CORA Cork Open Research Archive
Cartlann Taighde Oscailte Chorcaí

Title	Finite type minimal annuli in S2 x R
Authors	Hauswirth, L.;Kilian, Martin;Schmidt, M. U.
Publication date	2013
Original Citation	Hauswirth, L., Kilian, M. and Schmidt, M. U. (2013) 'Finite type minimal annuli in S2 x R', Illinois Journal of Mathematics, 57(3), pp. 697-741. doi: 10.1215/ijm/1415023507
Type of publication	Article (peer-reviewed)
Link to publisher's version	10.1215/ijm/1415023507
Rights	© 2013, University of Illinois.
Download date	2026-11-11 15:08:34
Item downloaded from	https://hdl.handle.net/10468/15093

UCC
University College Cork, Ireland
Coláiste na hOllscoile Corcaigh

Illinois Journal of Mathematics
Volume 57, Number 3, Fall 2013, Pages 697–741
S 0019-2082

FINITE TYPE MINIMAL ANNULI IN $\mathbb{S}^2 \times \mathbb{R}$

L. HAUSWIRTH, M. KILIAN AND M. U. SCHMIDT

Abstract. We study minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ of finite type by relating them to harmonic maps $\mathbb{C} \to \mathbb{S}^2$ of finite type. We rephrase an iteration by Pinkall–Sterling in terms of polynomial Killing fields. We discuss spectral curves, spectral data and the geometry of the isospectral set. We consider polynomial Killing fields with zeroes and the corresponding singular spectral curves, bubbletons and simple factors. We investigate the differentiable structure on the isospectral set of any finite type minimal annulus. We apply the theory to a 2-parameter family of embedded minimal annuli foliated by horizontal circles.

Introduction

In the last decade, there has been an interest in extending minimal surface theory to the target space $\mathbb{S}^2 \times \mathbb{R}$ [26], [22], [23], [12], [15]. A minimal surface that is conformally immersed in $\mathbb{S}^2 \times \mathbb{R}$ is essentially described by a harmonic map $G : \Omega \subset \mathbb{C} \to \mathbb{S}^2$. There is an important subclass of such harmonic maps that have an algebraic description, the so-called harmonic maps of finite type. For example, all constant mean curvature (CMC) tori in $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ have harmonic Gauss maps of finite type [14], [24]. The description of conformally immersed proper minimal annuli of finite type is analogous to the well known theory of constant mean curvature tori of finite type [14], [24], [2], [4], [9], [10], [7], [20]. Thus, periodic minimal immersions $X : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ of finite type can be described by algebraic data. Up to some finite dimensional and compact degrees of freedom, the immersion is determined by the so-called *spectral data* (a, b). This consists of two polynomials of degree $2g$ respectively $g + 1$ for some $g \in \mathbb{N}$. The polynomial $a(\lambda)$ encodes a hyperelliptic Riemann surface called the *spectral curve*. The genus of the spectral curve is called *spectral*

Received March 12, 2013; received in final form April 11, 2013.
L. Hauswirth was partially supported by the ANR-11-IS01-0002 Grant.
2010 *Mathematics Subject Classification.* 53A10.

©2014 University of Illinois

genus. The other polynomial $b(\lambda)$ encodes the extrinsic closing conditions. This correspondence is called the algebro-geometric correspondence. In particular, we characterize those algebraic curves which are the spectral curves of minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$. The spectral data is the starting point for a deformation theory to be used in [13].

Let us briefly outline this paper. In Section 1, we provide a short introduction to the local theory of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and prove a Sym-Bobenko type formula. In Section 2, we discuss polynomial Killing fields, how these are related to the Pinkall–Sterling iteration and define the spectral curve. In Section 3, we invoke the Iwasawa factorization of the underlying loop groups, mention the generalized Weierstrass representation, and briefly discuss the Symes method [28], [5], [6]. In Section 4, we treat the spectral curve, the isospectral action, and prove some properties of the isospectral set. In particular, we discuss singular spectral curves and how to remove singularities with the isospectral action. The isospectral set $I(a)$ of a minimal annulus consists of all minimal annuli with the same spectral data (a, b). We decompose $I(a)$ with the help of a group action into orbits and identify each orbit smoothly with a commutative Lie group. These orbits have different dimensions. The lower dimensional orbits are in the closures of the higher dimensional ones. Our main results are:

(1) The isospectral sets $I(a)$ are compact.
(2) If the $2g$-roots of a are pairwise distinct, then there is one orbit diffeomorphic to
$$I(a) \cong \left(\mathbb{S}^1\right)^g.$$
(3) If a has a double root $\alpha_0 \in \mathbb{S}^1$, then there is a diffeomorphism
$$I(a) \cong I(\tilde{a}) \quad \text{with } a(\lambda) = \bar{\alpha}_0(\lambda - \alpha_0)^2 \tilde{a}(\lambda).$$
(4) If a has a double root $\alpha_0 \notin \mathbb{S}^1$ there are two invariant submanifolds diffeomorphic to
$$I(a) \cong I(\tilde{a}) \cup G \times I(\tilde{a}) \quad \text{with}$$
$$G = \mathbb{C} \text{ or } \mathbb{C}^\times \quad \text{and} \quad a(\lambda) = (\lambda - \alpha_0)^2(1 - \lambda\bar{\alpha}_0)^2 \tilde{a}(\lambda).$$

In Section 5, we turn to periodic finite type harmonic maps, spectral data and how the closing conditions are encoded in the spectral curve. In Section 6, we encounter bubbletons, simple factors and prove a factorization theorem for polynomial Killing fields with zeroes. In particular, we show that changing the line in the simple factor preserves the period, and that for some choices of lines the nodal singularity on the spectral curve disappears. In Section 7, we compute the spectral data of minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ that are foliated by circles. These low spectral genus examples will play an important role in a forthcoming paper [13]. This paper presents a mostly self-contained account of the integrable systems approach to minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ of finite type, but may also serve as a companion for other integrable surface geometries.

1. Minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$

We study conformal minimal immersions $X : \Omega \subset \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ where Ω is a simply connected domain of $\mathbb{C}$. We write $X = (G, h)$ for $G : \Omega \to \mathbb{S}^2$ and $h : \Omega \to \mathbb{R}$, and call G the *horizontal*, and h the *vertical* components of X. If we denote by $(\mathbb{C}, \sigma^2(u)|du|^2)$ the complex plane with the metric induced by the stereographic projection of $\mathbb{S}^2$ ($\sigma^2(u) = 4/(1+|u|^2)^2$), the map $G : \Omega \to \mathbb{C}$ satisfies

$$G_{z\bar{z}} + \frac{2\bar{G}}{1+|G|^2} G_z G_{\bar{z}} = 0.$$

The holomorphic *quadratic Hopf differential* associated to the harmonic map G is given by

$$Q(G) = \big(\sigma \circ G(z)\big)^2 G_z \bar{G}_z (dz)^2 := \phi(z)(dz)^2.$$

The function ϕ depends on z, whereas $Q(G)$ does not.

By conformality the induced metric is of the form $ds^2 = \rho^2(z)|dz|^2$, and writing $z = x + \mathbb{i}\, y$ the partial derivatives satisfy $|X_x|^2 = |X_y|^2$ and $X_x \perp X_y$. Conformality reads

$$|G_x|^2_\sigma + (h_x)^2 = |G_y|^2_\sigma + (h_y)^2 \quad \text{and} \quad \langle G_x, G_y \rangle_\sigma + (h_x)(h_y) = 0,$$

hence $(h_z)^2 (dz)^2 = -Q(G)$. The zeroes of Q are double, and we can define η as the holomorphic 1-form $\eta = \pm 2i\sqrt{Q}$. The sign is chosen so that

$$h = \operatorname{Re} \int \eta.$$

The unit normal vector n in $\mathbb{S}^2 \times \mathbb{R}$ has third coordinate

$$\left\langle n, \frac{\partial}{\partial t} \right\rangle = n_3 = \frac{|g|^2 - 1}{|g|^2 + 1}, \quad \text{where } g^2 := -\frac{G_z}{\overline{G_{\bar{z}}}}.$$

We define the real function $\omega : \mathbb{C} \to \mathbb{R}$ by

$$n_3 := \tanh \omega.$$

We express the differential dG independently of z by

$$dG = G_{\bar{z}}\, d\bar{z} + G_z\, dz = \frac{1}{2\sigma \circ G} \overline{g^{-1}\eta} - \frac{1}{2\sigma \circ G} g\eta,$$

and the metric ds^2 is given in a local coordinate z by

$$ds^2 = \big(|G_z|_\sigma + |G_{\bar{z}}|_\sigma\big)^2 |dz|^2 = \frac{1}{4}\big(|g|^{-1} + |g|\big)^2 |\eta|^2 = 4 \cosh^2 \omega |Q|.$$

We remark that the zeroes of Q correspond to the poles of ω, so that the immersion is well defined. Moreover, the zeroes of Q are points where the tangent plane is horizontal. The Jacobi operator is

$$\mathcal{L} = \frac{1}{4|Q| \cosh^2 \omega} \big(\partial_x^2 + \partial_y^2 + \operatorname{Ric}(n) + |dn|^2\big)$$

and can be expressed in terms of Q and ω by

$$\mathcal{L} = \frac{1}{4|Q|\cosh^2\omega}\left(\partial_x^2 + \partial_y^2 + 4|Q| + \frac{2|\nabla\omega|^2}{\cosh^2\omega}\right). \tag{1.1}$$

Since $n_3 = \tanh\omega$ is a Jacobi field obtained by vertical translation in $\mathbb{S}^2 \times \mathbb{R}$, we have $\mathcal{L}\tanh\omega = 0$ and

$$\Delta\omega + 4|Q|\sinh(\omega)\cosh(\omega) = 0,$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian of the flat metric.

Minimal annuli. Consider a minimal annulus A properly immersed in $\mathbb{S}^2 \times \mathbb{R}$. If A is tangent to a horizontal plane $\{x_3 = t\}$, the set $A \cap \{x_3 = t\}$ defines on A a set of analytic curves with isolated singularities at points where the tangent plane of A is horizontal. Near such a singularity q, there are $2k+2$ smooth branches meeting at equal angles, for some integer $k \geq 1$.

Annuli are transverse to every horizontal plane $\mathbb{S}^2 \times \{t\}$. To see that we claim that $A \setminus \{x_3 = t\}$ defines at least three connected components. At most two of them are non-compact because the immersion is proper. Hence there is a compact disk in A with boundary in $\{x_3 = t\}$, a contradiction to the maximum principle. To prove the claim, consider $A_1, A_2, \ldots, A_{2n}$ distinct local components at q of $A \setminus \{x_3 = t\}$. We know that the A_i alternate between $\{x_3 \geq t\}$ and $\{x_3 \leq t\}$. If A_1 and A_3 are not in the same component of $\{x_3 \geq t\}$, then A_2 yields a third component and $A \setminus \{x_3 = t\}$ has at least three connected components. If A_1 and A_3 are in the same component of $\{x_3 \geq t\}$, we can construct a cycle α_{13} in $\{x_3 \geq t\}$ which meets $\mathbb{S}^2 \times \{t\}$ only at q. We consider α_0 in $\{x_3 > 0\}$ joining a point x of A_1 and y of A_2. Then join x to y by a local path α_1 in A going through q. Let $\alpha_{13} = \alpha_0 \cap \alpha_1$. If A_2 and A_4 were in the same component we could find a cycle α_{24} on A which meets α_{13} in a single point, which is impossible since the genus of A is zero. If A_2 and A_4 are not in the same component of $\{x_3 \leq t\}$, then A_1 yields a third component.

Hence, properly immersed annuli are transverse to horizontal planes and the third coordinate map $h : A \to \mathbb{R}$ is a proper harmonic function on each end of A with $dh \neq 0$. Then each end of A is parabolic and the annulus can be conformally parameterized by $\mathbb{C}/\tau\mathbb{Z}$. We will consider in the following conformal minimal immersions $X : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ with $X(z+\tau) = X(z)$.

Since $dh \neq 0$, the Hopf differential Q has no zeroes. If h^* is the harmonic conjugate of h, we can use the holomorphic map $\mathbb{i}(h + \mathbb{i}h^*) : \mathbb{C} \to \mathbb{C}$ to parameterize the annulus by the conformal parameter $z = x + \mathbb{i}y$. In this parametrization, the period of the annulus is $\tau \in \mathbb{R}$ and

$$X(z) = \big(G(z), y\big) \quad \text{with } X(z+\tau) = X(z).$$

We say that we have parameterized the surface by its *third component*. We remark that $Q = \frac{1}{4}(dz)^2$ and ω satisfies the sinh-Gordon equation

$$\Delta\omega + \sinh(\omega)\cosh(\omega) = 0. \tag{1.2}$$

REMARK 1.1. We relax the condition $\tau \in \mathbb{R}^\times$ to $\tau \in \mathbb{C}^\times$, but we will parameterize our annuli conformally such that $Q = \phi\, dz^2$ is constant, independent of z and $4|\phi| = 1$. This means that the third coordinate is linear.

In summary we have proven the following theorem.

THEOREM 1.2. *A proper minimal annulus is parabolic and $X : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2 \times \mathbb{R}$ has conformal parametrization $X(z) = (G(z), h(z))$ with*

(1) *Harmonic map $G : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{S}^2$, and $h(z) = \operatorname{Re}(-\mathbb{i}\, e^{\mathbb{i}\Theta/2} z)$.*
(2) *Constant Hopf differential $Q = \frac{1}{4}\exp(\mathbb{i}\Theta)\, dz^2$.*
(3) *The metric of the immersion is $ds^2 = \cosh^2(\omega)\, dz \otimes d\bar{z}$.*
(4) *The third coordinate of the unit normal vector is $n_3 = \tanh\omega$.*
(5) *The function $\omega : \mathbb{C}/\tau\mathbb{Z} \to \mathbb{R}$ is a solution of (1.2).*

Sym-Bobenko type formula. We use a description of harmonic maps into the symmetric space $\mathbb{S}^2$ in terms of SU_2-valued frames.

Identify $\mathfrak{su}_2 \cong \mathbb{R}^3, \left(\begin{smallmatrix} \mathbb{i}w & u+\mathbb{i}v \\ -u+\mathbb{i}v & -\mathbb{i}w \end{smallmatrix}\right) \cong (u, v, w)$, so $\mathbb{S}^2 \subset \mathbb{R}^3$ consists of length $\|X\| = \sqrt{\det X} = 1$ elements in $\mathfrak{su}_2$ and $\langle X_1, X_2\rangle = -\frac{1}{2}\operatorname{tr}(X_1 X_2)$. Pick

$$\sigma_3 = \begin{pmatrix} \mathbb{i} & 0 \\ 0 & -\mathbb{i} \end{pmatrix} \in \mathbb{S}^2.$$

Let $\mathbb{T}$ denote the stabiliser of σ_3 under the adjoint action of SU_2 on $\mathfrak{su}_2$, so that $\mathbb{S}^2 \cong \mathrm{SU}_2/\mathbb{T}$. If $\pi : \mathrm{SU}_2 \to \mathrm{SU}_2/\mathbb{T}$ is the coset projection, then a map $F : \Omega \subset \mathbb{C} \to \mathrm{SU}_2$ with $G(z) = \pi \circ F(z)$ is called a *frame* of G, and we have $G(z) = F(z)\sigma_3 F^{-1}(z)$.

Harmonic maps come in $\mathbb{S}^1$-families (associated families), associate to the same solution of the sinh-Gordon equation. Here $\lambda \in \mathbb{S}^1$ parameterizes an associated family of harmonic maps, and the frame of such an associated family is called an *extended frame*.

A method to obtain an extended frame is to write down an integrable 1-form which integrates to an extended frame by solving the integrability condition. We present a result which is similar to results in Bobenko [3]. We next specify for a real solution of the sinh-Gordon equation (1.2) a minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ with a particularly simple vertical component.

THEOREM 1.3. *Let $\omega : \mathbb{C} \to \mathbb{R}$ be a solution of the sinh-Gordon equation. Let $\delta, \gamma \in \mathbb{S}^1$ be arbitrary but fixed, and $F_\lambda(z)$ the solution of $F_\lambda^{-1}\, dF_\lambda = \alpha(\omega), F_\lambda(0) = \mathbb{1}$ where*

$$\alpha(\omega) = \frac{1}{4}\begin{pmatrix} 2\omega_z & \mathbb{i}\lambda^{-1}\delta e^{\omega} \\ \mathbb{i}\gamma e^{-\omega} & -2\omega_z \end{pmatrix} dz + \frac{1}{4}\begin{pmatrix} -2\omega_{\bar{z}} & \mathbb{i}\bar{\gamma} e^{-\omega} \\ \mathbb{i}\lambda\bar{\delta} e^{\omega} & 2\omega_{\bar{z}} \end{pmatrix} d\bar{z}. \tag{1.3}$$

Then the map $X_\lambda(z) = (F_\lambda(z)\sigma_3 F_\lambda(z)^{-1}, \mathrm{Re}(-\mathbb{i}\sqrt{\gamma\delta\lambda^{-1}}z))$ *with* $\lambda \in \mathbb{S}^1$ *defines an associate family of conformal minimal immersions* $X_\lambda : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ *with metric*

$$ds^2 = \cosh^2\omega\, dz \otimes d\bar{z},$$

and Hopf differential

$$Q = \frac{1}{4}\gamma\delta\lambda^{-1}\, dz^2.$$

REMARK 1.4 (Reality condition). For $\lambda \in \mathbb{S}^1$, $\alpha_\lambda = \alpha(\omega)$ in (1.3) takes values in $\mathfrak{su}_2$, so F_λ takes values in SU_2. For general $\lambda \in \mathbb{C}^\times$ we have $F_\lambda \in \mathrm{SL}_2(\mathbb{C})$. From the relation $\overline{\alpha_{1/\bar\lambda}}^t = -\alpha_\lambda$, the solution of $F_\lambda^{-1}\, dF_\lambda = \alpha(\omega)$ satisfies the reality condition

$$\overline{F_{1/\bar\lambda}}^t = F_\lambda^{-1}. \tag{1.4}$$

Proof of Theorem 1.3. Decomposing $\alpha(\omega) = \alpha'\, dz + \alpha''\, d\bar{z}$ into $(1,0)$ and $(0,1)$ parts, we compute

$$\alpha'_{\bar z} = \frac{1}{4}\begin{pmatrix} 2\omega_{z\bar z} & \mathbb{i}\lambda^{-1}\omega_{\bar z}\delta e^{\omega} \\ -\mathbb{i}\gamma\omega_{\bar z}e^{-\omega} & -2\omega_{z\bar z}\end{pmatrix},$$
$$\alpha''_z = \frac{1}{4}\begin{pmatrix} -2\omega_{z\bar z} & -\mathbb{i}\bar\gamma\omega_z e^{-\omega} \\ \mathbb{i}\lambda\omega_z\bar\delta e^{\omega} & 2\omega_{z\bar z}\end{pmatrix}.$$

Using $F_\lambda^{-1}\, dF_\lambda = \alpha$ in the integrability condition $(F_\lambda)_{z\bar z} = (F_\lambda)_{\bar z z}$ gives $\alpha'_{\bar z} - \alpha''_z = [\alpha', \alpha'']$, and a direct computation shows this is equivalent to the sinh-Gordon equation (1.2). Hence, we can integrate $dF_\lambda = F_\lambda\alpha$ to obtain a map $F_\lambda : \mathbb{C} \to \mathrm{SL}_2(\mathbb{C})$ and define $G_\lambda = F_\lambda\sigma_3 F_\lambda^{-1}$. The vertical component is $h = \frac{\mathbb{i}}{2}(-(\gamma\delta)^{1/2}\lambda^{-1/2}z + (\gamma\delta)^{-1/2}\lambda^{1/2}\bar z)$. Its partial derivatives of are $h_z = -\frac{\mathbb{i}}{2}(\gamma\delta)^{1/2}\lambda^{-1/2}$ and $h_{\bar z} = \frac{\mathbb{i}}{2}(\gamma\delta)^{-1/2}\lambda^{1/2}$. Then

$$\big\langle (X_\lambda)_z, (X_\lambda)_z\big\rangle = -\frac{1}{2}\,\mathrm{tr}\big(\big[\alpha', \sigma_3\big]^2\big) + (h_z)^2 = 0,$$
$$\big\langle (X_\lambda)_{\bar z}, (X_\lambda)_{\bar z}\big\rangle = -\frac{1}{2}\,\mathrm{tr}\big(\big[\alpha'', \sigma_3\big]^2\big) + (h_{\bar z})^2 = 0,$$

and the conformal factor computes to

$$\begin{aligned} 2\big\langle (X_\lambda)_z, (X_\lambda)_{\bar z}\big\rangle &= 2\big\langle \big[\alpha', \sigma_3\big], \big[\alpha'', \sigma_3\big]\big\rangle + 2(h_z)(h_{\bar z}) \\ &= \frac{1}{2}\cosh(2\omega) + \frac{1}{2} = \cosh^2(\omega). \end{aligned}$$

As $Q = \langle (G_\lambda)_z, (G_\lambda)_z\rangle (dz)^2 = -\frac{1}{2}\,\mathrm{tr}([\alpha', \sigma_3]^2)(dz)^2 = \frac{1}{4}\gamma\delta\lambda^{-1}(dz)^2$ is holomorphic, we conclude that G is harmonic. □

REMARK 1.5 (Isometric normalisation 1). By conformal parametrization, we can choose $4|\phi| = 1$ (the annulus is transverse to horizontal planes). We have constants $\delta, \gamma \in \mathbb{S}^1$ which are related to the Hopf differential, namely $4Q = \delta\gamma\lambda^{-1}\, dz^2$. We can normalize the parametrization with $\delta = 1$ and a

constant $|\gamma| = 1$. For a given extended frame F_λ which satisfies the equation $F_\lambda^{-1}\, dF_\lambda = \alpha(\omega), F_\lambda(0) = \mathbb{1}$, we consider the $U(1)$-gauge

$$g(\delta) = \begin{pmatrix} \delta^{1/2} & 0 \\ 0 & \delta^{-1/2} \end{pmatrix} \in \mathbb{T}. \tag{1.5}$$

Then $\tilde{F}_\lambda = g(\delta)^{-1} F_\lambda g(\delta)$ induces the immersion $\tilde{X}$ which differs from X by a rotation in $\mathbb{S}^2$. The third coordinate $h = \tilde{h}$ is preserved while $\tilde{G}_\lambda(z) = \tilde{F}_\lambda \sigma_3 \tilde{F}_\lambda^{-1} = g(\delta)^{-1} G_\lambda(z) g(\delta)$ and for $\alpha(\omega)$ as in (1.3) with $\tilde{\delta} = 1$ and $\tilde{\gamma} = \delta\gamma$ have

$$\tilde{F}_\lambda^{-1}\, d\tilde{F}_\lambda = g(\delta)^{-1} \alpha(\omega) g(\delta).$$

2. Polynomial Killing fields and spectral curves

We explain in this section how solutions of the sinh-Gordon equation give rise to polynomial Killing fields as solutions of a Lax equation. Polynomial Killing fields in turn define spectral curves, which are hyperelliptic Riemann surfaces.

If ω is a solution of the sinh-Gordon equation, we consider a deformation $\omega_t = \omega + tu + O(t^2)$. If ω_t is a one parameter family of solutions of the sinh-Gordon equation, then the variational function $u : \mathbb{C} \to \mathbb{R}$ satisfies the *linearized sinh-Gordon equation*

$$\Delta u + u \cosh(2\omega) = 0. \tag{2.1}$$

Definition 2.1. A solution ω of the sinh-Gordon equation is of *finite type* if there exist $g \in \mathbb{N}$ such that

$$\Phi_\lambda(z) = \frac{\lambda^{-1}}{4} \begin{pmatrix} 0 & \mathbb{i} e^{\omega} \\ 0 & 0 \end{pmatrix} + \sum_{n=0}^{g} \lambda^n \begin{pmatrix} u_n(z) & e^{\omega} \tau_n(z) \\ e^{\omega} \sigma_n(z) & -u_n(z) \end{pmatrix} \tag{2.2}$$

is a solution of the Lax equation

$$d\Phi_\lambda = \big[\Phi_\lambda, \alpha(\omega)\big] \tag{2.3}$$

for some functions $u_n, \tau_n, \sigma_n : \mathbb{C} \to \mathbb{C}$, and some $\gamma \in \mathbb{S}^1$, and $\delta = 1$ in $\alpha(\omega)$.

