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Non-existence of time-dependent three-dimensional gravity water flows with constant non-zero vorticity

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We show that in a three-dimensional gravity water flow with a constant non-vanishing vorticity vector $(\Omega_1, \Omega_2, \Omega_3)$, the free surface, the pressure, and the velocity field present no variations in the direction orthogonal to the direction of motion. In addition, the second component of the velocity field is constant throughout the flow. Moreover, we prove that the vertical component, Ω_3, of the vorticity vector has to vanish. This latter fact turns out to be of crucial importance in proving the absence of variations of the flow in the direction that is orthogonal to the direction of the surface wave propagation. Our results are obtained under general assumptions: both the free surface and the flow beneath are allowed to be time dependent in the most general way. *Published by AIP Publishing.* https://doi.org/10.1063/1.5048580

I. INTRODUCTION

The last decades have witnessed an outburst of studies concerning rotational water flows, that is, flows exhibiting vorticity, which is an extremely important quantity that measures the rotationality of a fluid and describes the interactions of waves with non-uniform currents. Among the most staggering examples of wave-currents interactions, we quote those at the eastern coast of South Africa (where 6 m high sea waves from southwest meeting the Agulhas current lead to many oil tankers' wreckages) and those at the Columbia River entrance, one of the insecurest navigational region in the world, given the doubling of the wave height in just a few hours, cf. Ref. 23. The strength of the interaction is determined—as documented by studies of Jonsson,[23] Peregrine,[28] and Thomas and Klopman[31]—by the vertical structure of the current profile whose description is realized by the vorticity: zero vorticity models irrotational flows as well as currents which are uniform with depth, the simplest rotational setting being that of linearly sheared currents of constant non-zero vorticity, cf. Ref. 22.

From a historical perspective, the mathematical theory of rotational water waves has its roots at the beginning of the 19th century, when Gerstner [1809] constructed a solution in Lagrangian coordinates describing an explicit family of periodic two-dimensional travelling gravity water waves with non-zero vorticity, cf. Ref. 17. While Gerstner's solution was found in homogeneous fluid, Dubreil-Jacotin[15] showed that a heterogeneous fluid can also be accommodated; for modern presentations of Gerstner's wave solution, we refer the reader to the studies of Constantin[1,4] and Henry.[19] Nevertheless, substantial rigorous analytical results in the area of rotational water flows appeared relatively recently after the ground-breaking work of Constantin and Strauss,[2] in which the existence of large-amplitude periodic water waves over two-dimensional flows with arbitrary (continuous) vorticity was proved; the same authors extended the result to water flows with discontinuous vorticity, cf. Ref. 6.

The vast majority of studies on water flows exhibiting vorticity pertains to two dimensional flows. That is, while allowing variations in the vertical direction, the flows analyzed in the previous mentioned papers present no variation in the direction orthogonal to the direction of propagation. Nevertheless, the scenario of rotational two-dimensional flows displays significant relevance since it is known that (constant) vorticity models wave-current interactions,[4,31] being also a prerequisite for the emergence of critical layers, cf. Refs. 7, 10, 16, and 32.

Moreover, recent results by Constantin,[5] Constantin and Kartashova,[3] and Martin[26] show that gravity, capillary, and capillary-gravity-wave trains at the surface of water in a flow with constant non-zero vorticity with a flat bed can only occur if the flow is two dimensional and if the vorticity vector has only one non-vanishing component that points in the horizontal direction orthogonal to the direction of wave propagation. For similar results concerning solitary waves, we refer the reader to the studies of Craig[13] and Stuhlmeier.[29] A further significant result showing the meaninglessness of three-dimensional water flows presenting a constant non-vanishing vorticity vector was obtained by Wahlén who showed that the assumption of a free surface that has a steady behavior in both horizontal directions over a flow assumed to be time independent results in a flow whose free surface, velocity field, and pressure are uniform in the y-direction, while the second component of the velocity vector is constant.

By contrast, for the situation of irrotational three-dimensional gravity water flows, Iooss and Plotnikov[20,21] have showed the existence of double periodic waves using Nash-Moser theory.

