ON THE SYMMETRY OF STEADY EQUATORIAL WIND WAVES

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Abstract. In this paper we prove symmetry results for two-dimensional steady periodic equatorial wind-waves, without stagnation points, and with general vorticity distributions.

1. Introduction

Equatorial waves are highly fascinating from both physical and mathematical perspectives. The equator acts as a natural waveguide, with equatorially trapped waves decaying exponentially away from the Equator [2, 4, 12, 13, 16, 23, 30]. Furthermore, the geophysical dynamics at the Equator incorporates large-scale currents, such as the Equatorial Undercurrent (EUC), coupled with wave-current interactions [31]. The EUC extends practically throughout the zonal breadth of the Pacific Ocean, 13000km, and it is confined to a shallow layer which is less than 200m beneath the surface, and which is symmetric about the Equator (typically about 200km in width). Indeed, the El Niño and La Niña phenomena have recently been ascribed to the interplay between equatorial currents in the ocean and atmosphere [21]. The EUC manifests itself when the flow changes direction at a depth of tens of meters beneath the surface of the ocean. The EUC flows eastwards although the wind-generated surface waves propagate predominantly westwards due to the prevailing direction of equatorial winds [3, 31, 33].

Mathematically, the complexity of geophysical waves leads to a number of interesting challenges. Since the Coriolis forces vanish at the Equator, we may apply the $f-$plane approximation to the full geophysical governing equations in the equatorial region [12, 14]. It has recently been shown that a number of qualitative analytical features, which apply to gravity waves with vorticity, similarly apply to equatorial waves [3, 5, 6, 11, 17, 18, 20, 22, 27]. Furthermore, following the derivation in [4] of an explicit exact solution to the geophysical governing equations in the $\beta-$plane, a number of further explicit exact solutions have been derived for different physical settings [2, 10, 16, 23–25].

The aim of this paper is to prove that the strong symmetry results which exist for gravity water waves, both irrotational [29] and with vorticity [6–9, 26, 28], can be extended to the geophysical setting for equatorial water waves. In [19] the authors have used bifurcation theory to prove the existence of equatorial waves which possess a given general vorticity

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distribution in the absence of stagnation points. Geophysical flows are generally rotational, but in [19] it was also shown that in the equatorial region it is possible to have irrotational flows, representing uniform currents or indeed an absence of underlying currents. The symmetry result we present in Theorem 3.1 below applies to equatorial wind waves which prescribe any general vorticity distribution in the absence of stagnation points. Our main tools are sharp elliptic maximum principles together with a symmetrization procedure which makes use of the moving plane method.

The outline of the paper is as follows. In Section 2 we present the governing equations for the $f-$plane approximation of geophysical flows. The main result Theorem 3.1, is stated at the beginning of Section 3, with the remainder of this section being devoted to its proof.

2. The governing equations

In a rotating frame with the origin at a point on earth's surface, we choose the $x-$axis to be horizontally due east, the $y-$axis horizontally due north and the $z-$axis upward. Due to the zonal character of equatorial water waves, we analyze two-dimensional flows which are independent of the $y-$coordinate. The upper fluid layer which is confined to the effects of the wind and the underlying current is bounded from above by the graph of the free surface η and from below by the line $z = -d$. We denote the layer by

$$\Omega_\eta = \{(x, z) \,:\, x \in \mathbb{R} \text{ and } -d < z < \eta(x)\},$$

and we assume that the mean-level of the free-surface is given by $z = 0$. Additionally we assume the fluid flow does not permeate $z = -d$, and so we choose d to be sufficiently large that there is no vertical motion at this depth. We consider travelling waves, with the velocity field (u, w), the pressure P, and the free surface η exhibiting an $(t, x)-$dependence of the form $(x - ct)$, where $c < 0$ is the westward propagation speed of the wave surface. The governing equations in the $f-$plane approximation near the Equator are expressed by the nonlinear free-boundary problem

