{c}_{N_M}, \mathbf{A}_{P_M}, \mathbf{b}_{P_M}, \mathbf{c}_{P_M})$ which minimises Eq. (6.22).

Unfortunately the RARL2 technique implemented in Chapter 3 cannot be utilised here to minimise the criterion function in Eq. (6.22) as it cannot be transformed into a rational approximation problem. However, the 2-EPT distribution of empirical daily returns has already been estimated with the 2-EPT density function under the real world measure which has the realization $(\mathbf{A}_{N_S}, \mathbf{b}_{N_S}, \mathbf{c}_{N_S}, \mathbf{A}_{P_S}, \mathbf{b}_{P_S}, \mathbf{c}_{P_S})$. Using a multi-dimensional optimisation algorithm over the vectors $(\mathbf{b}_N, \mathbf{b}_P, \mathbf{c}_N, \mathbf{c}_P)$ we can obtain the realization $(\mathbf{A}_{N_S}, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_{P_S}, \mathbf{b}_P, \mathbf{c}_P)$ for the 2-EPT risk neutral density function for daily log-returns. Hence we assume that $\mathbf{A}_N = \mathbf{A}_{N_S}$ and $\mathbf{A}_P = \mathbf{A}_{P_S}$. By keeping $\mathbf{A}_N$ and $\mathbf{A}_P$ fixed, stability is assured and the unique dominant poles in $\mathbf{A}_{N_S}$ and $\mathbf{A}_{P_S}$ are also preserved which is crucial for testing for non-negativity of the risk neutral density.

6.4.1. Calibration Algorithm

The 2-EPT density for the daily log-returns of the DJIA under the real world measure was approximated in Chapter 3 and given explicitly in Eqs. (3.60)

and (3.60). This density is used as the initial guess of the 2-EPT risk neutral density. The algorithm to find the risk neutral then takes the following steps. Assume the current estimate of the risk neutral 2-EPT density has a realization $(\mathbf{A}_N, \mathbf{b}_{N,i}, \mathbf{c}_{N,i}, \mathbf{A}_P, \mathbf{b}_{P,i}, \mathbf{c}_{P,i})$. The criterion function at step i is then given by

$$\mathcal{C}_i = \sum_j \left(C(S, K_j, \tau, (\mathbf{A}_{N_{M,i}}, \mathbf{b}_{N_{M,i}}, \mathbf{c}_{N_{M,i}}, \mathbf{A}_{P_{M,i}}, \mathbf{b}_{P_{M,i}}, \mathbf{c}_{P_{M,i}})) - P(S, K_i, \tau)\right)^2 \quad (6.23)$$

where the 2-EPT density with realization $(\mathbf{A}_{N_{M,i}}, \mathbf{b}_{N_{M,i}}, \mathbf{c}_{N_{M,i}}, \mathbf{A}_{P_{M,i}}, \mathbf{b}_{P_{M,i}}, \mathbf{c}_{P_{M,i}})$ is the M-fold convolution of the density with realization $(\mathbf{A}_N, \mathbf{b}_{N,i}, \mathbf{c}_{N,i}, \mathbf{A}_P, \mathbf{b}_{P,i}, \mathbf{c}_{P,i})$ with itself.

1. Compute $(\mathbf{b}_{N,i+1}, \mathbf{c}_{N,i+1}, \mathbf{b}_{P,i+1}, \mathbf{c}_{P,i+1})$ by perturbing the vectors $\mathbf{b}_{N,i}, \mathbf{c}_{N,i}, \mathbf{b}_{P,i}, \mathbf{c}_{P,i}$ in a direction in an effort to decrease the criterion function in Eq. (6.23)

2. Ensure continuity at the origin by scaling such that $\mathbf{c}_{N,i+1}\mathbf{b}_{N,i+1} = \mathbf{c}_{P,i+1}\mathbf{b}_{P,i+1}$

3. Scale 2-EPT function $(\mathbf{A}_N, \mathbf{b}_{N,i+1}, \mathbf{c}_{N,i+1}, \mathbf{A}_P, \mathbf{b}_{P,i+1}, \mathbf{c}_{P,i+1})$ such that integration to unity holds

4. Define $\mathcal{B}_1(\tilde{\mathbf{A}_N}, \hat{\mathbf{c}_{P,i+1}})$ and $\mathcal{B}_1(\tilde{\mathbf{A}_P}, \hat{\mathbf{c}_{N,i+1}})$ as shown in Section 3.7. If $\mathbf{b}_{N,i+1} \notin \mathcal{B}_1(\tilde{\mathbf{A}_N}, \hat{\mathbf{c}}_{N,i+1})$ or $\mathbf{b}_{P,i+1} \notin \mathcal{B}_1(\tilde{\mathbf{A}_P}, \hat{\mathbf{c}}_{P,i+1})$ revert to step 1.

5. Calculate $T_N < 0$ and $T_P > 0$ as defined in Section 3.7 such that checking for non-negativity of the 2-EPT function $(\mathbf{A}_N, \mathbf{b}_{N,i+1}, \mathbf{c}_{N,i+1}, \mathbf{A}_P, \mathbf{b}_{P,i+1}, \mathbf{c}_{P,i+1})$ on $[T_N, T_P]$ is sufficient for testing non-negativity on $(-\infty, \infty)$. If density is negative for some point on the interval $[T_N, T_P]$ then revert to step 1.

6. Compute the M-fold convolution of the 2-EPT probability density function with realization $(\mathbf{A}_N, \mathbf{b}_{N,i+1}, \mathbf{c}_{N,i+1}, \mathbf{A}_P, \mathbf{b}_{P,i+1}, \mathbf{c}_{P,i+1})$ with itself and evaluate the criterion function in Eq. (6.23). If the criterion function has decreased then accept $(\mathbf{b}_{N,i+1}, \mathbf{c}_{N,i+1}, \mathbf{b}_{P,i+1}, \mathbf{c}_{P,i+1})$ and begin at step 1 by computing $(\mathbf{b}_{N,i+2}, \mathbf{c}_{N,i+2}, \mathbf{b}_{P,i+2}, \mathbf{c}_{P,i+2})$. Otherwise, if the criterion function has increased then revert to step 1 and re-calculate $(\mathbf{b}_{N,i+1}, \mathbf{c}_{N,i+1}, \mathbf{b}_{P,i+1}, \mathbf{c}_{P,i+1})$

6.4.2. Calibration Performance

The above algorithm was implemented in MATLAB using the `fmincon` multi-dimensional search routine. This routine applies the Nelder-Mead search algorithm. Letting $M = 30$ (i.e. 30 days to maturity) and using the option price data from Appendix A.1 we can assess the performance of the calibration. As mentioned in Appendix A.1, the initial value of the DJIA was 13860.65 while the annualized interest rate was r=0.2%. Using the density under the real world measure, the criterion function gives an error of 75587.74. However once the above the calibration algorithm is implemented the error is reduced by over 92% to 5617.57. Fig 6.1 illustrates positive the effect of the calibration.

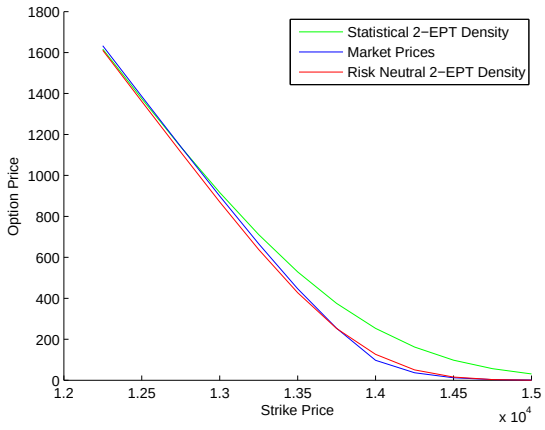

Figure 6.1: 2-EPT Call Option Prices with Densities under Real World and Risk Neutral Measures compared to observed market prices.

The improvement in the quality of the fit is obvious especially among the options with higher strikes.

6.4.3. Risk Neutral 2-EPT Density

The risk neutral daily 2-EPT density calibrated using the algorithm in Section 6.4.1 for a set of European Call Options with 30 days to maturity is given by

$$\left(\begin{array}{c|c} \mathbf{A}_N & \mathbf{b}_N \\ \hline \mathbf{c}_N & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 99.5597 & 0.0000 & 0.0000 & 0.5038 \\ 0.0000 & 61.4545 & -97.6067 & 3.6093 \\ 0.0000 & 370.1559 & 302.1875 & 8.9295 \\ \hline 11.1558 & 4.1030 & 1.8061 & 0 \end{array}\right) \tag{6.24}$$

and

$$\left(\begin{array}{c|c} \mathbf{A}_P & \mathbf{b}_P \\ \hline \mathbf{c}_P & 0 \end{array}\right) = \left(\begin{array}{ccc|c} -119.5601 & 0.0000 & 0.0000 & 0.0029 \\ 0.0000 & -33.0987 & 160.9465 & 1.5717 \\ 0.0000 & -426.6732 & -533.3337 & 14.3277 \\ \hline 40.1293 & 13.2936 & 0.99900 & \end{array}\right) \tag{6.25}$$

Section 6.4.3.2 utilises the methodology of Chapter 5 to show that the 2-EPT probability density with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ is infinitely divisible and therefore the risk neutral measure $\mathbb{Q}$ can be defined. According to Section 6.2, it is necessary to first prove that price process under real world measure $\mathbb{P}$ is a semi-martingale. This is proven in Section 6.4.3.1 where we show the price process under $\mathbb{P}$ is driven by a Lévy process.

Let X_1 be the random variable representing the daily log-return of the asset under the risk neutral measure $\mathcal{Q}$. Then X_{30} represents the 30-day log return of the underlying asset (the DJIA) and is of McMillan degree 90. Using the convolution formula from Section 2.2.8, the realization of X_{30} can be derived and by construction is infinitely divisible since the sum of infinitely divisible random variables is infinitely divisible. Model reduction could be employed to reduce the dimensions of the 2-EPT function but MATLAB is able to perform calculations with matrices of this size in negligible times.

6.4.3.1. Infinite Divisibility of 2-EPT Density under $\mathbb{P}$

The Laplace transform of the 2-EPT probability density function with realization from Eqs. (3.59) and (3.60) is given by

$$\phi(s) \;=\; -\mathbf{c}_{N_S}(s\mathbf{I} - \mathbf{A}_{N_S})^{-1}\mathbf{b}_{N_S} + \mathbf{c}_{P_S}(s\mathbf{I} - \mathbf{A}_{P_S})^{-1}\mathbf{b}_{P_S} \tag{6.26}$$

ϕ is a strictly proper rational function of McMillan degree 3 which can be expressed as the product of two rational functions ϕ_- and ϕ_+

$$\begin{aligned}\phi(s) &= \phi_-(s)\phi_+(s) \qquad (6.27)\\ &= \left(\tilde{\mathbf{c}}_{N_S}(s\mathbf{I}-\tilde{\mathbf{A}}_{N_S})^{-1}\tilde{\mathbf{b}}_{N_S}+\tilde{\mathbf{d}}_{N_S}\right)\left(\tilde{\mathbf{c}}_{P_S}(s\mathbf{I}-\tilde{\mathbf{A}}_{P_S})^{-1}\tilde{\mathbf{b}}_{P_S}+\tilde{D}_{P_S}\right)\end{aligned}$$

such that the poles and zeros of ϕ_- are located in the open right half plane. Likewise the poles and zeros of ϕ_+ are located in the open left half plane.

