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Trapped Surfaces in Spherical Stars

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We give necessary and sufficient conditions for the existence of trapped surfaces in spherically symmetric spacetimes. These conditions show that the formation of trapped surfaces depends on both the degree of concentration and the average flow of the matter. The result can be considered as a partial validation of the cosmic-censorship hypothesis.

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It is a common assumption in general relativity that if a sufficient amount of matter is concentrated in a small enough volume the system will collapse under its own weight. However, it is difficult to demonstrate this even in the spherically symmetric case. It is clear that if the Schwarzschild criterion $2M/R < 1$ is violated the system will collapse to form a black hole. This does not directly resolve the issue, because the M in question is the total mass which includes the negative binding energy and so cannot be directly related to the amount of matter in the collapsing star, while the R is the radius in Schwarzschild coordinates and so is a very poor measure of the enclosed volume.

In studies of gravitational collapse the most interesting object is the event horizon.[1] However, this is a global property of the spacetime and is very hard to identify in practice. A more useful, local, property is the "trapped surface." This is a closed two-surface at any instant of time which has the property that the outgoing light rays from it are convergent. In the spherically symmetric case the existence of a trapped surface guarantees that the system must collapse to a black hole assuming only that the matter has positive energy density (the weak energy condition).[2]

One explicit formulation of the idea that matter concentration causes gravitational collapse is the so-called "trapped-surface conjecture," i.e., that a trapped surface forms if a sufficient amount of matter is packed into a small enough volume.[3] In this Letter we will derive relationships in the spherically symmetric case which are necessary and sufficient conditions for the existence of trapped surfaces. These conditions will depend only on the amount of matter inside a sphere and the proper radius of the sphere. Thus we give an explicit realization and proof of the trapped-surface conjecture.

Let us have a spherical distribution of matter of varying density μ which is instantaneously at rest. Consider a spherical surface $\delta\Omega$ which encloses a volume Ω. Let L be the proper radius of $\delta\Omega$, the proper distance from $\delta\Omega$ to the center. If

$$\tilde{M} = \int_\Omega \mu \, dv \geq L, \tag{1}$$

then Ω must contain a trapped surface and so must collapse gravitationally. The only assumption is that μ is nonnegative; for example, we place no condition on the equation of state.

This is a "sharp" inequality in the sense that given any $\epsilon > 0$ we can find a distribution which satisfies

$$\int \mu \, dv = (1-\epsilon)L, \tag{2}$$

but which does not have a trapped surface.

If we are given a spherical distribution of matter which is not at rest but which has a radial current density $\mathbf{j}(r)$, a sufficient condition for the existence of a trapped surface is that

$$\int_\Omega (\mu - \mathbf{j}\cdot\mathbf{n}) dv \geq \tfrac{7}{6} L, \tag{3}$$

where n is the unit outward radial normal. Here we have assumed that the constant time slice is a maximal slice of the spacetime; i.e., the trace of the extrinsic curvature is zero.

In the case where the matter is at rest, we can find equivalent necessary conditions for the existence of trapped surfaces. If for a spherical volume Ω we have

$$\int_\Omega \mu \, dv < L/2, \tag{4}$$

then $\delta\Omega$ itself cannot be a trapped surface. We can also show that if

$$\mu_{\max} L^2 < 5/8\pi, \tag{5}$$

where $\mu_{\max}$ is the maximum value of the density in Ω, then there exist no trapped surfaces in Ω. We can show that (4) is sharp, but we do not believe that (5) is. An upper limit to the constant is $3\pi/32$. We feel that this may well be the best value.

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Initial data for the Einstein equations consist of a Riemannian three-metric g_{ab} and a symmetric tensor K^{ab} (the extrinsic curvature, the time derivative of g_{ab}).[4] These cannot be given independently, but must satisfy the constraints

$$^{(3)}R - K_{ab}K^{ab} + (g_{ab}K^{ab})^2 = 16\pi\mu, \tag{6}$$

$$[K^{ab} - g^{ab}(g_{cd}K^{cd})]_{;a} = 8\pi j^b, \tag{7}$$

where μ is the source energy density and j^b is the source current density. The semicolon denotes covariant derivative with respect to the Reimannian metric g.

