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On the Interaction of Rate-induced and Bifurcation-induced Tipping in Ecosystems



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This dissertation is submitted for the degree of
Doctor of Philosophy

October 2019

Declaration

This is to certify that the work I am submitting is my own and has not been submitted for another degree, either at University College Cork or elsewhere. All external references and sources are clearly acknowledged and identified within the contents. I have read and understood the regulations of University College Cork concerning plagiarism.

Paul E. O'Keeffe
October 2019

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Abstract

We analyse nonlinear tipping phenomena in a bi-stable ecosystem model, defined as sudden and unexpected transitions from the herbivore-dominating equilibrium to the plant-only equilibrium, which are triggered by environmental changes represented by time-varying parameters [Scheffer et al. *Ecosystems* 11 2008].

We obtain simple criteria for tipping in terms of properties of the autonomous system with fixed in time parameters. Specifically, we use classical bifurcation analysis to identify a codimension-three degenerate Bogdanov-Takens bifurcation: the organising centre for bifurcation-induced tipping (B-tipping) and the source of a dangerous subcritical Hopf bifurcation. We introduce basin instability analysis to identify parameter paths along which rate-induced tipping (R-tipping) is guaranteed to occur without crossing any bifurcation. We then produce tipping diagrams for the non-autonomous system with time-varying parameters in the plane of the magnitude and rate of a parameter shift to reveal tipping-tracking transitions due to maximal-canard solutions that, perhaps surprisingly, follow unstable states for infinite time. We also uncover non-trivial dynamics arising from the interaction between B-tipping and R-tipping.

Given a monotone parameter shift that causes tipping, we ask if tipping can be prevented upon a parameter trend reversal. The ensuing analysis of non-monotone parameter shifts reveals an intriguing tipping diagram with a single critical level and multiple critical rates, indicating that the system switches from tipping to tracking and back to tipping again as the rate of the parameter shift increases. In the diagram, we identify points of no return where tipping cannot be prevented by the parameter trend reversal and points of return tipping where tipping is inadvertently induced by the parameter trend reversal. The results give new insight into the sensitivity of ecosystems to the magnitudes and rates of environmental change. A comparison of the ecosystem model with modified saddle-node and subcritical Hopf normal forms shows that the tipping diagram is characteristic of a non-monotone passage through a basin instability boundary and a generic dangerous bifurcation in nonlinear systems in general.

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Chapter 1

Introduction

Tipping points are strongly nonlinear phenomena which can be described in layman's terms as large, sudden and often unexpected changes in the state of a system, caused by small and slow changes in the external inputs [1, 2]. The notion of a tipping point was popularised by Gladwell [3] and has since been used in a wide range of applications including climate science [4–6] and ecology [1, 7–12]. Scientists have identified interesting questions in relation to different tipping mechanisms [2, 13], generic early warning signals near a tipping point [14–17], and the possibility of preventing tipping [18–22], that need to be addressed in more rigorous terms. For example, Article 2 of the 1992 United Nations Framework Convention on Climate Change (UNFCCC) pointed out two *critical factors*: the *level* and the *time frame* for changing greenhouse gas concentrations [23], suggesting that there are at least two tipping mechanisms of great importance to the contemporary climate. More generally, tipping phenomena can be classified by a type of instability and analysed in more depth, although this often requires mathematical techniques beyond traditional stability theory [2, 24–27].

Earlier mathematical models described tipping points as dangerous bifurcations that occur at *critical levels* of an input parameter [28–30]. Such bifurcations have a discontinuity in the branch of stable states (attractors) at the bifurcation point, which explains why a system can remain near one stable state up to a critical level, but is destined to transition to a different state past the critical level [31]. However, tipping points are not just bifurcations. Some systems have *critical rates* of parameter change, meaning that they are very sensitive to how fast external conditions or inputs change. Such systems can tip to a different state, despite the absence of any classical bifurcation, when the input parameter varies slowly but fast enough [7, 22, 25, 32, 33]. Ashwin et al.

used the framework of non-autonomous dynamical systems to identify three different tipping mechanisms [2]. Bifurcation-induced tipping (B-tipping) occurs when the changing parameter passes through a *critical level* or a (*dangerous*) *bifurcation*, at which point the stable state loses stability or simply disappears. In other words, B-tipping describes the adiabatic effects of a parameter change. Rate-induced tipping (R-tipping) occurs when the parameter changes faster than some *critical rate* and the system deviates from the moving stable state sufficiently far to cross some tipping threshold, e.g. the boundary of the domain of attraction. In other words, R-tipping describes the non-trivial non-adiabatic effects of a parameter change. Noise-induced tipping (N-tipping) occurs when noisy fluctuations drive the system past some tipping threshold.¹ Shi et al. gave an alternative but similar classification of tipping mechanisms based on relative timescales of the input and of the noisy system alone [13]. Additionally, tipping points can be described as either reversible or irreversible, depending on whether or not the system returns to the original stable state in the long term [27]. So far, B-tipping and R-tipping have been discussed in isolation in the literature. However, real-world tipping phenomena will often involve different critical factors and different tipping mechanisms. Motivated by this observation, we use classical bifurcation analysis [34] in conjunction with the concepts of *parameter paths* and *basin instability* [2, 27] to analyse the effects of the rate of parameter change near the two generic dangerous bifurcations of equilibria: saddle-node and subcritical Hopf bifurcations [31]. In this way, we give new insight into testable criteria for R-tipping and reveal non-trivial phenomena such as multiple critical rates that arise from the interaction between B-tipping and R-tipping.

Ecological models appear to be a perfect test bed for this type of study. B-tipping has been observed and studied extensively in different ecosystems [4, 35–39], although the concept of a “global tipping point” in the context of planetary boundaries has recently received some criticism [40]. Ecologists speak of a “regime shift” when the bifurcation is safe or explosive, and of a “critical transition” when the bifurcation is dangerous [1]; we refer to [31] for the classification of bifurcations into safe, explosive and dangerous. Similarly, there is great and rapidly growing interest in R-tipping in the context of ecological dynamics [39, 41]. To the best of our knowledge, the first examples of R-tipping were reported in ecosystems [7, 10–12, 32]. More precisely, R-tipping conceptualises a failure to adapt to changing environments [42] in the sense that the stable state is continuously available but the system is unable to adjust to

¹In a certain sense, N-tipping can be thought of as a special case of R-tipping.

its changing position when the change happens too fast. This raises the interesting research question of whether tipping phenomena observed in nature are predominantly rate induced. What is more, the related question of whether tipping can be avoided or prevented has recently received much attention in the ecosystem literature [18–21]. Proper mathematical analysis of the interaction between critical levels and critical rates, or between B-tipping and R-tipping, is exactly what is needed to gain more insight into these questions. Lastly, there is a strong need to better understand whether ecosystems are sensitive to the magnitudes of environmental change, the rates of environmental change, or to both. This is of particular importance in view of a highly variable contemporary climate, intensifying human activity and rapidly declining resources.

1.1 Population Models

Before we describe the bistable predator-prey model introduced by Scheffer et al. [7], which is the focus of this thesis, we give a brief overview of earlier predator-prey models. The following examples will be discussed in terms of mass concentrations of prey $X \geq 0$ and predator $Y \geq 0$, and the two main input parameters, r the growth rate of the prey, and m the mortality of the predator.

The first predator-prey model was independently conceived by Lotka [43] and Volterra [44],

$$\frac{dX}{dt} = rX - Yg(X), \quad (1.1)$$

$$\frac{dY}{dt} = (Eg(X) - m)Y. \quad (1.2)$$

Here,

$$g(X) = cX,$$

describes a linear consumption with the food intake constant c . Such $g(X)$ is known as a type I functional response according to Holling's classifications [45]. The parameter E quantifies how well the predator converts food into energy for growth and reproduction, otherwise known as an assimilation efficiency parameter. This system has two equilibrium solutions: a trivial zero population solution $(X, Y) = (0, 0)$, which is a saddle, and a harmonious coexistence population at $(X, Y) = (m/Ec, r/c)$, which is a centre. For $X, Y > 0$, the two populations oscillate, with any changes to r and

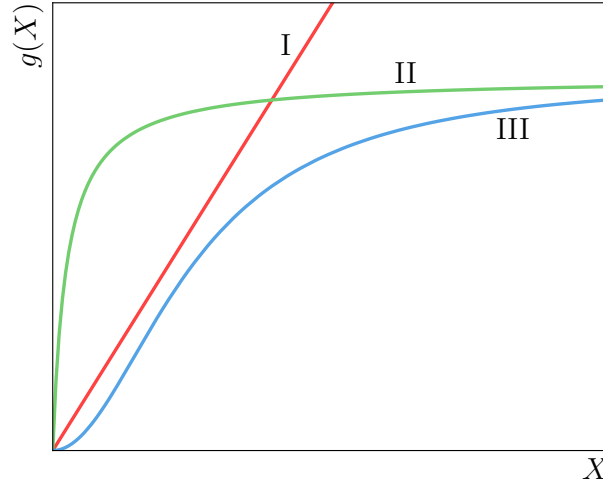


Figure 1.1 Holling's type I, II and III functional responses [45].

m doing nothing more than affecting the period of the oscillations. There is also an unrealistic, unbounded exponential growth of the prey in the absence of predators.

The Rosenzweig-MacArthur's predator-prey system [46] builds upon the Lotka-Volterra model, adding realism with both logistic growth of the prey, and a limit to the consumption rate of the predator,

$$\frac{dX}{dt} = rX - CX^2 - Yg(X), \quad (1.3)$$

$$\frac{dY}{dt} = (Eg(X) - m)Y. \quad (1.4)$$

Here,

$$g(X) = c_{max} \frac{X}{1 + X},$$

describes a nonlinear consumption, which grows linearly with X when X is small, and saturates at the maximum food intake c_{max} when X is large. Such $g(X)$ is known as a Holling type II functional response [Fig. 1.1]. The other difference is that, in the absence of the predator Y , the prey X grows over time to a carrying capacity r/C , where C is a competition factor for the prey. Both modifications take into account limited resources and eliminate unbounded growth. This system has at most three equilibrium solutions, a trivial zero population solution $(X, Y) = (0, 0)$, which is a saddle, a prey only solution $(X, Y) = (r/C, 0)$, which is also a saddle, and a coexistence equilibrium that, upon changes to r or m , can change from stable to unstable via a

supercritical Hopf bifurcation, giving rise to a stable limit cycle. The stability of its steady states shows that this model is monostable.

There is a third form of the consumption function as described by Holling, the type III functional response,

$$g(X) = c_{max} \frac{X^2}{X^2 + a^2}.$$

This is a sigmoid curve [Fig. 1.1] whose gradient increases at first, and then decreases with increasing prey concentrations, so that $g(X)$ saturates at the maximum food intake c_{max} , where a is a half-saturation constant.

Irreversible R-tipping, which is the topic of this thesis, requires bistability – two stable equilibrium points for the same parameter values. Bistability appears in population models when additional effects are taken into account. One example is the Allee effect, where the trivial zero equilibrium becomes stable and the prey concentration has to exceed a certain level to grow towards the coexistence state, which is the other stable state [47]. Another effect, discussed in more detail in the next chapter, involves a modification of the type III functional response with an exponential term that has a negative exponent to account for the reduced quality of food if food biomass is high. This modification allows the system to have a fourth equilibrium solution, and under the right conditions, bistability.

1.2 Methods

As many of the problems applied mathematicians face are nonlinear, exact solutions are a rarity. More often, one must rely on approximating solutions analytically or numerically. What follows are brief descriptions of the methods of approximation that are used throughout the thesis. For more details and background on nonlinear systems and bifurcation analysis, we refer to [48, 34].

1.2.1 Asymptotic Approximation

Perturbation methods are developed for problems that depend on a small parameter ε , and simplify (e.g. become linear or exactly solvable) when $\varepsilon = 0$. They encompass techniques for finding an approximate solution to a problem with $0 < \varepsilon \ll 1$ by taking a known solution to the unperturbed problem with $\varepsilon = 0$, and adding suitable corrections. This is achieved by seeking solutions in the form of an asymptotic expansion in different

powers of ε :

$$x(\varepsilon) = x_0 + \varepsilon^\alpha x_1 + \varepsilon^{2\alpha} x_2 + \dots,$$

where x_0 is the solution to the (exactly solvable) unperturbed problem, while $\varepsilon^\alpha x_1, \varepsilon^{2\alpha} x_2, \dots$ represent the higher-order corrections which get progressively smaller as $\varepsilon \rightarrow 0$.

We employ these methods in Chapter 3 in order to get useful information about the dependence of the equilibrium solutions on certain parameters. For more details and background on perturbation theory, we refer to [49].

1.2.2 Numerical Continuation

The software package AUTO [50] employs numerical continuation methods to continue equilibria and limit cycles in the system parameters and detect bifurcations. What follows is a brief illustration of how this is implemented [51].

Consider the following ODE,

$$\dot{x} = f(x, \lambda), \tag{1.5}$$

where $x \in \mathbb{R}$ is the state variable, and λ is the bifurcation parameter. We also assume that f is C^1 -smooth for all x and λ .

The equilibrium equation

$$f(x_e, \lambda) = 0, \tag{1.6}$$

gives rise to a λ -dependent equilibrium solution $x_e(\lambda)$. Differentiating this with respect to λ gives

$$\frac{\partial f}{\partial x_e} \frac{dx_e}{d\lambda} + \frac{\partial f}{\partial \lambda} = 0,$$

then rearranging for $dx_e/d\lambda$ gives

$$\frac{dx_e}{d\lambda} = -\frac{\partial f / \partial \lambda}{\partial f / \partial x_e}. \tag{1.7}$$

Equation (1.7) needs to be solved together with Eq. (1.6). Equation (1.7) is used to progress the equilibrium solution in λ (e.g. using Euler's method for solving initial value problems), and Eq. (1.6) is used to correct to the actual position of the equilibrium at the new value of λ (e.g. using Newton's root-finding method). The stability of the solution $x_e(\lambda)$ can be determined from the sign of $\partial f / \partial x_e$, and the singularity $\partial f / \partial x_e = 0$ at some $\lambda = \lambda^*$ corresponds to a bifurcation of $x_e(\lambda)$. Numerical continuation has an

advantage over numerical integration: it can follow both stable and unstable states and their bifurcations in a precise and efficient way. Examples of implementation of numerical continuation are illustrated in Appendix A.

1.2.3 Numerical Integration and The Tipping-Tracking Transition Condition

We use a classic fourth-order Runge-Kutta method to solve initial value problems, we compared our results with Matlab's own ordinary differential equation solver ODE45 for accuracy. The threshold between an input parameter shifting, such that the system fails to adapt to its changing environment and tips, and where the system is able to adapt and tracks, is found numerically. The tipping-tracking transition condition makes use of a modified bisection method, where the conditions for tipping and tracking replace the traditional plus and minus values. Examples of computer code can be found in Appendix C.

1.3 Terminology

Throughout the thesis, we extend the common indicators of criticality in Hopf and pitchfork bifurcations to describe saddle-node, transcritical, and homoclinic bifurcations. A visualisation of the following can be seen in Table 1.1.

A saddle-node bifurcation occurs at the point where either a stable and an unstable equilibrium becomes degenerate, or when two unstable equilibria become degenerate. We describe the former as a supercritical saddle-node bifurcation, and the latter as a subcritical saddle-node bifurcation.

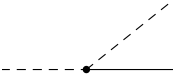
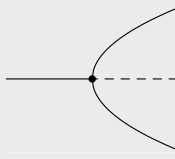
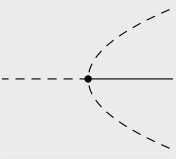
In the case of a transcritical bifurcation involving a trivial zero equilibrium solution, we only consider the positive part of the branch of non-zero equilibrium solutions. The bifurcation is supercritical (subcritical) when the non-zero branch is stable (unstable).

In a supercritical (subcritical) homoclinic bifurcation the bifurcating limit cycle is stable (unstable).

1.4 Thesis Layout

The thesis is organised as follows. Chapter 2 introduces the ecological model given by two non-autonomous ordinary differential equations and discusses the key nonlinearity

Table 1.1 Bifurcation Criticality.

Bifurcation	Supercritical	Subcritical
Saddle-node		
Transcritical		
Hopf		

due to a modified type-III functional response. It also introduces the concepts of a parameter path and a moving equilibrium. In Chapter 3 we perform classical bifurcation analysis of the corresponding autonomous system with fixed in time parameters, obtain two-dimensional bifurcation diagrams in the parameter plane of the plant growth rate and herbivore mortality rate, and uncover a codimension-three degenerate Bogdanov-Takens bifurcation as the organising centre for B-tipping and the source of a dangerous subcritical Hopf bifurcation. We give simple criteria for B-tipping in the non-autonomous system in terms of dangerous bifurcations in the autonomous system. In Chapter 4 we introduce the concept of basin instability for the corresponding autonomous system to give testable criteria for R-tipping in the non-autonomous system. We superimpose regions of basin instability on classical bifurcation diagrams to highlight rate-induced instabilities that cannot be captured by classical bifurcation analysis. We then obtain two-dimensional R-tipping diagrams in the parameter plane of the *rate* and *magnitude* of parameter shift for monotone and non-monotone parameter shifts, uncover R-tipping tongues with two critical rates, draw parallels between R-tipping tongues and resonance tongues, and demonstrate that tracking-tipping transitions correspond to special maximal canard solutions that track moving unstable states for infinite time. In Chapter 5 we describe non-trivial tipping phenomena arising from the interaction between B-tipping and R-tipping such as tipping diagrams with S-shaped tipping-tracking transition curves and multiple critical rates, which we explain in terms

of different timescales and bifurcation delays. In Chapter 6 we partition the tipping diagrams into “points of tracking”, “points of return”, “points of no return” and “points of return tipping” to give new insight into the problem of preventing tipping by a parameter trend reversal. We then depart from the ecological model and produce tipping diagrams capturing both B-tipping and R-tipping for modified (tilted) normal forms of the two generic dangerous bifurcations of equilibria namely saddle-node and subcritical Hopf. By comparison with the ecological model we show that the tipping diagram from Chapter 5 appears to be typical for non-monotone parameter shifts that cross a basin instability boundary and a generic dangerous bifurcation and then turn around. Chapter 7 provides an outlook for future work with a brief introduction to compactification. As analysis of nonautonomous systems is impeded by the lack of compact invariant sets, one can transform the system to an extended autonomous compactified system. We do this with the ecological model and identify the critical rates of the system as heteroclinic orbits from our starting point, a stable equilibrium, to an unstable equilibrium, whose stable invariant manifold is the basin boundary between the two stable states in the model. Furthermore, we draw comparisons between the non-autonomous system and the autonomous compactified system, and show a close correlation between the critical rates of both models. Chapter 8 summarises our findings.

The material throughout the thesis, with the exception of Chapter 7, is based on a paper that is currently being revised for SIAM Journal on Applied Dynamical Systems: P.E. O’Keeffe and S. Wiczorek. Tipping Phenomena and Points of No Return in Ecosystems: Beyond Classical Bifurcations.

Chapter 2

A Simple Autonomous Ecosystem Model

We consider a simple ecosystem model, where the time evolution of plant $P \geq 0$ and herbivore $H \geq 0$ biomass concentrations is modelled using two coupled autonomous ordinary differential equations [7]:

$$\frac{dP}{dt} = rP - CP^2 - Hg(P), \quad (2.1)$$

$$\frac{dH}{dt} = (Ee^{-bP}g(P) - m)H, \quad (2.2)$$

together with eight parameters listed in Table 2.1. The first two terms on the right-hand side (r.h.s.) of Eq. (2.1) describe logistic plant growth from 0 to the carrying capacity r/C . The third term describes grazing with a nonlinear dependence on the plant biomass P . Specifically, the functional response in units inverse day

$$g(P) = c_{max} \frac{P^2}{P^2 + a^2} e^{-b_c P}, \quad (2.3)$$

is a modification of the classical monotone and strictly-increasing type-III functional response $c_{max}P^2/(P^2 + a^2)$ [45] with an exponential factor $e^{-b_c P}$ to account for a decline in foraging at high plant biomass. The resulting non-monotone $g(t)$, shown in Fig. 2.1(a) for different predation efficiency b_c , is believed to describe a wide range of terrestrial and aquatic ecosystems; see [52, 7] and references therein. For example, rabbits graze more with faster-growing plants as long as the plants are small enough, but avoid overgrown bushes for fear of predators, and are unable to graze on plants that have grown too tall. Similarly, in aquatic ecosystems, phytoplankton can be

Table 2.1 Description of the system parameters and their values [7].

Symbol	Description	Units	Default value
$C > 0$	Competition factor of plants	$\text{m}^2\text{g}^{-1}\text{d}^{-1}$	0.02
$a > 0$	Half-saturation constant of functional response	g m^{-2}	10
$b \geq 0$	Exponent determining the reduced quality of food if food biomass is too high	m^2g^{-1}	0 - 0.04
$b_c \geq 0$	Exponent determining the predation efficiency of herbivores at high food biomass	m^2g^{-1}	0 - 0.04
$E > 0$	Assimilation efficiency of herbivores	dimensionless	0.4
$c_{max} > 0$	Maximum food intake of herbivores when $b_c = 0$	d^{-1}	1
$m > 0$	Herbivore mortality rate	d^{-1}	0 - 0.2
$r > 0$	Maximum plant growth rate	d^{-1}	0 - 2.5

heavily consumed at early life stages by herbivorous zooplankton, but higher-density phytoplankton colonies become less prone to exploration and foraging. Moving on to the herbivore dynamics, the first term on the r.h.s. of Eq. (2.2) describes an increase in herbivore biomass. The increase term consists of three factors: reproduction and grazing $g(P)$, herbivore assimilation efficiency E , and exponential decline e^{-bP} due to reduced food quality at high plant biomass. The last term on the r.h.s. of Eq. (2.2) represents herbivore death at the constant rate m .

In mathematical terms, the ecosystem model (2.1)–(2.2) with the modified functional response (2.3) is a singular perturbation problem because it has a different number of equilibrium solutions for $b + b_c = 0$ and $0 < b + b_c \ll 1$. To see that, consider the net per-capita herbivore growth shown in Fig. 2.1(b):

$$h(P) = \frac{dH/dt}{H} = E c_{max} \frac{P^2}{P^2 + a^2} e^{-(b+b_c)P} - m, \quad (2.4)$$

whose roots correspond to non-zero herbivore equilibrium concentrations. When $b + b_c = 0$, the net per-capita herbivore growth is a strictly-increasing function of P with a single root P_3 [Fig.2.1(b)].

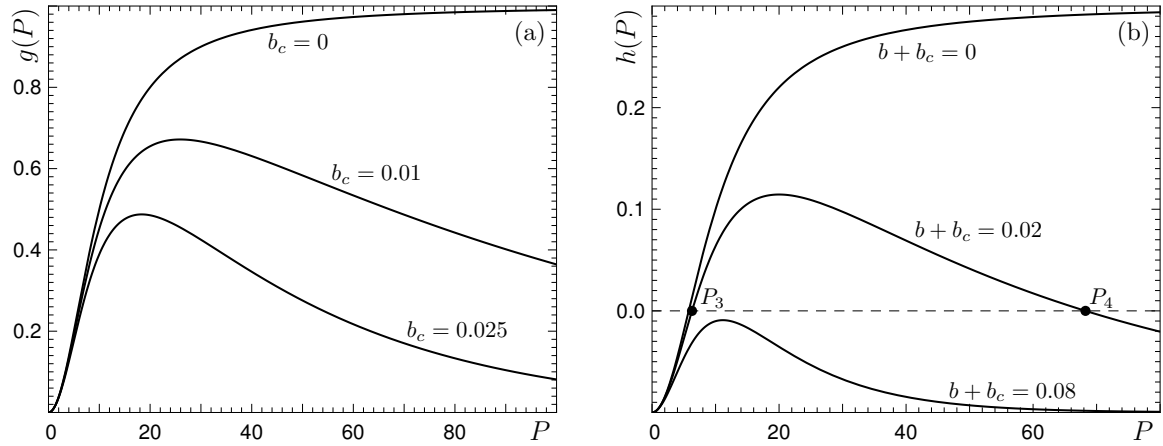


Figure 2.1 (a) The functional response $g(P)$ with dependence on b_c . (b) The key system nonlinearity: For $b + b_c > 0$, the net per-capita herbivore growth $h(P) = (dH/dt)/H$ has optimal plant biomass P_{opt} where the growth is maximal, and may change sign twice at P_3 and P_4 ; $m = 0.1$.

The optimal plant biomass P_{opt} , which defines the maximum herbivore growth (2.4), satisfies

$$\epsilon P_{opt}^3 + \epsilon a^2 P_{opt} - a^2 = 0, \quad (2.5)$$

where $\epsilon = (b + b_c)/2$ is the small parameter. This singular perturbation problem can be made regular by introducing a stretched variable $\tilde{P}_{opt} = \epsilon^{\frac{1}{3}} P_{opt}$:

$$\tilde{P}_{opt}^3 + \epsilon^{\frac{2}{3}} a^2 \tilde{P}_{opt} - a^2 = 0. \quad (2.6)$$

We now seek solutions in the form of an asymptotic expansion

$$\tilde{P}_{opt} = \tilde{P}_{opt,0} + \epsilon^{\frac{1}{3}} \tilde{P}_{opt,1} + \epsilon^{\frac{2}{3}} \tilde{P}_{opt,2} + \dots$$

Substituting the expansion into Eq (2.6), and successively equating terms with the same powers of ϵ on both sides gives

$$\tilde{P}_{opt} = a^{\frac{2}{3}} - \frac{1}{3} \epsilon^{\frac{2}{3}} a^{\frac{4}{3}} + \frac{1}{81} \epsilon^2 a^{\frac{8}{3}} + \mathcal{O}(\epsilon^{\frac{7}{3}}).$$

Going back to the original variable gives

$$P_{opt} = \epsilon^{-\frac{1}{3}} \left(a^{\frac{2}{3}} - \frac{1}{3} \epsilon^{\frac{2}{3}} a^{\frac{4}{3}} + \frac{1}{81} \epsilon^2 a^{\frac{8}{3}} + \mathcal{O}(\epsilon^{\frac{7}{3}}) \right),$$

which can be rewritten in terms of the original parameters

$$P_{opt} = \left(\frac{2a^2}{b+b_c} \right)^{\frac{1}{3}} + \frac{1}{3} \left(\frac{a^4(b+b_c)}{2} \right)^{\frac{1}{3}} + \frac{1}{81} \left(\frac{a^8(b+b_c)^5}{2} \right)^{\frac{1}{3}} + \mathcal{O} \left(\left(\frac{b+b_c}{2} \right)^2 \right).$$

So, when $0 < b + b_c \ll 1$, the net per-capita herbivore growth has a maximum at the optimal plant biomass

$$P_{opt} \approx \left(\frac{2a^2}{b+b_c} \right)^{\frac{1}{3}},$$

and can have no roots at all, one double root, or two distinct roots at $P_3 < P_{opt}$ and $P_4 > P_{opt}$ [Fig.2.1(b)]. The additional nonlinearity of $h(p)$ that arises from a decline in foraging at high plant biomass ($b_c > 0$), from reduced food quality at high plant biomass ($b > 0$), or from a combination of both [Fig.2.1(b)], is key to our study. Throughout the thesis, we refer to $b + b_c$ as the *nonlinearity parameter*, and work with different but fixed-in-time values of b and b_c , as indicated in Table 2.1.

2.1 The Non-Autonomous Model Reflecting a Changing Environment

Ecosystems are open systems that are inevitably subject to changing environmental conditions. These include climatic changes and weather anomalies, disease outbreaks, decline in resources or habitat quality, and human activity. In the model, environmental changes can be described by time-dependent plant growth rate $r(t)$ and herbivore mortality rate $m(t)$, which are the *input parameters* for this study. Specifically, we fix six of the system parameters to the values given in Table 2.1, and make $r(t)$ or $m(t)$ vary smoothly in time from one asymptotic value to another. For example, such $r(t)$ could describe the occurrence of a wet season, owing to a weather anomaly or El Niño Southern Oscillations (ENSO), while $m(t)$ could describe a disease outbreak among herbivores. This gives the non-autonomous ecosystem model

$$\frac{dP}{dt} = r(t)P - CP^2 - Hg(P), \quad (2.7)$$

$$\frac{dH}{dt} = (E e^{-bP} g(P) - m(t))H. \quad (2.8)$$

The exact time-dependence of $r(t)$ and $m(t)$ is specified in Chapters 4 and 6 ahead.

Our analysis of tipping points in the non-autonomous system (2.7)–(2.8) with time-varying environmental conditions is motivated by the need to better understand if ecosystems are sensitive to the magnitude of environmental change, the rate of environmental change, or to both. In particular, the question of critical rates of environmental change is relevant for ecosystems subject to a highly variable contemporary climate and to growing human activity.

2.1.1 The Moving Equilibrium

An equilibrium or a steady state for the autonomous system (2.1)–(2.2) is a pair

$$e(r, m) = (P, H),$$

for which $dP/dt = dH/dt = 0$. Typically, the position of an equilibrium depends on the input parameters r and/or m . When the input parameters vary over time, $e(r, m)$ changes its position in the (P, H) phase space, and we speak of a *moving equilibrium*

$$e(t) = e(r(t), m(t)),$$

also known as a quasistatic equilibrium [2]. Note that $e(t)$ is a property of the autonomous system (2.1)–(2.2) and the changing environment, but it is not a solution to the non-autonomous system (2.7)–(2.8).

As the input parameters $r(t)$ and $m(t)$ evolve smoothly over time, they trace out a continuous *parameter path* in the two-dimensional (r, m) parameter plane. We use the notions of a moving equilibrium and a parameter path to discuss the differences and interaction between B-tipping and R-tipping.

Chapter 3

Classical Bifurcation Analysis of the Autonomous System

In the non-autonomous system (2.7)–(2.8), *B-tipping* occurs when the input parameters pass through a dangerous bifurcation of the corresponding autonomous system (2.1)–(2.2). “Dangerous” means that, in a one-parameter bifurcation diagram like the ones shown in Fig. 3.3, there is a discontinuity in the branch of attractors at a bifurcation point [29]. Thus, B-tipping mechanisms in system (2.7)–(2.8) can be identified using classical bifurcation analysis in conjunction with singular perturbations of the autonomous system (2.1)–(2.2). Specifically, in this chapter we treat the input parameters r and m as fixed-in-time *bifurcation parameters*, compute bifurcation curves in the (r, m) parameter plane, and uncover the two generic dangerous bifurcations of equilibria; saddle-node and sub-critical Hopf bifurcations. In this way, given a parameter path of environmental change, we identify *critical levels* of r and m along the path whenever that path crosses a dangerous bifurcation.