Removing the assumption of a steady free surface wave utilized in Refs. 5 and 33 and that of a time-independent flow made in Ref. 33, we find that all the potential solutions to the time-dependent water wave problem exhibiting a constant non-vanishing vorticity vector [formulated below in (2.1)–(2.5)] present no variation in the direction orthogonal to

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the direction of motion. Moreover, we prove that the third component of the vorticity vector necessarily vanishes. Our results are true not only for inviscid flows but also for the viscous situation.

The non-existence of three-dimensional water flows exhibiting a constant non-vanishing vorticity vector is confirmed also by the recent studies of Constantin[8,9] and Constantin and Johnson.[11] Indeed, Refs. 8, 9, and 11 show that geophysical water flows that display a non-constant vorticity vector are inherently three-dimensional. Somewhat concurring with our conclusion (of two dimensionality of water flows with a constant non-vanishing vorticity) is the study by Xia and Francois[34] showing that in thick fluid layers, large-scale coherent structures can shear off the vertical eddies and reinforce the planarity of the flow.

Our study, among the few contributions aimed at the deepening of the analytical understanding of three-dimensional water flows,[5,8,9,11,20,21,33] takes into consideration the full nonlinear governing equations and their boundary conditions. Indeed, recent interesting studies (on three-dimensional water flows) perform numerical analyses of linearized Euler equations[18] or consider steady quasi-three-dimensional flows with vanishing vertical velocity, cf. Ref. 30.

II. THE THREE-DIMENSIONAL EULER EQUATIONS

We recall the governing equations for three-dimensional time-dependent water waves (see the studies of Johnson[24] and Constantin[4]).

Denoting with

$$\mathbf{u}(x,y,z,t)=(u(x,y,z,t),v(x,y,z,t),w(x,y,z,t))$$

the velocity field, with $P(x,\ y,\ z,\ t)$ the pressure within the fluid, and with g the gravitational acceleration, the equations of motion of a homogeneous, incompressible, and inviscid water flow are the Euler equations

$$\begin{aligned} u_t+uu_x+vu_y+wu_z &= -P_x,\\ v_t+uv_x+vv_y+wv_z &= -P_y,\\ w_t+uw_x+vw_y+ww_z &= -P_z-g \end{aligned} \tag{2.1}$$

and the equation of mass conservation

$$u_x+v_y+w_z=0, \tag{2.2}$$

which are valid in the fluid domain denoted as D_η bounded below by the flat bed $z=-d$ and above by the free surface $z=\eta(x,y,t)$, where η is a continuously differentiable function of $x,\ y,\ t$.

The water wave problem is completely formulated when we specify the boundary conditions

$$w=\eta_t+u\eta_x+v\eta_y \quad \text{on} \quad z=\eta(x,y,t) \tag{2.3}$$

and

$$w=0 \quad \text{on} \quad z=-d, \tag{2.4}$$

together with the dynamic boundary condition

$$P=P_{atm} \quad \text{on} \quad z=\eta(x,y,t). \tag{2.5}$$

The grasp into the swirling motion of the water is realized through the vorticity vector $\Omega(x,\ y,\ z,\ t)$, defined as the curl of the velocity field, that is,

$$\Omega=(w_y-v_z,u_z-w_x,v_x-u_y)=:(\Omega_1,\Omega_2,\Omega_3). \tag{2.6}$$

We will assume that the vorticity vector is constant. Under these assumptions, the following holds.

Theorem 2.1. *The third component of the vorticity vector, Ω_3, vanishes.*

Proof. To prove the claim made above, we note that the evolution of the vorticity is governed by the equation

$$\Omega_t+(\mathbf{u}\cdot\nabla)\Omega=(\Omega\cdot\nabla)\mathbf{u}, \tag{2.7}$$

cf. Refs. 4 and 25. Since the vector Ω is constant, we infer from the vorticity equation (2.7) that

$$(\Omega\cdot\nabla)\mathbf{u}=0, \tag{2.8}$$

which implies that $\Omega\cdot(\nabla w)=0$. Since $\Omega_3\neq 0$, we infer from the latter that w is constant in a non-horizontal direction. By means of (2.4), making use of the analyticity of w, and arguing as in Ref. 5, we conclude that $w=0$ throughout the entire fluid domain. This immediately implies that

$$P_z=-g \tag{2.9}$$

holds throughout the fluid domain.