Proposition 2.2. *Suppose Φ_λ is of the form (2.2) for some arbitrary $\omega : \mathbb{C} \to \mathbb{R}$, and that Φ_λ solves the Lax equation (2.3) with $\alpha(\omega)$ as in (1.3), $\delta = 1$ and $|\gamma| = 1$. Then:*

1. *The function ω is a solution of the sinh-Gordon equation (1.2).*
2. *The functions u_n are solutions of the linearized sinh-Gordon equation (2.1).*
3. *The following iteration gives a formal solution of $d\Phi_\lambda = [\Phi_\lambda, \alpha(\omega)]$. Let $u_n, \sigma_n, \tau_{n-1}$, with u_n solution of (2.1) be given. Now solve the system*

$$\tau_{n;\bar{z}} = \frac{1}{2} \mathbb{i} \bar{\gamma} e^{-2\omega} u_n, \qquad \tau_{n;z} = 4\mathbb{i}\bar{\gamma}\omega_z u_{n;z} - 2\mathbb{i}\bar{\gamma} u_{n;zz}$$

for $\tau_{n;z}$ and $\tau_{n;\bar z}$. Then define u_{n+1} and σ_{n+1} by

$$u_{n+1} = -2\mathbb{i}\tau_{n;z} - 4\mathbb{i}\omega_z\tau_n, \qquad \sigma_{n+1} = \gamma e^{2\omega}\tau_n + 4\mathbb{i}\gamma u_{n+1;\bar z}.$$

(4) *Each τ_n is defined up to a complex constant c_n, so u_{n+1} is defined up to $-4\mathbb{i}c_n\omega_z$.*

(5) *$u_0 = \omega_z, u_{g-1} = c\omega_{\bar z}$ for some $c \in \mathbb{C}$, and $\lambda^g\overline{\Phi_{1/\bar\lambda}}^t$ also solves (2.3).*

Proof. Inserting (2.2) into (2.3) and comparing coefficients yields

$$4u_{n;z} + \mathbb{i}e^{2\omega}\sigma_{n+1} - \mathbb{i}\gamma\tau_n = 0, \tag{2.4a}$$

$$4u_{n;\bar z} + \mathbb{i}\bar\gamma\sigma_n - \mathbb{i}e^{2\omega}\tau_{n-1} = 0, \tag{2.4b}$$

$$4\omega_z\tau_n + 2\tau_{n;z} - \mathbb{i}u_{n+1} = 0, \tag{2.4c}$$

$$2e^{\omega}\tau_{n;\bar z} - \mathbb{i}\bar\gamma e^{-\omega}u_n = 0, \tag{2.4d}$$

$$2e^{\omega}\sigma_{n;z} + \mathbb{i}\gamma e^{-\omega}u_n = 0, \tag{2.4e}$$

$$4\omega_{\bar z}\sigma_n + 2\sigma_{n,\bar z} + \mathbb{i}u_{n-1} = 0. \tag{2.4f}$$

(1) Solving (2.4b) for σ_{n+1}, (2.4c) for u_{n+1}, (2.4d) for $\tau_{n;\bar z}$, and inserting these, and $u_{n+1;\bar z}$ and $\tau_{n;\bar z z}$ into (2.4a) gives $e^{2\omega}\mathbb{i}\gamma\tau_n(16\omega_{z\bar z} - e^{-2\omega} + e^{2\omega}) = 0$, which implies (1.2) if $\tau_n \neq 0$.

(2) $\bar\partial$(2.4a) $-\frac{1}{2}\mathbb{i}e^{2\omega}$(2.4f) $+\frac{1}{2}\mathbb{i}\gamma e^{-\omega}$(2.4d) reads

$$4u_{n;z\bar z} + \frac{1}{2}u_n\big(e^{2\omega} + \gamma\bar\gamma e^{-2\omega}\big) = 0.$$

(3) The equation for $\tau_{n;\bar z}$ is (2.4d). Taking the z-derivative of (2.4a) and using (2.4a), (2.4c) and (2.4e) gives $\tau_{n;z} = 4\mathbb{i}\bar\gamma\omega_z u_{n;z} - 2\mathbb{i}\bar\gamma u_{n;zz}$. The equations for u_{n+1}, σ_{n+1} are given by (2.4c) respectively (2.4b).

(4) In the iteration (3), the function τ_n is determined up to an integration constant. This gives an additional term ω_z in u_{n+1}.

(5) Left to the reader. □

Pinkall–Sterling [24] constructed a series of special solutions of the induction of Proposition 2.2(3) via an auxiliary function ϕ as follows: For a given solution u_n of the linearized sinh-Gordon equation (2.1), consider the function $\phi : \mathbb{C} \to \mathbb{C}$ defined by

$$\phi_{n;z} = 4\omega_z u_{n;z}, \qquad \phi_{n;\bar z} = -u_n \sinh\omega\cosh\omega. \tag{2.5}$$

Then $\tau_n = 2\mathbb{i}\bar\gamma(\frac{1}{2}\phi_n - u_{n;z})$ and $u_{n+1} := (u_n)_{zz} - \omega_z\phi_n$. This defines a hierarchy of solutions of (2.1). Applying this iteration to the trivial solution $u_0 \equiv 0$ yields the sequence, whose first four terms are

$$\begin{aligned} u_0 &= 0,\\ u_1 &= \omega_z,\\ u_2 &= \omega_{zzz} - 2\omega_z^3,\\ u_3 &= \omega_{zzzzz} - 10\omega_{zzz}\omega_z^3 - 10\omega_{zz}^2\omega_z + 6\omega_z^5. \end{aligned}$$

This infinite sequence produces solutions of the linearized sinh-Gordon equation on $\mathbb{C}$. These come from the iteration (2.5), and Pinkall–Sterling prove in Proposition 3.1 [24], that ϕ_n depends only on ω and its kth derivatives with $k \leq 2n+1$. The fact that we consider on $\mathbb{C}$ a uniformly bounded solution of the sinh-Gordon equation $\omega : \mathbb{C} \to \mathbb{R}$, implies by Schauder estimates that each u_n is uniformly bounded on $\mathbb{C}$.

PROPOSITION 2.3. *A proper minimal annulus A immersed in $\mathbb{S}^2 \times \mathbb{R}$ with bounded curvature and linear area growth has a metric $ds^2 = \cosh^2 \omega |ds|^2$, where $\omega : \mathbb{C}/\tau\mathbb{C} \to \mathbb{R}$ is a finite type solution of the sinh-Gordon equation. We say that the annulus is of finite type.*

Proof. A first step is to prove that the function $\omega : \mathbb{C}/\tau\mathbb{C} \to \mathbb{R}$ is uniformly bounded. Consider a sequence of points p_n in A such that $\omega(p_n)$ is diverging to infinity and consider a sequence of translations $t_n e_3$ such that $A + t_n e_3$ is a sequence of annuli with $p_n + t_n e_3$ points of $\mathbb{S}^2 \times \{0\}$. Then by the bounded curvature hypothesis there is a sub-sequence converging locally to A_0, a properly immersed minimal surface. The linear area growth assumption assure that A_0 is an annulus. But our hypothesis leads to a pole occurring at the height $t = 0$ since $|\omega| \to \infty$. The limit normal vector $n_3(p_n) = \tanh \omega_n(p_n) \to \pm 1$ and the annulus A_0 would be tangent to the height $\mathbb{S}^2 \times \{0\}$, a contradiction to the maximum principle. Thus

$$\sup_{z \in A} |\omega| \leq C_0.$$

Now we apply Schauder estimates to the sinh-Gordon equation to obtain a $\mathcal{C}^{k,\alpha}$ estimate on the solution of the sinh-Gordon equation on $\mathbb{C}/\tau\mathbb{Z}$. There exists a constant $C_0 > 0$ such that for any $k \in \mathbb{N}$

$$|\omega|_{A,k,\alpha} \leq C_0.$$

Meeks–Perez–Ros [21] provide us with the following theorem.

THEOREM 2.4 ([21]). *An elliptic operator $Lu = \Delta u + qu$ on a cylinder $\mathbb{S}^1 \times \mathbb{R}$ has for bounded and continuous q a finite dimensional kernel on the space of uniformly bounded $\mathcal{C}^2$-functions on $\mathbb{S}^1 \times \mathbb{R}$.*

Since solutions $u_0, u_1, u_2, \ldots$ of the linearized sinh-Gordon equation are solutions depending only on ω and its higher derivatives, this family is a finite dimensional family by Theorem 2.4. Thus, there is a $g \in \mathbb{N}$, and there exist $a_i \in \mathbb{C}$ such that

$$\sum_{i=1}^{g} a_i u_i = 0.$$

This algebraic relation implies that ω is of finite type, and ensures the existence of a polynomial Killing field Φ_λ of degree g. To achieve that, one has to prescribe the right constants $c_0, c_1, \ldots, c_g$ in the iteration procedure, and set the $g+1$-coefficient to zero. □

Potentials and polynomial Killing fields. To parameterize real solutions of the sinh-Gordon equation, we make the following.

DEFINITION 2.5. The set of potentials is

$$\mathcal{P}_g = \left\{ \xi_\lambda = \sum_{d=-1}^{g} \hat{\xi}_d \lambda^d \;\middle|\; \hat{\xi}_{-1} \in \begin{pmatrix} 0 & \mathbb{i}\mathbb{R}^+ \\ 0 & 0 \end{pmatrix}, \operatorname{tr}(\hat{\xi}_{-1}\hat{\xi}_0) \neq 0, \right.$$
$$\left. \hat{\xi}_d = -\overline{\hat{\xi}_{g-1-d}}^{\,t} \in \mathfrak{sl}_2(\mathbb{C}) \text{ for } d = -1, \dots, g \right\}.$$

REMARK 2.6 (Isometric normalization 2). For $\delta \in \mathbb{S}^1$, we denote by $\mathcal{P}_g(\delta)$, potentials with residues

$$\hat{\xi}_{-1} \in \begin{pmatrix} 0 & \mathbb{i}\delta\mathbb{R}^+ \\ 0 & 0 \end{pmatrix}.$$

These correspond to the normalization of Remark 1.4 and there is an isomorphism $\mathcal{P}_g(\delta) \to \mathcal{P}_g$, given by

$$\xi_\lambda \mapsto g(\delta)^{-1} \xi_\lambda g(\delta).$$

Each $\xi_\lambda \in \mathcal{P}_g$ satisfies the *reality condition*

$$\lambda^{g-1} \overline{\xi_{1/\bar{\lambda}}}^{\,t} = -\xi_\lambda. \tag{2.6}$$

In other words, for $\xi_\lambda \in \mathcal{P}_g$, we have a map $\mathbb{S}^1 \to \mathfrak{su}_2, \lambda \mapsto \lambda^{\frac{1-g}{2}} \xi_\lambda$. The polynomial

$$a(\lambda) := -\lambda \det \xi_\lambda$$

then satisfies the reality condition

$$\lambda^{2g} \overline{a(1/\bar{\lambda})} = a(\lambda). \tag{2.7}$$

On $\mathfrak{su}_2$ the determinant is the square of a norm, thus we have for $\lambda \in \mathbb{S}^1$ that

$$\lambda^{-g} a(\lambda) \leq 0 \quad \text{for } \lambda \in \mathbb{S}^1. \tag{2.8}$$

When g is even, $\hat{\xi}_0, \dots, \hat{\xi}_{\frac{g}{2}-1}$ are independent 2×2 traceless complex matrices. For odd g, $\hat{\xi}_0, \dots, \hat{\xi}_{\frac{g-3}{2}}$ are independent 2×2 traceless complex matrices and $\hat{\xi}_{\frac{g-1}{2}} \in \mathfrak{su}_2$. Thus the space of potentials $\mathcal{P}_g$ of real finite type solutions of the sinh-Gordon equation is an open subset of a $3g+1$ dimensional real vector space. The condition $\operatorname{tr}(\hat{\xi}_{-1}\hat{\xi}_0) \neq 0$ implies that $a(0) \neq 0$ and by symmetry the highest coefficient of a is therefore non-zero. Thus, $\lambda \mapsto a(\lambda)$ is a polynomial of degree $2g$ with complex coefficients, and we denote such by $\mathbb{C}^{2g}[\lambda]$. Define

$$\begin{aligned} \mathcal{M}_g &= \left\{ a \in \mathbb{C}^{2g}[\lambda] \mid a(\lambda) = -\lambda \det \xi_\lambda \text{ with } \xi_\lambda \in \mathcal{P}_g \right\} \\ &= \left\{ a \in \mathbb{C}^{2g}[\lambda] \mid a(0) \neq 0, \lambda^{2g} \overline{a(1/\bar{\lambda})} = a(\lambda) \text{ and } \right. \\ &\qquad \left. \lambda^{-g} a(\lambda) \leq 0 \text{ for } \lambda \in \mathbb{S}^1 \right\}, \\ \mathcal{M}_g^0 &= \left\{ a \in \mathcal{M}_g \mid \lambda^{-g} a(\lambda) < 0 \text{ for } |\lambda| = 1 \right\}. \end{aligned} \tag{2.9}$$

Thus, $\mathcal{M}_g^0$ is an open subset of the $2g+1$ dimensional real vector space $\mathcal{M}_g$.

DEFINITION 2.7. Polynomial Killing fields are maps $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g$ (see Definition 2.5) which solve the Lax equation

$$d\zeta_\lambda = \big[\zeta_\lambda, \alpha(\omega)\big] \quad \text{with } \zeta_\lambda(0) = \xi_\lambda \in \mathcal{P}_g.$$

We use solutions Φ_λ of the Lax equation to construct polynomial Killing related to a finite type solution $\omega : \mathbb{C} \to \mathbb{R}$ of the sinh-Gordon equation.

LEMMA 2.8. *For a solution Φ_λ of (2.3) with $\delta = 1$ in $\alpha(\omega)$ (see formula (1.3)), there exists constants $\gamma \in \mathbb{S}^1$ and $k \in \mathbb{R}^+$ such that*

$$\zeta_\lambda(z) = k\Phi_\lambda(z) - k\lambda^{g-1}\overline{\Phi_{1/\bar\lambda}(z)}^t \quad \text{and} \quad \zeta_\lambda(0) = \xi_\lambda$$

is a polynomial Killing field.

Proof. The map ζ_λ satisfies the reality condition (2.6) so that

$$\lambda^{g-1}\overline{\zeta_{1/\bar\lambda}(z)}^t = -\zeta_\lambda(z).$$

It remains to prove that the residues $\hat\zeta_{-1}$ and $\hat\xi_{-1}$ are upper triangular with purely imaginary non-zero coefficient and $\operatorname{tr}(\hat\xi_{-1}\hat\xi_0) \neq 0$. We use the following.

REMARK 2.9 (Isometric normalization 3). We write $\alpha_{\lambda,\delta,\gamma}(\omega) := \alpha(\omega)$ for the 1-form (1.3) and $\Phi_{\lambda,\delta,\gamma}(z) := \Phi_\lambda(z)$ the associate solution (2.2) of the Lax equation (2.3). We use the unitary matrix $g(\delta)$ defined in (1.5). Now $\Phi_{\lambda\delta,1,\gamma}$ solves (2.3) with $\alpha_{\lambda\delta,1,\gamma}(\omega) = \alpha_{\lambda,\delta^{-1},\gamma}(\omega)$, so $g(\delta)\Phi_{\lambda\delta,1,\gamma}g(\delta)^{-1}$ solves (2.3) with $\alpha(\omega) = g(\delta)\alpha_{\lambda,\delta^{-1},\gamma}g(\delta)^{-1} = \alpha_{\lambda,1,\delta^{-1}\gamma}$. We conclude that

$$\Phi_{\lambda,1,\delta^{-1}\gamma} = g(\delta)\Phi_{\lambda\delta,1,\gamma}g(\delta)^{-1}.$$

In particular, if $\xi_\lambda \in \mathcal{P}_g(\delta)$, then a solution of the Lax equation of Definition 2.7 satisfies $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g(\delta)$. This can be deduced from Proposition 2.2, where $\tau_{-1}(z) = \tau_{-1}(0)$.

Remark 2.9 proves that changing γ by $\delta^{-1}\gamma$ changes the highest coefficient of Φ_λ by

$$\sigma_g\big(\delta^{-1}\gamma\big) = \delta^{g-1}\sigma_g(\gamma).$$

We compute the residue $\hat\xi_{-1}$ at $z=0$ with $\tilde\gamma = \delta^{-1}\gamma$, and choose a unimodular number δ, such that

$$\frac{1}{4}\mathbb{i}e^{\omega} - \bar\sigma_g(\tilde\gamma)e^{\omega} = \frac{1}{4}\mathbb{i}e^{\omega} - \delta^{1-g}\bar\sigma_g(\gamma)e^{\omega} \in \mathbb{i}\mathbb{R}^+.$$

We choose k to normalize the residue with $k^{-1} = 1 + 4\mathbb{i}\delta^{1-g}\bar\sigma_g(\gamma) \in \mathbb{R}^+$. There remains to compute with $\sigma_0 = \mathbb{i}\gamma e^{-2\omega}/4$,

$$\operatorname{tr}(\hat\xi_{-1}\hat\xi_0) = \left(\frac{1}{4}\mathbb{i}e^{\omega}\right)\left(\frac{1}{4}\mathbb{i}\gamma e^{-\omega}\right) = -\gamma/16 \neq 0.$$

Since $\xi_\lambda \in \mathcal{P}_g$, the Lax equation assures that $\zeta_\lambda(z) \in \mathcal{P}_g$ (see Remark 3.3). □

Spectral curve. Suppose $\xi_\lambda \in \mathcal{P}_g$ and ζ_λ is the polynomial Killing field with $\zeta_\lambda(0) = \xi_\lambda$. Suppose further that the polynomial $a(\lambda) = -\lambda \det \xi_\lambda$ has $2g$-pairwise distinct roots. Define

$$\begin{aligned}\Sigma^* &= \big\{(\nu,\lambda) \in \mathbb{C}^2 \mid \det(\nu \mathbb{1} - \zeta_\lambda) = 0\big\} \\ &= \big\{(\nu,\lambda) \in \mathbb{C}^2 \mid \nu^2 = -\det \xi_\lambda = \lambda^{-1} a(\lambda)\big\}.\end{aligned} \tag{2.10}$$

By construction, we have a map $\lambda : \Sigma^* \to \mathbb{C}^\times$ of degree 2, which is branched at the $2g$ simple roots of the polynomial a. By declaring the points over $\lambda = 0, \infty$ to be two further branch points, we then have $2g+2$ branch points. This 2-point compactification Σ is called the *spectral curve* of the polynomial Killing field ζ_λ.

The Riemann–Hurwitz formula gives that the spectral curve Σ is a hyperelliptic Riemann surface of genus g, and its genus is called the *spectral genus*. It has three involutions

$$\begin{aligned}\sigma &: (\lambda,\nu) \mapsto (\lambda,-\nu), \\ \varrho &: (\lambda,\nu) \mapsto \big(\bar\lambda^{-1}, -\bar\lambda^{1-g}\bar\nu\big), \\ \eta &: (\lambda,\nu) \mapsto \big(\bar\lambda^{-1}, \bar\lambda^{1-g}\bar\nu\big).\end{aligned} \tag{2.11}$$

The involution σ is called the hyperelliptic involution. Note that η has no fixed points ($a(1) \in \mathbb{R}^-$) and ϱ fixes $\mathbb{S}^1$ pointwise. In particular, roots of a are symmetric with respect to inversion across the unit circle so that $a(\alpha_i) = 0 \Leftrightarrow a(1/\bar\alpha_i) = 0$.

3. Harmonic maps and Weierstrass representation

The generalized Weierstrass representation [8] gives a correspondence between harmonic maps, extended frames and potentials. To formulate the generalized Weierstrass representation for harmonic maps into $\mathbb{S}^2$ we need various loop groups and a loop group factorization.

For real $r \in (0,1]$, denote the circle $\mathbb{S}_r = \{\lambda \in \mathbb{C} \mid |\lambda| = r\}$, the disk $I_r = \{\lambda \in \mathbb{C} \mid |\lambda| < r\}$ and the annulus $A_r = \{\lambda \in \mathbb{C} \mid r < |\lambda| < 1/r\}$. The loop group of $\mathrm{SL}_2(\mathbb{C})$ is the infinite dimensional Lie group $\Lambda_r \mathrm{SL}_2(\mathbb{C}) = \mathcal{O}(\mathbb{S}_r, \mathrm{SL}_2(\mathbb{C}))$ of analytic maps $\mathbb{S}_r \to \mathrm{SL}_2(\mathbb{C})$.

We need two subgroups. The first is

$$\Lambda_r \mathrm{SU}_2 = \big\{F_\lambda \in \mathcal{O}(A_r, \mathrm{SL}_2(\mathbb{C}) \mid F_{\lambda\in\mathbb{S}^1} \in \mathrm{SU}_2\big\}.$$

Thus

$$F_\lambda \in \Lambda_r \mathrm{SU}_2 \quad \Longleftrightarrow \quad \overline{F_{1/\bar\lambda}}^t = F_\lambda^{-1}. \tag{3.1}$$

The second subgroup that participates in the Iwasawa decomposition is

$$\Lambda_r^+ \mathrm{SL}_2(\mathbb{C}) = \left\{ B_\lambda \in \mathcal{O}\big(I_r \cup \mathbb{S}_r, \mathrm{SL}_2(\mathbb{C})\big) \;\middle|\; B_0 = \begin{pmatrix} \rho & c \\ 0 & 1/\rho \end{pmatrix} \text{ for } \rho \in \mathbb{R}^+ \text{ and } c \in \mathbb{C} \right\}.$$

The normalization that B_0 is upper-triangular with real diagonals ensures that

$$\Lambda_r \mathrm{SU}_2 \cap \Lambda_r^+ \mathrm{SL}_2(\mathbb{C}) = \{\mathbb{1}\}.$$

The following important result is due to Pressley–Segal [25], and generalized by McIntosh [19].

THEOREM 3.1. *Multiplication* $\Lambda_r \mathrm{SU}_2 \times \Lambda_r^+ \mathrm{SL}_2(\mathbb{C}) \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ *is a real analytic diffeomorphism onto. The unique splitting of* $\phi_\lambda \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$ *into* $\phi_\lambda = F_\lambda B_\lambda$ *with* $F_\lambda \in \Lambda_r \mathrm{SU}_2$ *and* $B_\lambda \in \Lambda_r^+ \mathrm{SL}_2(\mathbb{C})$ *is called the* r-Iwasawa decomposition *of* ϕ_λ *or just* Iwasawa decomposition *when* $r = 1$.

Before specializing to the finite type theory of harmonic maps $G : \mathbb{C} \to \mathbb{S}^2$, let us briefly recall the generalized Weierstrass representation [8]. Set

$$\Lambda_{-1}^\infty \mathfrak{sl}_2(\mathbb{C}) = \left\{ \xi_\lambda \in \mathcal{O}\big(\mathbb{C}^\times, \mathfrak{sl}_2(\mathbb{C})\big) \;\middle|\; (\lambda \xi_\lambda)|_{\lambda=0} \in \begin{pmatrix} 0 & \mathbb{C}^\times \\ 0 & 0 \end{pmatrix} \right\}.$$

A *potential* is a holomorphic 1-form on $\mathbb{C}$ with values in $\Lambda_{-1}^\infty \mathfrak{sl}_2(\mathbb{C})$. Suppose that we have such a potential $\xi_\lambda\, dz$ with $\xi_\lambda \in \Lambda_{-1}^\infty \mathfrak{sl}_2(\mathbb{C})$. To obtain a corresponding extended frame F_λ is a two step procedure:

(1) Solve the holomorphic ODE $d\phi_\lambda = \phi_\lambda \xi_\lambda$ to obtain a map $\mathbb{C} \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$, $z \mapsto \phi_\lambda(z)$.
(2) The r-Iwasawa factorization $\phi_\lambda(z) = F_\lambda(z) B_\lambda(z)$ at each $z \in \mathbb{C}$ gives an extended frame $\mathbb{C} \to \Lambda_r \mathrm{SU}_2, z \mapsto F_\lambda(z)$.