$$\left\{\begin{array}{rcll} (u-c)u_x + wu_z + 2\omega w &=& -P_x/\rho & \text{in } \Omega_\eta, \\ (u-c)w_x + uw_x + ww_z - 2\omega u &=& -P_z/\rho - g & \text{in } \Omega_\eta, \\ u_x + w_z &=& 0 & \text{in } \Omega_\eta, \\ P &=& P_0 & \text{on } z = \eta(t,x), \\ w &=& (u-c)\eta_x & \text{on } z = \eta(t,x), \\ w &=& 0 & \text{on } z = -d. \end{array}\right. \tag{2.1}$$

Here P_0 is the constant atmospheric pressure, $\omega = 73 \cdot 10^{-6} rad/s$ is the (constant) rotational speed of the Earth[1] round the polar axis towards the east, ρ is the (constant) density of the water, and $g = 9.8 m/s^2$ is the (constant) gravitational acceleration at the Earth's surface. The solutions we consider are periodic in the variable x, that is u, w, P, η

[1]Taken to be a perfect sphere of radius $6371 km$.

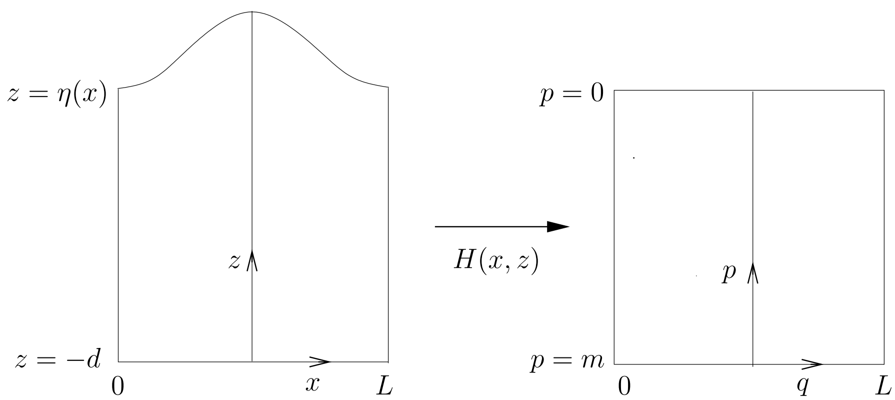

FIGURE 1. The hodograph transformation

are all $L-$ periodic in x, and stagnation points are excluded from the flow. The latter property is satisfied if we assume that

$$c < \min_{\overline{\Omega}_\eta} u. \tag{2.2}$$

Moreover, we require a priori that the solutions have the following regularity

$$\eta \in C^2(\mathbb{R}) \text{ and } (u, w, P) \in \left(C^1(\overline{\Omega}_\eta)\right)^3. \tag{2.3}$$

2.1. Reformulation of the problem (2.6). We may define the stream function $\psi \in C^2(\overline{\Omega}_\eta)$ up to a constant by

$$\psi_z = u - c, \quad \psi_x = -w, \tag{2.4}$$

and we fix the constant by setting $\psi = 0$ on $z = \eta(x)$. The last two relations of (2.1) imply that ψ is constant on both boundaries of Ω_η, and so it follows from integrating (2.4) and using (2.2) that $\psi = m$ on $z = -d$, where

$$m = \int_{-d}^{\eta(x)} (c - u(x,z))dz < 0.$$

The above expression is usually referred to as the relative mass flux. Since

$$\psi(x,z) = m + \int_{-d}^{z} (u(x,s) - c)ds,$$

we deduce that ψ is a periodic function, with period L, and the level sets of the stream function ψ describe the streamlines of the flow. Condition (2.2) enables us also to introduce new variables by means of the Dubreil-Jacotin hodograph transformation $\mathcal{H} : \Omega_\eta \to \Omega$, defined by (see Figure 1)

$$\mathcal{H}(x,z) := (q,p)(x,z) := (x, \psi(x,z)) \qquad \text{for } (x,z) \in \overline{\Omega}_\eta.$$

