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Optical scalars and singularity avoidance in spherical spacetimes

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Consider a spherically symmetric spacelike slice through a spacetime. One can derive universal bounds on any such slice assuming that the matter sources satisfy an energy condition and that the slice be regular. These bounds are used to derive the horizon formation conditions and to show how a regular spacelike slicing may avoid singularities. The results hold true even when the matter has a distribution on a shell or blows up at the origin so as to give a conical singularity.

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Relativists, especially those who are numerically inclined [1], have long known that regular spacelike slices often wrap around singularities rather than approaching them. In this Rapid Communication we derive a new and remarkable relation for spherical geometries which shows how regular slices may be prevented from coming close to singularities.

In a general spacetime the behavior of beams of light rays is described by specifying a number of functions which describe the expansion and shear of the rays. These are called the optical scalars. In a spherically symmetric spacetime there are two such functions.

Consider a spacelike slice through spacetime. The geometry of this slice cannot be chosen at will; it must satisfy the constraint equations. In the spherically symmetric case these constraints can be written as equations for the optical scalars. These equations, on a regular slice, force the optical scalars to remain bounded over the entire slice. The optical scalars are objects which we expect to become unboundedly large as one approaches a singularity. Thus regular spacelike slices are excluded from regions near singularities. This bound also has a more immediate use. Over the years, we have been interested in developing criteria to determine when and if apparent horizons form [2,3]. In spherically symmetric systems the existence of an apparent horizon implies the existence of a black hole [4,5]. These bounds on the optical scalars allow us sharpen significantly our condition for the formation of apparent horizons. They also can be used to gain insight into the global behavior of self-gravitating matter and/or to prove the existence of a global Cauchy solution.

We define a spherically symmetric spacetime as one having the metric

$$ds^2=-\alpha^2(r,t)dt^2+a(r,t)dr^2+b(r,t)r^2d\Omega^2 \tag{1}$$

where $0\leqslant\phi<2\pi$ and $0\leqslant\theta\leqslant\pi$ are the standard angle variables such that $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$. The initial data for the Einstein equations are prescribed by giving the spatial geometry at $t=0$, i.e., by specifying the functions $a(r,0)$ and $b(r,0)$, and by giving the extrinsic curvature (again at $t=0$)

$$K_r^r=\frac{\partial_t a}{2a\alpha}, \quad K_\theta^\theta=K_\phi^\phi=\frac{\partial_t R}{\alpha R}, \quad \mathrm{tr}K=\frac{\partial_t(\sqrt{a}b)}{\sqrt{a}b\alpha} \tag{2}$$

where the areal (Schwarzschild) radius R is defined as

$$R=r\sqrt{b}. \tag{3}$$

It is useful to define the mean curvature of a centered sphere in the initial hypersurface by

$$p=2\partial_r R/\sqrt{a}R. \tag{4}$$

The two optical scalars can be expressed in terms of the initial data on any spacelike slice. They are the divergence of future directed light rays,

$$\theta=\frac{2}{R}\frac{d}{\alpha dt_{\mathrm{out}}}R=p-K_r^r+\mathrm{tr}K, \tag{5}$$

and the divergence of past directed light rays,

$$\theta'=\frac{-2}{R}\frac{d}{\alpha dt_{\mathrm{in}}}R=p+K_r^r-\mathrm{tr}K, \tag{6}$$

where $d/\alpha dt_{\mathrm{in}}=\partial_t/\alpha-\partial_r/\sqrt{a}$ and $d/\alpha dt_{\mathrm{out}}=\partial_t/\alpha+\partial_r/\sqrt{a}$ are the full derivatives in the direction of radial ingoing and outgoing null rays, respectively. In flat spacetime both quantities are positive and equal to $2/R$; hence, each of the products $R\theta$ and $R\theta'$ equals 2.