Consider any two-surface in the three-manifold. It will have a unit normal n^a. The expansion of the outgoing null rays from any point on this surface is given by

$$n^a{}_{;a} + K^{ab}n_a n_b - g_{ab}K^{ab}. \tag{8}$$

A trapped surface is a closed two-surface which satisfies

$$n^a{}_{;a} + K^{ab}n_a n_b - g_{ab}K^{ab} \leq 0 \tag{9}$$

on every point of the surface.[1]

Let us now restrict ourselves to spherical symmetry; further, let us assume that the time slice on which we define the initial data is a maximal slice, one on which $\mathrm{tr}K = g_{ab}K^{ab} = 0$. The constraint (6) can be written

$$^{(3)}R = 16\pi\mu + K_{ab}K^{ab}. \tag{10}$$

We have a spherically symmetric three metric and so we know that there exist isotropic coordinates in which the metric is conformally flat,

$$g_{ab} = \phi^4(r)\delta_{ab}, \tag{11}$$

where δ_{ab} is the flat (Cartesian) metric. The scalar curvature $^{(3)}R$ takes a simple form for such a manifold

$$^{(3)}R = -8\phi^{-5}\nabla^2\phi, \tag{12}$$

where ∇^2 is the flat-space Laplacian. If we demand that the energy density μ be nonnegative, we get from (10) that

$$\nabla^2\phi \leq 0. \tag{13}$$

The min-max principle tells us that ϕ has its only maximum at the origin and so

$$d\phi/dr \leq 0. \tag{14}$$

Let us now consider a spherical surface of coordinate radius r. On such a surface it is easy to show that

$$n^a{}_{;a} = (r^2\phi^6)^{-1}\, d(r^2\phi^4)/dr. \tag{15}$$

If it is *not* a trapped surface then we must have

$$d(r\phi^2)/dr + \tfrac{1}{2}r\phi^4 K^{ab}n_a n_b > 0. \tag{16}$$

Let us have a surface $\delta\Omega$, of coordinate radius r_0, which encloses a volume Ω. Take the momentum constraint [from (7)]

$$K^{ab}{}_{;a} = 8\pi j^b,$$

multiply by the unit radial normal, and integrate over Ω to give

$$8\pi\int_\Omega j^b n_b\, dv = \int n_b K^{ab}{}_{;a}\, dv = \oint n_b K^{ab}\, dS_a - \int K^{ab}n_{a;b}\, dv = 4\pi r_0^2\phi^4(r_0)K^{ab}n_a n_b\,|_{r_0} - \int K^{ab}n_{a;b}\, dv. \tag{17}$$

Now integrate the Hamiltonian constraint (10) over Ω to give

$$\begin{aligned} 16\pi\int_\Omega \mu\, dv &= \int {}^{(3)}R\, dv - \int K_{ab}K^{ab}\, dv = -8\int \phi\nabla^2\phi\, d^3x - \int K_{ab}K^{ab}\, dv \\ &= -32\pi r_0\phi\, d\phi/dr\,|_{r_0} + 32\pi\int_0^{r_0} r^2(d\phi/dr)^2\, dr - \int K_{ab}K^{ab}\, dv \\ &= -16\pi r_0 d(r\phi^2)/dr\,|_{r_0} + 16\pi\int_0^{r_0}[d(r\phi^2)/dr + 2r^2(d\phi/dr)^2]dr - \int K_{ab}K^{ab}\, dv. \end{aligned} \tag{18}$$

Now subtract twice (17) from (18) to get

$$\begin{aligned} 16\pi\int_\Omega(\mu - \mathbf{j}\cdot\mathbf{n})dv = &-16\pi r_0[d(r\phi^2)/dr + \tfrac{1}{2}r\phi^4K^{ab}n_a n_b]_{r_0} + 16\pi\int[d(r\phi^2)/dr + 2r^2(d\phi/dr)^2]dr \\ &+ \int(2K^{ab}n_{a;b} - K^{ab}K_{ab})dv. \end{aligned} \tag{19}$$

