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Diffusion of the electromagnetic energy due to the backscattering off Schwarzschild geometry

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Electromagnetic waves propagate in the Schwarzschild spacetime like in a nonuniform medium with a varying refraction index. A fraction of the radiation scatters off the curvature of the geometry. The energy of the backscattered part of an initially outgoing pulse of electromagnetic radiation can be estimated, in the case of dipole radiation, by a compact formula depending on the initial energy, the Schwarzschild radius, and the pulse location. The magnitude of the backscattered energy depends on the frequency spectrum of the initial configuration. This effect becomes negligible in the short-wave limit, but it can be significant in the long-wave regime. Similar results hold for the massless scalar fields and are expected to hold also for weak gravitational waves.

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I. INTRODUCTION

Backscattering is a phenomenon that prevents waves from being transmitted exclusively along null cones. That aspect of waves propagation has been investigated for a long time for various wave equations (see, for instance, [1]). It has been established that solutions of the Klein-Gordon equation with nonuniform coefficients generically do exhibit backscattering [1]. This topic has been investigated in general relativity since the early 1960s [2,3,4,5,6]; a comprehensive bibliography can be found in [6]. The propagation of electromagnetic waves and of the resulting tails were studied in the early 1970s [4,7] and recently by Ching *et al.* [8] in the context of Schwarzschild spacetime and by Hod [9] in the context of Kerr spacetime. The backscattering effect can be understood as the result of wave propagation in a nonuniform medium with a varying refraction index [10].

In Ref. [11] a classical aspect of the phenomenon that was not previously studied has been assessed—the energy diffusion through null cones—in the example of a spherically symmetric massless scalar field propagating in the Schwarzschild geometry. The novel aspect of that work was a compact estimate of the magnitude of the backscattered energy in terms of the energy of initial data.

This paper is dedicated to the investigation of propagation of electromagnetic fields in a background Schwarzschild spacetime. Similar to [11] the main attention is focused on obtaining bounds on the backscattered fraction of the radiation energy, in terms of initial data. From the notional point of view the present paper parallels [11], with three notable exceptions. First, the crucial technical points of the former work could have been applied only to spherically symmetric fields. In order to overcome this difficulty, the electromagnetic fields have to be split, with the extraction of a known part which defines initial data. Then the standard expansion in terms of vector spherical harmonics leads to a problem that can be tackled with methods applied earlier in [11]. Second, an energy inequality is proved. Third, this paper shows that the energy diffusion depends on the frequency of the radiation. An example of a dipole radiation allows one to characterize this quantitatively. The magnitude of the backscattering can be characterized as the ratio of the backscattered energy versus the initial energy of outgoing waves. This is vanishingly small in the short-wave regime but it can be quite significant in the long part of the radiation spectrum. This kind of dependence on the frequency can be expected to hold also for higher multipoles. The scale is essentially set by the gravitational radius of the gravity source. All results of this paper hold true for any material sources of the Schwarzschild geometry—including stars, white dwarves, neutron stars, and black holes—although the effects can really matter only in the two latter classes of objects.

The order of the remaining parts of this paper is following. The next section defines notation, basic equations and a decomposition of the electromagnetic potential. The subsequent sections of this work deal only with dipole radiation. In Sec. III is derived an energy estimate. Section IV is dedicated to the derivation of a bound, depending on the initial energy, of the backscattered part of the potential. Section V is devoted to the derivation of useful estimates of a pair of null-line integrals. In Sec. VI the equations are formulated in the language of characteristics. Previously found restrictions on the backscattered part of the potential allow one to estimate radiation intensities. Section VII brings an improved estimate of the backscattered potential, again based on the method of characteristics. The next section proves the main results—a bound on that fraction of the energy that can diffuse due to the backscatter off the Schwarzschild geometry curvature. Section IX shows that in the case of short-wave radiation the dipole radiation backscatter is negligible. In contrast, in the long-wave regime the effect can be significant. Section X discusses how the effect depends on a distance and evaluates the exactness of the obtained criteria. The last section presents a short summary and conclusions.

II. FORMALISM

The spherically symmetric geometry outside matter is given by a Schwarzschildean geometry line element:

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$$ds^2=-\left(1-\frac{2m}{R}\right)dt^2+\frac{1}{1-2m/R}\,dR^2+R^2d\Omega^2, \tag{2.1}$$

where t is a time coordinate, R is a radial coordinate that coincides with the areal radius, and $d\Omega^2=d\theta^2+\sin^2\theta\,d\phi^2$ is the line element on the unit sphere, $0\leqslant\phi<2\pi$ and $0\leqslant\theta\leqslant\pi$.