Here

$$\Omega := \{(q,p) : m < p < 0, q \in \mathbb{R}\}.$$

The mapping $\mathcal{H}$ is a diffeomorphism and, as in [6], one can show that the vorticity of the flow

$$\gamma := u_z - w_x$$

satisfies $\partial_q(\gamma \circ \mathcal{H}^{-1}) = 0$ in Ω, which means that the vorticity is prescribed as a function of the streamlines $\gamma(x,z) = \gamma(\psi(x,z))$ for all $(x,z) \in \Omega_\eta$. Finally, if we define the hydraulic head by the expression

$$E := \frac{(u-c)^2 + w^2}{2} + (g - 2\omega c)z + \frac{P}{\rho} - 2\omega\psi + \int_0^\psi \gamma(s)\,ds \qquad \text{in } \Omega_\eta,$$

we can show that E is constant in Ω_η by taking the curl of the first two equations of (2.1). This is the $f-$plane version of Bernoulli's law. In terms of the stream function, the velocity formulation (2.1) is expressed as the following free boundary value problem

$$\left\{\begin{array}{rcll} \Delta\psi &=& \gamma(\psi) & \text{in } \Omega_\eta, \\ |\nabla\psi|^2 + 2(g-2\omega c)z &=& Q & \text{on } z = \eta(x), \\ \psi &=& 0 & \text{on } z = \eta(x), \\ \psi &=& m & \text{on } z = -d, \end{array}\right. \tag{2.5}$$

the condition (2.2) being equivalent to

$$\min_{\overline{\Omega}_\eta} \psi_z > 0. \tag{2.6}$$

To present a further formulation of the problem, we introduce the height function $h : \Omega \to \mathbb{R}$ by the relation

$$h(q,p) = z \circ H^{-1} + d \qquad \text{for } (q,p) \in \Omega,$$

and we remark, from the definition of $\mathcal{H}$, that

$$h_q \circ H^{-1} = -\frac{\psi_x}{\psi_z} = \frac{w}{u-c} \qquad \text{and} \qquad h_p \circ H^{-1} = \frac{1}{\psi_z} = \frac{1}{u-c}.$$

Assumption (2.2) implies then that

$$\min_{\overline{\Omega}} h_p > 0. \tag{2.7}$$

It follows readily from the definition of h and of the coordinates transformation $\mathcal{H}$ that h solves the following equations

$$\left\{\begin{array}{rcll} (1+h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} + \gamma(p)h_p^3 &=& 0 & \text{in } \Omega, \\ 1 + h_q^2 + [2(g-2\omega c)(h-d) - Q]h_p^2 &=& 0 & \text{on } p = 0, \\ h &=& 0 & \text{on } p = m. \end{array}\right. \tag{2.8}$$

Note that $h \in C^2(\overline{\Omega})$ has period L in the $q-$variable. The problem (2.8) is a quasilinear elliptic equation, since $h_p > 0$, with a nonlinear boundary condition. In [19] the authors have proven the existence of solutions to (2.8), which satisfy (2.7), for general vorticity distributions. We remark that if we know $h(q,0)$, then we know also the free surface $z = \eta(x)$, since $h(q,0) = \eta(q)+d$ for all $q \in \mathbb{R}$. The formulation (2.8) with (2.7) is equivalent to the stream function formulation. Our main result, which we state below, takes advantage of the last two formulations of the problem, in order to present a sufficient criterion for the symmetry of solutions of the water wave problem in the $f-$plane approximation.

3. The main result

Our goal is to prove the following theorem on the symmetry of steady periodic geophysical surface water waves which propagate westwards without stagnation points in equatorial oceanic regions.