Another direct consequence of the finding $w=0$ (when considering the definition of the vorticity vector) is that

$$u_z=\Omega_2 \quad \text{and} \quad v_z=-\Omega_1. \tag{2.10}$$

Therefore, there exists functions $\tilde{u}=\tilde{u}(x,y,t)$ and $\tilde{v}=\tilde{v}(x,y,t)$ such that

$$\begin{aligned} u(x,y,z,t)&=\tilde{u}(x,y,t)+\Omega_2 z,\\ v(x,y,z,t)&=\tilde{v}(x,y,t)-\Omega_1 z. \end{aligned} \tag{2.11}$$

Another upshot of the finding $w=0$ is obtained via the equation of mass conservation. Indeed, we obtain from (2.2) the existence of a stream function $\psi(x,\ y,\ t)$ such that

$$\tilde{u}=\psi_y, \quad \tilde{v}=-\psi_x. \tag{2.12}$$

Equation (2.12) facilitates the writing of the third component of the vorticity vector $\Omega_3=v_x-u_y$ as the equation

$$\Delta\psi=-\Omega_3. \tag{2.13}$$

Employing (2.11) and (2.12), we derive that

$$\Delta_{(x,y,z)}u=\Delta_{(x,y)}\tilde{u}=(\Delta\psi)_y=0$$

and

$$\Delta_{(x,y,z)}v=\Delta_{(x,y)}\tilde{v}=-(\Delta\psi)_x=0, \tag{2.14}$$

where we have also used (2.13). Moreover, since $P(x,\ y,\ \eta(x,\ y,\ t))=P_{atm}$ for all $(x,\ y,\ t)$, we have that

$$P_x(x,y,\eta(x,y,t))+P_z(x,y,\eta(x,y,t))\eta_x(x,y,t)=0. \tag{2.15}$$

From $P(x,\ y,\ \eta(x,\ y,\ t))=P_{atm}$ for all $(x,\ y,\ t)$, we have that

$$P_y(x,y,\eta(x,y,t))+P_z(x,y,\eta(x,y,t))\eta_y(x,y,t)=0. \tag{2.16}$$

We claim now the following:

Claim. The free surface $z=\eta(x,\ y,\ t)$ is flat at all times t. We prove that the functions

$$(x,y,t)\to\eta_x(x,y,t) \quad \text{and} \quad (x,y,t)\to\eta_y(x,y,t)$$

do not depend on x, y. To this end, note that the first two components of the vorticity equation (2.8) can be written as

$$\begin{aligned}\Omega_1 u_x + \Omega_2 u_y + \Omega_3 u_z &= 0,\\ \Omega_1 v_x + \Omega_2 v_y + \Omega_3 v_z &= 0,\end{aligned} \tag{2.17}$$

the system that can be reformulated as

$$\begin{aligned}\Omega_1 u_x + \Omega_2 v_x &= 0,\\ \Omega_1 u_y + \Omega_2 v_y &= 0,\end{aligned} \tag{2.18}$$

where we used the definition of Ω_3 and the already established relations

$$u_z = \Omega_2, \quad v_z = -\Omega_1.$$

Obviously, the previous formulas on the vertical derivatives of u and v imply that

$$\Omega_1 u_z + \Omega_2 v_z = 0,$$

a relation which, combined with (2.18), implies that the space gradient of the function

$$(x, y, z) \to \Omega_1 u(x, y, z, t) + \Omega_2 v(x, y, z, t)$$

vanishes at each fixed time t. Thus, there exists a function $t \to f(t)$ such that the equality

$$\Omega_1 u(x, y, z, t) + \Omega_2 v(x, y, z, t) = f(t) \tag{2.19}$$

holds throughout the fluid domain at all times t. Multiplying now the first equation in (2.1) by Ω_1 and the second by Ω_2 and adding the results, we obtain, upon using also (2.18) and (2.19), that

$$f'(t) = -\Omega_1 P_x - \Omega_2 P_y \text{ within } D_\eta. \tag{2.20}$$

Since

$$P_x\big|_{z=\eta(x,y,t)} = g\eta_x \quad \text{and} \quad P_y\big|_{z=\eta(x,y,t)} = g\eta_y,$$

we obtain from (2.20) that

$$\Omega_1 \eta_x(x, y, t) + \Omega_2 \eta_y(x, y, t) = -\frac{f'(t)}{g} \text{ for all } (x, y, t). \tag{2.21}$$