By factorizing Eq. (6.27) and normalising such that $\phi_-(0) = \phi_+(0) = 1$ we find the state space realizations

$$\left(\begin{array}{c|c}\tilde{\mathbf{A}}_N & \tilde{\mathbf{b}}_N\\ \hline \tilde{\mathbf{c}}_N & \tilde{\mathbf{d}}_N\end{array}\right) = \left(\begin{array}{ccc|c} 0.0000 & 1.0000 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 1.0000 & 0.0000\\ 5445978.796 & -90904.665 & 463.202 & 1.0000\\ \hline -5414489.319 & 46062.589 & -81.752 & 0.00578\end{array}\right) \qquad (6.28)$$

and

$$\left(\begin{array}{c|c}\tilde{\mathbf{A}}_P & \tilde{\mathbf{b}}_P\\ \hline \tilde{\mathbf{c}}_P & 0\end{array}\right) = \left(\begin{array}{ccc|c} 0.0000 & 1.0000 & 0.0000 & 0.0000\\ 0.0000 & 0.0000 & 1.0000 & 0.0000\\ -10320924.501 & -154046.885 & -685.994 & 1.0000\\ \hline 10320924.501 & 68814.011 & 0 & 0.0000\end{array}\right) \qquad (6.29)$$

From the realizations given in Eqs. (6.28) and (6.29), the Lévy measures $\nu'_P(x)$ and $\nu'_N(x)$, defined on $(0,\infty)$ and $(-\infty,0)$, can be calculated explicitly using the formulae in Chapter 5 and we see that

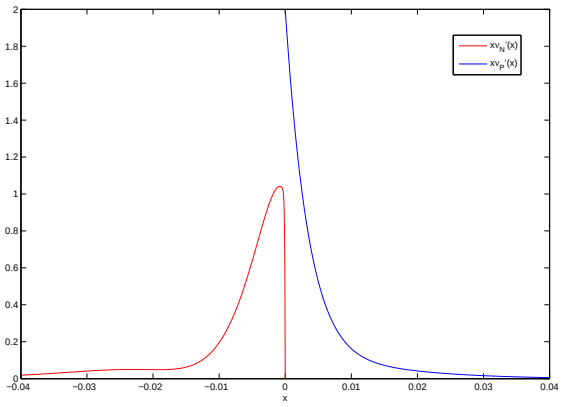

Figure 6.2: Plot of $x\nu'_N(x)$ and $x\nu'_P(x)$

where

$$\nu'(x) = \begin{cases} \nu'_N(x) & \text{if } x < 0 \\ \\ \nu'_P(x) & \text{if } x > 0 \end{cases} \tag{6.30}$$

Using the Budan-Fourier algorithm it is possible to conclude that the both $x\nu'_N(x)$ and $x\nu'_P(x)$ are non-negative on $(-\infty, 0)$ and $(0, \infty)$ respectively implying that the 2-EPT probability density function with realization $(\mathbf{A}_{N_S}, \mathbf{b}_{N_S}, \mathbf{c}_{N_S}, \mathbf{A}_{P_S}, \mathbf{b}_{P_S}, \mathbf{c}_{P_S})$ is infinitely divisible.

6.4.3.2. Infinite Divisibility of 2-EPT Density under $\mathbb{Q}$

The Laplace transform of the 2-EPT probability density function with realization from Eqs. (6.24) and (6.25) is given by

$$\begin{aligned} \phi(s) &= -\mathbf{c}_N(s\mathbf{I} - \mathbf{A}_N)^{-1}\mathbf{b}_N + \mathbf{c}_P(s\mathbf{I} - \mathbf{A}_P)^{-1}\mathbf{b}_P & (6.31) \\ &= \phi_-(s)\phi_+(s) \\ &= \left(\tilde{\mathbf{c}}_N(s\mathbf{I} - \tilde{\mathbf{A}}_N)^{-1}\tilde{\mathbf{b}}_N + \tilde{\mathbf{d}}_N\right)\left(\tilde{\mathbf{c}}_P(s\mathbf{I} - \tilde{\mathbf{A}}_P)^{-1}\tilde{\mathbf{b}}_P + \tilde{\mathbf{d}}_P\right) & (6.32) \end{aligned}$$

such that the poles and zeros of ϕ_- are located in the open right half plane. Likewise the poles and zeros of ϕ_+ are located in the open left half plane.

The location of the poles and zeros of ϕ are given in Section 3.12. It should also be clear that $\{\sigma(\mathbf{A}_N) \cup \sigma(\mathbf{A}_P)\} = \{\sigma(\tilde{\mathbf{A}_N}) \cup \sigma(\tilde{\mathbf{A}_P})\}$.

By factorizing Eq. (6.31) and normalising such that $\phi_-(0) = \phi_+(0) = 1$ we find the state space realizations

$$\left(\begin{array}{c|c} \tilde{\mathbf{A}}_N & \tilde{\mathbf{b}}_N \\ \hline \tilde{\mathbf{c}}_N & \tilde{\mathbf{d}}_N \end{array}\right) = \left(\begin{array}{ccc|c} 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ 5445978.796 & -90904.665 & 463.202 & 1.0000 \\ \hline -5284181.29 & 47996.19 & -72.37 & 0.0297 \end{array}\right) \tag{6.33}$$

and

$$\left(\begin{array}{c|c} \tilde{\mathbf{A}}_P & \tilde{\mathbf{b}}_P \\ \hline \tilde{\mathbf{c}}_P & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -10320924.501 & -154046.885 & -685.994 & 1.0000 \\ \hline 10320924.500 & 69153.74 & 0 & 0.0000 \end{array}\right) \tag{6.34}$$

The poles and zeros of ϕ_- and ϕ_+ can be computed exactly and we note that

	$\phi_-(u)$	$\phi_+(u)$
Poles	99.5597, 181.8210±147.1108i	-119.5601, -283.2162±78.1844i
Zeros	138.42, 214.29, 617.98	-149.24 $-1.12.10^{18}$

Table 6.1: Locations of Poles and Zeros for $\phi_-(u)$ and $\phi_+(u)$

Examining the locations of the poles and zeros of the rational functions ϕ_- and ϕ_+ in Table 6.1 we conclude that both rational functions are indeed of minimum phase. The strip of convergence $\mathcal{I} = \{u+iv | u \in (-119.5601, 99.5597), v \in \mathbb{R}\}$ contains no zeros which is a necessary condition for infinitely divisible rational functions.

From the realizations given in Eqs. (6.33) and (6.34), the Lévy measures $\nu'_P(x)$ and $\nu'_N(x)$, defined on $(0, \infty)$ and $(-\infty, 0)$, can be calculated explicitly using

the formulae in Chapter 5 and we see that

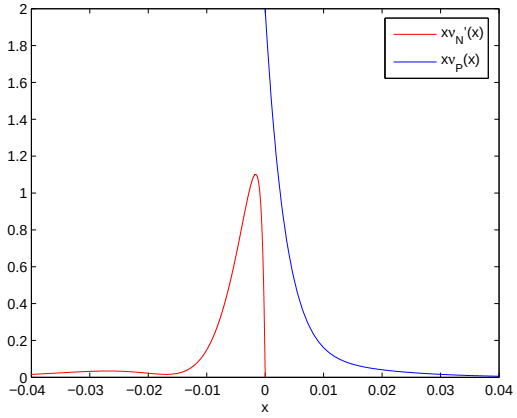

Figure 6.3: Plot of $x\nu'_N(x)$ and $x\nu'_P(x)$

where

$$\nu'(x) = \begin{cases} \nu'_N(x) & \text{if } x < 0 \\ \\ \nu'_P(x) & \text{if } x > 0 \end{cases} \tag{6.35}$$

Using the Budan-Fourier algorithm it is possible to conclude that the both $x\nu'_N(x)$ and $x\nu'_P(x)$ are non-negative on $(-\infty, 0)$ and $(0, \infty)$ respectively implying that the 2-EPT probability density function with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ is infinitely divisible.

6.4.4. Option Greeks Computations

Using the minimal realization for the 30-day log return (X_{30}) of an asset S derived in Section 6.4.3 we will examine the values of the European Call Option Greeks. Letting the annual risk free rate be constant at 0.2% and the strike price of the option $K = 13860.65$. A range of initial asset prices $S(0) \in [11750, 16000]$ is considered and Figure 6.4 illustrates the values of a European Call Option Delta and Gamma obtained using the option pricing formulae of Eqs. (6.20) and (6.21). Similar formulae to Eqs. (6.20) and (6.21) can be derived for $d > 0$.

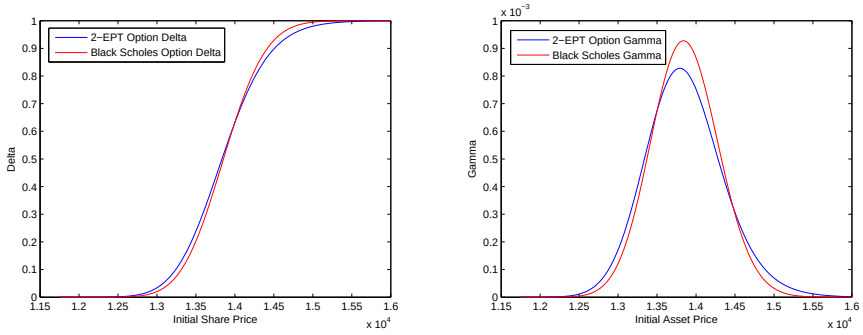

Figure 6.4: 2-EPT and Black Scholes Call Option Delta and Gamma

The Black-Scholes values in Fig 6.4 are computed using an annual volatility of $\sigma = 9.13\%$. This value of σ can be derived as risk neutral volatility parameter which minimises the squared distance between the observed market option prices in Appendix A.1 and the corresponding Black-Scholes option prices with a constant volatility. From Fig 6.4 it is clear the 2-EPT Pricing Model has a higher value of Delta for out-of-the-money options and a lower value of Delta for in-the-money-options compared to the Black-Scholes model. This feature stems from the heavier tails of the 2-EPT distribution where there is a higher probability of extreme asset returns.

6.5. Lookback Options

Lookback options are a type of exotic option which are weakly path dependent, that is, the payoff depends on the maximum or minimum underlying asset's price occurring over the life of the option. There are two types of Lookback Options, those with floating strikes and those with fixed strikes. A Lookback Option with floating strike means the strike is determined at maturity by the terminal asset price. A Lookback Option with fixed strike sets the exercise price when the option is initiated. A brief description of the motivation for adopting a 2-EPT approach to pricing Lookback Options was provided in Section 1.4.5.

6.5.1. Minimum of Discrete Time 2-EPT Process

A discrete time 2-EPT process is defined as the sum of independent and identically distributed 2-EPT random variables. The 2-EPT process of fixed length T is given by

$$L(T) \;=\; \sum_{i=0}^{T} Y_i \tag{6.36}$$

where Y_i for all $i > 0$ are independent and identically distributed 2-EPT random variables and $Y_0 = 0$. The probability density function for $L(t)$ for all $t = \{1, 2, ..., T\}$ is 2-EPT and can be found using the convolution formula given in Section 2.2.8. In Section 5.2 it is shown how the generalised EPT density can be computed for a discrete time 2-EPT process as in Eq. (6.36).

6.5.2. Discrete Time 2-EPT Price Process

We begin by defining the 2-EPT price under the real world measure $\mathbb{P}$ as in Section 6.2 and using the same arguments the 2-EPT price process under the risk neutral measure $\mathbb{Q}$ can similarly be defined as in Eq. (6.2). The discretely monitored 2-EPT price process under the measure $\mathbb{P}$ can be represented by

$$S(T) \;=\; S(t)e^{\sum_{i=0}^{N} Y_i} \tag{6.37}$$

where Y_i are the independent and identically distributed 2-EPT random variables with minimal realization $(\mathbf{A}_{N_S}, \mathbf{b}_{N_S}, \mathbf{c}_{N_S}, \mathbf{A}_{P_S}, \mathbf{b}_{P_S}, \mathbf{c}_{P_S})$ for all $i = \{1, 2, ..., N\}$, $\tau = T - t$, $Y_0 = 0$ and $S(t)$ is the current asset price. Y_i for $i = \{1, 2, ..., N\}$ represents the log-return for the asset S over the period δt, where $\delta t = \tau/N$. It is clear that the extrema of the price process under $\mathbb{P}$ can be computed directly from the extrema of the 2-EPT process $\sum_i Y_i$. Using identical arguments to those in 6.2, the discretely monitored risk neutral 2-EPT price process is given by

$$S(T) \;=\; S(t)e^{\sum_{i=0}^{N} (X_i + (r+\omega)\delta t)} \;=\; S(0)e^{\sum_{i=0}^{t} Z_i} \tag{6.38}$$

The extrema of the price process under $\mathbb{Q}$ can be computed using the extrema of the process $\sum_i Z_i$. However the probability density function of $X + (r+\omega)\delta t$ is not 2-EPT, as was seen in Section (2.2.1) where it is proven that a translated 2-EPT random variable is not 2-EPT.

Therefore we will only value Lookback options under real world measure $\mathbb{P}$. The discrete time process of interest is given by Eq. (6.36). The generalised EPT probability density function of the minimum of $(0, L(1), L(2), ..., L(T))$ is denoted $M(T)$ which has minimal realization $(\mathbf{A}_M, \mathbf{b}_M, \mathbf{c}_M, \mathbf{d}_M)$ and can be determined using the recursive algorithm in Section 6.5.1. Under the real world measure $\mathbb{P}$ the minimum of the discretely monitored price process is given by

$$S_{\min}(T) \;=\; \min_{j \,\in\, \{0,1,2,\ldots,N\}} S(t) e^{\sum_{i=0}^{j} Y_i} \;=\; S(t) e^{M(T)} \tag{6.39}$$

Closed form formula for discretely monitored Lookback options and their Greeks with fixed and floating strikes can then be derived in terms of the generalised EPT probability density function with realization $(\mathbf{A}_M, \mathbf{b}_M, \mathbf{c}_M, \mathbf{d}_M)$.