3.1 Existence of Equilibrium Solutions

There are at most four equilibrium solutions for the autonomous system (2.1)–(2.2), which are pairs $e = (P, H)$ of non-negative P and H , that satisfy the following conditions:

$$rP - CP^2 - Hg(P) = 0, \tag{3.1}$$

$$(Ee^{-bP}g(P) - m)H = 0. \tag{3.2}$$

When $H = 0$, there are at most two equilibria: a *trivial* equilibrium e_1 , and a *plant-only* equilibrium e_2 :

$$e_1 = (0, 0), \quad e_2 = (r/C, 0).$$

When $H \neq 0$, the equilibrium conditions (3.1)–(3.2) become

$$H = \frac{(r - CP)(P^2 + a^2)}{c_{max} P e^{-b_c P}}, \quad (3.3)$$

$$h(P) = E c_{max} \frac{P^2 e^{-(b+b_c)P}}{P^2 + a^2} - m = 0. \quad (3.4)$$

Note that condition (3.4), which gives the P -component of equilibrium solutions, depends on the nonlinearity parameter $b + b_c$ rather than on b and b_c individually, which simplifies the discussion. What is more, the condition and thus the P -components of the ensuing equilibrium solutions are r -independent. Independently of $b + b_c$, the net per-capita herbivore growth $h(P)$ equals $-m$ for $P = 0$. When $b + b_c = 0$, the herbivore growth $h(P)$ is strictly increasing and levels off at $E c_{max} - m > 0$ for large P [Fig. 2.1(b)]. Thus, Eq. (3.4) has one positive root, giving at most three equilibrium solutions for the system. However, the maximum number of equilibrium solutions increases in the presence of the key nonlinearity. When $b + b_c > 0$, the herbivore growth $h(P)$ has a global maximum at $P = P_{opt} > 0$, and tends back to $-m$ for large P [Fig. 2.1(b)]. Thus, Eq. (3.4) has at most two positive roots, giving at most four equilibrium solutions for the system. Although the roots of Eq. (3.4) cannot be expressed in a closed-form, one can take advantage of the small nonlinearity parameter $0 < b + b_c \ll 1$ and use perturbation theory to obtain closed-form approximations in terms of an asymptotic expansion in different powers of $b + b_c$.

3.1.1 Analytical Approximation for the herbivore-dominating equilibrium e_3

Analytical approximations to the roots of Eq.(3.4) when $b + b_c$ is small enough can be obtained using perturbation techniques. Rewriting Eq.(3.4) in the form

$$P_3^2 e^{-\epsilon P_3} - P_3^2 \tilde{c} - \tilde{d} = 0, \quad (3.5)$$

where P_3 is the P -component of the herbivore-dominating equilibrium e_3 , $\tilde{c} = m/E c_{max}$, $\tilde{d} = a^2 \tilde{c}$ and the small parameter $\epsilon = b + b_c$. We expand $e^{-\epsilon P_3}$ up to $\mathcal{O}(\epsilon^2)$ to give the

polynomial equation

$$-\epsilon P_3^3 + (1 - \tilde{c})P_3^2 - \tilde{d} + \mathcal{O}(\epsilon^2) = 0, \quad (3.6)$$

Using regular perturbation methods we seek solutions in the form of an asymptotic expansion

$$P_3 = P_{3,0} + \epsilon P_{3,1} + \mathcal{O}(\epsilon^2).$$

Substituting this into Eq.(3.6) and successively equating terms with the same powers of ϵ on both sides gives

$$P_3 = \sqrt{\frac{\tilde{d}}{1 - \tilde{c}}} + \frac{\tilde{d}}{2(1 - \tilde{c})^2}\epsilon + \mathcal{O}(\epsilon^2), \quad (3.7)$$

and with original parameters

$$P_3 = \sqrt{\frac{a^2 m}{E c_{max} - m}} + \frac{a^2 m E c_{max}}{2(E c_{max} - m)^2}(b + b_c) + \mathcal{O}((b + b_c)^2), \quad (3.8)$$

where $\mathcal{O}(\epsilon^n)$ is the error term of order n as $\epsilon \rightarrow 0$, and

$$e_3 = \left(\sqrt{\frac{a^2 m}{E c_{max} - m}} + \mathcal{O}(b + b_c), \frac{(r - C P_3)(P_3^2 + a^2)}{c_{max} P_3 e^{-b_c P_3}} \right). \quad (3.9)$$

3.1.2 Analytical Approximation for the plant-dominating equilibrium e_4

We use singular perturbation methods to find the other positive root of Eq.(3.4). Rescaling P in Eq.(3.5) such that $\tilde{P} = \epsilon P$, we obtain

$$(e^{-\tilde{P}_4} - \tilde{c})\tilde{P}_4^2 = \epsilon^2 \tilde{d}, \quad (3.10)$$

where P_4 is the P -component of the plant-dominating equilibrium e_4 . We now seek solutions in the form of an asymptotic expansion

$$\tilde{P}_4 = \tilde{P}_{4,0} + \epsilon \tilde{P}_{4,1} + \epsilon^2 \tilde{P}_{4,2} + \mathcal{O}(\epsilon^3),$$

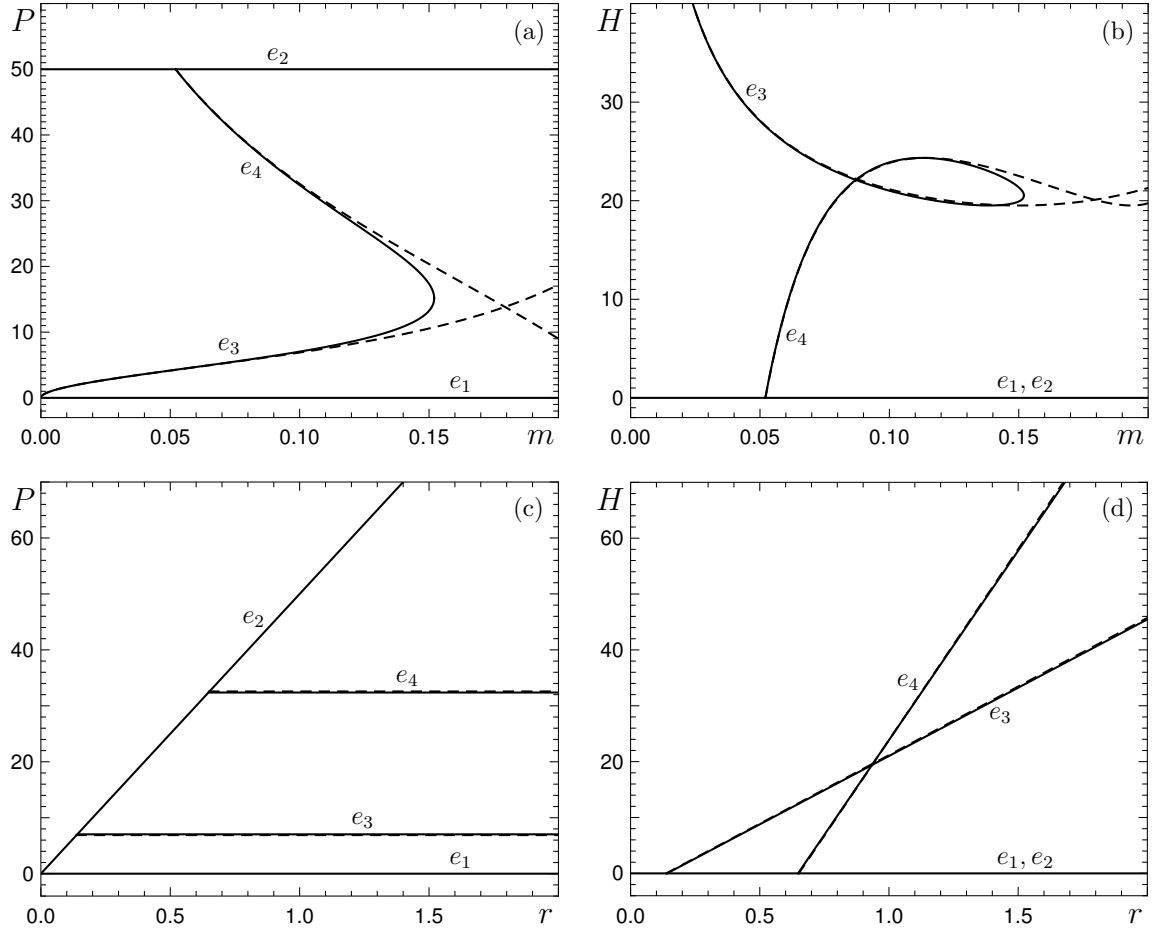


Figure 3.1 P and H components of the herbivore-dominating equilibrium e_3 and the plant-dominating equilibrium e_4 obtained from (solid curve) numerically solving Eqs. (3.3)–(3.4), and from (dashed curves) first-order asymptotic approximations. Panels (a) and (b) show the dependence on m for fixed $r = 1$, and panels (c) and (d) show the dependence on r for fixed $m = 0.1$; e_1 and e_2 are included for reference. $b = b_c = 0.02$, see Table 1 for other parameter values.

and successively equate for different powers of ϵ to give

$$\tilde{P}_4 = \ln \frac{1}{\tilde{c}} - \epsilon^2 \frac{\tilde{d}}{\tilde{c}(\ln \frac{1}{\tilde{c}})^2} + \mathcal{O}(\epsilon^3),$$

Thus,

$$P_4 = \frac{1}{\epsilon} \left(\ln \frac{1}{\tilde{c}} - \epsilon^2 \frac{\tilde{d}}{\tilde{c}(\ln \frac{1}{\tilde{c}})^2} + \mathcal{O}(\epsilon^3) \right),$$

and with original parameters

$$P_4 = \frac{\ln(E c_{max}/m)}{b + b_c} - \frac{a^2(b + b_c)}{(\ln(E c_{max}/m))^2} + \mathcal{O}((b + b_c)^2), \quad (3.11)$$

and

$$e_4 = \left(\frac{\ln(E c_{max}/m)}{b + b_c} + \mathcal{O}(b + b_c), \frac{(r - CP_4)(P_4^2 + a^2)}{c_{max} P_4 e^{-b_c P_4}} \right). \quad (3.12)$$

The solid curves in Fig. 3.1 show the numerically computed e_3 and e_4 , and the dashed curves show the first-order approximations for e_3 and e_4 using the P -formulas (3.8) and (3.11) with $\mathcal{O}((b + b_c)^2) = 0$, and the H -formula (3.3). The main advantage of the closed-form approximations is the information about the dependence of the equilibrium positions on the system parameters. In particular, the effect of the nonlinearity parameter $b + b_c$ can now be discussed in qualitative terms. First of all, R-tipping from the herbivore-dominating equilibrium e_3 to the plant-only equilibrium e_2 requires that both equilibria are stable for the same parameter settings (bistability between e_2 and e_3). This, in turn, requires one additional equilibrium e_4 that ‘separates’ the two stable equilibria e_2 and e_3 . It is clear from Eq. (3.11) that non-monotone herbivore growth due to $b + b_c > 0$ together with $m > 0$ are both necessary for the additional equilibrium e_4 to exist.

Figure 3.1 shows that e_3 and e_4 may become degenerate with each other, or each of them may become degenerate with e_2 . These degeneracies are indicative of transcritical and saddle-node bifurcations.

3.1.3 Degeneracy of e_2 with e_3 or e_4 via a transcritical bifurcation

Equilibrium e_3 or e_4 becomes degenerate with e_2 in a transverse crossing if $P = r/C$ in Eqs. (3.3)–(3.4). Thus, substituting $P = r/C$ into Eq. (3.4) defines a curve of transcritical bifurcations in the (r, m) parameter plane

$$T = \left\{ (r, m) : m = \frac{E c_{max} e^{-(b+b_c)r/C}}{(a C/r)^2 + 1} \right\}. \quad (3.13)$$

If $b + b_c = 0$, curve T emerges from the origin and levels off at $m = E c_{max}$ for large r [Fig.3.2(a)]. Equilibrium e_3 that bifurcates from e_2 exists below T . If $b + b_c > 0$, curve T emerges from the origin, has a maximum ST , and approaches $m = 0$ from above for large r [Fig.3.2(b)]. Now, T consists of two different branches separated by ST .

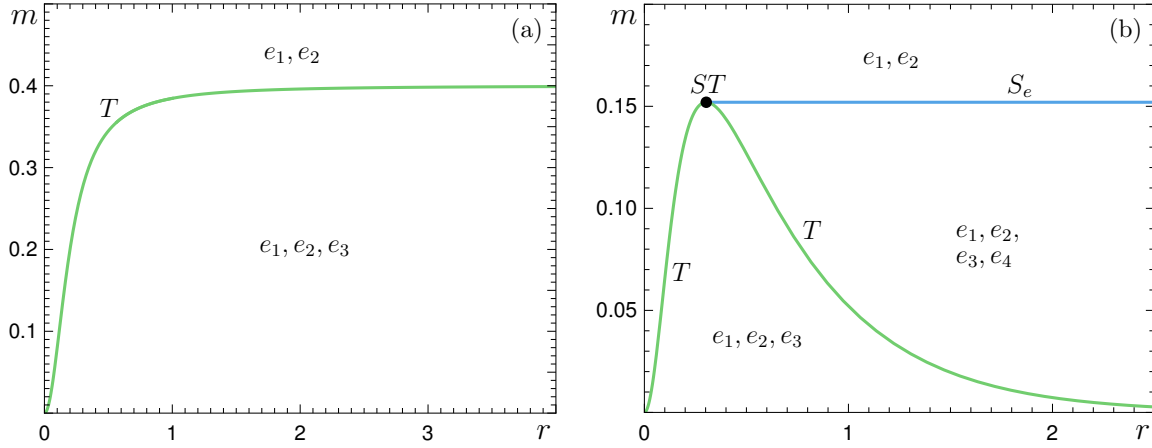


Figure 3.2 Two-parameter bifurcation diagrams obtained analytically for (a) $b = b_c = 0$ and (b) $b + b_c = 0.04$, showing the curve T of transcritical bifurcations, and the half-line S_e of saddle-node bifurcations. S_e emerges from a codimension-2 saddle-node-transcritical bifurcation point ST . T and S_e divide the (r, m) -parameter plane into regions with different numbers of equilibria. See Table 1 for other parameter values.

Equilibrium e_3 , that bifurcates from e_2 along the left-hand branch of T , exists below the left-hand branch of T . In contrast, equilibrium e_4 , that bifurcates from e_2 along the right-hand branch of T , exists above the right-hand branch of T .

3.1.4 Degeneracy of e_3 with e_4 in a saddle-node bifurcation

Equilibria e_3 and e_4 become degenerate in a quadratic tangency when r -independent Eq. (3.4) has a non-negative double root, meaning that

$$h(P) = \frac{dh}{dP} = 0 \quad \text{and} \quad \frac{d^2h}{dP^2} \neq 0 \quad \text{for} \quad P \geq 0, \quad (3.14)$$

and the corresponding H from Eq.(3.3) is non-negative, meaning that

$$r \geq CP. \quad (3.15)$$

Conditions (3.14) give the cubic equation for P :

$$q(P) = (b + b_c)P^3 + a^2(b + b_c)P - 2a^2 = 0. \quad (3.16)$$

A positive root of $q(P)$ is used in Eq. (3.4) to determine the value of m at which e_3 and e_4 become degenerate

$$m = \frac{E c_{max} e^{-(b+b_c)P}}{(a/P)^2 + 1}, \quad (3.17)$$

and in Eq. (3.15) to determine the corresponding range of r where e_3 and e_4 become degenerate. Thus, conditions (3.15)–(3.17) define a half-line of saddle-node bifurcations of equilibria in the (r, m) parameter plane

$$S_e = \left\{ (r, m) : q(P) = 0, r \geq CP \text{ and } m = \frac{E c_{max} e^{-(b+b_c)P}}{(a/P)^2 + 1} \right\}. \quad (3.18)$$

If $b + b_c = 0$, then $q(P)$ has no roots, meaning that there is no saddle-node bifurcation of equilibria [Fig.3.2(a)]. If $b + b_c > 0$, $q(P)$ is negative for $P = 0$, monotonically increasing, and positive for P large enough, meaning that Eq. (3.16) has a unique positive root. This root corresponds to a unique saddle-node half-line S_e . Equilibria e_3 and e_4 exist below S_e , and become degenerate and disappear along S_e .

What is more, conditions (3.13) and (3.18) become identical when $r = CP$, meaning that the end of the half-line S_e lies on T . Indeed, Fig.3.2(b) shows that S_e emerges from the special saddle-node-transcritical bifurcation point ST , where all three equilibria e_2 , e_3 and e_4 become degenerate.

3.2 Stability and Bifurcation Analysis

Linear stability of equilibria can be determined from the eigenvalues of the Jacobian matrix

$$J = \begin{pmatrix} r - 2CP - Hg'(P) & -g(P) \\ Ee^{-bP}(g'(P) - bg(P))H & Eg(P)e^{-bP} - m \end{pmatrix}, \quad (3.19)$$

where

$$g'(P) = -g(P) \left(\frac{2P}{P^2 + a^2} - \frac{2}{P} + b_c \right).$$

At the trivial equilibrium $e_1 = (0, 0)$, the Jacobian matrix has eigenvalues $\lambda_1 = r > 0$ and $\lambda_2 = -m < 0$, meaning that e_1 is always a saddle. At the plant-only equilibrium $e_2 = (r/C, 0)$, the Jacobian matrix has eigenvalues $\lambda_1 = -r < 0$ and $\lambda_2 = Ec_{max} e^{-(b+b_c)r/C} / ((aC/r)^2 + 1) - m$. Hence, e_2 is a stable node when $\lambda_2 < 0$, a saddle when $\lambda_2 > 0$, and undergoes a transcritical bifurcation whenever $\lambda_2 = 0$. (One can verify that $\lambda_2 = 0$ gives the transcritical bifurcation condition (3.13).) Stability

and bifurcations of the herbivore-dominating equilibrium e_3 and the plant-dominating equilibrium e_4 are obtained numerically. The Jacobian matrix shows that the stability of e_3 and e_4 depends on the nonlinearity parameter $b+b_c$ as well as on b_c alone, meaning that it needs to be discussed with dependence on b and b_c individually.

To showcase different types of dynamics and bifurcations in the autonomous ecosystem (2.1)–(2.2), we plot one-dimensional bifurcation diagrams in Fig. 3.3 for two types of parameter paths. In the left column we fix r and consider a range of $m \in (0, 0.2]$. In the right column we fix m and consider a range of $r \in (0, 2]$. In addition to the transcritical T and saddle-node S_e bifurcations of equilibria identified in the previous section, there are sub- and super-critical Hopf bifurcations H_e . In a super-critical Hopf bifurcation, a stable equilibrium turns unstable and gives rise to a stable limit cycle [Fig. 3.3(e)–(f)]. Thus, this bifurcation is safe or non-dangerous. In a sub-critical Hopf bifurcation, an unstable limit cycle shrinks onto a stable equilibrium and the equilibrium becomes unstable [Fig. 3.3(b)–(c)]. The sub-critical Hopf bifurcation is dangerous because it gives rise to a discontinuity in the attracting set that is the branch of stable equilibria. What is more, a limit cycle can connect to the saddle equilibrium e_4 and disappear in a homoclinic bifurcation h [Fig. 3.3(b)–(c) or (e)–(f)]. For more details and background on bifurcation theory, we refer to [34].

To provide a systematic bifurcation analysis, we obtain two-dimensional (r, m) bifurcation diagrams for different but fixed values of b and b_c [Fig. 3.4]. We plot codimension-one super-critical bifurcations as solid curves and sub-critical bifurcations as dashed curves. Along a solid (dashed) transcritical bifurcation, the bifurcating branch of equilibria is stable (of saddle type); along a solid (dashed) saddle-node bifurcation, a saddle collides with an attractor (repellor); and along a solid (dashed) Hopf and homoclinic bifurcations, the bifurcating limit cycle is attracting (repelling). Transcritical and saddle-node bifurcations of equilibria are obtained using conditions (3.13) and (3.18), respectively. Hopf, homoclinic and saddle-node bifurcations of limit cycles are computed using the numerical continuation software AUTO [50]. For each bifurcation diagram, we identify regions with qualitatively different dynamics and illustrate these dynamics with examples of phase portraits in the (P, H) phase plane [Fig. 3.5]. It turns out that there are at least four qualitatively different (r, m) bifurcation diagrams, depending on the settings of b and b_c .

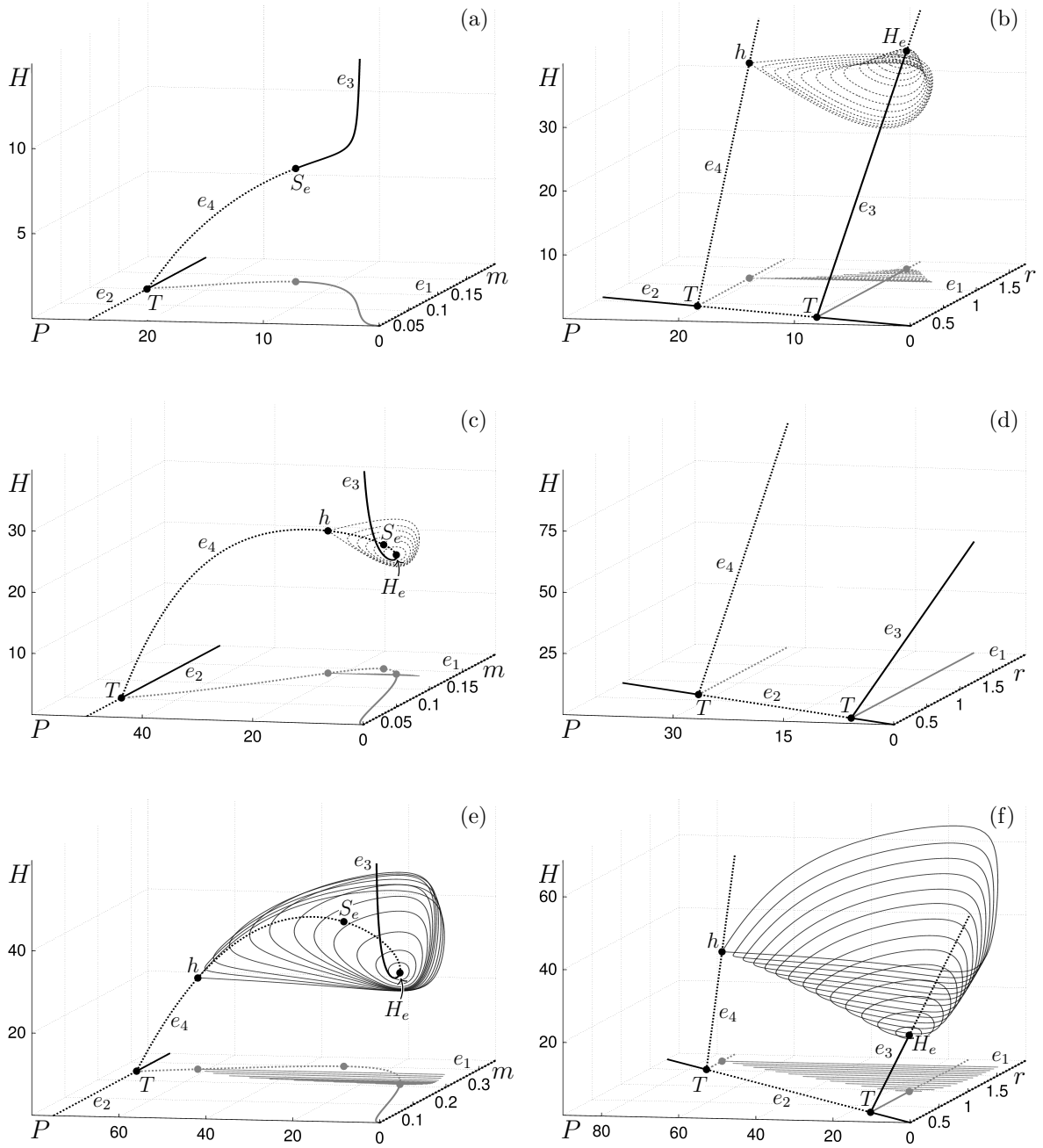


Table 3.1 Glossary of terms for bifurcation diagrams.

Symbol	Description
T	Transcritical bifurcation
S_e	Saddle-node of equilibria bifurcation
ST	Saddle-node-transcritical bifurcation
H_e	Hopf bifurcation
h	Homoclinic bifurcation
$BT_{I(II)}$	Bogdanov-Takens type I(II) bifurcation
GH	Generalised Hopf (Bautin) bifurcation
S_{lc}	Saddle-node of limit cycles bifurcation
h_{res}	Resonant homoclinic bifurcation
BI	Basin Instability

In the absence of the key nonlinearity, that is when $b + b_c = 0$, there are just two bifurcation curves: curve T of super-critical transcritical bifurcations, and curve H_e of super-critical Hopf bifurcations [Fig. 3.4(a)]. These two curves do not interact, and separate the (r, m) parameter plane into three distinct regions with qualitatively different dynamics [Fig. 3.5, 1–3]. In particular, H_e gives rise to a stable limit cycle in region 3, which represents stable but oscillatory coexistence between plants and herbivores. These simple dynamics change drastically in the presence of the key nonlinearity.

When $b + b_c$ becomes small but non-zero, a number of qualitative changes take place in the bifurcation diagram, which is expected from the singular perturbation nature of the problem. There are three additional co-dimension one bifurcation curves, and four special codimension-two bifurcation points [Fig. 3.4(b)]. Understanding the new bifurcation diagram is reminiscent of assembling a jigsaw-puzzle. Firstly, a half-line S_e of saddle-node bifurcations of equilibria appears. S_e emerges from the saddle-node-transcritical bifurcation point ST on T , where T changes from super- to sub-critical. Secondly, H_e is no longer unbounded at both ends, but emerges from the Bogdanov-Takens bifurcation point BT_I on S_e , where S_e changes from super- to sub-critical. There are two possible types of Bogdanov-Takens bifurcation, and BT_I is type-I according to the classification in [34, Sec.8.4]. It is known from the unfolding of a Bogdanov-Takens bifurcation that an additional bifurcation curve, namely the curve of homoclinic bifurcations h , must emerge from BT_I . Along h , the limit cycle originating from H_e becomes a connecting orbit to the saddle equilibrium e_3 and disappears [Fig. 3.5 h]. Thirdly, there is a generalised Hopf (or Bautin) bifurcation point GH on

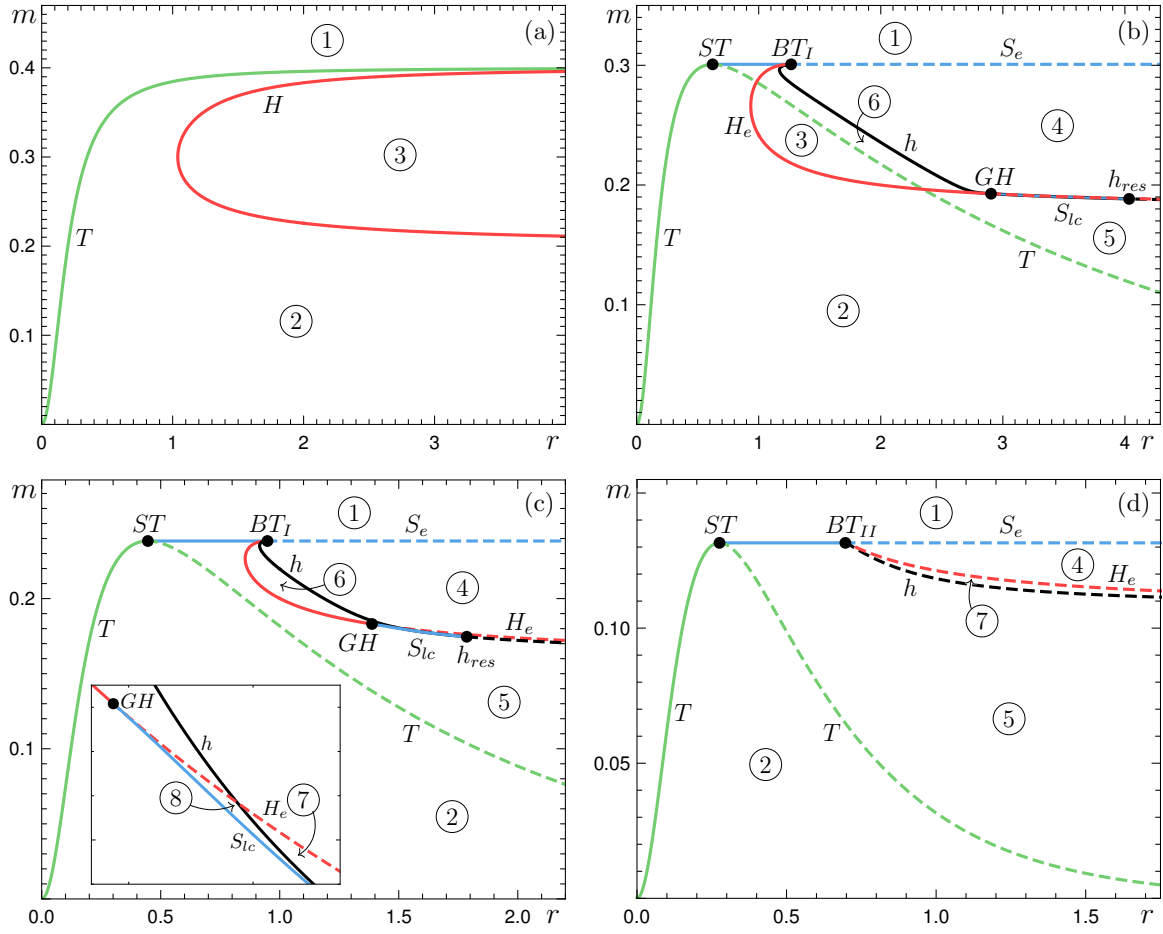


Figure 3.4 Examples of four qualitatively different (r, m) -bifurcation diagrams obtained for different but fixed $(b, b_c) =$ (a) $(0, 0)$, (b) $(0.001, 0.005)$, (c) $(0.005, 0.01)$, (d) $(0.025, 0.025)$. Super-critical (sub-critical) bifurcations are plotted as solid (dashed) curves. For the labelling of different bifurcations See Table 3.1.