We will prove in the sequel that there is a function $t \to \tilde{f}(t)$ such that

$$\Omega_2 \eta_x(x, y, t) - \Omega_1 \eta_y(x, y, t) = \tilde{f}(t) \text{ for all } (x, y, t). \tag{2.22}$$

To reach the latter goal, we notice that the relation $\Omega_3 = v_x - u_y$ can be written as

$$\psi_{xx} + \psi_{yy} = -\Omega_3,$$

while the vorticity equations (2.17) become

$$\begin{aligned}\Omega_1 \psi_{xy} + \Omega_2 \psi_{yy} + \Omega_2 \Omega_3 &= 0,\\ \Omega_1 \psi_{xx} + \Omega_2 \psi_{xy} + \Omega_1 \Omega_3 &= 0,\end{aligned} \tag{2.23}$$

from which we infer that

$$\psi_{xx} = -\frac{\Omega_1^2 \Omega_3}{\Omega_1^2 + \Omega_2^2}, \quad \psi_{xy} = -\frac{\Omega_1 \Omega_2 \Omega_3}{\Omega_1^2 + \Omega_2^2}, \quad \psi_{yy} = -\frac{\Omega_2^2 \Omega_3}{\Omega_1^2 + \Omega_2^2}. \tag{2.24}$$

Therefore, the stream function ψ has the shape

$$\psi(x, y, t) = A_1 y^2 + A_2 xy + A_3 x^2 + a(t)x + b(t)y + c(t), \tag{2.25}$$

for some functions a, b, c and with

$$A_1 = -\frac{\Omega_2^2 \Omega_3}{2(\Omega_1^2 + \Omega_2^2)}, \quad A_2 = -\frac{\Omega_1 \Omega_2 \Omega_3}{\Omega_1^2 + \Omega_2^2}, \quad A_3 = -\frac{\Omega_1^2 \Omega_3}{2(\Omega_1^2 + \Omega_2^2)}. \tag{2.26}$$

Upon utilization of the formulas (2.11), (2.12), (2.25), and (2.26), we find that

$$\begin{aligned}\Omega_2 u - \Omega_1 v &= (2A_1\Omega_2 + A_2\Omega_1)y + (A_2\Omega_2 + 2A_3\Omega_1)x\\ &\quad + (\Omega_1^2 + \Omega_2^2)z + \Omega_2 b(t) + \Omega_1 a(t)\\ &= -\Omega_2\Omega_3 y - \Omega_1\Omega_3 x + (\Omega_1^2 + \Omega_2^2)z\\ &\quad + \Omega_2 b(t) + \Omega_1 a(t).\end{aligned} \tag{2.27}$$

Again, from the Euler equation, we obtain, upon utilization of (2.19) and (2.27), that

$$\begin{aligned}-\Omega_2 P_x + \Omega_1 P_y &= \Omega_2 u_t - \Omega_1 v_t + u(\Omega_2 u_x - \Omega_1 v_x)\\ &\quad + v(\Omega_2 u_y - \Omega_1 v_y)\\ &= \Omega_2 b'(t) + \Omega_1 a'(t) - \Omega_1\Omega_3 u - \Omega_2\Omega_3 v\\ &= \Omega_2 b'(t) + \Omega_1 a'(t) - \Omega_3 f(t).\end{aligned} \tag{2.28}$$

Hence,

$$\begin{aligned}-\Omega_2 \eta_x(x, y, t) + \Omega_1 \eta_y(x, y, t) &= \frac{\Omega_2 b'(t) + \Omega_1 a'(t) - \Omega_3 f(t)}{g}\\ &=: \tilde{f}(t)\end{aligned} \tag{2.29}$$

holds for all x, y, t. Upon recalling (2.21), we have the system

$$\begin{aligned}\Omega_1 \eta_x(x, y, t) + \Omega_2 \eta_y(x, y, t) &= -\frac{f'(t)}{g},\\ -\Omega_2 \eta_x(x, y, t) + \Omega_1 \eta_y(x, y, t) &= \tilde{f}(t),\end{aligned} \tag{2.30}$$

which holds for all x, y, t. To complete the proof, we divide it now into two cases.