Theorem 3.1. *Let (η, u, v, P) be a solution of (2.1)-(2.2) and having regularity (2.3), such that the associated vorticity function $\gamma \in C([m,0])$ is Lipschitz continuous on $[m_0, 0]$ with $m_0 \in (m, 0)$ arbitrary.*

If the surface profile is monotonic between troughs and crests, then it is symmetric.

Remark 3.2. If we choose a period to be the interval delimited by two trough lines, by monotonicity we require only that in this period the surface profile be non-decreasing to the left of the global crest line and non-increasing to its right. In the case of flat lines at the trough or the crest, we chose the middle point of the line to be the actual trough or the crest of the wave.

We also emphasize that the Lipschitz continuity of the vorticity function close to the wave surface (in fact on $[m,0]$) is satisfied when considering more regular solutions [1]

$$\eta \in C^{3-}(\mathbb{R}) \text{ and } (u, w, P) \in \left(C^{2-}(\overline{\Omega}_\eta)\right)^3.$$

3.1. Maximum principles. The proof of Theorem 3.1 is based on elliptic maximum principles and on the moving plane method, cf. [1]. The moving plane method consists in this particular setting in reflecting the wave surface with respect to vertical lines close to the line where the global trough is attained and moving this line until an extremal position is reached. Then, using sharp elliptic maximum principles, cf. Lemmas 3.3 and 3.4 below, we show that the limiting line is the crest line and that the wave is symmetric with respect to it.

Lemma 3.3. *Let $R \subset \mathbb{R}^2$ be an open rectangle and $\mathcal{H} \in C^2(\overline{R})$ satisfy $L\mathcal{H} = 0$ for some uniformly elliptic operator $L = a_{ij}\partial_{ij} + b_i\partial_i$ with continuous coefficients in $\overline{R}$. Then, the following hold.*

- *(i) The weak maximum principle: $\mathcal{H}$ attains its maximum (minimum) on ∂R.*
- *(ii) The strong maximum principle: If $\mathcal{H}$ attains its maximum (minimum) in R, then $\mathcal{H}$ is constant in R.*

[1]The notation C^{k-}, $k \geq 1$, is used for spaces consisting of functions with Lipschitz continuous derivatives of order $k-1$.

(*iii*) *Hopf's maximum principle: Let Q be a point on ∂R, different from the corners of the rectangle $\overline{R}$. If $\mathcal{H}(Q) > \mathcal{H}(X)$ [resp. $\mathcal{H}(Q) < \mathcal{H}(X)$] for all X in R, then $\partial_\nu \mathcal{H}(Q) \neq 0$.*

Proof. For the proof see Theorems 3.1 and 3.5, and Lemma 3.4 in [15]. □

Besides these classical elliptic maximum principles, we shall employ the following edge-point lemma.

Lemma 3.4 (Serrin's edge-point lemma). *Let*

$$R := \{(x,z) \in \mathbb{R}^2 \,:\, a < x < b,\, m < z < \xi(x)$$

where $a < b$, $\xi \in C^2([a,b])$, $\xi > m$, and $\xi'(a) = 0$ [resp. $\xi'(b) = 0$]. Let further $\mathcal{H} \in C^2(\overline{R})$ satisfy $\mathcal{L}\mathcal{H} \geq 0$ in R for the elliptic operator $\mathcal{L} := \Delta + b_i\partial_i$ with continuous coefficients in $\overline{R}$. If the edge point $Q = (a, \xi(a))$ [resp. $Q = (b, \xi(b))$] satisfies $\mathcal{H}(Q) = 0$ and if $\mathcal{H} < 0$ in R, then either

$$\frac{\partial \mathcal{H}}{\partial s} < 0 \qquad or \qquad \frac{\partial^2 \mathcal{H}}{\partial s^2} < 0 \qquad at\ Q,$$

where $s \in R^2$ is any direction at Q that enters R non-tangentially.