H_e , where H_e changes from super- to sub-critical [34, Sec.8.3]. It is known from the unfolding of a generalised Hopf bifurcation that an additional bifurcation curve, namely the curve of saddle-node of limit cycles S_{lc} , must emerge from GH . Along S_{lc} , two limit cycles of which one is stable and the other repelling collide and disappear. Finally, S_{lc} terminates on h at a resonant homoclinic bifurcation point h_{res} , where h changes from super- to sub-critical. This new bifurcation structure has five additional regions 4–8 with qualitatively different dynamics; for regions 7–8 see the inset in Fig. 3.4(c). We would like to point out the appearance of adjacent regions 5 and 7 with bistability between the plant-only equilibrium e_2 and the herbivore-dominating equilibrium e_3 . When the combination of b and b_c is increased further, bifurcation points GH and

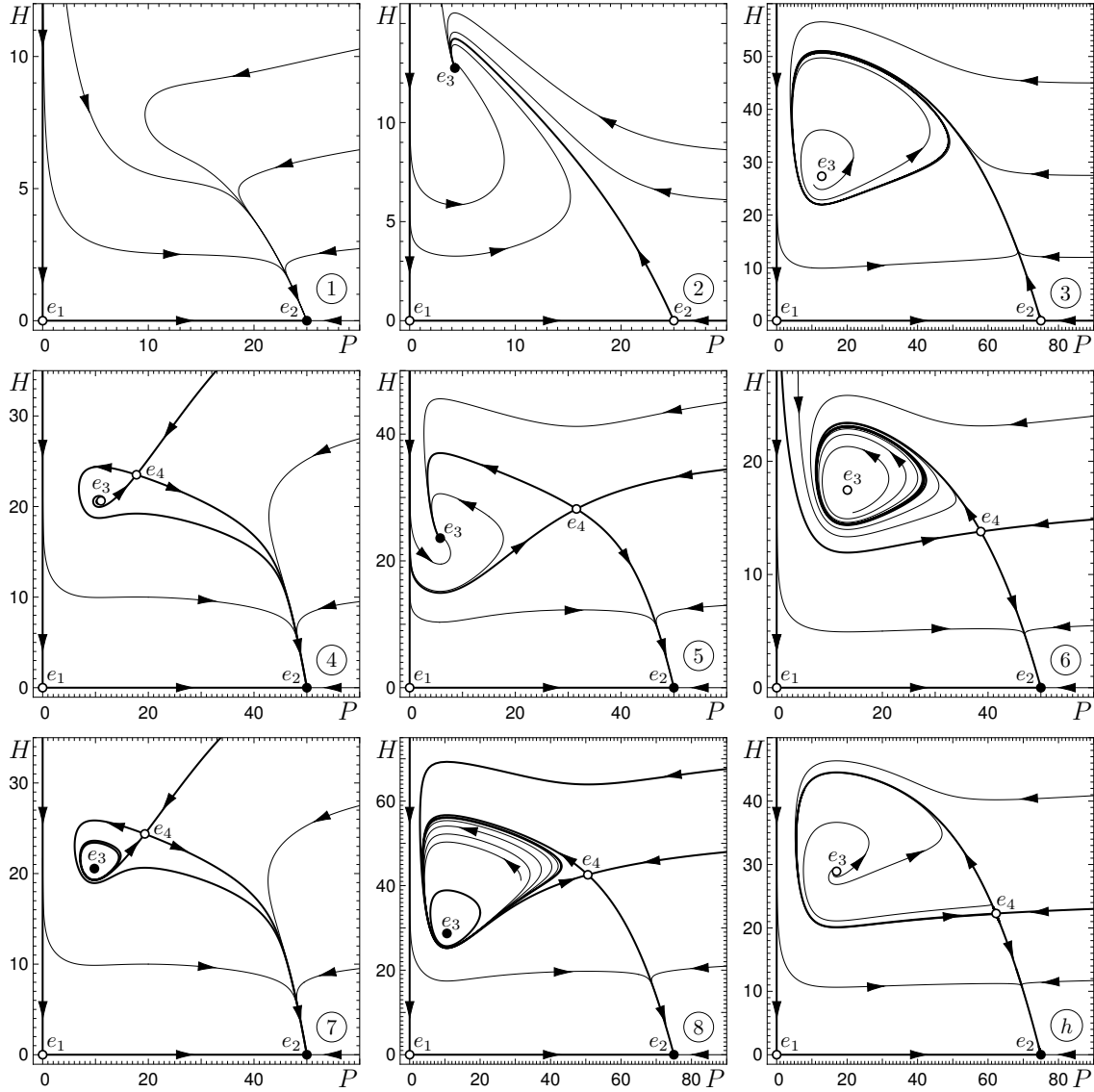


Figure 3.5 Examples of qualitatively different (P, H) -phase portraits for the autonomous system (2.1)–(2.2) showing (filled circles) stable equilibria, (open circles) unstable equilibria, (thick curves) limit cycles and stable/unstable invariant manifolds of saddle equilibria, and (thin curves) examples of typical trajectories. Note the stable limit cycle in regions 3 and 6, the unstable limit cycle in region 7, and two limit cycles in region 8. See Table 3.2 for parameter values.

h_{res} approach BT_I [Fig. 3.4(c)]. In the process, region 3 with stable self-sustained oscillations disappears, while bistable region 5 becomes noticeably larger. Then, there are special combinations of b and b_c , where GH and h_{res} collide simultaneously with BT_I and disappear in a codimension-three degenerate Bogdanov-Takens bifurcation

Table 3.2 Parameter values chosen for phase portraits in Fig.3.5

Phase Portrait	r	m	b	b_c
1	0.5	0.14	0.025	0.025
2	0.5	0.05	0.025	0.025
3	1.5	0.23	0.001	0.005
4	1	0.125	0.025	0.025
5	1	0.075	0.025	0.025
6	1	0.21	0.005	0.01
7	1	0.12	0.025	0.025
8	1.5	0.18025	0.005	0.01
h	1.5	0.2684	0.001	0.005

(not shown in the figure). This collision eliminates S_{lc} together with the super-critical part of H_e and regions 6 and 8. What is more, the Bogdanov-Takens bifurcation point changes to type II. The difference from BT_I is that H_e and h emerging from BT_{II} swap their relative positions and become sub-critical [Fig. 3.4(d)].

Past the special combination of b and b_c , there are four bifurcation curves, including the two dangerous bifurcations of equilibria that are of interest for B-tipping: the (solid) half-line S_e of super-critical saddle-node bifurcations, and the (dashed) curve H_e of sub-critical Hopf bifurcations. Additionally, there are two special bifurcation points, one of which is the type-II Bogdanov-Takens bifurcation point BT_{II} . H_e gives rise to a repelling limit cycle in region 7, which becomes a connecting orbit to the saddle equilibrium e_4 and disappears in a homoclinic bifurcation along h . Finally, a substantial part of the diagram is occupied by adjacent bistable regions 5 and 7. In these two regions, the plant-only equilibrium e_2 and the herbivore-dominating equilibrium e_3 are both stable, which is of interest for R-tipping from e_3 to e_2 studied in the next chapter.

3.2.1 Type I and II Bogdanov-Takens Bifurcation

The aim of this section is to illustrate the two types of codimension-two Bogdanov-Takens bifurcation using its normal form. There are different versions of the normal form, for example compare the version derived by Takens [34, Sec.8.4 Eq.(8.49)] and Bogdanov [34, Sec.8.4 Eq.(8.48)]. Here, we follow Ref. [34] and use the following form

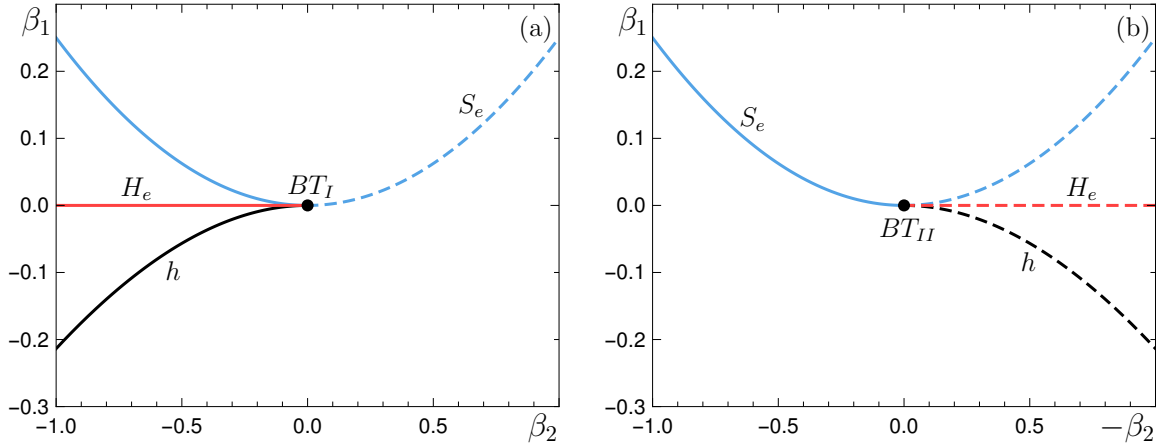


Figure 3.6 Two qualitatively different $(\pm\beta_2, \beta_1)$ -bifurcation diagrams using the normal form equations for a Bogdanov-Takens bifurcation Eq.(3.20)–(3.21) with (a) codimension-2 type I Bogdanov-Takens bifurcation point BT_I , $s = -1$. (b) codimension-2 type II Bogdanov-Takens bifurcation point BT_{II} , $s = +1$ and $\beta_2 \mapsto -\beta_2$. Supercritical (subcritical) bifurcations are plotted as solid (dashed) curves. For the labelling of different bifurcations See Table 3.1.

of the normal form [34, Sec.8.4]:

$$\dot{\eta}_1 = \eta_2, \quad (3.20)$$

$$\dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 + s\eta_1\eta_2, \quad (3.21)$$

where η_1 and η_2 are the state variables, β_1 and β_2 are the bifurcation parameters and $s = \pm 1$ is the unfolding parameter.

Existence of Equilibria

Equilibrium solutions of Eqs.(3.20)–(3.21) are a pair of (η_1^*, η_2^*) which satisfy the following conditions:

$$\eta_2^* = 0, \quad (3.22)$$

$$\beta_1 + \beta_2\eta_1^* + \eta_1^{*2} + s\eta_1^*\eta_2^* = 0, \quad (3.23)$$

these conditions become

$$f(\eta_1) = \beta_1 + \beta_2\eta_1^* + \eta_1^{*2} = 0, \quad (3.24)$$

giving $e_1 = \left(\frac{-\beta_2}{2} + \sqrt{\frac{\beta_2^2}{4} - \beta_1}, 0 \right)$ and $e_2 = \left(\frac{-\beta_2}{2} - \sqrt{\frac{\beta_2^2}{4} - \beta_1}, 0 \right)$.

Condition for Degeneracy of e_1 and e_2 – Saddle-Node Bifurcation

A saddle-node bifurcation occurs when Eq.(3.24) has a double root, that is when

$$f(\eta_1) = 0 \text{ and } \frac{df}{d\eta_1} = 0, \quad (3.25)$$

this gives the condition

$$S_e = \{(\beta_1, \beta_2) : 4\beta_1 - \beta_2^2 = 0\}, \quad (3.26)$$

a parabola with its minimum at the origin.

Linear Stability of Equilibria

Linear stability analysis, valid sufficiently close to equilibria (η_1^*, η_2^*) is determined by the eigenvalues of the Jacobian matrix of Eqs.(3.20)–(3.21)

$$J|_{(\eta_1^*, \eta_2^*)} = \begin{pmatrix} 0 & 1 \\ \beta_2 + 2\eta_1^* & s\eta_1^* \end{pmatrix}, \quad (3.27)$$

which are

$$\lambda_{1,2} = \frac{s\eta_1^*}{2} \pm \sqrt{\frac{\eta_1^{*2}}{4} + \beta_2 + 2\eta_1^*}. \quad (3.28)$$

The unfolding parameter $s = \pm 1$ has a clear purpose. Its location in the real part of Eq.(3.28) means that going from $s = -1$ to $s = +1$ will have the effect of changing the stability of the equilibrium solution (η_1^*, η_2^*) . This is identical to the effect that would be achieved by the substitution $t \mapsto -t$.

Condition for Hopf Bifurcation

A Hopf bifurcation requires imaginary eigenvalues, applying this consideration to Eq.(3.28) gives the condition

$$H_e = \{(\beta_1, \beta_2) : \beta_1 = 0, \beta_2 < 0\}, \quad (3.29)$$

a half-line extending from the origin along the negative β_2 -axis.

Figure 3.6 illustrates both (a) type I ($s = -1$) and (b) type II ($s = +1$), codimension-2 Bogdanov-Takens bifurcations. Both the saddle-node of equilibria bifurcation curves S_e and the Hopf bifurcation H_e curves are obtained analytically while the homoclinic bifurcation h curves are computed using the numerical continuation software AUTO [50]. For $s = -1$ we observe the orientation and supercriticality of the Hopf and homoclinic curves (this is comparable to Fig. 3.4(b)–(c)). When $s = +1$, the Hopf and homoclinic curves switch their relative position and change their criticality, while the two parts of the saddle-node bifurcation curve change their criticality. To maintain the same orientation of the bifurcation curves as in Fig. 3.4(d), we plot β_1 vs. $-\beta_2$ in Fig. 3.6(b).

Codimension-Three Degenerate Bogdanov-Takens Bifurcation

To conclude the bifurcation analysis, we quantify in Fig. 3.7 “the special combinations of b and b_c ” that give rise to a codimension-three degenerate Bogdanov-Takens bifurcation. In the (b, b_c) parameter plane, these “special combinations” lie on a curve which separates the regions with type I and type II Bogdanov-Takens bifurcation points. In other words, Fig. 3.7 shows the projection of a codimension-three bifurcation curve from the four-dimensional (r, m, b, b_c) parameter space onto the (b, b_c) parameter plane. Points labeled (a)–(d) in Fig. 3.7 refer to the values of b and b_c chosen for the (r, m) bifurcation diagrams in Fig. 3.4. The asterisk indicates the values of b and b_c used in Ref. [7].

3.3 Summary of B-tipping: Simple Criteria and Robustness

In summary, the classical bifurcation analysis describes *quasistatic effects* of the plant growth rate r , the herbivore mortality rate m , and the decline in herbivore growth at high plant biomass quantified by the nonlinearity parameter $b + b_c$. When $b + b_c = 0$, we do not expect any B-tipping owing to the lack of dangerous bifurcations. However, when $b + b_c > 0$, meaning that there is a decline in herbivore growth at high plant biomass, a number of different critical transitions appear in the ecosystem. The two most dominant are the generic dangerous bifurcations of equilibria, namely super-critical saddle-node and sub-critical Hopf bifurcations. Additionally, but less importantly for this study, there is a sub-critical transcritical bifurcation [Fig. 3.3(b), (d) and (f)], a super-critical saddle-node bifurcation of limit cycles [e.g. see the inset in Fig. 3.4(c)],

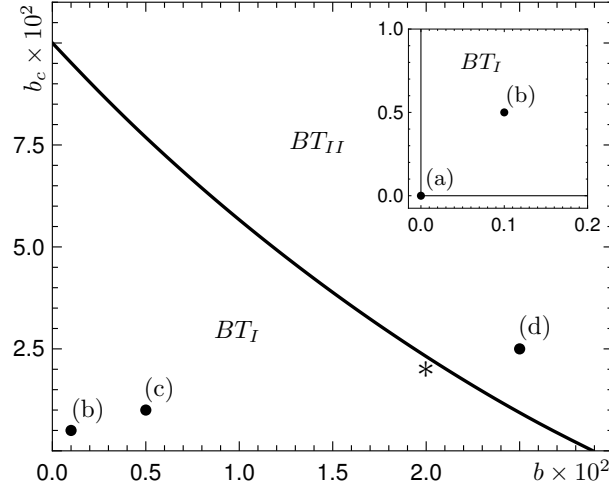


Figure 3.7 Projection of codimension-three degenerate Bogdanov-Takens bifurcation curve onto the (b, b_c) -parameter plane. Also shown are the (b, b_c) pairs used for generating diagrams (a)–(d) in Fig.3.4. The asterisk indicates the values of b and b_c used in Ref. [7].

and a super-critical homoclinic bifurcation [Fig. 3.3(e)–(f)]. To identify B-tipping in the non-autonomous system (2.7)–(2.8), it is sufficient to consider a (r, m) bifurcation diagram for the autonomous system (2.1)–(2.2) with a prescribed parameter path.

3.3.1 Testable criterion for B-tipping

If a parameter path in a (r, m) bifurcation diagram crosses one of the dangerous bifurcations, then there is a time-varying external input $\Lambda(t) = (r(t), m(t))$ that traces out this path and gives rise to B-tipping.

Figure 3.8 shows an example of such a path, denoted with Δ_m in panel (a), together with the dynamics of the non-autonomous system (2.7)–(2.8), where $m(t)$ drifts slowly along the path [panel (b)]. If the system starts near the stable equilibrium e_3 at the lower end p_1 of the path and $m(t)$ increases over time, then the non-autonomous system tracks the moving stable equilibrium $e_3(t)$ up to the point of the dangerous bifurcation S_e [Figure 3.8(b)]. As $m(t)$ passes through the bifurcation, which defines the *critical level* of m , the system undergoes a sudden and abrupt transition to the other stable equilibrium $e_2(t)$. Such an instability is also described as a *dynamic* or *adiabatic bifurcation* [53, 54]. To discuss robustness of B-tipping, we can invoke the notions of genericity and co-dimension of a bifurcation [34, Sec.2.5]. Specifically, B-tipping due to generic co-dimension one bifurcations is unaffected by small perturbations to the

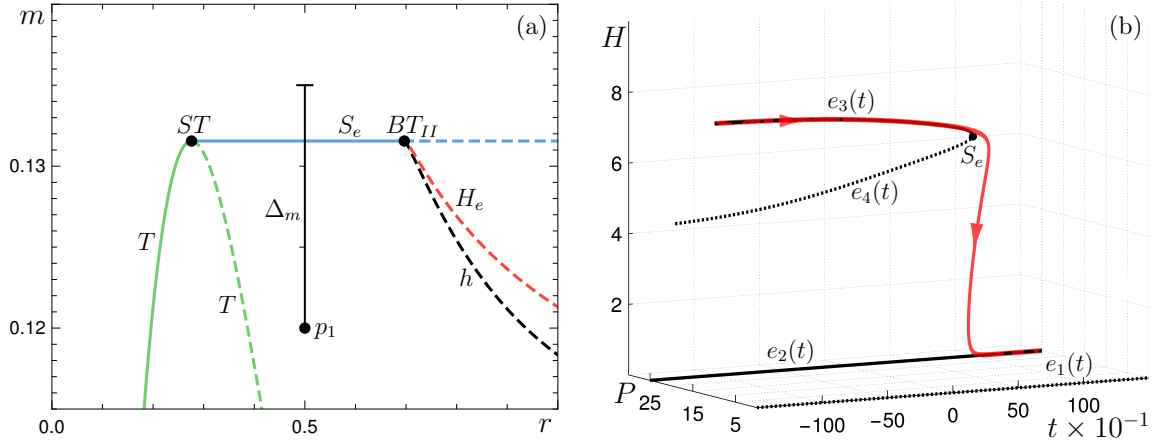


Figure 3.8 (a) Example of a parameter path that crosses a saddle-node bifurcation of equilibria S_e in the (r, m) -bifurcation diagram. (b) As $m(t)$ is increased from $p_1 = (0.5, 0.12)$ along the path, the non-autonomous system initially follows the moving stable equilibrium $e_3(t)$, but then undergoes B-tipping from $e_3(t)$ to $e_2(t)$ as $m(t)$ passes through S_e . $b = b_c = 0.025$, and $m(t) = 0.12 + 0.015(\tanh(10^{-3}t) + 1)/2$. The moving equilibria are obtained for $\varepsilon = 10^{-3}$.

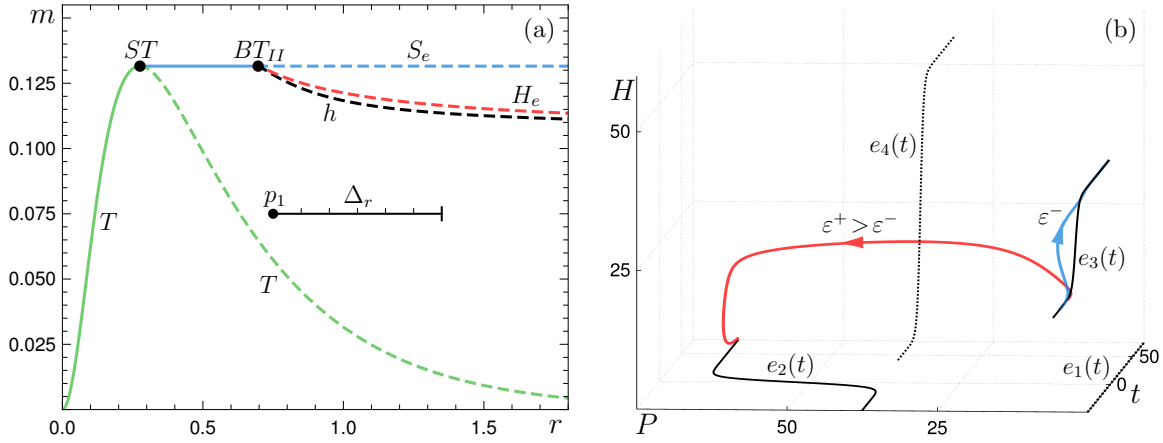


Figure 3.9 (a) Example of a parameter path Δ_r that does not cross any bifurcation curves in the (r, m) -bifurcation diagram. (b) As $r(t)$ is increased from $p_1 = (0.75, 0.075)$ along the path at a rate ε^- , (blue trajectory) the non-autonomous system tracks the moving stable equilibrium $e_3(t)$. However, for a faster rate $\varepsilon^+ > \varepsilon^-$, (red trajectory) there is irreversible R-tipping from $e_3(t)$ to $e_2(t)$ even though $e_3(t)$ never disappears or loses stability. $b = b_c = 0.025$, and $r(t) = 0.75 + 0.6(\tanh(\varepsilon t) + 1)/2$, with $\varepsilon^- = 0.1$ and $\varepsilon^+ = 0.2$. The moving equilibria are obtained for $\varepsilon \approx 0.14$.

system or to the parameter path. We refer to [55] for a more general and precise definition of B-tipping, and for rigorous criteria for B-tipping.

Chapter 4

Irreversible R-Tipping: Beyond Classical Bifurcations

In this chapter, we consider genuine non-autonomous instabilities that arise solely from the time variation of the input parameters r and m , and cannot be captured by classical bifurcation analysis. Specifically, we ask : *Are there parameter paths in the (r, m) bifurcation diagram that do not cross any bifurcations of the stable equilibrium e_3 but give rise to tipping?* The answer is yes. This was demonstrated in [7], and is shown here more explicitly. Consider the (r, m) bifurcation diagram with a parameter path Δ_r that does not cross any bifurcations in Figure 3.9(a). If the non-autonomous system starts at the stable equilibrium e_3 near the lower end p_1 of the path, and $r(t)$ increases slowly enough along the path, then the non-autonomous system is able to adapt to the changing environment and track the moving stable equilibrium $e_3(t)$ along the entire path [blue trajectory in Fig. 3.9(b)]. However, if $r(t)$ increases slowly but faster than some critical rate, the non-autonomous system fails to adapt to the changing environment and undergoes a critical transition from $e_3(t)$ to the other stable equilibrium $e_2(t)$ [red trajectory in Fig. 3.9(b)]. This happens even though $e_3(t)$ never loses stability along the path. Such instability is called *irreversible R-tipping*.¹

¹This is in contrast to the transient phenomenon of *reversible R-tipping*, where the system fails to track the moving stable state, suddenly moves to a different state, but then returns to and tracks the original stable state [25, 27].

4.1 The Vicious Cycle

Intuitively, R-tipping is the result of a *vicious cycle* that could potentially tip the system to a different state if the input parameters vary too fast. In the ecosystem model, the vicious cycle arises from the key nonlinearity identified in Chapter 2, that is from non-monotone herbivore growth (2.4) that changes sign from positive to negative at high plant biomass $P = P_4$ [see Fig.2.1(b)].

The effect can be understood as follows. Consider a stable herbivore population with a lower than optimal plant biomass P_3 for some $r = r_-$. Then, consider a smooth increase in the plant growth rate from r_- to r_+ . The increase results in faster growing plants, and moves the stable equilibrium to a larger herbivore population with the same plant biomass P_3 . If $r(t)$ increases slowly enough, herbivores manage to graze and grow fast enough so that the larger herbivore population is able to maintain the same plant biomass at larger $r = r_+$. However, if $r(t)$ increases too fast, herbivores may be unable to keep up and prevent the plant biomass from growing past its optimal value P_{opt} . This, in turn, triggers the vicious cycle: past the optimal plant biomass, the larger the plant biomass gets, the less the herbivores graze and grow. The ultimate result is negative net herbivore growth causing a sudden collapse of the herbivore population. This is accompanied by a sudden increase in the plant biomass to P_4 . Even though there is no classical bifurcation along the parameter path between r_- and r_+ , the rate of change of $r(t)$ alone prevents the system from adapting to the modified stable equilibrium. In the proceeding sections, we perform a systematic mathematical analysis of the vicious cycle mechanism that gives rise to irreversible R-tipping as shown in Fig. 3.9(b).

4.2 Simple Criteria for Irreversible R-tipping

It turns out that, similarly to B-tipping, much can be understood about irreversible R-tipping in the non-autonomous system (2.7)–(2.8) from certain properties of the corresponding autonomous system (2.1)–(2.2). The difference is that R-tipping is related to global, rather than local, properties of the stable state (an attractor). Specifically, we need the following ingredients to give testable criteria for irreversible R-tipping:

- (i) A *stable base equilibrium* $e(p)$ whose position depends on the input parameter(s) p . For example, stable equilibrium e_3 for the ecosystem model can be given in terms of r and m by the asymptotic expansion formula (3.9).
- (ii) *Bistability or multistability* - at least one additional attractor a that coexists with e for the same setting of the input parameters. For example, there is bistability between e_3 and e_2 in the (r, m) parameter regions 5 and 7.
- (iii) A continuous *parameter path* Δ in the space of the input parameters, that does not cross any dangerous bifurcations of the base equilibrium e . For example, the horizontal path Δ_r in the (r, m) bifurcation diagram from Fig. 4.1(a) does not cross any bifurcations.
- (iv) *The basin of attraction* of the base equilibrium, defined as the set of all initial states (P_0, H_0) that converge to the stable base equilibrium $e(p)$ in time

$$B(e, p) = \{(P_0, H_0) \in \mathbb{R}^2 : (P(t), H(t)) \rightarrow e(p), t \rightarrow +\infty\},$$

together with the evolution of $B(e, p)$ along the chosen parameter path. For example, Fig. 4.1(b)–(d) shows the (blue shaded) basin of attraction of e_3 for three different settings of r along the parameter path Δ_r . The boundary between the basins of attraction of e_3 and e_2 is given by the stable invariant manifold of the saddle equilibrium e_4 .

- (v) *Basin instability on a path*. Let $\overline{B(e, p)}$ denote the basin of attraction of $e(p)$ together with its basin boundary. Then, we say that the stable base equilibrium $e(p)$ is basin unstable on a parameter path Δ if there are two points on the path, $p_1, p_2 \in \Delta$, such that $e(p_1)$ is outside the basin of attraction of $e(p_2)$:

$$e(p_1) \notin \overline{B(e, p_2)}.$$

For example, consider Δ_r from Fig. 4.1(a) and pick r_- . The stable equilibrium $e_3(r_-)$ is contained within the basin of attraction of $e_3(r)$ for all $r_- < r < r_*$, lies on the basin boundary of $e_3(r_*)$ [Fig. 4.1(c)], and is outside the basin of attraction of $e_3(r)$ for all $r_* < r \leq r_+$ [Fig. 4.1(d)]. Thus, e_3 is basin unstable on the path Δ_r because, for $r_1 = r_-$ and any $r_2 \in (r_*, r_+]$, we have that $e_3(r_1) \notin \overline{B(e_3, r_2)}$.

Testable criterion for irreversible R-tipping. If a stable equilibrium $e(p)$ is basin unstable on a parameter path Δ , meaning that there are $p_1, p_2 \in \Delta$ such that $e(p_1) \notin$

$\overline{B(e, p_2)}$, then there is an external input $\Lambda(t) = (r(t), m(t))$ that traces out the path from p_1 to p_2 and gives irreversible R-tipping from $e(p)$. More generally, basin instability is necessary and sufficient to observe irreversible R-tipping in one-dimensional systems [55], and sufficient but not necessary to observe irreversible R-tipping in higher-dimensional systems [56, 27, 57].

The criterion above can be understood intuitively using the example from Fig. 4.1. Suppose the system starts in the basin of attraction and near the stable equilibrium e_3 at $r = r_-$, and undergoes a monotone parameter shift from r_- to any $r \in (r_*, r_+]$. We choose a shift from r_- to r_+ , and consider two extreme scenarios. If $r(t)$ varies sufficiently slowly, meaning that the speed of the moving equilibrium $|\dot{e}(t)|$ is much slower than the natural timescales of H and P , the non-autonomous system is guaranteed to closely track (adiabatically follow) the moving stable equilibrium $e_3(t)$ along the path [55, 27]. However, the dynamics are different when $r(t)$ shifts smoothly but abruptly from r_- to r_+ at some point in time, remains almost constant otherwise, and the speed $|\dot{e}(t)|$ during the shift is much faster than the natural timescales of H and P . Initially, the system approaches $e_3(r_-)$ because $r(t)$ is almost constant. Then comes the shift from r_- to r_+ , the stable equilibrium $e(t)$ changes its position, but the system is too slow to respond. Thus, just after the shift, the system is still at its earlier position near $e_3(r_-)$, which now lies outside the basin of attraction of $e_3(r_+)$ and inside the basin of attraction of $e_2(r_+)$ [Fig. 4.1(d)]. As $r(t)$ remains almost constant from now on, the system approaches the other stable equilibrium $e_2(r_+)$. These two qualitatively different scenarios indicate that there is an intermediate critical rate of change of $r(t)$, above which the system R-tips from e_3 to e_2 . We refer to [27, 57] for more general and precise definitions of basin instability, for rigorous statements of the sufficient criteria for irreversible R-tipping, and for extension of these ideas to *threshold instability* that captures both reversible and irreversible R-tipping.