Note that while ϕ_λ is holomorphic in $z \in \mathbb{C}$, the resulting extended frame is not, since it also depends on $\bar{z}$ by the reality condition (1.4). It is proven in [8] that each extended frame can be obtained from a potential ξ_λ by the Iwasawa decomposition. Hence for any conformal minimal immersion $X = (G, h) : \mathbb{C} \to \mathbb{S}^2 \times \mathbb{R}$ there is a potential and corresponding extended frame which frames G.

An extended frame is of (semi-simple) *finite type*, if there exists a $g \in \mathbb{N}$, and it has a corresponding potential $\xi_\lambda\, dz$ with $\xi_\lambda \in \mathcal{P}_g \subset \Lambda_{-1}^\infty \mathfrak{sl}_2(\mathbb{C})$. Hence harmonic maps of finite type come from constant (1, 0)-forms with values in the finite dimensional space $\mathcal{P}_g$, and thus have an algebraic description. In the finite type case, the first step in the above two step procedure is explicit, since then $\phi_\lambda = \exp(z\xi_\lambda)$. Thus, extended frames of finite type are obtained by factorizing $\exp(z\xi_\lambda) = F_\lambda B_\lambda$ with $\xi_\lambda \in \mathcal{P}_g$. This step can be made explicit in terms of theta functions on the spectral curve (see Bobenko [2]).

Expanding a polynomial Killing field $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g$ as

$$\zeta_\lambda(z) = \begin{pmatrix} 0 & \beta_{-1}(z) \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \alpha_0(z) & \beta_0(z) \\ \gamma_0(z) & -\alpha_0(z) \end{pmatrix} + \cdots + \begin{pmatrix} \alpha_g(z) & \beta_g(z) \\ \gamma_g(z) & -\alpha_g(z) \end{pmatrix} \lambda^g \tag{3.2}$$

we associate a matrix 1-form defined by

$$\alpha(\zeta_\lambda) = \begin{pmatrix} \alpha_0(z) & \beta_{-1}(z)\lambda^{-1} \\ \gamma_0(z) & -\alpha_0(z) \end{pmatrix} dz - \begin{pmatrix} \overline{\alpha_0(z)} & \overline{\gamma_0(z)} \\ \overline{\beta_{-1}(z)}\lambda & -\overline{\alpha_0(z)} \end{pmatrix} d\bar{z}. \tag{3.3}$$

The following proposition is well known, and the correspondence between potentials, polynomial Killing fields and extended frames is known as 'Symes method' [5], [6], [28].

PROPOSITION 3.2. *For each $\xi_\lambda \in \mathcal{P}_g$ there is a unique polynomial Killing field $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g$ solving $d\zeta_\lambda = [\zeta_\lambda, \alpha(\zeta_\lambda)]$ with $\zeta_\lambda(0) = \xi_\lambda$. The unitary factor $F_\lambda : \mathbb{C} \to \Lambda\mathrm{SU}_2$ of the Iwasawa decomposition $\exp(z\xi_\lambda) = F_\lambda B_\lambda$ is a solution of $F_\lambda^{-1}\, dF_\lambda = \alpha(\zeta_\lambda)$ with initial value $F_\lambda(0) = \mathbb{1}$ and $\zeta_\lambda(z) = B_\lambda(z)\xi_\lambda B_\lambda^{-1}(z) = F_\lambda^{-1}(z)\xi_\lambda F_\lambda(z)$.*

Proof. Clearly $\zeta_\lambda = B_\lambda(z)\xi_\lambda B_\lambda^{-1}(z) = F_\lambda^{-1}(z)\xi_\lambda F_\lambda(z)$ uniquely solves $d\zeta_\lambda = [\zeta_\lambda, F_\lambda^{-1}\, dF_\lambda]$ with $\zeta_\lambda(0) = \xi_\lambda$, so it remains to show $F_\lambda^{-1}\, dF_\lambda = \alpha(\zeta_\lambda)$.

Now $F_\lambda^{-1}\, dF_\lambda = B_\lambda\xi_\lambda B_\lambda^{-1} - dB_\lambda B_\lambda^{-1}$, so by the reality condition (1.4) we have $F_\lambda^{-1}\, dF_\lambda = \mathfrak{a}_{-1}\lambda^{-1} + \mathfrak{a}_0 + \mathfrak{a}_1\lambda$. If we decompose $\mathfrak{a}_j = \mathfrak{a}_j'\, dz + \mathfrak{a}_j''\, d\bar{z}$ into (1, 0) and (0, 1) parts, then we have in addition that

$$\mathfrak{a}_0'' = -\overline{\mathfrak{a}_0'}^t, \qquad \mathfrak{a}_1'' = -\overline{\mathfrak{a}_{-1}'}^t, \qquad \mathfrak{a}_{-1}'' = -\overline{\mathfrak{a}_1'}^t.$$

Now $F_\lambda^{-1}(F_\lambda)_z = \zeta_\lambda - (B_\lambda)_z B_\lambda^{-1}$, and $(B_\lambda)_z B_\lambda^{-1}$ is holomorphic at $\lambda = 0$. Hence $\mathfrak{a}_{-1}' = \hat{\zeta}_{-1}$. Further, $F_\lambda^{-1}(F_\lambda)_{\bar{z}} = -(B_\lambda)_{\bar{z}} B_\lambda^{-1}$ implies $\mathfrak{a}_{-1}'' = 0$. It remains to determine $\mathfrak{a}_0$.

Expand $B_\lambda = \hat{B}_0 + \hat{B}_1\lambda + \cdots$. Now $\mathfrak{a}_0' = \zeta_0 - (\hat{B}_0)_z \hat{B}_0^{-1}$. Since $\hat{B}_0$ is upper-triangular, then so is $(\hat{B}_0)_z \hat{B}_0^{-1}$. Hence the lower-diagonal term of $\mathfrak{a}_0'$ is γ_0, and the upper-diagonal term of $\mathfrak{a}_0''$ is $-\bar{\gamma}_0$. Also $\mathfrak{a}_0'' = -(\hat{B}_0)_{\bar{z}} \hat{B}_0^{-1}$ is upper-triangular, so the lower-diagonal entry of $\mathfrak{a}_0''$ is zero, and consequently also the upper-diagonal entry of $\mathfrak{a}_0'$ is zero.

Finally, writing $B(0) = \hat{B}_0 = \left(\begin{smallmatrix} \rho & c \\ 0 & 1/\rho \end{smallmatrix}\right)$, and $\mathfrak{a}_0' = \left(\begin{smallmatrix} u & v \\ w & -u \end{smallmatrix}\right)$, then $\bar{u} = \rho^{-1}\rho_{\bar{z}}$ and $u = \alpha_0 - \rho^{-1}\rho_z$. These two equations, and since ρ is real analytic, give $2u = \alpha_0$. □

REMARK 3.3. With initial data $\beta_{-1}(0) \in \mathbb{i}\mathbb{R}^+$, the Lax equation gives $\beta_{-1}(z) \in \mathbb{i}\mathbb{R}^+$. For a given potential ξ_λ and polynomial Killing field $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g$ we define $\omega : \mathbb{C} \to \mathbb{R}$ by setting $4\beta_{-1} := \mathbb{i}e^\omega$. The iteration implies that $2\alpha_0 = \omega_z$. To express γ_0 in terms of α_0 and β_{-1}, we consider the Lax equation and find $(\gamma_0)_{\bar{z}} = -2\bar{\alpha}_0\gamma_0$. Then $\gamma_0 = qe^{-\omega}$ where q is a holomorphic function. The term q is constant. The reason is that along the parameter z, we have $a(\lambda) = -\lambda \det \zeta_\lambda(z) = -\lambda \det \xi_\lambda$ and $a(0) = \beta_{-1}\gamma_0 = q/4$. Coefficients of ζ_λ depend only on higher derivatives of ω point wise in z, and $\alpha(\zeta_\lambda) = \alpha(\omega)$.

REMARK 3.4 (Isometric normalization 4). With initial data $\beta_{-1}(0) \in \mathbb{i}\delta\mathbb{R}^+$ (i.e., $\xi_\lambda \in \mathcal{P}_g(\delta)$), the Iwasawa decomposition gives a solution of the Lax equation $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g(\delta)$ given by $\zeta_\lambda(z) = F_\lambda^{-1}(z)\xi_\lambda F_\lambda(z)$.

4. Isospectral set

The set $I(a)$ consists of all initial conditions ξ_λ which give rise to the same spectral curve Σ and the same off-diagonal product $a(0) = \beta_{-1}\gamma_0$.

DEFINITION 4.1. Define for polynomial Killing field $\zeta_\lambda : \mathbb{C} \to \mathcal{P}_g(\delta)$ as in (3.2), and $a(\lambda) = -\lambda \det \zeta_\lambda$ the set

$$I_\delta(a) := \left\{ \xi_\lambda \in \mathcal{P}_g(\delta) \;\middle|\; \lambda \det \xi_\lambda = -a(\lambda) \text{ and } \beta_{-1}\gamma_0 = a(0) = -\frac{1}{16} e^{\mathbb{i}(1-g)\theta} := -\frac{1}{16} e^{\mathbb{i}\Theta} \right\}.$$

When $\delta = 1$, we write $I(a)$. The set $I(a)$ is called the *isospectral set* of the polynomial Killing field ζ_λ.

We next define the *isospectral action* $\pi : \mathbb{C}^g \times I(a) \to I(a)$.

DEFINITION 4.2. Let $\xi_\lambda \in I(a)$ and $t = (t_0, \dots, t_{g-1}) \in \mathbb{C}^g$, and

$$\exp\left(\xi_\lambda \sum_{i=0}^{g-1} \lambda^{-i} t_i \right) = F_\lambda(t) B_\lambda(t) \tag{4.1}$$

the Iwasawa factorization. Define the map $\pi(t) : I(a) \to I(a)$ by

$$\pi(t)\xi_\lambda = B_\lambda(t)\xi_\lambda B_\lambda^{-1}(t).$$

Since $F_\lambda(t)B_\lambda(t)$ commutes with ξ_λ we have

$$\pi(t)\xi_\lambda = B_\lambda(t)\xi_\lambda B_\lambda^{-1}(t) = F_\lambda^{-1}(t)\xi_\lambda F_\lambda(t).$$

PROPOSITION 4.3. *The map $\pi(t) : I(a) \to I(a)$ defines a commutative group action*

$$\pi\big(t + t'\big) = \pi(t)\pi\big(t'\big) = \pi\big(t'\big)\pi(t).$$

Proof. For $t, t' \in \mathbb{C}^g$, we have

$$F_\lambda\big(t + t'\big) B_\lambda\big(t + t'\big) = F_\lambda(t) B_\lambda(t) F_\lambda\big(t'\big) B_\lambda\big(t'\big) = F_\lambda\big(t'\big) B_\lambda\big(t'\big) F_\lambda(t) B_\lambda(t).$$

Hence,

$$\begin{aligned} \exp\big(B_\lambda(t)\xi_\lambda B_\lambda^{-1}(t)\Sigma\lambda^{-i}t_i'\big) &= B_\lambda(t) F_\lambda\big(t'\big) B_\lambda\big(t'\big) B_\lambda^{-1}(t) \\ &= F_\lambda^{-1}(t) F_\lambda\big(t + t'\big) B_\lambda\big(t + t'\big) B_\lambda^{-1}(t), \\ \exp\big(B_\lambda\big(t'\big)\xi_\lambda B_\lambda^{-1}\big(t'\big)\Sigma\lambda^{-i}t_i\big) &= B_\lambda\big(t'\big) F_\lambda(t) B_\lambda(t) B_\lambda^{-1}\big(t'\big) \\ &= F_\lambda^{-1}\big(t'\big) F_\lambda\big(t + t'\big) B_\lambda\big(t + t'\big) B_\lambda^{-1}\big(t'\big). \end{aligned}$$

Set $\hat{B}_\lambda(t) = B_\lambda(t+t')B_\lambda^{-1}(t')$ and $\tilde{B}_\lambda(t') = B_\lambda(t+t')B_\lambda^{-1}(t)$. Then

$$\begin{aligned}\pi\big(t+t'\big)\xi_\lambda &= B_\lambda\big(t+t'\big)\xi_\lambda B_\lambda^{-1}\big(t+t'\big)\\ &= \tilde{B}_\lambda\big(t'\big)B_\lambda(t)\xi_\lambda B_\lambda^{-1}(t)\tilde{B}_\lambda^{-1}\big(t'\big)\\ &= \tilde{B}_\lambda\big(t'\big)\pi(t)\xi_\lambda\tilde{B}_\lambda^{-1}\big(t'\big) = \pi\big(t'\big)\pi(t)\xi_\lambda\\ &= \hat{B}_\lambda(t)B_\lambda\big(t'\big)\xi_\lambda B_\lambda^{-1}\big(t'\big)\hat{B}_\lambda^{-1}(t)\\ &= \hat{B}_\lambda(t)\pi\big(t'\big)\xi_\lambda\hat{B}_\lambda^{-1}(t) = \pi(t)\pi\big(t'\big)\xi_\lambda. \qquad \square\end{aligned}$$

Removing singularities on Σ. Elements of different isospectral sets may give the same extended frame up to conjugation by $g \in \mathbb{T}$ (see Remark 1.4 and below). This is the case in particular if an initial value ξ_λ has a root at some $\lambda = \alpha_0 \in \mathbb{C}^\times$. Then the corresponding polynomial Killing field ζ_λ also has a root at $\lambda = \alpha_0$ for all $z \in \mathbb{C}$. In this case, we may reduce the order of ξ_λ and ζ_λ without changing the immersion. This situation corresponds to a singular spectral curve because then the polynomial $a(\lambda) = -\lambda \det \xi_\lambda$ has a root of order at least two at α_0. We can remove such a singularity by changing the surface by an isometry. We describe this change below.

PROPOSITION 4.4. *Suppose a polynomial Killing field ζ_λ has roots in $\lambda \in \mathbb{C}^\times$. Then there is a polynomial $p(\lambda)$ such that $\zeta_\lambda/p(\lambda)$ has no roots in $\lambda \in \mathbb{C}^\times$. If F_λ and $\breve{F}_\lambda$ are the extended frames of ζ_λ respectively $\zeta_\lambda/p(\lambda)$, then $\breve{F}_\lambda(p(0)z) = F_\lambda(z)$. Hence, there is $g(\delta) \in \mathbb{T}$ with $\tilde{\zeta}_\lambda := g(\delta)^{-1}(\zeta_\lambda/p(\lambda))g(\delta) : \mathbb{C} \to \mathcal{P}_g$ and $\tilde{F}(z) := g(\delta)^{-1}\breve{F}_\lambda(p(0)z)g(\delta)$ induces a minimal surface $\tilde{X}_\lambda$ congruent to X_λ in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof. Suppose the polynomial Killing field $\zeta_\lambda(z) = F_\lambda^{-1}(z)\xi_\lambda F_\lambda(z)$ has a root at $\lambda = \alpha_0$. Define

$$p(\lambda) = \begin{cases} \overline{\sqrt{-\alpha_0}}\lambda + \sqrt{-\alpha_0} & \text{if } \alpha_0 \in \mathbb{S}^1,\\ (\lambda - \alpha_0)(1 - \bar{\alpha}_0\lambda) & \text{if } \alpha_0 \in \mathbb{C}^\times \setminus \mathbb{S}^1.\end{cases}$$

If ζ_λ has a simple root at $\lambda = \alpha_0 \in \mathbb{C}^\times$, then $\zeta_\lambda/p(\lambda) : \mathbb{C} \to \Lambda_{-1}^{g-\deg p}\mathfrak{sl}_2(\mathbb{C})$ does not vanish at α_0. Then there is a map $C : \mathbb{C} \to \Lambda^+\mathrm{SL}_2(\mathbb{C})$ such that

$$\begin{aligned}F_\lambda(z)B_\lambda(z) &= \exp(z\xi_\lambda) = \exp\big(p(\lambda)z\breve{\xi}_\lambda\big) = \exp\big(p(0)z\breve{\xi}_\lambda\big)C(z)\\ &= \breve{F}_\lambda\big(p(0)z\big)\breve{B}_\lambda\big(p(0)z\big)\end{aligned}$$

and

$$F_\lambda(z) = \breve{F}_\lambda\big(p(0)z\big) = \begin{cases}\breve{F}_\lambda(\sqrt{-\alpha_0}z) & \text{if } \alpha_0 \in \mathbb{S}^1,\\ \breve{F}_\lambda(-\alpha_0 z) & \text{if } \alpha_0 \in \mathbb{C}^\times \setminus \mathbb{S}^1.\end{cases}$$

We consider $\delta = \overline{\sqrt{-\alpha_0}}$ when $|\alpha_0| = 1$ and $\delta = -\bar{\alpha}_0/|\alpha_0|$ in the other case. We conjugate $\zeta_\lambda/p(\lambda)$ by $g(\delta) \in \mathbb{T}$. Hence by Remarks 1.5, 2.6, 2.9, 3.4, the

immersion $\tilde{X}_\lambda$ obtained from $\tilde{F}_\lambda(z) = g(\delta)^{-1}\breve{F}_\lambda(p(0)z)g(\delta)$ is congruent to the immersion X_λ. □

Hence amongst all polynomial Killing fields that give rise to an extended frame of finite type there is one of smallest possible degree. We say that a polynomial Killing field has *minimal degree* if and only if it has neither roots nor poles in $\lambda \in \mathbb{C}^\times$. We summarize two results by Burstall–Pedit [5], [6].

PROPOSITION 4.5. *For an extended frame of finite type there exists a unique polynomial Killing field of minimal degree. There is a smooth* 1–1 *correspondence between the set of extended frames of finite type and the set of polynomial Killing fields without zeroes.*

Proof. We briefly outline how to prove the existence and uniqueness of a minimal element. If the initial value ξ_λ gives rise to an extended frame F_λ, then the corresponding polynomial Killing field ζ_λ can be modified according to Proposition 4.4 so that $\tilde{\zeta}_\lambda$ is of minimal degree, and still giving rise to F_λ. Hence, there exists a polynomial Killing field $\zeta_\lambda \in \mathcal{P}_g$ of least degree giving rise to F_λ.

For the uniqueness, let ζ_λ and $\tilde{\zeta}_\lambda$ both solve $d\zeta_\lambda + [\alpha(\omega), \zeta_\lambda] = 0$, with $\deg \zeta_\lambda \geq \deg \tilde{\zeta}_\lambda$. We can assume that $\zeta_\lambda, \tilde{\zeta}_\lambda$ have no roots (if not, we simplify the polynomial Killing field using Proposition 4.4). We use the iteration of Proposition 2.2. We prove that there is a polynomial $q \in \mathbb{C}^k[\lambda]$ such that

$$\zeta_\lambda(z) = q(\lambda)\tilde{\zeta}_\lambda(z). \tag{4.2}$$

The polynomial q is constructed recursively by considering coefficients u_n, σ_n, τ_{n-1} and $\tilde{u}_n$, $\tilde{\sigma}_n$, $\tilde{\tau}_{n-1}$. Since $\tau_{-1}, \tilde{\tau}_{-1}$, are constant, there is q_0 with $\tilde{\tau}_{-1} = q_0\tau_{-1}$. This implies that $\tilde{u}_0 = q_0 u_0$, $\tilde{\sigma}_0 = q_0\sigma_0$ and there is q_1 such that $\tilde{\tau}_0 = q_0\tau_0 + q_1\tau_{-1}$. By the iteration, if there are constants $q_0, q_1, \ldots, q_\ell$ with

$$\begin{aligned}
&\tilde{u}_0 = q_0 u_0, && \tilde{u}_1 = q_0 u_1 + q_1 u_0, && \ldots, && \tilde{u}_\ell = q_0 u_\ell + \cdots + q_\ell u_0,\\
&\tilde{\sigma}_0 = q_0\sigma_0, && \tilde{\sigma}_1 = q_0\sigma_1 + q_1\sigma_0, && \ldots, && \tilde{\sigma}_\ell = q_0\sigma_\ell + \cdots + q_\ell\sigma_0,\\
&\tilde{\tau}_{-1} = q_0\tau_{-1}, && \tilde{\tau}_0 = q_0\tau_0 + q_1\tau_{-1}, && \ldots, && \tilde{\tau}_{\ell-1} = q_0\tau_{\ell-1} + \cdots + q_\ell\tau_{-1},
\end{aligned}$$

the iteration implies that there is $q_{\ell+1}$ such that $\tilde{\tau}_\ell = q_0\tau_\ell + \cdots + q_{\ell+1}\tau_{-1}$ and this proves (4.2). Now, since $\zeta_\lambda, \tilde{\zeta}_\lambda$ have no roots, the polynomial $q(\lambda) = q_0$ is constant and since the residues coincide, we conclude that $\zeta_\lambda = \tilde{\zeta}_\lambda$. □

REMARK 4.6. Since the Iwasawa factorization is a diffeomorphism, and all other operations involved in obtaining an extended frame from a polynomial Killing field are smooth, the resulting minimal surface depends smoothly on the entries of the polynomial Killing field, and thus also smoothly on the entries of its initial value.

For $\xi_\lambda \in I(a)$, the polynomial $a(\lambda) = -\lambda \det \xi_\lambda$ has the form

$$a(\lambda) = b \prod_{i=1}^{g} (\lambda - \alpha_i)(1 - \lambda \bar{\alpha}_i), \quad b \in \mathbb{R}^-. \tag{4.3}$$

The condition (2.8) implies $a(1) \leq 0$ so that $b \in \mathbb{R}^-$.