6.5.2.1. Discrete Time Risk Neutral Price Process

One possible solution to the problem of working in the real world measure $\mathbb{P}$ as opposed to the risk neutral measure $\mathbb{Q}$ would be to approximate the distribution $Z = X + (r+\omega)\delta t$ with a 2-EPT probability density function g with minimal realization $(\mathbf{A}_{N_L}, \mathbf{b}_{N_L}, \mathbf{c}_{N_L}, \mathbf{A}_{P_L}, \mathbf{b}_{P_L}, \mathbf{c}_{P_L})$. This approximation could be carried out using the rational approximation software RARL2 under the constraint that g is non-negative as described in Chapter 3. The criterion to minimised is

$$||g(x) \;-\; f(x + (r+\omega)\delta t)||_2^2 \tag{6.40}$$

such that g is non-negative and f is the 2-EPT probability density function with minimal realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. The error on the approximation

should be small enough to ensure that

$$\mathbb{E}_{\mathbb{Q}}[e^{-r\delta t}S(t)e^{Z}] \;=\; S(t) \tag{6.41}$$

which implies that the discounted asset price under the measure $\mathbb{Q}$ is a martingale. It is also necessary that the 2-EPT distribution of Z is infinitely divisible which can be determined using the methods of Chapter 5.

6.5.3. State Space Realization of Risk Neutral Process

For the Lookback option pricing example it is assumed the options will be monitored daily using the official closing price. The underlying asset will be the DJIA as given in Section 6.4.3 where the daily log-returns (based on closing prices) Y_1 have a 2-EPT probability density function with minimal realization given by Eqs. (3.59) and (3.60). The 2-EPT realization for the annual risk neutral log return, denoted Y_{252}, can be computed using the convolution formula. The distribution of minimum $M(252)$ denoted $(\mathbf{A}_M, \mathbf{b}_M, \mathbf{c}_M, \mathbf{d}_M)$ is plotted in Figure 6.5.

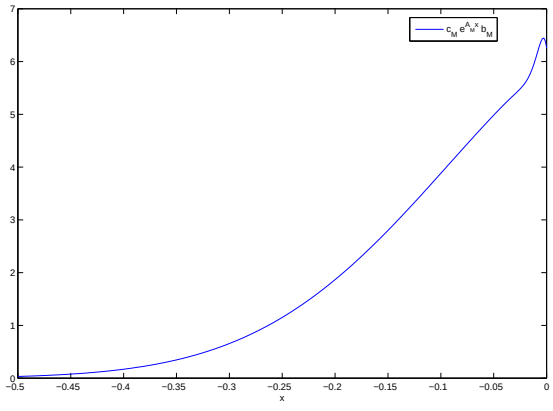

Figure 6.5: Plot of $M(252)$, the minimum of the process $L(252)$ as given in Eq. (6.36). There is a point mass at zero which is not visible

6.5.4. Lookback Option with Fixed Strike

We will now consider pricing a Lookback Option at time $t = 0$ on an asset S whose discrete risk neutral price process is described in Section 6.4.3.2. The option has a fixed strike K and a payoff at maturity T according to

$$L(S(T), T; T, K) \;=\; \max\{K \;-\; S_{\min}(T),\; 0\} \tag{6.42}$$

The asset price is monitored daily, 252 times over a one year period.

If $K \;<\; S(0)$ we can see that the price of the Lookback Option at time $t_0 = 0$ is given by

$$\begin{aligned} L(S(0), 0; T, K) \quad &= \quad e^{-rT} \int_0^K (K - S_{\min}(T))\; f(S_{\min}(T)) dS_{\min}(T) \\ &= \quad e^{-rT} \int_{-\infty}^{d} (K - S(0)e^{M(T)}) \mathbf{c}_M e^{\mathbf{A}_M M(T)} \mathbf{b}_M dM(T) \\ &= \quad K e^{-rT} \mathbf{c}_M \mathbf{A}_M^{-1} e^{\mathbf{A}_M d} \mathbf{b}_M \;-\; S(0) e^{-rT} \mathbf{c}_M (\mathbf{A}_M + \mathbf{I})^{-1} e^{(\mathbf{A}_M + \mathbf{I})d} \mathbf{b}_M \end{aligned}$$

where $d \;=\; \log(K/S(0))$. Similarly if $d \;>\; 0$ we would have that

$$\begin{aligned} L(S(0), 0; T, K) \quad &= \quad K e^{-rT} \mathbf{d} \;+\; e^{-rT} \int_{-\infty}^{0} (K - S(0)e^{M(T)}) \mathbf{c}_M e^{\mathbf{A}_M M_T} \mathbf{b}_M dM(T) \\ &= \quad K e^{-rT} (\mathbf{c}_M \mathbf{A}_M^{-1} \mathbf{b}_M + \mathbf{d}_M) \;-\; S(0) e^{-rT} \mathbf{c}_M (\mathbf{A}_M + \mathbf{I}_M)^{-1} \mathbf{b}_M \\ &= \quad K e^{-rT} \;-\; S(0) e^{-rT} (\mathbf{c}_M (\mathbf{A}_M + \mathbf{I})^{-1} \mathbf{b}_M + \mathbf{d}_M) \end{aligned}$$

Using the density of $M(252)$ calculated in Section 6.5.3 the values of $L(S(0), 0; 1, 100)$ over a range of initial asset prices $S(0) \in [70, 130]$ are illustrated in Figure 6.6.

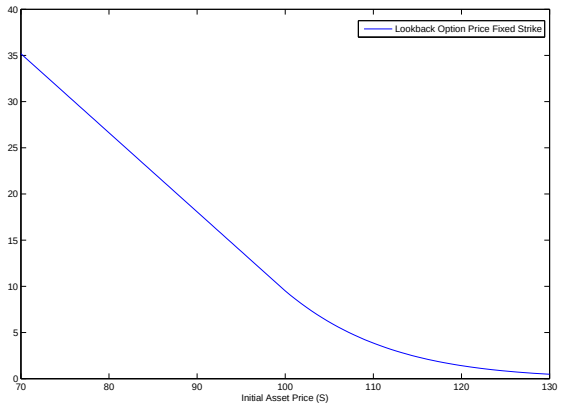

Figure 6.6: Lookback Option prices with payoff as in Eq. (6.42) over a range of initial asset prices. The parameters $K = 100$, $T = 1$ and $r = 0.05$ are fixed.

The Greeks, delta (Δ_L) and gamma (Γ_L), for such an option can also be computed analytically. For $d = \log(K/S) < 0$

$$\begin{aligned} \Delta_L &= \frac{\partial L}{\partial S} = e^{-rT}\Big(- \frac{-K}{S}\mathbf{c}_M e^{\mathbf{A}_M d}\mathbf{b}_M \\ &- \mathbf{c}_M(\mathbf{A}_M + \mathbf{I})^{-1}e^{(\mathbf{A}_M+\mathbf{I})d}\mathbf{b}_M + \mathbf{c}_M e^{(\mathbf{A}_M+\mathbf{I})d}\mathbf{b}_M\Big) \end{aligned} \tag{6.43}$$

while for $d > 0$ we have $\Delta_L = e^{-rT}$. It is obvious that the gamma of such an option when $d > 0$ is zero. However for $d < 0$

$$\begin{aligned} \Gamma_L &= \frac{\partial^2 L}{\partial S^2} = \frac{e^{-rT}}{S}\Big(\frac{K}{S}\mathbf{c}_M e^{\mathbf{A}_M d}\mathbf{b}_M \\ &+ \frac{K}{S}\mathbf{c}_M\mathbf{A}_M e^{\mathbf{A}_M d}\mathbf{b}_M + \mathbf{c}_M e^{(\mathbf{A}_M+\mathbf{I})d}\mathbf{b}_M - \mathbf{c}_M(\mathbf{A}_M + \mathbf{I})e^{(\mathbf{A}_M+\mathbf{I})d}\mathbf{b}_M\Big) \end{aligned} \tag{6.44}$$

In Figure 6.7 the Delta and Gamma of the Lookback Option with Fixed Strike are depicted.

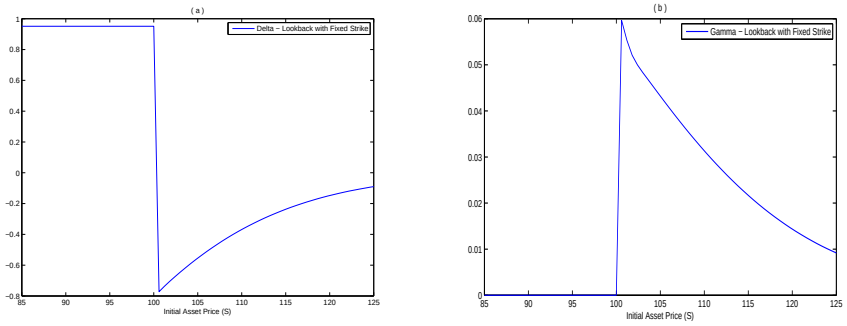

Figure 6.7: (a) Delta and (b) Gamma for a Lookback with Fixed Strike. The parameters $K = 100$, $T = 1$ and $r = 0.05$ are fixed.

For Figure 6.7, the distribution of $M(252)$ was computed using the methodology of Section 6.5.3.

6.5.5. Lookback Option with Floating Strike

A Lookback option with floating strike is now considered. It is assumed the option has the following payoff at maturity $T > 0$

$$L(S(T), T; T) \;=\; S(T) \;-\; S_{\min}(T) \tag{6.45}$$

which is always non-negative. Again the asset price is monitored at the daily close price over a one year period (252 trading days). Assuming the risk neutral dynamics of the asset price process are given by Eq. (6.38) the price of such a contract at time 0 is

$$\begin{aligned} L(S(0); 0, T) &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[S(T) \;-\; S_{\min}(T)] \\ &= S(0) \;-\; e^{-rT}\mathbb{E}_{\mathbb{Q}}[S_{\min}(T)] \end{aligned}$$

by the Fundamental Theory of Asset Pricing. We have that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}[S_{\min}(T)] &= \int_0^{S(0)} S_{\min}(T) f(S_{\min}(T)) dS_{\min}(T) \\ &= S(0) \int_{-\infty}^{0} e^{M(T)} (\mathbf{c}_M e^{\mathbf{A}_M M_T} \mathbf{b}_M + \mathbf{d}_M) d(T) \\ &= S(0)(\ \mathbf{c}_M (\mathbf{A}_M + \mathbf{I})^{-1} \mathbf{b}_M \ +\ \mathbf{d}_M)\end{aligned}$$

The price of the option at time 0 is given by

$$L(S(0); 0, T) \ = \ S(0)\Big(1 \ - \ e^{-rT}(\mathbf{c}_M(\mathbf{A}_M + \mathbf{I})^{-1}\mathbf{b}_M \ + \ \mathbf{d}_M)\Big) \tag{6.46}$$

6.6. 2-Period Bermudan Option

We will examine pricing a 2-Period Bermudan Put Option although in Section 6.6.3 we show that the methodology can be applied to price compound options i.e. an option on an option. A Bermudan Option is an option in which the holder has the right to exercise the option early at some pre-specified times (the case of one early exercise opportunity is treated here). The option is so-called as it is a combination of a European and an American option. We consider valuing the option at time $t_0 = 0$ which expires at $t_2 = 2\tau$ and the holder has one early exercise opportunity at time $t_1 = \tau$. We assume that the risk neutral asset price process is given

$$S(t_j) \ = \ S(t_{j-1}) e^{(r+\omega)\tau + X_\tau} \qquad , \qquad j \ = \ 1, 2 \tag{6.47}$$

where X_τ is an infinitely divisible 2-EPT random variable whose distribution has a minimal realization given by $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. The times t_1 and t_2 could be generalised provided the underlying asset had log returns over the periods $[0, t_1]$ and $[t_1, t_2]$ with infinitely divisible 2-EPT distributions.

$P(S(t_1); t_1, t_2, K)$ will denote the price of a European Put Option at time t_1

expiring at time t_2 with strike K on an asset with price $S(t_1)$. At time t_1 the holder of the Bermudan Option may choose to exercise the option early or hold the option until expiry. Hence the value of the Bermudan Option at time $t_0 = 0$ is

$$B(S(t_0), t_0; t_2, t_1, K) \;=\; e^{-r\tau} \int_0^{\infty} \max\{K \;-\; S_{t_1},\; P(S(t_1); t_1, t_2, K)\}\; f(S(t_1))\; dS(t_1) \tag{6.48}$$

where $f(S(t_1))$ is distribution of $S(t_1)$. We cannot evaluate Eq. (6.48) directly due to the max operator. However due to the payoff structure of early exercise component and put option value we can derive a critical share price S_K^* such that

$$K \;-\; S(t_1) \quad \leq \quad P(S(t_1); t_1, t_2, K) \qquad\qquad \forall\; S(t_1) \;\geq\; S_K^* \tag{6.49}$$

$$K \;-\; S(t_1) \quad > \quad P(S(t_1); t_1, t_2, K) \qquad\qquad \forall\; S(t_1) \;<\; S_K^* \tag{6.50}$$

If $S(t_1) < S_K^*$ then it is optimal to exercise the option early at t_1.
The integral in Eq. (6.48) can be broken up and written as

$$\begin{aligned} B(S(t_0); t_2, t_1, K) \quad &= \quad \underbrace{e^{-r\tau} \int_0^{S_K^*} (K \;-\; S(t_1))\; f(S(t_1))\; dS(t_1)}_{EE(0)} \\ &+ \quad \underbrace{e^{-r\tau} \int_{S_K^*}^{\infty} P(S(t_1); t_1, t_2, K)\; f(S(t_1))\; dS(t_1)}_{H(0)} \end{aligned} \tag{6.51}$$

The value $B(S(t_0); t_2, t_1, K)$ in Eq. (6.51) is decomposed into its early exercise component ($EE(0)$) and the value derived from holding the contract until maturity denoted $H(0)$. These components, $EE(0)$ and $H(0)$, are evaluated separately in Sections 6.6.1 and 6.6.2 respectively.