Proof. This lemma is a particular version of [32, Lemma 2]. □

Remark 3.5. In order to apply these maximum principles to our setting, we observe that given two solutions $h, k \in C^2(\overline{\Omega})$ of the problem (2.8), with $h_p > 0$, the difference $h - k$ is a solution of $\mathcal{L}(h-k) = 0$ in $\overline{\Omega}$. Hereby, we defined the uniformly elliptic operator $\mathcal{L}$, with continuous coefficients in $\overline{\Omega}$, by

$$\begin{aligned}\mathcal{L} =& (1+h_q^2)\partial_p^2 - 2h_ph_q\partial_p\partial_q + h_p^2\partial_q^2 \\ &+ [k_{qq}(h_p+k_p) - 2k_qk_{pq} + \gamma(p)(h_p^2 + h_pk_p + k_p^2)]\partial_p \\ &+ [k_{pp}(h_p+k_p) - 2h_ph_{pq}]\partial_q.\end{aligned}$$

3.2. Proof of the main theorem. We assume, since the system (2.8) invariant in the $q-$variable, that the horizontal position of the wave trough is at $q = 0$. We introduce now, for the reflection parameter $\lambda \in (0, L/2]$, the function

$$\mathcal{H}^\lambda(q,p) := h(q,p) - h(2\lambda - q, p), \qquad (q,p) \in [\lambda, 2\lambda] \times [m, 0].$$

which measures the difference between the height of the fluid particle corresponding to (q,p) to that determined by $(2\lambda - q, p)$. We notice that the map $\mathcal{H}^\lambda$ satisfies the boundary conditions

$$\begin{cases} \mathcal{H}^\lambda(\lambda, p) &= 0 \qquad \text{for } p \in [m, 0], \\ \mathcal{H}^\lambda(q, m) &= 0 \qquad \text{for } q \in [\lambda, 2\lambda]. \end{cases} \tag{3.1}$$

Indeed, the first property follows immediately from the definition of $\mathcal{H}$ and the second one from the fact that $h = 0$ on $p = m$. As mentioned before, we shall assume throughout this proof that the surface profile is non-decreasing from trough to crest. Therefore, we

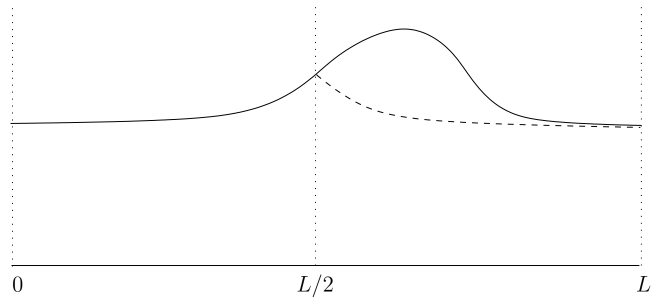

FIGURE 2. Symmetrization in Case 1

additionally have $\mathcal{H}^\lambda(q,0) \geq 0$ for all $q \in [\lambda, 2\lambda]$ provided that $2\lambda \leq L/2$. Moreover, there exists an extremal position λ_0 for λ, which is defined as

$$\lambda_0 := \max\{\lambda \in (0, L/2] : \mathcal{H}^\lambda(q,0) \geq 0 \text{ for all } q \in [\lambda, 2\lambda]\}.$$

Only the following two alternatives can occur:

Case 1: $\lambda_0 = L/2$ (see Figure 2);

Case 2: $\lambda_0 \in (0, L/2)$ and there exists $q_0 \in (\lambda_0, 2\lambda_0]$ for which $\mathcal{H}^{\lambda_0}(q_0, 0) = 0$. Moreover, at the point q_0 the graphs of the functions $q \mapsto h(q,0)$ and $q \mapsto h(2\lambda_0 - q, 0)$ are tangent to each other (see Figure 3).