LEMMA 4.7. (1) *If a has a double root $\alpha_0 \in \mathbb{S}^1$, then we have an isomorphism*

$$I(a) \cong I\big(-\alpha_0(\lambda - \alpha_0)^{-2} a\big).$$

(2) *If a has double root $\alpha_0 \notin \mathbb{S}^1$, then $I(a) = \{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} \neq 0\} \cup \{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} = 0\}$ and there is an isomorphism*

$$\big\{\xi_\lambda \in I(a) \mid \xi_{\alpha_0} = 0\big\} \cong I\big((\lambda - \alpha_0)^{-1}(1 - \bar{\alpha}_0 \lambda)^{-1} a\big).$$

Proof. (1) If a has a double root at $\alpha_0 \in \mathbb{S}^1$ then for any $\xi_\lambda \in I(a)$, we have $\xi_{\alpha_0} = 0$. We can remove this root by Proposition 4.4, with $\delta = \overline{\sqrt{-\alpha_0}}$, to obtain the isomorphism

$$\xi_\lambda \mapsto (\overline{\sqrt{-\alpha_0}}\lambda + \sqrt{-\alpha_0})^{-1} g(\delta)^{-1} \xi_\lambda g(\delta) \in I\big(-\alpha_0(\lambda - \alpha_0)^{-2} a\big).$$

(2) Suppose a has a double root at $\alpha_0 \notin \mathbb{S}^1$. Then the isospectral set splits into a part which contains potentials with a zero at α_0, which we can remove, and the set of potentials not zero at α_0. But in this last case, this means that ξ_{α_0} is a nilpotent matrix. We use Proposition 4.4, with $\delta = -\bar{\alpha}_0/|\alpha_0|$, to obtain the isomorphism

$$\xi_\lambda \mapsto \frac{1}{\lambda - \alpha_0} \frac{1}{1 - \bar{\alpha}_0 \lambda} g(\delta)^{-1} \xi_\lambda g(\delta) \in I\left(\frac{1}{\lambda - \alpha_0} \frac{1}{1 - \bar{\alpha}_0 \lambda} a\right). \qquad \square$$

THEOREM 4.8. *Isospectral sets are compact. If the $2g$-roots of the polynomial $a(\lambda) = -\lambda \det \xi_\lambda$ for $\xi_\lambda \in \mathcal{P}_g$ are pairwise distinct, then $I(a) \cong (\mathbb{S}^1)^g$ is a connected smooth g-dimensional manifold diffeomorphic to a g-dimensional real torus.*

The proof of Theorem 4.8 follows in several steps. The compactness is a consequence of the next Proposition 4.9. The second statement follows from Propositions 4.10 and 4.11. We shall prove several properties of the map

$$A : \mathcal{P}_g \to \mathcal{M}_g, \quad \xi_\lambda \mapsto -\lambda \det \xi_\lambda. \tag{4.4}$$

PROPOSITION 4.9. *The map A in (4.4) is proper.*

Proof. Since A is continuous it suffices to show that pre-images of compact sets are bounded. Let $K \subset \mathcal{M}_g$ be compact. Then the map $\mathbb{S}^1 \times K \to \mathbb{R}, (\lambda, a) \mapsto \lambda^{-g} a(\lambda)$ is bounded. For the compactness of the pre-image it suffices to show that all Laurent-coefficients of a $\xi_\lambda \in \mathcal{P}_g$ are bounded, if $A(\xi_\lambda) \in K$. Fix a polynomial $a \in K$ and consider the isospectral set $I(a)$ as a closed subset of the $(3g+1)$-dimensional vector space $\mathcal{P}_g$. For $d = (1-g)/2$

the map $\lambda^d \xi_\lambda$ is traceless and skew-hermitian for $|\lambda| = 1$. The determinant of traceless skew-Hermitian 2×2 matrices is the square of a norm $\|\cdot\|$. The Laurent-coefficients of $\xi_\lambda = \sum_{i=-1}^{g} \lambda^i \hat{\xi}_i$ are

$$\hat{\xi}_i = \frac{1}{2\pi \mathbb{i}} \int_{\mathbb{S}^1} \lambda^{-i} \xi_\lambda \frac{d\lambda}{\lambda}.$$

Using the norm gives

$$\|\hat{\xi}_i\| \leq \frac{1}{2\pi \mathbb{i}} \int_{\mathbb{S}^1} \left\|\lambda^{(1-g)/2} \xi_\lambda\right\| \frac{d\lambda}{\lambda} \leq \sup_{\lambda \in \mathbb{S}^1} \sqrt{-\lambda^{-g} a(\lambda)}.$$

Thus each entry of $\hat{\xi}_i$ is bounded on $\mathbb{S}^1$, so A is proper and $I(a)$ therefore compact. □

PROPOSITION 4.10. *Suppose $\xi_\lambda \in \mathcal{P}_g$ has no roots in $\lambda \in \mathbb{C}^\times$. Then the map A in (4.4) has maximal rank $2g+1$. Let $a(\lambda) = -\lambda \det \xi_\lambda$. Then $I(a)$ is a g-dimensional sub-manifold of $\mathcal{P}_g$.*

Proof. Since det is the square of a norm on $\mathfrak{su}_2$, at all roots of a on $\mathbb{S}^1$, the corresponding $\xi_\lambda \in I(a)$ has to vanish. If ξ_λ is without roots, then $a \in \mathcal{M}_g^0$. We show that the derivative of the map A has rank $2g+1$ at a potential ξ_λ without roots, and then invoke the implicit function theorem. Hence it suffices to prove that for all roots α_0 of a of order n, and all $\beta \in \mathbb{C}$ there exists a tangent vector $\dot{\xi}_\lambda$ along $\mathcal{P}_g$ at ξ_λ, such that the corresponding derivative of a is equal to

$$\dot{a}(\lambda) = -\lambda \det \xi_\lambda \operatorname{tr}\left(\xi_\lambda^{-1} \dot{\xi}_\lambda\right) = \frac{\beta a(\lambda)}{(\lambda - \alpha_0)^m} + \frac{\lambda^m \bar{\beta} a(\lambda)}{(1 - \bar{\alpha}_0 \lambda)^m}$$

with $m = 1, \ldots, n$. Such a vector field fixes all roots of a except α_0. The set of these vector fields form a $2g$-dimensional real vector space. Besides the roots of $a(\lambda)$ also the coefficient b in (4.3) can be changed by this variation. We consider $\dot{\xi}_\lambda = t\xi_\lambda$ with

$$\dot{a}(\lambda) = -\lambda \det \xi_\lambda \operatorname{tr}\left(\xi_\lambda^{-1} \dot{\xi}_\lambda\right) = -2\lambda t \det \xi_\lambda.$$

This vector field preserves all roots of $a(\lambda)$, but changes $a(1)$ with variational field $\dot{a}(1) = ta(1)$, and $b \in \mathbb{R}^-$ is changing non-trivially. This will prove the theorem.

Now we construct vector fields $\dot{\xi}_\lambda$. If α_0 is a root of a of order n, then $\det \xi_{\alpha_0}$ vanishes, and ξ_{α_0} is nilpotent. For a non-zero nilpotent 2×2-matrix ξ_{α_0} there exists a matrix $Q \in \mathfrak{su}_2$ such that $\xi_{\alpha_0} = [Q, \xi_{\alpha_0}]$. To prove this remark, observe that it holds if $\xi_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, by setting $Q_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The general statement now follows since there exists $g \in \mathrm{SU}_2$ with $\xi_{\alpha_0} = g^{-1} \xi_0 g$, and setting $Q = g^{-1} Q_0 g$.

We need the following basic fact: For $A, B \in \mathfrak{sl}_2(\mathbb{C})$ and $A \neq 0$, we have

$$\operatorname{tr}(AB) = 0 \quad \Longleftrightarrow \quad B = [C, A] \text{ with some } C \in \mathfrak{sl}_2(\mathbb{C}). \tag{4.5}$$

For any $\xi_\lambda \in \mathcal{P}_g$, we have $\xi_\lambda^2 = -\det(\xi_\lambda)\mathbb{1}$ and at a root α_0 of $a(\lambda)$ of order n, there exists for any $m = 1, \dots, n$, a matrix

$$\hat{Q}_\lambda(m) = Q_0 + Q_1(\lambda - \alpha_0) + \cdots + Q_{m-1}(\lambda - \alpha_0)^{m-1}$$

with $Q_0, \dots, Q_{n-1} \in \mathfrak{sl}_2(\mathbb{C})$, such that $\xi_\lambda - [\hat{Q}_\lambda(m), \xi_\lambda]$ has at α_0 a root of order m. The matrix $\hat{Q}_\lambda(m)$ is constructed inductively using (4.5). We remark that at α_0, the function $\lambda \mapsto \operatorname{tr}(\xi_\lambda^2)$ has a root of order n. Then there is Q_0 such that $\xi_\lambda - [Q_0, \xi_\lambda] = (\lambda - \alpha_0)\xi_{\lambda,1}$ and $\operatorname{tr}(\xi_\lambda(\xi_\lambda - [Q_0, \xi_\lambda])) = (\lambda - \alpha_0)\operatorname{tr}(\xi_\lambda \xi_{\lambda,1})$ has a root at α_0 of order n. Then there is a matrix Q_1 with $\xi_{\lambda,1} - [Q_1, \xi_\lambda] = (\lambda - \alpha_0)Q_2$. Now we define for $m = 1, \dots, n$

$$\tilde{Q}_\lambda(m) = -{}^t\bar{Q}_0 - {}^t\bar{Q}_1(1 - \bar{\alpha}_0\lambda)\lambda^{-1} - \cdots - {}^t\bar{Q}_{m-1}(1 - \bar{\alpha}_0\lambda)^{m-1}\lambda^{m-1}.$$

Then $\xi_\lambda - [\tilde{Q}_\lambda(m), \xi_\lambda]$ has at $\bar{\alpha}_0^{-1}$ a root of order m. Now we define

$$q_m(\lambda) = \frac{\beta}{(\lambda - \alpha_0)^m} + \frac{\lambda^m \bar{\beta}}{(1 - \alpha_0\lambda)^m},$$
$$Q_\lambda(m) = \frac{\beta}{(\lambda - \alpha_0)^m}\hat{Q}_\lambda(m) + \frac{\lambda^m \bar{\beta}}{(1 - \alpha_0\lambda)^m}\tilde{Q}_\lambda(m).$$

There exists some $P \in \mathfrak{su}_2$ such that

$$\dot{\xi}_\lambda = q_m(\lambda)\xi_\lambda - \big[Q_\lambda(m), \xi_\lambda\big] + [P, \xi_\lambda] \in T_{\xi_\lambda}\mathcal{P}_g.$$

To see that we need to check $\dot{\hat{\xi}}_{-1} \in \mathbb{i}\mathbb{R}^+ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $|\dot{a}|(0) = 0$. Note that if $A \in \mathfrak{su}_2$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, then $[A, B] = \left(\begin{smallmatrix} \alpha & \mathbb{i}x \\ 0 & -\alpha \end{smallmatrix}\right)$ with $\alpha \in \mathbb{C}, x \in \mathbb{R}$. Then we can choose $P \in \mathfrak{su}_2$ such that

$$\dot{\hat{\xi}}_{-1} = \frac{\beta\xi_{-1} - \beta[Q_0(m), \xi_{-1}]}{(-\alpha_0)^n} + [P, \xi_{-1}] \in \mathbb{i}\mathbb{R}^+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For the second condition we have $(\dot{a}\bar{a} + a\dot{\bar{a}})(0) = |a|^2(0)(-\alpha_0)^{-n}(\beta + \bar{\beta})$, and we can choose $\beta \in \mathbb{C}$ such that the variational field keeps $|a(0)|$ unchanged along the deformation. This proves that $\dot{\xi}_\lambda \in T_{\xi_\lambda}\mathcal{P}_g$, and such vector fields span a $2g+1$ dimensional real vector space of vector fields in the complement of the kernel of the map. This proves that around ξ_λ without zeroes, $I(a)$ is a real g-dimensional manifold. □

PROPOSITION 4.11. *For all $\xi_\lambda \in \mathcal{P}_g$ without roots the vector fields of the isospectral group action generate at ξ_λ a real g-dimensional subspace of the tangent space $\mathcal{P}_g$ at ξ_λ.*

Proof. The vector field $(t_0, \dots, t_{g-1})$ of the isospectral action at ξ_λ takes the values

$$\dot{\xi}_\lambda = \left[\left(\sum_{i=0}^{g-1} \lambda^{-i} t_i \xi_\lambda\right)^+, \xi_\lambda\right] = -\left[\left(\sum_{i=0}^{g-1} \lambda^{-i} t_i \xi_\lambda\right)^-, \xi_\lambda\right].$$

Here

$$\sum_{i=0}^{g-1} \lambda^{-i} t_i \xi_\lambda = \left(\sum_{i=0}^{g-1} \lambda^{-i} t_i \xi_\lambda \right)^+ + \left(\sum_{i=0}^{g-1} \lambda^{-i} t_i \xi_\lambda \right)^-$$

is the Lie algebra decomposition of the Iwasawa decomposition. For $A \in \mathfrak{sl}_2(\mathbb{C})$ with $A \neq 0$ we have that $\{B \in \mathfrak{sl}_2(\mathbb{C}) \mid [A, B] = 0\} = \{xA \mid x \in \mathbb{C}\}$. Hence the vector field corresponding to $(t_0, \dots, t_{g-1})$ vanishes at ξ_λ if and only if there exists a decomposition of the polynomial $\sum_{i=0}^{g-1} \lambda^{-i} t_i = f^+(\lambda) + f^-(\lambda)$ into complex functions such that

$$\left(\sum_{i=0}^{g-1} \lambda^{-i} t_i \xi_\lambda \right)^+ = f^+ \xi_\lambda \quad \text{and} \quad \left(\sum_{i=0}^{g-1} \lambda^{-i} t_i \xi_\lambda \right)^- = f^- \xi_\lambda.$$

Hence $\overline{f^+(\bar{\lambda}^{-1})} = \lambda^g f^+(\lambda)$ and $f^-(0) = 0$.

The polynomial $\sum_{i=0}^{g-1} \lambda^{-i} t_i$ is a linear combination of such functions if and only if $t_{g-1-i} = \bar{t}_i$. The subspace of such $(t_0, \dots, t_{g-1})$ is a real g-dimensional subspace of $\mathbb{C}^g$. This implies the proposition. □

Recall $\mathcal{M}_g$ from (2.9), and define

$$\mathcal{M}_g^1 = \{a \in \mathcal{M}_g \mid a \text{ has } 2g\text{-pairwise distinct roots}\},$$
$$\mathcal{P}_g^1 = \{\xi_\lambda \in \mathcal{P}_g \mid a(\lambda) = -\lambda \det \xi_\lambda \in \mathcal{M}_g^1\}.$$

PROPOSITION 4.12. *For all $a \in \mathcal{M}_g^1$, the isospectral action $\pi : \mathbb{R}^g \times I(a) \to I(a)$ acts transitively on $I(a)$ and the mapping $A : \mathcal{P}_g^1 \to \mathcal{M}_g^1$ is a principal bundle with fibre $I(a) = (\mathbb{S}^1)^g$.*

Proof. At all roots of $\xi_\lambda \in \mathcal{P}_g$, the determinant $\det \xi_\lambda$ has a higher order root. Hence all $\xi_\lambda \in \mathcal{P}_g^1$ have no roots on $\mathbb{C}^\times$. Proposition 4.10 implies that $A : \mathcal{P}_g^1 \to \mathcal{M}_g^1$ has maximal rank $2g + 1$ and induces a fibre bundle, whose fibres are real g-dimensional manifolds. The isospectral action preserves the determinant and thus the fibres. Proposition 4.11 implies that for all $\xi_\lambda \in \mathcal{P}_g^1$ the corresponding orbit of the isospectral action is an open submanifold of the corresponding fibre. If $\pi(t_n)\xi_\lambda$ converges to $\tilde{\xi}_\lambda \in \mathcal{P}_g^1$ for a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^g$, then the orbit of $\tilde{\xi}_\lambda$ is again an open submanifold of the corresponding fibre. Thus $\pi(t_n)\xi_\lambda$ belongs to the orbit of $\tilde{\xi}_\lambda$ for sufficiently large $n \in \mathbb{N}$. Then $\pi(t_n)\xi_\lambda = \pi(t'_n)\tilde{\xi}_\lambda$ and $\tilde{\xi}_\lambda = \pi(t_n - t'_n)\xi_\lambda$ is in the orbit of ξ_λ. This shows that the orbits of the isospectral set $I(a)$ are open and closed submanifolds of the fibre. Due to Proposition 4.9, the fibers are compact. Therefore all orbits are compact as well. Due to Proposition 4.11, for all $\xi_\lambda \in \mathcal{P}_g^1$ the stabilizer subgroup

$$\Gamma_{\xi_\lambda} = \{t \in \mathbb{R}^g \mid \pi(t)\xi_\lambda = \xi_\lambda\} \tag{4.6}$$

is discrete and $\mathbb{R}^g / \Gamma_{\xi_\lambda}$ is diffeomorphic to the connected component of the fibre of ξ_λ. We conclude that Γ_{ξ_λ} is a lattice in $\mathbb{R}^g$ isomorphic to $\mathbb{Z}^g$. This

shows that $A : \mathcal{P}^1_g \to \mathcal{M}^1_g$ is a fibre bundle, whose fibres have connected components all isomorphic to $(\mathbb{S}^1)^g$.

It remains to prove that $I(a)$ has only one connected component. We first show that $\mathcal{M}^0_g$ (see (2.9)) is path connected: If $a, \tilde{a} \in \mathcal{M}^0_g$ satisfy

$$t\tilde{a}(0) + (1-t)a(0) \neq 0 \quad \text{for all } t \in [0,1] \tag{4.7}$$

then $t\tilde{a} + (1-t)a \in \mathcal{M}^0_g$ for all $t \in [0,1]$. If $a, \tilde{a} \in \mathcal{M}^0_g$ do not satisfy (4.7), then modifying $\lambda \mapsto a(\lambda)$ by the rotation $\lambda \mapsto e^{-\mathbb{i}g\theta} a(e^{\mathbb{i}\theta}\lambda)$ for some suitable $\theta \in \mathbb{R}$ we can ensure that (4.7) holds. Hence, $\mathcal{M}^0_g$ is path connected.

Since $\mathcal{M}^1_g$ is an open subset of $\mathcal{M}^0_g$, whose complement $\mathcal{M}^0_g \setminus \mathcal{M}^1_g$ has codimension at least 2, we conclude that $\mathcal{M}^1_g$ is connected and $A : \mathcal{P}^1_g \to \mathcal{M}^1_g$ has maximal rank $2g+1$ at any point. Hence it remains to show that there exists at least one $a \in \mathcal{M}^1_g$ for which $I(a)$ has only one connected component.

Denote the entries of $\xi_\lambda \in \mathcal{P}_g$ by polynomials α, β, γ so that

$$\xi_\lambda = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}.$$

Then $a(\lambda) = -\lambda \det \xi_\lambda = \lambda\alpha^2 + \lambda\beta\gamma$. Here α is a polynomial of degree at most $g-1$. For $|\lambda| = 1$ the polynomial $\lambda^{\frac{1-g}{2}}\alpha \in \mathbb{i}\mathbb{R}$ and $\lambda^{1-g}\alpha^2 \leq 0$, and therefore $\lambda^{1-g}\beta\gamma \in \mathbb{R}$.

Now we consider a potential ξ_λ with $\lambda\beta = \gamma$, and γ has only roots on $|\lambda| = 1$. We claim in this case that $I(a)$ is connected. This condition on β implies that $a(\lambda) = \lambda\alpha^2 + \gamma^2$ and $\lambda^g\overline{\gamma(\bar{\lambda}^{-1})} = -\lambda\beta(\lambda) = -\gamma(\lambda)$. Now observe that $\lambda^{-g}\gamma^2 \leq 0$ and $\lambda^{-g}a(\lambda) \leq \lambda^{1-g}\alpha^2 \leq 0$.

Let $\tilde{\xi}_\lambda \in I(a)$ with entries $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. We construct a family $\xi_{\lambda,t} = \left(\begin{smallmatrix} \alpha_t & \beta_t \\ \gamma_t & -\alpha_t \end{smallmatrix}\right)$ with $\alpha_t = t\alpha + (1-t)\tilde{\alpha}$. We prove that there exist polynomials β_t, γ_t uniquely defined such that $\xi_{\lambda,t} \in I(a)$ for all $t \in [0,1]$, with $\beta_1 = \beta$, $\gamma_1 = \gamma$ and $\beta_0 = \tilde{\beta}$, $\gamma_0 = \tilde{\gamma}$.

Since $\lambda^{1-g}\tilde{\alpha}^2 \leq 0$ for $|\lambda| = 1$ we have $\lambda^{1-g}\alpha_t^2 \leq 0$ for all $t \in [0,1]$. Now we consider the polynomial $p_t(\lambda) = a(\lambda) - \lambda^{1-g}\alpha_t^2$. If α_0 is a root of $p_t(\lambda)$, then $\bar{\alpha}_0^{-1}$ is a root of $p_t(\lambda)$. For $t = 0$, we know that $p_0 = \lambda\tilde{\beta}\tilde{\gamma}$ where roots of $\tilde{\beta}$ are symmetric to the roots of $\tilde{\gamma}$. At roots of p_t we can define β_t and γ_t with $\beta_1 \in \mathbb{i}\mathbb{R}^+$. At $t = 1$, we have $p = \gamma^2$ and all the roots of p_1 are double roots on the unit circle. Then γ is defined uniquely, and $I(a)$ is connected, if we can find such a path with $a \in \mathcal{M}^1_g$.

Therefore, we consider $\gamma = \lambda^g + 1$ and $\alpha = \mathbb{i}k(\lambda^{g-1} + 1)$ with $k \in \mathbb{R}^+$. Then $0 = \lambda^g + 1 = \lambda^{g-1} + 1$ implies $\lambda^{g-1}(1-\lambda) = 0$. Then the polynomials γ and α do not have common roots. Hence at $k = 0$ we have $\lambda^{-g}a(\lambda) = \lambda^{1-g}\alpha^2 + \lambda^{-g}\gamma^2 \leq 0$ for $|\lambda| = 1$, and a has only double roots on the unit circle. For $k > 0$ small enough, the roots change. But there are no roots on the unit circle and the roots are simple and conjugate, so that $a \in \mathcal{M}^1_g$. □

5. Periods

Suppose $X_\lambda = (G_\lambda, h_\lambda)$ is an associated family of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$. For one member of this family to be periodic, say for $\lambda = 1$ with period $\tau \in \mathbb{C}^\times$, this means that $X_1(z+\tau) = X_1(z)$ for all $z \in \mathbb{C}$, or equivalently $G_1(z+\tau) = G_1(z)$ and $h_1(z+\tau) = h_1(z)$ for all $z \in \mathbb{C}$. If $G_\lambda(z) = F_\lambda(z)\sigma_3 F_\lambda(z)^{-1}$ and $h_\lambda(z) = \operatorname{Re}(-4\mathbb{i}\sqrt{\beta_{-1}\gamma_0\lambda^{-1}}z) = \operatorname{Re}(-\mathbb{i}e^{\mathbb{i}\Theta/2}\sqrt{\lambda^{-1}}z)$ (where $Q = \frac{1}{4}e^{\mathbb{i}\Theta}(dz)^2$) then periodicity reads

$$\big[F_1^{-1}(z)F_1(z+\tau), \sigma_3\big] = 0 \quad \text{and} \quad \operatorname{Re}\big(-\mathbb{i}e^{\mathbb{i}\Theta/2}\tau\big) = 0.$$

The *monodromy* of an extended frame F_λ with respect to the period τ is the matrix

$$M_\lambda(\tau) = F_\lambda(z)^{-1}F_\lambda(z+\tau).$$

Thus periodicity of the horizontal part reads $[M_\lambda(\tau), \sigma_3] = 0$. Due to (1.3), the monodromy $\mathbb{C}^\times \to \mathrm{SL}_2(\mathbb{C}), \lambda \mapsto M_\lambda(\tau)$ is a holomorphic map with essential singularities at $\lambda = 0, \infty$.

For a periodic immersion its conformal factor ω is periodic, and hence also $\alpha(\omega)$ in (1.3) is periodic. This in turn implies that $dM_\lambda(\tau) = 0$ so that $M_\lambda(\tau)$ does not depend on z. Hence, $M_\lambda(\tau) = F_\lambda(0)^{-1}F_\lambda(\tau) = F_\lambda(\tau)$ since $F_\lambda(0) = \mathbb{1}$.

Let ζ_λ be a periodic solution of the Lax equation (2.3) with initial value $\xi_\lambda \in \mathcal{P}_g$, with period τ so that $\zeta_\lambda(z+\tau) = \zeta_\lambda(z)$ for all $z \in \mathbb{C}$. Then also the corresponding $\alpha(\zeta_\lambda)$ in (3.3) is τ-periodic. Let $dF_\lambda = F_\lambda\alpha(\zeta_\lambda), F_\lambda(0) = \mathbb{1}$ and $M_\lambda(\tau) = F_\lambda(\tau)$ be the monodromy with respect to τ. Then for $z = 0$ we have $\xi_\lambda = \zeta_\lambda(0) = \zeta_\lambda(\tau) = F_\lambda^{-1}(\tau)\xi_\lambda F_\lambda(\tau) = M_\lambda^{-1}(\tau)\xi_\lambda M_\lambda(\tau)$ and thus

$$\big[M_\lambda(\tau), \xi_\lambda\big] = 0.$$

The monodromy takes values in SU_2 for $|\lambda| = 1$. The monodromy depends on the choice of base point, but its conjugacy class and hence eigenvalues μ, μ^{-1} do not. The eigenspaces of $M(\lambda, \tau)$ depend holomorphically on (ν, λ). The eigenvalues of ξ_λ and $M_\lambda(\tau)$ are different functions on the spectral curve Σ.

PROPOSITION 5.1. *Let* $M_\lambda = \left(\begin{smallmatrix} A & C \\ B & D \end{smallmatrix}\right) \in \Lambda\mathrm{SU}_2$ *and* $\xi_\lambda = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \in \mathcal{P}_g$ *with* $[M_\lambda, \xi_\lambda] = 0$. *Assume* $\nu \neq 0$ *and* $\mu^2 \neq 1$. *Then* M_λ *and* ξ_λ *have same eigenvectors* $\psi_+ = (1, (\nu-\alpha)/\beta), \psi_- = (1, -(\nu+\alpha)/\beta)$ *with*

$$2\xi_\lambda\psi_+ = \nu\psi_+ \quad \text{and} \quad M\psi_+ = \mu\psi_+,$$
$$\xi_\lambda\psi_- = -\nu\psi_- \quad \text{and} \quad M\psi_- = \mu^{-1}\psi_-.$$

The involution $\eta\colon (\lambda, \nu) \to (\bar\lambda^{-1}, \bar\lambda^{1-g}\bar\nu)$ *acts on* μ *by* $\eta^*\mu = \bar\mu$.