6.6.1. Early Exercise Component

Firstly we evaluate the early exercise component and note that

$$\begin{aligned} EE_A = \int_0^{S_K^*} (K - S(t_1))\, f(S(t_1))dS(t_1) &= e^{-r\tau} \int_{-\infty}^{x_K^*} (K - S(t_0)e^{(r+\omega)\tau+x})\, \mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N\, dx \\ &= K\mathbf{c}_N\mathbf{A}_N^{-1}e^{\mathbf{A}_N x_K^*}\mathbf{b}_N \\ &- S(t_0)e^{(r+\omega)\tau}\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})e^{(\mathbf{A}_N+\mathbf{I})x_K^*}\mathbf{b}_N \end{aligned} \tag{6.52}$$

where $x_K^* = \log(S_K^*/S(t_0)) - (r+\omega)\tau < 0$. For $x_K^* \geq 0$ we have that

$$\begin{aligned} EE_B = \int_0^{S_K^*} (K - S(t_1))\, f(S(t_1))dS(t_1) &= \int_{-\infty}^{0} (K - S(t_1)e^{(r+\omega)\tau+x})\, \mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N\, dx \\ &+ \int_0^{x_K^*} (K - S(t_0)e^{(r+\omega)\tau+x})\, \mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P\, dx \\ &= K\mathbf{c}_N\mathbf{A}_N^{-1}\mathbf{b}_N - S(t_0)e^{(r+\omega)\tau}\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})\mathbf{b}_N \\ &+ K\mathbf{c}_P\mathbf{A}_P^{-1}e^{\mathbf{A}_P x_K^*}\mathbf{b}_P \\ &- S(t_0)e^{(r+\omega)\tau}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})e^{(\mathbf{A}_P+\mathbf{I})x_K^*}\mathbf{b}_P \\ &- K\mathbf{c}_P\mathbf{A}_P^{-1}\mathbf{b}_P - S(t_0)e^{(r+\omega)\tau}\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})\mathbf{b}_P \end{aligned}$$

The value at time t_0 of exercising the option at t_1 is

$$EE(t_0) = e^{-r\tau}\left(EE_A\mathbb{I}_{[x_K^*<0]} + EE_B\mathbb{I}_{[x_K^*\geq 0]}\right) \tag{6.53}$$

where $\mathbb{I}_{[x_K^*<0]}$ is an indicator function equal to 1 if $x_K^* < 0$ and zero otherwise.

6.6.2. Holding Value

We now consider valuing the contribution of not exercising the option at t_1 but inheriting a European Put Option at t_1 with maturity t_2 and $S(t_1) > S_K^*$. This implies evaluating the second integral of Eq. (6.51).

A similar calculation to that provided in Section 6.3.1 implies that the value of

a European Put Option $P(S(t_1); t_1, t_2, K)$ is given by

$$\begin{aligned} \underline{P}(S(t_1); t_1, t_2, K) &= Ke^{-r\tau}(1 + \mathbf{c}_P \mathbf{A}_P^{-1} e^{-\mathbf{A}_P d} \mathbf{b}_P) \\ &- S(t_1)e^{\omega\tau}\Big(\lambda + \mathbf{c}_P(\mathbf{A}_P + \mathbf{I})^{-1} e^{-(\mathbf{A}_P+\mathbf{I})d}\mathbf{b}_P\Big) \end{aligned} \quad (6.54)$$

for $d = \log(S(t_1)/K) + (r+\omega)\tau \leq 0$ and $\lambda = \mathbf{c}_N(\mathbf{A}_N + \mathbf{I})^{-1}\mathbf{b}_N - \mathbf{c}_P(\mathbf{A}_P + \mathbf{I})^{-1}\mathbf{b}_P$. Similarly the price of a European Put Option with $d > 0$ is

$$\begin{aligned} \bar{P}(S(t_1); t_1, t_2, K) &= Ke^{-r\tau}\mathbf{c}_N \mathbf{A}_N^{-1} e^{-\mathbf{A}_N d}\mathbf{b}_N \\ &- S(t_1)e^{\omega\tau}\mathbf{c}_N(\mathbf{A}_N + \mathbf{I})^{-1} e^{-(\mathbf{A}_N+\mathbf{I})d}\mathbf{b}_N \end{aligned} \quad (6.55)$$

$\underline{P}(d)$ and $\bar{P}(d)$ denote the value of the Put Option for $d \leq 0$ and $d > 0$ respectively.

The asset price at t_1, $S(t_1)$ is unknown so the option value is expressed in terms of $S(t_0)$ and the log return x over $[0, t_1]$

$$\begin{aligned} \underline{P}(x; t_1, t_2, K) &= Ke^{-r\tau}(1 + \mathbf{c}_P \mathbf{A}_P^{-1} e^{-\mathbf{A}_P d_0} e^{-\mathbf{A}_P x}\mathbf{b}_P) \\ &- S(t_0)e^{(r+2\omega)\tau + x}\Big(\lambda + \mathbf{c}_P(\mathbf{A}_P + \mathbf{I})^{-1} e^{-(\mathbf{A}_P+\mathbf{I})d_0} e^{-(\mathbf{A}_P+\mathbf{I})x}\mathbf{b}_P\Big) \end{aligned} \quad (6.56)$$

where $d_0 = \log(S(t_0)/K) + 2(r+\omega)\tau$ and $x \leq -d_0$. For $x > -d_0$ we have

$$\begin{aligned} \bar{P}(x; t_1, t_2, K) &= Ke^{-r\tau}\mathbf{c}_N \mathbf{A}_N^{-1} e^{-\mathbf{A}_N d_0} e^{-\mathbf{A}_N x}\mathbf{b}_N \\ &- S(t_0)e^{(r+2\omega)\tau + x}\mathbf{c}_N(\mathbf{A}_N + \mathbf{I})^{-1} e^{-(\mathbf{A}_N+\mathbf{I})d_0} e^{-(\mathbf{A}_N+\mathbf{I})x}\mathbf{b}_N \end{aligned} \quad (6.57)$$

The final term of Eq. (6.51) can be written as

$$\int_{S_K^*}^{\infty} P(S(t_1); t_1, t_2, K)\, f(S(t_1))\, dS(t_1) = \int_{x_K^*}^{\infty} P(x; t_1, t_2, K) f(x) dx \quad (6.58)$$

The above integral must also be split up according to the sign of x and d_0 so there are three possible cases to be considered.

Case	Parameter Ordering
1	$x_K^* < -d_0 < 0$
2	$x_K^* < 0 < -d_0$
3	$0 < x_K^* < -d_0$

Table 6.2: Parameter Ordering for Integrating Domains

We can make the assumption that $x_K^* < -d_0$ as

$$\begin{aligned} \log(S_K^*/S(t_0)) - (r+\omega)\tau &< -(\log(S(t_0)/K) + 2(r+\omega)\tau) \\ \log(S_K^*) &< \log(K) + (r+\omega)\tau \end{aligned} \tag{6.59}$$

If $x_K^* \geq -d_0$ we set $-d_0 = x_K^*$ implying that cases 1 and 2 are equal. Based on Table 6.2 we can write

$$\int_{x_K^*}^{\infty} P(x;t_1,t_2,K)f(x)dx = I_1 \mathbb{I}_{[x_K^* < -d_0 < 0]} + I_2 \mathbb{I}_{[x_K^* < 0 < -d_0]} + I_3 \mathbb{I}_{[0 < x_K^* < -d_0]} \tag{6.60}$$

where we treat the integrals I_1, I_2 and I_3 individually.

6.6.2.1. Holding Value: Case I

For case 1 above we have that the integral in Eq. (6.58) can be computed as

$$\begin{aligned} I_1 = \int_{x_K^*}^{\infty} P(x;t_1,t_2,K)f(x)dx &= \underbrace{\int_{x_K^*}^{-d_0} \underline{P}(x;t_1,t_2,K)\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx}_{I_{1_A}} \\ &+ \underbrace{\int_{-d_0}^{0} \bar{P}(x;t_1,t_2,K)\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx}_{I_{1_B}} \\ &+ \underbrace{\int_{0}^{\infty} \bar{P}(x;t_1,t_2,K)\mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx}_{I_{1_C}} \end{aligned} \tag{6.61}$$

The integrals must be evaluated separately and for the term I_{1_A}

$$\begin{aligned} I_{1_A} &= Ke^{-r\tau}\int_{x_K^*}^{-d_0}(1+\mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_P d_0}e^{-\mathbf{A}_P x}\mathbf{b}_P)\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx \\ &- S(t_0)e^{(r+2\omega)\tau}\int_{x_K^*}^{-d_0}(\lambda e^x+\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_P+\mathbf{I})d_0}e^{-(\mathbf{A}_P+\mathbf{I})(x+\mathbf{I})}\mathbf{b}_P)\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx \\ &= Ke^{-r\tau}\Big[\mathbf{c}_N\mathbf{A}_N^{-1}e^{\mathbf{A}_N x}\mathbf{b}_N+\mathbf{c}_1\mathbf{A}_1^{-1}e^{\mathbf{A}_1 x}\mathbf{b}_1\Big]_{x_K^*}^{-d_0} \\ &- S(t_0)e^{(r+2\omega)\tau}\Big[\lambda\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^{(\mathbf{A}_N+\mathbf{I})x}\mathbf{b}_N+\mathbf{c}_2\mathbf{A}_1^{-1}e^{\mathbf{A}_1 x}\mathbf{b}_1\Big]_{x_K^*}^{-d_0} \end{aligned} \tag{6.62}$$

where

$$\begin{aligned} \mathbf{A}_1 &= -\mathbf{A}_P\oplus\mathbf{A}_N \\ \mathbf{b}_1 &= \mathbf{b}_P\otimes\mathbf{b}_N \\ \mathbf{c}_1 &= \mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_{P_1}d_0}\otimes\mathbf{c}_N \\ \mathbf{c}_2 &= \mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_{P_1}+\mathbf{I})d_0}\otimes\mathbf{c}_N \end{aligned} \tag{6.63}$$

Next we evaluate integral I_{1_B}

$$\begin{aligned} I_{1_B} &= Ke^{-r\tau}\int_{-d_0}^{0}\mathbf{c}_N\mathbf{A}_N^{-1}e^{-\mathbf{A}_N d_0}e^{-\mathbf{A}_N x}\mathbf{b}_N\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx \\ &- S(t_0)e^{(r+2\omega)\tau}\int_{-d_0}^{0}\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^{-(\mathbf{A}_N+\mathbf{I})d_0}e^{-\mathbf{A}_N x}\mathbf{b}_N\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx \\ &= \Big[Ke^{-r\tau}\mathbf{c}_3\mathbf{A}_3^{-1}e^{\mathbf{A}_3 x}\mathbf{b}_3-S(t_0)e^{(r+2\omega)\tau}\mathbf{c}_4\mathbf{A}_3^{-1}e^{\mathbf{A}_3 x}\mathbf{b}_3\Big]_{-d_0}^{0} \end{aligned} \tag{6.64}$$

where

$$\begin{aligned} \mathbf{A}_3 &= -\mathbf{A}_N\oplus\mathbf{A}_N \\ \mathbf{b}_3 &= \mathbf{b}_N\otimes\mathbf{b}_N \\ \mathbf{c}_3 &= \mathbf{c}_N\mathbf{A}_N^{-1}e^{-\mathbf{A}_{N_1}d_0}\otimes\mathbf{c}_N \\ \mathbf{c}_4 &= \mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^{-(\mathbf{A}_{N_1}+\mathbf{I})d_0}\otimes\mathbf{c}_N \end{aligned} \tag{6.65}$$

However the matrix $\mathbf{A}_3$ is singular so care must be taken integrating the

$\mathbf{c}_3 e^{\mathbf{A}_3 x}\mathbf{b}_3$ and $\mathbf{c}_4 e^{\mathbf{A}_3 x}\mathbf{b}_3$ terms. The reader is referred to Section 6.6.4 for a more detailed look at this problem.

