Combining Two Choice Functions and Enforcing Natural Properties

Nic Wilson [1]

Abstract. This paper considers the problem of combining two choice functions (CFs), or setwise optimisation functions, based on use of intersection and composition. Each choice function represents preference information for an agent, saying, for any subset of a set of alternatives, which are the preferred, and which are the sub-optimal alternatives. The aim is to find a combination operation that maintains good properties of the choice function. We consider a family of natural properties of CFs, and analyse which hold for different classes of CF. We determine relationships between intersection and composition operations, and find out which properties are maintained by these combination rules. We go on to show how the most important of the CF properties can be enforced or restored, and use this kind of procedure to define combination operations that then maintain the desirable properties.

1 INTRODUCTION

Two of my colleagues, Ian and Jane, are heading for lunch, and need to decide which canteen to choose. Jane says she'd be happy with anywhere but Brookfield. Ian doesn't want to walk too far. I would like to help them with their decision making process, by suggesting a set of options. There are some obvious ways of doing this. Firstly, one could just consider the union of their preferred sets of options; however, this is rather cautious, and it might well be better to narrow down the options more. Instead one could consider the intersection of their sets of options, i.e., the set of canteens which are close, excepting Brookfield. However, this set may be empty, i.e., there are no such canteens currently available; in this case, one might instead return the union. Another approach is to give one of the agents priority, e.g., Jane, so Ian gets to choose from all options except Brookfield. Or, to be fairer, one could take the union of this set with that resulting from giving Ian priority.

This paper considers this kind of task. Each of the two agents is assumed to have an associated choice function (CF) that will return a subset (their preferred options) of any set of alternatives, and we would like to generate a combined CF. There are certain desirable properties one would like for a CF, and therefore, we would like a combination operation that maintains these properties.

Although we do not *a priori* make assumptions about the form of the CF, our approach is mainly motivated by situations in which the CF is not very definite, i.e., the options are typically reduced by the CF, but not to a singleton set. An individual decision maker's preferences might be, for instance, a total pre-order over the set of alternatives. However, we will often only have partial information about this, such as a number of constraints on the set of total pre-orders based on the decision maker's answers to preference elicitation queries. So our choice function for that decision maker will be based on this restricted set of total pre-orders, often represented implicitly (and compactly) with a set of constraints.

We consider various ways that choice functions can be generated, in particular, from partial information about a decision maker's preferences, and from a binary relation on the set of alternatives. We discuss a number of natural properties that one might expect of a choice function, and we analyse which of these properties hold for different forms of choice function. We show how desirable properties of a CF can be enforced or restored.

We also analyse which of the properties are maintained by basic combination operations: union, intersection and composition. In particular, we point out that the very natural intersection combination operation can cause some of the natural properties to be lost. However, this issue can be dealt with by our restoration methods.

Section 2 gives the basic definitions and considers important properties that one might expect of a choice function. In Section 3, we consider various important classes of CFs, related to different notions of optimality. The basic combination operations of union, intersection and composition are considered in Section 4, and in Section 5 we analyse which of the properties of choice functions are maintained by the basic combination operations. We consider how to enforce (or restore) the desirable properties for a CF in Section 6, and this is applied to combination operations in Section 7. Section 8 concludes.

2 DESIRABLE PROPERTIES OF CFS

We start with definitions of CFs, and then consider classic desirable properties, along with some weaker versions of the properties.

2.1 Basic definitions of choice functions

Let Ω be a finite set, which is intended to represent a set of alternatives, i.e., alternative choices in a decision making problem. We define a *Choice Function (CF)* Op *over* Ω to be a function from 2^Ω to 2^Ω satisfying the following property:

(Sub): for all $A \subseteq \Omega$, $\mathrm{Op}(A) \subseteq A$.

In this paper, the main intended interpretation of a choice function Op will be that it represents (what we know about) the preferred alternatives (e.g., of some agent) in a particular decision making problem. For set of alternatives $A \subseteq \Omega$, the set $\mathrm{Op}(A)$ represents the set of optimal, i.e., best alternatives among A.

[1] Insight Centre for Data Analytics, School of Computer Science and IT, University College Cork, Cork, Ireland, email: nic.wilson@insight-centre.org

Given choice function Op over Ω, we also consider the complementary function $\overline{\mathrm{Op}}$ over Ω, given by $\overline{\mathrm{Op}}(A) = \Omega \setminus A$ for $A \subseteq \Omega$. If $\mathrm{Op}(A)$ represents the optimal elements in A, then $\overline{\mathrm{Op}}(A)$ is the set of sub-optimal elements.

We define the identity function ID on 2^Ω by $\mathrm{ID}(A) = A$ for all $A \subseteq \Omega$. This choice function we also refer to as the vacuous CF over Ω, since, for all A it eliminates no alternatives: no alternative is sub-optimal. For mathematical reasons, it is also helpful to consider the null CF EMP on 2^Ω defined by $\mathrm{EMP}(A) = \emptyset$ for all $A \subseteq \Omega$. This is a choice function that always eliminates all the alternatives.

We define the fixed points $\mathit{Fix}(\mathrm{Op})$ of choice function Op to be $\{A \subseteq \Omega : \mathrm{Op}(A) = A\}$. These are the sets for which Op is equal to the vacuous CF. If $A \in \mathit{Fix}(\mathrm{Op})$ then Op is, in a sense, uninformative about A: none of elements of A are preferred to the others.

2.2 Properties and their relationships

Not all choice functions represent sensible decision making attitudes. There are natural properties that one might assume on a CF. The main properties of choice functions, that we focus on, are the following three:

(NE): For non-empty A, $\mathrm{Op}(A) \neq \emptyset$.
(SMA): If $B \subseteq A$ then $\mathrm{Op}(A) \cap B \subseteq \mathrm{Op}(B)$.
(IIA): If $\mathrm{Op}(A) \subseteq B \subseteq A$ then $\mathrm{Op}(A) = \mathrm{Op}(B)$.

Property (NE) (being non-empty) might be viewed as a kind of consistency requirement: that at least one alternative is optimal (i.e., not excluded). Mostly in the social choice literature, a (social) choice function is defined to be a function 2^Ω to 2^Ω satisfying (Sub) and property (NE). We consider functions not satisfying (NE) for a number of reasons; in particular, because this looser definition makes the set of choice functions closed under intersection, and because certain natural notions of optimality, such as being necessarily optimal, and being possibly strictly optimal (see Section 3.1), do not satisfy (NE).

Property (SMA) can be more simply viewed in terms of the correspond sub-optimality function, in the following equivalent form:

If $B \subseteq A$ then $\overline{\mathrm{Op}}(A) \supseteq \overline{\mathrm{Op}}(B)$.

That is, an element is sub-optimal in A if it is sub-optimal in a subset of A. Sub-optimality is maintained if we add elements to A, which is why we use the abbreviation (SMA).

In property (IIA), writing B as $A \setminus C$ we obtain the following equivalent form:

If $C \subseteq A$ and $\mathrm{Op}(A) \cap C = \emptyset$ then $\mathrm{Op}(A) = \mathrm{Op}(A \setminus C)$.

That is, if every element of C is sub-optimal in A, then deleting C from A does not change the optimal elements. Thus, (IIA) is a form of *independence of irrelevant alternatives*, hence the abbreviation.