Proof. From the reality condition $\lambda^{g-1}\overline{\xi_{1/\bar\lambda}}^t = -\xi_\lambda$ or equivalently $\xi_{1/\bar\lambda} = -\bar\lambda^{1-g}\overline{\xi_\lambda}^t$ and with $\tilde\psi_- = (1, -(\bar\nu+\bar\alpha)/\bar\gamma)$ we have

$$\xi_{1/\bar\lambda}\tilde\psi_- = -\bar\lambda^{1-g}\begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & -\bar\alpha \end{pmatrix}\tilde\psi_- = \bar\lambda^{1-g}\bar\nu\tilde\psi_-.$$

Since $\overline{M_{1/\bar\lambda}}^{t-1} = M_\lambda$, and $M_\lambda\psi_+ = \mu\psi_+$, and $\psi_+ = (1,(\nu-\alpha)/\beta) = (1,\gamma/(\nu+\alpha))$, we obtain the system

$$A + C\frac{\gamma}{\nu+\alpha} = \mu \quad \text{and} \quad B + D\frac{\gamma}{\nu+\alpha} = \mu\frac{\gamma}{\nu+\alpha}.$$

This implies by direct computation that $M_{1/\bar\lambda}\tilde\psi_- = \overline{M_\lambda^{-1}}^t\tilde\psi_- = \bar\mu\tilde\psi_-$. □

At $\lambda = 0$ and $\lambda = \infty$ a monodromy $M_\lambda(\tau)$ has essential singularities. Next, we study the behavior of $\mu = \mu(\lambda,\tau)$ at these two points.

LEMMA 5.2. *Let $\psi = \psi(\lambda,z) = (1,h(\lambda,z))$ be an eigenvector of $\zeta_\lambda(z) = F_\lambda(z)^{-1}\xi_\lambda F_\lambda(z)$. Then there exists a complex function $f = f(\lambda,z)$ such that*

$$f(\lambda,z)\psi(\lambda,z) = F_\lambda^{-1}(z)\psi(\lambda,0) \tag{5.1}$$

which satisfies

(1) $f^{-1}\,df = -\frac{\mathbb{i}}{4}\lambda^{-1/2}\exp(\mathbb{i}\Theta/2)\,dz + O(1)$ *in a neighborhood of* $\lambda = 0$,

(2) $f^{-1}\,df = -\frac{\mathbb{i}}{4}\lambda^{1/2}\exp(-\mathbb{i}\Theta/2)\,d\bar z + O(1)$ *in a neighborhood of* $\lambda = \infty$.

Proof. Note that $h(\lambda,z)$ and $f(\lambda,z)$ are 2-valued in λ because they depend on the choice of eigenvalue. By Proposition 5.1, an eigenvector of

$$\zeta_\lambda(z) = \begin{pmatrix}\alpha(\lambda,z) & \beta(\lambda,z)\\ \gamma(\lambda,z) & -\alpha(\lambda,z)\end{pmatrix}$$

associate to the eigenvalue $\nu = \sqrt{a(\lambda)\lambda^{-1}}$ is given by $\psi(\lambda,z) = (1,(\nu-\alpha)/\beta)$. Now using $\zeta_\lambda(z) = F_\lambda^{-1}(z)\xi_\lambda F_\lambda(z)$ we see that $F_\lambda^{-1}(z)\psi(\lambda,0)$ is an eigenvector of $\zeta_\lambda(z)$ and it is collinear to $\psi(\lambda,z)$. This defines the function $f(\lambda,z)$.

Differentiating (5.1) reads $df\psi + f\,d\psi = -\alpha(\zeta_\lambda)F_\lambda^{-1}\psi|_{z=0}$, and then

$$f^{-1}\,df\psi = -\alpha(\zeta_\lambda)\psi - d\psi. \tag{5.2}$$

In a neighborhood of $\lambda = 0$, we have $\nu^2 = -\det\zeta_\lambda = \frac{\lambda\alpha^2+\lambda\beta\gamma}{\lambda} = \frac{\beta_{-1}\gamma_0}{\lambda} + O(1) = \frac{-e^{\mathbb{i}\Theta}}{16\lambda} + O(1)$. Expanding at $\lambda = 0$

$$\alpha(\zeta_\lambda) = \begin{pmatrix}\alpha_0 & \beta_{-1}\lambda^{-1}\\ \gamma_0 & -\alpha_0\end{pmatrix}dz + O(1),$$

and considering the first entry of the vector equation (5.2) yields

$$\begin{aligned} f^{-1}\,df &= -\alpha_0\,dz - \beta_{-1}\lambda^{-1}\frac{\nu-\alpha(z)}{\beta(z)}\,dz + O(1)\\ &= -\nu\,dz + O(1) = \frac{-\mathbb{i}e^{\mathbb{i}\Theta/2}}{4\sqrt{\lambda}}\,dz + O(1).\end{aligned}$$

In a neighborhood of $\lambda = \infty$, we have

$$\alpha(\zeta_\lambda) = \begin{pmatrix}-\bar\alpha_0 & -\bar\gamma_0\\ -\bar\beta_{-1}\lambda & \bar\alpha_0\end{pmatrix}d\bar z + O(1)$$

and we obtain (ii) by considering the eigenvector $\psi = (\frac{\beta}{\nu-\alpha}, 1)$ and looking at the second entry gives

$$f^{-1}\,df = \bar\beta_{-1}\lambda\frac{\beta}{\nu-\alpha}\,d\bar z + O(1) = \frac{\bar\beta_{-1}\beta_{g-1}}{\sqrt{-a_{2g}}}\sqrt{\lambda}\,d\bar z + O(1)$$
$$= \frac{-\mathbb{i}e^{-\mathbb{i}\Theta/2}\sqrt{\lambda}}{4}\,d\bar z + O(1).$$

Further, $\hat\xi_d = -\bar{\hat\xi}^t_{g-1-d}$ implies $\beta_{g-1} = -\bar\gamma_0$ and $a_{2g} = -\beta_{g-1}\gamma_g = -\bar\gamma_0\bar\beta_{-1} = e^{-\mathbb{i}\Theta}/16$. □

Using these properties, we compute the local behavior of $\mu(\lambda,\tau)$ near $\lambda = 0$ and $\lambda = \infty$.

PROPOSITION 5.3. *Let $X : A \to \mathbb{S}^2 \times \mathbb{R}$ be an immersed finite type minimal cylinder with spectral curve Σ. Then there exists a meromorphic differential $d\ln\mu$ on Σ with second order poles without residues at $\lambda = 0, \infty$ so that $d\ln\mu - \frac{1}{4}\mathbb{i}\tau\exp(\mathbb{i}\Theta/2)\,d\sqrt{\lambda}^{-1}$ extends holomorphically to $\lambda = 0$, and $d\ln\mu - \frac{1}{4}\mathbb{i}\bar\tau\exp(-\mathbb{i}\Theta/2)\,d\sqrt{\lambda}$ extends holomorphically to $\lambda = \infty$.*

This differential is the logarithmic derivative of a function $\mu : \Sigma \to \mathbb{C}$ which transforms under the involutions (2.11) as $\sigma^*\mu = \mu^{-1}$, $\varrho^*\mu = \bar\mu^{-1}$ and $\eta^*\mu = \bar\mu$. Further $\mu = \pm 1$ at each branch point of Σ.

Proof. If $\nu \neq 0$, the eigenspace of ζ_λ is a complex 1-dimensional vector space and since $[M_\lambda, \zeta_\lambda] = 0$, every eigenvector ψ of ζ_λ with associated eigenvalue $\nu \neq 0$ is an eigenvector of M_λ with eigenvalue μ which depends only on (ν,λ). If $\nu = 0$ then $\mu = \pm 1$. Note that μ is a non-zero holomorphic function on Σ^*. At $\lambda = 0$ and $\lambda = \infty$, the monodromy has essential singularities and we thus need the local analysis of Lemma 5.2. If τ is the period of the annulus, we have $\psi(\tau) = \psi(0)$ and $f(\tau)\psi(0) = F_\lambda^{-1}\psi(0)$. Then $\mu = f^{-1}(\tau)$. This proves that at $\lambda = 0$, $d\ln\mu - \frac{\mathbb{i}}{4}\exp(\mathbb{i}\Theta/2)\tau\,d\sqrt{\lambda}^{-1}$ is holomorphic, and at $\lambda = \infty$ the differential $d\ln\mu - \frac{\mathbb{i}}{4}\exp(-\mathbb{i}\Theta/2)\bar\tau\,d\sqrt{\lambda}$ extends holomorphically. The differential $d\ln\mu$ has second order poles without residues at $\lambda = 0$ and at $\lambda = \infty$.

If ψ_+ and ψ_- are eigenvectors associated to eigenvalues $\pm\nu$ of ζ_λ, the corresponding eigenvalues $\mu_\pm$ of M_λ satisfy $\mu_+\mu_- = 1$. To see how the involution η acts on μ, we note that since M_λ satisfies (1.4), we have that $\bar\mu$ is the corresponding eigenvalue of $M_{1/\bar\lambda}$ associated to μ by Proposition 5.1. Thus, $\eta^*\mu = \bar\mu$. Similarly $\varrho^*\mu = \bar\mu^{-1}$.

If $\nu = 0$, then $\det\zeta_\lambda = 0$, so let $(\tilde e_1, \tilde e_2) \in \mathbb{C}^2$ such that $\zeta_\lambda(\tilde e_1) = 0$ and $\zeta_\lambda(\tilde e_2) = \delta\tilde e_1$. Since $M_\lambda \in \mathrm{SL}_2(\mathbb{C})$, let (e_1, e_2) be a basis of eigenvectors of M_λ associate to μ and μ^{-1} and $\zeta_\lambda(e_1) = \kappa_1\delta\tilde e_1$, $\zeta_\lambda(e_2) = \kappa_2\delta\tilde e_1$. Inserting this in $M_\lambda\zeta_\lambda(e_i) = \zeta_\lambda M_\lambda(e_1)$ proves that $\mu = \mu^{-1}$. This proves that the holomorphic function μ takes values ± 1 at each branch point of Σ. □

Next, we relate the eigenvalues μ to the isospectral action. We prove in particular that the existence of such a holomorphic function is a sufficient condition to close the period of a polynomial Killing field with any initial potential.

PROPOSITION 5.4. *The stabilizer Γ_{ξ_λ} in (4.6) depends only on the orbit of ξ_λ. If $\gamma \in \Gamma_{\xi_\lambda}$ satisfies $F_\lambda(\gamma) = \pm\mathbb{1}$ for some $\xi_\lambda \in \mathcal{P}_g$ then the same is true for every element in the orbit of ξ_λ. The period τ is related to $t = (\tau, 0, \dots, 0) \in \Gamma_{\xi_\lambda}$.*

Proof. This follows from the commuting property of the isospectral action, since

$$\pi(\gamma)\pi(t)\xi_\lambda = \pi(t)\pi(\gamma)\xi_\lambda = \pi(t)\xi_\lambda. \qquad \square$$

PROPOSITION 5.5. *Assume $\xi_\lambda \in \mathcal{P}_g$ has no roots. Then $\gamma \in \Gamma_{\xi_\lambda}$ if and only if there exists on Σ a function μ which satisfies the following properties*:

(1) *μ is holomorphic on Σ^* and there exist holomorphic functions f, g on $\mathbb{C}^\times$ with $\mu = f\nu + g$.*
(2) *$\sigma^*\mu = \mu^{-1}, \varrho^*\mu = \bar{\mu}^{-1}, \eta^*\mu = \bar{\mu}$ and $\mu = \pm 1$ at branch points of Σ.*
(3) *$d\ln\mu$ is a meromorphic 1-form with $d\ln\mu - d(\sum_{i=0}^{g-1} \gamma_i \lambda^{-i}\nu)$ holomorphic in a neighborhood of $\lambda = 0$ and $d\ln\mu + d(\sum_{i=0}^{g-1} \bar{\gamma}_i \lambda^{i+1-g}\nu)$ is holomorphic at $\lambda = \infty$.*

Proof. For $\gamma \in \mathbb{R}^g$, we write $\exp(\sum_{i=0}^{g-1} \gamma_i \lambda^{-i}\xi_\lambda) = F_\lambda(\gamma)B_\lambda(\gamma)$ for the Iwasawa decomposition. Then $\gamma \in \Gamma_{\xi_\lambda}$ if and only if $[F_\lambda(\gamma), \xi_\lambda] = [B_\lambda(\gamma), \xi_\lambda] = 0$. Hence, $F_\lambda(\gamma)$ and $B_\lambda(\gamma)$ act trivially on ξ_λ and $\xi_\lambda = \zeta_\lambda(0) = \zeta_\lambda(\gamma) = F_\lambda^{-1}(\gamma)\xi_\lambda F_\lambda(\gamma) = B_\lambda(\gamma)\xi_\lambda B_\lambda^{-1}(\gamma)$. Since $F_\lambda(\gamma)$ and $B_\lambda(\gamma)$ commute with ξ_λ, we have by (4.5) that $F_\lambda(\gamma) = f(\lambda)\xi_\lambda + \frac{1}{2}\operatorname{tr}(F_\lambda(\gamma))\mathbb{1}$ and $B_\lambda(\gamma) = e(\lambda)\xi_\lambda + \frac{1}{2}\operatorname{tr}(B_\lambda(\gamma))\mathbb{1}$ with functions e, f depending only on λ. In this case we define on the spectral curve Σ the eigenvalue function $\mu(\lambda)$ of $F_\lambda(\gamma)$ by

$$\mu(\lambda) = f(\lambda)\nu + g(\lambda) = f(\lambda)\nu + \frac{1}{2}\operatorname{tr}\big(F_\lambda(\gamma)\big).$$

We prove that μ satisfies properties (1), (2) and (3). Since $\lambda \mapsto F_\lambda(\gamma)$ is holomorphic, the function $\lambda \mapsto \operatorname{tr}(F_\lambda(\gamma))$ is holomorphic on Σ^*. Since $F_\lambda(\gamma)$ and ξ_λ commute, can write $f\xi_\lambda = F_\lambda(\gamma) - \frac{1}{2}\operatorname{tr}(F_\lambda(\gamma))\mathbb{1})$, and since ξ_λ has no zeroes conclude that $\lambda \mapsto f(\lambda)$ is holomorphic. Hence, $\mu : \tilde{\Sigma} \to \mathbb{C}$ is holomorphic and f, g have no poles on $\mathbb{C}^\times$, proving (1).

Property (2) follows since μ is the eigenvalue of $F_\lambda(\gamma)$ and F_λ satisfies (1.4).

At $\lambda = 0$, the eigenvalue of the matrix $B_\lambda(\gamma) = e(\lambda)\xi_\lambda + \frac{1}{2}\operatorname{tr}(B_\lambda(\gamma))\mathbb{1}$ is holomorphic, so $e(0) = 0$ (ξ_λ has a pole at $\lambda = 0$). Hence, $B_0(\gamma) = \mathbb{1}$. Since

$$\exp\left(\sum_{i=0}^{g-1} \gamma_i \lambda^{-i}\nu\right)$$

is the product of eigenvalues of the product $F_\lambda(\gamma)B_\lambda(\gamma)$, then at $\lambda = 0$, this is precisely the value of μ, which proves (3). Similarly, using $\varrho^* d\ln\mu = \overline{d\ln\mu}$ allows to deal with the point $\lambda = \infty$.

Conversely, assume $\mu : \Sigma \to \mathbb{C}$ satisfies conditions (1), (2) and (3). Assume there are two holomorphic function $f, g : \mathbb{C}^\times \to \mathbb{C}$ with $\mu = f\nu + g$. Hence, $F = f\xi_\lambda + g\mathbb{1}$ is holomorphic on Σ^* and belongs to the first factor of the Iwasawa decomposition (a consequence of (2)). Due to (3), the matrix

$$B_\lambda = F_\lambda^{-1} \exp\left(\sum_{i=0}^{g-1} \gamma_i \lambda^{-i} \xi_\lambda\right)$$

is holomorphic in a neighborhood of $\lambda = 0$ and $[B_\lambda, \xi_\lambda] = 0$. Since $B_\lambda - \frac{1}{2}\operatorname{tr}(B_\lambda)\mathbb{1}$ is at $\lambda = 0$ proportional to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, B_λ belongs to the second factor of the Iwasawa decomposition. This shows that $F_\lambda B_\lambda$ is the Iwasawa decomposition of

$$\exp\left(\sum_{i=0}^{g-1} \gamma_i \lambda^{-i} \xi_\lambda\right) = F_\lambda B_\lambda.$$

Since $[F_\lambda(\gamma), \xi_\lambda] = 0$, we conclude that $\gamma \in \Gamma_{\xi_\lambda}$. □

REMARK 5.6. Holomorphic functions $f, g : \mathbb{C}^\times \to \mathbb{C}$ are given by

$$\mu = \frac{1}{2}\left(\frac{\mu - \sigma^*\mu}{\nu}\right)\nu + \frac{1}{2}\left(\mu + \sigma^*\mu\right) = f\nu + g.$$

Theorem 8.2 in Forster [11] assures that f, g extend to holomorphic functions on $\mathbb{C}^\times$. At fixed points of the involution σ (zeroes of ν), the function $\mu - \sigma^*\mu$ has zeroes.

In case ν has higher order roots, the function $(\mu - \sigma^*\mu)/\nu$ may have a pole. Then the condition that $\mu = f\nu + g$ with f, g holomorphic is stronger than μ being holomorphic on Σ^*.

The 1-form $d\ln\mu$ is meromorphic on Σ, and changes sign under the hyperelliptic involution.

The closing conditions for regular spectral curves. In the following, we restrict to the case where a has only simple roots. Then the spectral curve Σ is a hyperelliptic curve without singularities. The closing condition is simplest as we only need to check the existence of the holomorphic ν by Remark 5.6.

PROPOSITION 5.7. *Let $a \in \mathbb{C}^{2g}[\lambda]$ satisfy (2.7) and (2.8). Then on Σ there exist for all $\tau \in \mathbb{C}^\times$ a unique meromorphic differential ϕ such that*

(1) $\sigma^*\phi = -\phi$, $\varrho^* = -\bar{\phi}$, $\eta^*\phi = \bar{\phi}$,
(2) *ϕ has second order poles at 0 and ∞ without residues, and no other poles,*
(3) $\phi = \frac{1}{4}\mathbb{i}\tau e^{\mathbb{i}\Theta/2}\, d\sqrt{\lambda}^{-1} + O(1)$ *at* $\lambda = 0$ *and* $\phi = \frac{1}{4}\mathbb{i}\bar{\tau} e^{-\mathbb{i}\Theta/2}\, d\sqrt{\lambda} + O(1)$ *at* $\lambda = \infty$,

(4) $\int_{\alpha_i}^{1/\bar{\alpha}_i} \phi = \mathrm{Re}(\int_{\alpha_i}^{1/\bar{\alpha}_i} \phi) = 0$ *for all roots* α_i *of* a *where the integral is computed along the line segment* $[\alpha_i, 1/\bar{\alpha}_i]$.

In conclusion, there exists a unique $b \in \mathbb{C}^{g+1}[\lambda]$ *with* $\phi = \frac{b\,d\lambda}{\nu\lambda^2}$ *which satisfies* $\lambda^{g+1}\bar{b}(\bar{\lambda}^{-1}) = -b(\lambda)$.

Proof. We make the Ansatz that $\phi = \frac{b\,d\lambda}{\nu\lambda^2}$. The conditions (1), (2) and (3) fix the highest and lowest coefficient of b, so there remain g real free coefficients of b. These coefficients correspond to holomorphic differentials satisfying condition (4) $\mathrm{Re}(\int_{\alpha_i}^{1/\bar{\alpha}_i} \phi) = 0$. The first equality comes from the reality condition (1) and reads

$$\int_{\alpha_i}^{\alpha_i/|\alpha_i|} \phi = \overline{\int_{\alpha_i/|\alpha_i|}^{1/\bar{\alpha}_i} \phi}.$$

Concerning uniqueness, holomorphic differentials whose integrals along cycles are imaginary, are zero by Riemann's bilinear relations. □

Definition 5.8. We define a compact Riemann surface with boundary by $\hat{\Sigma} = \Sigma \setminus \bigcup \gamma_i$ where γ_i are closed cycles over the straight lines connecting the branch points α_i and $1/\bar{\alpha}_i$.

Corollary 5.9. *Let* $a \in \mathbb{C}^{2g}[\lambda]$ *satisfy* (2.7) *and* (2.8).

(1) *If there is* $\tau \in \mathbb{C}$ *and* b *which satisfies* (1)–(4) *of Proposition* 5.7, *then there exists a unique meromorphic function* $h(\lambda) : \hat{\Sigma} \to \mathbb{C}$ *such that* $\sigma^* h(\lambda) = -h(\lambda)$ *and* $dh = \frac{b\,d\lambda}{\nu\lambda^2}$.

(2) $I(a)$ *corresponds to minimal annuli in* $\mathbb{S}^2 \times \mathbb{R}$ *if and only if there exists* $\tau \in \mathbb{C}^\times$ *with* $\tau e^{\mathbb{i}\Theta/2} \in \mathbb{R}^\times$, *and such that the polynomial* b *defining the function* $h(\lambda) : \hat{\Sigma} \to \mathbb{C}$, *satisfies* $\sigma^* h(\lambda) = -h(\lambda)$ *and* $dh = \frac{b\,d\lambda}{\nu\lambda^2}$. *This function continuously extends to boundary segments connecting* α_i *and* $1/\bar{\alpha}_i$ *and then takes values on* $\mathbb{i}\pi\mathbb{Z}$ *at all roots of* $(\lambda - 1)a(\lambda)$.

Proof. (1) In a small neighborhood of Σ over $\lambda = 0$, the function h is uniquely determined by $dh = \frac{b\,d\lambda}{\nu\lambda^2}$ up to some additive constant. This constant is determined by $\sigma^* h(\lambda) = -h(\lambda)$ in this small neighborhood. By conditions on b, we have $\int_{\gamma_i} dh = 0$, so

$$\int_{\gamma_i} \frac{b\,d\lambda}{\nu\lambda^2} = 2\int_{\alpha_i}^{1/\bar{\alpha}_i} \frac{b\,d\lambda}{\nu\lambda^2} = 0.$$

Now we can uniquely extend the function h to $\hat{\Sigma}$.

(2) For an immersed annulus

$$X_1(z) = \big(F_1(z)\sigma_3 F_1^{-1}(z), \mathrm{Re}\big(-\mathbb{i}\, e^{\mathbb{i}\Theta/2} z\big)\big)$$

in $\mathbb{S}^2 \times \mathbb{R}$ the extended frame $F_\lambda(z)$ admits a period $\tau \in \mathbb{C}^\times$ with $\tau e^{\mathbb{i}\Theta/2} \in \mathbb{R}$, and periodic Killing field $\zeta_\lambda(z + \tau) = \zeta_\lambda(z)$. This implies that $d\ln\mu$ is a

meromorphic differential with second order pole at 0 and ∞ without residues and no other poles and satisfies condition (1)–(3) of Proposition 5.7. The integrals of $d\ln\mu$ along closed cycles are integer multiples of $2\pi\mathbb{i}$, since the function μ is globally single-valued by condition (4). Since the extended frame $F_1(z)$ is periodic then $\mu(1) = \pm 1$. Since $\mu^2 = 1$ at branch points α_i, it is equivalent to the condition $\ln\mu(\alpha_i) \in \mathbb{i}\pi\mathbb{Z}$. Hence, there is a polynomial b associated to $\tau \in \mathbb{C}$ such that $d\ln\mu = \frac{b\,d\lambda}{\nu\lambda^2}$.