The initial data must satisfy the constraints. These constraints, expressed in terms of θ and θ', can be written as

$$\partial_l(\theta R)=-8\pi R\left(\rho-\frac{j_r}{\sqrt{a}}\right)-\frac{1}{4R}[\theta^2R^2-4-4\theta\,\mathrm{tr}KR^2 + \theta R(\theta R-\theta' R)], \tag{7}$$

$$\partial_l(\theta' R)=-8\pi R\left(\rho+\frac{j_r}{\sqrt{a}}\right)-\frac{1}{4R}[\theta'^2R^2-4+4\theta'\mathrm{tr}KR^2 + \theta' R(\theta' R-\theta R)], \tag{8}$$

where l is the proper distance from the center, i.e., $dl=\sqrt{a}dr$. ρ and j_r are the energy density and the current density of the sources that generate the gravitational field. Note that $j_r/\sqrt{a}$ equals $j\cdot n$ where n is the unit normal in the radial direction. We will assume that the sources satisfy the dominant energy condition, $\rho\geq|j|$. If the origin is regular, local flatness forces both optical scalars to satisfy the conditions $\lim_{R\to 0}\theta R=\lim_{R\to 0}\theta' R=2$. Asymptotic flatness also gives $\lim_{R\to\infty}\theta R=\lim_{R\to\infty}\theta' R=2$.

Our primary result is a proof that if θR, $\theta' R$ are bounded at the origin and at infinity they are bounded on the entire hypersurface. Define $B=4\sup_{0\leq r\leq\infty}(|R\,\mathrm{tr}K|)$. We prove the following:

Lemma 1. Given spherical initial data that are regular at the origin and at infinity with sources that satisfy the dominant energy condition, both optical scalars are bounded on the entire hypersurface by

$$-2-B\leq\theta R,\quad \theta' R\leq 2+B. \tag{9}$$

Proof. Let us assume that $\theta R\geq 2+B$ and $\theta R\geq\theta' R$. Consider the nonsource part of Eq. (7), i.e., $[\theta^2R^2-4-4\theta\,\mathrm{tr}KR^2+\theta R(\theta R-\theta' R)]$. Since $\theta R\geq 2+B$, the first three terms are non-negative, while $\theta R\geq\theta' R$ means that the last term is non-negative. Therefore Eq. (7) implies that $\partial_l(\theta R)\leq 0$. Also, if $\theta' R\geq 2+B$ and $\theta' R\geq\theta R$, a similar analysis of Eq. (8) gives $\partial_l(\theta' R)\leq 0$. However, the first derivative must vanish at an interior maximum. Since the value at the end points is 2, we get $\max(\theta R,\theta' R)\leq 2+B$.

The argument for the lower bound works in exactly the same way. Let us assume that $\theta R\leq -2-B$ and that $\theta R\leq\theta' R$. Again Eq. (7) means that $\partial_l(\theta R)\leq 0$. Thus either $\min(\theta R,\theta' R)\geq -2-B$ or it is unbounded below.

We know that R is a four-dimensional scalar, a property of the spacetime, while B, θ, and θ' are three scalars which depend on a particular spacelike slice. On the other hand, the product $\theta\theta'$, and $\theta\theta' R^2$, is a four scalar under transformations preserving spherical symmetry. We can combine Eqs. (7) and (8) to give

$$\partial_l(\theta R\theta' R)=-8\pi R[\rho(\theta R+\theta' R)+j(\theta R-\theta' R)]$$
$$-(1/4R)(\theta R\theta' R-4)(\theta R+\theta' R). \tag{10}$$

Let us assume that $\theta R\theta' R>4$. If both are positive we have that the right hand side of Eq. (10) is strictly negative and if both are negative, the right hand side is positive. Thus we have

$$\theta\, R\theta' R\leq 4 \tag{11}$$

independent of any foliation. We can define the Hawking mass of any spherical surface via

$$1-M_H/2R=\theta\, R\theta' R/4. \tag{12}$$

Thus Eq. (11) guarantees that the Hawking mass is positive for any spherical surface [6]. We can combine Eqs. (9) and (11) to finally give

$$-(2+B)^2\leq\theta R\theta' R\leq 4. \tag{13}$$

There are only two allowed topologies for globally regular, asymptotically flat, spherically symmetric, spacelike three manifolds. They can either have R^3 topology with a regular center and one asymptotic end or $R\times S^2$ topology with two asymptotic ends, as in the extended Schwarzschild geometry [7]. We also can have manifolds where we identify points on a sphere [8]. Lemma 1 and Eq. (13) hold in all these cases. They also hold for any compact manifold.