These properties have been explored for (social) choice functions including, for instance, their relationship with path independence (union decomposition); see the survey article [16], and e.g., [1, 8]. Property (SMA) corresponds with Condition H of [1], Sen's Condition α [18], and Moulin's Chernoff Condition [16]. Property (IIA)is called Condition 0 in [1] and relates with Moulin's Aizerman property [16]. It is shown in [1] that Path Independence is equivalent to properties (SMA) and (IIA). In [22], computational properties are analysed of choice functions satisfying properties (SMA) and (IIA).

In addition, we also consider the following weaker properties related to (SMA) and (IIA).

(IIAa): If $\mathrm{Op}(A) \subseteq B \subseteq A$ then $\mathrm{Op}(A) \supseteq \mathrm{Op}(B)$.
(IIAb): If $\mathrm{Op}(A) \subseteq B \subseteq A$ then $\mathrm{Op}(A) \subseteq \mathrm{Op}(B)$.
(Idem): Idempotence: $\mathrm{Op}(\mathrm{Op}(A)) = \mathrm{Op}(A)$.
(SIdem): Strong Idempotence: If $A \subseteq \mathrm{Op}(B)$ then $\mathrm{Op}(A) = A$.

Clearly, property (IIA) is equivalent to the conjunction of properties (IIAa) and (IIAb). Property (IIA) is equivalent to (IIAa) if one assumes (SMA), because (SMA) implies (IIAb).

Idempotence (Idem) is saying that the best elements among the best elements in A are just the best elements in A. Idempotence of Op is equivalent to the statement that its set of fixed points is equal to the image of Op, i.e., $\mathit{Fix}(\mathrm{Op}) = \{\mathrm{Op}(A) : A \subseteq \Omega\}$. (SIdem) is a stronger form of idempotence, which additionally assumes that any subset of a fixed point is a fixed point. One reason for considering (SIdem) is that it is preserved under composition: see Lemma 7.

Property (SMA) implies strong idempotence; specifically, (SIdem) corresponds to the cases of (SMA) when one adds the condition $\mathrm{Op}(A) \supseteq B$ to the antecedent of (SMA). Similarly, (IIAb) corresponds to the special case of (SMA) in which we additionally assume $\mathrm{Op}(A) \subseteq B$, and idempotence corresponds to the special case in which $\mathrm{Op}(A) = B$. Therefore, both (SIdem) and (IIAb) lie between (SMA) and idempotence.

Neither of (SIdem) or (IIAb) imply the other. Consider Op over $\Omega = \{a, b, c, d\}$ defined by $\mathrm{Op}(\{a, b, c, d\}) = \{a, b\}$, and $\mathrm{Op}(\{a, b, c\}) = \{a\}$, and for all other $C \subseteq \Omega$, $\mathrm{Op}(C) = C$. This satisfies strong idempotence (SIdem), but not (IIAb), since $\mathrm{Op}(\{a, b, c, d\}) \subseteq \{a, b, c\} \subseteq \{a, b, c, d\}$, and $\mathrm{Op}(\{a, b, c, d\}) \not\subseteq \mathrm{Op}(\{a, b, c\})$. Below in Section 3.2 we give example of a choice function that satisfies (IIAb) but not strong idempotence.

Some consequences of property (SMA)

Below we list some basic consequences of property (SMA).

Lemma 1 *Suppose that* Op *is a choice function over* Ω *satisfying property (SMA), and let* Y *and* Z *be arbitrary subsets of* Ω. *Then*
(i) Op *satisfies property (IIAb) and is strongly idempotent, and thus also idempotent.*
(ii) $\mathrm{Op}(Y \cap Z) \supseteq \mathrm{Op}(Y) \cap \mathrm{Op}(Z)$.
(iii) $\mathrm{Op}(Y \cup Z) \subseteq \mathrm{Op}(Y) \cup \mathrm{Op}(Z)$.

Regarding (iii), property (SMA) implies $\mathrm{Op}(Y \cup Z) \cap Y \subseteq \mathrm{Op}(Y)$ and $\mathrm{Op}(Y \cup Z) \cap Z \subseteq \mathrm{Op}(Z)$, so $\mathrm{Op}(Y \cup Z) \cap (Y \cup Z) \subseteq \mathrm{Op}(Y) \cup \mathrm{Op}(Z)$. Condition (Sub) then implies (iii).

In particular, Lemma 1(iii) implies that $\mathrm{Op}^{-1}(\emptyset)$ is closed under union, and so has a unique largest element (where $\mathrm{Op}^{-1}(\emptyset) = \{Z \subseteq \Omega : \mathrm{Op}(Z) = \emptyset\}$). Also, if $\mathrm{Op}(Z) = \emptyset$ then for any $Y \subseteq \Omega$, $\mathrm{Op}(Y \cup Z) \subseteq \mathrm{Op}(Y)$, so, in particular, $\mathrm{Op}(Y \cup Z) \subseteq Y$ (by Condition (Sub)). If we in addition have property (IIA) then we have $\mathrm{Op}(Y \cup Z) \subseteq Y \subseteq Y \cup Z$, so $\mathrm{Op}(Y \cup Z) = \mathrm{Op}(Y)$. In particular, for arbitrary $X \subseteq \Omega$ we can choose $Y = X \setminus Z$ leading to $\mathrm{Op}(X) = \mathrm{Op}(X \setminus Z)$.

This leads to the following result concerning the structure of the set of sets A such that $\mathrm{Op}(A)$ is empty, i.e., no best alternatives are returned. This structure will enable methods for restoring the (NE) property: see Section 6.2.

Proposition 1 *Suppose that* Op *is a choice function over* Ω *satisfying property (SMA). Then* $\mathrm{Op}^{-1}(\emptyset)$ *is closed under union, and so has a unique largest element, i.e., there is a unique largest subset* Z *of* Ω *such that* $\mathrm{Op}(Z) = \emptyset$. *Let us call this* Z, Z_{Op}. *For all* $X \subseteq \Omega$, *and for all* $Z \in \mathrm{Op}^{-1}(\emptyset)$, $\mathrm{Op}(X \cup Z) \subseteq \mathrm{Op}(X)$.

If, in addition, Op *satisfies property (IIA), then, for all* $Y \subseteq \Omega$, $\mathrm{Op}(Y) = \emptyset \iff Y \subseteq Z_{\mathrm{Op}}$. *Also, for all* $Y \subseteq \Omega$ *and* $Z \subseteq Z_{\mathrm{Op}}$,

$\mathrm{Op}(Y) = \mathrm{Op}(Y \setminus Z)$*, and* $\mathrm{Op}(Y) \cap Z_{\mathrm{Op}} = \emptyset$*. For any* $\alpha \in \Omega \setminus Z_{\mathrm{Op}}$ *and* $\beta \in Z_{\mathrm{Op}}$*, we have* $\mathrm{Op}(\{\alpha, \beta\}) = \{\alpha\}$*.*

3 SOME CLASSES OF CHOICE FUNCTIONS AND THEIR PROPERTIES

We consider ways of generating choice functions; firstly, in Section 3.1, based on different notions of optimality when we have partial information about a decision maker's preferences; secondly, in Section 3.2, when we have a binary preference relation between alternatives that may not be transitive. In both cases, we analyse which of the properties of a CF necessarily hold, for such a class of choice functions.

3.1 CFs based on a set of total pre-orders

A natural and common way of modelling a decision maker's preferences over a set of alternatives Ω, is as a total pre-order over Ω. However, we will often only have partial information about their preferences; this can lead to a restricted set of total pre-orders on Ω, e.g., all those compatible with the elicited preference information. Given a set of total pre-orders on Ω, there are quite a number of different ways of generating a CF over Ω, in particular, the choice functions PO_Θ, UD_Θ, POUD_Θ, PSO_Θ, MPO_Θ and NO_Θ, defined below.