Conversely, consider (a, b) such that on the spectral curve Σ the meromorphic differential $\phi = \frac{b\,d\lambda}{\nu\lambda^2}$ satisfies $\int_1^{\alpha_i} \phi \in \pi\mathbb{i}\mathbb{Z}$ and $\int_{\alpha_i}^{\alpha_j} \phi \in \pi\mathbb{i}\mathbb{Z}$. While $h(\lambda) = \int_1^{\lambda} \phi$ is a multiple-valued function on Σ, the function $e^h : \Sigma \to \mathbb{C}$ is again holomorphic. It is described in Proposition 5.5 where τ is given by the residue of ϕ at 0 and ∞. Then $\zeta_\lambda(z+\tau) = \zeta_\lambda(z)$, and we can integrate the extended frame F_λ. There remains to prove that $F_1(\tau) = \pm\mathbb{1}$. We remark that a solution of the characteristic equation of a solution of $F_\lambda^{-1}\,dF_\lambda = \alpha(\zeta_\lambda)$ defines a meromorphic differential $d\ln\mu$ which satisfies (1)–(3) of Proposition 5.7. By uniqueness of such differentials, $d\ln\mu = \phi$ and $\mu(1) = e^{h(1)} = \pm 1$ so that F_1 is τ-periodic. □

Definition 5.10. The spectral data of a minimal cylinder of finite type in $\mathbb{S}^2 \times \mathbb{R}$ with sym point at $\lambda = 1$ is a pair $(a, b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ such that

(i) $\lambda^{2g}\overline{a(\bar\lambda^{-1})} = a(\lambda)$, $\lambda^{-g}a(\lambda) \le 0$ for all $\lambda \in S^1$ and $a(0) = -\frac{1}{16}e^{\mathbb{i}\Theta}$,

(ii) $\lambda^{g+1}\overline{b(\bar\lambda^{-1})} = -b(\lambda)$,

(iii) $b(0) = \frac{\tau e^{\mathbb{i}\Theta}}{32} \in e^{\mathbb{i}\Theta/2}\mathbb{R}$ (closing condition of the third coordinate),

(iv) $\operatorname{Re}(\int_{\alpha_i}^{1/\bar\alpha_i} \frac{b\,d\lambda}{\nu\lambda^2}) = 0$ for all roots α_i of a where the integral is computed along the straight segment $[\alpha_i, 1/\bar\alpha_i]$,

(v) the unique function $h : \tilde\Sigma \to \mathbb{C}$, where $\tilde\Sigma = \Sigma \setminus \bigcup\gamma_i$ and γ_i are closed cycles over the straight lines connecting α_i and $1/\bar\alpha_i$, satisfies

$$\sigma^* h(\lambda) = -h(\lambda) \quad \text{and} \quad dh = \frac{b\,d\lambda}{\nu\lambda^2}.$$

This function continuously extends to boundary segments connecting α_i and $1/\bar\alpha_i$ and then takes values on $\mathbb{i}\pi\mathbb{Z}$ at all roots of $(\lambda - 1)a(\lambda)$,

(vi) when a has higher order roots then $e^h = f\nu + g$ for holomorphic $f, g : \mathbb{C}^\times \to \mathbb{C}$ with $f(1) = 0$.

6. Bubbletons

The term 'bubbleton' is due to Sterling–Wente [27]. They are the solitons of the theory, and finite type solutions of the sinh-Gordon equations with bubbletons have singular spectral curves. For more details on the relationship between bubbletons, Bianchi–Bäcklund transformations, simple factors and CMC surfaces we refer to [17], [16], [18] and the references therein. By Proposition 4.4, we can discard roots of a potential ξ_λ without changing the

surface, but it remains to discuss higher order roots of $\det\xi_\lambda$ where ξ_λ is not semi-simple.

We now consider potentials ξ_λ for which the polynomial $a(\lambda) = -\lambda\det\xi_\lambda$ has higher order roots. Roots of ξ_λ come in symmetric pairs $\alpha_0, 1/\bar\alpha_0 \in \mathbb{C}^\times$ and we set in this section $\delta = -\bar\alpha_0/|\alpha_0|$. Because the polynomial $\lambda \mapsto a(\lambda)$ is homogeneous of degree 2, such roots generate even roots of order at least 2 in the polynomial a. Then there exists a polynomial $\tilde a$ with

$$a(\lambda) = (\lambda-\alpha_0)^2(1-\bar\alpha_0\lambda)^2\tilde a(\lambda).$$

We study such ξ_λ and show that they can be factored into a product of *simple factors* (see below) and a potential $\tilde\xi_\lambda \in I(\tilde a)$. If $\xi_{\alpha_0} \neq 0$ but $\operatorname{ord}_{\alpha_0}\det\xi_\lambda \geq 2$, then the matrix ξ_{α_0} is nilpotent and defines a complex line

$$L = \ker\xi_{\alpha_0} = \operatorname{im}\xi_{\alpha_0} \in \mathbb{CP}^1.$$

Let $v_1 \in \mathbb{C}^2$ be a unit vector which spans L, and complement v_1 to an orthonormal basis (v_1, v_2) of $\mathbb{C}^2$. Set

$$p(\lambda) = (\lambda-\alpha_0)(1-\bar\alpha_0\lambda)$$

and let $Q_L \in \mathrm{SU}_2$ with $Q_L(e_1) = v_1$. The entries of ξ_λ in the basis (v_1, v_2) are given by

$$Q_L^{-1}\xi(\lambda)Q_L := \begin{pmatrix} \tilde\alpha(\lambda)p(\lambda) & \tilde\beta(\lambda)(1-\bar\alpha_0\lambda)^2 \\ \tilde\gamma(\lambda)(\lambda-\alpha_0)^2 & -\tilde\alpha(\lambda)p(\lambda)\end{pmatrix}, \tag{6.1}$$

where $\lambda\tilde\alpha(\lambda)^2 - \lambda\tilde\beta(\lambda)\tilde\gamma(\lambda) = \tilde a(\lambda)$. If $(u_1,u_2) \in \ker\xi_{\alpha_0}$ then $(-\bar u_2, \bar u_1) \in \ker\xi_{1/\bar\alpha_0}$.

Simple factors. Define for $\lambda \neq \alpha_0, \bar\alpha_0^{-1}$ the $\mathrm{SL}_2(\mathbb{C})$-valued maps

$$\pi_{\alpha_0}(\lambda) := \begin{pmatrix} \sqrt{\frac{\lambda-\alpha_0}{1-\bar\alpha_0\lambda}} & 0 \\ 0 & \sqrt{\frac{1-\bar\alpha_0\lambda}{\lambda-\alpha_0}}\end{pmatrix} \quad\text{and}\quad \pi_L := Q_L\pi_{\alpha_0}Q_L^{-1}.$$

At $\lambda = 0$, we apply the QR-decomposition to get

$$\pi_L(0) = Q_L\pi_{\alpha_0}(0)Q_L^{-1} = Q_{0,L}R_{0,L} = R_{1,L}Q_{1,L},$$

where $Q_{0,L}, Q_{1,L} \in \mathrm{SU}_2$ and $R_{0,L}, R_{1,L}$ are upper triangular matrices of the form $\left(\begin{smallmatrix}\rho & r\\ 0 & \rho^{-1}\end{smallmatrix}\right)$ for $\rho \in \mathbb{R}^+$ both depending on α_0 and L.

Definition 6.1. Let $L \in \mathbb{CP}^1$ and $\alpha_0 \in \mathbb{C}$ with $r < \min\{|\alpha_0|, 1/|\alpha_0|\}$. A *left simple factor* is a map

$$g_{L,\alpha_0}(\lambda) = \pi_L^{-1}(\lambda)Q_{0,L}.$$

Then $H^\ell_{\alpha_0} = \{g_{L,\alpha_0} \mid L \in \mathbb{CP}^1\} \cong \mathbb{CP}^1$, and $H^\ell_{\alpha_0} \subset \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$.

A *right simple factor* is a map

$$h_{L,\alpha_0}(\lambda) = Q_{1,L}\pi_L^{-1}(\lambda).$$

Then $H^r_{\alpha_0} = \{h_{L,\alpha_0} \mid L \in \mathbb{CP}^1\} \cong \mathbb{CP}^1$, and $H^r_{\alpha_0} \subset \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$.

LEMMA 6.2. *There is a one-to-one correspondence between $H^{\ell}_{\alpha_0}$ and $H^{r}_{\alpha_0}$ via*

$$g_{L,\alpha_0} = \pi_L^{-1} Q_{0,L} = Q_{0,L} \pi_{L'}^{-1} = h_{L',\alpha_0}, \tag{6.2}$$

where

$$L' = Q_{0,L}^{-1} L \quad \text{and} \quad Q_{1,L'} = Q_{0,L}.$$

Conversely, if L' is given then we can define a QR-decomposition $\pi_{L'} = R_{1,L'} Q_{1,L'}$ with

$$L = Q_{1,L'} L'.$$

Proof. For any $A \in \mathrm{SU}_2$ and $L' = AL$ we have $Q_{AL} = A Q_L$. Hence $\pi_{L'} = Q_{AL} \pi_{\alpha_0} Q_{AL}^{-1} = A \pi_L A^{-1}$. Now we define $L' = Q_{0,L}^{-1} L$ and we have $g_{L,\alpha_0}^{-1} = Q_{0,L}^{-1} \pi_L = \pi_{L'} Q_{0,L}^{-1}$. With $\pi_L(0) = Q_{0,L} R_{0,L}$, we observe that

$$\pi_{L'}(0) = Q_{0,L}^{-1} \pi_L(0) Q_{0,L} = R_{0,L} Q_{0,L} = R_{1,L'} Q_{1,L'}$$

and hence $Q_{1,L'} = Q_{0,L}$. Then $g_{L,\alpha_0}^{-1}(\lambda) = \pi_{L'} Q_{0,L}^{-1} = \pi_{L'} Q_{1,L'}^{-1} = h_{L',\alpha_0}^{-1}(\lambda)$. If $L' = Q_{0,L}^{-1} L$, so that $L = Q_{0,L} L' = Q_{1,L'} L'$. □

Now we consider the matrix $\breve{\xi}_\lambda = p^{-1}(\lambda) g_{L,\alpha_0}^{-1} \xi_\lambda g_{L,\alpha_0}$.

PROPOSITION 6.3. *Let $\xi_\lambda \in I(a)$ with $\xi_{\alpha_0} \neq 0$ and $\operatorname{ord}_{\alpha_0} \det \xi_\lambda \geq 2$. Let $\delta = -\bar{\alpha}_0/|\alpha_0|$. Then ξ_λ uniquely factorizes as*

$$\xi_\lambda = p(\lambda) g_{L,\alpha_0} \breve{\xi}_\lambda g_{L,\alpha_0}^{-1}$$

with $L = \ker \xi_{\alpha_0}$ and $\breve{\xi}_\lambda \in \mathcal{P}_g(\delta)$ with $\tilde{\xi}_\lambda = g(\delta)^{-1} \breve{\xi}_\lambda g(\delta) \in I(\tilde{a})$ (i.e., $\breve{\xi}_\lambda \in I_\delta(\tilde{a})$).

Proof. The matrix ξ_{α_0} uniquely defines the line L. Consider

$$\breve{\xi}_\lambda = p^{-1}(\lambda) g_{L,\alpha_0}^{-1} \xi_\lambda g_{L,\alpha_0}.$$

We need to prove that $\breve{\xi}_\lambda$ has no poles at $\alpha_0, \bar{\alpha}_0^{-1}$ and $\breve{\xi}_\lambda \in I_\delta(\tilde{a})$. First, define

$$Q_L^{-1} \xi_\lambda Q_L := \begin{pmatrix} \tilde{\alpha}(\lambda) p(\lambda) & \tilde{\beta}(\lambda)(1 - \bar{\alpha}_0 \lambda)^2 \\ \tilde{\gamma}(\lambda)(\lambda - \alpha_0)^2 & -\tilde{\alpha}(\lambda) p(\lambda) \end{pmatrix}.$$

The following matrix has neither pole nor zero at $\lambda = \alpha_0, \bar{\alpha}_0^{-1}$:

$$Q_L^{-1} Q_{0,L} \breve{\xi}_\lambda Q_{0,L}^{-1} Q_L = \frac{\pi_{\alpha_0}}{\sqrt{p}} Q_L^{-1} \xi_\lambda Q_L \frac{\pi_{\alpha_0}^{-1}}{\sqrt{p}} = \begin{pmatrix} \tilde{\alpha}(\lambda) & \tilde{\beta}(\lambda) \\ \tilde{\gamma}(\lambda) & -\tilde{\alpha}(\lambda) \end{pmatrix}.$$

The residue of $\breve{\xi}_\lambda$ at $\lambda = 0$ is $R_{0,L} \delta \hat{\xi}_{-1} R_{0,L}^{-1}$. As $R_{0,L} \in \Lambda^+ \mathrm{SL}_2(\mathbb{C})$, we conclude that the residue takes values in $\mathbb{i}\delta\mathbb{R}^+ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore $\breve{\xi}_\lambda \in \mathcal{P}_{g-2}(\delta)$. By Remark 2.6 on Isometric normalization, we conclude that $\tilde{\xi}_\lambda = g(\delta)^{-1} \breve{\xi}_\lambda g(\delta) \in I(\tilde{a})$. □

Using the relation (6.2) between left and right factors immediately yields

COROLLARY 6.4. *Let* $\xi_\lambda \in I(a)$ *with* $\xi_{\alpha_0} \neq 0$ *and* $\operatorname{ord}_{\alpha_0} a(\lambda) \geq 2$. *Then* ξ_λ *uniquely factorizes as*

$$\xi_\lambda = p(\lambda) h_{L',\alpha_0} \breve{\xi}_\lambda h_{L',\alpha_0}^{-1}$$

with $(L', \breve{\xi}_\lambda) \in \mathbb{CP}^1 \times I_\delta(\tilde{a})$ *where* $L' = Q_{0,L}^{-1} L$ *and* $L = \ker \xi_{\alpha_0}$.

REMARK 6.5. The factorizations of Proposition 6.3 and Corollary 6.4 give rise to pairs in $(L', \breve{\xi}_\lambda) \in \mathbb{CP}^1 \times I_\delta(\tilde{a})$, and we say that we *decompose* such ξ_λ into $(L, \breve{\xi}_\lambda)$ or $(L', \breve{\xi}_\lambda)$ or into $(\tilde{L}, \tilde{\xi}_\lambda)$ or $(\tilde{L}', \tilde{\xi}_\lambda)$ where $\tilde{\xi}_\lambda = g(\delta)^{-1} \breve{\xi}_\lambda g(\delta) \in I(\tilde{a})$ and $\tilde{L} = g(\delta)^{-1} L, \tilde{L}' = g(\delta)^{-1} L'$ depending on the context.

A special situation occurs when $L'^{\perp}$ is an eigenline of $\breve{\xi}_{\alpha_0}$.

PROPOSITION 6.6. *Suppose* ξ_λ *decomposes into* $(L', \breve{\xi}_\lambda)$ *and* $\breve{\xi}_{\alpha_0} L'^{\perp} = L'^{\perp}$. *Then* ξ_λ *has zeroes at* $\lambda = \alpha_0, 1/\bar{\alpha}_0$. *Furthermore the singularity of the spectral curve is removable and up to a conformal change of coordinate the potentials* ξ_λ *and* $\breve{\xi}_\lambda$ *induce the same extended frame* $\breve{F}_\lambda(p(0)z) = F_\lambda(z)$.

Proof. We prove that

$$\dim_{\mathbb{C}} \ker \xi_{\alpha_0} = \dim_{\mathbb{C}} \ker p(\alpha_0) \pi_{L'}^{-1}(\alpha_0) \breve{\xi}_{\alpha_0} \pi_{L'}(\alpha_0) = 2.$$

If $L'^{\perp} = \langle v_2 \rangle$ is an eigenline of $\breve{\xi}_{\alpha_0}$, then $\pi_{L'}(\lambda) v_2 = \sqrt{\frac{1-\bar{\alpha}_0 \lambda}{\lambda - \alpha_0}} v_2$ and

$$\pi_{L'}^{-1}(\alpha_0) \breve{\xi}_{\alpha_0} \pi_{L'}(\alpha_0) v_2 = \mu v_2 \quad \text{and} \quad \xi_{\alpha_0} v_2 = 0.$$

If $L' = \langle v_1 \rangle$, we have $\pi_{L'}(\lambda) v_1 = \sqrt{\frac{\lambda - \alpha_0}{1-\bar{\alpha}_0 \lambda}} v_1$ and

$$p(\alpha_0) \pi_{L'}^{-1}(\alpha_0) \breve{\xi}_{\alpha_0} \pi_{L'}(\alpha_0) v_1 = 0.$$

Then ξ_λ has a zero at $\lambda = \alpha_0$ and we can remove it without changing the extended frame by Proposition 4.4. □

Terng–Uhlenbeck formula. Let $\xi_\lambda \in \mathcal{P}_g$ with $\xi_{\alpha_0} \neq 0, \operatorname{ord}_{\alpha_0} a(\lambda) \geq 2$ for some $\alpha_0 \in \mathbb{C}^\times \setminus \mathbb{S}^1$. Suppose ξ_λ decomposes into $(L', \breve{\xi}_\lambda)$, and let $0 < r < \min\{|\alpha_0|, 1/|\alpha_0|\}$. Now consider the unitary factor $F_\lambda : \mathbb{R}^2 \to \Lambda_r \mathrm{SU}_2(\mathbb{C})$ of the r-Iwasawa decomposition

$$\exp(z\xi_\lambda) = F_\lambda B_\lambda$$

and define $\breve{F}_\lambda : \mathbb{R}^2 \to \Lambda_r \mathrm{SU}_2(\mathbb{C})$ to be the unitary factor of the r-Iwasawa decomposition

$$\exp\big(z p(\lambda) \breve{\xi}_\lambda\big) = \breve{F}_\lambda \breve{B}_\lambda.$$

Terng–Uhlenbeck [29] obtained a relationship between F_λ and $\breve{F}_\lambda$ and found

$$F_\lambda(z) = h_{L',\alpha_0} \breve{F}_\lambda(z) h_{L'(z),\alpha_0}^{-1} \quad \text{with } L'(z) = {}^t\overline{\breve{F}}_{\alpha_0}(z) L'.$$

We provide a proof of this in the Appendix. We next show that closing conditions are preserved when changing the first factor in the factorization $(L', \breve{\xi}_\lambda)$.

PROPOSITION 6.7. *If there is* $(L_0', \breve{\xi}_\lambda) \in \mathbb{CP}^1 \times I_\delta(\tilde{a})$, *such that*

$$\xi_{0,\lambda} = p(\lambda) h_{L_0',\alpha_0} \breve{\xi}_\lambda h_{L_0',\alpha_0}^{-1}$$

induces a minimal annulus with period τ, *then for any* $L_1' \in \mathbb{CP}^1$, *the extended frame associate to* $\xi_{1,\lambda} = p(\lambda) h_{L_1',\alpha_0} \breve{\xi}_\lambda h_{L_1',\alpha_0}^{-1}$ *produces a* τ*-periodic minimal annulus.*

Proof. The annulus induced by $F_{0,\lambda}$ is periodic at $\lambda = 1$, so $F_{0,1}(z+\tau) = F_{0,1}(z)$. In particular $F_{0,1}(\tau) = \mathbb{1}$ and the solution of sinh-Gordon equation is periodic $\omega_0(z+\tau) = \omega_0(z)$. This period condition on ω_0 implies $\zeta_{0,\lambda}(z+\tau) = \zeta_{0,\lambda}(z)$ for any $\lambda \in \mathbb{C}^\times$ and $z \in \mathbb{C}$, since entries of $\zeta_{0,\lambda}$ depend only on ω_0 and its derivatives. Now we remark that the associate polynomial Killing field $\zeta_{0,\lambda}(z) \in I(a)$ decomposes uniquely as $(L_0'(z), \breve{\zeta}_\lambda(z)) \in \mathbb{CP}^1 \times I_\delta(\tilde{a})$ where $L_0'(z) = Q_{0,L_0(z)}^{-1} L_0(z)$ and $L_0(z) = \ker \zeta_{0,\alpha_0}(z)$. Thus, $\zeta_{0,\alpha_0}(z)$ periodic implies that $L_0(z)$ and $L_0'(z)$ are periodic. The decomposition of $\zeta_{0,\lambda}(z)$ is given by

$$\zeta_{0,\lambda}(z) = p(\lambda) g_{L_0(z),\alpha_0} \breve{\zeta}_\lambda(z) g_{L_0(z),\alpha_0}^{-1} = p(\lambda) h_{L_0'(z),\alpha_0} \breve{\zeta}_\lambda(z) h_{L_0'(z),\alpha_0}^{-1}$$

and $\breve{\zeta}_\lambda(z+\tau) = \breve{\zeta}_\lambda(z)$ for any $\lambda \in \mathbb{C}^\times$. We can recover explicitly this relation by using the decomposition with $L' = Q_{0,L}^{-1} L$ and the formula

$$F_\lambda(z) = h_{L_0',\alpha_0} \breve{F}_\lambda(z) h_{L_0'(z),\alpha_0}^{-1},$$

where $L_0'(z) = {}^t\overline{\breve{F}}_{\alpha_0}(z) L_0'$. Then

$$\begin{aligned} \zeta_{0,\lambda}(z) &= F_\lambda(z)^{-1} \xi_{0,\lambda} F_\lambda(z) \\ &= h_{L_0'(z),\alpha_0} \breve{F}_\lambda^{-1}(z) h_{L_0',\alpha_0}^{-1} \xi_{0,\lambda} h_{L_0',\alpha_0} \breve{F}_\lambda(z) h_{L_0'(z),\alpha_0}^{-1}. \end{aligned}$$

Since $L_0(z+\tau) = L_0(z)$, we have $L_0'(z+\tau) = L_0'(z)$. Periodicity $F_1(\tau) = \mathbb{1}$ implies $\breve{F}_1(\tau) = \mathbb{1}$, and then there is a τ-periodic extended frame without a bubbleton associate to the polynomial Killing field $\breve{\zeta}_\lambda$. Now we consider $\xi_{1,\lambda}$ associate to $(L_1', \breve{\xi}_\lambda)$ and we prove that its extended frame has the same period τ. To see that it remains to prove that $\breve{F}_{\alpha_0}(\tau) = \mathbb{1}$ to conclude the periodicity $L_1'(\tau + z) = {}^t\overline{\breve{F}}_{\alpha_0}(\tau) L_1'(z) = L_1'(z)$.

If $L_0'(\tau) = L_0'$, then $L_0' = L_0'(\tau) = {}^t\bar{\breve{F}}_{\alpha_0}(\tau) L_0'$ and L_0' is an eigenvalue of ${}^t\overline{\breve{F}}_{\alpha_0}(\tau) = \breve{F}_{1/\bar{\alpha}_0}^{-1}(\tau)$, hence $(L_0')^\perp$ is an eigenline of $\breve{F}_{\alpha_0}(\tau)$.

Now $\breve{\zeta}_\lambda(\tau) = \breve{\zeta}_\lambda(0)$ for all $\lambda \in \mathbb{C}^\times$ implies $[\breve{\xi}_{0,\alpha_0}, \breve{F}_{\alpha_0}(\tau)] = 0$. Thus $(L_0')^\perp$ is not an eigenline of $\breve{\xi}_{0,\alpha_0}$, since the polynomial Killing field ξ_{0,α_0} would have a zero and we could remove the singularity of the spectral curve.

Recall that if $A \in \mathrm{SL}_2(\mathbb{C})$ and $B \in \mathrm{sl}_2(\mathbb{C})$, then $[A,B] = 0$ implies $A = xB + y\mathbb{1}$. This implies $\breve{F}_{\alpha_0}(\tau) = x\breve{\xi}_{0,\alpha_0} + y\mathbb{1}$. Now $\breve{\xi}_{0,\alpha_0} L_0^\perp \neq L_0^\perp$ implies $x = 0$ and thus $\breve{F}_{\alpha_0}(\tau) = \pm\mathbb{1}$. □

Group action on bubbletons. We prove that there is a group action which acts transitively on the first factor $\tilde{L}'$ of the decomposition ξ_λ into $(\tilde{L}', \tilde{\xi}_\lambda)$. For $\xi_\lambda \in \mathcal{P}_g$ and $\beta \in \mathbb{C}$, define

$$m(\beta)\xi_\lambda = \begin{cases} (\frac{\beta}{\lambda-\alpha_0} + \frac{\bar{\beta}\lambda}{1-\bar{\alpha}_0\lambda})\lambda^{\frac{1-g}{2}}\xi_\lambda, & \text{when } g = 2k+1, \\ (\frac{\beta}{\lambda-\alpha_0} + \frac{\bar{\beta}\lambda}{1-\bar{\alpha}_0\lambda})(\lambda^{\frac{-g}{2}} + \lambda^{1-\frac{g}{2}})\xi_\lambda, & \text{when } g = 2k. \end{cases}$$

Let $\exp\big(m(\beta)\xi_\lambda\big) = F_\lambda(\beta)B_\lambda(\beta)$ be the r-Iwasawa factorization for $r < \min\{|\alpha_0|, 1/|\alpha_0|\}$. We have a complex 1-dimensional isospectral group action $\tilde{\pi} : \mathbb{C} \times I(a) \to I(a)$ defined by

$$\xi_\lambda(\beta) := \tilde{\pi}(\beta)\xi_\lambda = F_\lambda(\beta)^{-1}\xi_\lambda F_\lambda(\beta) = B_\lambda(\beta)\xi_\lambda B_\lambda^{-1}(\beta). \tag{6.3}$$

The potential $\xi_\lambda(\beta)$ has a decomposition $\big(\tilde{L}'(\beta), \tilde{\xi}_\lambda(\beta)\big)$, and we prove that $\tilde{\xi}_\lambda(\beta) = \tilde{\xi}_\lambda$ is invariant under this action.