4.3 Beyond Traditional Bifurcation Diagrams: Basin Instability

To examine the robustness of irreversible R-tipping from e_3 , we need to examine the persistence of its basin instability on different parameter paths in the (r, m) bifurcation diagram. To this end, we fix $b = b_c = 0.025$ close to the values used in Ref. [7] [point (d) in Fig. 3.7], and explore different parameter paths within adjacent regions 5 and

Table 4.1 Parameter values chosen to generate regions of basin instability in Fig.4.3

Figure	(b, b_c)	p_1
(a)	(0.015, 0.1)	(0.275, 0.06)
(b)	(0.015, 0.1)	(0.5, 0.0225)
(c)	(0.015, 0.1)	(0.8, 0.0075)
(d)	(0.015, 0.1)	(0.8, 0.05)
(e)	(0, 0.04)	(0.5, 0.14)
(f)	(0, 0.04)	(1, 0.065)
(g)	(0, 0.04)	(1.8, 0.015)
(h)	(0, 0.04)	(1.8, 0.1)
(i)	(0.02, 0)	(0.7, 0.2)
(j)	(0.02, 0)	(1.45, 0.11)
(k)	(0.02, 0)	(2.5, 0.04)
(l)	(0.02, 0)	(2.5, 0.15)
(m)	(0.005, 0.01)	(0.775, 0.225)
(n)	(0.005, 0.01)	(1.5, 0.14)
(o)	(0.005, 0.01)	(2.5, 0.07)
(p)	(0.005, 0.01)	(2.5, 0.15)

7 from Fig. 3.4(d). Recall that e_3 remains stable and does not bifurcate within or across the boundary between these two regions. Specifically, we choose four different points p_1 within region 5, and mark them with a black dot on different panels in Fig. 4.2. For each p_1 , we identify the set of points p_2 within regions 5 and 7 such that $e_3(p_1) \notin \overline{B(e_3, p_2)}$, and shade this set in grey in Fig. 4.2. We speak of the *region of basin instability*

$$BI(e_3, p_1) = \{p_2 \in \textcircled{5} \cup \textcircled{7} : e_3(p_1) \notin \overline{B(e_3, p_2)}\}. \quad (4.1)$$

In other words, the stable equilibrium e_3 is basin unstable on any parameter path that stays within regions 5 and 7 and connects p_1 to some $p_2 \in BI(e_3, p_1)$. The analysis of $BI(e_3, p_1)$ unveils a robust region of basin instability that can occupy almost the entirety of regions 5 and 7, and is in line with the intuitive vicious cycle discussion from Section 4.1. R-tipping is expected for a variety of parameter paths, even for small-magnitude shifts in r . Basin instability is easily achieved for increasing r , less easy to achieve for increasing m , and appears impossible to achieve for decreasing r alone [Fig. 4.2(a)–(d)]. Moving p_1 to a different position in the (r, m) diagram modifies

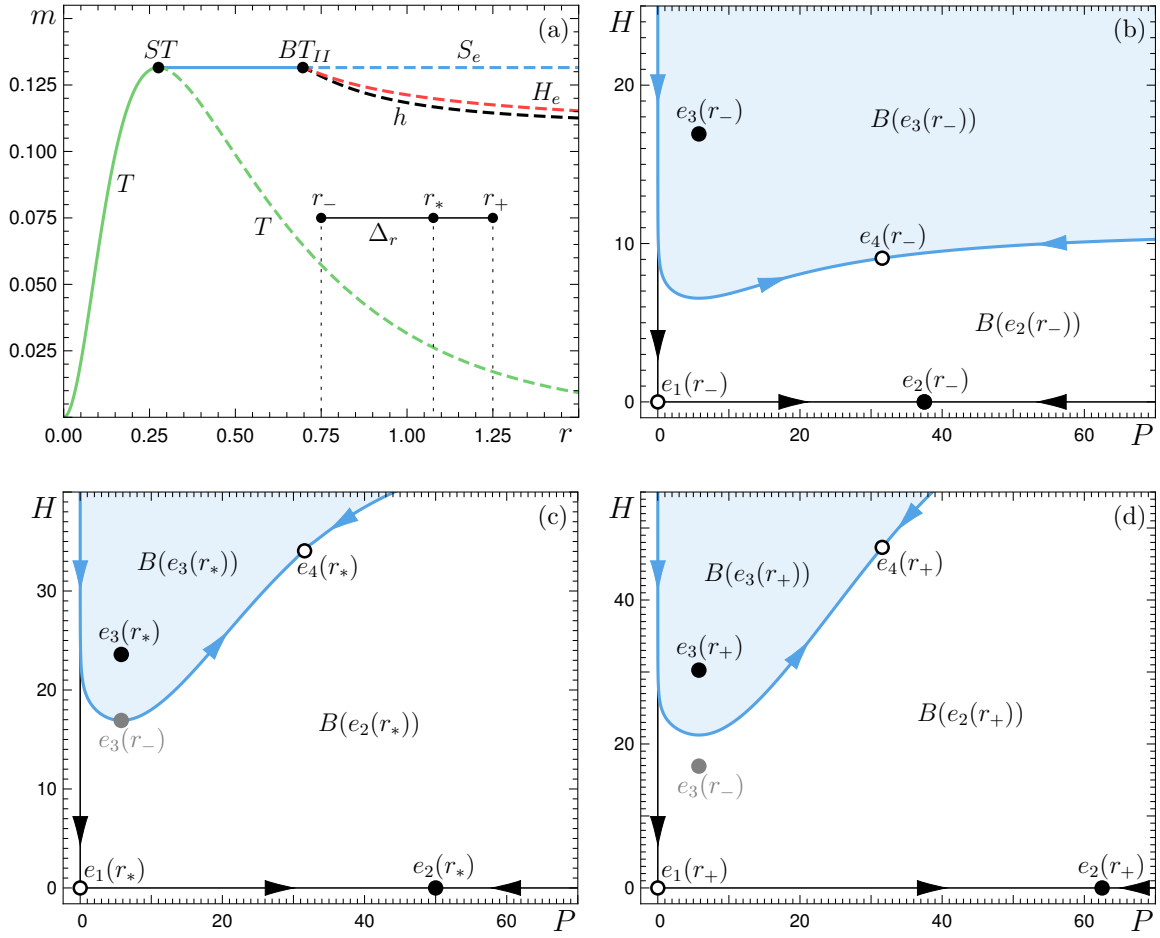


Figure 4.1 (a) A two-dimensional bifurcation diagram in the (r, m) -parameter plane for $b = b_c = 0.025$ with a parameter path Δ_r . (b)–(d) Phase portraits for system (2.1)–(2.2) at three different points along the path Δ_r revealing basin instability of e_3 on Δ_r . Blue shading indicates the basin of attraction of stable equilibrium e_3 for (b) $r = r_- = 0.75$, (c) $r = r_* \approx 1.07672$ and (d) $r = r_+ = 1.25$. $e_3(r_-)$ is shown in (c)–(d) for reference to demonstrate that $e_3(r_-)$ lies on the basin boundary of $e_3(r_*)$ and outside the basin boundary of $e_3(r_+)$.

the region of basin instability in different ways. For example, starting at lower values of m gives a small section of basin instability for shifts in m alone [Fig. 4.2(b)]. Overall, the region of basin instability persists upon moving p_1 to near the upper [Fig. 4.2(c)] or the lower [Fig. 4.2(d)] boundary of region 5. Figure 4.3 shows that the region of basin instability also persists for all reasonable values of b and b_c .

In addition to dangerous magnitudes of environmental change, the ecosystem model appears to be particularly sensitive to how fast the plant growth rate r increases over time. The basin instability analysis quantifies this effect in terms of different settings

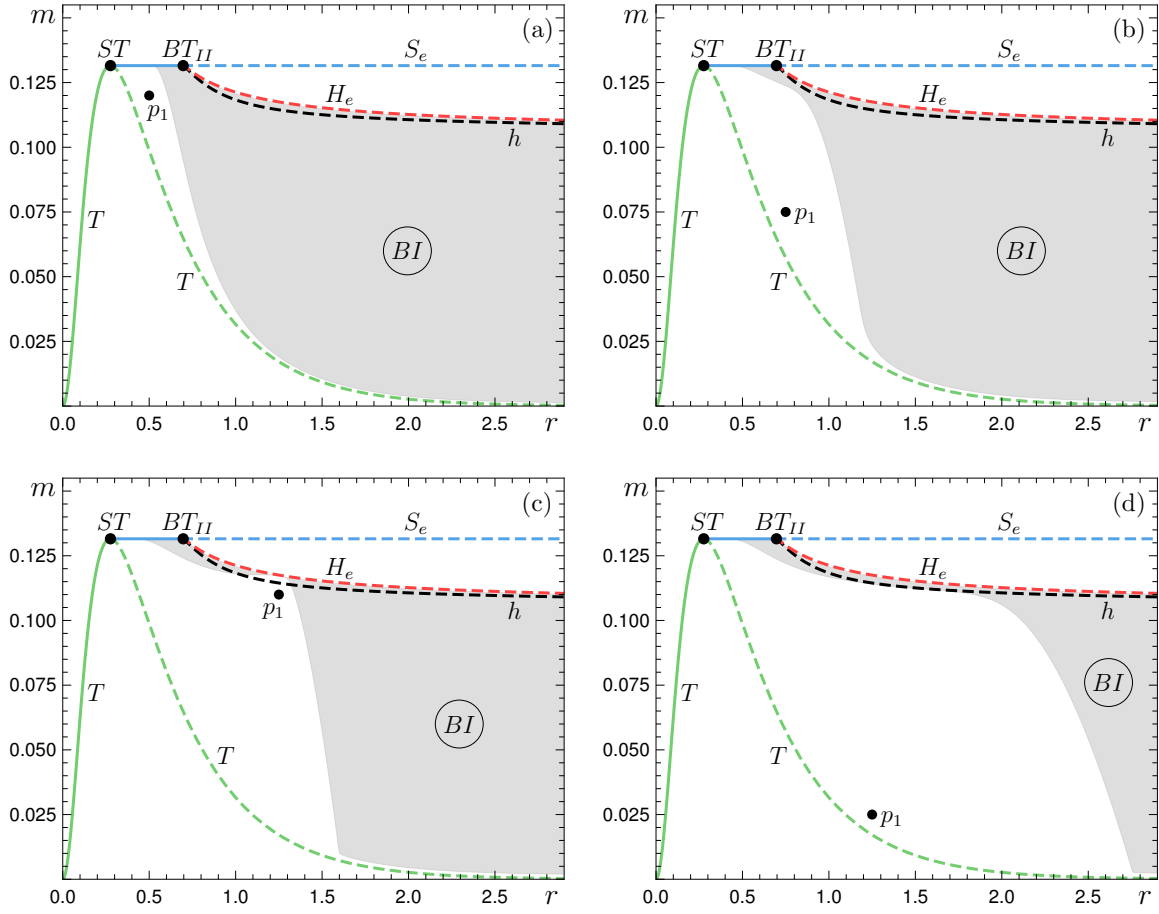


Figure 4.2 (Shading) Region of basin instability $BI(e_3, p_1)$ in the (r, m) bifurcation diagram for stable equilibrium e_3 along continuous parameter paths from p_1 consists of all points p_2 such that $e_3(p_1) \notin \bar{B}(e_3(p_2))$ (or $e_3(p_1) \in B(e_2(p_2))$) as defined by Eq. (4.1). p_1 is chosen to be at $(r, m) =$ (a) $(0.5, 0.12)$, (b) $(0.75, 0.075)$, (c) $(1.25, 0.025)$ and (d) $(1.25, 0.11)$. $b = b_c = 0.025$.

of (r, m) , and shows that sensitivity to the rate of environmental change becomes greatly enhanced at higher predator mortality rates m . Beyond the specific ecosystem model, our approach can be used to analyse tipping phenomena in nonlinear systems in general. Specifically, the region of basin instability can be superimposed on a classical bifurcation diagram to indicate rate-induced instabilities that cannot be captured by classical bifurcation analysis. What is more, the additional information about rate-induced instabilities can be made more specific in different ways. For example, one can specify the shape of the parameter shift $\Lambda(t) = (r(t), m(t))$, and use colour-scale instead of plane gray to indicate different critical rates of this $\Lambda(t)$ for each point p_2 within $BI(e_3, p_1)$.

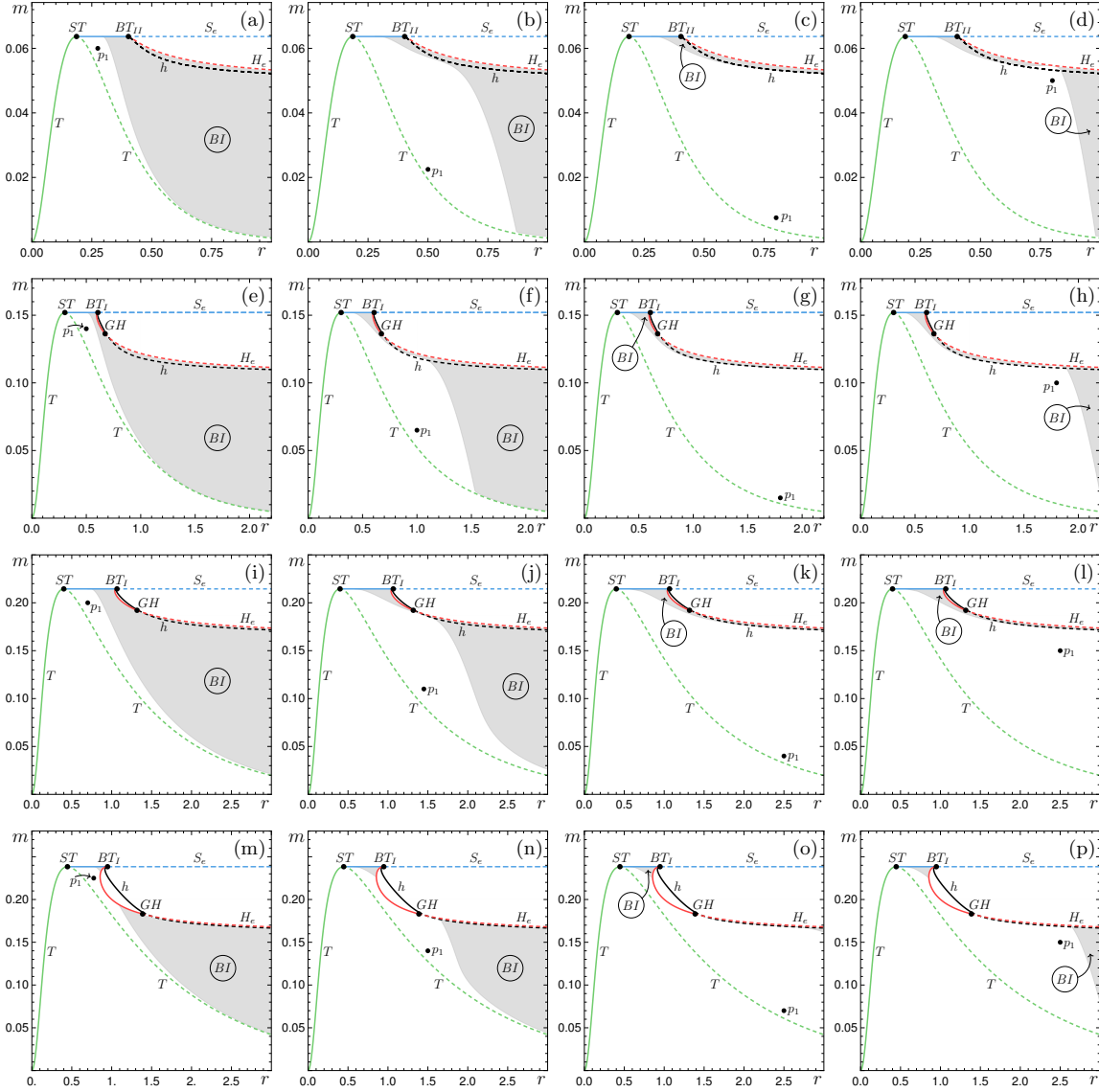


Figure 4.3 (Shading) Region of basin instability $BI(e_3, p_1)$ for a range of values of (b, b_c) in the (r, m) bifurcation diagram for stable equilibrium e_3 along continuous parameter paths from p_1 consists of all points p_2 such that $e_3(p_1) \notin B(e_3(p_2))$ (or $e_3(p_1) \in B(e_2(p_2))$) as defined by Eq. (4.1). See Table 4.1 for the values of (b, b_c) and p_1 .

4.4 Tipping Diagrams for Parameter Shifts

Now, consider irreversible R-tipping from $e_3(t)$ in the non-autonomous system (2.7)–(2.8). Guided by the basin instability analysis performed in the previous section, we focus on shifts in the plant growth rate r . Specifically, we analyse the system response

to two shapes of $r(t)$, each of which is parameterized by its *magnitude* Δ_r and *rate* ε . Firstly, we consider a monotone shift

$$r(t) = r_- + \frac{\Delta_r}{2} (\tanh(\varepsilon t) + 1), \quad (4.2)$$

from r_- to $r_+ = r_- + \Delta_r$ at the rate ε in units inverse day, and with $\dot{r}_{max} = \varepsilon \Delta_r / 2$ in units inverse day squared [$c^\uparrow$ in Fig. 4.4(a)]. Secondly, we consider a non-monotone shift

$$r(t) = r_- + \Delta_r \operatorname{sech}(\varepsilon t), \quad (4.3)$$

from r_- to $r_+ = r_- + \Delta_r$ and then back to r_- at the rate ε in units inverse day, and with $\dot{r}_{max} = \varepsilon \Delta_r / 2$ in units inverse day squared [$c^\uparrow$ in Fig. 4.4]. Such a setting enables parametric study in the form of two-dimensional (Δ, ε) or $(\Delta, \dot{r}_{max})$ *tipping diagrams* to identify *critical rates* ε_c at which the system switches between tracking and irreversible R-tipping.

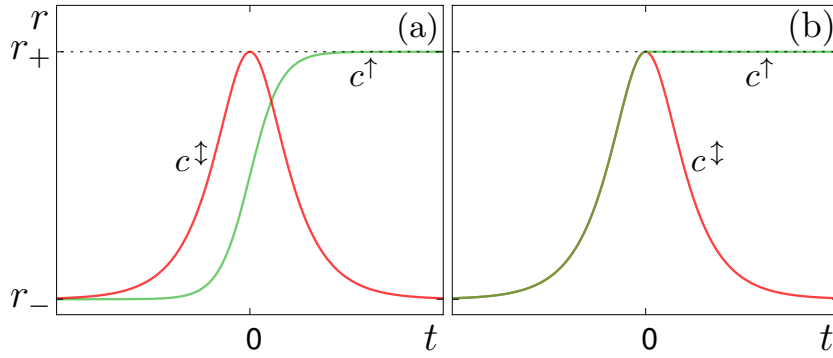


Figure 4.4 (a) ($c^\uparrow$) Monotone (4.2) and ($c^\downarrow$) non-monotone (4.3) parameter shifts $r(t)$ used in Chapter 4. (b) ($c^\uparrow$) Monotone (6.1) and ($c^\downarrow$) non-monotone (4.3) parameter shifts $r(t)$ used in Chapter 6. The dashed horizontal line indicates $r_+ = r_- + \Delta_r$.

In all instances, the non-autonomous system (2.7)–(2.8) is initialised at the moving stable equilibrium

$$(P(t_0), H(t_0)) = e_3(t_0),$$

at some initial time t_0 such that

$$r(t_0) = r_- + \delta,$$

is δ -close to r_- , and $\dot{r}(t_0) \ll \dot{r}_{max}$ to ensure that the speed of the moving equilibrium $|\dot{e}_3(t_0)|$ is sufficiently small. (Note that initiating the system at $r_0 = r_-$ would require

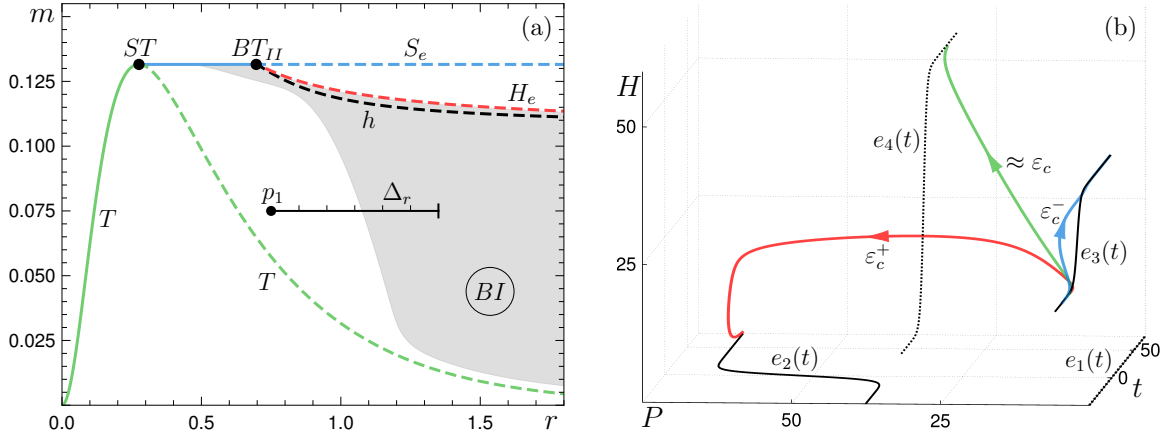


Figure 4.5 The same as Fig. 3.9 but with (a) the extended path Δ_r and the addition of the shaded region of basin instability $BI(e_3, p_1)$ for $p_1 = (0.75, 0.075)$ as defined by Eq. (4.1), and (b) the addition of the green trajectory for $\varepsilon \approx \varepsilon_c$ that (rather surprisingly) follows the unstable moving equilibrium $e_4(t)$.

$t_0 = -\infty$.) Henceforth, we set $\delta = \Delta_r/10^3$. This gives

$$t_0(\varepsilon) = \frac{1}{\varepsilon} \tanh^{-1}(-0.998) \approx -\frac{3.453}{\varepsilon},$$

with $\dot{r}(t_0) \approx 0.004 \dot{r}_{max}$ for the monotone input (4.2), and

$$t_0(\varepsilon) = \frac{1}{\varepsilon} \operatorname{sech}^{-1}(10^{-3}) \approx -\frac{7.6}{\varepsilon},$$

with $\dot{r}(t_0) \approx 0.002 \dot{r}_{max}$ for the non-monotone input (4.3).

4.5 Monotone Parameter Shifts: Single Critical Rate

Figure 4.5 sheds more light on the R-tipping from Fig. 3.9. Firstly, the stable equilibrium e_3 can be basin unstable upon increasing r from p_1 along the path Δ_r [Fig. 4.5(a)]. Secondly, there is a critical rate ε_c which defines the transition between tracking and R-tipping. When $\varepsilon = \varepsilon_c$, the non-autonomous system neither tracks $e_3(t)$ nor R-tips to $e_2(t)$ but, rather surprisingly, follows the unstable equilibrium $e_4(t)$ [Fig. 4.5(b)]. This behaviour is akin to so-called canard trajectories that follow unstable slow manifolds in slow-fast systems [58, 59]. It is interesting to note that critical-rate canard trajectories can follow different unstable states, depending on the basin boundary for the future-limit autonomous system. For example, Fig. 5.1(a) shows a parameter path Δ_r that

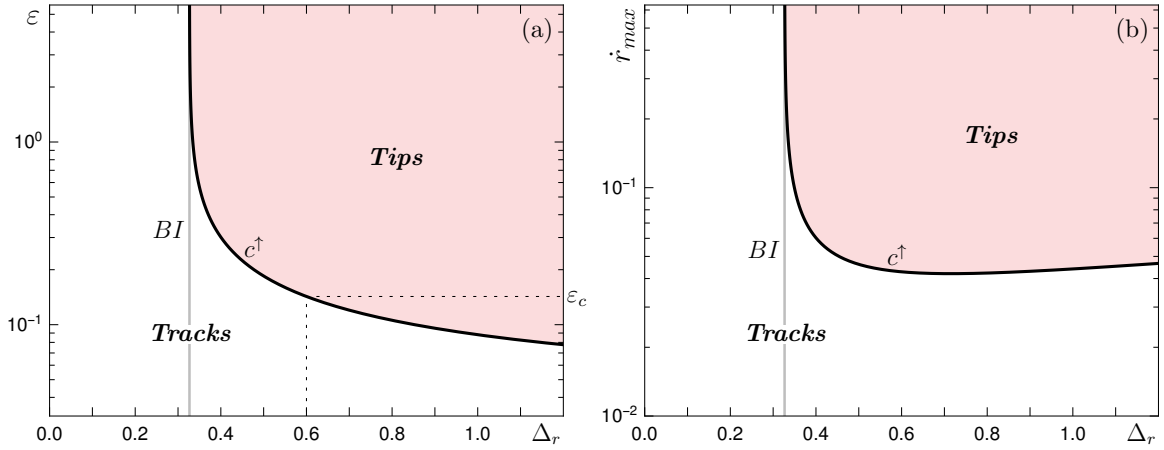


Figure 4.6 Tipping diagrams in the (a) (Δ_r, ε) and (b) $(\Delta_r, \dot{r}_{max})$ parameter plane for monotone shifts (4.2) from $p_1 = (0.75, 0.075)$ along the extended parameter path Δ_r from Fig. 4.5(a). The tipping-tracking transition curve $c^\uparrow$ separates the diagram into regions of (white) tracking and (pink) irreversible R-tipping. BI indicates the boundary of the basin instability region $BI(e_3, p_1)$. The critical rate ε_c corresponds to Fig. 4.5(b). $b = b_c = 0.025$.

starts at p_1 in region 5 and extends past the (non-dangerous) homoclinic bifurcation h to region 7. Along this path, equilibrium e_3 is smoothly stable, but its basin boundary changes from the stable invariant manifold of saddle e_4 (region 5) to a repelling limit cycle (region 7) [Fig. 3.5]. Given a monotone shift from p_1 along Δ_r and across h , there is R-tipping due to basin instability. The difference from Fig. 4.5(b) is the (green) critical-rate canard trajectory which now follows the unstable limit cycle [Fig. 5.1(b)].

A systematic analysis of R-tipping for monotone shifts (4.2) from p_1 along the path Δ_r from Fig. 4.5(a) gives the (Δ_r, ε) and $(\Delta_r, \dot{r}_{max})$ tipping diagrams [Fig. 4.6]. In the diagrams, the tracking-tipping transitions occur along the curve $c^\uparrow$. This curve divides the tipping diagram into separate regions of (white) tracking and (pink) irreversible R-tipping [Fig. 4.6]. The entire R-tipping region appears to be located past the basin instability boundary BI . As Δ_r decreases, the $c^\uparrow$ curve is asymptotic to BI from the right. This suggests that basin instability is both sufficient and necessary for irreversible R-tipping in the ecosystem model. In general, this need not be the case in higher than one dimension, where basin instability is guaranteed to be sufficient but not necessary for R-tipping; see [56, 57] for examples of irreversible R-tipping in two dimensions in the absence of basin instability. What is more, as Δ_r increases, the $c^\uparrow$ curve appears to level off at $\dot{r}_{max} \approx 0.045$. In other words, R-tipping in the ecosystem model requires sufficiently large $\dot{r}_{max}$, rather than ε , independently of Δ_r .

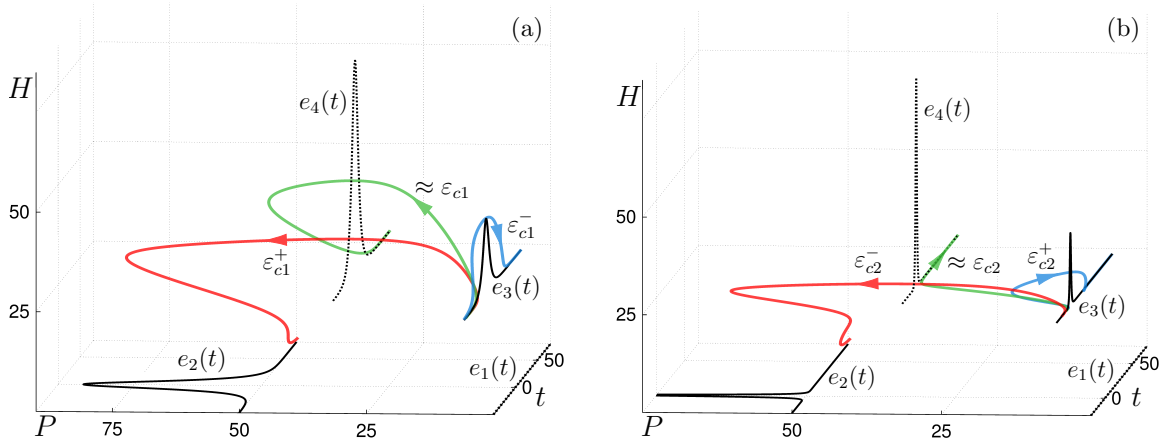


Figure 4.7 Trajectories of the non-autonomous system (2.7)–(2.8) with non-monotone parameter shifts (4.3) from $p_1 = (1.0, 0.075)$ along a path Δ_r with fixed $m = 0.075$ and time-varying $r > 1$. (a) The system (blue) tracks the moving stable equilibrium $e_3(t)$ when $\varepsilon = \varepsilon_{c1}^- = 0.1 < \varepsilon_{c1}$, (green) rather surprisingly follows the moving unstable equilibrium $e_4(t)$ when $\varepsilon = 0.166491526823788 \approx \varepsilon_{c1}$, and (red) R-tips when $\varepsilon = \varepsilon_{c1}^+ = 0.2 > \varepsilon_{c1}$. (b) Upon further increase in the rate, the system (green) again rather surprisingly follows the moving unstable equilibrium when $\varepsilon = 1.049396269470948 \approx \varepsilon_{c2}$, and (blue) switches back to tracking $e_3(t)$ when $\varepsilon = \varepsilon_{c2}^+ = 1.5 > \varepsilon_{c2}$. The moving equilibria are obtained for (a) $\varepsilon = \varepsilon_{c1}$ and (b) $\varepsilon = \varepsilon_{c2}$. $b = b_c = 0.025$.

Thus, one can give simple approximate conditions for irreversible R-tipping along this path in terms of Δ_r exceeding the boundary BI and $\dot{r}_{max} \gtrsim 0.045$. Finally, we say that this R-tipping is unique, meaning that there is a unique critical rate ε_c for every fixed magnitude Δ_r that exceeds the boundary BI .