Let Θ be a non-empty set of total pre-orders on Ω. Given $\succcurlyeq$ in Θ, and $A \subseteq \Omega$, we say that α is optimal in A, written $\alpha \in \mathrm{O}_{\succcurlyeq}(A)$, if $\alpha \in A$ and for all $\beta \in A$, $\alpha \succcurlyeq \beta$. We say that α is possibly optimal in A, written $\alpha \in \mathrm{PO}_\Theta(A)$, if there exists $\succcurlyeq$ in Θ such that $\alpha \in \mathrm{O}_{\succcurlyeq}(A)$. Thus, $\mathrm{PO}_\Theta(A) = \bigcup_{\succcurlyeq \in \Theta} \mathrm{O}_{\succcurlyeq}(A)$.

Define the pre-order $\succcurlyeq_\Theta$ to be the intersection of all relations in Θ, with $\alpha \succcurlyeq_\Theta \beta$ if and only if $\alpha \succcurlyeq \beta$ for all relations $\succcurlyeq$ in Θ. Let $\succ_\Theta$ be the strict part of $\succcurlyeq_\Theta$, so that $\alpha \succ_\Theta \beta$ if and only if $\alpha \succcurlyeq_\Theta \beta$ and it is not the case that $\beta \succcurlyeq_\Theta \alpha$. We define $\equiv_\Theta$ to be the corresponding equivalence relation on Ω, given by $\alpha \equiv_\Theta \beta$ iff $\alpha \succcurlyeq_\Theta \beta$ and $\beta \succcurlyeq_\Theta \alpha$.

We define $\mathrm{UD}_\Theta(A)$ to be the set of elements of A that are undominated (with respect to elements in A). Thus $\alpha \in \mathrm{UD}_\Theta(A)$ if and only if $\alpha \in A$ and there does not exist $\beta \in A$ with $\beta \succ_\Theta \alpha$. We also define $\mathrm{POUD}_\Theta(A)$ to be $\mathrm{PO}_\Theta(A) \cap \mathrm{UD}_\Theta(A)$, the elements that are both possibly optimal and undominated.

We define α to be necessarily optimal in A, written $\alpha \in \mathrm{NO}_\Theta(A)$, if $\alpha \in A$ and for all $\beta \in A$, $\alpha \succcurlyeq_\Theta \beta$.

Given Θ and $\succcurlyeq$ in Θ, we say that α is strictly optimal in A w.r.t. $\succcurlyeq$ if $\alpha \in A$ and for all $\beta \in A$, we have $\beta \succcurlyeq \alpha$ only if $\beta \equiv_\Theta \alpha$. We say that α is possibly strictly optimal in A (given Θ), written $\alpha \in \mathrm{PSO}_\Theta(A)$, if there exists some $\succcurlyeq$ in Θ such that α is strictly optimal in A w.r.t. $\succcurlyeq$.

For $\alpha \in A$, let $\mathrm{Opt}^A_\Theta(\alpha)$ be the orderings $\succcurlyeq$ in Θ that make α optimal, i.e., are such that $\alpha \in \mathrm{O}_{\succcurlyeq}(A)$. We say that $\alpha \in \mathrm{MPO}_\Theta(A)$ (α is maximally possibly optimal) if there does not exist $\beta \in A$ such that $\mathrm{Opt}^A_\Theta(\beta)$ is a strict superset of $\mathrm{Opt}^A_\Theta(\alpha)$, i.e., there is no β that is optimal with respect to more orderings than α.

The set $\mathrm{UD}_\Theta(A)$, a natural generalisation of the Pareto-optimal elements, appears in many contexts, e.g., [15, 12]. Possibly optimal (also known as *potentially optimal*) elements have been considered in many publications (and sometimes also, the necessarily optimal elements) such as [11, 3, 9, 10, 22, 5, 4, 6]; The Possibly Strictly Optimal function $\mathrm{PSO}_\Theta(A)$ and the $\mathrm{MPO}_\Theta(A)$ have been considered much less, such as in [21, 17, 20, 19].

Properties of instance classes

For any Θ, choice functions PO_Θ, UD_Θ and POUD_Θ satisfy (NE), (SMA) and (IIA) (see e.g., Proposition 3 of [22]) and thus also [strong] idempotence.

The choice function PSO_Θ satisfies (SMA) (see Lemma 2), and thus, (IIAb) and (SIdem) and (Idem). It does not generally satisfy (NE) nor (IIAa) (see the examples below).

The choice function MPO_Θ satisfies (NE), (Idem) and (IIA). However, it does not generally satisfy strong idempotence and thus not (SMA) either.

The choice function NO_Θ satisfies (SMA) (see Lemma 3), and thus, (IIAb) and (SIdem) and (Idem). (NE) doesn't generally (or even usually) hold. Property (IIAa) is equivalent to the conjunction of two properties:

(IIAa)(i): If $\emptyset \neq \mathrm{Op}(A) \subseteq B \subseteq A$ then $\mathrm{Op}(A) \supseteq \mathrm{Op}(B)$.
(IIAa)(ii): If $\mathrm{Op}(A) = \emptyset$ and $B \subseteq A$ then $\mathrm{Op}(B) = \emptyset$.

Choice function NO_Θ satisfies (IIAa)(i) but not generally (IIAa)(ii).

Lemma 2 *Let* Ω *be a finite set, and let* Θ *be an arbitrary non-empty set of total pre-orders* Θ *on* Ω*. The choice function* PSO_Θ *over* Ω *satisfies property (SMA).*

Lemma 3 *Let* Ω *be a finite set, and let* Θ *be an arbitrary non-empty set of total pre-orders* Θ *on* Ω*. The choice function* NO_Θ *over* Ω *satisfies properties (SMA) and (IIAa)(i).*

Example

We first give examples that show that NO_Θ does not necessarily satisfy (NE) or (IIAa)(ii), (and so neither (IIAa) nor (IIA)). The same example can be used to show that MPO_Θ does not necessarily satisfy strong idempotence ((SIdem)), and thus not property (SMA) either.

Let $\Omega = \{a, b, c\}$. Define total pre-order $\succcurlyeq_1$ as $(\{a, b\}, c)$, meaning that a and b are equivalent, and both are preferred to c. Thus, $\succcurlyeq_1$ consists of the pairs $\{(a, b), (b, a), (a, c), (b, c)\}$. Define $\succcurlyeq_2$ to be the total order (c, a, b). Let $\Theta_{12} = \{\succcurlyeq_1, \succcurlyeq_2\}$. Then, $\mathrm{NO}_{\Theta_{12}}(\{a, b, c\}) = \emptyset$ and $\mathrm{NO}_{\Theta_{12}}(\{a, b\}) = \mathrm{NO}_{\Theta_{12}}(\{a\}) = \{a\}$, which demonstrates that $\mathrm{NO}_{\Theta_{12}}$ does not satisfy (NE) or (IIAa)(ii), (and so neither (IIAa) nor (IIA)).

In addition, $\mathrm{MPO}_{\Theta_{12}}(\{a, b, c\}) = \{a, b, c\}$, but $\mathrm{MPO}_{\Theta_{12}}(\{a, b\}) = \{a\}$, which shows that $\mathrm{MPO}_{\Theta_{12}}$ does not satisfy strong idempotence, property (SIdem).