As concerns the electromagnetic fields, it is convenient to assume that the scalar component of the electromagnetic potential vanishes while the vector potential satisfies the Coulomb gauge condition. Using a multipole expansion of the electromagnetic vector potential in terms of vector spherical harmonics, one obtains [12]

$$(-\partial_0^2+\partial_{r*}^2)\Psi_l=\left(1-\frac{2m}{R}\right)\frac{l(l+1)}{R^2}\Psi_l. \tag{2.2}$$

Ψ's should be essentially two-index functions Ψ_{lM} (where M is the projection of the angular momentum), but since the evolution equation is φ independent, the index M is suppressed. The variable $r^*\equiv R+2m\ln(R/2m-1)$ is the Regge-Wheeler tortoise coordinate. The backreaction exerted by the electromagnetic field onto the metric has been neglected in the present analysis. That is readily justified for any gravitational sources other than black holes. In the case of a black hole this approximation holds true some distance away from its horizon [13].

Consider a set of functions of the form

$$\tilde{\Psi}_l(t,r^*)=\sum_{s=0}^{l}\frac{\Psi_{ls}(r^*-t)}{R^s}, \tag{2.3}$$

where the functions Ψ_{ls} are given by the recurrence relations

$$\partial_{r*}\Psi_{l1}=-\frac{l(l+1)}{2}\Psi_{l0},$$

$$\partial_{r*}\Psi_{l(s+1)}=\frac{1}{2(s+1)}[(s(s+1)-l(l+1))\Psi_{ls}-2m(s^2-1)\Psi_{l(s-1)}]. \tag{2.4}$$

In [2] is shown a dipole solution of this type [14]. In Minkowski space-time ($m=0$), $\tilde{\Psi}_l$ solves Eq. (2.2); it represents a purely outgoing electromagnetic radiation.

Let a function $\tilde{\Psi}_l$ be given by Eqs. (2.3) and (2.4) and assume that (for spacelike sections with $t\geqslant 0$) its support is compact and located entirely in the vacuum region outside some radius $a>2m$, i.e., outside the Schwarzschild radius. Let the initial data of a solution Ψ_l of Eq. (2.2) coincide with $\tilde{\Psi}_l$ at $t=0$. Thus initially Ψ_l is a purely outgoing partial wave. It should be noted that the assumption that initial data are (initially) purely outgoing is made in this paper only for the sake of clear presentation. The propagation of electromagnetic waves is a linear process as far as the backreaction can be neglected. Therefore the propagation of the initially outgoing radiation (or even of a fraction of the outgoing radiation) is independent of whether or not the ingoing radiation is present.

It will be convenient to decompose the sought solution $\Psi_l(r^*,t)$ into the known part $\tilde{\Psi}_l$ and an unknown function δ_l:

$$\Psi_l=\tilde{\Psi}_l+\delta_l. \tag{2.5}$$

Initially, $\delta_l=\partial_0\delta_l=0$. A similar splitting is done in [4], who then seek a series expansion of δ_l. This will be avoided in this paper, in favor of finding a number of estimates of δ_l that would provide the needed information about the backscattered part of the radiation.

In the rest of this paper only the dipole radiation Ψ_1 will be considered. Consequently, all angular-momentum-related subscripts will be omitted.

III. ENERGY ESTIMATE

The dipole term constitutes the most important part of the electromagnetic radiation. Assume dipole-type initial data

$$\tilde{\Psi}(x(R))=-\partial_{r*}f(x(R))+\frac{f(x(R))}{R(r^*)}, \tag{3.1}$$

with the initial support (a,∞) of a C^2-differentiable f and $x(R)\equiv r^*(R)-r^*(a)$. The differentiability of f guarantees that the initial energy density is continuous and vanishes on the boundary a.