4.6 Non-monotone Parameter Shifts: Two Critical Rates

Now, consider system (2.7)–(2.8) with non-monotone $r(t)$ tracing out the path Δ_r in Fig. 4.5(a) from p_1 at $r_- = 0.75$ to $r_- + \Delta_r$ and then back to p_1 . The six solutions for different values of ε shown in Fig. 4.7(a)–(b) highlight the main difference from the monotone shift: two different critical rates for the same Δ_r . Specifically, the system tracks $e_3(t)$ below the first critical rate $\varepsilon < \varepsilon_{c1}$, then switches from tracking to irreversible R-tipping when $\varepsilon = \varepsilon_{c1}$, R-tips for a range of rates $\varepsilon_{c1} < \varepsilon < \varepsilon_{c2}$, then switches back from irreversible R-tipping to tracking when $\varepsilon = \varepsilon_{c2}$, and continues to track $e_3(t)$ for $\varepsilon > \varepsilon_{c2}$. In the (Δ_r, ε) and $(\Delta_r, \dot{r}_{max})$ tipping diagrams, tracking-tipping

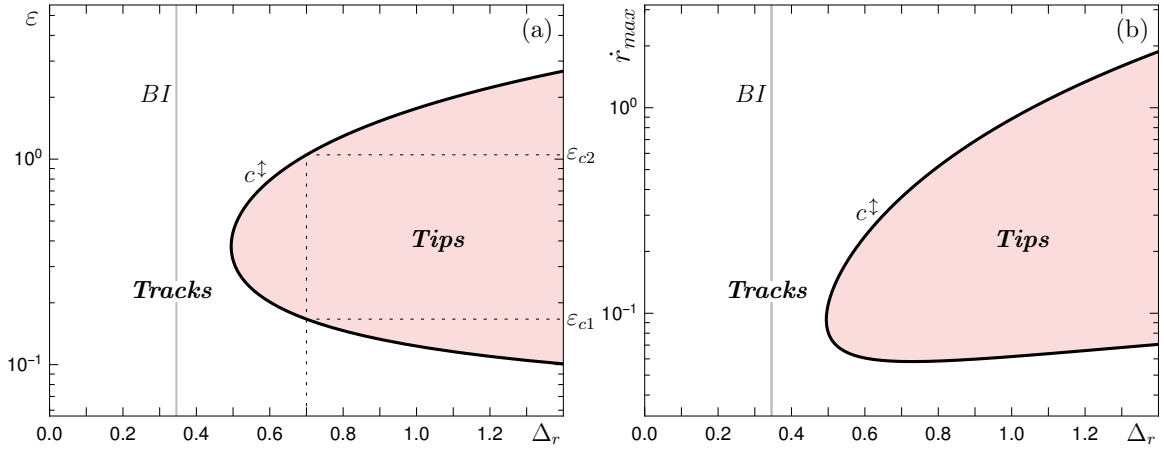


Figure 4.8 Tipping diagrams in the (a) (Δ_r, ε) and (b) $(\Delta_r, \dot{r}_{max})$ parameter plane for non-monotone shifts (4.3) from $p_1 = (1.0, 0.075)$ along a path Δ_r with a fixed $m = 0.075$ and varied $r > 1$. The tipping-tracking transition curve $c^\uparrow$ separates the diagram into regions of (white) tracking and (pink) irreversible R-tipping. BI indicates the boundary of the basin instability region $BI(e_3, p_1)$. The critical rates ε_{c1} and ε_{c2} correspond to Fig. 4.7(a) and (b), respectively. $b = b_c = 0.025$.

transitions occur along the curve $c^\uparrow$. This curve divides the tipping diagram into two separate regions of (white) tracking and (pink) irreversible R-tipping. The region of irreversible R-tipping is located past the basin instability boundary BI , and is tongue-shaped. The R-tipping tongue is reminiscent of a resonance tongue [60] in the sense that the system exhibits a strongly enhanced response to external inputs with optimal timing. This tongue shape can be understood in terms of relative time scales. At high ε , the natural timescales of $H(t)$ and $P(t)$ are slower than $e_3(t)$. Thus, the system is unable to respond to a short impulse $r(t)$. As ε is decreased, the natural timescales of $H(t)$ and $P(t)$ get closer to $e_3(t)$, the system starts to react to the input and R-tips due to basin instability. This transition is marked by the higher critical rate. As ε is decreased further, the natural timescales of $H(t)$ and $P(t)$ become comparable to $e_3(t)$, giving rise to a strongly enhanced response in the form of the tipping tongue. As ε is decreased even further, the natural timescales of $H(t)$ and $P(t)$ become faster than $e_3(t)$, and the system starts to closely track $e_3(t)$. This transition is marked by the lower critical rate. In summary, for a fixed Δ_r past the BI boundary, the ε -interval of irreversible R-tipping can be bounded by two critical rates, ε_{c1} from below and ε_{c2} from above [Fig. 4.8(a)–(b)]. We speak of non-unique R-tipping.

Chapter 5

Interaction Between R-tipping and B-tipping: Multiple Critical Rates

So far, we have discussed B-tipping and R-tipping in isolation. At the same time, we recognise that real-world tipping phenomena will often involve both mechanisms, although the ensuing nonlinear dynamics is less understood. This chapter discusses three different types of interplay between critical levels and critical rates, and reveals intriguing tipping diagrams.

5.1 Monotone Shift Across Basin Instability and Dangerous Bifurcation

Figure 5.1 shows a monotone parameter shift from p_1 along the parameter path Δ_r across the (non-dangerous) homoclinic bifurcation h in order to demonstrate the effects of a qualitative change in the basin boundary along the path. To study the interaction between B-tipping and R-tipping, we extend this parameter path away from p_1 and past the (dangerous) subcritical Hopf bifurcation [Fig. 5.4(a)]. The ensuing (Δ_r, ε) and $(\Delta_r, \dot{r}_{max})$ tipping diagrams for monotone shifts (4.2) from p_1 along the extended path are shown in Fig.5.2. The tipping-tracking transition curve $c^\uparrow$ consists of two distinct parts, which correspond to two different tipping mechanisms. The upper part, that has a Δ_r -dependent critical rate and is asymptotic to the basin instability boundary BI as Δ_r decreases, corresponds to unique R-tipping due to basin instability. The lower part, that is the vertical line along $\Delta_r = H_e$, does not have any critical rates in the following sense. Critical transitions occur past the critical level $\Delta_r = H_e$ independently

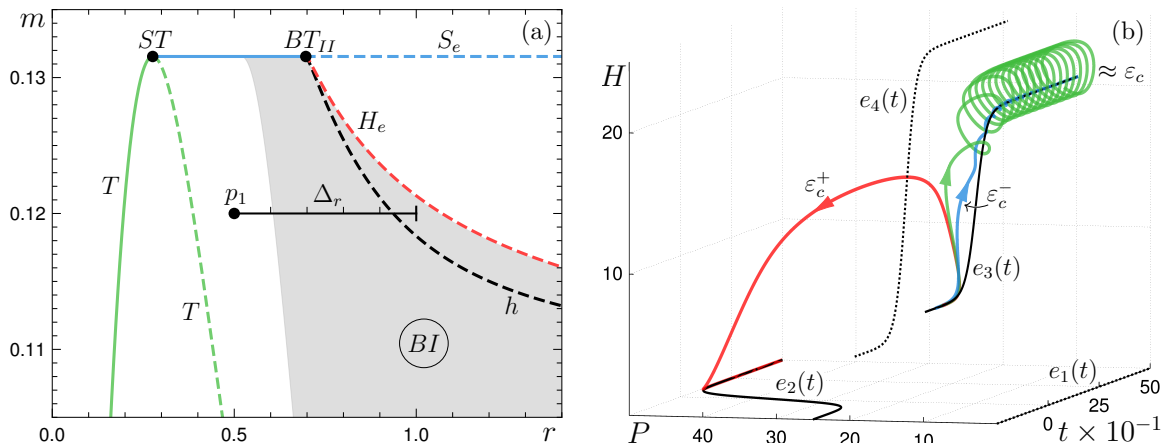


Figure 5.1 (a) Example of a parameter path Δ_r across the homoclinic bifurcation h separating regions 5 and 7 together with (shading) the region of basin instability $BI(e_3, p_1)$ for $p_1 = (0.5, 0.12)$ in the (r, m) -bifurcation diagram. (b) The non-autonomous system (2.7)–(2.8) with monotone parameter shift (4.2) from p_1 along Δ_r (blue) tracks the moving stable equilibrium when $\varepsilon = \varepsilon_c^- = 0.0175 < \varepsilon_c$, (green) (rather surprisingly) tracks the repelling limit cycle from region 7 when $\varepsilon = 0.020342768468207 \approx \varepsilon_c$, and (red) R-tips when $\varepsilon = \varepsilon_c^+ = 0.021 > \varepsilon_c$. The moving equilibria are obtained for $\varepsilon = \varepsilon_c$. $b = b_c = 0.025$.

of the rate, meaning that this part corresponds to B-tipping due to the dangerous bifurcation H_e . The separation between the two tipping mechanisms is particularly clear-cut in the $(\Delta_r, \dot{r}_{max})$ tipping diagram. When $\dot{r}_{max} > 0.005$, unique R-tipping for $\Delta_r \gtrsim BI$ is the tipping mechanism. As $\dot{r}_{max}$ is decreased, there is an abrupt transition near $\dot{r}_{max} \approx 0.005$ to a different tipping mechanism. When $\dot{r}_{max} < 0.005$, B-tipping for $\Delta_r > H_e$ is the tipping mechanism. An indication of the length of time needed for the tipping-tracking transition curve $c^\uparrow$ to converge to H_e is given in Fig. 5.3.

We would like to point out that the slow passage through a Hopf bifurcation gives rise to a bifurcation delay that does not vanish if the rate of parameter change tends to zero [61–63]. This means that trajectories follow the unstable equilibrium past the bifurcation point for a noticeable time even if the rate of parameter change tends to zero [61]. Whereas the bifurcation delay has no effect on the tipping diagram for monotone shifts (4.2), it is expected to manifest itself for non-monotone shifts (4.3) that are considered in the next section.

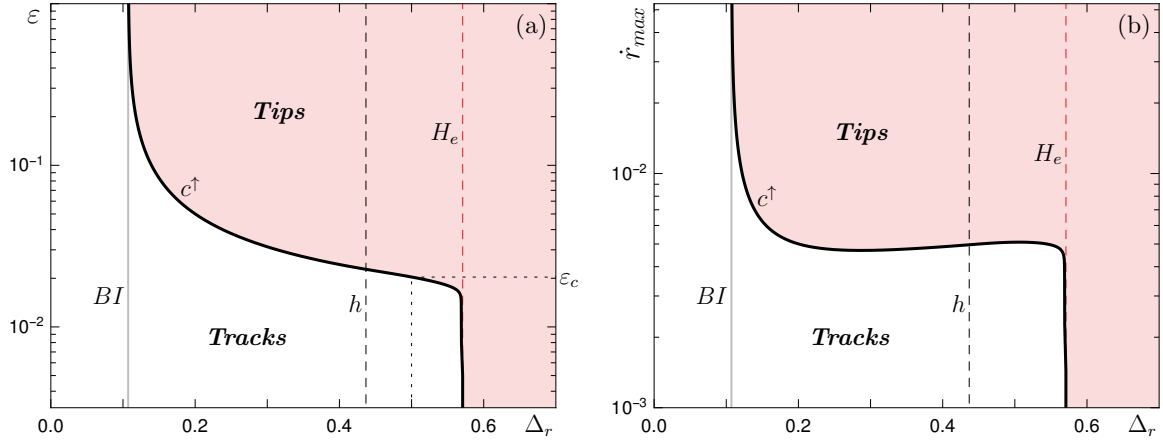


Figure 5.2 Tipping diagrams in the (a) (Δ_r, ε) - and (b) $(\Delta_r, \dot{r}_{max})$ -parameter plane for monotone shifts (4.2) from $p_1 = (0.5, 0.12)$ along the parameter path Δ_r from Fig. 5.4(a). The tipping-tracking transition curve $c^\uparrow$ separates the diagram into regions of (white) tracking and (pink) tipping. BI indicates the boundary of the basin instability region $BI(e_3, p_1)$, h indicates the homoclinic bifurcation, and H_e indicates the (dangerous) subcritical Hopf bifurcation of e_3 . $b = b_c = 0.025$. The critical rate ε_c indicated in (a) is the value of (Δ_r, ε) that was used to calculate the critical (green) trajectory in Fig. 5.1(b).

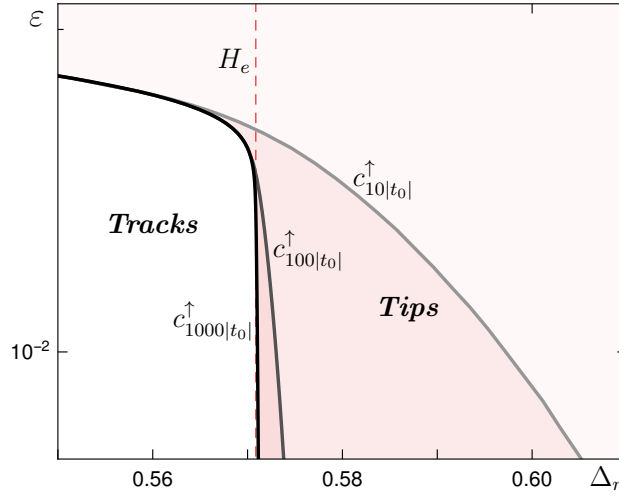


Figure 5.3 Tipping diagram in the (Δ_r, ε) -parameter plane for monotone shifts (4.2) from $p_1 = (0.5, 0.12)$ along the parameter path Δ_r from Fig. 5.4(a), showing tipping-tracking transition curves $c^\uparrow$ converging to the Hopf bifurcation H_e , where the system is run for time spans of $10|t_0|$, $100|t_0|$ and $1000|t_0|$, the initial time $t_0 \approx -3.453/\varepsilon$.

5.2 Non-monotone Shift Across Basin Instability and Dangerous Bifurcation

Monotone parameter shifts across basin instability and a dangerous bifurcation give rise to an intuitive tipping diagram with two distinct regimes: B-tipping for low rates and unique R-tipping for higher rates. Now, we consider non-monotone shifts along the same path Δ_r from Fig. 5.4(a). Specifically, $r(t)$ increases from p_1 , passes through the basin instability boundary BI and through the (dangerous) subcritical bifurcation H_e , but then turns around and tends back to p_1 . This turning around allows the system to avoid tipping even if it goes past the subcritical bifurcation H_e . What is more, the bifurcation delay gives the system additional time to turn back before tipping occurs. The nine solutions for different values of ε shown in Fig. 5.4(b)–(d) highlight the main difference from the monotone shift: three different critical rates for the same Δ_r . The system tips from $e_3(t)$ to $e_2(t)$ below the first critical rate $\varepsilon < \varepsilon_{c1}$, then switches from tipping to tracking when $\varepsilon = \varepsilon_{c1}$ [Fig. 5.4(b)], tracks $e_3(t)$ for a range of rates $\varepsilon_{c1} < \varepsilon < \varepsilon_{c2}$, switches back to tipping when $\varepsilon = \varepsilon_{c2}$ [Fig. 5.4(c)], tips for a range of rates $\varepsilon_{c2} < \varepsilon < \varepsilon_{c3}$, and switches again from tipping to tracking when $\varepsilon = \varepsilon_{c3}$ [Fig. 5.4(d)]. The nine solutions for different values of ε are repeated in Fig. 5.5(b)–(d), with two-dimensional projections of Fig. 5.4(b)–(d) on to the (t, H) -plane, indicating the times at which $r(t)$ encounters subcritical bifurcations h and H_e .

The ensuing (Δ_r, ε) and $(\Delta_r, \dot{r}_{max})$ tipping diagrams for non-monotone shifts (4.3) from p_1 are shown in Fig. 5.6. Although the separation between different tipping mechanisms is less obvious now, the tipping-tracking transition curve $c^\dagger$ still consists of two different parts that can be associated with the two different tipping mechanisms. At high ε and between BI and H_e , we replicate the distinctive tongue-shaped tipping region from Fig. 4.8. Thus, we attribute this part of the tipping diagram to irreversible R-tipping. As ε is decreased, we observe two new features. Firstly, the curve $c^\dagger$ forms a deep wedge whose tip delineates the change from R-tipping to B-tipping. Secondly, as $\varepsilon \rightarrow 0$, the curve $c^\dagger$ approaches the critical level H_e for B-tipping. However, this approach is ‘slow’, which is in stark contrast to Fig. 4.8. The new features can be explained in terms of relative timescales and bifurcation delay.

As ε decreases below the tipping tongue, the natural timescales of $H(t)$ and $P(t)$ start to exceed the timescale of $e_3(t)$, meaning that the system becomes more able to follow the moving stable equilibrium $e_3(t)$. On the one hand, we start to lose R-tipping. On the other hand, the system acquires some characteristics of a slow passage through

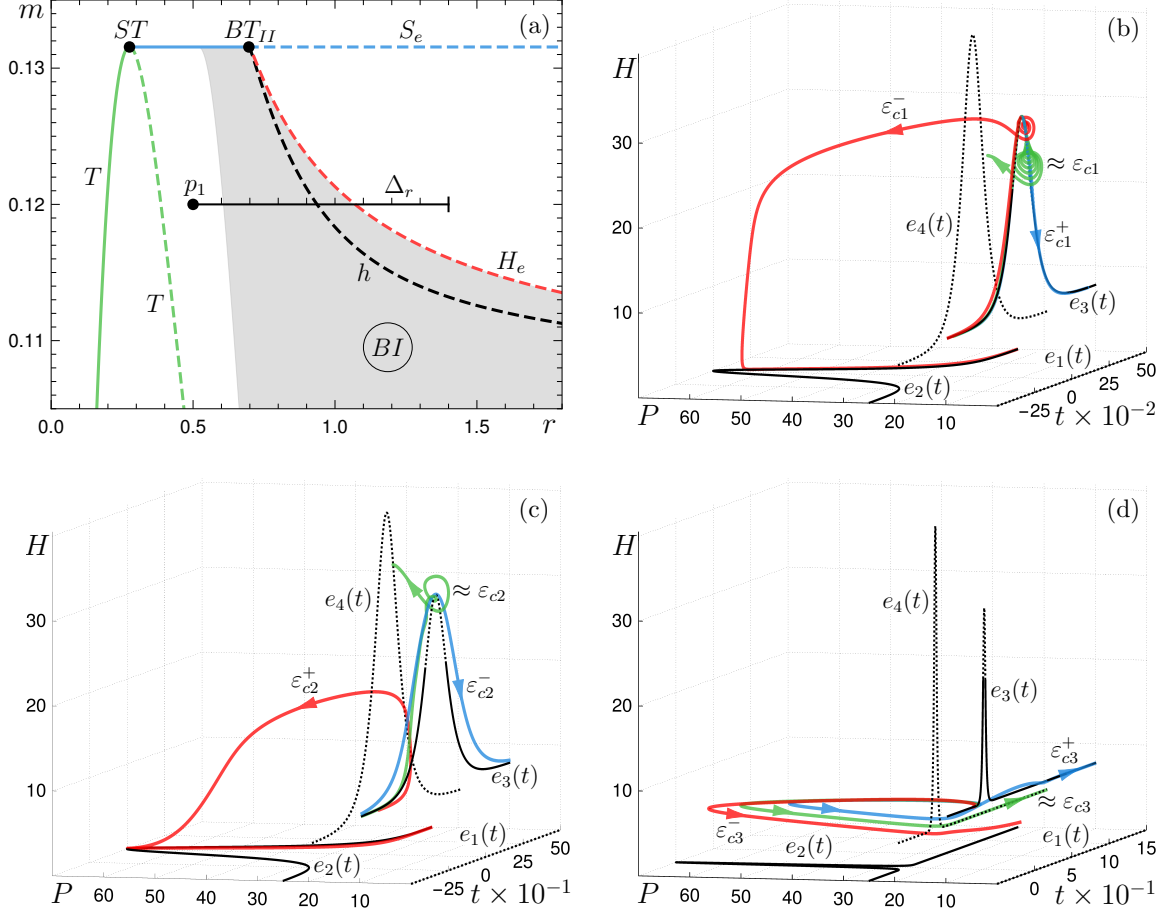


Figure 5.4 (a) Example of a parameter path Δ_r across the basin instability region $BI(e_3, p_1)$ and (dangerous) subcritical Hopf bifurcation H_e in the (r, m) bifurcation diagram. The non-autonomous system (2.7)–(2.8) with non-monotone parameter shift (4.3) from $p_1 = (0.5, 0.12)$ along Δ_r (b) tips below the first critical rate $\varepsilon < \varepsilon_{c1}$, then switches from tipping to tracking when $\varepsilon = 0.00165601005 \approx \varepsilon_{c1}$ for a range of rates $\varepsilon_{c1} < \varepsilon < \varepsilon_{c2}$, (c) switches back to tipping when $\varepsilon = 0.014830495837 \approx \varepsilon_{c2}$, tips for a range of rates $\varepsilon_{c2} < \varepsilon < \varepsilon_{c3}$, and (d) switches again from tipping to tracking when $\varepsilon = 0.700596344828672 \approx \varepsilon_{c3}$. The moving equilibria are obtained for (b) $\varepsilon = \varepsilon_{c1}$, (c) $\varepsilon = \varepsilon_{c2}$ and (d) $\varepsilon = \varepsilon_{c3}$. $b = b_c = 0.025$. The three different critical rates ε_{c1} , ε_{c2} and ε_{c3} are indicated in the following tipping diagram Fig.5.6(a) where one can observe the interaction between the bifurcation-induced ε_{c1} and the rate-induced ε_{c2} and ε_{c3} .

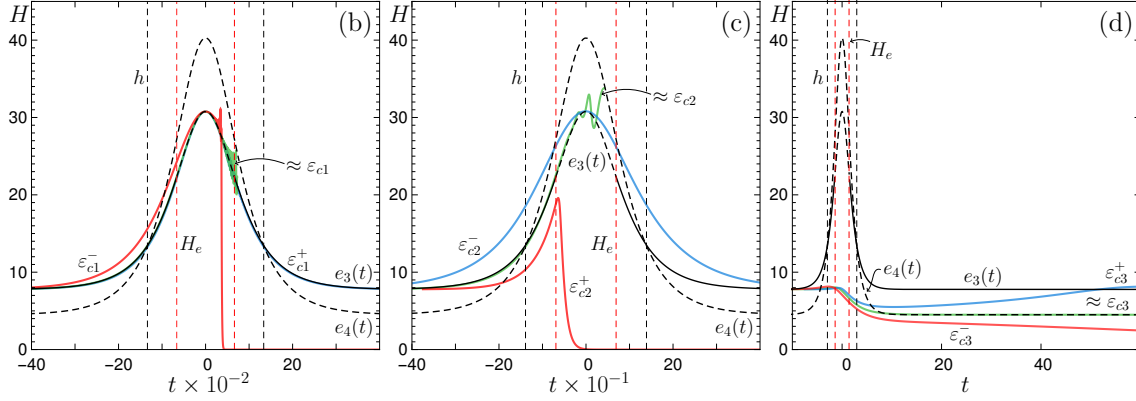


Figure 5.5 Two-dimensional projections of Fig. 5.4 (b), (c) and (d) on the (t, H) -plane. Included are where Hopf H_e and homoclinic h bifurcations occur.

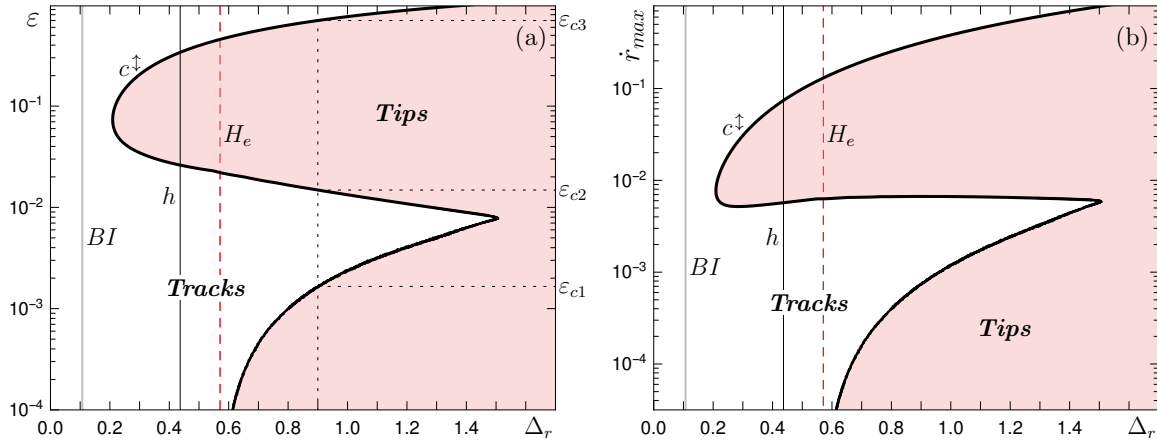


Figure 5.6 Tipping diagrams in the (a) (Δ_r, ε) and (b) $(\Delta_r, \dot{r}_{max})$ parameter plane for non-monotone shifts (4.3) from $p_1 = (0.5, 0.12)$ along the parameter path Δ_r from Fig. 5.4(a). The tipping-tracking transition curve $c^\dagger$ separates the diagram into regions of (white) tracking and (pink) tipping. BI indicates the boundary of the basin instability region $BI(e_3, p_1)$, h indicates the homoclinic bifurcation, and H_e indicates the (dangerous) subcritical Hopf bifurcation of e_3 . The critical rates ε_{c1} , ε_{c2} and ε_{c3} in panel (a) correspond to Fig. 5.4 (b), (c) and (d), respectively. $b = b_c = 0.025$.

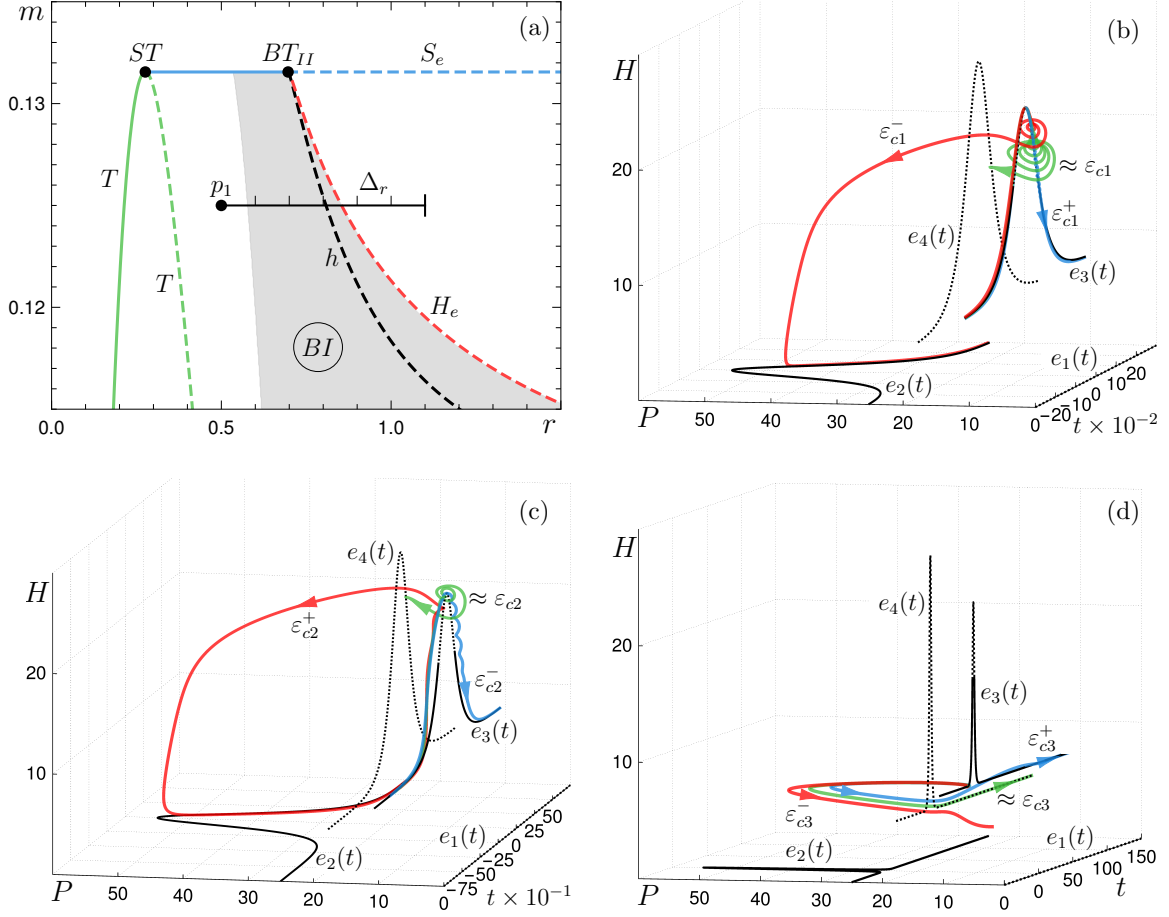


Figure 5.7 (a) Example of a parameter path Δ_r across the basin instability region $BI(e_3, p_1)$ and (dangerous) subcritical Hopf bifurcation H_e in the (r, m) bifurcation diagram. The non-autonomous system (2.7)–(2.8) with non-monotone parameter shift (4.3) from $p_1 = (0.5, 0.125)$ along Δ_r (b) tips below the first critical rate $\epsilon < \epsilon_{c1}$, then switches from tipping to tracking when $\epsilon = 0.0018514939 \approx \epsilon_{c1}$ for a range of rates $\epsilon_{c1} < \epsilon < \epsilon_{c2}$, (c) switches back to tipping when $\epsilon = 0.009917768 \approx \epsilon_{c2}$, tips for a range of rates $\epsilon_{c2} < \epsilon < \epsilon_{c3}$, and (d) switches again from tipping to tracking when $\epsilon = 0.837674126 \approx \epsilon_{c3}$. The moving equilibria are obtained for (b) $\epsilon = \epsilon_{c1}$, (c) $\epsilon = \epsilon_{c2}$ and (d) $\epsilon = \epsilon_{c3}$. $b = b_c = 0.025$. The three different critical rates $\epsilon_{c1}, \epsilon_{c2}$ and ϵ_{c3} are indicated in the following tipping diagram Fig.5.6(a) where one can observe the interaction between the bifurcation-induced ϵ_{c1} and the rate-induced ϵ_{c2} and ϵ_{c3} .

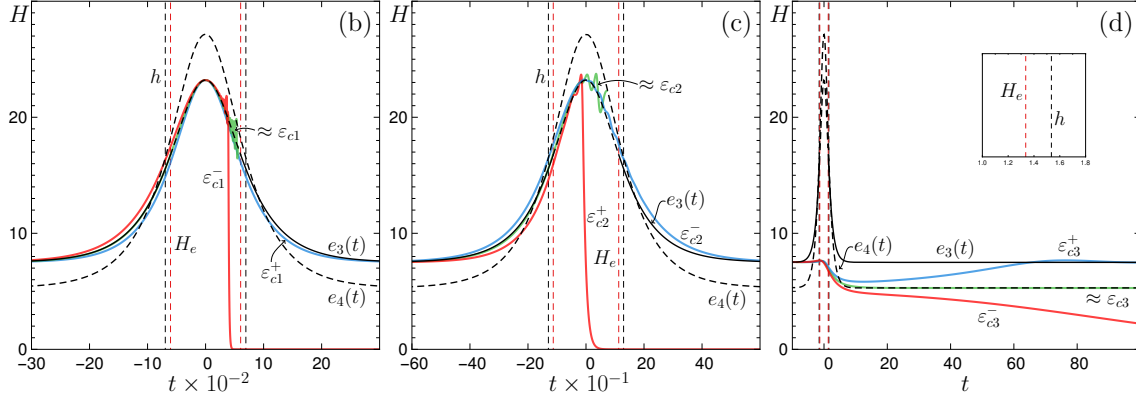


Figure 5.8 Two-dimensional projections of Fig. 5.7 (b), (c) and (d) on the (t, H) -plane. Included are where Hopf H_e and homoclinic h bifurcations occur.