Case 1: $\Omega_1^2 + \Omega_2^2 > 0$.

For this scenario, we infer from the system (2.30) that $\eta_x(x, y, t)$ and $\eta_y(x, y, t)$ do not depend on x and y at all times t. This shows that the outward pointing normal vector $(-\eta_x(x, y, t), -\eta_y(x, y, t), 1)$ at the point $(x, y, \eta(x, y, t))$ of the free surface $z = \eta(x, y, t)$ does not depend on (x, y) at all times t. The latter implies that the surface $z = \eta(x, y, t)$ is a plane at all times t. Owing to the boundedness of η, we infer that the free surface is, in fact, a horizontal plane, $z = F(t)$, at each time t.

We prove in the following that u and v are constant throughout the flow. We perform the proof for u and assume for the sake of contradiction that u is not constant. We note first that from $(\Omega\cdot\nabla)\mathbf{u} = 0$, we have that

$$\Omega_1 u_x + \Omega_2 u_y + \Omega_3 u_z = 0, \tag{2.31}$$

that is, the horizontal velocity u is constant in the direction of the oblique vector $(\Omega_1, \Omega_2, \Omega_3)$. Owing to the fact that the free surface is a horizontal plane, any straight line with the direction vector $(\Omega_1, \Omega_2, \Omega_3)$ starting at a point on the surface and ending at a point on the bed $z = -d$ is completely contained in the fluid domain. The former conclusion, Eq. (2.31), and the Phragmen-Lindelöf maximum principle imply that, at any time t, u achieves its maximum on the surface as well as on the bed

$z=-d$. Were $(x_1, y_1, \eta(x_1, y_1, t_0))$ and $(x_2, y_2, -d)$ such points, at some time t_0, we would have by the Hopf boundary point lemma that

$$u_z(x_1, y_1, \eta(x_1, y_1, t), t_0)u_z(x_2, y_2, -d, t_0) < 0,$$

obviously a contradiction with (2.10). Hence, u is constant throughout the flow. Analogously, from

$$\Omega_1 v_x + \Omega_2 v_y + \Omega_3 v_z = 0,$$

we can prove that v is constant throughout the flow. This implies that $\Omega_3 = v_x - u_y = 0$ within the flow, which is a contradiction with our assumption.

We are now left with the case when both Ω_1 and Ω_2 vanish.

Case 2: $\Omega_1 = \Omega_2 = 0$

Assuming now that $\Omega_1 = \Omega_2 = 0$ and under the assumption that Ω_3 is constant, we proceed as in the beginning of the proof and conclude that u and v are bounded harmonic functions, satisfying

$$u_z = v_z = 0 \text{ within the flow.} \tag{2.32}$$

We assume for the sake of contradiction that u is not a constant function. Then, the maximum of u is achieved on the boundary of the fluid domain. Due to the Hopf boundary point lemma, the maximum of u cannot be achieved on the bed $z=-d$. Therefore the maximum of u is assumed on the surface $z=\eta(x, y, t)$. Let $(x_0, y_0, \eta(x_0, y_0, t_0))$ be a point on the surface where u achieves its maximum at some time t_0. But then, owing to (2.32), we obtain that

$$u(x_0, y_0, \eta(x_0, y_0, t_0), t_0) = u(x_0, y_0, -d, t_0),$$

which shows that u also takes on its maximum on the bed $z=-d$. The previous conclusion is a contradiction. Hence, u is a constant function throughout the fluid domain. Analogously, we show that v is also a constant within the water flow. Thus, as before, $\Omega_3 = v_x - u_y = 0$ which is again a contradiction with the assumption $\Omega_3 \neq 0$. Hence, $\Omega_3 = 0$. □