Finally we consider the integral I_{1_C}

$$\begin{aligned} I_{1_C} = &= Ke^{-r\tau}\int_0^\infty \mathbf{c}_N \mathbf{A}_N^{-1} e^{-\mathbf{A}_N d_0} e^{-\mathbf{A}_N x}\mathbf{b}_N \mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx \\ &- S(t_0)e^{(r+2\omega)\tau}\int_0^\infty \mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1} e^{-(\mathbf{A}_N+\mathbf{I})d_0} e^{-\mathbf{A}_N x}\mathbf{b}_N \mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx \\ &= -Ke^{-r\tau}\mathbf{c}_5\mathbf{A}_5^{-1}\mathbf{b}_5 + S(t_0)e^{(r+2\omega)\tau}\mathbf{c}_6\mathbf{A}_5^{-1}\mathbf{b}_5 \end{aligned} \tag{6.66}$$

where

$$\begin{aligned} \mathbf{A}_5 &= -\mathbf{A}_N \oplus \mathbf{A}_P \\ \mathbf{b}_5 &= \mathbf{b}_N \otimes \mathbf{b}_P \\ \mathbf{c}_5 &= \mathbf{c}_N \mathbf{A}_N^{-1} e^{-\mathbf{A}_{N_1} d_0} \otimes \mathbf{c}_P \\ \mathbf{c}_6 &= \mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1} e^{-(\mathbf{A}_{N_1}+\mathbf{I})d_0} \otimes \mathbf{c}_P \end{aligned} \tag{6.67}$$

6.6.2.2. Holding Value: Case II

Now we consider Case 2 as shown in Table 6.2 and the integral in Eq. (6.58) is given by

$$I_2 = \int_{x_K^*}^\infty P(x;t_1,t_2,K)f(x)dx = \underbrace{\int_{x_K^*}^0 \underline{P}(x;t_1,t_2,K)\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx}_{I_{2_A}} \tag{6.68}$$

$$+ \underbrace{\int_0^{-d_0} \underline{P}(x;t_1,t_2,K)\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx}_{I_{2_B}}$$

$$+ \underbrace{\int_{-d_0}^\infty \bar{P}(x;t_1,t_2,K)\mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx}_{I_{2_C}} \tag{6.69}$$

Using the notation given in Eq. (6.63) we have that I_{2_A} is given by

$$\begin{aligned} I_{2_A} &= = Ke^{-r\tau}\Big[\mathbf{c}_N\mathbf{A}_N^{-1}e^{\mathbf{A}_N x}\mathbf{b}_N + \mathbf{c}_1\mathbf{A}_1^{-1}e^{\mathbf{A}_1 x}\mathbf{b}_1\Big]_{x_K^*}^{0} \\ &- S(t_0)e^{(r+2\omega)\tau}\Big[\lambda\mathbf{c}_N(\mathbf{A}_N+\mathbf{I})^{-1}e^{(\mathbf{A}_N+\mathbf{I})x}\mathbf{b}_N + \mathbf{c}_2\mathbf{A}_1^{-1}e^{\mathbf{A}_1 x}\mathbf{b}_1\Big]_{x_K^*}^{0} \end{aligned} \tag{6.70}$$

Some new notation is required for the integral I_{2_B} as

$$\begin{aligned} I_{2_B} &= Ke^{-r\tau}\int_0^{-d_0}(1+\mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_P d_0}e^{-\mathbf{A}_P x}\mathbf{b}_P)\mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx \\ &- S(t_0)e^{(r+2\omega)\tau}\int_0^{-d_0}(\lambda e^x + \mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_P+\mathbf{I})d_0}e^{-(\mathbf{A}_P+\mathbf{I})(x+\mathbf{I})}\mathbf{b}_P)\mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx \\ &= Ke^{-r\tau}\Big[\mathbf{c}_P\mathbf{A}_P^{-1}e^{\mathbf{A}_P x}\mathbf{b}_N + \mathbf{c}_7\mathbf{A}_7^{-1}e^{\mathbf{A}_7 x}\mathbf{b}_7\Big]_0^{-d_0} \\ &- S(t_0)e^{(r+2\omega)\tau}\Big[\lambda\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{(\mathbf{A}_P+\mathbf{I})x}\mathbf{b}_P + \mathbf{c}_8\mathbf{A}_7^{-1}e^{\mathbf{A}_7 x}\mathbf{b}_7\Big]_0^{-d_0} \end{aligned} \tag{6.71}$$

where

$$\begin{aligned} \mathbf{A}_7 &= -\mathbf{A}_P \oplus \mathbf{A}_P \\ \mathbf{b}_7 &= \mathbf{b}_P \otimes \mathbf{b}_P \\ \mathbf{c}_7 &= \mathbf{c}_P\mathbf{A}_P^{-1}e^{-\mathbf{A}_{P_1}d_0} \otimes \mathbf{c}_P \\ \mathbf{c}_8 &= \mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{-(\mathbf{A}_{P_1}+\mathbf{I})d_0} \otimes \mathbf{c}_P \end{aligned} \tag{6.72}$$

Similar to $\mathbf{A}_3$ above the matrix $\mathbf{A}_7$ is singular therefore the integral I_{2_B} must be treated with caution so the reader is again referred to Section 6.6.4 while lastly for I_{2_C} we have

$$I_{2_C} = -Ke^{-r\tau}\mathbf{c}_5\mathbf{A}_5^{-1}e^{-\mathbf{A}_5 d_0}\mathbf{b}_5 + S(t_0)e^{(r+2\omega)\tau}\mathbf{c}_6\mathbf{A}_5^{-1}e^{-\mathbf{A}_5 d_0}\mathbf{b}_5 \tag{6.73}$$

6.6.2.3. Holding Value: Case III

Finally we examine Case 3 where we have

$$\begin{aligned} I_3 = \int_{x_K^*}^{\infty} P(x;t_1,t_2,K)f(x)dx &= \underbrace{\int_{x_K^*}^{-d_0} \underline{P}(x;t_1,t_2,K)\mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx}_{I_{3_a}} \\ &+ \underbrace{\int_{-d_0}^{\infty} \bar{P}(x;t_1,t_2,K)\mathbf{c}_P e^{\mathbf{A}_P x}\mathbf{b}_P dx}_{I_{3_b}} \end{aligned} \tag{6.74}$$

where the integral I_{3_B} is clearly equivalent to the integral I_{2_C} above. Hence we must only consider evaluating I_{3_A}

$$\begin{aligned} I_{3_A} &= Ke^{-r\tau}\Big[\mathbf{c}_P\mathbf{A}_P^{-1}e^{\mathbf{A}_P x}\mathbf{b}_N + \mathbf{c}_7\mathbf{A}_7^{-1}e^{\mathbf{A}_7 x}\mathbf{b}_7\Big]_{x_K^*}^{-d_0} \\ &- S(t_0)e^{(r+2\omega)\tau}\Big[\lambda\mathbf{c}_P(\mathbf{A}_P+\mathbf{I})^{-1}e^{(\mathbf{A}_P+\mathbf{I})x}\mathbf{b}_P + \mathbf{c}_8\mathbf{A}_7^{-1}e^{\mathbf{A}_7 x}\mathbf{b}_7\Big]_{x_K^*}^{-d_0} \end{aligned} \tag{6.75}$$

The value at time t_0 of holding the Bermudan Option until maturity t_2 as $S_{t_1} > S_K^*$ is

$$H(t_0) = e^{-r\tau}\Big(I_1\mathbb{I}_{[x_K^*<-d_0<0]} + I_2\mathbb{I}_{[x_K^*<0<-d_0]} + I_3\mathbb{I}_{[0<x_K^*<-d_0]}\Big) \tag{6.76}$$

6.6.3. Compound Options

This methodology may also be used to value a compound option (i.e. an option on an option). An example of a call-on-put compound option would have two strike prices K_1 and K_2 associated with two expiry dates t_1 and t_2 respectively such that $t_1 < t_2$. It is assumed that the underlying asset has a 2-EPT price process as defined in Section 6.2. Also the log-returns of the asset over the periods $[0, t_1]$ and $[t_1, t_2]$ have a known (although possibly different) infinitely divisible 2-EPT probability density function. The holder of the contract would then have the right at time t_1 to buy a put option for K_1 which matures at t_2 and strike K_2. The value of such an option is given by

$$CoP(S(t_0); t_2, K_2, t_1, K_1) \;=\; e^{-r\tau} \int_0^{\infty} \max\{P((t_1); t_1, t_2, K_2) - K_1,\; 0\} f(S(t_1)) dS(t_1) \tag{6.77}$$

Similar to the Bermudan option we can derive the share price S_K^* such that

$$P(S(t_1); t_1, t_2, K_2) \;-\; K_1 \quad \leq \quad 0 \qquad\qquad \forall\, S(t_1) \;\geq\; S_K^* \tag{6.78}$$

$$P(S(t_1); t_1, t_2, K_2) \;-\; K_1 \quad > \quad 0 \qquad\qquad \forall\, S(t_1) \;<\; S_K^* \tag{6.79}$$

Hence the value of the call-on-put option can be written as

$$CoP(S(t_1); t_2, K_2, t_1, K_1) \;=\; e^{-r\tau} \int_0^{S_K^*} \big(P(S(t_1); t_1, t_2, K_2) - K_1\big) f(S(t_1)) dS(t_1) \tag{6.80}$$

which is very similar to Eq. (6.58) and can be solved using the same approach.

6.6.4. Integrating EPT functions with Singular Matrices

Consider the problem of integrating an EPT function with minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ where $\mathbf{A}$ is singular. It is clear that

$$\int_{x_0}^{x_1} \mathbf{c} e^{\mathbf{A}x} \mathbf{b}\, dx \;=\; \mathbf{c}\mathbf{A}^{-1} e^{\mathbf{A}x} \mathbf{b} \,\Big]_{x_0}^{x_1} \tag{6.81}$$

cannot be computed as $\mathbf{A}$ is not invertible. Using a basis transformation and solving the appropriate Sylvester equation $\mathbf{A}$ can be split into a singular block $\mathbf{A}_S$ and a non-singular block $\mathbf{A}_{NS}$ such that

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \\ \hline \mathbf{c} & 0 \end{array}\right) \;=\; \left(\begin{array}{cc|c} \mathbf{A}_{NS} & 0 & \mathbf{b}_{NS} \\ 0 & \mathbf{A}_S & \mathbf{b}_S \\ \hline \mathbf{c}_{NS} & \mathbf{c}_S & 0 \end{array}\right) \tag{6.82}$$

The integral above can then be decomposed as follows

$$\begin{aligned}\int_{x_0}^{x_1} \mathbf{c}e^{\mathbf{A}x}\mathbf{b}\,dx &= \underbrace{\int_{x_0}^{x_1} \mathbf{c}_{NS}e^{\mathbf{A}_{NS}x}\mathbf{b}_{NS}\,dx}_{Non-Singular} + \underbrace{\int_{x_0}^{x_1} \mathbf{c}_S e^{\mathbf{A}_S x}\mathbf{b}_S\,dx}_{Singular} \\ &= \mathbf{c}_{NS}\mathbf{A}_{NS}^{-1}e^{\mathbf{A}_{NS}x}\mathbf{b}_{NS}\,\Big]_{x_0}^{x_1} + \int_{x_0}^{x_1} \mathbf{c}_S e^{\mathbf{A}_S x}\mathbf{b}_S\,dx\end{aligned}$$

It remains to integrate the singular part of the EPT function but since $\sigma(\mathbf{A}_S) \equiv 0$ we know that $\mathbf{A}_S$ is nilpotent and there exists a N such that $(\mathbf{A}_S)^N = 0$. Hence the integral of the "Singular" term above can be computed exactly using the expansion

$$\begin{aligned}\int_{x_0}^{x_1} \mathbf{c}_S e^{\mathbf{A}_S x}\mathbf{b}_S\,dx &= \mathbf{c}_S\Big(\sum_{n=1}^{N}\frac{\mathbf{A}_S^{n-1}x^n}{n!}\Big)\mathbf{b}_S\,\Big]_{x_0}^{x_1} \qquad (6.83)\\ &= \mathbf{c}_S x\mathbf{b}_S + \frac{\mathbf{c}_S\mathbf{A}_S x^2\mathbf{b}_S}{2} + \frac{\mathbf{c}_S\mathbf{A}_S^2 x^3\mathbf{b}_S}{3!} + ... + \frac{\mathbf{c}_S\mathbf{A}_S^{N-1}x^N\mathbf{b}_S}{N!}\Big]_{x_0}^{x_1}\end{aligned}$$

6.7. Value-at-Risk and Expected Shortfall

Traditionally Value-at-Risk (VaR) has been used in risk management practice. Consider an asset with risk neutral dynamics given by Eq. (6.2) such that for some fixed $\tau > 0$, $X_\tau \sim 2 - EPT(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. The loss L which corresponds to a certain pre-defined probability p of being exceeded during the timeframe τ can be determined. Hence one must solve for L such that

$$\begin{aligned}p &= \mathbb{P}[S(T) < L] \qquad (6.84)\\ &= \mathbb{P}[S(t)e^{((r+\omega)\tau + X_\tau)} < L]\\ &= \mathbb{P}[X_\tau < ln\Big(\frac{L}{S(t)}\Big) - (r+\omega)\tau]\\ &= \int_{-\infty}^{d_L} \mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N\,dx\\ &= \mathbf{c}_N(\mathbf{A}_N)^{-1}e^{\mathbf{A}_N\,d_L}\mathbf{b}_N\end{aligned}$$

where $d_L = \log\left(\frac{L}{S(t)}\right) - (r+\omega)\tau < 0$ and $\tau = T - t$.