Lemma 1. Define $I_{a,\epsilon}(R)$:

$$I_{a,\epsilon}(R)\equiv\int_a^R dr\,\frac{f^2(x(r))}{r^{4+2\epsilon}}, \tag{3.2}$$

and

$$\beta_a(R)\equiv\int_a^R dr\,\frac{\tilde{\Psi}^2(x(r))}{r^2}, \tag{3.3}$$

where $0<2\epsilon<1$. Then for $a>2m(1+1/\sqrt{1+2\epsilon})$ the following inequality holds:

$$I_{a,\epsilon}(R)\leqslant\frac{\beta_a(R)}{\epsilon a^{2\epsilon}}\,\frac{1-(a/R)^{2\epsilon}}{(1+2\epsilon)(1-2m/a)^2-4m^2/a^2}. \tag{3.4}$$

Remark. The integral $\beta_a(R)$ is bounded above by the electromagnetic energy $E_R(t)/(4\pi)$ defined later. Therefore,

$$I_{a,\epsilon}(R)\leqslant\frac{E_a(R,t)}{4\pi\epsilon a^{2\epsilon}}\,\frac{1-(a/R)^{2\epsilon}}{(1+2\epsilon)(1-2m/a)^2-4m^2/a^2}.$$

Proof. Notice that

$$\frac{f^2(x(r))}{r^2}=\left[\int_a^r ds\,\partial_s\frac{f(s)}{s}\right]^2$$
$$=\left[\int_a^r ds\,\frac{1}{1-2m/s}\left(-\frac{\tilde{\Psi}}{s}+\frac{2mf}{s^3}\right)\right]^2$$
$$\leqslant 2\left[\int_a^r ds\,\frac{-\tilde{\Psi}}{s(1-2m/s)}\right]^2$$
$$+2\left[\int_a^r ds\,\frac{2mf}{s^3(1-2m/s)}\right)\Bigg]^2; \tag{3.5}$$

the second inequality follows from $(A+B)^2\leqslant 2A^2+2B^2$. The factor $1/(1-2m/s)$ that appears in the integrands can be bounded above by $1/\eta_a$, where

$$\eta_a\equiv 1-\frac{2m}{a}. \tag{3.6}$$

Subsequently, the use of the Schwarz inequality and simple integrations yield

$$\frac{f^2(x(r))}{r^2}\leqslant\frac{2\beta_a(r)(r-a)}{\eta_a^2}+\frac{8m^2I_{a,\epsilon}(r)}{\eta_a^2(1-2\epsilon)}\left(\frac{1}{a^{1-2\epsilon}}-\frac{1}{r^{1-2\epsilon}}\right). \tag{3.7}$$

The insertion of Eq. (3.7) into the integral of Eq. (3.2) gives

$$I_{a,\epsilon}(R)\leqslant\int_a^R dr\,\frac{1}{r^{2+2\epsilon}}\left[\frac{2\beta_a(r)(r-a)}{\eta_a^2}\right.$$
$$\left.+\frac{8m^2I_{a,\epsilon}(r)}{\eta_a^2(1-2\epsilon)}\left(\frac{1}{a^{1-2\epsilon}}-\frac{1}{r^{1-2\epsilon}}\right)\right]. \tag{3.8}$$

$\beta_a(r)$ and $I_{a,\epsilon}(r)$ are nondecreasing functions: therefore, taking them in front of the appropriate integrals would not make the corresponding terms smaller. Straightforward integration of the obtained expressions yields

$$I_{a,\epsilon}(R)\leqslant\frac{\beta_a(R)}{\eta_a^2a^{2\epsilon}}\left\{\frac{1}{2\epsilon}\left[1-\left(\frac{a}{R}\right)^{2\epsilon}\right]\right.$$
$$\left.+\frac{1}{1+2\epsilon}\left[-1+\left(\frac{a}{R}\right)^{1+2\epsilon}\right]\right\}$$
$$+\frac{2I_{a,\epsilon}(R)}{1-2\epsilon}\left(\frac{2m}{a-2m}\right)^2\left\{\frac{1}{1+2\epsilon}\left[1-\left(\frac{a}{R}\right)^{1+2\epsilon}\right]\right.$$
$$\left.-\frac{1}{2}\left[1-\left(\frac{a}{R}\right)^2\right]\right\}. \tag{3.9}$$

One should note that the expression inside the first pair of curly brackets can be estimated as follows:

$$\frac{1}{2\epsilon}\left[1-\left(\frac{a}{R}\right)^{2\epsilon}\right]+\frac{1}{1+2\epsilon}\left[-1+\left(\frac{a}{R}\right)^{1+2\epsilon}\right]$$
$$\leqslant\frac{1}{\epsilon(1+2\epsilon)}\left[1-\left(\frac{a}{R}\right)^{2\epsilon}\right],$$

while the expression inside the second pair of curly brackets is bounded above by

$$\frac{1-2\epsilon}{2(1+2\epsilon)}\left[1-\left(\frac{a}{R}\right)^{1+2\epsilon}\right].$$