The spherical symmetry allows us to make a further simplification. Since K^{ab} is a spherically symmetric tracefree tensor it must be of the form

$$K_{ab} = (n_a n_b - \tfrac{1}{3}g_{ab})K(r). \tag{20}$$

This means

$$\int K^{ab}K_{ab}\, dv = \tfrac{2}{3}\int K^2\phi^6 d^3x = \tfrac{8}{3}\pi\int_0^{r_0} K^2\phi^6 r^2\, dr, \tag{21}$$

$$\int K^{ab}n_{a;b}\, dv = -\tfrac{1}{3}\int K n^a{}_{;a}\, dv = -\tfrac{4}{3}\pi\int K[d(r^2\phi^4)/dr]dr, \tag{22}$$

and the condition that a surface not be trapped, (16), becomes

$$d(r\phi^2)/dr + \tfrac{1}{3}K\phi^4 r > 0. \tag{23}$$

Equation (19) now reads

$$16\pi\int_\Omega(\mu - \mathbf{j}\cdot\mathbf{n})dv = -16\pi r_0[d(r\phi^2)/dr + \tfrac{1}{3}K\phi^4 r]_{r_0} + 16\pi\int_0^{r_0}\{d(r\phi^2)/dr + 2(d\phi/dr)^2 r^2 - \tfrac{1}{6}K^2\phi^6 r^2 - \tfrac{1}{3}Kr\phi^4 - \tfrac{2}{3}Kr^2\phi^3 d\phi/dr\}dr. \tag{24}$$

The integral on the right-hand side of (24) can now be written as

$$16\pi\int\{\tfrac{7}{6}\phi^2 - \tfrac{1}{6}[\phi + 6r\,d\phi/dr + K\phi^3 r]^2 + 4r(d\phi/dr)[\phi + 2r\,d\phi/dr + \tfrac{1}{3}K\phi^3 r]\}dr. \tag{25}$$

If (14) and (23) are valid everywhere in Ω, we get that the last two terms in (25) and the surface term in (24) are all negative. In this case we immediately deduce from (24) that

$$16\pi\int_\Omega(\mu - \mathbf{j}\cdot\mathbf{n})dv < 16\pi\int_0^{r_0}\tfrac{7}{6}\phi^2 dr. \tag{26}$$

Of course, $\int\phi^2 dr = L$, the proper radius of $\delta\Omega$, and so we deduce that, in the absence of spherical trapped surfaces,

$$\int_\Omega(\mu - \mathbf{j}\cdot\mathbf{n})dv < \tfrac{7}{6}L. \tag{27}$$

This is the desired result because if we find a set Ω for which (27) is not satisfied, i.e., for which

$$\int_\Omega(\mu - \mathbf{j}\cdot\mathbf{n})dv \geq \tfrac{7}{6}L, \tag{28}$$

there must be a trapped surface in Ω.

This formula shows that the matter flow plays an important role in the formation of trapped surfaces, in addition to the energy density. If the matter flow is inward the likelihood of gravitational collapse is increased, whereas an outflow of matter makes it more difficult to form a horizon. If we were to consider the formation of past trapped surfaces, the analysis would go through exactly as before, except the sign of the $\mathbf{j}\cdot\mathbf{n}$ term. We will have a past trapped surface if we can find a volume which satisfies

$$\int_\Omega(\mu + \mathbf{j}\cdot\mathbf{n})dv \geq \tfrac{7}{6}L. \tag{29}$$

Using this technique, we can find an even stronger result for the moment-of-time-symmetry case, i.e., when $K^{ab}=0$ and the matter is instantaneously at rest. Looking at (24), when $K=0$, the only terms that remain in the integral are

$$\phi^2 + 2r\phi\frac{d\phi}{dr} + 2\left(\frac{d\phi}{dr}\right)^2 = \phi^2 + r\phi\frac{d\phi}{dr} + r\left(\frac{d\phi}{dr}\right)\left[\phi + 2r\frac{d\phi}{dr}\right] < \phi^2 \tag{30}$$

if we have no trapped surfaces and if (14) is satisfied. Thus, if we have a spherical volume for which

$$\int_\Omega\mu\,dv > L \tag{31}$$

for spherical, moment-of-time-symmetry data, there must be a spherical trapped surface inside Ω. Static solutions, for example, must have moment-of-time-symmetry slices. Therefore, the converse of (31) must hold for every spherically symmetric static star.