Case 1: Because by assumption $\lambda_0 = L/2$ and recalling that h is $L-$periodic in the q variable, we obtain additionally the following boundary properties

$$\begin{cases} \mathcal{H}^{\lambda_0}(L/2, p) &= 0 \qquad \text{for } p \in [m, 0], \\ \mathcal{H}^{\lambda_0}(q, 0) &\geq 0 \qquad \text{for } q \in [L/2, L]. \end{cases} \tag{3.2}$$

Let R be the rectangle

$$R := (L/2, L) \times (m, 0).$$

Having the boundary conditions (3.1)-(3.2) at hand, that is $\mathcal{H}^{\lambda_0} \geq 0$ on ∂R, and since $\mathcal{H}^{\lambda_0} \in C^2(\overline{R})$ is a solution of $\mathcal{L}\mathcal{H}^{\lambda_0} = 0$ (with $k := h(2\lambda_0 - \cdot, \cdot)$) we conclude from Lemma 3.3 (i) that $\mathcal{H}^{\lambda_0} \geq 0$ in R. Lemma 3.3 (ii) implies now that there cannot exist an interior point $(q,p) \in R$ with $\mathcal{H}^{\lambda_0}(q,p) = 0$ unless $\mathcal{H}^{\lambda_0}$ vanishes everywhere in R.

In the following we assume that

$$\mathcal{H}^{\lambda_0}(q,p) > 0 \qquad \text{in } R, \tag{3.3}$$

otherwise, if $\mathcal{H}^{\lambda_0}$ vanishes completely, we have symmetry. Using Serrin's edge point lemma we obtain next a contradiction to (3.3). Because of the special form of the elliptic operator $\mathcal{L}$ we cannot apply Lemma 3.4 directly. Instead, we define the function $\Psi^{\lambda_0} : \overline{R_\eta^{\lambda_0}} \to \mathbb{R}$ by setting $R_\eta^{\lambda_0} := \{(x,z) \in \Omega_\eta \,:\, 0 < x < \lambda_0\}$ and

$$\Psi^{\lambda_0}(x,z) = \psi(2\lambda_0 - x, z) - \psi(x,z).$$

Then, if $(x,y) \in R_\eta^{\lambda_0}$, we let $p_1 := \psi(x,z)$ and $p_2 := \psi(2\lambda_0 - x, z)$. By the definition of h we then find, $h(x,p_1) = h(2\lambda_0 - x, p_2)$. But, since $(2\lambda_0 - x, p_2) \in R$, we obtain from (3.3) that $h(2\lambda_0 - x, p_2) > h(x,p_2)$, so that $h(x,p_1) > h(x,p_2)$. Recalling that $h_p > 0$, we conclude that $p_1 > p_2$. This shows that $\Psi^{\lambda_0} < 0 = \Psi^{\lambda_0}(Q)$ in $R_\eta^{\lambda_0}$ if we define Q to be the upper left edge of $R_\eta^{\lambda_0}$, that is $Q := (0, \eta(0))$. Moreover, for all $(x,y) \in \overline{R_\eta^{\lambda_0}}$ we have

$$\Delta\Psi^{\lambda_0}(x,y) = \gamma(\psi(2\lambda_0 - x, y)) - \gamma(\psi(x,y)) = c(x,y)\Psi^{\lambda_0}(x,y),$$

whereby $c : \overline{R_\eta^{\lambda_0}} \to \mathbb{R}$ is the function given by

$$c(x,y) = \begin{cases} \dfrac{\gamma(\psi(2\lambda_0 - x, y)) - \gamma(\psi(x,y))}{\psi(2\lambda_0 - x, y) - \psi(x,y)}, & \text{if } \psi(x,y) \neq \psi(2\lambda_0 - x, y)), \\ 0, & \text{if } \psi(x,y) = \psi(2\lambda_0 - x, y)). \end{cases}$$