For the same asset the expected shortfall for a given loss $L < 0$ could also be cimputed. The expected shortfall is defined as

$$\mathbb{E}[L - S(T)|S(T) < L] = L - \mathbb{E}[S(T)|S(T) < L] \tag{6.85}$$

Assuming that $L < 0$

$$\begin{aligned} \mathbb{E}[S(T)|S(T) < L] &= \int_{-\infty}^{L} S(t)e^{(r+\omega)\tau+x}\mathbf{c}_N e^{\mathbf{A}_N x}\mathbf{b}_N dx \\ &= S(t)e^{(r+w)\tau}\mathbf{c}_N(\mathbf{A}_N + I)^{-1}e^{(\mathbf{A}_N+I)L}\mathbf{b}_N \end{aligned} \tag{6.86}$$

The expected shortfall is then

$$\mathbb{E}[L - S(T)|S(T) < L] = L - S(t)e^{(r+w)\tau}\mathbf{c}_N(\mathbf{A}_N + I)^{-1}e^{(\mathbf{A}_N+I)L}\mathbf{b}_N \tag{6.87}$$

Chapter 7

2-EPT Matlab Functionality

7.1. Introduction

This penultimate chapter discusses MATLAB functionality available to assist implementing some of the techniques described in the previous chapters. MATLAB (**MAT**rix **LAB**oratory) was designed to allow easy matrix manipulation and is an ideal computing environment to perform 2-EPT calculations. Firstly an introduction to some inbuilt MATLAB commands is given which takes advantage of the Control Systems Toolbox to implement some well known results from the system theory.

The website "www.2-ept.com" was designed in conjunction with this research to be a source for 2-EPT related software and literature. The site contains beta versions of all scripts described in this chapter. The most significant and perhaps important algorithm available is the Budan-Fourier method of Hanzon and Holland (2010) to locate the sign-changing zeros of an EPT function on a finite interval. Another significant algorithm provided, is the convex optimisation algorithm described in Section 3.8. A number of additional scripts have been made available on the website including a piece of code to additively decompose a proper rational function, with minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$, into

the sum of two proper rational functions with poles located in the open left and open right half planes. A programme is also available to compute the convolution of two 2-EPT functions. An N-Fold convolution script utilises the convolution function to perform an N-Fold convolution of a 2-EPT function with itself. These functions are necessary for the Process Minimum / Maximum script which computes the minimal realization of the generalised EPT probability density function of the maximum or minimum of a fixed length discrete time 2-EPT process as described in Section 5.2.
The rational approximation software RARL2 is available only from INRIA directly.

7.2. Inbuilt Matlab Functions

It should be noted that all of the functions mentioned in this section require the Control System Toolbox and are applicable to EPT functions only. The first and most important function is the `ss` routine. This function allows the user to create a state space system by inputting the $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ realization of an EPT function. The $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ realization can also be found from the transfer function using the `tf` command by supplying the coefficients of the numerator and the denominator of the rational transfer function. Alternatively the transfer function can be given by the `zpk` routine where the inputs are the zeros, poles and gain of the transfer function.

Once the state space model is defined its minimal realization can be obtained using the `minreal` command. The `balreal` function computes the balanced realization of a state space system. The order of the system can then be reduced using the `balred` or `modred` scripts where the desired number of states in the output model is specified. `balred` eliminates states with the smallest Hankel singular values while `modred` eliminates particular states.

The H_2 norm of system can be calculated using the `norm` command. The `lyap`

solves continuous time Lyapunov equations which includes Sylvester equations. Finally the `gram` routine returns the observability or controllability grammian of the input pair $(\mathbf{A}, \mathbf{c})$ or $(\mathbf{A}, \mathbf{b})$ respectively.

7.3. Budan-Fourier Technique

This code implements the Budan-Fourier technique of Hanzon and Holland (2010) to identify all sign-changing zeros of the EPT function $\mathbf{c}e^{\mathbf{A}x}\mathbf{b}$ of McMillan degree n on an interval $[0, T]$. The method computes a generalised Budan-Fourier sequence, denoted $p^{(k)}(t)$, for a real EPT function and a set of associated boundary points. A partition of the interval $[0, T]$ allows the sign-changing zeros of $p^{(k)}(t)$ to be identified by bisection. Each $p^{(k)}(t)$ has a pre-defined set of boundary points (known as the extended set of boundary points) which are dependent on the k^{th} eigenvalue in $\mathbf{A}$. Using all sign-changing zeros of $p^{(k)}(t)$ on $[0, T]$ with the extended set of boundary points for $p^{(k-1)}(t)$, the sign-changing zeros of $p^{(k-1)}(t)$ on $[0, T]$ can be determined. This method is implemented by beginning with $k = n - 1$ and decreasing until $k = 1$. It will be seen that $p^{(0)}(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b}$. The output from the script is a plot of the EPT function identifying any sign-changing zeros on the interval $[0, T]$ and also returning the location of these points. If there are no sign-changing zeros on the interval then a statement indicating so is printed. There are six m-files required to run this algorithm.

A simple interval $[a, b]$ is so-called if there exists at most one sign-changing zero on the interval. A sign-changing zero exists on the simple interval $[a, b]$ if

$$\lim_{\epsilon \to 0} \text{Sign}[f(a + \epsilon)] \times \lim_{\epsilon \to 0} \text{Sign}[f(b - \epsilon)] < 0 \tag{7.1}$$

For programming purposes $\epsilon = K(b - a)$ where K is some small positive number set to default as 10^{-8}. For notation purposes the set of sign-changing zeros of $p^{(k)}(t)$ on $[0, T]$ will be denoted $\mathcal{S}_k$ while the extended set of boundary points

for $p^{(k)}(t)$ will be denoted by $\mathcal{E}_k$. A simple grid, $\mathcal{P}_k$, for $p^{(k)}(t)$ is a partition of the interval $[0, T]$ into simple intervals $[t_0, t_1, ..., t_N]$ where $t_0 = 0$ and $t_N = T$ such that there is at most a single sign-changing zero of $p^{(k)}(t)$ in each simple interval (t_i, t_{i+1}) for all $i = \{0, 1, ..., N-1\}$. It is seen in Hanzon and Holland (2010) that the partition for the simple grid of $p^{(k)}(t)$ is given by the ordered set of boundary points $\mathcal{P}_k = \{\mathcal{S}_{k+1} \cup \mathcal{E}_k\}$.

Zeros_EPT.m

This is the front end m-file of the Budan-Fourier algorithm. The inputs necessary are the $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ minimal realization of the EPT function of McMillan degree n and $T > 0$ is also required to define the interval $[0, T]$ on which to check for sign-changing zeros. The structure of the spectrum of $\mathbf{A}$ is then examined. Depending on the eigenvalues of $\mathbf{A}$, a different sub-routine is called to determine the sign-changing zeros of the EPT function on $[0, T]$. If all eigenvalues of $\mathbf{A}$ are zero implying the function is a polynomial then `Budan_Fourier_P` is called while if the eigenvalues of $\mathbf{A}$ are real, and not all identically zero, `Budan_Fourier_EP` is utilised indicating an exponential-polynomial (EP) function is being examined. If complex eigenvalues are present in the spectrum of $\mathbf{A}$ then `Budan_Fourier_EPT` is called. Each of the Budan Fourier algorithms above return a vector containing the sign-changing zeros of EPT function on $[0, T]$. The EPT function is plotted on $[0, T]$ illustrating any sign-changing zeros on the interval.

Budan_Fourier_P.m

If all eigenvalues of $\mathbf{A}$ are zero such that the EPT function is a polynomial then the function `Budan_Fourier_P` returns the sign-changing zeros of the polynomial function. For the "P", polynomial, case the Budan-Fourier sequence is given by

$$p^{(k)}(t) \;=\; \mathbf{c}\mathbf{A}^k e^{\mathbf{A}t}\mathbf{b} \tag{7.2}$$

The algorithm is a loop for i decreasing from $n-1$ to 0. As $\mathbf{A}$ is nilpotent it is clear that $p^{(n)}(t) \equiv 0$ and $\mathcal{S}_n$ is the empty set. For $i = k$ the simple grid for

$p^{(k)}(t)$ has the partition given by the ordered boundary points $\mathcal{P}_k = \{\mathcal{S}_{k+1} \cup \mathcal{E}_k\}$ where $\mathcal{E}_k = \{0, T\}$ for all $k = \{1, 2, ..., n-1\}$. A nested loop over the elements of $\mathcal{P}_k$ identifies the sign-changing zeros of $p^{(k)}(t)$ denoted $\mathcal{S}_k$. If the simple interval (t_i, t_{i+1}) contains a sign-changing zero then the bisection algorithm `Bisection_EP` is implemented to locate it. The algorithm continues in this manner until the sign-changing zeros of $p^{(0)}(t) = \mathbf{c}e^{\mathbf{A}t}\mathbf{b}$ are identified.

Budan_Fourier_EP.m

If all eigenvalues of $\mathbf{A}$ are real and not all identically zero, `Budan_Fourier_EP` locates the sign-changing zeros of the EP function. The Budan-Fourier sequence for the EP function is found using the Cayley-Hamilton result which is given for all $k = \{0, 1, 2, ..., n\}$ by

$$p^{(k)}(t) = \mathbf{c}(\mathbf{A} - \lambda_k \mathbf{I})(\mathbf{A} - \lambda_{k-1}\mathbf{I})...(\mathbf{A} - \lambda_1 \mathbf{I})e^{\mathbf{A}t}\mathbf{b} \tag{7.3}$$

The algorithm here is identical to that shown for `Budan_Fourier_P` only that the $\mathbf{A}^k$ term of $p^{(k)}(t)$ in Eq. (7.2) is replaced by $(\mathbf{A} - \lambda_k \mathbf{I})(\mathbf{A} - \lambda_{k-1}\mathbf{I})...(\mathbf{A} - \lambda_1 \mathbf{I})$. Similarly $p^{(n)}(t) \equiv 0$.

Budan_Fourier_EPT.m

When the spectrum of $\mathbf{A}$ contains complex elements a more involved method is required to located the sign-changing zeros of the EPT function. The eigenvalues are first ordered such that the complex conjucates come in pairs $\lambda_k = \theta_k + iv_k$, $\lambda_{k+1} = \theta_k - iv_k$ with $v_k > 0$ for all $k = 1, 2, ..., n$. The Budan-Fourier sequence is dependent on the sign of the complex part of λ_k or if λ_k is real. If $\lambda_k \in \mathbb{R}$ (equivalent to $v_k = 0$) or if $\lambda_k = \theta_k - iv_k$ then the Budan-Fourier sequence is given by

$$p^{(k)}(t) = \mathbf{c}(\mathbf{A} - \lambda_k \mathbf{I})(\mathbf{A} - \lambda_{k-1}\mathbf{I})...(\mathbf{A} - \lambda_1 \mathbf{I})e^{\mathbf{A}t}\mathbf{b} \tag{7.4}$$

However, if $\lambda_k = \theta_k + iv_k$, then

$$p^{(k)}(t) \;=\; \mathrm{Im}\Big(e^{-\bar{\lambda_k}t}\mathbf{c}(\mathbf{A}-\lambda_k\mathbf{I})(\mathbf{A}-\lambda_{k-1}\mathbf{I})...(\mathbf{A}-\lambda_1\mathbf{I})e^{\mathbf{A}t}\mathbf{b}\Big) \tag{7.5}$$

The set of extended boundary points $\mathcal{E}_k$ for each iteration, k, depends on whether the eigenvalue λ_k is real or complex.

$$\mathcal{E}_k = \begin{cases} \{0, T\} & \text{if } \lambda_k \in \mathbb{R} \\ \\ \{0, \frac{\pi}{v_k}, \frac{\pi}{v_k}, ..., \lfloor \frac{T}{\pi/v_k} \rfloor \frac{\pi}{v_k}, T\}; & \text{if } \lambda_k \in \mathbb{C} \end{cases} \tag{7.6}$$

where the entier of the real number x is $\lfloor x \rfloor = \max\{m \in \mathbb{N} | m \leq x\}$.