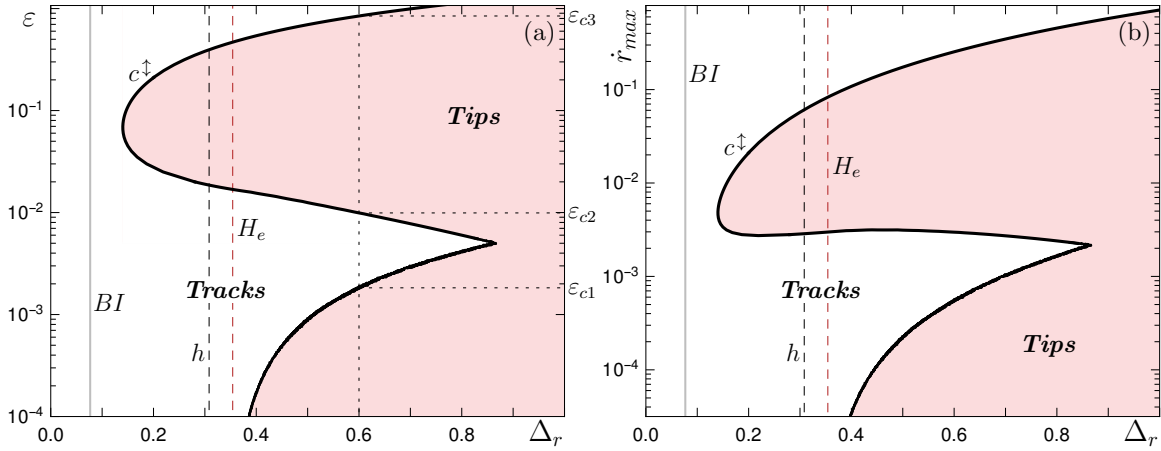


Figure 5.9 Tipping diagrams in the (a) (Δ_r, ε) and (b) $(\Delta_r, \dot{r}_{max})$ parameter plane for non-monotone shifts (4.3) from $p_1 = (0.5, 0.125)$ along the parameter path Δ_r from Fig. 5.7(a). The tipping-tracking transition curve $c^{\downarrow}$ separates the diagram into regions of (white) tracking and (pink) tipping. BI indicates the boundary of the basin instability region $BI(e_3, p_1)$, h indicates the homoclinic bifurcation, and H_e indicates the (dangerous) subcritical Hopf bifurcation of e_3 . The critical rates ε_{c1} , ε_{c2} and ε_{c3} in panel (a) correspond to Fig. 5.7 (b), (c) and (d), respectively. $b = b_c = 0.025$.

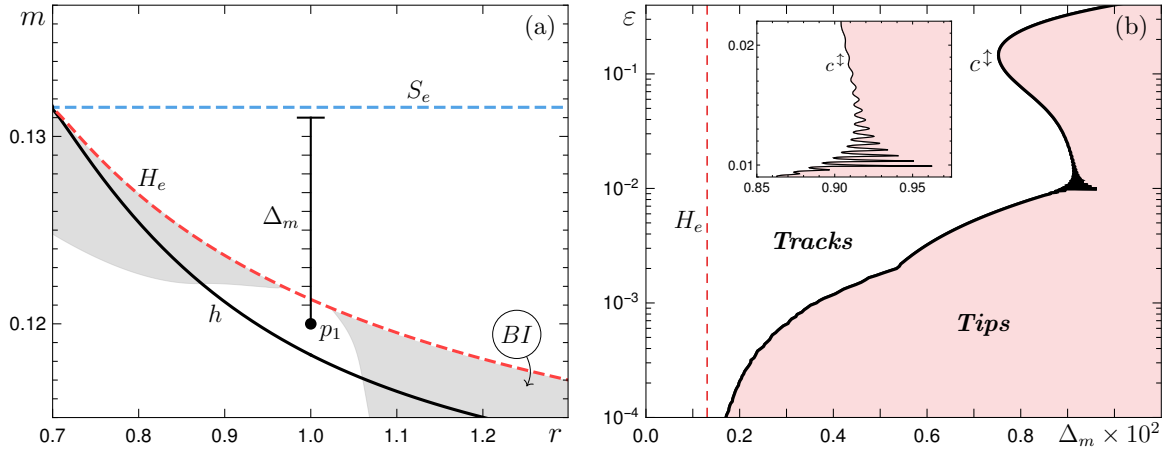


Figure 5.10 (a) Example of a parameter path Δ_m across the dangerous bifurcation H_e , together with (shading) the region of basin instability $BI(e_3, p_1)$ for $p_1 = (1, 0.12)$ in the (r, m) bifurcation diagram. (b) The tipping diagram in the (Δ_m, ε) parameter plane for non-monotone shifts (4.3) from p_1 along the parameter path Δ_m . The tipping-tracking transition curve $c^\dagger$ separates the diagram into regions of (white) tracking and (pink) tipping. H_e indicates a (dangerous) subcritical Hopf bifurcation. $b = b_c = 0.025$.

a Hopf bifurcation. In particular, there is a bifurcation delay that allows the system to spend a noticeable time past the critical level H_e before B-tipping actually occurs. As a result, the tracking-tipping transition due to H_e is shifted to a much larger Δ_r . Hence the deep wedge in $c^\dagger$. As ε is decreased further, $H(t)$ and $P(t)$ become much faster than $e_3(t)$, and start to closely track $e_3(t)$. We move into the regime of a slow passage through a Hopf bifurcation, which is characterised by a noticeable bifurcation delay that does not vanish when $\varepsilon \rightarrow 0$. Thus, the ‘slow’ approach of $c^\dagger$ towards H_e as $\varepsilon \rightarrow 0$ is attributed to this bifurcation delay. In summary, the intricate tipping diagram captures different aspects of the interaction between B-tipping and R-tipping, and explains the non-unique tipping with three critical rates from Fig. 5.4.

Figures 5.7– 5.9 replicate the results of Figs 5.4– 5.6, but for $p_1 = (0.5, 0.125)$.

5.3 Non-monotone Shift Across Dangerous Bifurcation

The third type of interaction arises during a non-monotone passage through a dangerous bifurcation, and is more of an interplay between critical levels and critical rates rather than between B-tipping and R-tipping. To be more specific, we consider a parameter

path that crosses a dangerous bifurcation, but does not involve any basin instability. The path Δ_m through a subcritical Hopf bifurcation H_e from Fig. 5.10(a) is an example of such a path. The difference from the first two types of interaction is that neither basin instability nor pure R-tipping occur along this path. Nonetheless, the system response is expected to depend on the rate ε . For example, the system may avoid tipping despite going past the dangerous bifurcation if it turns around fast enough [20, 21]. Thus, in addition to the critical level, we also expect critical rate(s).

We fix $r = 1$, consider non-monotone shifts in the herbivore death rate along Δ_m :

$$m(t) = 0.12 + \Delta_m \sinh(\varepsilon t), \quad (5.1)$$

and initiate the non-autonomous system (2.7)–(2.8) at the stable equilibrium $e_3(t_0)$ at time $t_0(\varepsilon) = \text{sech}^{-1}(10^{-3})/\varepsilon \approx -7.6/\varepsilon$. The resulting tipping-tracking transition curve $c^\dagger$ in the (Δ_m, ε) tipping diagram shows a complicated rate dependence and is far from trivial [Fig. 5.10(b)]. Owing to the absence of basin instability and R-tipping, it is expected that Δ_m has to exceed the critical level H_e for tipping to occur. What is less obvious is the presence of multiple critical rates. Past H_e , there is a range of shift magnitudes Δ_m with a unique critical rate. However, for larger Δ_m , the curve $c^\dagger$ has a ‘bump’ that gives rise to three critical rates for a fixed Δ_m . One can think of this ‘bump’ as a remnant of the R-tipping tongue found for paths Δ_m starting at lower values of m . Most interestingly, there is an interval of Δ_m where the ‘oscillating’ part of $c^\dagger$ gives rise to several critical rates for the same Δ_m [inset in Fig. 5.10(b)].

Chapter 6

Points of Return, Points of No Return, Points of Return Tipping

Tipping is often defined as a large, sudden and possibly unexpected change in the state of the system, caused by a slow or small change in the external input (e.g. environmental conditions). Although “sudden” and “unexpected” suggest that foreseeing and preventing tipping may be difficult, it should in general be possible [19]. In this chapter, we are guided by the question: *Given a monotone parameter shift that gives tipping, when can tipping be prevented by a parameter-shift reversal?* Certain aspects of this question have been explored in the context of B-tipping near a saddle-node bifurcation. For example, Hughes et al. [19] speak of “living dangerously on borrowed time” to describe a window of opportunity for ecosystems to return to safer conditions before an otherwise inevitable tipping occurs. Biggs et al. [18] ask whether early-warning indicators for tipping provide sufficient warning to modify the ecosystem’s management and avert undesired regime shifts by “turning back from the brink”. Gandhi et al. [58, 64] consider non-monotone parameter shifts through the global saddle-node bifurcation (saddle-node on a limit cycle) to identify a new resonance mechanism in the context of spatially localised patterns. Ritchie et al. [21] model systems near a saddle-node bifurcation and analyse relations between the time and amplitude of a saddle-node crossing to avoid B-tipping. More recently, Alkhayoun et al. investigate “avoided” B-tipping and R-tipping near a subcritical Hopf bifurcation in the box model of the Atlantic Meridional Overturning Circulation (AMOC) in the context of collapse of the AMOC and climate change mitigation [22].

Here, we extend the existing literature on avoiding B-tipping to (i) analyse a subcritical Hopf bifurcation (ii) obtain additional results on a saddle-node bifurcation,

and (iii) describe R-tipping effects for shifts that start away from a bifurcation point. What is more, we compare the ecosystem model results, our analysis of canonical forms for the two generic dangerous bifurcations of equilibria: saddle-node and subcritical Hopf bifurcations, and the recent theoretical predictions for saddle-node bifurcation from Ref. [21]. The canonical forms are modified (‘tilted’) normal forms to capture B-tipping near the bifurcation point as well as R-tipping away from the bifurcation point.

Specifically, we consider paths in one parameter μ . A path starts at $\mu = \mu_-$ and may traverse the bifurcation at $\mu = \mu_b$. Along a parameter path, we consider modified monotone shifts that reach a maximum in finite time [green in Fig. 4.4(b)]:

$$\mu(t) = \begin{cases} \mu_- + \Delta_\mu \operatorname{sech}(\varepsilon t), & t \leq 0, \\ \mu_- + \Delta_\mu, & t > 0, \end{cases} \quad (6.1)$$

and are parametrised by the magnitude Δ_μ and rate $\varepsilon > 0$. The parameter shift reversal of (6.1) can, in general, have two additional parameters $\rho, \tau > 0$:

$$\mu(t) = \begin{cases} \mu_- + \Delta_\mu \operatorname{sech}(\varepsilon t), & t \leq 0, \\ \mu_- + \Delta_\mu, & 0 < t < \tau, \\ \mu_- + \Delta_\mu \operatorname{sech}(\rho\varepsilon(t - \tau)), & t \geq \tau, \end{cases} \quad (6.2)$$

where $\rho \neq 1$ allows for different rates of shifting back and forth, and $\tau > 0$ allows for some ‘waiting time’ before turning around. Here, we consider a special case, obtained by setting $\rho = 1$ and $\tau = 0$ in (6.2), which corresponds to parameter shifts (4.3) used in the previous chapter [red in Fig. 4.4(b)]. For each path, we obtain $(\Delta_\mu, \varepsilon)$ combinations where tipping can or cannot be prevented by the parameter-shift reversal. In this way, we uncover four possible regions in the $(\Delta_\mu, \varepsilon)$ tipping diagram:

- *Points of tracking* are defined as $(\Delta_\mu, \varepsilon)$ settings where the system avoids tipping for monotone and non-monotone shifts. This is the safe region of tracking, also referred to as the “safe operating space” [65].
- *Points of return* are defined as $(\Delta_\mu, \varepsilon)$ settings where the system tips for monotone shifts, but does not tip for non-monotone shifts. Here, an otherwise imminent tipping is prevented by the parameter-shift reversal.

- *Points of no return* are defined as $(\Delta_\mu, \varepsilon)$ settings where the system tips for monotone and non-monotone shifts. Here, tipping is not prevented by the parameter-shift reversal.
- *Points of return tipping* are defined as $(\Delta_\mu, \varepsilon)$ settings where the system does not tip for monotone shifts, but tips for non-monotone shifts. Here, the parameter shift reversal inadvertently induces tipping in an otherwise safe situation.

Note that the existence, shape and location of the four regions in the $(\Delta_\mu, \varepsilon)$ tipping diagram will, in general, depend on the geometric form of the shift $\mu(t)$, on the difference between the rates for shifting back and forth ($\rho \neq 0$), and on the waiting time ($\tau > 0$). See Appendix B for a brief exploration of the effect of these dependencies on the tipping diagrams.

To facilitate comparisons with other works that compute the exceedance time t_e , which is the time the system spends past a dangerous bifurcation, we give the formula for t_e in terms of the magnitude Δ_μ and rate ε of the shift (6.2):

$$t_e = \frac{\rho + 1}{\rho \varepsilon} \operatorname{sech}^{-1} \left(\frac{\mu_b - \mu_-}{\Delta_\mu} \right) + \tau, \quad (6.3)$$

where $\operatorname{sech}^{-1} x \geq 0$ for $0 < x \leq 1$.

6.1 The Ecosystem Model

For the ecosystem model (2.7)–(2.8), we consider two different parameter paths giving rise to two different diagrams in Fig. 6.1. The (Δ_r, ε) tipping diagram in Fig. 6.1(a) is obtained for a parameter path with a fixed $m = 0.075$, $r_- = 1$, and $r(t) > 1$ such that the path crosses the boundary BI of the basin instability $BI(e_3, p_1)$, but does not cross any bifurcations. Thus, Fig. 6.1(a) describes points of return and no return for R-tipping alone. Points of no return are bounded by the tipping-tracking transition curve $c^\dagger$ for the non-monotone shift (4.3). Points of return are located between $c^\dagger$ and the tipping-tracking transition curve $c^\uparrow$ for the monotone shift (6.1) with $\mu = r$. At higher ε , (green) points of return extend over the entire Δ_r interval. This is indicative of R-tipping occurring after the input $r(t)$ reaches its maximum. Here, the natural timescales of $H(t)$ and $P(t)$ are slower than $e_3(t)$, and the system is slow to respond to changes in $r(t)$. However, as ε is decreased, $c^\uparrow$ and $c^\dagger$ approach each other so that the (green) points of return shrink and appear to vanish at $\varepsilon \approx 0.2$. Overlapping $c^\uparrow$ and

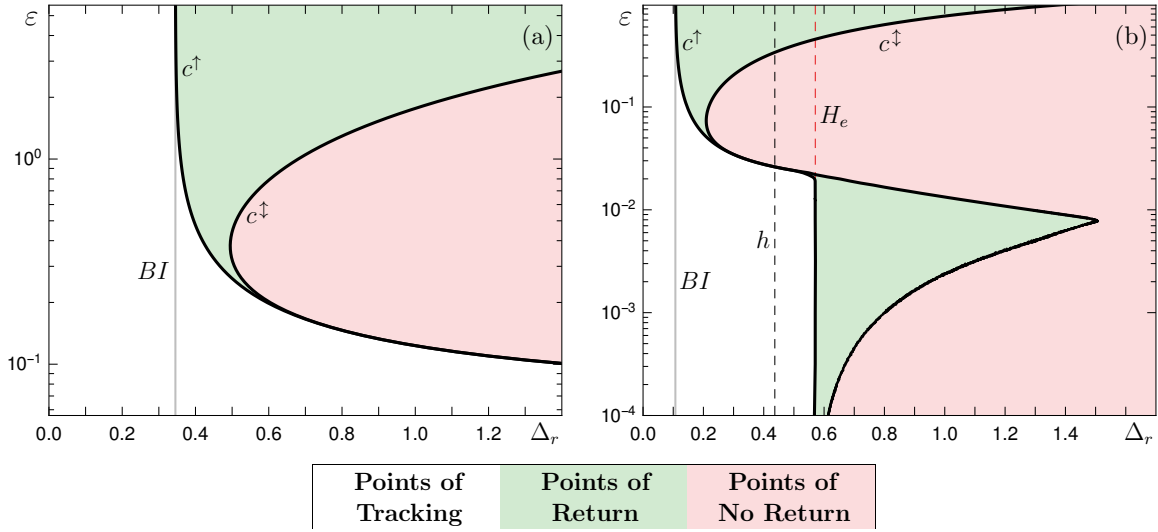


Figure 6.1 Tipping diagrams in the (Δ_r, ε) -plane for (a) R-tipping alone and (b) R- and B-tipping are partitioned into (white) “points of tracking”, (green) “points of return” and pink “points of no return”. (a) The tipping-tracking transition curves $c^\uparrow$ and $c^\downarrow$ are obtained using monotone Eq. (6.1) and non-monotone Eq. (4.3) parameter shifts respectively, from $p_1 = (1.0, 0.075)$ along the parameter path Δ_r from Fig. 4.8. (b) The tipping-tracking transition curves $c^\uparrow$ and $c^\downarrow$ are obtained using monotone Eq. (6.1) and non-monotone Eq. (4.3) parameter shifts respectively, from $p_1 = (0.5, 0.12)$ along the parameter path Δ_r from Fig. 5.2(a) and Fig. 5.6(a). $b = b_c = 0.025$.

$c^\downarrow$ give rise to direct transitions from (white) tracking to (pink) points of no return. This is indicative of R-tipping occurring before the input $r(t)$ reaches its maximum. Here, the natural timescales of $H(t)$ and $P(t)$ become comparable to $e_3(t)$, the system R-tips to $e_2(t)$ during the upshift in $r(t)$, and the parameter-shift reversal has no effect on the overall response of the system. Note that $e_2(t)$ is basin stable on any parameter path within regions 5 and 7.

The (Δ_r, ε) tipping diagram in Fig. 6.1(b) is obtained for the parameter path Δ_r from Fig. 5.4(a) with a fixed $m = 0.12$, $r_- = 0.5$, and $r(t) > 0.5$ such that the path crosses the boundary BI of the basin instability $BI(e_3, p_1)$ as well as the (dangerous) subcritical Hopf Bifurcation H_e . Thus, Fig. 6.1(a) describes points of return and no return for the interplay between B-tipping and R-tipping. At higher ε , R-tipping is the dominant tipping mechanism. Indeed, the part of the tipping diagram between BI and H_e at higher ε is the same as in Fig. 6.1(a), including the vanishing (green) region with points of return. At intermediate ε , the competition between B-tipping and R-tipping gives rise to a deep wedge in $c^\downarrow$, which opens up another (green) region with points of

return. At lower ε , B-tipping is the dominant tipping mechanism. Here, the natural timescales of $H(t)$ and $P(t)$ become faster than $e_3(t)$, and the problem resembles a slow passage through a Hopf bifurcation. The associated bifurcation delay is responsible for the lower boundary of the deep wedge in $c^\dagger$, and for the ‘slow’ convergence of $c^\dagger$ to $c^\uparrow$ (or to H_e) as $\varepsilon \rightarrow 0$.

Overall, the intricate (Δ_r, ε) tipping diagram for the ecosystem model is partitioned into regions of tracking, points of return and points of no return. In particular, there appears to be two different regions of points of return separated by direct transitions from tracking to points of no return. This leads us to the final question: *How typical is the intricate tipping diagram from Fig. 6.1(b)?* To answer this question we analyse tipping diagrams for a (slow) passage through the two generic dangerous bifurcations of equilibria, namely saddle-node and subcritical Hopf bifurcations.

6.2 The Two Generic Dangerous Bifurcations of Equilibria

From among different dangerous bifurcations of equilibria, only saddle-node and subcritical Hopf bifurcations are generic in the sense that they persist under arbitrarily small perturbations of the vector field. Here, we consider modified (‘tilted’) versions of the saddle-node and subcritical Hopf normal forms to study ‘typical’ effects of non-monotone shifts across a dangerous bifurcation. The modification is an additional parameter s that quantifies the ‘tilt’ of the branches of solutions in the one-parameter bifurcation diagram; see Fig 6.2. Both bifurcations occur at $\mu_b = 0$, and the regular normal forms are recovered when $s = 0$. As there is no basin instability in the regular normal forms, there can be no R-tipping from the stable equilibrium when $s = 0$.¹ However, the dynamics change when $s \neq 0$. In particular, R-tipping can be observed when the ‘tilt’ is sufficient enough to give basin instability along the chosen parameter path. In the following, we use μ_* to denote the basin instability boundary.

¹For the subcritical Hopf normal form, both B-tipping and R-tipping from the stable equilibrium can be excluded because the branch of equilibria is a flow-invariant line in the (z, t) phase space of the nonautonomous system. For the saddle-node normal form, R-tipping can be excluded because the system is one-dimensional and the stable equilibrium is basin stable [2].

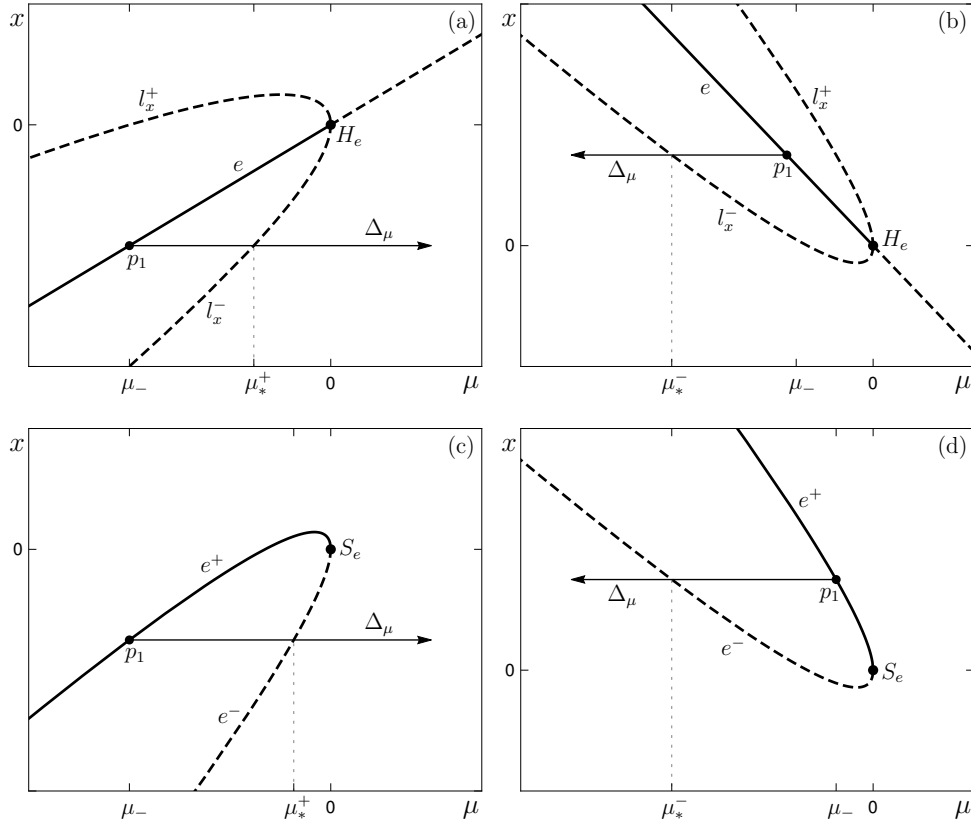


Figure 6.2 One-parameter bifurcation diagrams for the modified (tilted) subcritical Hopf normal form Eq. (6.4) with (a) $s > 0$ and (b) $s < 0$, and for the modified (tilted) saddle-node normal form Eq. (6.6) with (c) $s > 0$ and (d) $s < 0$. Shown are branches of (solid) stable and (dashed) unstable equilibria e , branches of the maxima l_x^+ and minima l_x^- of the x -component of the unstable limit cycle, parameter paths Δ_μ from $p_1 = \mu_-$, and the corresponding basin instability boundary $\mu_*^\pm$ of (a-b) $BI(e, \mu_-)$ and (c-d) $BI(e^+, \mu_-)$.

6.2.1 Generic Subcritical Hopf Bifurcation

To make comparisons with the ecosystem model, we begin with analysis of the generic subcritical Hopf bifurcation. Consider a system in $\mathbb{R}^2$ akin to the normal form of the subcritical Hopf bifurcation [34, Sec.3.4] written in terms of a complex variable $z = x + iy$:

$$\dot{z} = (\mu + i[\omega + \alpha|z - \mu s|^2])(z - \mu s) + |z - \mu s|^2(z - \mu s). \quad (6.4)$$

where μ is the bifurcation parameter, ω is the angular frequency of small-amplitude oscillations, α quantifies the amount of shear or amplitude-phase coupling and s is

the ‘tilt’ parameter. The subcritical Hopf normal form is recovered when we set $s = 0$ and apply a change of coordinates to transform away the term proportional to α [34, Sec.3.4]. There is one branch of equilibria

$$e(\mu, s) = \mu s + 0i,$$

that is stable for $\mu < 0$ and unstable for $\mu > 0$, and one branch of unstable limit cycles

$$l(\mu, s, t) = \mu s + \sqrt{-\mu} e^{i(\omega - \alpha\mu)t},$$

that exists for $\mu < 0$. The real part of the limit cycle solution oscillates between

$$l_x^-(\mu, s) = -\sqrt{-\mu} + \mu s \quad \text{and} \quad l_x^+(\mu, s) = \sqrt{-\mu} + \mu s,$$

as shown in Fig. 6.2(a)–(b). For every $s \neq 0$, there are two basin instability boundaries. They are obtained by fixing μ_- and solving

$$\text{Re}[e(\mu_-, s)] = l_x^-(\mu_*, s) \quad \text{and} \quad \text{Re}[e(\mu_-, s)] = l_x^+(\mu_*, s),$$

for μ_* , which gives

$$\mu_*^- = \mu_- - \frac{1 + \sqrt{1 - 4s^2\mu_-}}{2s^2} < \mu_- \quad \text{and} \quad \mu_*^+ = \mu_- - \frac{1 - \sqrt{1 - 4s^2\mu_-}}{2s^2} > \mu_-.$$

Since we restrict to small enough and positive shift magnitudes $\Delta_\mu > 0$, the relevant basin instability boundary is $\mu_*^+ > \mu_-$; see Fig. 6.2(a).

Now, consider the corresponding nonautonomous system

$$\dot{z} = (\mu(t) + i[\omega + \alpha|z - s\mu(t)|^2])(z - s\mu(t)) + |z - s\mu(t)|^2(z - s\mu(t)), \quad (6.5)$$

initialised at

$$z(t_0) = e(\mu(t_0), s), \quad t_0 = \frac{1}{\varepsilon} \text{sech}^{-1}(10^{-3}).$$

Firstly, we analyse R-tipping for non-monotone $\mu(t)$ given by Eq. (6.2) with $\mu_- = -1$, $\Delta_\mu > 0$, $\rho = 1$, $\tau = 0$ and different values of s [Fig. 6.3(a)]. Note that the line $e = \mu s + 0i$ is flow-invariant when $s = 0$, but not when $s \neq 0$. Therefore, tipping from the stable equilibrium e requires nonzero s . For $s = 10^{-4}$, we obtain $\mu_*^+ \approx -10^{-8}$, meaning that the region of basin instability between μ_*^+ and H_e is negligible. The only

tipping that occurs in the nonautonomous system is B-tipping for $\Delta_\mu > 1$, as evidenced by the tipping-tracking transition curve $c^\dagger$ in the $(\Delta_\mu, \varepsilon)$ tipping diagram. When $s = 0.5$, the basin instability boundary moves to $\mu_*^+ = 2\sqrt{2} - 3 \approx -0.17$ or $\Delta_\mu \approx 0.83$, and the region of basin instability becomes non-negligible. As a result, the curve $c^\dagger$ deviates from the case $s = 0$ in different ways. While R-tipping still does not occur, basin instability gives rise to a fold on $c^\dagger$ and a range of shift magnitudes Δ_μ with three critical rates. When the ‘tilt’ is increased to $s = 2$, the basin instability boundary moves to $\mu_*^+ \approx -0.61$ or $\Delta_\mu \approx 0.39$. Now, in addition to B-tipping and a range of Δ_μ with three critical rates, there is R-tipping for $\Delta_\mu < 1$. The tracking-tipping transition curve $c^\dagger$ closely resembles the tipping diagram for the ecosystem model from Fig. 5.6. The R-tipping tongue at higher rates is the result of basin instability. The ‘slow’ approach (and possibly lack of convergence) of the $c^\dagger$ curves towards H_e as $\varepsilon \rightarrow 0$ is the result of a surprising property of the slow passage through a Hopf bifurcation. Namely, the distance the solution tracks the unstable equilibrium past the bifurcation point is independent of the rate of parameter change and does not tend to zero as $\varepsilon \rightarrow 0$ [61–63]². In other words, the system tracks the unstable equilibrium past the bifurcation point for a noticeable amount of time, making it possible to turn around and avoid tipping even for vanishing rates of parameter change. The most noticeable difference from the ecosystem model is the absence of the “deep wedge” at the intermediate rates. Instead, there is a characteristic kink on the $c^\dagger$ curves near $\varepsilon = 10^{-2}$ in Fig. 6.3(a), possibly with ‘oscillations’ such as those shown in Fig. 5.10(b). The origin of the kink and the ‘oscillations’, as well as the scaling law for $c^\dagger$ in the limit $\varepsilon \rightarrow 0$, are left for future study.

The agreement with the ecosystem model extends to “points of return” and “points of no return” as shown in Figs. 6.1(b) and 6.4(b1), where the tracking-tipping transition curve $c^\dagger$ is obtained for the monotone parameter shift (6.1). Interestingly, for sufficiently high ‘tilt’ parameter s , a new region of “points of return tipping” appears in the diagram [Fig. 6.4(c1)] that is not present in the ecosystem model. This means that, in general, all four regions identified in the beginning of Chapter 6 can be present for a non-monotone passage through a subcritical Hopf bifurcation. What is more, the $c^\dagger$ and $c^\ddagger$ curves need not approach each other like they do in the ecosystem model in Fig. 6.1(b). Finally, the rotational symmetry in the phase space of the (modified) Hopf normal

²A slow passage through a Hopf bifurcation is a singular perturbation problem since there is no delay for the static case $\varepsilon = 0$ and there is a noticeable delay for arbitrarily small but nonzero ε .

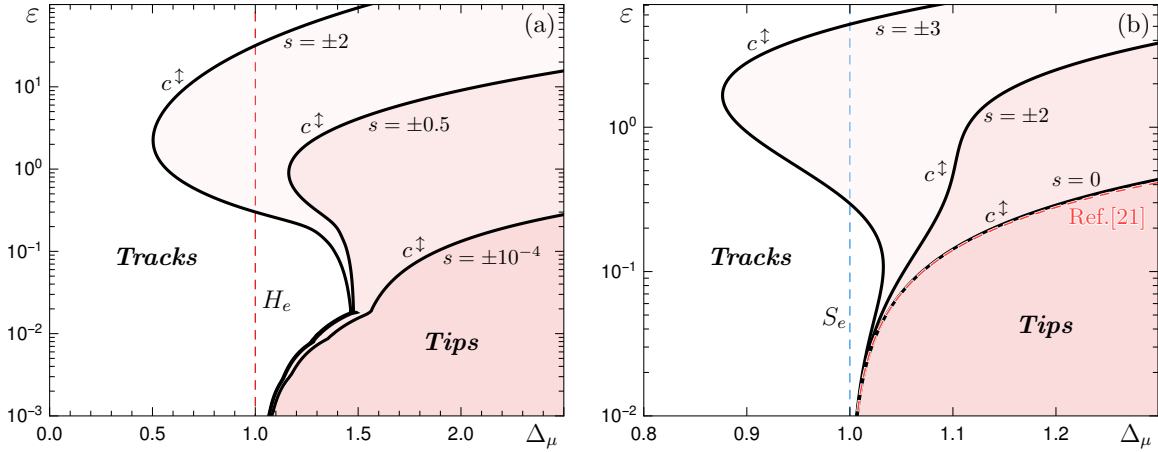


Figure 6.3 Tipping diagrams in the $(\Delta_\mu, \varepsilon)$ parameter plane for non-monotone shifts (6.2) with $\tau = 0$, $\rho = 1$ and $\mu_- = -1$ from $p_1 = \mu_-$ along the parameter path Δ_μ from Fig. 6.2(a) and (c). (a) Tipping-tracking transition curves $c^\ddagger$ for the modified (tilted) subcritical Hopf normal form Eq. (6.5) with $\alpha = 1$, $\omega = 1$ and different values of s . (b) Tipping-tracking transition curves $c^\ddagger$ for the modified (tilted) saddle-node normal form Eq. (6.9) with different values of s . The dashed red curve in (b) is the approximation to $c^\ddagger$ obtained in Ref. [21] for $s\varepsilon$ small enough.

form implies a symmetry in the basin instability boundaries

$$\mu_*^\pm(s) = \mu_*^\pm(-s),$$

meaning that the the system has the same basin instability properties for s and $-s$. According to the R-tipping criterion from Chapter 4.2, given a suitable $\mu(t)$ that increases over time, R-tipping for s and $-s$ requires the same shift magnitude. Similarly, given a suitable $\mu(t)$ that decreases over time, R-tipping for s and $-s$ requires the same shift magnitude. Thus, we obtain the same tipping diagrams for s and $-s$ in the left column of Fig. 6.4. For a fixed $s \neq 0$, R-tipping for an increasing $\mu(t)$ requires a smaller shift magnitude than R-tipping for the decreasing $\mu(-t)$. This is why the region of “points of return tipping” in Fig. 6.4(c1) is small.