We next choose constants $\delta \in (0,\lambda)$ and $\theta \in (m, \eta(\lambda))$ such that for all $(x,z) \in R_{\delta,\theta}^{\lambda_0}$ we have $\psi(x,z), \psi(2\lambda_0 - x, z) \in [m_0, 0]$, with $m_0 \in (m, 0)$ being such that $\gamma \in C^{1-}([m_0, 0])$. Hereby, $R_{\delta,\theta}^{\lambda_0}$ is the subset of $R_\eta^{\lambda_0}$ defined by

$$R_{\delta,\theta}^{\lambda_0} := \{(x,z) \,:\, 0 < x < \delta,\, \theta < z < \eta(x)\},$$

and having Q as a upper left edge. With this choice, $c : \overline{R_{\delta,\theta}^{\lambda_0}} \to \mathbb{R}$ is bounded. We choose $M^2 > \sup_{R_{\delta,\theta}^{\lambda_0}} c$ and define $\Psi : \overline{R_{\delta,\theta}^{\lambda_0}} \to \mathbb{R}$ by $\Psi(x,z) := e^{Mx}\Psi^{\lambda_0}(x,z)$. Clearly, we have $\Psi < 0 = \Psi(Q)$ in $R_{\delta,\theta}^{\lambda_0}$, and moreover

$$\Delta\Psi - 2M\psi_x = (c - M^2)\Psi \geq 0 \qquad \text{in } R_{\delta,\theta}^{\lambda_0}.$$

Applying the Lemma 3.4 at the edge Q of $R_{\delta,\theta}^{\lambda_0}$, we know that at least one of the first or second order derivatives of Ψ must not vanish at Q. On the other hand, since $x = 0$ is the trough line we have $\eta'(0) = 0$, and therewith $\psi_x(Q) = 0$. Hence, we immediately see that $\Psi^{\lambda_0}(Q) = \Psi_x^{\lambda_0}(Q) = \Psi_y^{\lambda_0}(Q) = \Psi_{xx}^{\lambda_0}(Q) = \Psi_{yy}^{\lambda_0}(Q) = 0$. Moreover, differentiating the second relation of (2.5) with respect to x shows that

$$\psi_x(\psi_{xx} + \eta'\psi_{xy}) + \psi_y(\psi_{xy} + \eta'\psi_{yy}) + (g - 2\omega c)\eta' = 0 \qquad \text{on } z = \eta(x),$$

hence $\psi_{xy}(Q) = 0$. We conclude that also $\Psi_{xy}^{\lambda_0}(Q) = 0$. But then it follows readily from the definition of Ψ that all its first and second order derivatives vanish at Q, which gives a contradiction. Therefore, $\mathcal{H}^{\lambda_0} = 0$, and this is equivalent to

$$h(q,p) = h(L - q, p) \qquad \text{for all } (q,p) \in R,$$

which means that the wave is symmetric around the crest located at $q = L/2$.

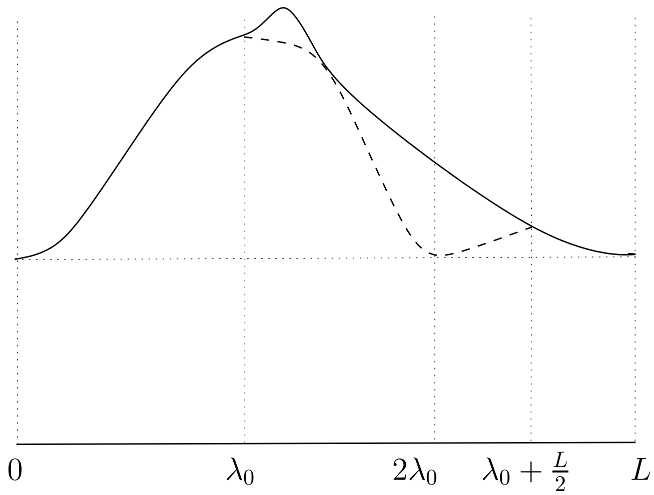

FIGURE 3. Symmetrization in Case 2

Case 2: We enlarge now the definition domain of $\mathcal{H}^{\lambda_0}$ by setting