As a consequence of $\Omega_3 = 0$, we have that the vorticity equation becomes

$$\begin{aligned} \Omega_1 u_x + \Omega_2 u_y &= 0, \\ \Omega_1 v_x + \Omega_2 v_y &= 0, \\ \Omega_1 w_x + \Omega_2 w_y &= 0. \end{aligned} \tag{2.33}$$

We will take in the sequel a closer look at three-dimensional water flows. Our attempt will greatly benefit from the already established fact concerning the vanishing of Ω_3. More precisely, we will show in the remaining part of the paper that the possible solutions of the water wave equations (2.1)–(2.5) have a two-dimensional character. This fact is the object of the main result of the paper, which is stated below.

Theorem 2.2. *Assume that η, u, v, w, and P represent a bounded solution of the water wave problem (2.1)–(2.5) with a constant vorticity vector $\Omega \neq 0$. In addition, we assume that*

$$\sup_{(x,y)\in\mathbb{R}^2} P_z(x, y, \eta(x, y)) < 0. \tag{2.34}$$

Then v is constant and u, w, P, and the free surface $z=\eta(x, y)$ present no variations in the y-direction.

Proof. To begin with, we note that due the invariance of the water wave problem under rotations around the z-axis, we can assume without loss of generality that one of the horizontal components of the vorticity vector vanishes. We consider the case $\Omega_1 = 0$, $\Omega_2 \neq 0$. Under this assumption, Eq. (2.33) delivers

$$u_y = v_y = w_y = 0 \quad \text{within the flow.} \tag{2.35}$$

Since $\Omega_3 = 0$, we conclude from above that also $v_x \equiv 0$, while from $\Omega_1 = 0$ we infer that $v_z \equiv 0$. The latter inferences show that the horizontal component of the velocity, v, is only a function of the time t. From the second of Euler's equations, we conclude that

$$v'(t) = -P_y \quad \text{within the flow.}$$

We now claim that

$$v'(t) = 0 \quad \text{for all } t. \tag{2.36}$$

To prove the claim, we note that the previous equality implies that there is a function $(x, z) \to f(x, z)$ such that

$$P(x, y, z, t) = -v'(t)y + f(x, z) \quad \text{for all } x, y, z, t, \tag{2.37}$$

with the property that (x, y, z) lies in the fluid domain at time t. We assume for the sake of contradiction that there is t_0 such that $v'(t_0) \neq 0$.

Using now the boundedness of η, we have that for some x_0 there are y_1, y_2 with $y_1 \neq y_2$, $\eta(x_0, y_1) = \eta(x_0, y_2)$ and such that the segment joining the points $(x_0, y_1, \eta(x_0, y_1))$ and $(x_0, y_2, \eta(x_0, y_2))$ lies entirely within the fluid domain. Since P is constant on the free surface, we have that

$$P(x_0, y_1, \eta(x_0, y_1)) = P(x_0, y_2, \eta(x_0, y_2)) = P_{atm},$$

the equality that in conjunction with (2.37) delivers

$$v'(t_0)(y_1 - y_2) = 0,$$

which, due to $y_1 \neq y_2$, is only possible if $v'(t_0) = 0$. The claim (2.36) is now proved. Thus, P_y vanishes throughout the flow. Differentiating now the kinematic boundary condition with respect to y, we obtain that

$$P_y(x, y, \eta(x, y)) + P_z(x, y, \eta(x, y))\eta_y(x, y) = 0$$

for all x, y. Taking into account the vanishing of P_y, we see that

$$P_z(x, y, \eta(x, y))\eta_y(x, y) = 0,$$

for all x, y. The assumption (2.34) yields now that $\eta_y(x, y) = 0$ for all x, y. □

III. THE VISCOUS CASE

We will use the same notation from Sec. II concerning the velocity field, the pressure, and the definition of the fluid domain. Then, the motion of a viscous, incompressible, and homogeneous water flow in the domain D_η is described by the equation of momentum conservation

$$\begin{aligned} u_t + uu_x + vu_y + wu_z &= -P_x - \nu\Delta u, \\ v_t + uv_x + vv_y + wv_z &= -P_y - \nu\Delta v, \\ w_t + uw_x + vw_y + ww_z &= -P_z - g - \nu\Delta w, \end{aligned} \tag{3.1}$$