We now show that PSO_Θ does not necessarily satisfy (NE) or (IIAa)(i) or (IIAa)(ii), (and so neither (IIAa) nor (IIA)). $\mathrm{PSO}_{\Theta_{12}}(\{a, b, c\}) = \{c\}$ and $\mathrm{PSO}_{\Theta_{12}}(\{a, c\}) = \{a, c\}$, showing that $\mathrm{PSO}_{\Theta_{12}}$ does not satisfy (IIAa)(i). Define $\succcurlyeq_3$ to be the total pre-order $(\{a, c\}, b)$, and let $\Theta_{13} = \{\succcurlyeq_1, \succcurlyeq_3\}$. Then, $\mathrm{PSO}_{\Theta_{13}}(\{a, b, c\}) = \emptyset$ and $\mathrm{PSO}_{\Theta_{13}}(\{a, b\}) = \mathrm{PSO}_{\Theta_{13}}(\{a\}) = \{a\}$, and so $\mathrm{PSO}_{\Theta_{13}}$ does not satisfy (NE) or (IIAa)(ii), (and so neither (IIAa) nor (IIA)).

3.2 Class of CFs based on an arbitrary binary relation on Ω

Given relation R on Ω, for any $A \subseteq \Omega$ we can consider the restriction of R to A (i.e., all pairs $(\alpha, \beta) \in R$ with $\alpha, \beta \in A$) and its reflexive and transitive closure R_A, and let R'_A be strict part of R_A, and let $\equiv^R_A$ be the symmetric part. We can then consider the elements of A that are undominated with respect to R'_A, i.e., the elements α of A

that are such that there does not exist $\beta \in A$ with $\beta R'_A \alpha$. Call this $\mathrm{UD}_R(A)$. choice function UD_R satisfies (NE), because $\mathrm{UD}_R(A)$ is always non-empty. Clearly, if $\alpha, \beta \in A$ and $\alpha \equiv_A^R \beta$ then $\alpha \in \mathrm{UD}_R(A) \iff \beta \in \mathrm{UD}_R(A)$.

We can consider cycles in A w.r.t. R; specifically, let an R-cycle in A be a sequence $\alpha_1, \ldots, \alpha_k$ ($k \geq 2$) of elements of A such that $\alpha_k = \alpha_1$ and for each $i = 1, \ldots, k-1$, $(\alpha_i, \alpha_{i+1}) \in R$. For each elements α_i and α_j in an R-cycle in R we have $\alpha_i \equiv_A^R \alpha_j$. Thus, if any α_i is in $\mathrm{UD}_R(A)$ then every element in the R-cycle is in $\mathrm{UD}_R(A)$.

Note that if $\alpha \in \mathrm{UD}_R(A)$ and $(\beta, \alpha) \in R_A$ (or, in particular, if $(\beta, \alpha) \in R$ and $\beta \in A$) then $\alpha \equiv_A^R \beta$ so $\beta \in \mathrm{UD}_R(A)$. Also, there exists an R-cycle in A containing α and β.

Lemma 4 *For any R the choice function UD_R satisfies property (IIAb), and thus also idempotence.*

The two examples below show that UD_R does not necessarily satisfy property (IIAa) (and thus not (IIA) either), and does not necessarily satisfy property (SIdem) and thus not (SMA) either.

Let R_1 be the set of pairs $\{(\alpha, \beta), (\beta, \gamma)\}$. Then $\mathrm{UD}_{R_1}(\{\alpha, \beta, \gamma\}) = \{\alpha\}$, and $\mathrm{UD}_{R_1}(\{\alpha, \gamma\}) = \{\alpha, \gamma\}$. This shows that UD_{R_1} does not satisfy property (IIAa) and thus not (IIA) either.

Let R_2 be the set of pairs $\{(\alpha, \beta), (\beta, \gamma), (\gamma, \alpha)\}$. Then $\mathrm{UD}_{R_2}(\{\alpha, \beta, \gamma\}) = \{\alpha, \beta, \gamma\}$, and $\mathrm{UD}_{R_2}(\{\alpha, \beta\}) = \{\alpha\}$. This shows that UD_{R_2} does not satisfy property (SIdem) and thus not (SMA) either.

In summary, choice function UD_R satisfies (NE), (IIAb) and thus idempotence. When R is not transitive, it does not necessarily satisfy property (IIAa) (and thus (IIA) neither); and it need not satisfy property (SIdem) (and thus not (SMA) either). On the other hand, if R is transitive then R equals $\succcurlyeq_\Theta$, where Θ is the set of total pre-orders extending R (see Section 3.1), and UD_R equals UD_Θ, and thus the choice function UD_R satisfies (NE), (SMA) and (IIA).

4 BASIC COMBINATIONS

The most straight-forward ways of combining choice functions is in terms of taking the intersections of the sets, taking the union of the sets, and applying one choice function and then the other. In this section we explore the basic properties and connections between these simple combination operations.

Union and intersection of choice functions: We can define the union and intersection of functions Op in the obvious way ('pointwise'). For instance, we define $\mathrm{Op}_1 \cup \mathrm{Op}_2$ by $(\mathrm{Op}_1 \cup \mathrm{Op}_2)(A) = \mathrm{Op}_1(A) \cup \mathrm{Op}_2(A)$ for each $A \subseteq \Omega$.

We can also extend the subset relation to such functions, by applying it for each subset of Ω. Thus, $\mathrm{Op}_1 \subseteq \mathrm{Op}_2$ means for all $A \subseteq \Omega$, $\mathrm{Op}_1(A) \subseteq \mathrm{Op}_2(A)$. We say that Op_1 then *strengthens* Op_2, because Op_1 gives a stronger (or at least as strong) result than Op_2, i.e., for each set A, Op_1 finds as least as many elements as Op_2 to be suboptimal.

We have $\mathrm{Op}_1 \subseteq \mathrm{Op}_2 \iff \mathrm{Op}_1 \cup \mathrm{Op}_2 = \mathrm{Op}_2 \iff \mathrm{Op}_1 \cap \mathrm{Op}_2 = \mathrm{Op}_1$.

Composition of choice functions: For choice functions Op_1 and Op_2 we define the composition choice function $\mathrm{Op}_1 \circ \mathrm{Op}_2$ (meaning Op_1 followed by Op_2) by $(\mathrm{Op}_1 \circ \mathrm{Op}_2)(A) = \mathrm{Op}_2(\mathrm{Op}_1(A))$ for each $A \subseteq \Omega$.

In the composition $\mathrm{Op}_1 \circ \mathrm{Op}_2$, choice function Op_1 is given priority over Op_2, since the first step is to eliminate alternatives with Op_1. Work on combining preference information using priority includes, for instance, a general framework for combining preference relations using priority [2], voting rules based on sequential elimination of alternatives [7]; and a computational technique for preference inference based on composition of lexicographic orders [20].

4.1 Relationships between intersection and composition

The intersection and the two compositions are often very different from each other. However, they do have the same set of fixed points, which is equal to the intersection of the fixed points of the CFs. (This doesn't require any additional assumption on the CFs.)