Lemma 1 has a number of interesting consequences. Let us assume, for a moment, that the trace of the extrinsic curvature vanishes, i.e., that the initial data define a maximal slice. This means that $B\equiv 0$ and lemma 1 implies that $|\theta R|,|\theta' R|\leq 2$. A surface on which $\theta<0$ is called a trapped surface; such surfaces play a key role in the singularity theorems of general relativity. Equation (7) can be used to derive

$$\partial_l(\theta R^2)=-8\pi R^2(\rho-j_r/\sqrt{a})+1$$
$$+\tfrac{1}{4}\theta R(2\theta' R-\theta R). \tag{14}$$

Let $L(S)$ be the geodesic (proper) radius of a sphere S; $R(S)$ its areal radius; $M(S)=\int_{V(S)}\rho dV$ the total mass inside S; and $P(S)=\int_{V(S)}(j_r/\sqrt{a})dV$ be the total radial momentum. Integrating (14), noting that $4\pi R^2dl=4\pi\sqrt{a}R^2dr$ is the proper volume, we get

$$(\theta R^2)(S)=-2(M-P)(S)+L(S)$$
$$+\frac{1}{4}\int_0^{L(S)}\theta R(2\theta' R-\theta R)dl. \tag{15}$$

We can see that $\frac{1}{4}\int_0^L dl\,\theta R(2\theta' R-\theta R)\leq\frac{1}{4}\int_0^L dl(\theta' R)^2\leq L$, where the first inequality comes from the trivial estimate $2ab-a^2\leq b^2$ and the second from lemma 1. Therefore

$$(\theta R^2)(S)\leq -2(M-P)(S)+2L(S) \tag{16}$$

for any surface S. In particular, if $M-P\geq L$ at any given sphere S, then $\theta(S)$ must be negative. Thus we have proven the following:

Theorem 1. Under conditions of lemma 1, assuming $\mathrm{tr}K\equiv 0$, if the difference between the total rest mass $M(S)$ and the radial momentum $P(S)$ exceeds the proper radius $L(S)$ of a sphere S, $M(S)-P(S)>L(S)$, then S is trapped.

This theorem improves our earlier result [2], in which we got a similar result but with L replaced by $\frac{7}{6}L$ and the weaker conclusion that there exists a trapped surface inside S. The difference is due to the fact that we now impose the somewhat stronger condition that $\rho-|j|\geq 0$, whereas in [2] we used $\rho+(3/32\pi)(K_r^r)^2\geq 0$. Since the new conditions in theorem 1 eliminate tachyons this is a real difference. The constant $\frac{7}{6}$ also appears in our criteria for the formation of cosmological black holes [3]; we believe that these can also be improved to 1.

The meaning of theorem 1 is transparent. Radially ingoing matter $j_r\leq 0$ helps form apparent horizons. The presence of outgoing matter, i.e., when $P(S)$ becomes positive, has to be compensated for by a greater matter density. In the extremal case of radially outgoing photons, when $M(S)=P(S)$, apparent horizons cannot form. This follows from our theorem 2 below.

Theorem 1 is sharp in the sense that there exists an initial value configuration when the inequality saturates. This is a three geometry created by a shell of moving matter; the explicit calculation will be done elsewhere. The case in which $P=0$ was discussed in [2] and the corresponding criterion (with the same constant 1, as above) was shown to be the best possible.

It is interesting that we obtain an exact criterion with the constant 1; this suggests that theorem 1 is part of a more complex true statement that can be formulated for general nonspherical spacetimes. It suggests also that $M(S)$ is a sensible measure of the energy of a gravitational system that might appear as a part of a quasilocal energy measure in nonspherical systems.

We also obtain a necessary condition for the formation of apparent horizons. In [9] we found a criterion based on asymptotic data outside a collapsing system. Reference [2] states that $M(S)>L/2$ must be satisfied if S is trapped in the case of moment of time symmetry data. The same holds true if the matter is moving under some stringent conditions on the sign of the momentum density [10]. Here we will derive a different estimate. The most important assumption we make is that θ' is everywhere positive on the initial hypersurface. Just as $\theta\leqslant 0$ guarantees a singularity to the future, $\theta'\leqslant 0$ guarantees a singularity to the past. Therefore, data which arise from a regular past must have positive θ'.