The algorithm is then based on a loop starting from $k = n-1$ and decreasing to $i = 0$ as $p^{(n)}(t) \equiv 0$. At each step $k = \{n-1, n-2, ..., 2, 1\}$ the set of extended boundary points $\mathcal{E}_k$ is computed depending whether λ_k is real or complex as given in Eq. (7.6). The partition of the interval $[0, T]$ which forms a simple grid is then computed as $\mathcal{P}_k = \{\mathcal{S}_{k+1} \cup \mathcal{E}_k\}$ where $\mathcal{S}_{k+1}$ are the sign changing zeros from the previous iteration. $\mathcal{S}_n = \emptyset$ as $p^{(n)}(k) \equiv 0$. A nested loop then runs through the partition points of $[0, T]$ given by the set $\mathcal{P}_k$ and determines all sign-changing zeros of $p^{(k)}(t)$ on $[0, T]$. The bisection algorithm required to locate a sign-changing zero on a simple interval when $\lambda_k = \theta_k + iv_k$ for $v_k > 0$ is `EPT_Bisection` compared with the bisection algorithm `EP_Bisection` which can be applied when $\lambda_k \in \mathbb{R}$ or $\lambda_k = \theta_k - iv_k$.

EP_Bisection.m

The inputs are the triple of the EPT function and also the lower and upper boundaries, "l" and "r" resp., of the simple interval $[l, r]$. The algorithm assumes the function has a single sign-changing zero on the interval $[l, r]$ and locates that zero up a tolerance of 10^{-6}. The zero on the interval is returned. The tolerance, "*tol*", can be changed easily.

EPT_Bisection.m

The inputs are the triple of the EPT function where the spectrum of $\mathbf{A}$ is

complex and also the lower and upper boundaries, "l" and "r" resp., of the simple interval $[l, r]$. The eigenvalue $\lambda_k = \theta_k + iv_k$, with $v_k > 0$, associated with the current iteration of the Budan-Fourier algorithm is also passed as an input. The algorithm assumes there is at most one zero on the interval $[l, r]$ and locates that zero up a tolerance of 10^{-6}. The zero on the interval is returned. The bisection is performed on the EPT function $p^{(k)}(t)$ which is of the form of Eq. (7.5).

7.3.1. Budan-Fourier Algorithm Example

An example locating the sign-changing zeros completes this analysis of the algorithm. The EPT function to be considered is given by

$$f(x) \;=\; \mathbf{c}e^{\mathbf{A}x}\mathbf{b} \;=\; 2 + \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix} + \frac{1}{2}e^{i\pi x} + \frac{1}{2}e^{-i\pi x} - 20e^{-x} \qquad (7.7)$$

The MATLAB code required to input the minimal realization of $f(x)$ can be given by

```
v=[0;i;-i;i*pi;-i*pi;-1];
A=diag(v);
b = [2;1/2;1/2;1/2;1/2;-20];
c = [1 1 1 1 1 1 ];
```

It is clear that the spectrum of $\mathbf{A}$ is complex and given by $\sigma(\mathbf{A}) = (0, i, -i, \pi i, -\pi i, -1)$. We will locate the sign-changing zeros of the EPT function on $[0, 10]$. The Budan-Fourier algorithm is then called using the command

```
Zeros_EPT(A,b,c,10)
```

A screenshot of the written output is given in Figure 7.1 below.

```
>> Zeros_EPT(A,b,c,10)
The "A" matrix has complex Eigenvalues given by

Eigenvalues =

        0
        0 + 1.0000i
        0 - 1.0000i
        0 + 3.1416i
        0 - 3.1416i
  -1.0000

The sign changing zeros of the function are located at

x =

    2.1050    2.2092    3.3854

Elapsed time is 0.970167 seconds.
>>
```

Figure 7.1: *A screenshot of the written output from the Budan-Fourier Algorithm giving the exact locations of the sign-changing zeros on* $[0, 10]$

The plot in Figure 7.2 accompanies the above output illustrating the EPT function given in Eq. (7.7). A snapshot of the function $f(x)$ over the interval $[2.0, 2.3]$ has also been included showing the accuracy of the algorithm.

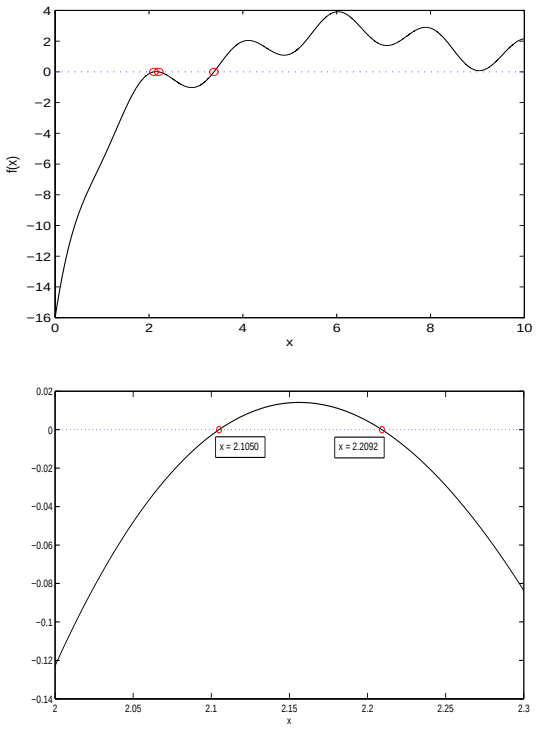

Figure 7.2: *Plot of EPT Function* $f(x)$ *from Eq.* (7.7) *identifying sign-changing zeros. Also a snapshot of* $f(x)$ *on the interval* $[2.0, 2.3]$

7.4. Convex Optimisation Algorithm

The Convex Optimisation Algorithm is based on the method described in Section 3.8. Consider an EPT function $f(x)$ defined on $[0, \infty)$ given by

$$f(x) \;=\; \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}}^{*} + \mu_M e^{\lambda_M x} \;=\; \mathbf{c} e^{\mathbf{A}x} \mathbf{b}^{*} \tag{7.8}$$

for which $f(x) < 0$ for some $x \geq 0$ and $\sigma(\mathbf{A}) \subset \{s | Re(s) < \lambda_M\}$. The objective is to approximate $f(x)$ with a non-negative function $g(x)$ which minimises the L_2 criterion

$$||f(x) - g(x)||_2^2 \tag{7.9}$$

where $g(x)$ is of the form

$$g(x) \;=\; \hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}} + \mu_M e^{\lambda_M x} \;=\; \mathbf{c} e^{\mathbf{A}x} \mathbf{b} \tag{7.10}$$

Hence it is clear from Section 3.8 that Eq. (7.9), with the non-negativity constraint, can be re-written as

$$\min_{\hat{\mathbf{b}} \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})} ||\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} (\hat{\mathbf{b}}^{*} - \hat{\mathbf{b}})||_2^2 \tag{7.11}$$

where the convex set $\mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ is defined in Section 3.6 such that if $\hat{\mathbf{b}} \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ then

$$\hat{\mathbf{c}} e^{\hat{\mathbf{A}}x} \hat{\mathbf{b}} \geq -\mu_M e^{\lambda_M x} \qquad , \qquad x \;\geq\; 0 \tag{7.12}$$

and therefore $g(x)$ is non-negative on $[0, \infty)$. We denote $\hat{\mathbf{b}}_C^{*} \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ as the vector satisfying the criterion in Eq. (7.11). The algorithm consists of ten m-files in total although six of these refer to the Budan-Fourier technique described previously.

EPT_ConvexOP_MinSearch.m

The inputs include the optimal triple $(\hat{\mathbf{A}}, \hat{\mathbf{b}}^{*}, \hat{\mathbf{c}})$ of McMillan degree $(n-1)$, the dominant pole and coefficient, λ_M and μ_M respectively. An initial guess of

$\hat{\mathbf{b}}_0 \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ must also be provided. The code first transforms the $\hat{\mathbf{A}}$ matrix such that $\tilde{\mathbf{A}} = \hat{\mathbf{A}} - \lambda \mathbf{I}$ transforming the non-negativity constraint into

$$\hat{\mathbf{c}} e^{\tilde{\mathbf{A}} x} \hat{\mathbf{b}} \geq -\mu_M \qquad , \qquad x \geq 0$$

The `gram` command is then used to calculate the observability grammian $\mathbf{Q}$ for the pair $(\hat{\mathbf{A}},\ \hat{\mathbf{c}})$. `Finite_T` as described below calculates $T > 0$, such that examining non-negativity of the EPT function with the unique dominant pole on $[0,\ T]$ is sufficient for checking for sign-changing zeros of the EPT function on the real half line $[0, \infty)$.

The convex optimisation procedure is then conducted using the MATLAB direct search algorithm `fminsearch` while `Convex_Optimisation` calculates the value function at each step to be minimised and returns the final $\hat{\mathbf{b}}_C^*$. A plot of the resulting EPT functions is then given.

Convex_Optimisation.m

`Convex_Optimisation` is used in conjunction with `fminsearch` to calculate the value function

$$||\hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} (\hat{\mathbf{b}}^* - \hat{\mathbf{b}}_i)||_2^2 \ = \ (\hat{\mathbf{b}}^* - \hat{\mathbf{b}}_i)^T \mathbf{Q} (\hat{\mathbf{b}}^* - \hat{\mathbf{b}}_i) \tag{7.13}$$

at each step i. Letting $\hat{\mathbf{b}}_{i-1} \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ is a fair assumption as the zero vector $\hat{\mathbf{b}}_0 \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$. `fminsearch` updates the step from $\hat{\mathbf{b}}_{i-1}$ to $\hat{\mathbf{b}}_i$ attempting to minimise the criterion in Eq. (7.13). Using the Budan-Fourier algorithm, `Zeros_EPT_DP`, it is necessary to check if $\hat{\mathbf{b}}_i \in \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ which is equivalent to testing if

$$\hat{\mathbf{c}} e^{\hat{\mathbf{A}} x} \hat{\mathbf{b}} + \mu_M e^{\lambda_M x} \geq 0 \qquad , \qquad \forall\ x \in [0, T] \tag{7.14}$$

If $\hat{\mathbf{b}}_i \notin \mathcal{B}(\hat{\mathbf{A}}, \hat{\mathbf{c}})$ the value function is penalised significantly such that `fminsearch` reverts to $\hat{\mathbf{b}}_{i-1}$ and recalculates a different $\hat{\mathbf{b}}_i$. This procedure continues until

the criterion cannot be minimised any further and $\hat{\mathbf{b}}_C^*$, the global minimum of Eq. (7.11), is returned.

Finite_T.m

The inputs required are the triple $(\hat{\mathbf{A}},\ \hat{\mathbf{b}}^*,\ \hat{\mathbf{c}})$ of the EPT function with its dominant pole and coefficient, λ_M and μ_M. Following the notation and logic in Section 3.7 the values $\lambda_{\min},\ \tilde{\lambda}_{\min}$ and D^2 are calculated yielding $\epsilon = \mu\lambda_{\min}\tilde{\lambda}_{\min}/D^2$. A large T_1 is constructed such that $V(T_1) < \epsilon$ and a bisection technique is used to solve for $T \in [0, T_1]$ such that $V(T) = \epsilon$.

Zeros_EPT_DP.m

The m-file accepts as inputs an EPT function defined using the triple $(\hat{\mathbf{A}}, \hat{\mathbf{b}}, \hat{\mathbf{c}})$ and dominant pole, λ_M, with coefficient μ_M. $T > 0$ is also required an input. The Budan-Fourier algorithm as described in and Hanzon and Holland (2010) is used to test whether the EPT function with dominant real pole has any sign-changing zeros on the finite interval $[0, T]$. This file is very similar to `Zeros_EPT` from the Budan-Fourier algorithm of Section 7.3 except it simply returns 0 if the EPT function is non-negative on $[0, T]$ or 1 if the EPT function has sign-changing zeros on $[0, T]$.

7.5. Additional 2-EPT MATLAB Scripts

A number of additional scripts are now considered which can be used to perform some 2-EPT computations.

7.5.1. Additive Decomposition

The purpose of the `Additivie_Decomposition` code is to decompose a proper rational function ϕ given by the minimal realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ of order n, with poles in either half plane, into the sum of two rational functions, ϕ_P and ϕ_N,

while $\mathbf{d}$ remains unchanged.

$$\begin{aligned}\phi(s) &= \mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}+\mathbf{d}\\ &= \phi_P(s)+\phi_N(s)+\mathbf{d}\\ &= \mathbf{c}_P(s\mathbf{I}-\mathbf{A}_P)^{-1}\mathbf{b}_P+\mathbf{c}_N(s\mathbf{I}-\mathbf{A}_N)^{-1}\mathbf{b}_N+\mathbf{d}\end{aligned}$$

$\phi_N(s)$ is a proper rational function with realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ whose poles are located in the open right plane and $\phi_P(s)$ is a proper rational function with poles located in the open left half plane with realization $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. The `cschur` function within the script computes a unitary similarity transformation of $\mathbf{A}$ which is used as a basis change. The resultant $\mathbf{T} = \mathbf{V}^T\mathbf{A}\mathbf{V}$ matrix is upper triangular with eigenvalues (i.e. diagonal elements) sorted in ascending order according to their real part. The matrix is then transformed to block diagonal form by solving the appropriate sylvester equation using `lyap` function. The basis change is performed on the complete triple, yielding $(\mathbf{V}^T\mathbf{A}\mathbf{V}, \mathbf{V}^T\mathbf{b}, \mathbf{c}\mathbf{V})$. The output is two minimal triples $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ with $\sigma(\mathbf{A}_N) \subset \mathbb{H}_+$ and $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ with $\sigma(\mathbf{A}_P) \subset \mathbb{H}_-$. Although $\mathbf{d}$ can be given as an input it plays no part in the additive decomposition and is returned unchanged.