Proposition 2 *Consider choice functions Op_1 and Op_2 over Ω. Then Fix$(\mathrm{Op}_1 \circ \mathrm{Op}_2) =$ Fix$(\mathrm{Op}_2 \circ \mathrm{Op}_1) =$ Fix$(\mathrm{Op}_1 \cap \mathrm{Op}_2) =$ Fix$(\mathrm{Op}_1) \cap$ Fix(Op_2).*

A clear disadvantage of the composition as a combination operation is that it is not symmetric between the two CFs (i.e., composition is not commutative); $\mathrm{Op}_1 \circ \mathrm{Op}_2$ is giving priority to Op_1, and thus typically gives effectively more importance to Op_1 than to Op_2. Because of that, it is natural to consider the intersection of the two compositions, as this restores symmetry between Op_1 and Op_2. The result below shows that this doesn't lead to a new combination operation, at least when each choice function satisfies property (SMA): the intersection of the two compositions is just equal to the intersection of the CFs.

Proposition 3 *When choice functions Op_1 and Op_2 over Ω satisfy property (SMA) then $\mathrm{Op}_1 \cap \mathrm{Op}_2 = (\mathrm{Op}_1 \circ \mathrm{Op}_2) \cap (\mathrm{Op}_2 \circ \mathrm{Op}_1)$, i.e., for all $A \subseteq \Omega$,*

$$\mathrm{Op}_1(A) \cap \mathrm{Op}_2(A) = \mathrm{Op}_2(\mathrm{Op}_1(A)) \cap \mathrm{Op}_1(\mathrm{Op}_2(A)).$$

For CFs satisfying property (SMA), Proposition 3 implies that Op_1 and Op_2 commute if and only if both compositions are equal to the intersection. We now explore some situations when the two choice functions commute, i.e., when the two compositions are equal. Proposition 4 shows that this happens when the intersection satisfies property (IIA). Proposition 5 gives another sufficient condition for the CFs to commute.

Proposition 4 *Let Op_1 and Op_2 be choice functions over Ω, and suppose that their intersection satisfies property (IIA) and that Op_1 is idempotent. Then $\mathrm{Op}_1 \circ \mathrm{Op}_2 = \mathrm{Op}_1 \cap \mathrm{Op}_2$. Thus, if Op_2 is also idempotent then Op_1 and Op_2 commute: $\mathrm{Op}_1 \circ \mathrm{Op}_2 = \mathrm{Op}_2 \circ \mathrm{Op}_1$.*

An example of this is when, for some Θ, $\mathrm{Op}_1 = \mathrm{PO}_\Theta$ and $\mathrm{Op}_2 = \mathrm{UD}_\Theta$ (see Section 3.1). The intersection POUD_Θ satisfies property (IIA) which implies, by this proposition, that choice functions PO_Θ and UD_Θ commute.

Proof: Let $\mathrm{Op} = \mathrm{Op}_1 \cap \mathrm{Op}_2$. Consider any $A \subseteq \Omega$. We have $\mathrm{Op}(A) \subseteq \mathrm{Op}_1(A) \subseteq A$, using Condition (Sub). Applying property (IIA) for Op gives $\mathrm{Op}(\mathrm{Op}_1(A)) = \mathrm{Op}(A)$, i.e., $\mathrm{Op}_1(\mathrm{Op}_1(A)) \cap \mathrm{Op}_2(\mathrm{Op}_1(A)) = \mathrm{Op}(A)$. Using idempotence of Op_1 and Condition (Sub) for Op_2 gives $\mathrm{Op}_2(\mathrm{Op}_1(A)) = \mathrm{Op}(A)$. By the same argument, if Op_2 is also idempotent then $\mathrm{Op}_2 \circ \mathrm{Op}_1 = \mathrm{Op}_1 \cap \mathrm{Op}_2$ and so $\mathrm{Op}_1 \circ \mathrm{Op}_2 = \mathrm{Op}_2 \circ \mathrm{Op}_1$. $\square$

Proposition 5 *Assume that choice functions* Op_1 *and* Op_2 *over* Ω *satisfy properties (SMA) and (IIA), and that for all* $A \subseteq \Omega$*, either* $\mathrm{Op}_1(A) \subseteq \mathrm{Op}_2(A)$ *or* $\mathrm{Op}_1(A) \supseteq \mathrm{Op}_2(A)$*. Then* $\mathrm{Op}_1 \circ \mathrm{Op}_2 = \mathrm{Op}_2 \circ \mathrm{Op}_1 = \mathrm{Op}_1 \cap \mathrm{Op}_2$.

4.2 Further iteration

One might wonder if anything new is obtained with iterated application of two choice functions, and application of intersection. Thus turns out not to the case (assuming the CFs satisfy property (SMA)). Strong idempotence implies that $\mathrm{Op}_1(\mathrm{Op}_1(A) \cap \mathrm{Op}_2(A))$ equals $\mathrm{Op}_1(A) \cap \mathrm{Op}_2(A)$, and $\mathrm{Op}_1(\mathrm{Op}_2(\mathrm{Op}_1(A))) = \mathrm{Op}_2(\mathrm{Op}_1(A))$. Also, idempotence implies $\mathrm{Op}_2(\mathrm{Op}_2(\mathrm{Op}_1(A))) = \mathrm{Op}_2(\mathrm{Op}_1(A))$.

Let Op_1 and Op_2 be CFs over Ω, both satisfying strong idempotence (SIdem). Consider a sequence $\mathrm{Op} = \mathrm{Op}_{m_1} \circ \cdots \circ \mathrm{Op}_{m_k}$, (with $k \geq 1$) where each $m_i \in \{1, 2\}$. If each $m_i = 1$ then $\mathrm{Op} = \mathrm{Op}_1$; if each $m_i = 2$ then $\mathrm{Op} = \mathrm{Op}_2$; else, if $m_1 = 1$ then $\mathrm{Op} = \mathrm{Op}_1 \circ \mathrm{Op}_2$; and otherwise, if $m_1 = 2$, then $\mathrm{Op} = \mathrm{Op}_2 \circ \mathrm{Op}_1$.

4.3 Some further desirable properties of a combination

As well as the properties on choice functions, there are further natural properties that one would hope for with a combination operation for CFs. Suppose that we have Op as some kind of combination of Op_1 and Op_2, i.e., $\mathrm{Op} = \mathrm{Op}_1 \otimes \mathrm{Op}_2$, for some combination operation $\otimes$.

- commutativity of combination, i.e., $\mathrm{Op}_1 \otimes \mathrm{Op}_2 = \mathrm{Op}_2 \otimes \mathrm{Op}_1$.

- $\mathrm{Op}(A) \subseteq \mathrm{Op}_1(A) \cup \mathrm{Op}_2(A)$.

This is equivalent to $\overline{\mathrm{Op}}(A) \supseteq \overline{\mathrm{Op}}_1(A) \cap \overline{\mathrm{Op}}_2(A)$, i.e., if Op_1 and Op_2 both say that α is sub-optimal in A, then α should be sub-optimal in A with Op.

Conversely, we can posit the following property: if Op_1 and Op_2 both say that α is optimal in A then α should be optimal in A with Op:

- $\mathrm{Op}(A) \supseteq \mathrm{Op}_1(A) \cap \mathrm{Op}_2(A)$.

The conjunction of the two previous properties is:

- $\mathrm{Op}_1(A) \cap \mathrm{Op}_2(A) \subseteq \mathrm{Op}(A) \subseteq \mathrm{Op}_1(A) \cup \mathrm{Op}_2(A)$.

Below are two consequences of this pair of properties.