Theorem 2. Assume a regular maximal slice on which the sources satisfy the dominant energy condition. Let S be the innermost trapped surface and let $(R\theta')>\epsilon>0$ inside S. Then

$$M(S)-P(S)\geqslant \tfrac{1}{2}\epsilon L.$$

Proof. As before, we consider (15), which reads

$$\theta R^2=-2(M-P)+L+\frac{1}{4}\int_0^L dl\,\theta R(2\theta' R-\theta R). \quad (17)$$

Inside S, $R\theta$ is positive. We seek a lower bound on the last term on the right hand side of (17). Let $t=R\theta, u=R\theta'$; from lemma 1 we know $|t|,|u|\leqslant 2$, so our task consists in estimating $2tu-t^2$ for $0\leqslant t\leqslant 2, \epsilon\leqslant u\leqslant 2$. We know that $2tu-t^2\geqslant F(t)=2t\epsilon-t^2$. The only extremum of $F(t)$ is a maximum at $t=\epsilon$. The minimum must occur at the end points and it is easy to show that $2tu-t^2\geqslant F(t)\geqslant 4\epsilon-4$. Inserting this into (17) yields

$$\theta(S)R^2\geqslant -2(M-P)(S)+L(S)+\frac{1}{4}\int_0^{L(S)} dl(4\epsilon-4)$$

$$=-2(M-P)(S)+\epsilon L, \quad (18)$$

that is, since $\theta(S)=0$,

$$M(S)-P(S)\geqslant \epsilon L/2.$$

Hence theorem 2 is proven.

The inequality of theorem 2 becomes an equality in the case of a spherical shell. The geometry inside the shell is flat and $\theta' R=2$. The necessary condition that the shell be trapped is that $M-P>L$. In [2] we proved this in the special case when $P=0$.

It is clear that the analysis performed here can include cases where the sources are distributions rather than classical functions; in particular, we have no difficulty with shells of matter. All we get on crossing the shell is a downward step in θ and θ'. More interestingly, we can extend the analysis to include weak singularities at the origin.

Let us begin by considering a conical singularity [11]. Consider a metric of the form

$$dS^2=dr^2+a^2r^2d\Omega^2. \quad (19)$$

The scalar curvature of this metric is ${}^{(3)}R=2(1-a^2)/r^2$. A moment of time symmetry data set is one for which j^i and $K^{ij}\equiv 0$. For such data sets the constraints reduce to ${}^{(3)}R=16\pi\rho$. For the above metric we get $\rho=(1-a^2)/8\pi r^2$. The dominant energy condition reduces to the positivity of ρ, which implies $a^2\leqslant 1$. For this metric we can also compute the mean curvature p, which in this case equals both θ and θ', to get $p=2/r=2a/R$. Hence we get $|pR|\leqslant 2$. However, the argument of lemma 1 only requires that $\theta R,\theta' R$ be bounded at the origin. Therefore we have shown that lemma 1 holds for moment of time symmetry data with a conical singularity at the origin. The conical singularity in question is determined by the deficit of the solid angle $4\pi(1-a^2)$. We will show that a similar result holds true for general nonmaximal data.

Let us consider initial data such that tr$K$ is finite while $R\theta\to X$ and $R\theta'\to Y$ as $R\to 0$. Let us also assume that $\partial_l(R\theta)$ and $\partial_l(R\theta')$ are finite at $R=0$. There are terms on the right hand side of Eqs. (7) and (8) which seem to diverge like $1/R$. The source term will have the same sort of $1/R$ divergence if $8\pi R^2\rho\to\alpha$ and $8\pi R^2 j_r/\sqrt{a}\to\beta$, just as in the case of the conical singularity. The coefficient of this $1/R$ term must vanish. This gives us a pair of equations, one from (7) and one from (8):