THEOREM 6.8. *Suppose a potential ξ_λ decomposes into $\big(\tilde{L}', \tilde{\xi}_\lambda\big) \in \mathbb{CP}^1 \times I(\tilde{a})$. Then the action (6.3) on ξ_λ preserves the second term $\tilde{\xi}_\lambda$. If $\det \tilde{\xi}_{\alpha_0} \neq 0$, then $\mathbb{C}$ acts on $\tilde{L}' \in \mathbb{CP}^1 \setminus \big\{\tilde{L}'_1, \tilde{L}'_2\big\}$ transitively where $\big(\tilde{L}'_1\big)^\perp, \big(\tilde{L}'_2\big)^\perp$ are eigenlines of $\tilde{\xi}_{\alpha_0}$ and fixed points of the action. If $\det \tilde{\xi}_{\alpha_0} = 0$, then $\mathbb{C}$ acts on $\tilde{L}' \in \mathbb{CP}^1 \setminus \big\{\tilde{L}'_3\big\}$ transitively where $\big(\tilde{L}'_3\big)^\perp$ is the eigenline of $\tilde{\xi}_{\alpha_0}$.*

Proof. Using Remark 6.5, we prove the theorem with $L' = g(\delta)\tilde{L}'$ and $\breve{\xi}_\lambda = g(\delta)\tilde{\xi}_\lambda g(\delta)^{-1}$. Now $\tilde{L}'$ is an eigenline of $\tilde{\xi}_{\alpha_0}$ if and only if L' is an eigenline of $\breve{\xi}_\lambda$. We consider for $\beta \in \mathbb{C}$ the map

$$\psi(\lambda, \beta) = \begin{cases} (\frac{\beta}{\lambda-\alpha_0} + \frac{\bar{\beta}\lambda}{1-\bar{\alpha}_0\lambda})\lambda^{\frac{1-g}{2}}p(\lambda)\breve{\xi}_\lambda, & \text{when } g = 2k+1, \\ (\frac{\beta}{\lambda-\alpha_0} + \frac{\bar{\beta}\lambda}{1-\bar{\alpha}_0\lambda})(\lambda^{\frac{-g}{2}} + \lambda^{1-\frac{g}{2}})p(\lambda)\breve{\xi}_\lambda, & \text{when } g = 2k. \end{cases}$$

Then ψ satisfies ${}^t\overline{\psi(1/\bar{\lambda}, \beta)} = -\psi(\lambda, \beta)$ and $\psi \in \Lambda_r\mathfrak{su}_2$. We have by the r-Iwasawa decomposition $\exp(\psi(\lambda, \beta)) = \breve{F}_\lambda(\beta)$. Further, since $\breve{B}_\lambda(\beta) = \mathbb{1}$ we have

$$\breve{F}_\lambda^{-1}(\beta)\breve{\xi}_\lambda\breve{F}_\lambda(\beta) = \breve{B}_\lambda(\beta)\breve{\xi}_\lambda\breve{B}_\lambda^{-1}(\beta) = \breve{\xi}_\lambda.$$

Suppose $\xi_\lambda(\beta)$ has a decomposition $(L'(\beta), \breve{\xi}_\lambda(\beta))$. We prove next that $\breve{\xi}_\lambda(\beta) = \breve{\xi}_\lambda$ is invariant by the group action $m(\beta)\xi_\lambda$.

From $F_\lambda(\beta)B_\lambda(\beta) = h_{L',\alpha_0}\exp(\psi(\lambda, \beta))h_{L',\alpha_0}^{-1} = h_{L',\alpha_0}\breve{F}_\lambda(\beta)h_{L',\alpha_0}^{-1}$, and by the Terng–Uhlenbeck formula we obtain

$$F_\lambda(\beta) = h_{L',\alpha_0}\breve{F}_\lambda(\beta)h_{L'(\beta),\alpha_0}^{-1}, \quad \text{where } L'(\beta) = {}^t\overline{\breve{F}}_{\alpha_0}(\beta)L'.$$

Applying the action and the invariance of the conjugation of $\breve{\xi}_\lambda$ by $\breve{F}_\lambda(\beta)$ we have

$$\xi_\lambda(\beta) = \tilde{\pi}(\beta)\xi_\lambda = p(\lambda)h_{L'(\beta),\alpha_0}\breve{\xi}_\lambda h_{L'(\beta),\alpha_0}^{-1}.$$

This proves that $\breve{\xi}_\lambda$ is invariant while $L'(\beta)$ changes under the group action. We consider now the map $\beta \mapsto L'(\beta)$, and prove that this map spans $\mathbb{CP}^1$ (if $L'(0) \neq L'_0$ and $L'(0) \notin \{\tilde{L}'_1, \tilde{L}'_2, \tilde{L}'_3\}$).

First, we assume $\det \breve{\xi}_{\alpha_0} \neq 0$ (so also $\det \breve{\xi}_{1/\bar{\alpha}_0} \neq 0$). We denote by v_1, v_2 the eigenvectors of $\breve{\xi}_{1/\bar{\alpha}_0}$, and by ν_1, ν_2 the corresponding eigenvalues. Since $\operatorname{tr} \breve{\xi}_\lambda = 0$, we have $\nu_2 = -\nu_1$. Now we have for

$$\gamma_0 = \begin{cases} \bar{\alpha}_0^{g/2-5/2}(1-|\alpha_0|^2) & \text{for } g \text{ odd}, \\ \bar{\alpha}_0^{g/2-3}(1-|\alpha_0|^2) & \text{for } g \text{ even}, \end{cases}$$

that ${}^t\overline{\psi(\alpha_0,\beta)} = -\psi(1/\bar{\alpha}_0,\beta) = -\bar{\beta}\gamma_0 \breve{\xi}_{1/\bar{\alpha}_0}$ and ${}^t\overline{\breve{F}}_{\alpha_0}(\beta) = \exp(-\psi(1/\bar{\alpha}_0,\beta))$. With $L' = \langle xv_1 + yv_2 \rangle$ we obtain

$$\begin{aligned} L'(\beta) &= {}^t\overline{\breve{F}}_{\alpha_0}(\beta)(xv_1+yv_2) = \exp\bigl(-\psi(1/\bar{\alpha}_0,\beta)\bigr)(xv_1+yv_2) \\ &= \exp(-\bar{\beta}\gamma_0\nu_1)xv_1 + \exp(-\bar{\beta}\gamma_0\nu_2)yv_2 = w. \end{aligned}$$

Hence,

$${}^t\overline{\breve{F}}_{\alpha_0}(\beta)L' = \langle w \rangle$$

and the map $L' : \mathbb{CP}^1 \to \mathbb{CP}^1$ given by $L'(\beta) = \langle w \rangle$ is surjective since

$$\beta : \mathbb{CP}^1 \to \bigl\langle \exp(-\bar{\beta}\gamma_0\nu_1)xv_1 + \exp(\bar{\beta}\gamma_0\nu_1)yv_2 \bigr\rangle \in \mathbb{CP}^1$$

is surjective.

Now assume $\det \breve{\xi}_{\alpha_0} = \det \breve{\xi}_{1/\bar{\alpha}_0} = 0$. There exists a basis (v_1, v_2) with $\breve{\xi}_{1/\bar{\alpha}_0} v_1 = 0$ and $\breve{\xi}_{1/\bar{\alpha}_0} v_2 = v_1$, and then

$$\exp\bigl(-\psi(1/\bar{\alpha}_0,\beta)\bigr)(xv_1+yv_2) = (x - \bar{\beta}\gamma_0 y)v_1 + yv_2.$$

For $y \neq 0$, the map $\beta : \mathbb{CP}^1 \to \langle (x-\bar{\beta}\gamma_0 y)v_1 + yv_2 \rangle \in \mathbb{CP}^1$ is surjective. When $y = 0$ then $L' = v_1$ is an eigenvector of $\breve{\xi}_{\alpha_0}$, and there is no bubbleton. □

7. Spectral curves of the Riemann family

The Riemann family consists of embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ that are foliated by horizontal constant curvature curves of $\mathbb{S}^2$. From [12], these annuli can be conformally parameterized by their third coordinate with $Q = \frac{1}{4}(dz)^2$ and the metric $ds^2 = \cosh^2\omega |dz|^2$ is obtained from real-analytic solutions of the Abresch system [1]

$$\begin{cases} \Delta\omega + \sinh\omega\cosh\omega = 0, \\ \omega_{xy} - \omega_x\omega_y \tanh\omega = 0. \end{cases} \tag{7.1}$$

The second equation is the condition that the curve $x \mapsto (G(x,y), y)$ has constant curvature. This condition induces a separation of variables of the sinh-Gordon equation, and solutions can be described by two elliptic functions

$$f(x) = \frac{-\omega_x}{\cosh\omega} \quad \text{and} \quad g(y) = \frac{-\omega_y}{\cosh\omega} \tag{7.2}$$

of real variables x and y respectively, and for $c<0, d<0$ solve the system

$$\begin{aligned} -(f_x)^2 &= f^4+(1+c-d)f^2+c, & -f_{xx} &= 2f^3+(1+c-d)f,\\ -(g_y)^2 &= g^4+(1+d-c)g^2+d, & -g_{yy} &= 2g^3+(1+d-c)g. \end{aligned}$$

We can then recover the function ω by

$$\sinh\omega = \left(1+f^2+g^2\right)^{-1}(f_x+g_y). \tag{7.3}$$

The spectral curves of members of the Riemann family have spectral genus 0, 1 or 2. The spectral genus zero case consists of flat annuli $\gamma\times\mathbb{R}$, where $\gamma\subset\mathbb{S}^2$ is a great circle. In this case, the solution of the sinh-Gordon equation is the trivial solution $\omega\equiv 0$. The spectral genus 1 case consists of solutions of the sinh-Gordon equation that only depend on one real variable. The corresponding minimal annuli are analogous to associate family members of Delaunay surfaces. In particular, the spectral genus 1 case contains the rotational annuli and helicoids, and these are foliated by circles. Amongst the spectral genus 2 surfaces, the corresponding minimal annuli are again foliated by circles, but no longer have rotational symmetry. This condition endows the spectral curve with an additional symmetry.

Spectral genus 0. We first study annuli with spectral curves of genus 0. Inserting the trivial solution of the sinh-Gordon equation $\omega_0\equiv 0$ into (1.3), and setting $\gamma=1$, gives

$$\alpha(\omega_0) = \frac{1}{4}\begin{pmatrix} 0 & \mathbb{i}\lambda^{-1}\\ \mathbb{i} & 0\end{pmatrix}dz + \frac{1}{4}\begin{pmatrix} 0 & \mathbb{i}\\ \mathbb{i}\lambda & 0\end{pmatrix}d\bar z. \tag{7.4}$$

The solution of $F_\lambda^{-1}\,dF_\lambda=\alpha(\omega_0), F_\lambda(0)=\mathbb{1}$ is given by

$$\begin{aligned} F_\lambda &= \exp\left(\frac{\mathbb{i}}{4}\begin{pmatrix} 0 & \lambda^{-1}z+\bar z\\ z+\lambda\bar z & 0\end{pmatrix}\right)\\ &= \begin{pmatrix} \cos(\frac{1}{4}(\frac{z}{\sqrt\lambda}+\bar z\sqrt\lambda)) & \mathbb{i}\,\frac{\sin((1/4)(z/\sqrt\lambda+\bar z\sqrt\lambda))}{\sqrt\lambda}\\ \mathbb{i}\sqrt\lambda\sin(\frac{1}{4}(\frac{z}{\sqrt\lambda}+\bar z\sqrt\lambda)) & \cos(\frac{1}{4}(\frac{z}{\sqrt\lambda}+\bar z\sqrt\lambda))\end{pmatrix}. \end{aligned} \tag{7.5}$$

The horizontal part of $(F_\lambda\sigma_3F_\lambda^{-1}, \mathrm{Re}(-\mathbb{i}\lambda^{-1/2}z))$ computes to

$$F_\lambda\sigma_3F_\lambda^{-1} = \begin{pmatrix} \mathbb{i}\cos(\mathrm{Re}(z\lambda^{-1/2})) & \lambda^{-1/2}\sin(\mathrm{Re}(z\lambda^{-1/2}))\\ -\lambda^{1/2}\sin(\mathrm{Re}(z\lambda^{-1/2})) & -\mathbb{i}\cos(\mathrm{Re}(z\lambda^{-1/2}))\end{pmatrix}.$$

Identifying $\mathfrak{su}_2\cong\mathbb{R}^3$, $\left(\begin{smallmatrix}\mathbb{i}w & u+\mathbb{i}v\\ -u+\mathbb{i}v & -\mathbb{i}w\end{smallmatrix}\right)\cong(u,v,w)$, evaluating the associated family at $\lambda=1$, and writing $z=x+\mathbb{i}y$, we obtain the conformal minimal immersion $\mathbb{C}\to\mathbb{S}^2\times\mathbb{R}\subset\mathbb{R}^4$ given by

$$X_1(x,y) = (\sin x, 0, \cos x, y).$$

Restricting X_1 to the strip $(x, y) \in [0, 2\pi] \times \mathbb{R}$ then gives an embedded minimal flat annulus in $\mathbb{S}^2 \times \mathbb{R}$. Evaluating the associated family at some other point $\lambda_0 \in \mathbb{S}^1$, then τ-periodicity requires that $\operatorname{Re}(\tau \lambda_0^{-1/2}) \in 2\pi\mathbb{Z}$.

We next compute the corresponding spectral data (a, b). Since

$$F_\lambda = \exp\bigl(z\xi_\lambda - \bar{z}\overline{\xi_{1/\lambda}}^t\bigr) \quad \text{for } \xi_\lambda = \frac{\mathbb{i}}{4}\begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix}$$

coincides with the extended flat frame (7.5) computed above, we conclude that ξ_λ is a potential for the flat surface. Hence, $a(\lambda) = -\lambda \det \xi_\lambda = -1/16$, and the spectral curve (2.10) is the 2-point compactification of $\{(\nu, \lambda) \mid \nu^2 = -\lambda^{-1}/16\}$. The flat annulus has the simplest possible spectral curve. It is a genus zero hyperelliptic curve, so a double cover of $\mathbb{CP}^1$ with two branch points.

The eigenvalues of F_λ in (7.5) are $\exp(\pm\frac{\mathbb{i}}{4}(z\lambda^{-1/2} + \bar{z}\lambda^{1/2}))$. Therefore, the logarithmic eigenvalue (up to sign) of the monodromy with respect to the translation $z \mapsto z + 2\pi$ is

$$\ln \mu(\lambda) = \frac{\pi\mathbb{i}}{2}\bigl(\lambda^{-1/2} + \lambda^{1/2}\bigr).$$

Then

$$d\ln\mu = \frac{\pi\mathbb{i}(\lambda - 1)}{4\lambda^{3/2}}\,d\lambda = \frac{\pi(1-\lambda)}{16\lambda^2\nu}\,d\lambda$$

since $\lambda^{3/2} = -4\mathbb{i}\lambda^2\nu$. Thus, $b(\lambda) = \frac{\pi}{16}(1 - \lambda)$.

Spectral genus 1. We apply the Pinkall–Sterling iteration to the case where ω satisfies $\alpha\omega_z + \beta\omega_{\bar{z}} = 0$ for $\alpha, \beta \in \mathbb{C}$. The relation implies that $|\alpha| = |\beta|$ and up to a change of coordinate we assume without loss of generality that $\omega_x = 0$. Then $\omega_z = -\omega_{\bar{z}}$, and we are exactly in the setting of Abresch's system [1] and there is a constant $d < 0$ with $-b_y^2 = (b^2 + 1)(b^2 + d)$ where

$$b(y) = \frac{-\omega_y}{\cosh\omega} \quad \text{and} \quad \sinh\omega = \frac{b_y}{1 + b^2}. \tag{7.6}$$

We now use the Pinkall–Sterling iteration to compute the polynomial Killing field. Starting with $u_{-1} = \sigma_{-1} = 0$ and $\tau_{-1} = \mathbb{i}/4$, and using $4\omega_{z\bar{z}} = -\frac{1}{2}\sinh(2\omega)$, gives

$$u_0 = -4\mathbb{i}\omega_z\tau_{-1} = \omega_z, \qquad \sigma_0 = \gamma e^{2\omega}\tau_{-1} + 4\mathbb{i}\gamma u_{0;\bar{z}} = \frac{1}{4}\mathbb{i}\gamma e^{-2\omega}.$$

We use the function ϕ_0 to compute $\tau_0 = 2\mathbb{i}\bar{\gamma}(\frac{1}{2}\phi_0 - u_{0;z})$. We have $u_0 = -\omega_{\bar{z}}$, and then

$$\phi_{0;z} = -4\omega_z\omega_{z\bar{z}} = \frac{1}{4}\bigl(\cosh(2\omega)\bigr)_z,$$

$$\phi_{0;\bar{z}} = -\omega_{\bar{z}}\sinh\omega\cosh\omega = \frac{1}{4}\bigl(\cosh(2\omega)\bigr)_{\bar{z}}.$$

Then $\tau_0 = 2\mathbb{i}\bar{\gamma}(\omega_{z\bar{z}} + \frac{1}{8}\cosh(2\omega)) = \frac{1}{4}\mathbb{i}\bar{\gamma}e^{-2\omega}$.

At the next step, we find $u_1 = -2\mathbb{i}\tau_{o;z} - 4\mathbb{i}\omega_z\tau_0 = 0$, $\sigma_1 = \gamma e^{2\omega}\tau_0 = \frac{\mathbb{i}}{4}$ and $\tau_1 = 0$. This gives the polynomial Killing field (2.2) as

$$\zeta_\lambda = \frac{\mathbb{i}}{4}\begin{pmatrix} -2\omega_y & e^{\omega}\lambda^{-1} + \bar\gamma e^{-\omega} \\ \gamma e^{-\omega} + e^{\omega}\lambda & 2\omega_y \end{pmatrix}.$$

Then $a(\lambda) = -\lambda \det\zeta_\lambda = -\frac{1}{16}(\gamma + (2\cosh(2\omega) + 4\omega_y^2)\lambda + \bar\gamma\lambda^2)$. Using (7.6) gives

$$2\omega_y^2 + \cosh 2\omega = 2b^2\cosh^2\omega + 1 + 2\sinh^2\omega = 1 - 2d$$

so that $a(\lambda) = -\frac{1}{16}(\gamma + 2(1-2d)\lambda + \bar\gamma\lambda^2)$. Its roots are $(2d - 1 \pm 2\sqrt{d^2-d})\gamma$. If $\gamma = \pm 1$, then this polynomial satisfies the additional symmetry

$$\lambda^{2g} a(1/\lambda) = a(\lambda) \tag{7.7}$$

for $g = 1$, and it has two real roots. The corresponding spectral curve is a double cover of $\mathbb{CP}^1$ branched at 4 points, so a hyperelliptic curve of genus 1. To close the surface, we have to close the third coordinate with $Q = \frac{1}{4}\gamma\lambda_0^{-1}(dz)^2 = \frac{1}{4}(dz)^2$. Riemann annuli of spectral genus 1 satisfy $\omega_x = 0$ or $\omega_y = 0$. This means that $\lambda_0 = \pm\gamma$. We can parameterize the annulus in such a way that $\lambda_0 = 1$, and apply the iteration with $\gamma = 1$. This corresponds to the case where $y \mapsto \omega(y)$ depends only on its third coordinate and describes rotational examples. In the other case where $\lambda_0 = 1$ and $\gamma = -1$, this corresponds to the helicoidal surfaces, where the surface is foliated by horizontal geodesics. In this case, the function ω depends only on the variable x.

Symmetric spectral genus 2. We next consider general real-analytic solutions of Abresch's system. In this case, we first prove that they correspond to spectral genus 2 surfaces.

LEMMA 7.1. *Every solution $\omega : \mathbb{C} \to \mathbb{R}$ of Abresch's system (7.1) satisfies*

$$\omega_{zzz} - 2\omega_z^3 = -\frac{1}{4}\omega_{\bar z} + \frac{1}{2}(c-d)\omega_z. \tag{7.8}$$

Proof. Differentiating $\omega_z = -\frac{1}{2}(f - \mathbb{i}g)\cosh(\omega)$ gives

$$\omega_{zz} = \frac{1}{4}(f - \mathbb{i}g)^2\sinh(\omega)\cosh(\omega) - \frac{1}{2}(f - \mathbb{i}g)_z\cosh(\omega).$$

Now using the equation of the system, we have

$$2(f - \mathbb{i}g)_z = f_x - g_y = \frac{(1 + f^2 + g^2)(g^2 - f^2 + d - c)}{f_x + g_y} = \frac{g^2 - f^2 + d - c}{\sinh(\omega)}.$$

Then $\omega_{zz} = \tanh(\omega)\omega_z^2 - \frac{1}{4}\coth(\omega)(g^2 - f^2 + d - c)$ and thus

$$\tanh(\omega)\omega_{zz} - \tanh^2(\omega)\omega_z^2 + \frac{1}{4}\big(g^2 - f^2\big) = \frac{1}{4}(c-d) \in \mathbb{R}$$

since the imaginary part is $\tanh(\omega)(\omega_{xy} - \tanh(\omega)\omega_x\omega_y) = 0$. Using the expression for $f_x + g_y$ and $g_y - f_x$ we obtain

$$\begin{aligned} 2f_x &= \big(1 + f^2 + g^2\big)\sinh(\omega) - \big(f^2 - g^2 + c - d\big)\sinh^{-1}(\omega), \\ 2g_y &= \big(1 + f^2 + g^2\big)\sinh(\omega) + \big(f^2 - g^2 + c - d\big)\sinh^{-1}(\omega). \end{aligned}$$

To understand higher order derivative, we write

$$\begin{aligned} -2\big(g^2 - f^2 + (d-c)\big)_z &= (2ff_x + 2\mathbb{i}gg_y) \\ &= (f + \mathbb{i}g)\big(1 + f^2 + g^2\big)\sinh(\omega) \\ &\quad - (f - \mathbb{i}g)\big(f^2 - g^2 + c - d\big)\sinh^{-1}(\omega). \end{aligned}$$

We can check that

$$\begin{aligned} \frac{1}{8}(f + \mathbb{i}g)\cosh(\omega) &= -\frac{1}{4}\omega_{\bar{z}}, \\ \frac{1}{8}(f + \mathbb{i}g)\big(f^2 + g^2\big)\cosh(\omega) &= \frac{\cosh(\omega)}{8}(f - \mathbb{i}g)^3 + \frac{\cosh(\omega)}{2}\mathbb{i}fg(f - \mathbb{i}g) \\ &= -\frac{\omega_z^3}{\cosh^2(\omega)} - \mathbb{i}fg\omega_z. \end{aligned}$$