- If A is a fixed point of Op_1 then $\mathrm{Op}(A) = \mathrm{Op}_2(A)$.
- If $\mathrm{Op}_1(A) = \mathrm{Op}_2(A)$ then $\mathrm{Op}(A) = \mathrm{Op}_2(A)$.

5 COMBINATIONS MAINTAINING PROPERTIES

We will consider a number of different combination operations for CFs: intersection, union, two compositions and the union of compositions. (We do not further consider the intersection of the two compositions, since, it is equal to the intersection when the input choice functions satisfy property (SMA), by Proposition 3.)

We will analyse which of the properties described in Section 2.2 are maintained by these different combinations. Formally, a property P on CFs (over a set of alternatives Ω) is *maintained by a combination operation* $\otimes$ (on CFs) if $\mathrm{Op}_1 \otimes \mathrm{Op}_2$ satisfies property P whenever both Op_1 and Op_2 satisfy P.

Lemma 5 shows that the union combination maintains properties (SMA), (IIA), and the property of being a subset of a given CF. Union maintains also the non-empty property (NE). Perhaps surprisingly, union does not maintain strong idempotence, as shown with an example at the end of this section.

Lemma 5 *Consider choice functions* Op_1, Op_2 *and* Op'*. Then the following hold:*

1. *If* $\mathrm{Op}_1 \subseteq \mathrm{Op}'$ *and* $\mathrm{Op}_2 \subseteq \mathrm{Op}'$ *then* $\mathrm{Op}_1 \cup \mathrm{Op}_2 \subseteq \mathrm{Op}'$.
2. *If* Op_1 *and* Op_2 *both satisfy property (SMA) then* $\mathrm{Op}_1 \cup \mathrm{Op}_2$ *satisfies property (SMA).*
3. *If* Op_1 *and* Op_2 *both satisfy property (IIA) then* $\mathrm{Op}_1 \cup \mathrm{Op}_2$ *satisfies property (IIA).*
4. *If* Op_1 *and* Op_2 *both satisfy property (IIAa) then* $\mathrm{Op}_1 \cup \mathrm{Op}_2$ *satisfies property (IIAa).*
5. *If* Op_1 *and* Op_2 *both satisfy property (NE) then* $\mathrm{Op}_1 \cup \mathrm{Op}_2$ *satisfies property (NE).*

Intersection maintains property (SMA), and strong idempotence, but not (NE) or (IIA) (see the examples below). In fact, Proposition 4 suggests that the intersection will not usually satisfy property (IIA), since, it implies (under the very weak assumption of idempotence) that if the intersection satisfies (IIA) then the CFs commute, which presumably only rather rarely occurs.

Lemma 6 *Intersection of choice functions maintains property (SMA), and maintains strong idempotence.*

Proof: Let Op_1 and Op_2 be CFs over Ω, with $B, A \subseteq \Omega$. First assume that $B \subseteq A$ and property (SMA) holds for Op_1 and Op_2. Then, for $i = 1, 2$, $\mathrm{Op}_i(A) \cap B \subseteq \mathrm{Op}_i(B)$. Thus, $\mathrm{Op}_1(A) \cap \mathrm{Op}_2(A) \cap B \subseteq \mathrm{Op}_1(B) \cap \mathrm{Op}_2(B)$, showing that (SMA) holds also for the intersection.

Now assume that strong idempotence holds for Op_1 and Op_2, and that $A \subseteq \mathrm{Op}_1(B) \cap \mathrm{Op}_2(B)$. Then, $A \subseteq \mathrm{Op}_1(B)$ and $A \subseteq \mathrm{Op}_2(B)$, so, by strong idempotence of Op_1 and Op_2, $\mathrm{Op}_1(A) = A$ and $\mathrm{Op}_2(A) = A$. Thus, $\mathrm{Op}_1(A) \cap \mathrm{Op}_2(A) = A$, showing strong idempotence for the intersection. □

It is clear that composition maintains (NE). The following result shows that strong idempotence is maintained by composition. Examples below show that composition does not maintain properties (SMA) or (IIA).

Lemma 7 *Suppose that both choice functions* Op_1 *and* Op_2 *satisfy strong idempotence (SIdem). Then the composition* $\mathrm{Op}_1 \circ \mathrm{Op}_2$ *also satisfies (SIdem). In particular, the composition is idempotent.*

Examples showing properties not being maintained

In the examples below, we use only total orderings on $\Omega = \{a, b, c\}$, and abbreviate a total ordering such as (b, a, c) to just bac. For instance, we can consider a set of orderings $\Theta = \{abc, cab\}$, and the associated CF based on possibly optimal alternatives, $\mathrm{PO}_{\{abc,cab\}}$ (see Section 3.1). Recall that for any Θ, the choice function PO_Θ satisfies the properties in Section 2.2, in particular, properties (NE), (SMA) and (IIA). Now, a is the top element in the total order abc, and c is the top element in cab, so the set of possibly optimal elements in $\Omega = \{a, b, c\}$, $\mathrm{PO}_{\{abc,cab\}}(\{a, b, c\})$, is $\{a, c\}$. If we are interested in $\{a, b\}$ then we consider the restrictions of the two orderings to this set, giving the pair of orderings $\{ab, ab\}$, so a is better than b in both orderings, and hence, $\mathrm{PO}_{\{abc,cab\}}(\{a,b\})$, is $\{a\}$.

Let Op_1 be $\mathrm{PO}_{\{abc,bca\}}$, and let Op_2 be $\mathrm{PO}_{\{abc,cab\}}$. We write $\mathrm{Op}_{1\cap 2}$ for $\mathrm{Op}_1 \cap \mathrm{Op}_2$, and $\mathrm{Op}_{1\circ 2}$ for $\mathrm{Op}_1 \circ \mathrm{Op}_2$, and $\mathrm{Op}_{2\circ 1}$ for $\mathrm{Op}_2 \circ \mathrm{Op}_1$. It can be shown that $\mathrm{Op}_{1\circ 2} \subseteq \mathrm{Op}_{2\circ 1}$ and so, because of Proposition 3, $\mathrm{Op}_{1\cap 2} = \mathrm{Op}_{1\circ 2}$. It can be seen that $\mathrm{Op}_{1\circ 2}(\{a,b,c\}) = \{a\}$, and $\mathrm{Op}_{1\circ 2}(\{a,c\}) = \{a,c\}$, which implies that neither the composition $\mathrm{Op}_{1\circ 2}$, nor the intersection $\mathrm{Op}_{1\cap 2}$ satisfy property (IIA) (specifically, (IIAa)). $\mathrm{Op}_{2\circ 1}(\{b,c\}) = \{b\}$, and, $\mathrm{Op}_{2\circ 1}(\{a,b,c\}) = \{a,c\}$, and thus, $\mathrm{Op}_{2\circ 1}(\{a,b,c\}) \cap \{b,c\} = \{c\}$, which implies that $\mathrm{Op}_{2\circ 1}$ does not satisfy property (SMA).

Now let $\mathrm{Op}_3 = \mathrm{PO}_{\{abc,acb\}}$ and let $\mathrm{Op}_4 = \mathrm{PO}_{\{bac,cab\}}$. $\mathrm{Op}_{3\cap 4}(\{a,b,c\}) = \{a\} \cap \{b,c\} = \emptyset$, so $\mathrm{Op}_{3\cap 4}$ does not satisfy (NE). Because $\mathrm{Op}_{3\cap 4}(\{a,b\}) = \{a\} \cap \{a,b\} \neq \emptyset$, so $\mathrm{Op}_{3\cap 4}$ does not satisfy property (IIA) (more specifically, it doesn't satisfy property (IIAa)(ii)).