$$\mathcal{H}^{\lambda_0}(q,p) := h(q,p) - h(2\lambda_0 + L - q, p) \quad \text{for} \quad (q,p) \in \left[2\lambda_0, \lambda_0 + \frac{L}{2}\right] \times [m, 0],$$

and redefine R as

$$R := \left(\lambda_0, \lambda_0 + \frac{L}{2}\right) \times (m, 0).$$

Because of the $L-$periodicity of h with respect to q, we have $\mathcal{H}^{\lambda_0} \in C^2(\overline{R})$. Additionally, by construction and due to the fact that the wave profile is monotone between crests and troughs, the line $2\lambda_0$ lies on the right-hand side of the crest line. Therefore, in this case we have that $h(\cdot, p)$ is non-increasing for $[2\lambda_0, L]$, property which leads us to

$$\mathcal{H}^{\lambda_0}(q, 0) \geq 0 \qquad \text{for all } q \in \left[\lambda_0, \lambda_0 + \frac{L}{2}\right].$$

Additionally, the function $\mathcal{H}$ satisfies also the following boundary conditions

$$\begin{aligned} \mathcal{H}^{\lambda_0}(\lambda_0, p) &= \mathcal{H}^{\lambda_0}\left(\lambda_0 + \tfrac{L}{2}, p\right) = 0 && \text{for } p \in [m, 0], \\ \mathcal{H}^{\lambda_0}(q, m) &= 0 && \text{for } q \in \left[\lambda_0, \lambda_0 + \tfrac{L}{2}\right]. \end{aligned}$$

Applying Lemma 3.3 (i) we get that $\mathcal{H}^{\lambda_0} \geq 0$ in $\overline{R}$. Assuming by contradiction that $\mathcal{H}^{\lambda_0}$ is not identically zero in $\overline{R}$, we then obtain from Lemma 3.3 (ii) that

$$\mathcal{H}^{\lambda_0} > 0 \qquad \text{in } R.$$

Then, from the tangency condition at $(q_0, 0)$ we obtain that $\mathcal{H}^{\lambda_0}(q_0, 0) = 0$, and additionally

$$h_q(q_0, 0) = -h_q(2\lambda_0 - q_0, 0).$$

Since $(q_0, 0) \in \partial R$ is not an edge, Hopf's maximum principle, cf. Lemma 3.3 (iii), implies now that $\mathcal{H}_p^{\lambda_0}(q_0, 0) \not\equiv 0$. On the other hand, the second equation of (2.8) evaluated at the points $(q_0, 0)$ and $(2\lambda_0 + L - q_0, 0)$, respectively, together with the tangency property yields that

$$h_p^2(q_0, 0) = h_p^2(2\lambda_0 + L - q_0, 0).$$

Recalling that $h_p > 0$, we conclude that $h_p(q_0, 0) = h_p(2\lambda_0 + L - q_0, 0)$, and therewith $\mathcal{H}_p^{\lambda_0}(q_0, 0) = 0$. This contradicts Hopf's principle and additionally our assumption that $\mathcal{H}^{\lambda_0} \not\equiv 0$.

In order to finish the proof we must analyze the case when $\mathcal{H} \equiv 0$ in R. Since $h(\cdot, 0)$ is non-increasing on the interval $[2\lambda_0, L]$ we have that

$$h(q, 0) = h(L, 0) \qquad \text{for } 2\lambda_0 \le q \le L.$$

Furthermore, for $q \in [0, 2\lambda_0]$ we have that $h(\cdot, 0)$ is symmetric around λ_0, while by periodicity

$$h(q, 0) = h(L, 0) \qquad \text{for } 2\lambda_0 - L \le q \le 0.$$

Hence, we can conclude that h is symmetric around the crest line $q = \lambda_0$, and the proof is completed.

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