The code was originally written by Prof. Wolfgang Scherrer but altered to return both minimal triples $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N)$ and $(\mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ which may each have a different McMillan degree.

7.5.2. Convolution

The `Convolution` command requires the realizations, $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{d}_1)$ and $(\mathbf{A}_2, \mathbf{b}_2, \mathbf{c}_2, \mathbf{d}_2)$, of two generalised 2-EPT functions f and g respectively as inputs. The minimal realizations of these realizations are then found using the `minreal` command. The minimal realization $(\mathbf{A}_3, \mathbf{b}_3, \mathbf{c}_3, \mathbf{d}_3)$ of the convolution of the functions f and g is computed using the formulae given in Section 2.2.8 and outputted.

7.5.3. N-Fold Convolution

This function performs an N-Fold convolution of a 2-EPT function with itself. The inputs required are the realization $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ of the 2-EPT function and an integer N indicating the number of times the convolution will be performed. The minimal realization of $(\mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is first calculated. Then a loop from 1 to N uses the `Convolution` function above to perform the N-Fold Convolution. The minimal realization of the resulting function is then returned.

7.5.4. Generalised EPT Realizations of 2-EPT Process Extrema

The function `Process_2EPT_Min` calculates the minimal realization of the generalised EPT probability density function for the minimum of a 2-EPT discrete time process of length T. The inputs required are the 2-EPT realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ for the distribution of the increments of the process. An integer $T > 0$ is also required to specify the length of the 2-EPT process. The recursive algorithm described in Section 5.2 is used to compute the minimum of the process and is given by

$$M(T) \;=\; \min\{0\;,\; X_1 \;+\; \min\{0,\; X_2 \;+\; \min\{0,\; X_3 \;+\; ...\;\}\}\} \tag{7.15}$$

where $M(0) = 0$ and $X_i \sim 2 - EPT(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ for all $i = \{1, 2, ..., T\}$. The first step of the algorithm is to represent the 2-EPT realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$ in a minimal triple $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ as shown in Eqs. (2.6) and (2.7). The probability density function of the minimum of process of length $T = 1$, denoted $M(1)$, is first computed as $(\mathbf{A}_1, \mathbf{b}_1, \mathbf{c}_1, \mathbf{d}_1)$ where $\mathbf{d}_1 = 1 - \mathbf{c}_1\mathbf{A}_1^{-1}\mathbf{b}_1$ and $\mathbf{A}_1 = \mathbf{A}_N,\ \mathbf{b}_1 = \mathbf{b}_N,\ \mathbf{c}_1 = \mathbf{c}_N$. A loop for i from 2 to T is then entered calculating the generalised EPT probability density function $(\mathbf{A}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i)$ for the minimum of the 2-EPT process of length $i = \{2, 3, ..., T\}$.

The recursive formula for the minimum $M(i+1)$ in terms of $M(i)$ is exploited

$$M(i+1) \stackrel{d}{=} \min\{0, \underbrace{X_{i+1} + M(i)}_{Z}\} \tag{7.16}$$

The generalised EPT probability density function for $M(i)$ is given by the realization $(\mathbf{A}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i)$. The 2-EPT probability density function for Z has a realization $(\mathbf{A}_{N_Z}, \mathbf{b}_{N_Z}, \mathbf{c}_{N_Z}, \mathbf{A}_{P_Z}, \mathbf{b}_{P_Z}, \mathbf{c}_{P_Z})$ and is found by convolving the density functions with minimal realizations $(\mathbf{A}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i)$ and $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ together. The 2-EPT density function for Z is non-zero for certain intervals on both halflines $(-\infty, 0)$ and $(0, \infty)$. Applying the $\min\{0, Z\}$ operator is equivalent to replacing density of Z with support on the open right half plane $(0, \infty)$ with a pointmass at zero. Hence

$$\mathbf{d}_{i+1} \;=\; \mathbb{P}[Z > 0] \;=\; 1 - \mathbf{c}_{i+1}\mathbf{A}_{i+1}^{-1}\mathbf{b}_{i+1} \tag{7.17}$$

where $\mathbf{A}_{i+1} = \mathbf{A}_{N_Z}$, $\mathbf{b}_{i+1} = \mathbf{b}_{N_Z}$ and $\mathbf{c}_{i+1} = \mathbf{c}_{N_Z}$. The algorithm continues in this manner until the realization of the EPT density of $M(T)$ given by $(\mathbf{A}_T, \mathbf{b}_T, \mathbf{c}_T, \mathbf{d}_T)$ is returned.

A similar function `Process_2EPT_Max` is also available to calculate the realization of the generalised EPT probability density function for the maximum of a 2-EPT process of length T.

7.5.5. Generating 2-EPT Random Variables

`Generate_2EPT_rv` generates a single 2-EPT random variable whose probability density function is given by the realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$. The script requires the 2-EPT realization as an input and also a uniform $(0, 1)$ random variable, denoted p. The cumulative 2-EPT distribution function is given by

$$F(x) = \begin{cases} \mathbf{c}_N \mathbf{A}_N^{-1} e^{\mathbf{A}_N x} \mathbf{b}_N & \text{if } \mathrm{x} \leq 0 \\ \\ 1 + \mathbf{c}_P \mathbf{A}_P^{-1} e^{\mathbf{A}_P\, x}\, \mathbf{b}_P & \text{if } \mathrm{x} > 0 \end{cases} \tag{7.18}$$

Bisection is used to solve for x_p such that $F(x_p) = p$. There is a separate bisection technique for both of the cases, $p \leq \mathbf{c}_N \mathbf{A}_N^{-1} \mathbf{b}_N$ and $p > \mathbf{c}_N \mathbf{A}_N^{-1} \mathbf{b}_N$. It should be clear that the distribution of x_p is 2-EPT with minimal realization $(\mathbf{A}_N, \mathbf{b}_N, \mathbf{c}_N, \mathbf{A}_P, \mathbf{b}_P, \mathbf{c}_P)$.

A function `Generate_EPT_rv` is also available to generate EPT random variables on $[0, \infty)$.

Chapter 8

Conclusion

8.1. Concluding Remarks

This research seeks to take advantage of the recent advances in computing power which allow for calculations involving matrices of high dimensions to be executed rapidly. It is also illustrated that many results from Systems Theory have extremely worthwhile applications within Financial Modelling. It is hoped that this link can be enhanced with further research.

The thesis introduces the flexible class of 2-EPT Probability Density Functions. On $[0, \infty)$ as well as $(-\infty, 0]$ these probability density functions are non-negative EPT functions, which were discussed in Hanzon and Holland (2010). It is seen that phase-type and matrix exponential distributions are both subsets of the 2-EPT class of distributions. Once the 2-EPT probability density functions are formulated mathematically it is illustrated that the class is closed under a variety of operations thereby demonstrating the benefits of adopting a 2-EPT approach to stochastic modelling. Based on this property, the applications of 2-EPT probability density functions within a financial modelling context are considered.

A natural question to pose, is how a 2-EPT probability density function can

be fitted to an empirically observed distribution. This problem is tackled using the approximation software RARL2. Ensuring the 2-EPT density function is non-negative was the most difficult problem encountered in the approximation procedure. Non-negativity was imposed using a convex optimisation algorithm after an unconstrained approximation. The order of the approximation can be increased until the associated error is sufficiently small up to a certain tolerance.

A discrete time 2-EPT process is the sum of independent and identically distributed 2-EPT random variables. It was proven that the distribution of the minimum and maximum of such a discrete time 2-EPT process have a generalised EPT density which can be computed exactly. Typically, numerical techniques would be required to derive such distributions for these processes.

The Variance Gamma distribution is popular for modelling the log-returns of an asset. It can be seen that the Variance Gamma distribution is infinitely divisible and hence the risk neutral Variance Gamma asset price process can be derived. However, the density is typically specified in terms of a special function which means numerical techniques are required to compute results, including deriving option prices, which involve the density function. It is shown that the Variance Gamma density is a 2-EPT density function under a parameter restriction and under this restriction it can be seen that closed form formulae for European, discretely monitored Lookback and 2-Period Bermudan Options can be derived over certain time frames. Analytic formulae also exist for the associated option Greeks.

Using results from Feller (1971), Widder (1941) and Khintchine (1937) a necessary and sufficient condition was derived to characterise infinitely divisible EPT and 2-EPT functions. It follows that every infinitely divisible 2-EPT distribution generates a unique 2-EPT Lévy process. Assuming an assets log-returns over a fixed period τ can be modelled with an infinitely divisible 2-EPT probability density function, the 2-EPT asset price process can be defined as an exponential 2-EPT Lévy process. As in the unconstrained Variance Gamma

case, closed form formulae do not usually exist for option prices when the log returns of an asset are modelled as a Lévy process. Under these 2-EPT assumptions the risk neutral 2-EPT price process can be defined for the asset and it is possible to derive closed form formulae for European Options (with time to maturity equal to integer multiplies of τ) and their Greeks. Analytic expressions for option prices and their Greeks can also be determined for discretely monitored Lookback Options and the 2-Period Bermudan Option.

A chapter of the thesis is dedicated to describing the software, made available by the author, to assist with 2-EPT calculations. We illustrate that financial modelling using 2-EPT density functions is straightforward to implement in practice. This is demonstrated by fitting a 2-EPT density function to a set of asset returns and using this approximation, various option prices and their Greeks are computed using the closed form 2-EPT formulae derived.

8.2. Further Research Opportunities

Modelling with 2-EPT probability density functions is still in its infancy and there is significant potential for further research based on work completed to date. Here, we only consider some opportunities within the field of mathematical finance but 2-EPT functions may have applications in other disciplines. For example, it was already noted that matrix-exponential and phase-type distributions are subclasses of EPT density functions which immediately leads to applications in other areas. Alternatively, Zhang et al (2005) unknowingly use a subclass of EPT densities to model molecular population distributions in a system of first-order chemical reactions.

It is possible to define bi-variate 2-EPT probability density functions for independent 2-EPT random variables. Using such densities and the results of Sato (1999) relating to discretely monitored processes, it may be possible to price discretely monitored Barrier Options for an asset with a 2-EPT price process.

Letting $M(T)$ be the minimum or maximum of a discretely monitored Lévy process and $S(T)$ be the terminal value of the same process, Sato (1999) proves the independence of the quantities $(M(T) - S(T))$ and $S(T)$. The bivariate distribution of these random variables can then be used to price discretely monitored Barrier Options.

An alternative method to pricing options, when the log returns are modelled as a Lévy process, is to solve the associated Partial Integro-Differential Equation (PIDE). There are numerous methods available in the literature describing how to solve these PIDEs which include multinomial trees, finite difference methods, finite elements and Galerkin schemes. It would be of significant interest to examine these PIDEs and their solutions in the 2-EPT Lévy framework. These techniques also allow for American Option prices to be derived.

A potential pitfall of using 2-EPT densities to model an assets log returns under the physical measure (as done in Section 6.2) is that the 2-EPT density function may not be infinitely divisible. If the 2-EPT density function is not infinitely divisible then it is not be possible to construct the risk neutral measure and therefore, the risk neutral prices of options on the asset can not be derived. Hence, it would be convenient to be able to project onto the set of infinitely divisible 2-EPT probability density functions.

The class of geometrically infinitely divisible random variables was introduced by Klebanov,Maniya and Melamed (1984) and is a subset of the class of infinitely divisible random variables. Using some of the results provided here, including the composition formula, it may be possible to characterise geometrically infinitely divisible 2-EPT/EPT random variables.

Finally translated 2-EPT random variables do not have a 2-EPT probability density function. This property meant that in Chapter 6, Lookback options were priced under the real world measure $\mathbb{P}$ rather than under the risk neutral measure $\mathbb{Q}$. The concept of approximating the translated density with a 2-EPT density (as described in Section 6.5.2) via the RARL2 procedure could

be examined in detail.

Appendix A

Empirical Option Prices

A.1. Option Prices

Table A.1 gives the prices for European Call Options on the Dow Jones Industrial Average as of January 25^{th}, 2013 while the Index value was 13860.65. The options matured on March 8^{th}, 30 business later. The annualized risk free interest rate for the period was 0.2%.

Strike	Price
12250	1633.44
12500	1385.99
12750	1141.18
13000	899.38
13250	665.89
13500	446.69
13750	253.60
14000	96.99
14250	36.33
14500	12.13
14750	3.02
15000	0.58

Table A.1: European Call Option Prices on the DJIA on Jan 25 2013. Options Maturing on March 8 2013. Source Bloomberg.

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