Lemma 7 implies that $\mathrm{Op}_{3\circ 4}$ and $\mathrm{Op}_{4\circ 3}$ both satisfy strong idempotence. However, their union, $\mathrm{Op}_{34}^{\bigcup\circ} = \mathrm{Op}_{3\circ 4} \cup \mathrm{Op}_{4\circ 3}$ does not satisfy strong idempotence, because $\mathrm{Op}_{34}^{\bigcup\circ}(\{a,b,c\}) = \{a\} \cup \{b,c\} = \{a,b,c\}$, but $\mathrm{Op}_{34}^{\bigcup\circ}(\{a,b\}) = \{a\} \neq \{a,b\}$. Thus, the union combination does not maintain strong idempotence. Also, the union of compositions does not maintain properties (SMA) and strong idempotence.

We sum up these results as follows.

Theorem 1 • *Union maintains properties (NE), (SMA) and (IIA).*
- *Intersection maintains (SMA) and strong idempotence but not (NE) or (IIA).*
- *Composition maintains (NE) and strong idempotence, but not (SMA) or (IIA).*
- *Union of composition maintains (NE), but not strong idempotence, or (SMA).*

6 ENFORCING PROPERTIES AND RESTORING CONSISTENCY (NE)

Properties (SMA) and (IIA) are very desirable properties of a choice function, as is (NE). However, applying intersection or composition operations can mean that some of these properties can be lost, as shown in the last section. In addition, as shown in Section 3, some natural ways of generating CFs can lack one of more of these properties. It is therefore natural to consider ways of changing a CF to make it have such a property. For properties (SMA) and (IIA), it seems natural to strengthen the CF to enforce the property, as discussed in Section 6.1. This is not possible for property (NE), and instead we need a way of replacing an empty value of Op by a non-empty one: see Section 6.2. In Section 6.3 we consider how to combine enforcing (SMA) and (IIA) with restoring (NE), so that all three properties are satisfied.

6.1 Enforcing properties through maximal strengthenings

Suppose that a choice function Op does not satisfy a desirable property P. One can attempt to enforce this property by changing Op to Op' that does satisfy P. Our focus here is on strengthening Op to make it satisfy P; in this way, if α is viewed as sub-optimal in a set A w.r.t. Op (i.e., $\alpha \notin \mathrm{Op}(A)$), then it will be suboptimal w.r.t. Op' ($\alpha \notin \mathrm{Op}'(A)$).

Suppose that property P is such that (a) union maintains a property P and (b) the null choice function EMP satisfies P. Consider any arbitrary choice function Op. Let $\mathrm{Op}^{(P)}$ be the union of all choice functions Op' such that (i) Op' satisfies P and (ii) $\mathrm{Op}' \subseteq \mathrm{Op}$. (Since EMP satisfies P and EMP $\subseteq$ Op, there exists at least one such choice function Op'.)

Clearly, we have $\mathrm{Op}^{(P)} \subseteq \mathrm{Op}$. Also, since union maintains property P, choice function $\mathrm{Op}^{(P)}$ satisfies property P. Moreover, the definition implies that if any choice function Op' is such that (i) and (ii) hold, then $\mathrm{Op}' \subseteq \mathrm{Op}^{(P)}$. Thus, $\mathrm{Op}^{(P)}$ is the maximal strengthening of Op that satisfies property P, where *maximal* means *maximal with respect to the relation $\subseteq$ between CFs*. In other words, $\mathrm{Op}^{(P)}$ is the weakest strengthening of Op that satisfies property P. We say that $\mathrm{Op}^{(P)}$ is the *maximal P-strengthening of* Op. Note that if Op satisfies property P then $\mathrm{Op}^{(P)} = \mathrm{Op}$. In particular, enforcing P for $\mathrm{Op}^{(P)}$ leaves it unchanged; thus enforcing property P is an idempotent operation.

This notion of maximal P-strengthening can be applied to property (SMA), to property (IIA), and to the conjunction of properties (SMA) and (IIA). By Lemma 5, union maintains property (SMA) and property (IIA), and hence union maintains also the conjunction of properties (SMA) and property (IIA). EMP satisfies property (SMA) and property (IIA) (and hence also the conjunction of the two properties). Thus, we can consider, for choice function Op, its maximal strengthening $\mathrm{Op}^{(\mathrm{SMA})}$ satisfying (SMA), and $\mathrm{Op}^{(\mathrm{IIA})}$, its maximal strengthening satisfying (IIA), and $\mathrm{Op}^{(\mathrm{SMA\text{-}IIA})}$, its maximal strengthening satisfying both (SMA) and (IIA).

The definitions imply that $(\mathrm{Op}^{(\mathrm{SMA})})^{(\mathrm{SMA\text{-}IIA})} = \mathrm{Op}^{(\mathrm{SMA\text{-}IIA})}$. It can be shown that if Op satisfies (SMA) then enforcing (IIA) preserves (SMA), and also preserves (NE) if that holds as well. Thus, in fact, as stated by the proposition below, $(\mathrm{Op}^{(\mathrm{SMA})})^{(\mathrm{IIA})} = (\mathrm{Op}^{(\mathrm{SMA})})^{(\mathrm{SMA\text{-}IIA})} = \mathrm{Op}^{(\mathrm{SMA\text{-}IIA})}$, so to enforce both properties (SMA) and (IIA), we can first enforce (SMA) and then enforce (IIA).

Proposition 6 *For any choice function* Op,

$$(\mathrm{Op}^{(SMA)})^{(IIA)} = (\mathrm{Op}^{(SMA)})^{(SMA\text{-}IIA)} = \mathrm{Op}^{(SMA\text{-}IIA)}.$$

If Op satisfies property (SMA) then one can give a characterisation of $\mathrm{Op}^{(\mathrm{SMA\text{-}IIA})}$: briefly, $\mathrm{Op}^{(\mathrm{SMA\text{-}IIA})}(A)$ is equal to $\mathrm{Op}(C)$, where C is the (unique) largest set such that $\mathrm{Op}(C) \subseteq A \subseteq C$.

Enforcing property (SMA)

It is possible to give a simple explicit formula for $\mathrm{Op}^{(\mathrm{SMA})}$, the result of enforcing property (SMA) on choice function Op. To understand $\mathrm{Op}^{(\mathrm{SMA})}$, it is helpful to consider property (SMA) in terms of the associated sub-optimality function $\overline{\mathrm{Op}}$: if $B \subseteq A$ then $\overline{\mathrm{Op}}(A) \supseteq \overline{\mathrm{Op}}(B)$. Based on this, one can see that $\overline{\mathrm{Op}}^{(\mathrm{SMA})}$, the associated sub-optimality function for $\mathrm{Op}^{(\mathrm{SMA})}$, is given by $\overline{\mathrm{Op}}^{(\mathrm{SMA})}(A) = \bigcup_{B \subseteq A} \overline{\mathrm{Op}}(B)$, which leads to $\mathrm{Op}^{(\mathrm{SMA})}(A) = \bigcap_{B \subseteq A}(\mathrm{Opt}(B) \cup (A \setminus B))$.