Now we compute

$$\begin{aligned} \omega_{zzz} - 2\omega_z^3 &= -2\omega_z^3 + \omega_z^3\operatorname{sech}^2(\omega) + 2\tanh(\omega)\omega_z\omega_{zz} \\ &\quad + \frac{1}{4}\omega_z\operatorname{csch}^2(\omega)\big(g^2 - f^2 + d - c\big) \\ &\quad - \frac{1}{4}\coth(\omega)\big(g^2 - f^2 + d - c\big)_z \\ &= -2\omega_z^3 + \omega_z^3\operatorname{sech}^2(\omega) - \omega_z^3\operatorname{sech}^2(\omega) - \mathbb{i}fg\omega_z \\ &\quad + 2\omega_z\left(\tanh^2(\omega)\omega_z^2 + \frac{1}{4}\big(f^2 - g^2 + c - d\big)\right) \\ &\quad + \frac{1}{4}\omega_z\big(g^2 - f^2 + d - c\big)\operatorname{csch}^2(\omega) - \frac{1}{4}\omega_{\bar{z}} \\ &\quad - \frac{1}{8}(f - \mathbb{i}g)\big(f^2 - g^2 + c - d\big)\coth(\omega)\operatorname{csch}(\omega) \\ &= -\frac{1}{4}\omega_{\bar{z}} + \omega_z\big(2\tanh^2(\omega)\omega_z^2 - 2\omega_z^2\big) \\ &\quad + \frac{1}{2}(c-d)\omega_z + \frac{1}{2}(f - \mathbb{i}g)^2\omega_z \\ &= -\frac{1}{4}\omega_{\bar{z}} + \frac{1}{2}(c-d)\omega_z. \end{aligned}$$

□

Next, we use the iteration of Pinkall–Sterling to compute the spectral curve associated to the algebraic relation (7.8). It is a priori a one-parameter family of algebraic relations but ω itself encodes other invariant quantities than the

one found in the expression of ω_{zz}. The Pinkall–Sterling iteration gives

$$u_{-1}=u_2=\sigma_{-1}=\tau_2=0,\qquad u_0=\omega_z,\qquad u_1=\bar\gamma\omega_{\bar z},$$
$$\sigma_0=\frac{1}{4}\mathbb{i}\gamma e^{-2\omega},\qquad \sigma_1=2\mathbb{i}e^{-2\omega}\Big(\omega_{zz}+\omega_z^2+\frac{1}{4}(c-d)\Big),\qquad \sigma_2=-\frac{1}{4}\mathbb{i}\bar\gamma,$$
$$\tau_{-1}=\frac{1}{4}\mathbb{i},\qquad \tau_0=2\mathbb{i}\bar\gamma\Big(\omega_z^2-\omega_{zz}+\frac{1}{4}(c-d)\Big),\qquad \tau_1=-\frac{1}{4}\mathbb{i}\gamma^{-2}e^{-2\omega}.$$

This defines a solution of the Lax equation Φ by (2.2) of degree $N=2$. To obtain a polynomial Killing field we skew-symmetrize and define

$$\zeta_\lambda(z)=\frac{1}{2}\Phi_\lambda(z)-\frac{\lambda}{2}\overline{\Phi_{1/\bar\lambda}(z)}^t.$$

Then $\lambda\overline{\zeta_{1/\bar\lambda}}^t=-\zeta_\lambda$ and has λ^{-1}-coefficient $\frac{1}{8}\mathbb{i}e^\omega(1-\gamma)\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. For ζ_λ to be $\mathcal{P}_2$-valued we require that $\frac{1}{8}\mathbb{i}e^\omega(1-\gamma)\in\mathbb{i}\mathbb{R}^+$, which means that $\gamma=-1$. In this case, $Q=\frac{1}{4}\gamma\lambda^{-1}(dz)^2$ implies that the Sym point is at $\lambda=-1$ and the entries of the polynomial Killing field are

$$\begin{aligned}\alpha&=(\omega_z-\lambda\omega_{\bar z}),\\ \beta&=\frac{\mathbb{i}}{4}\lambda^{-1}e^\omega+\mathbb{i}\Big(e^{-\omega}\big(\omega_{\bar z}^2+\omega_{\bar z\bar z}\big)-e^\omega\big(\omega_z^2+\omega_{zz}\big)\\ &\qquad+\frac{1}{4}(c-d)\big(e^{-\omega}-e^\omega\big)\Big)-\frac{\mathbb{i}}{4}\lambda e^{-\omega},\\ \gamma&=-\frac{\mathbb{i}}{4}e^{-\omega}+\mathbb{i}\lambda\Big(e^{-\omega}\big(\omega_z^2+\omega_{zz}\big)-e^\omega\big(\omega_{\bar z}^2+\omega_{\bar z\bar z}\big)\\ &\qquad+\frac{1}{4}(c-d)\big(e^{-\omega}-e^\omega\big)\Big)+\frac{\mathbb{i}}{4}\lambda^2e^\omega.\end{aligned}$$

To compute the spectral curve, we have only to compute $a(\lambda)=\lambda(\alpha^2+\beta\gamma)$ at one point. We choose a point where $\omega(x_0,y_0)=\partial_xf(x_0)=\partial_yg(y_0)=0$. At this point $\omega_{zz}=\omega_{\bar z\bar z}=0$, thus

$$f(x_0)=f_0=-\partial_x\omega(x_0)=\frac{1}{2}(-1+d-c+\sqrt{\Delta}),$$
$$g(y_0)=g_0=-\partial_y\omega(y_0)=\frac{1}{2}(-1+c-d+\sqrt{\Delta}),$$

where $\Delta=(1+c-d)^2-4c=(1+d-c)^2-4d$.

Writing $a(\lambda)=\lambda(\alpha^2+\beta\gamma)=a_0+a_1\lambda+a_2\lambda^2+a_3\lambda^3+a_4\lambda^4$, and using $\omega_z(x_0,y_0)=-\frac{1}{2}(f_0-\mathbb{i}g_0)$ and $\omega_{\bar z}(x_0,y_0)=-\frac{1}{2}(f_0+\mathbb{i}g_0)$, a computation gives the real coefficients

$$\begin{aligned}a_0&=a_4=\frac{1}{16},\\ a_1&=a_3=\frac{1}{2}\big(\omega_z^2+\omega_{\bar z}^2\big)=\frac{1}{4}\big(f_0^2-g_0^2\big),\\ a_2&=\big(\omega_z^4+\omega_{\bar z}^4\big)-2\big(\omega_z\omega_{\bar z}+\omega_z^2\omega_{\bar z}^2\big)-\frac{1}{8}=-\frac{1}{8}-\frac{1}{2}g_0^2-\frac{1}{2}f_0^2-f_0^2g_0^2.\end{aligned}$$

The four real roots of $a(\lambda)$ are $-1-2f_0^2 \pm 2\sqrt{f_0^2+f_0^4}, 1+2g_0^2 \pm 2\sqrt{g_0^2+g_0^4}$, and $a(\lambda)$ satisfies the additional symmetry (7.7) for $g=2$. In summary the spectral data of the 2-parameter family of the Riemann family are given by

PROPOSITION 7.2. *The genus* 0 *spectral data of an embedded annulus with Sym point* $\lambda=1$ *is given by*

(1) $a(\lambda) = -\frac{1}{16}$ *and* $b(\lambda) = \frac{\pi}{16}(1-\lambda)$.

The genus 1 *spectral data of an embedded annulus with Sym point* $\lambda=1$ *is given by*

(2) $a(\lambda) = \frac{1}{16\alpha}(\lambda-\alpha)(\alpha\lambda-1)$ *for* $\alpha \in (0,1)$ *and* $b(\lambda) = \frac{b(0)}{\gamma}(\lambda-\gamma)(\gamma\lambda-1)$, *with* $\gamma \in (\alpha,1)$ *and* $b(0) \in \mathbb{i}\mathbb{R}$ *both determined by* α.
(3) $a(\lambda) = \frac{-1}{16\beta}(\lambda+\beta)(\beta\lambda+1)$ *for* $\beta \in (0,1)$ *and* $b(\lambda) = \frac{b(0)}{\gamma}(1-\lambda)(1+\lambda)$ *and* $b(0) \in \mathbb{R}$ *determined by* β.

The genus 2 *spectral data of an embedded annulus with Sym point* $\lambda=1$ *is given by*

(4) $a(\lambda) = \frac{1}{16\beta\alpha}(\lambda-\alpha)(\alpha\lambda-1)(\lambda+\beta)(\beta\lambda+1)$ *for* $\alpha,\beta \in (0,1)$ *and* $b(\lambda) = \frac{b(0)}{\gamma}(1+\lambda)(\lambda-\gamma)(\gamma\lambda-1)$ *for* $\gamma \in (\alpha,1)$ *and* $b(0) \in \mathbb{i}\mathbb{R}$ *both determined by* α *and* β.

In conclusion, the polynomial a *satisfies the additional symmetry* $\lambda^{2g}a(1/\lambda) = a(\lambda)$ *and*

(a) $\lambda^{g+1}b(1/\lambda) = b(\lambda)$ *if* a *has a root* $\alpha \in \mathbb{R}^+$ *and* $b(0) \in \mathbb{i}\mathbb{R}$;
(b) $\lambda^{g+1}b(1/\lambda) = -b(\lambda)$ *if* a *has only roots in* $\mathbb{R}^-$ *and* $b(0) \in \mathbb{R}$.

Proof. We have seen above that spectral curves of the Riemann family have an additional involution $(\lambda,\nu) \to (\lambda^{-1}, \lambda^{1-g}\nu)$, since in all cases $\lambda^{2g}a(1/\lambda) = a(\lambda)$. Now depending on $a(\lambda)$, we construct a function h which satisfies the closing condition of the annulus. We prove that there are constants γ and $b(0)$ such that b satisfies the closing condition of Proposition 5.7.

First, we remark that $\lambda^{g+1}\overline{b(1/\bar{\lambda})} = -b(\lambda)$ by construction. We look for h satisfying $\sigma^* h = -h$ and $dh = \frac{b\,d\lambda}{\nu\lambda^2}$. First we need to prove that h is well defined on $\hat{\Sigma}$ (Definition 5.8).

In cases (1) and (3), there is a root $\alpha \in (0,1)$. Along the segment $(\alpha, 1/\alpha)$, the polynomial $b(\lambda) \in \mathbb{i}\mathbb{R}$ and $\nu \in \mathbb{i}\mathbb{R}$. Since $b(\lambda)$ has exactly one root in the interval $(\alpha,1)$ at $\gamma \in (\alpha,1)$, there exists exactly one value of γ which cancels the following integral for a given α and β. Using the additional symmetry, there is a real $\gamma \in (\alpha,1)$ with

$$\int_\alpha^{1/\alpha} \frac{b}{\nu\lambda^2}\,d\lambda = 2\int_\alpha^1 \frac{b}{\nu\lambda^2}\,d\lambda = 0.$$

Moreover, in cases (2) and (3), we have by the reality condition that

$$\int_{-\beta}^{-1/\beta} \frac{b}{\nu\lambda^2}\, d\lambda = 0.$$

Now the function h with $dh = \frac{b\, d\lambda}{\nu\lambda^2}$ and $\sigma^* h = -h$ is well defined on $\hat{\Sigma}$, the curve with the two cycles around $(-1/\beta, -\beta)$ and $(\alpha, 1/\alpha)$ removed.

For $\kappa : (\lambda, \nu) \to (\bar{\lambda}, \bar{\nu})$ on $\tilde{\Sigma}$ have $\kappa^* h = -\bar{h}$, since $b(\bar{\lambda}) = \overline{b(\lambda)}$.

On the real axis between $\lambda = 0$ and $\lambda = \alpha$ (or $\lambda = 1$ in the case (2)), the polynomial a takes real positive values. Then the segment $(0, \alpha]$ (or $(0, 1]$ in the case (2)) is a set of fixed points for the involution κ. On this segment we deduce that $h = \kappa^* h$ and the function h is purely imaginary on this segment. Since the integral $\int_\alpha^1 d \ln \mu = 0$, the function h is imaginary at $\lambda = 1$ and $h(\alpha) = h(1) \in \mathbb{i}\mathbb{R}$.

The involution ϱ in (2.11) leaves $\mathbb{S}^1$ invariant, and we have $\varrho^*\, dh = -\overline{dh}$. Hence, $dh \in \mathbb{i}\mathbb{R}$ on $\mathbb{S}^1$. Thus on the unit circle h stays imaginary, so in particular $h \in \mathbb{i}\mathbb{R}$ at $\lambda = -1$.

The segment $(-1, -\beta)$ is a set of fixed points for κ and the function $h \in \mathbb{i}\mathbb{R}$ on this segment. Since on the real line the function $a(\lambda)$ changes sign and become real negative on $(-\beta, 0)$, the function $h \in \mathbb{R}$ on this segment. We can then deduce that $h(-\beta) = 0$ at this point. Now we can choose the value of $b(0)$ to get a multiple value of $\pi\mathbb{i}$ at the sym point $\lambda = 1$. This proves the closing condition and concludes the proof of the proposition. □

LEMMA 7.3. *If γ is a root of b, then the corresponding function $|\mu(\gamma)| \neq 1$.*

Proof. For $\lambda \in [\alpha, \bar{\alpha}^{-1}]$, the function $h = \ln \mu$ is real and $\int_\alpha^1 dh = 0$. Then γ is a root of dh and is contained in $(\alpha, 1)$. Since $\operatorname{Re} h(\alpha) = \operatorname{Re} h(1) = 0$, the value γ is the local critical point of h. Then $\operatorname{Re} h(\gamma) \neq 0$, and thus $|\mu| \neq 1$. □

Appendix: Terng–Uhlenbeck formula

PROPOSITION A.1. *Let $h_{L',\alpha_0} \in H^r_{\alpha_0}$ the simple factor with $\alpha_0 \in \mathbb{C}^\times \setminus \mathbb{S}^1$, with $r < \min\{|\alpha_0|, 1/|\alpha_0|\}$ and $L' \in \mathbb{CP}^1$. Then*

$$F_\lambda(z) = h_{L',\alpha_0} \breve{F}_\lambda(z) h^{-1}_{L'(z),\alpha_0} \quad \text{with } L'(z) = {}^t\overline{\breve{F}}_{\alpha_0}(z) L'.$$

Proof. By r-Iwasawa decomposition,

$$\begin{aligned} F_\lambda(z) B_\lambda(z) = \exp(z\xi_\lambda) &= \exp\bigl(zp(\lambda) h_{L',\alpha_0} \breve{\xi}_\lambda h^{-1}_{L',\alpha_0}\bigr) \\ &= h_{L',\alpha_0} \exp\bigl(zp(\lambda)\breve{\xi}_\lambda\bigr) h^{-1}_{L',\alpha_0} \in \Lambda_r \mathrm{SL}_2(\mathbb{C}) \end{aligned}$$

and $h_{L',\alpha_0} \in \Lambda^+_r \mathrm{SL}_2(\mathbb{C})$ ($h_{L',\alpha_0}(0) = Q_{1,L'} Q_{L'} \pi^{-1}_{\alpha_0}(0) Q^{-1}_{L'} = R^{-1}_{1,L'}$ at $\lambda = 0$ and π is holomorphic for $r < |\alpha_0|$). Then

$$\exp\bigl(zp(\lambda)\breve{\xi}_\lambda\bigr) h^{-1}_{L',\alpha_0} = \breve{F}_\lambda(z) \breve{B}'_\lambda(z) = \breve{F}_\lambda(z) \breve{B}_\lambda(z) h^{-1}_{L',\alpha_0}.$$

Now we have

$$\begin{aligned} F_\lambda(z)B_\lambda(z) &= h_{L',\alpha_0}\exp\bigl(zp(\lambda)\check{\xi}_\lambda\bigr)h_{L',\alpha_0}^{-1} = h_{L',\alpha_0}\check{F}_\lambda(z)\check{B}'_\lambda(z) \\ &= \bigl(h_{L',\alpha_0}\check{F}_\lambda(z)H\bigr)\bigl(H^{-1}\check{B}'_\lambda(z)\bigr). \end{aligned}$$

By uniqueness of the r-Iwasawa decomposition, we have only to prove that if $H = h_{L'(z),\alpha_0}^{-1} \in H^r_{\alpha_0}$ with $L'(z) = {}^t\bar{\check{F}}_{\alpha_0}(z)L'$ then $h_{L',\alpha_0}\check{F}_\lambda(z)H \in \Lambda_r\mathrm{SU}_2$. Clearly

$$h_{L',\alpha_0}\check{F}_\lambda(z)h_{L'(z),\alpha_0}^{-1} \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$$

is holomorphic on A_r away from α_0 and $1/\bar{\alpha}_0$, and SU_2-valued on $\mathbb{S}^1$. At the roots α_0 and $1/\bar{\alpha}_0$, we have simple poles and we have to study the residues of

$$G_\lambda(z) = \pi_{L'}^{-1}(\lambda)\check{F}_\lambda(z)\pi_{L'(z)}(\lambda).$$

Now we consider the simple factor $L'(z) = {}^t\bar{\check{F}}_{\alpha_0}(z)L'$. Let $(L'(z), L'(z)^\perp)$ be an orthonormal basis of $\mathbb{C}^2$. Note that $L'(z) = {}^t\bar{\check{F}}_{\alpha_0}(z)L' = \check{F}^{-1}_{1/\bar{\alpha}_0}(z)L'$ and $\check{F}_{\alpha_0}(z)^{-1}L'^\perp = L'(z)^\perp$. When $\lambda \to 1/\bar{\alpha}_0$, we have

$$\lim_{\lambda\to 1/\bar{\alpha}_0} G_\lambda L'(z) = \lim_{\lambda\to 1/\bar{\alpha}_0}\sqrt{\frac{\lambda-\alpha_0}{1-\bar{\alpha}_0\lambda}}Q_{L'}\pi_{\alpha_0}^{-1}Q_{L'}^{-1}\check{F}_\lambda\bar{\check{F}}^t_{\alpha_0}L'(z) = L',$$

$$\begin{aligned} \lim_{\lambda\to\bar{\alpha}_0^{-1}}(1-\bar{\alpha}_0\lambda)G_\lambda L'^\perp(z) &= \lim_{\lambda\to\bar{\alpha}_0^{-1}}(1-\bar{\alpha}_0\lambda)\sqrt{\frac{1-\bar{\alpha}_0\lambda}{\lambda-\alpha_0}}Q_{L'}\pi_{\alpha_0}^{-1}Q_{L'}^{-1}\check{F}_\lambda L'^\perp(z) \\ &= 0. \end{aligned}$$

When $\lambda \to \alpha_0$, we compute

$$\lim_{\lambda\to\alpha_0}(\lambda-\alpha_0)G_\lambda L'(z) = \lim_{\lambda\to\alpha_0}(\lambda-\alpha_0)\sqrt{\frac{\lambda-\alpha_0}{1-\bar{\alpha}_0\lambda}}Q_{L'}\pi_{\alpha_0}^{-1}Q_{L'}^{-1}\check{F}_\lambda L'(z) = 0,$$

$$\lim_{\lambda\to\alpha_0} G_\lambda L'^\perp(z) = \lim_{\lambda\to\alpha_0}\sqrt{\frac{1-\bar{\alpha}_0\lambda}{\lambda-\alpha_0}}Q_{L'}\pi_{\alpha_0}^{-1}Q_{L'}^{-1}\check{F}_\lambda\check{F}^{-1}_{\alpha_0}L'^\perp(z) = L'^\perp.$$

This proves the proposition. □

References

[1] U. Abresch, *Constant mean curvature tori in terms of elliptic functions*, J. Reine Angew. Math. **374** (1987), 169–192. MR 0876223

[2] A. I. Bobenko, *All constant mean curvature tori in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ in terms of theta-functions*, Math. Ann. **290** (1991), 209–245. MR 1109632

[3] A. I. Bobenko, *Constant mean curvature surfaces and integrable equations*, Russian Math. Surveys **46** (1991), 1–45.

[4] F. E. Burstall, D. Ferus, F. Pedit and U. Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. (2) **138** (1993), 173–212. MR 1230929

[5] F. E. Burstall and F. Pedit, *Harmonic maps via Adler–Kostant–Symes theory*, Harmonic maps and integrable systems, Aspects of Mathematics, vol. E23, Vieweg, Braunschweig, 1994. MR 1264189

[6] F. E. Burstall and F. Pedit, *Dressing orbits of harmonic maps*, Duke Math. J. **80** (1995), no. 2, 353–382. MR 1369397

[7] J. Dorfmeister and G. Haak, *On constant mean curvature surfaces with periodic metric*, Pacific J. Math. **182** (1998), 229–287. MR 1609603

[8] J. Dorfmeister, F. Pedit and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6** (1998), no. 4, 633–668. MR 1664887

[9] D. Ferus, F. Pedit, U. Pinkall and I. Sterling, *Minimal tori in* $\mathbb{S}^4$, J. Reine Angew. Math. **429** (1992), 1–47. MR 1173114

[10] A. P. Fordy and J. C. Wood (eds.), *Harmonic maps and integrable systems*, Aspects of Mathematics, vol. E23, Vieweg, Braunschweig, 1994. MR 1264179

[11] O. Forster, *Lectures on Riemann surfaces*, Graduate Texts in Mathematics, vol. 81, Springer, New York, 1991. MR 1185074

[12] L. Hauswirth, *Minimal surfaces of Riemann type in three-dimensional product manifolds*, Pacific J. Math. **224** (2006), no. 1, 91–117. MR 2231653

[13] L. Hauswirth, M. Kilian and M. U. Schmidt, *Properly embedded minimal annuli in* $\mathbb{S}^2 \times \mathbb{R}$, preprint, available at `arXiv:1210.5953`, 2012.

[14] N. Hitchin, *Harmonic maps from a 2-torus to the 3-sphere*, J. Differential Geom. **31** (1990), no. 3, 627–710. MR 1053342

[15] D. Hoffman and B. White, *Axial minimal surfaces in* $S^2 \times \mathbb{R}$ *are helicoidal*, J. Differential Geom. **87** (2011), no. 3, 515–523. MR 2819547

[16] M. Kilian, *Bubbletons are not embedded*, Osaka J. Math. **49** (2012), no. 3, 653–663. MR 2993061

[17] M. Kilian, N. Schmitt and I. Sterling, *Dressing CMC n-Noids*, Math. Z. **246** (2004), no. 3, 501–519. MR 2073454

[18] S.-P. Kobayashi, *Bubbletons in 3-dimensional space forms*, Balkan J. Geom. Appl. **9** (2004), no. 1, 44–68. MR 2205901

[19] I. McIntosh, *Global solutions of the elliptic 2D periodic Toda lattice*, Nonlinearity **7** (1994), no. 1, 85–108. MR 1260134

[20] I. McIntosh, *Harmonic tori and their spectral data*, Surveys on geometry and integrable systems, Adv. Stud. Pure Math., vol. 51, Math. Soc. Japan, Tokyo, 2008, pp. 285–314. MR 2509797

[21] W. H. Meeks III, J. Perez and A. Ros, *Properly embedded minimal planar domain*, to appear in Ann. of Math. (2).

[22] W. H. Meeks III and H. Rosenberg, *Stable minimal surfaces in* $M \times \mathbb{R}$, J. Differential Geom. **68** (2004), no. 3, 515–534. MR 2144539

[23] W. H. Meeks III and H. Rosenberg, *The theory of minimal surfaces in* $M \times \mathbb{R}$, Comment. Math. Helv. **80** (2005), no. 4, 811–858. MR 2182702

[24] U. Pinkall and I. Sterling, *On the classification of constant mean curvature tori*, Ann. of Math. (2) **130** (1989), 407–451. MR 1014929

[25] A. Pressley and G. Segal, *Loop groups*, Oxford Science Monographs, Oxford University Press, New York, 1988.

[26] H. Rosenberg, *Minimal surfaces in* $\mathbb{M}^2 \times \mathbb{R}$, Illinois J. Math. **46** (2002), no. 4, 1177–1195. MR 1988257

[27] I. Sterling and H. Wente, *Existence and classification of constant mean curvature multibubbletons of finite and infinite type*, Indiana Univ. Math. J. **42** (1993), no. 4, 1239–1266. MR 1266092

[28] W. W. Symes, *Systems of Toda type, inverse spectral problems, and representation theory*, Invent. Math. **59** (1980), no. 1, 13–51. MR 0575079

[29] C. Terng and K. Uhlenbeck, *Bäcklund transformations and loop group actions*, Comm. Pure Appl. Math. **LIII** (2000), 1–75. MR 1715533

L. Hauswirth, LAMA (UMR 8050), Université Paris-Est, UPEMLV, UPEC, CNRS, F-77454, Marne-la-Vallée, France

E-mail address: hauswirth@univ-mlv.fr

M. Kilian, Department of Mathematics, University College Cork, National University of Ireland, Cork, Ireland

E-mail address: m.kilian@ucc.ie

M. U. Schmidt, Institut für Mathematik, Universität Mannheim, A5, 6, 68131 Mannheim, Germany

E-mail address: schmidt@math.uni-mannheim.de