$$\alpha-\beta+\tfrac{1}{2}X^2-\tfrac{1}{4}XY-1=0, \quad (20)$$

$$\alpha+\beta+\tfrac{1}{2}Y^2-\tfrac{1}{4}XY-1=0. \quad (21)$$

By adding these equations we get

$$4\alpha=4-X^2-Y^2+XY, \quad (22)$$

and, by subtracting,

$$4\beta=X^2-Y^2. \quad (23)$$

Note that Eq. (22) implies that $\alpha\leqslant 1$. The weak energy condition gives $\alpha\geqslant|\beta|$. Let us assume that $\beta\geqslant 0$. Equation (23) now gives us $X^2\geqslant Y^2$ and $Y=\pm\sqrt{X^2-4\beta}$. Substitute this into Eq. (22) to give

$$[3X^2-4(1-\alpha+\beta)][X^2-4(1-\alpha+\beta)]+4X^2\beta^2=0. \quad (24)$$

The roots of this equation, if it has any, must lie in the range $4(1-\alpha+\beta)/3\leqslant X^2\leqslant 4(1-\alpha+\beta)$. Therefore we have shown that $2\geqslant|X|\geqslant|Y|$. If we assume $\beta<0$, we just reverse the roles of X and Y. Hence we obtain the following lemma.

Lemma 2. Given $\rho \geq |j|$ and if all of $\mathrm{tr}\, K, \theta R, \theta' R, \partial_l \theta R, \partial_l \theta' R, (8\pi \int_0^R \rho \tilde{R}^2 d\tilde{R})/R$ are finite in the limit $R=0$, then

$$2 \geq \lim_{R\to 0} |\theta R|, \lim_{R\to 0} |\theta' R|, \quad 1 \geq \frac{8\pi \int_0^R \rho \tilde{R}^2 d\tilde{R}}{R}. \tag{25}$$

From Eqs. (4)–(6) it is clear that

$$2\partial_l R = 2\partial_r R/\sqrt{a} = pR = (\theta R + \theta' R)/2. \tag{26}$$

This means that the spatial part of the metric (1) can be written, at least in a small neighborhood of $R=0$, as

$$[16/(R\theta + R\theta')^2] dR^2 + R^2 d\Omega^2. \tag{27}$$

The estimate derived in lemma 2 implies that, under the stated conditions, there can be at most a conical singularity at the origin, with solid angle deficit $4\pi\{1-[(X+Y)^2/16]\}$. Conical singularities have previously been investigated in 2+1 gravity [12]. In the 2+1 case the conical singularity can also be described by an angle deficit expressed in terms of the mean curvature: $2\pi(1-pR)$. However, in the 2+1 case the geometry is locally flat but globally nontrivial and the deficit angle is related to a total mass [12]. In our case, the deficit angle is a local phenomenon caused by a mildly singular mass distribution at the origin, where ρ diverges like r^{-2}.

Lemma 2 gives the desired bound $|\theta R|, |\theta' R| \leq 2$ at the origin so we get a generalized version of lemma 1.

Lemma 1'. Assume an asymptotically flat nonmaximal slice, satisfying the dominant energy condition, such that $4\sup_{0\leq R\leq\infty} |R\,\mathrm{tr}K| = B$ is finite. Let the conditions of lemma 2 be satisfied at the origin. Then

$$2 + B \geq |\theta R|, |\theta' R|. \tag{28}$$

Theorems 1 and 2 hold under similar conditions.

As we have mentioned earlier, the product $\theta R \theta' R$ is defined for any point in a spherically symmetric spacetime geometry, independent of any foliation or choice of time. One consequence of lemma 1 is that if a point exists in a spherical spacetime for which $|\theta R| \theta' R|$ is larger than 4 then we know that a regular, maximal, asymptotically flat slice cannot pass through this point.

Consider regular, asymptotically flat, spherically symmetric initial data which contain an apparent horizon. Let us now evolve the spacetime and look at the maximal Cauchy development of this data. We are guaranteed that a singularity will occur for a sufficiently large value of local proper time. If the singularity is such that $R\theta\, R\theta' \to -\infty$, as in the Schwarzschild solution, regular maximal slices (and any other slicing with bounded trace of the extrinsic curvature) do not cover the full Cauchy evolution. We get a "collapse of the lapse." The foliation can be chosen to continue for infinite time as seen by asymptotic observers but "freezes" in the interior.

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