(where ν is the coefficient of kinematic viscosity) and the equation of mass conservation

$$u_x + v_y + w_z = 0. \tag{3.2}$$

The equations of motions are completed by the boundary conditions. While the kinematic boundary condition on the surface is, as before,

$$w = \eta_t + u\eta_x + v\eta_y \quad \text{on} \quad z = \eta(x, y, t), \tag{3.3}$$

the kinematic condition on the bed reads

$$u = v = w = 0 \quad \text{on} \quad z = -d. \tag{3.4}$$

The dynamic boundary condition at the surface $z = \eta(x, y, t)$ is

$$P = P_{atm}, \tag{3.5}$$

and the wind stress is proportional to the normal derivative of the velocity at the surface (see Ref. 12); in our context, we do not discuss the transfer of energy from the wind to the water.

We now state the main result of this section.

Theorem 3.1. *Assume that η, u, v, w, and P represent a bounded solution of the water wave problem (3.1)–(3.5) with a constant vorticity vector $\Omega \neq 0$. In addition, we assume that*

$$\sup_{(x,y)\in\mathbb{R}^2} P_z(x, y, \eta(x, y)) < 0. \tag{3.6}$$

Then v is constant and u, w, P, and the free surface $z = \eta(x, y)$ present no variations in the y-direction.

Proof. We will first show that $\Omega_3 = 0$. Note that the vorticity equation (2.7) becomes, under the inclusion of viscous effects,

$$\Omega_t + (\mathbf{u} \cdot \nabla)\Omega = (\Omega \cdot \nabla)\mathbf{u} + \nu\Delta\Omega, \tag{3.7}$$

cf. Ref. 25. Since we work under the assumption that Ω is a constant vector, it follows that the vorticity obeys, as before, the equation

$$(\Omega \cdot \nabla)\mathbf{u} = 0, \tag{3.8}$$

which is a restatement of the fact that u, v, and w are constant in the direction of the vector $\Omega = (\Omega_1, \Omega_2, \Omega_3)$. We claim now that $\Omega_3 = 0$. Assuming, for the sake of contradiction, that $\Omega_3 \neq 0$, we obtain from (3.8) that u, v, and w are constant in the direction of the vector $(\Omega_1, \Omega_2, \Omega_3)$, which is not parallel to the flat bed, due to the assumption $\Omega_3 \neq 0$. Using now the kinematic condition on the bed (3.4), it follows that $u = v = w = 0$ throughout the flow. But, this implies immediately that $\Omega_3 = 0$, that is, we obtain a contradiction with the assumption $\Omega_3 \neq 0$. Therefore, $\Omega_3 = 0$. The latter implies

$$\Omega_1 u_x + \Omega_2 u_y = \Omega_1 v_x + \Omega_2 v_y = \Omega_1 w_x + \Omega_2 w_y = 0, \tag{3.9}$$

throughout the water flow. As in the proof of Theorem 2.2, we conclude that the velocity field (u, v, w) is y-independent and that v depends only on t, the fact which implies $\Delta v = 0$ within the fluid. Thus, using the second equation in (3.1), we obtain that

$$P_y = -v'(t) \quad \text{within the flow.} \tag{3.10}$$

This latter inference allows us to use the same arguments from the proof of Theorem 2.2 to prove the remaining statements, concerning the vanishing of v and the y-independence of η and P. □

We would like to mention that, as emphasized by daSilva and Peregrine,[14] inviscid theory is suitable for the study of water waves that are not near breaking. Indeed, cf. Ref. 14, the most appreciable effects of viscosity in the open sea are to produce wave-amplitude reduction, and diffusion of the deeper motions, over time scales and length scales that are far larger than those of the dynamical surface-processes. Moreover, the choice of constant vorticity is legitimate since for long waves propagating at the surface of water over a nearly flat bed (with wavelengths that exceed the mean water depth), the existence of a non-zero mean vorticity is more important than its specific distribution; see the discussions in the work of daSilva and Peregrine[14] and in the recent paper.[27]

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