6.2.2 Generic Saddle-Node Bifurcation

To make comparisons with the other generic dangerous bifurcation of equilibria, consider a system in $\mathbb{R}$ akin to the normal form of the saddle-node bifurcation [34, Sec.3.2]:

$$\dot{x} = -(x - \mu s)^2 - \mu, \quad (6.6)$$

where μ is the bifurcation parameter and s is the ‘tilt’ parameter. The branches of stable e^+ and unstable e^- equilibria exist for $\mu \leq 0$ and are given by

$$e^+(\mu, s) = \mu s + \sqrt{-\mu}, \quad \text{and} \quad e^-(\mu, s) = \mu s - \sqrt{-\mu}, \quad (6.7)$$

as shown in Fig. 6.2(c)–(d). The basin instability boundary is obtained by fixing μ_- and solving

$$e^+(\mu_-, s) = e^-(\mu_*, s),$$

for μ_* . The boundary exists for $s < 0$ or $s > 1/\sqrt{-\mu_-}$ and is given by

$$\mu_* = - \left(\sqrt{-\mu_-} - \frac{1}{s} \right)^2. \quad (6.8)$$

Now, consider the corresponding nonautonomous system

$$\dot{x} = -(x - \mu(t)s)^2 - \mu(t), \quad (6.9)$$

initialised at

$$x(t_0) = e^+(\mu(t_0), s), \quad t_0 = \frac{1}{\varepsilon} \text{sech}^{-1}(10^{-3}).$$

Firstly, we analyse R-tipping for non-monotone $\mu(t)$ given by Eq. (6.2) with $\mu_- = -1$, $\Delta_\mu > 0$, $\rho = 1$, $\tau = 0$ and different values of s [Fig. 6.3(b)]. When $s = 0$, there is no basin instability and R-tipping cannot occur. The only tipping that occurs for $s = 0$ is B-tipping for $\Delta_\mu > 1$.

Ritchie et al. [21] considered the problem where the system parameter μ drifts past the saddle-node bifurcation at $\mu = \mu_b$ and then turns around. For $s\varepsilon$ small enough, they obtained an asymptotic formula to prevent tipping in terms of the distance $\Delta_\mu + \mu_-$ the parameter travels past the bifurcation and the time t_e the system spends past the bifurcation. More specifically, their criterion corresponding to our tracking-tipping transition curve $c^\dagger$ has the form [21, Eq.(2.10)]:

$$d^b(\Delta_\mu + \mu_-)t_e^2 = 16, \quad (6.10)$$

where

$$d^b = \lim_{\mu \nearrow \mu^b} \frac{[\lambda(\mu)]^2}{\mu^b - \mu},$$

is the system-dependent factor and $\lambda(\mu)$ is the leading eigenvalue of the stable equilibrium.

To compare our computations of $c^\dagger$ with the result derived in Ref. [21] we obtain the exceedance time t_e for the non-monotone parameter shift Eq. (6.2) with $\mu_- = -1$, $\Delta_\mu > 0$, $\rho = 1$ and $\tau = 0$:

$$t_e = \frac{2}{\varepsilon} \operatorname{sech}^{-1}(-\mu_-/\Delta_\mu) > 0,$$

note that the modified saddle-node normal form (6.6) has $\mu_b = 0$ and $\lambda(\mu) = -2\sqrt{-\mu}$ which give

$$d^b = \lim_{\mu \nearrow \mu^b} \frac{[\lambda(\mu)]^2}{\mu^b - \mu} = 4,$$

and rewrite Eq. (6.10) in terms of ε and Δ_μ :

$$\varepsilon = \sqrt{\Delta_\mu + \mu_-} \operatorname{sech}^{-1}\left(\frac{-\mu_-}{\Delta_\mu}\right). \quad (6.11)$$

Condition (6.11) is plotted as a red dashed curve in Fig. 6.3(b). The tracking-tipping transition curve $c^\dagger$ in the $(\Delta_\mu, \varepsilon)$ tipping diagram is in very good agreement with the critical “exceedance time” formula derived in Ref. [21]:

$$t_e \approx \frac{2}{\sqrt{\Delta_\mu + \mu_-}}.$$

When $s = 2$, there is a basin instability boundary at $\mu_* = -1/4$, meaning that the stable equilibrium e^+ is basin unstable for $\Delta_\mu > \mu_* - \mu_- = 3/4$. Although the tracking-tipping transition curve $c^\dagger$ deviates noticeably from the case $s = 0$, especially at higher rates ε , the changes are quantitative and R-tipping still does not occur. When the ‘tilt’ is increased to $s = 3$, the basin instability boundary moves to $\mu_* = -4/9$, meaning that e^+ is basin unstable for $\Delta_\mu > \mu_* - \mu_- = 5/9$. This results in two significant changes to the tracking-tipping transition curve $c^\dagger$. Firstly, $c^\dagger$ develops two folds and becomes S-shaped, giving rise to a range of shift magnitudes Δ_μ with three different critical rates. Secondly, in addition to B-tipping, there is an R-tipping tongue for $\Delta_\mu < 1$. In contrast to the Hopf bifurcation, different $c^\dagger$ curves appear to converge to S_e as $\varepsilon \rightarrow 0$.

This is because the distance the solution overshoots the saddle-node bifurcation point vanishes as the rate of parameter change tends to zero [66, 67]. In other words, in the limit of a vanishing rate of parameter change, the solution undergoes an instantaneous

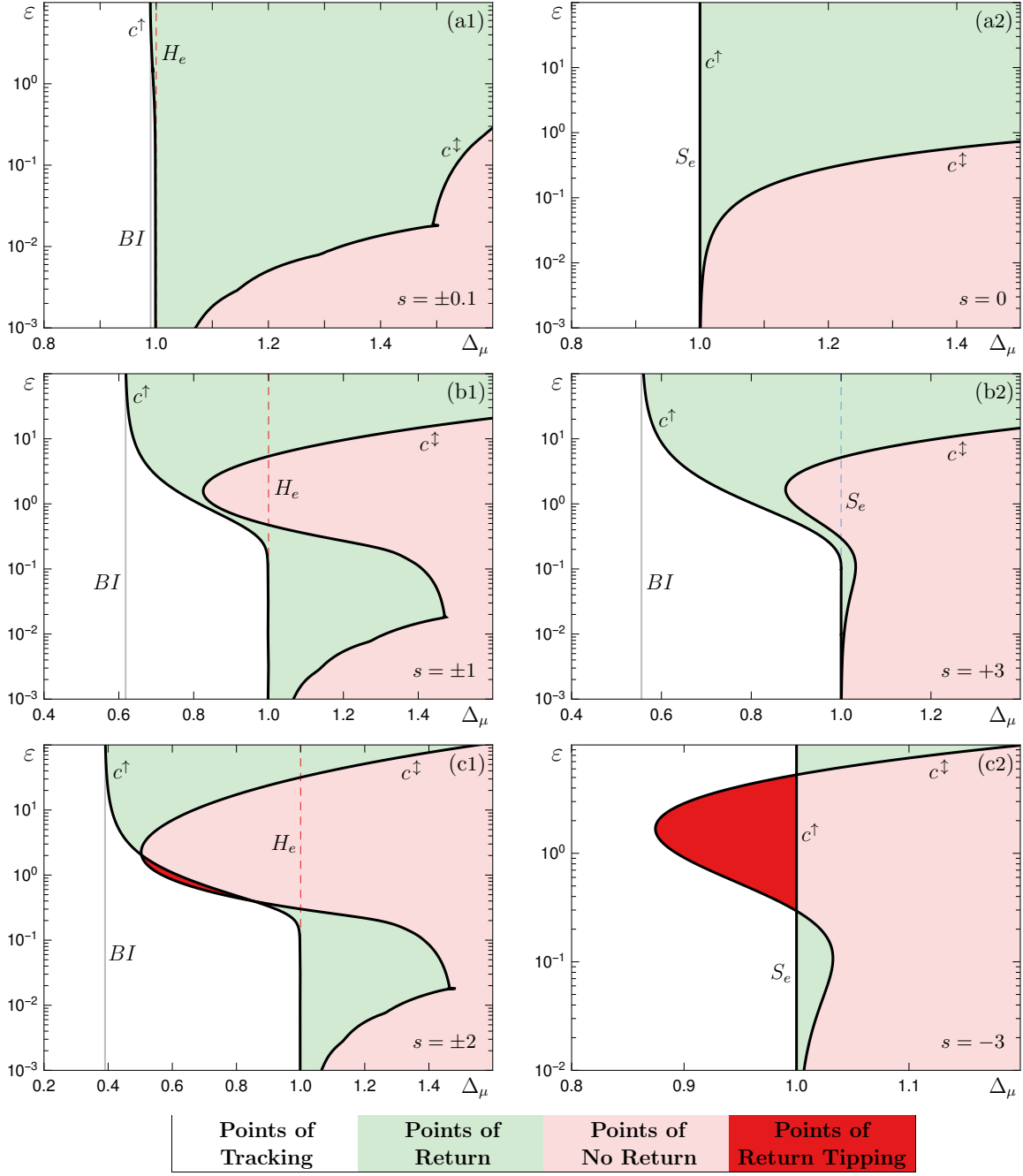


Figure 6.4 Tipping diagrams (a1–c1) for the modified (tilted) Hopf normal form Eq. (6.5) and different values of s and (a2–c2) for the modified (tilted) saddle-node normal form Eq. (6.5) are partitioned into (white) “points of tracking”, (green) “points of return”, (pink) “points of no return” and (red) “points of return tipping”. The tipping-tracking transition curves $c^\uparrow$ and $c^\downarrow$ are obtained for monotone Eq. (6.1) and non-monotone Eq. (6.2) parameter shifts, respectively, with $\tau = 0$, $\rho = 1$ and $\mu_- = -1$ along the parameter path Δ_μ from Fig. 6.2(a) and (c).

jump off the branch of stable equilibria at the bifurcation point, leaving no time to turn around and avoid tipping.

Apart from some differences at small ε owing to the different character of the bifurcation delay, the analysis of “points of return” and “points of no return” near a saddle-node bifurcation reveals much similarity to the subcritical Hopf bifurcation when $s > 0$ [Fig. 6.4(b2)]. However, the dynamics for $s < 0$ are rather different. The striking difference for $s = -3$ is the large region of “points of return tipping”, where there is R-tipping for non-monotone $\mu(t)$, but not for monotone increasing $\mu(t)$ [Fig. 6.4(c2)]. This difference is a consequence of asymmetry in the (modified) saddle-node normal form. To be more specific,

$$\mu_*(s) \neq \mu_*(-s),$$

meaning that the system has different basin instability properties for s and $-s$. According to the R-tipping criterion from Chapter 4.2, given a suitable $\mu(t)$ that increases over time, the system is guaranteed to R-tip for $s > 0$, but not for $s < 0$. Conversely, given a suitable $\mu(t)$ that decreases over time, the system is guaranteed to R-tip for $s < 0$, but not for $s > 0$. Thus, “points of return tipping” cannot occur for $s > 0$, and are expected to occur for $s < 0$, which explains the diagrams for $s = 3$ and $s = -3$ in Fig. 6.4(b2) and (c2).

6.3 Universal Properties of a Non-monotone Passage Through a Dangerous Bifurcation

A comparison between the tracking-tipping transition curves $c^\dagger$ for the modified subcritical Hopf [Fig. 6.3(a)] and saddle-node [Fig. 6.3(b)] normal forms reveals universal qualitative properties of a non-monotone passage through a dangerous bifurcation that are independent of the bifurcation type. In both systems, the tracking-tipping transition curve $c^\dagger$ becomes S-shaped, gives rise to three critical rates and develops an R-tipping tongue as the ‘tilt’ parameter s is increased. On the other hand, there are differences between the two systems that are also worth pointing out. Multiple critical rates and R-tipping are achieved for a smaller ‘tilt’ parameter s in the modified Hopf normal form, whereas the approach of $c^\dagger$ towards the bifurcation as $\varepsilon \rightarrow 0$ is much faster and follows a different scaling law in the modified saddle-node normal form. What is more, owing to the basin instability properties, a saddle-node bifurcation may give rise to a larger region of “points of return tipping”.

Chapter 7

Outlook: Compactification of the Ecosystem Model

Analysis of non-autonomous systems is hindered by the lack of compact invariant sets such as equilibria or limit cycles. In this chapter, we briefly discuss a suitable compactification technique, developed in Ref. [68], to reformulate the problem as an extended autonomous dynamical system that contains compact invariant sets. This is limited to non-autonomous inputs that ‘die out’ at infinity, but simplifies the analysis of R-tipping. Specifically, non-autonomous R-tipping instabilities can be studied in terms of heteroclinic connecting orbits in the extended compactified system. We refer to Wieczorek et al. [68] for more details on compactification.

7.1 The General Idea

If we consider a path Δ_λ in one parameter λ , we can limit to external inputs $\lambda(t)$ that trace out the path and are bi-asymptotically constant, meaning that

$$\lim_{t \rightarrow \infty} \lambda(t) = \lambda^+ \text{ and } \lim_{t \rightarrow -\infty} \lambda(t) = \lambda^-,$$

with λ^+ and $\lambda^- \in \mathbb{R}$ being the future and a past limit, respectively. This gives the autonomous future limit system

$$\dot{x} = f(x, \lambda^+),$$

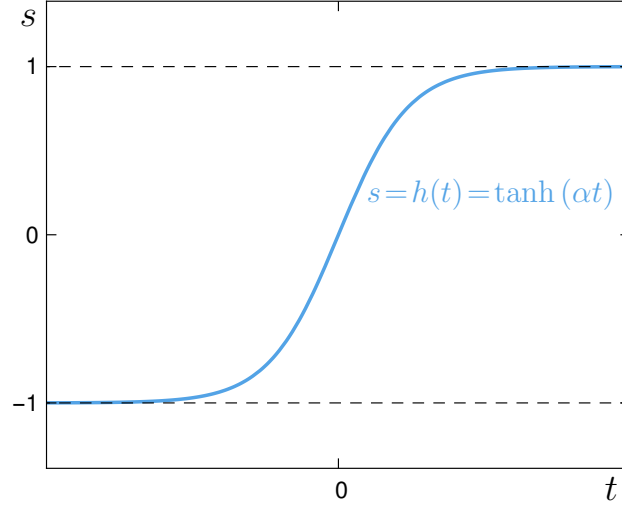


Figure 7.1 The nonlinear coordinate transformation $s = h(t) = \tanh \alpha t$, $\alpha > 0$.

and the autonomous past limit system

$$\dot{x} = f(x, \lambda^-).$$

The limit systems contain regular equilibria

$$e_i^\pm = \lim_{t \rightarrow \pm\infty} e_i(\lambda(t)) = e_i(\lambda^\pm).$$

So far, in examining the dynamics of the non-autonomous system

$$\dot{x} = f(x, \lambda(t)), \tag{7.1}$$

our approach has been to introduce an additional unbounded variable, $\nu \in \mathbb{R}$:

$$\begin{aligned} \dot{x} &= f(x, \lambda(\nu)), \\ \dot{\nu} &= 1, \quad \nu(t_0) = t_0. \end{aligned} \tag{7.2}$$

As the system above is not defined for $t = \pm\infty$, then neither are the equilibria $e_i^\pm$. However, if $\lambda(t)$ is bi-asymptotically constant, we can introduce a different dependent variable $s \in [-1, 1]$ that is bounded and allows us to include the equilibria $e_i^\pm$.

7.2 Compactifying the Ecosystem Model

In Chapter 4.4 we introduced the monotone forcing function (4.2) representing a shift from r_- to $r_- + \Delta_r$, and the non-monotone forcing function (4.3) representing a shift from r_- to $r_- + \Delta_r$ and back again to r_- . Because these are both bi-asymptotically constant, we can apply the compactification transformation [68]:

$$s = h(t) = \tanh(\alpha t),$$

with the inverse

$$t = \frac{1}{2\alpha} \ln \left(\frac{1+s}{1-s} \right), \quad (7.3)$$

where α is the compactification parameter. Then, the non-autonomous terms can be expressed in terms of s . Specifically, the monotone input (4.2) becomes

$$r(s) = r_- + \Delta_r \frac{(1+s)^{\varepsilon/\alpha}}{(1+s)^{\varepsilon/\alpha} + (1-s)^{\varepsilon/\alpha}}$$

and the non-monotone input (4.3) becomes

$$r(s) = r_- + 2\Delta_r \frac{(1-s^2)^{\varepsilon/2\alpha}}{(1+s)^{\varepsilon/\alpha} + (1-s)^{\varepsilon/\alpha}}.$$

We differentiate (7.3) with respect to s and use its reciprocal to obtain the dynamical equation for the time evolution of $s(t)$:

$$\frac{ds}{dt} = \alpha(1-s^2). \quad (7.4)$$

We can now write the ecosystem model (2.1)–(2.2) as an extended autonomous compactified system

$$\begin{aligned} \frac{dP}{dt} &= f(P, H, r(s)), \\ \frac{dH}{dt} &= g(P, H, m), \\ \frac{ds}{dt} &= \alpha(1-s^2), \end{aligned} \quad (7.5)$$

defined on $\mathbb{R}^2 \times [-1, 1]$.

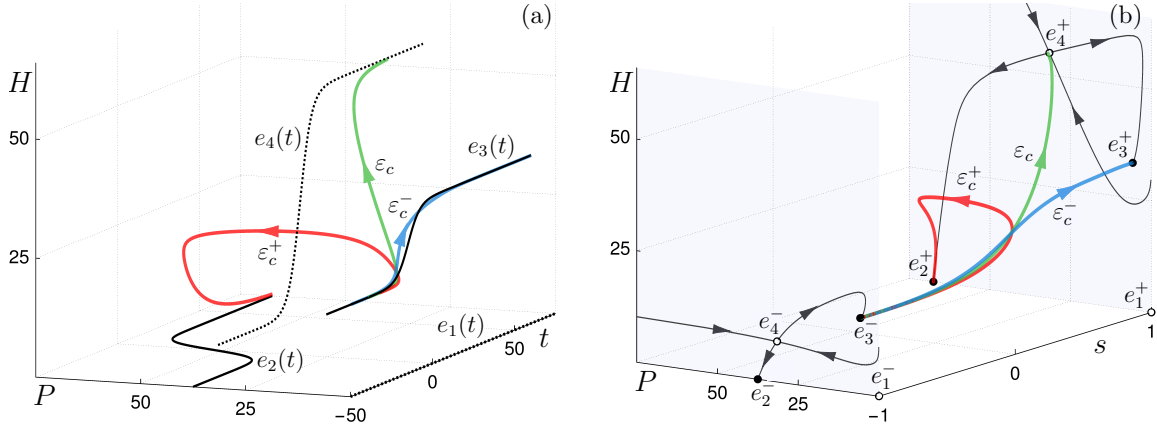


Figure 7.2 (a) The non-autonomous system (2.7)–(2.8) on $\mathbb{R}^2 \times \mathbb{R}$ with unbounded monotone forcing (4.2), identical to Fig. 4.5(b). (b) The autonomous compactified system (7.5) on $\mathbb{R}^2 \times [-1, 1]$ with compact monotone forcing (7.7).

The Jacobian matrix of system (7.5) is

$$J = \begin{pmatrix} f'(P) & f'(H) & r'(s) \\ g'(P) & g'(H) & 0 \\ 0 & 0 & -2\alpha s \end{pmatrix}, \quad (7.6)$$

where for monotonic forcing $r(s)$ (7.7),

$$r'(s) = 2\Delta_r \frac{\varepsilon(1-s^2)^{-1+\varepsilon/\alpha}}{\alpha((1-s)^{\varepsilon/\alpha} + (1+s)^{\varepsilon/\alpha})^2}.$$

This is only defined for $0 < \alpha < \varepsilon$. Choosing $\alpha = \varepsilon/2$, gives the bi-asymptotically constant, monotone

$$r(s) = r_{\min} + \Delta_r \frac{(1+s)^2}{(1+s)^2 + (1-s)^2}, \quad (7.7)$$

and bi-asymptotically constant, non-monotone

$$r(s) = r_{\min} + \Delta_r \frac{2(1-s^2)}{(1+s)^2 + (1-s)^2}. \quad (7.8)$$

The compactified system has two invariant subspaces $\{s = \pm 1\}$. Equilibria e_2^- and e_3^- are stable within the subspace $\{s = -1\}$, but gain one unstable direction when embedded in the extended phase space, turning e_2^- into a saddle and e_3^- into a saddle-focus. There is no change in the stability of unstable equilibria e_1^- and e_4^- : they

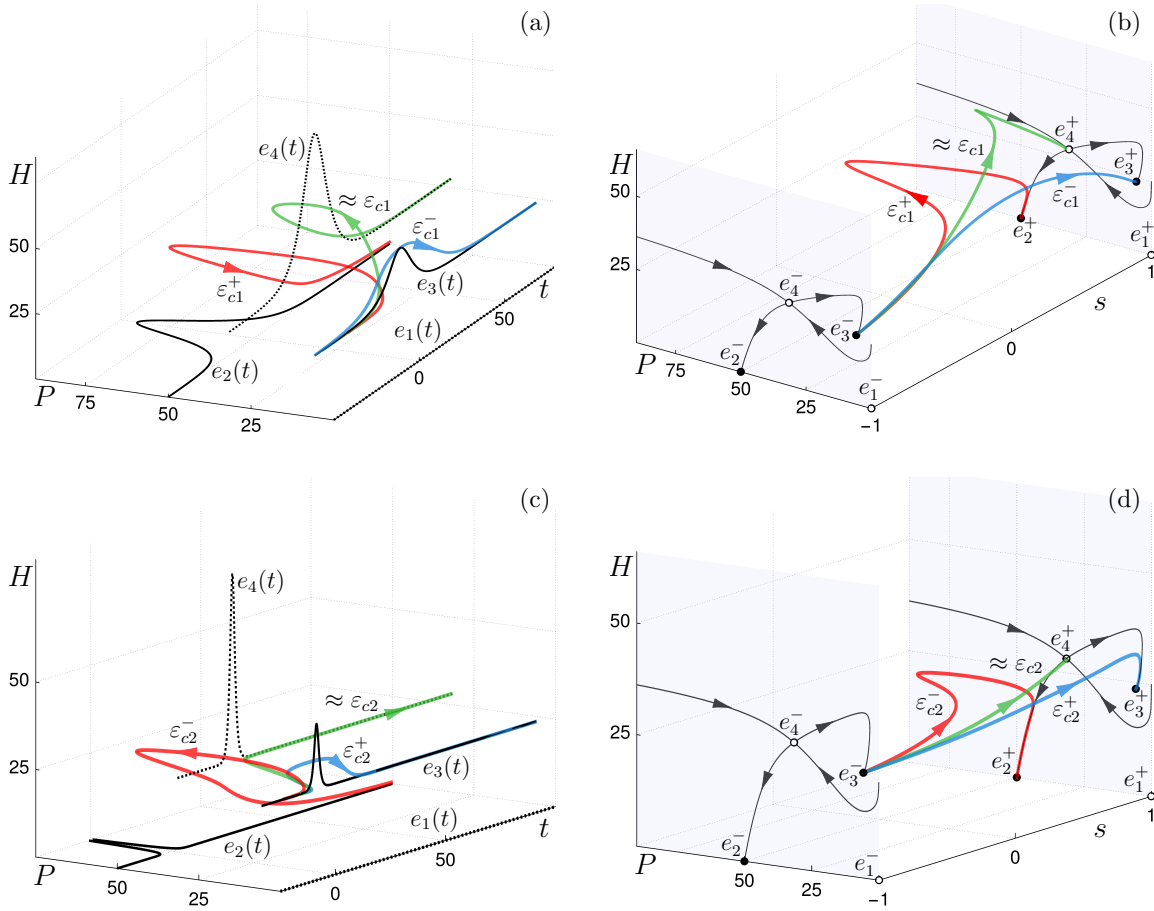


Figure 7.3 (a) and (c) The non-autonomous system (2.7)–(2.8) on $\mathbb{R}^2 \times \mathbb{R}$ with unbounded non-monotone forcing (4.3), identical to Fig. 4.7. (b) and (d). The extended autonomous compactified system (7.5) on $\mathbb{R}^2 \times [-1, 1]$ with compact non-monotone forcing (7.8).

remain saddles with one additional unstable eigendirection. All equilibria within the subspace $\{s = +1\}$ gain one additional stable direction, meaning that, e_3^+ remains a stable sink, e_1^+ and e_4^+ remain saddles, and e_2^+ remains a stable node [Figs. 7.2(b) and 7.3(b) and (d)].

Utilising the same numerical approximation techniques that were used in the previous chapters, we calculated the trajectories near critical rates, using the same initial conditions and parameter values for the trajectories in the non-autonomous system (2.7)–(2.8) as shown in Fig. 4.5(b) and Fig. 4.7.

Figs. 7.2(b) and 7.3(b) and (d) show that, in the compactified system, the trajectory for the critical rate ε_c is recognisable as a heteroclinic connecting orbit from the

saddle equilibrium e_3^- , to the saddle equilibrium e_4^+ . Numerical techniques, such as Lin's method [69, 70], are traditionally employed to detect and perform numerical continuation of these heteroclinic orbits, however, this is not explored here and is left for future study.

In Fig. 4.5 we found the critical rate $\varepsilon_c \approx 0.142675924301148$, in the non-autonomous system (2.7)–(2.8) with the unbounded monotone forcing function (4.2), $\Delta_r = 0.6$. In the compactified system for the same parameter values, with the compact monotone forcing function (7.7), we found $\varepsilon_c \approx 0.142691080668076$, this is close to just over a percentage error of 0.01%. Similarly, in Fig. 4.7 we found $\varepsilon_{c1} \approx 0.166491526823788$ and $\varepsilon_{c2} \approx 1.049396269470948$, with the unbounded non-monotone forcing function (4.3), $\Delta_r = 0.7$. For the same parameter values in the compactified system, with the compact non-monotone forcing function (7.8) we found $\varepsilon_{c1} = 0.166640599679522$, close to within a percentage error of 0.09% and $\varepsilon_{c2} = 1.048638628584058$, close to a percentage error of just above 0.07%.

Chapter 8

Conclusion

In this thesis we analyse nonlinear tipping phenomena in non-autonomous systems using examples of an ecological model [7] and modified saddle-node and subcritical Hopf normal forms with smooth parameter shifts. The mathematical work is motivated and inspired by two scientific concerns. One is Article 2 of the United Nations Framework Convention for Climate Change (UNFCCC) [23] highlighting two critical factors for real-world tipping points: *critical levels* and *critical rates (time frames)* of changing environmental conditions. This was later extended to become the Kyoto Protocol [71] and the current Paris Agreement [72]. The other is the question of whether tipping can be prevented by a parameter trend reversal. We combine classical bifurcation analysis with the concepts of *parameter paths* and *basin instability* to give new insight into critical rates, uncover non-trivial effects arising from the interplay between critical levels (B-tipping) and critical rates (R-tipping), and extend the existing literature on preventing tipping by a parameter trend reversal.

We begin with classical bifurcation analysis of the corresponding autonomous ecosystem model with fixed in time parameters and identify a codimension-three degenerate Bogdanov-Takens bifurcation as the organising centre for B-tipping and the source of a dangerous subcritical Hopf bifurcation. We give testable criteria for B-tipping in the non-autonomous system in terms of parameter paths that cross a subcritical Hopf in the corresponding autonomous system. Next, we perform basin instability analysis to reveal and give testable criteria for R-tipping in the non-autonomous system in terms of parameter paths that do not cross any bifurcation in the corresponding autonomous system. Finally, we produce a single diagram encompassing criteria for both B-tipping and R-tipping by superimposing regions of basin instability on a classical two-parameter bifurcation diagram of the plant growth rate vs. the herbivore mortality rate. This

approach gives new insight into system stability, beyond traditional bifurcation analysis and adiabatic effects of a parameter change. Specifically, it provides basic information on R-tipping which is a genuine non-autonomous instability, captures non-trivial non-adiabatic effects of a parameter change, and thus guides tipping analysis of the non-autonomous system.

In the non-autonomous system with time-varying parameters we obtain *tipping diagrams* in the plane of the rate and magnitude of parameter shift and show that:

- R-tipping transitions in the tipping diagram correspond to special solutions in the phase space that are akin to *maximal canard trajectories* which, rather surprisingly, track a moving unstable state for infinite time.
- R-tipping transition curves in the tipping diagram for non-monotone parameter shifts that cross a basin instability boundary alone and then turn around can form *R-tipping tongues* with two critical rates. This means that the system switches from tracking to tipping and back to tracking again as the rate of the parameter shift increases. R-tipping tongues are reminiscent of resonance tongues in the sense of enhanced response to optimally timed external inputs.
- The interplay between critical levels and critical rates (or between B-tipping and R-tipping) for non-monotone parameter shifts that cross a basin instability boundary and a dangerous bifurcation and then turn around gives rise to an *S-shaped tipping-tracking transition curve* in the tipping diagram with one critical level and multiple critical rates. This means the system exhibits inverted behaviour to an R-tipping tongue and switches from tipping to tracking and back to tipping again as the rate of the parameter shift increases.
- Given a monotone parameter shift and its non-monotone reversal, tipping diagrams can be partitioned into *points of tracking*, *points of return* where tipping can be prevented by the reversal, *points of no return* where tipping cannot be prevented by the reversal, and *points of return tipping* where tipping is inadvertently induced by the reversal. This partitioning provides an alternative way to categorise tipping phenomena.