6.2 Restoring consistency (non-empty property (NE))

Suppose that we have a choice function Op that does not satisfy property (NE), so that for some $A \subseteq \Omega$, $\mathrm{Op}(A) = \emptyset$. We cannot restore (NE) in the same way that we enforced the properties in Section 6.1, since if Op fails to satisfy (NE) and $\mathrm{Op}' \subseteq \mathrm{Op}$ then Op' fails to satisfy (NE) (since $\mathrm{Op}(A) = \emptyset$ implies $\mathrm{Op}'(A) = \emptyset$). Instead we need to weaken Op rather than strengthen it.

The basic idea of our approach here is to replace empty values by a default value; that is, if $\mathrm{Op}(A) = \emptyset$, we reset $\mathrm{Op}(A)$ to some default value for $\mathrm{Op}(A)$. Often this default value will just be A, i.e., we don't assume that any element of A is sub-optimal. In the case of the combination Op of two choice functions Op_1 and Op_2, we will make the default value equal to $\mathrm{Op}_1(A) \cup \mathrm{Op}_2(A)$, in order the satisfy a property in Section 4.3.

Consider a situation in which Op satisfies properties (SMA) and (IIA). Then Proposition 1 shows that there exists a unique largest set Z_{Op} with $\mathrm{Op}(Z_{\mathrm{Op}}) = \emptyset$, and also shows that $\mathrm{Op}(A) = \emptyset$ for all $A \subseteq Z_{\mathrm{Op}}$. The following result implies that we can restore consistency whilst preserving properties (SMA) and (IIA). The idea is to replace the empty values (relating to all subsets of Z_{Op}) by the values of a choice function Op' over Z_{Op}, where Op' satisfies (SMA), (IIA) and (NE). We call this *grafting* Op' onto Op, and Op' is called the *graft*.

Proposition 7 *Assume that choice function* Op *over* Ω *satisfies properties (SMA) and (IIA). Let* Z_{Op} *be the union of all subsets* Z *of* Ω *such that* $\mathrm{Op}(Z) = \emptyset$*, and let* Op' *be a choice function over* Z_{Op} *that satisfies properties (SMA), (IIA) and (NE).*

Define Op'' *by* $\mathrm{Op}''(B) = \mathrm{Op}'(B)$ *if* $B \subseteq Z_{\mathrm{Op}}$*, and otherwise, define* $\mathrm{Op}''(B) = \mathrm{Op}(B)$*. Then* Op'' *satisfies properties (SMA), (IIA) and (NE).*

Thus, if Op satisfies (SMA) and (IIA), and if the default values (Op') form a CF over Z_{Op} that satisfies (SMA), (IIA) and (NE), then grafting Op' onto Op preserves properties (SMA) and (IIA), and restores (NE). In particular, using a default value of A for A, corresponds to the graft Op' being the identity choice function ID on Z_{Op} (where ID satisfies (SMA), (IIA) and (NE)).

6.3 Simultaneously, enforcing (SMA) and (IIA), and restoring consistency

Putting the results of Section 6.1 and 6.2 together gives a general way of enforcing properties (SMA) and (IIA), and restoring consistency (NE) for an arbitrary CF: we first enforce (SMA), then enforce (IIA) (see Section 6.1) and then restore consistency (see Section 6.2). By Propositions 6 and 7, the result will satisfy (SMA), (IIA) and (NE). (Of course, if the initial CF already satisfies (SMA), then there is no need to enforce (SMA), and, as observed earlier, doing so will make not change.)

In the remainder of the section we briefly discuss variations of the general method described above. Note that enforcing (SMA) can mean that property (NE) is lost. An example of this is $\mathrm{Op}_{4\circ 3}$ in the example at the end of Section 5, which satisfies (NE) but we have $(\mathrm{Op}_{4\circ 3})^{(\mathrm{SMA})}(\{a,b,c\}) = \emptyset$ so that (NE) no longer holds after enforcing (SMA). Also, if the CF already has property (SMA), then restoring consistency (NE) can lose property (SMA). For instance, if we restore consistency to $(\mathrm{Op}_{4\circ 3})^{(\mathrm{SMA})}$, by grafting ID, then we lose property (SMA).

If one has a way of enforcing (SMA) at the same time as restoring consistency, then one can do so, and then enforce (IIA) on the result, and again we'll have all three of the properties. More generally, we could add extra operations before enforcing (SMA) and (IIA), and restoring consistency, i.e., pre-processing steps before the general method above. For instance we could first restore consistency and then apply (SMA); or we could apply (SMA) and then restore consistency.

7 COMBINATIONS THROUGH ENFORCING AND RESTORING PROPERTIES

Let us assume that input choice functions Op_1 and Op_2 satisfy properties (NE), (SMA) and (IIA). As shown in Section 5, the union combination maintains these three properties, as well as the properties in Section 4.3. However, union will often seem an excessively conservative combination operation. For approaches based on intersection and composition we will use the methods described in Section 6.

7.1 Intersection

The intersection combination operation maintains property (SMA) but not (IIA) or (NE) (see Theorem 1). One natural and simple approach is to first enforce (IIA), and then restore (NE) by grafting $\mathrm{Op}_1 \cup \mathrm{Op}_2$ (restricted to the part with empty values) onto the result, as shown by Proposition 7. This ensures that the three properties are satisfied, and the resulting CF is also a strengthening of the union $\mathrm{Op}_1 \cup \mathrm{Op}_2$ (see Section 4.3).

However, there are situations in which enforcing (IIA) to an inconsistent choice function Op leads to somewhat drastic results. In particular, if $\mathrm{Op}(\Omega) = \emptyset$ then enforcing (IIA) will lead to the choice function EMP, which returns always the empty set. So, then grafting on the union will lead to the final combination being just the union. For example, with Op_3 and Op_4 from Section 5, $\mathrm{Op}_{3\cap 4}(\{a,b,c\}) = \emptyset$, so $\mathrm{Op}_{3\cap 4}^{(\mathrm{IIA})}$ is just EMP over $\Omega = \{a,b,c\}$, leading to the combination being just union, which in this case is just the vacuous choice function ID over Ω.

An alternative approach for enforcing/restoring the properties is to first restore consistency to the intersection (by grafting on the union), then enforce properties (SMA) and (IIA), and (if necessary) restore consistency again. In the case of $\mathrm{Op}_{3\cap 4}$ this leads to a stronger result, with the combination ending up equal to Op_3.

7.2 Union of composition

Similar considerations and approaches apply to a combination based on the union of the two compositions. One can enforce (SMA) and (IIA), and (if necessary) restore consistency by grafting on the union. Or, one could first enforce (SMA) and restore consistency, before applying this sequence. In addition, one could apply such a sequence of operators to each composition separately, and take the union (which preserves the three properties).

8 DISCUSSION

We have derived methods, for the combination of a pair of choice functions, that maintain the desirable properties (SMA), (IIA) and (NE). There are many potential avenues of future research, for instance: deeper analysis and comparison of these different combination methods; computational issues, i.e., how one computes the value of the combination when applied to a given an input set of alternatives A; extending our approaches to the case of more than two agents. It would also be interesting to consider the effect of enforcing/restoring the desirable properties on the CFs based on certain optimality definitions (see Section 3), such as maximally possibly optimal and possibly strictly optimal.

It could be possible to apply belief merging approaches e.g., [14, 13], for each given subset of alternatives A, and then enforce/restore desirable properties to the resulting CF.

The current paper assumes that the preference inputs are choice functions. If instead, the preference inputs are binary preference relations, then one can let R be the union of these relations, generate combined choice function UD_R (see Section 3.2), and then enforce/restore the desirable properties, using the methods in Section 6.

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