To obtain a necessary condition for $\delta\Omega$ to be a trapped surface, return to (19) and assume that $\delta\Omega$ is trapped, i.e.,

$$(d/dr)(r\phi^2) + \tfrac{1}{2}r\phi^4 K^{ab}n_a n_b\,|_{r_0} < 0. \tag{32}$$

We then get (assuming $K^{ab}=0$)

$$\begin{aligned}\int_\Omega\mu\,dv &> r_0\phi^2|_{r_0} + 2\int_0^{r_0} r^2(d\phi/dr)^2 dr \\ &= \int[d(r\phi^2)/dr + 2r^2(d\phi/dr)^2]dr \\ &= \int[\tfrac{1}{2}\phi^2 + \tfrac{1}{2}\{\phi + 2r\,d\phi/dr\}^2]dr > \tfrac{1}{2}\int_0^{r_0}\phi^2 dr = \tfrac{1}{2}L.\end{aligned} \tag{33}$$

Thus, if we have a surface on which

$$\int_\Omega\mu\,dv < \tfrac{1}{2}L, \tag{34}$$

it cannot be trapped. It is straightforward to show that both (31) and (34) are sharp inequalities: For example, (31) can be saturated by our choosing $\phi = 1 - r^\alpha$ with α large.

We can derive another necessary condition in the moment-of-time-symmetry case. Start with

$$\int_0^{r_0}\phi^5 r^{3/2}\,{}^{(3)}R\,dr=-8\int_0^{r_0}r^{3/2}\nabla^2\phi\,dr=-8\int r^{-1/2}\left[\frac{d(r^2 d\phi/dr)}{dr}\right]dr=-8r^{3/2}\frac{d\phi}{dr}\Bigg|_{r_0}-4\int r^{1/2}\frac{d\phi}{dr}\,dr$$

$$=-4r^{1/2}\phi^{-1}\frac{d(r\phi^2)}{dr}\Bigg|_{r_0}+2\int r^{-1/2}\phi\,dr. \tag{35}$$

If the surface at r_0 is trapped, we have

$$\int_0^{r_0}\phi^5 r^{3/2}\,{}^{(3)}R\,dr\geq 2\int_0^{r_0}r^{-1/2}\phi\,dr. \tag{36}$$

Now we have

$$\tfrac{2}{5}\,{}^{(3)}R_{\max}L^{5/2}={}^{(3)}R_{\max}\int l^{3/2}dl$$

$$\geq {}^{(3)}R_{\max}\int\phi^5 r^{3/2}dr\geq\int\phi^5 r^{3/2}\,{}^{(3)}R\,dr\geq 2\int_0^{r_0}r^{-1/2}\phi\,dr=2\int r^{-1/2}\phi^{-1}\phi^2 dr$$

$$\geq 2\int l^{-1/2}dl=4L^{1/2}. \tag{37}$$

Hence a necessary condition for the surface at r_0 to be trapped is

$$\tfrac{2}{5}\,{}^{(3)}R_{\max}L^{5/2}>4L^{1/2} \tag{38}$$

or

$${}^{(3)}R_{\max}L^2>10 \tag{39}$$

or

$$\mu_{\max}L^2>5/8\pi. \tag{40}$$

Both $\mu_{\max}$ and L monotonically increase as r increases. Therefore, if at a particular radius $\mu_{\max}L^2>5/8\pi$, this will also be true for all interior surfaces and so they also cannot be trapped. It is hard to believe from the derivation that this is a sharp inequality; the best value that we have found is $3\pi/32$ [with the constant-density star, $\phi=(1+r^2)^{-1/2}$]. This may well be the best constant.

Condition (3) clearly demonstrates the validity of the trapped-surface conjecture. It also lends credence to a version of cosmic censorship—"massive naked singularities do not exist."[2,5] If we have a spherical collapse there may be a naked singularity at the origin.[6] That singularity, however, must be massless, otherwise (3) would be satisfied, and so we would have a horizon around it.

The key results, (1) and (3), have only been derived in the spherically symmetric case, but, of course, they are obviously valid (with some minor adjustments of the constants) for any data which are close to spherical symmetry.

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[1]S. W. Hawking, in *Black Holes,* edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, New York 1973).

[2]W. Israel, Phys. Rev. Lett. **56**, 789 (1986), and Can. J. Phys. **64**, 120 (1986).

[3]H. J. Seifert, Gen. Relativity Gravitation **10**, 1065 (1976).

[4]C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 21.

[5]A. Królak, Classical Quantum Gravity **3**, 267 (1986).

[6]D. Christodoulou, Commun. Math. Phys. **93**, 171 (1984).