Our results on the ecosystem model give new insight into the sensitivity of ecosystems to the magnitudes and rates of environmental change. More generally, the method of superimposing regions of *basin instability* on traditional bifurcation diagrams can be extended to regions of *threshold instability* for tipping thresholds that do not separate

the phase space into different basins of attraction [27, 57]. Such an approach would capture, in addition to B-tipping due to dangerous bifurcations, both *irreversible* and transient *reversible R-tipping*, and facilitate systematic in-depth analysis of tipping phenomena in any nonlinear system. This is evidenced further by a comparison of the ecosystem model with the modified saddle-node and subcritical Hopf normal forms that reveals some universal features of non-monotone parameter shifts that cross a basin instability boundary and a dangerous bifurcation and then turn around.

Finally in Chapter 7, we provide an outlook for further study. As there is yet to be universal consensus on the methods we employ in studying the dynamics of the non-autonomous systems, we convert the ecosystem model to an extended autonomous compactified system. Applying the same initial conditions and parameter values that were used for studying purely R-tipping in Chapter 4, we produce trajectories for critical rates in the compactified system. Although analysis of compactified systems usually employs numerical techniques such as Lin's Method, we calculated the trajectories using the same numerical approximation techniques we had used previously. Nonetheless, it is gratifying to find such a strong correlation between the non-autonomous and extended autonomous models.

Appendix A

Numerical Continuation Software AUTO

A.1 Finding Hopf H_e and homoclinic h Bifurcation Curves

In the autonomous ecosystem model (2.1)–(2.2) we used AUTO to find both the Hopf and homoclinic bifurcation curves in the (r, m) bifurcation plane. In using AUTO, we begin with an equilibrium solution for a specific set of parameter values. We then prepare two files (Sec. A.2 and A.3). The first is an ‘equations-file’ that can either be written in fixed-form (old-style) Fortran (xxx.f), free-form Fortran (xxx.f90) or in C (xxx.c), where xxx is a user supplied name. This file contains the Fortran routines for the required task, as well as the equations, parameter values of the system being investigated, and equilibrium solutions. The second file is a ‘constants-file’ named c.xxx. This file is where one specifies the types of bifurcations that are to be detected, the control parameters and error tolerances, etc. We refer to the AUTO manual [50] for the significance of each constant.

What follows is a demonstration of how this was done for Fig. 3.4(d), $b = b_c = 0.025$. In this example we solve the equilibrium equations for $(r, m) = (2.5, 0.05)$, we then write an ‘equations-file’ (Sec.A2) for the required task.

```
paulokeeffe > cp c.hopf.1 c.hopf
paulokeeffe > @r hopf
Starting hopf ...
```

BR	PT	TY	LAB	PAR(2)	L2-NORM	U(1)	U(2)
1	1	EP	1	5.00000E-02	7.44049E+01	4.28009E+00	7.42817E+01
1	87	HB	2	1.11236E-01	5.90236E+01	8.66639E+00	5.83839E+01
1	135	LP	3	1.31552E-01	6.74841E+01	1.37880E+01	6.60606E+01
1	200		4	1.24802E-01	8.04139E+01	1.78186E+01	7.84148E+01
1	337	EP	5	1.01519E-01	1.07807E+02	2.42940E+01	1.05034E+02

```
Total Time 0.420E-01
hopf ... done
paulokeeffe > @sv 1
Saving fort.7 as b.1 ... done
Saving fort.8 as s.1 ... done
Saving fort.9 as d.1 ... done
```

We write a ‘constants-file’ (c.hopf.1) telling AUTO to detect any bifurcations beginning at $(r, m) = (2.5, 0.05)$, keeping r fixed and moving in the positive m direction. We see that it detects a Hopf bifurcation HB at PAR(2), $m = 0.111236$ and a saddle-node bifurcation LP at PAR(2), $m = 0.131552$ (this we can confirm analytically).

```
paulokeeffe > cp c.hopf.4 c.hopf
paulokeeffe > @r hopf 1
Starting hopf ...
```

BR	PT	TY	LAB	PAR(1)	L2-NORM	U(1)	U(2)	PAR(2)
2	50		6	2.11000E+00	4.93802E+01	8.78523E+00	4.85924E+01	1.12301E-01
2	100		7	1.71272E+00	3.96327E+01	8.97850E+00	3.86023E+01	1.13960E-01
2	150		8	1.31705E+00	3.01025E+01	9.33951E+00	2.86170E+01	1.16825E-01
2	200		9	9.27454E-01	2.13186E+01	1.02825E+01	1.86749E+01	1.22936E-01
2	238	BT	10	6.96573E-01	1.86095E+01	1.37880E+01	1.24983E+01	1.31552E-01

```
Total Time 0.449E-01
Note: The following floating-point exceptions are signalling: IEEE_UNDERFLOW_FLAG IEEE_DENORMAL
hopf ... done
```

We then write a ‘constants-file’ (c.hopf.4) telling AUTO to begin at label LAB 2, allowing both r and m to vary, and track along the Hopf bifurcation curve in the negative r direction. AUTO does this and detects a Bogdanov-Takens bifurcation where the Hopf bifurcation curve terminates.

```
paulokeeffe > cp c.hopf.hom c.hopf
paulokeeffe > @r hopf 1
Starting hopf ...
```

BR	PT	TY	LAB	PAR(1)	L2-NORM	MAX U(1)	MAX U(2)	PERIOD
2	50		5	2.14732E+00	5.15154E+01	1.17088E+01	5.41387E+01	1.83287E+01
2	100		6	1.81179E+00	5.24671E+01	2.02668E+01	6.14454E+01	4.69660E+01
2	150		7	1.80857E+00	6.12186E+01	2.17642E+01	6.42716E+01	1.30187E+02
2	200	UZ	8	1.80857E+00	6.54543E+01	2.17659E+01	6.42744E+01	1.00000E+03

```
Total Time 0.118E+00
Note: The following floating-point exceptions are signalling: IEEE_UNDERFLOW_FLAG IEEE_DENORMAL
hopf ... done
paulokeeffe > @sv hom
Saving fort.7 as b.hom ... done
```

The presence of a Bogdanov-Takens bifurcation indicates that there is an accompanying homoclinic bifurcation curve. To detect this, we write another 'constants-file' (c.hopf.hom) beginning again at the Hopf bifurcation. This time we fix r and move in the positive m direction, detecting periodic orbits. As the period of the homoclinic orbit grows to infinity at a homoclinic bifurcation, we issue a stop command when the period reaches 1000 (PERIOD 1.00000E+03).

```
paulokeeffe > cp c.hopf.hom1 c.hopf
paulokeeffe > @r hopf hom
Starting hopf ...
```

BR	PT	TY	LAB	PAR(1)	L2-NORM	MAX U(1)	MAX U(2)	PAR(2)
2	50		11	2.00438E+00	7.42898E+01	2.19201E+01	7.36354E+01	1.10653E-01
2	100		12	1.47246E+00	5.06855E+01	2.13667E+01	4.83285E+01	1.12738E-01
2	150		13	9.23758E-01	2.84790E+01	1.92239E+01	2.28461E+01	1.20396E-01
2	177	MX	14	6.96629E-01	1.86065E+01	1.37908E+01	1.25001E+01	1.31552E-01

```
Total Time 0.107E+00
hopf ... done
```

Finally, we write a 'constant-file' (c.hopf.hom1) telling AUTO to begin at label LAB 8, allowing both r and m to vary, and track along those periodic orbits in the negative r direction. AUTO does this and terminates very close to where the Bogdanov-Takens bifurcation occurs.

A.2 The Equations-File xxx.f90

```
1  !-----
2  !-----
3  ! hopf : Ecosystem Model
4  !-----
5  !-----
6
7      SUBROUTINE FUNC(NDIM,U,ICP,PAR,IJAC,F,DFDU,DFDP)
8  ! -----
9
10 ! Evaluates the algebraic equations or ODE right hand side
11
12 ! Input arguments :
13 ! NDIM : Dimension of the ODE system
14 ! U : State variables
```

```

15 ! ICP : Array indicating the free parameter(s)
16 ! PAR : Equation parameters
17
18 ! Values to be returned :
19 ! F : ODE right hand side values
20
21 ! Normally unused Jacobian arguments : IJAC, DFDU, DFDP (see manual)
22
23     IMPLICIT NONE
24     INTEGER NDIM, IJAC, ICP(*)
25     DOUBLE PRECISION U(NDIM), PAR(*), F(NDIM), DFDU(*), DFDP(*)
26     DOUBLE PRECISION P,H,G,r,m,bc,cmax,a,rH,E,b,C
27
28     r=PAR(1)
29     m=PAR(2)
30     bc=PAR(3)
31     cmax=PAR(4)
32     a=PAR(5)
33     E=PAR(6)
34     b=PAR(7)
35     C=PAR(8)
36
37     P=U(1)
38     H=U(2)
39
40
41     G = (P**2/(P**2 + a**2))*EXP(-bc*P)
42
43     F(1) = r*P - C*P**2 - cmax*H*G
44     F(2) = (E*EXP(-b*P)*cmax*H*G - m*H)
45
46     END SUBROUTINE FUNC
47 !-----
48 !-----
49

```

```
50      SUBROUTINE STPNT(NDIM,U,PAR,T)
51      ! -----
52
53      ! Input arguments :
54      ! NDIM : Dimension of the ODE system
55
56      ! Values to be returned :
57      ! U : A starting solution vector
58      ! PAR : The corresponding equation-parameter values
59      ! T : Not used here
60
61      IMPLICIT NONE
62      INTEGER NDIM
63      DOUBLE PRECISION U(NDIM), PAR(*), T
64
65      ! Initialize the equation parameters
66      PAR(1)=2.5
67      PAR(2)=0.05
68      PAR(3)=0.025
69      PAR(4)=1.
70      PAR(5)=10.
71      PAR(6)=0.4
72      PAR(7)=0.025
73      PAR(8)=0.02
74      ! Initialize the solution
75      U(1)=4.28009
76      U(2)=74.2817
77
78      END SUBROUTINE STPNT
79      !-----
80      !-----
81      ! The following subroutines are not used here,
82      ! but they must be supplied as dummy routines
83
84      SUBROUTINE BCND
```

```

85      END SUBROUTINE BCND
86
87      SUBROUTINE ICND
88      END SUBROUTINE ICND
89
90      SUBROUTINE FOPT
91      END SUBROUTINE FOPT
92
93      SUBROUTINE PVLS
94      END SUBROUTINE PVLS
95      !-----
96      !-----

```

A.3 The Constants-File c.xxx

```

1  NDIM= 2, IPS = 1, IRS = 0, ILP = 1
2  ICP = [2]
3  NTST= 50, NCOL= 4, IAD = 3, ISP = 1, ISW = 1, IPLT= 0, NBC= 0, NINT= 0
4  NMX= 200, NPR= 200, MXBF= 10, IID = 2, ITMX= 8, ITNW= 5, NWTN= 3, JAC=
   ↪  0
5  EPSL= 1e-07, EPSU = 1e-07, EPSS = 1e-05
6  DS = 0.1, DSMIN= 0.01, DSMAX= 0.2, IADS= 1
7  NPAR= 3, THL = {11: 0.0}, THU = {}

```

Appendix B

Dependencies of the Non-monotone Forcing Function

In Chapter 6 we added more functionality to the non-monotone forcing function (6.2), transcribed here in terms of the rate of plant growth r in the ecosystem model,

$$r(t) = \begin{cases} r_- + \Delta_r \operatorname{sech}(\varepsilon t), & t \leq 0, \\ r_- + \Delta_r, & 0 < t < \tau \\ r_- + \Delta_r \operatorname{sech}(\rho \varepsilon(t - \tau)), & t \geq \tau, \end{cases} \quad (\text{B.1})$$

We now have control over both the rate at which the reversing shift occurs ($\rho \neq 0$), and the delay at the maximum of the parameter shift $r_- + \Delta_r$ before the reversal is initiated ($\tau > 0$).

B.1 Manipulating the Rate of Reversal

The (Δ_r, ε) tipping diagrams in Fig B.1 are obtained for a parameter path with fixed $m = 0.075$, $r_- = 1$, and $r(t) > 1$. The path crosses the boundary BI of the basin instability $BI(e_3, p_1)$, but does not cross any bifurcations. Points of no return are bounded by the tipping-tracking transition curve $c^\dagger$ for the non-monotone shift (B.1). Points of return are located between $c^\dagger$ and the tipping-tracking transition curve $c^\uparrow$ for the monotone shift (6.1). Panel (a) shows what occurs when the rate of reversal is halved ($\rho = 0.5$) and (b) shows when the rate is doubled ($\rho = 2$). The effect on the tipping diagrams is expected. A slower rate of reversal increases the area of “points of no return”. In other words, for each ε , where a recovery of the system is possible, there

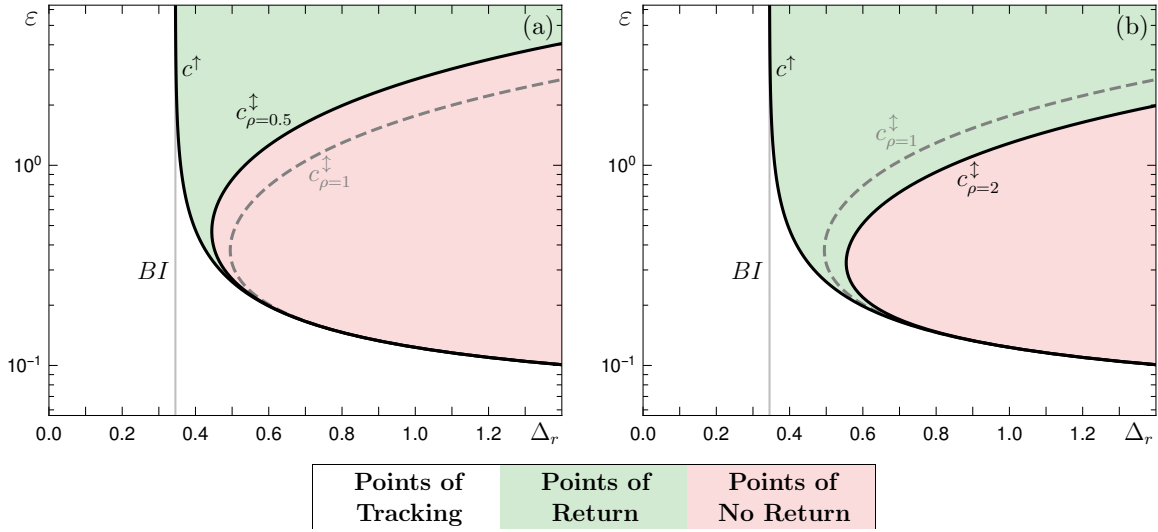


Figure B.1 Tipping diagrams in the (Δ_r, ε) -plane showing a side by side comparison of (a) the tipping-tracking transition curves $c^\uparrow$ and $c^\downarrow$ obtained using monotone Eq. (6.1) and non-monotone Eq. (B.1), where $\rho = 0.5$ and $\tau = 0$ and (b) the tipping-tracking transition curves $c^\uparrow$ and $c^\downarrow$ obtained using monotone Eq. (6.1) and non-monotone Eq. (B.1), where $\rho = 2$ and $\tau = 0$. $p_1 = (1.0, 0.075)$ along a path Δ_r with a fixed $m = 0.075$ and varied $r > 1$. $b = b_c = 0.025$. The dashed grey tipping-tracking transition curve for $\rho = 1$ and $\tau = 0$ is the same curve as was produced in Fig. 6.1(a).

is now a shorter length of Δ_r , by which the system can recover itself by a parameter trend reversal. Similarly, by increasing the rate of reversal, we now have a longer length of Δ_r in which to recover the system.

B.2 Manipulating the Delay

Figure B.2 shows the tipping-transition curve $c^\uparrow$ for the monotone forcing function (6.1) and tipping-tracking transition curves for the non-monotone forcing function (B.1) $c^\downarrow$ for a range of τ from 0 to 5, along a parameter path Δ_r with a fixed $m = 0.075$ and varied $r > 1$. Fixing $\rho = 1$ and beginning with $\tau = 0$ we have the same conditions as were used for the tipping diagram Fig. 6.1(a). As we increase τ , the area of “points of no return” increases, with the tipping-transition curve converging towards the line of basin instability BI . Dismissing the unrealistic values of $\varepsilon > 10$, the area of “points of return” becomes greatly reduced within a short time frame. In this context, it is worth bearing in mind the units of the system, this change in τ relates to delays of only 1, 3 and 5 days.

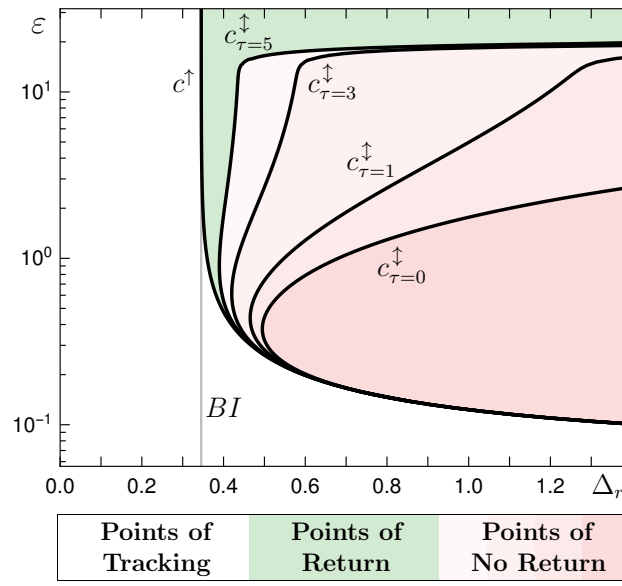


Figure B.2 Tipping diagram in the (Δ_r, ε) -parameter plane for shifts from $p_1 = (1.0, 0.075)$ along a path Δ_r with a fixed $m = 0.075$ and varied $r > 1$, showing tipping-tracking transition curves for the non-monotone function (B.1) $c^\uparrow$ for $\tau = 0, 1, 3$ and 5 and the tipping-tracking transition curve for the monotone function (6.1) $c^\uparrow$.

Appendix C

MATLAB Code

C.1 Finding Region of Basin Instability in the Autonomous Ecosystem Model

The following code found the region of Basin Instability in the autonomous ecosystem model (2.1)–(2.2) as plotted in Fig. 4.2(b).

```
1 function odefbsresvarym
2
3 fun1 = @root2d1;
4
5 C = 0.02;
6 a = 10;
7 b = 0.025;
8 bc = 0.025;
9 c_max = 1;
10 E = 0.4;
11
12 rr = 0.75;
13 mm = 0.075;
14 x0 = [5,10];
15 x = fsolve(fun1,x0);
16 initial_p = x(1);
17 initial_h = x(2);
18
```

```

19 Err = 1e-8;
20
21 mtest = 0.13;
22
23 r = 0.53:0.002:3;
24
25 for j = 1:length(r)
26
27     m_track = mtest - 0.005;
28     m_tip = mtest + 0.005;
29
30     h = 0.01;
31     t = 0:h:10000;
32
33     P = zeros(1, length(t));
34     H = zeros(1, length(t));
35
36     P(1) = initial_p;
37     H(1) = initial_h;
38
39     while(abs(m_tip - m_track) >= Err)
40         m = (m_tip + m_track)/2;
41
42         PHs_1 = @(t,P,H) (r(j)*P - C*P^2 - c_max*H*P^2*exp(-bc*P)/(P^2
43             ↪ + a^2));
44
45         PHs_2 = @(t,P,H) (E*exp(-b*P)*c_max*H*P^2*exp(-bc*P)/(P^2 + a^2)
46             ↪ - m*H);
47
48         for i = 1:(length(t))
49
50             k_1 = h*(PHs_1(t(i), P(i), H(i)));
51             l_1 = h*(PHs_2(t(i), P(i), H(i)));
52
53             k_2 = h*(PHs_1(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*
54                 ↪ l_1));

```

```

51         l_2 = h*(PHs_2(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*
           ↪ l_1));
52
53         k_3 = h*(PHs_1(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*
           ↪ l_2));
54         l_3 = h*(PHs_2(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*
           ↪ l_2));
55
56         k_4 = h*(PHs_1(t(i) + h, P(i) + h*k_3, H(i) + h*l_3));
57         l_4 = h*(PHs_2(t(i) + h, P(i) + h*k_3, H(i) + h*l_3));
58
59         P(i+1) = P(i) + (1/6)*(k_1 + 2*k_2 + 2*k_3 + k_4);
60         H(i+1) = H(i) + (1/6)*(l_1 + 2*l_2 + 2*l_3 + l_4);
61
62     end
63
64     hh = H(:);
65
66     if hh(length(hh)) < 0.2
67         m_tip = m;
68     else
69         m_track = m;
70     end
71 end
72
73 fbs = [r(j), m];
74 save('basin_instability.dat', 'fbs', '-ascii', '-append')
75 mtest = m;
76 end
77 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
78 function F = root2d1(x)
79
80     F(1) = E*exp(-b*x(1))*(x(1)^2/(x(1)^2+100))*exp(-bc*x(1))-mm;
81     F(2) = rr*x(1) - C*x(1)^2 - x(2)*(x(1)^2/(x(1)^2+100))*exp(-bc*
           ↪ x(1));

```

```

82
83     end
84 end

```

C.2 Finding ε_c , the Critical Rate with Non-monotonic Forcing Function

We use as our example the code for finding ε_{c1} , the critical rate with non-monotonic forcing function Eq. (4.3) in the non-autonomous ecosystem model Eq. (2.7)–(2.8) as plotted in Fig. 4.7(a). $(r, m) = (1.0 + \delta, 0.075)$, $b = b_c = 0.025$.

```

1 function odersechsingle
2
3 fun = @root2d;
4
5 Declare Variables see Sec.C1
6
7 m = 0.075;
8 rmin = 1;
9
10 A = 0.7;
11
12 Err = 1e-15;
13
14 eps_tip = 10^-0.77;
15 eps_track = 10^-0.79;
16
17 while(abs(eps_tip - eps_track) >= Err)
18     eps = (eps_track + eps_tip)/2;
19
20     del = A*10^-3;
21
22 x0 = [5,23];
23 x = fsolve(fun,x0);
24

```

```

25 initial_p = x(1);
26 initial_h = x(2);
27 initial_s = -asech(del/A)/eps;
28
29 h = 0.02;
30 t = 0:h:ceil(-5*initial_s);
31
32 P = zeros(1, length(t));
33 H = zeros(1, length(t));
34 s = zeros(1, length(t));
35
36 P(1) = initial_p;
37 H(1) = initial_h;
38 s(1) = initial_s;
39
40 PHs_1 = @(t,P,H,s) ((A*f(s)+rmin)*P - C*P^2 - r_H*c_max*H*P^2*exp(-bc*
    ↪ P)/(P^2 + a^2));
41 PHs_2 = @(t,P,H,s)((E*exp(-b*P)*c_max*H*P^2*exp(-bc*P)/(P^2 + a^2) - m*
    ↪ H)*r_H);
42 PHs_3 = @(t,P,H,s) 1;
43
44 for i = 1:(length(t))
45
46     k_1 = h*(PHs_1(t(i), P(i), H(i), s(i)));
47     l_1 = h*(PHs_2(t(i), P(i), H(i), s(i)));
48     m_1 = h*(PHs_3(t(i), P(i), H(i), s(i)));
49
50     k_2 = h*(PHs_1(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*l_1, s(
    ↪ i) + 0.5*h*m_1));
51     l_2 = h*(PHs_2(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*l_1, s(
    ↪ i) + 0.5*h*m_1));
52     m_2 = h*(PHs_3(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*l_1, s(
    ↪ i) + 0.5*h*m_1));
53

```

```

54     k_3 = h*(PHs_1(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*l_2, s(
        ↪ i) + 0.5*h*m_2));
55     l_3 = h*(PHs_2(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*l_2, s(
        ↪ i) + 0.5*h*m_2));
56     m_3 = h*(PHs_3(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*l_2, s(
        ↪ i) + 0.5*h*m_2));
57
58     k_4 = h*(PHs_1(t(i) + h, P(i) + h*k_3, H(i) + h*l_3, s(i) + h*m_3))
        ↪ ;
59     l_4 = h*(PHs_2(t(i) + h, P(i) + h*k_3, H(i) + h*l_3, s(i) + h*m_3))
        ↪ ;
60     m_4 = h*(PHs_3(t(i) + h, P(i) + h*k_3, H(i) + h*l_3, s(i) + h*m_3))
        ↪ ;
61
62     P(i+1) = P(i) + (1/6)*(k_1 + 2*k_2 + 2*k_3 + k_4);
63     H(i+1) = H(i) + (1/6)*(l_1 + 2*l_2 + 2*l_3 + l_4);
64     s(i+1) = s(i) + (1/6)*(m_1 + 2*m_2 + 2*m_3 + m_4);
65
66 end
67
68 if P(i) > 32 && H(i) < 1
69     eps_tip = eps;
70 else
71     eps_track = eps;
72 end
73 end
74 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
75 function fval = f(s)
76     fval = sech(eps*s);
77 end
78 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
79 function F = root2d(x)
80
81 F(1) = E*exp(-b*x(1))*(x(1)^2/(x(1)^2+100))*exp(-bc*x(1))-m;

```

```

82 F(2) = (rmin + del)*x(1) - C*x(1)^2 - x(2)*(x(1)^2/(x(1)^2+100))*exp(-
      ↪ bc*x(1));
83
84
85 end
86 end

```

C.3 Finding ε_c , the Critical Rate of the Heteroclinic Orbit from e_3^- to e_4^+ in the Autonomous Compactified System

The following code found ε_c , the critical rate of the heteroclinic orbit from e_3^- to e_4^+ in the autonomous compactified system (7.5) as plotted in Fig. 6.4(b).

```

1 function odercompactcanard
2
3 fun = @root2d;
4
5 Declare Variables (see Sec.B1).
6
7 m = 0.075;
8 rmin = 0.75;
9
10 A = 0.6;
11
12 Err = 1e-15;
13
14 eps_track = 10^-1.2;
15 eps_tip = 10^-1;
16
17 while(abs(eps_tip - eps_track) >= Err)
18     eps = (eps_track + eps_tip)/2;
19
20     alpha = eps/2;

```

```

21     s = -0.99;
22
23     x0 = [5,16];
24     x = fsolve(fun,x0)
25
26     initial_p = x(1);
27     initial_h = x(2);
28     initial_s = -0.99;
29
30     h = 0.02;
31
32     t = 0:h:200;
33
34     P = zeros(1, length(t));
35     H = zeros(1, length(t));
36     s1 = zeros(1, length(t));
37     s2 = zeros(1, length(t));
38
39
40     P(1) = initial_p;
41     H(1) = initial_h;
42     s1(1) = initial_s;
43     s2(1) = 0;
44
45     PHs_1 = @(t,P,H,s1,s2) ((A*f(s1)+rmin)*P - C*P^2 - c_max*H*P^2*exp
        ↪ (-bc*P)/(P^2 + a^2));
46     PHs_2 = @(t,P,H,s1,s2) (E*exp(-b*P)*c_max*H*P^2*exp(-bc*P)/(P^2 + a
        ↪ ^2) - m*H);
47     PHs_3 = @(t,P,H,s1,s2) (eps/2)*(1-s1^2);
48     PHs_4 = @(t,P,H,s1,s2) 1;
49
50     for i = 1:(length(t))
51
52         k_1 = h*(PHs_1(t(i), P(i), H(i), s1(i), s2(i)));
53         l_1 = h*(PHs_2(t(i), P(i), H(i), s1(i), s2(i)));

```

```

54     m_1 = h*(PHs_3(t(i), P(i), H(i), s1(i), s2(i)));
55     n_1 = h*(PHs_4(t(i), P(i), H(i), s1(i), s2(i)));
56
57     k_2 = h*(PHs_1(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*l_1,
        ↪ s1(i) + 0.5*h*m_1, s2(i) + 0.5*h*n_1));
58     l_2 = h*(PHs_2(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*l_1,
        ↪ s1(i) + 0.5*h*m_1, s2(i) + 0.5*h*n_1));
59     m_2 = h*(PHs_3(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*l_1,
        ↪ s1(i) + 0.5*h*m_1, s2(i) + 0.5*h*n_1));
60     n_2 = h*(PHs_4(t(i) + 0.5*h, P(i) + 0.5*h*k_1, H(i) + 0.5*h*l_1,
        ↪ s1(i) + 0.5*h*m_1, s2(i) + 0.5*h*n_1));
61
62     k_3 = h*(PHs_1(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*l_2,
        ↪ s1(i) + 0.5*h*m_2, s2(i) + 0.5*h*n_2));
63     l_3 = h*(PHs_2(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*l_2,
        ↪ s1(i) + 0.5*h*m_2, s2(i) + 0.5*h*n_2));
64     m_3 = h*(PHs_3(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*l_2,
        ↪ s1(i) + 0.5*h*m_2, s2(i) + 0.5*h*n_2));
65     n_3 = h*(PHs_4(t(i) + 0.5*h, P(i) + 0.5*h*k_2, H(i) + 0.5*h*l_2,
        ↪ s1(i) + 0.5*h*m_2, s2(i) + 0.5*h*n_2));
66
67     k_4 = h*(PHs_1(t(i) + h, P(i) + h*k_3, H(i) + h*l_3, s1(i) + h*
        ↪ m_3, s2(i) + h*n_3));
68     l_4 = h*(PHs_2(t(i) + h, P(i) + h*k_3, H(i) + h*l_3, s1(i) + h*
        ↪ m_3, s2(i) + h*n_3));
69     m_4 = h*(PHs_3(t(i) + h, P(i) + h*k_3, H(i) + h*l_3, s1(i) + h*
        ↪ m_3, s2(i) + h*n_3));
70     n_4 = h*(PHs_3(t(i) + h, P(i) + h*k_3, H(i) + h*l_3, s1(i) + h*
        ↪ m_3, s2(i) + h*n_3));
71
72     P(i+1) = P(i) + (1/6)*(k_1 + 2*k_2 + 2*k_3 + k_4);
73     H(i+1) = H(i) + (1/6)*(l_1 + 2*l_2 + 2*l_3 + l_4);
74     s1(i+1) = s1(i) + (1/6)*(m_1 + 2*m_2 + 2*m_3 + m_4);
75     s2(i+1) = s2(i) + (1/6)*(n_1 + 2*n_2 + 2*n_3 + n_4);
76

```

```

77     end
78
79     if P(i) > 32 && H(i) < 1
80         eps_tip = eps;
81     else
82         eps_track = eps;
83     end
84 end
85
86 pp = P(:);
87 hh = H(:);
88 ss = s1(:);
89
90 Aeps1 = [pp , hh, ss];
91 save('trajdel6a.dat','Aeps1','-ascii','-append')
92 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
93     function fval = f(s1)
94         fval = (1+s1)^2/((1+s1)^2+(1-s1)^2);
95     end
96 %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
97     function F = root2d(x)
98         F(1) = E*exp(-b*x(1))*(x(1)^2/(x(1)^2+100))*exp(-bc*x(1))-m;
99         F(2) = (rmin + A*((1+s)^2/((1+s)^2+(1-s)^2)))*x(1) - C*x(1)^2 -
           ↪ x(2)*(x(1)^2/(x(1)^2+100))*exp(-bc*x(1));
100     end
101 end

```

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