Equation (3.9) can be now written as

$$I_{a,\epsilon}(R)\leqslant\frac{\beta_a(R)[1-(a/R)^{2\epsilon}]}{\eta_a^2a^{2\epsilon}\epsilon(1+2\epsilon)}+\left(\frac{2m}{a-2m}\right)^2\frac{I_{a,\epsilon}(r)}{1+2\epsilon}. \tag{3.10}$$

Rearranging Eq. (3.10) so that the two terms with $I_{a,\epsilon}(R)$ are on the left-hand side, one obtains

$$I_{a,\epsilon}(R)\left[1-\left(\frac{2m}{a-2m}\right)^2\frac{1}{1+2\epsilon}\right]\leqslant\frac{\beta_a(R)[1-(a/R)^{2\epsilon}]}{\eta_a^2a^{2\epsilon}\epsilon(1+2\epsilon)}; \tag{3.11}$$

this gives the postulated bound of $I_{a,\epsilon}(R)$ if $a>2m(1+1/\sqrt{1+2\epsilon})$, as assumed above.

The obtained formula is not exact, but with the appropriate choice of f and ϵ the error is small. Take, for instance, $f=C$ within $(a+a_1,b-b_1),a_1,b_1\ll a,b\gg a$ (which obviously means that $b-a\gg a_1,b_1$), and let f be smoothly joined to zero outside (a, b) by some intermediary functions. Under those conditions, a direct calculation gives

$$\frac{I_{a,\epsilon}(b)}{\beta_a(b)}\approx\frac{3}{(3+2\epsilon)a^{2\epsilon}}, \tag{3.12}$$

as compared with $1/\epsilon(1+\epsilon)a^{2\epsilon}$, which follows from Eq. (3.11). If $\epsilon\approx 1/2$, then the exact result differs by less than 25% from the bound Eq. (3.11). Later on, the value $\epsilon=1/8$ will be used (which appears to be more economical in subsequent calculations), in which case the above estimate deteriorates significantly. The exact value of $I_{a,\epsilon}(b)/\beta_a(b)$ is then roughly 15% of that predicted by Eq. (3.11).

IV. ESTIMATING δ

δ is initially zero, and its evolution is governed by the following equation:

$$(-\partial_0^2+\partial_{r*}^2)\,\delta=\left(1-\frac{2m}{R}\right)\left[\frac{2}{R^2}\,\delta+\frac{6mf}{R^4}\right]. \tag{4.1}$$

Define $\tilde{\Gamma}_{(R,t)}$—a null geodesic that originates at (R, t) and is directed outward. If a point lies on the initial hypersurface, then $\tilde{\Gamma}_{(R,0)}\equiv\tilde{\Gamma}_R$. By $\tilde{\Gamma}_{(R_0,t_0),(R,t)})$ a segment of $\tilde{\Gamma}_{(R_0,t_0)}$ ending at (R, t) will be understood.

Later will be needed the following bound.

Theorem 2. Let the support of initial data be $(a,b),b\leqslant\infty$ and let $\tilde{\Gamma}_{R_0,(R,t)}$ be the outgoing null geodesic from $(R_0,t=0)$ to (R, t). Then

$$\frac{|\delta(R)|}{R}\leqslant mC_1\sqrt{\beta_a(b)}\frac{1}{a^\epsilon\sqrt{R\,\eta_R}}\left(\frac{1}{R_0^{1-\epsilon}}-\frac{1}{R^{1-\epsilon}}\right), \tag{4.2}$$

where

$$C_1 \equiv \frac{6\sqrt{2}}{\eta_a^{3/2}(1-\epsilon)} \sqrt{\frac{1}{\epsilon[(1+2\epsilon)\eta_a^2 - 4m^2/a^2]}} \tag{4.3}$$

and $\eta_R = 1-2m/R$.

Proof. Define an energy $H(R,t)$ of the field δ contained in the exterior of a sphere of a radius R:

$$H(R,t) = \int_R^\infty dr \left[\frac{(\partial_0\delta)^2}{1-\frac{2m}{r}} + \left(1-\frac{2m}{r}\right)(\partial_r\delta)^2 + \delta^2\frac{2}{r^2}\right]. \tag{4.4}$$

One can easily show that

$$\begin{aligned}(\partial_t + \partial_r^*)H(R,t) = &-\left(1-\frac{2m}{R}\right)\Bigg[\left(1-\frac{2m}{R}\right) \\ &\times\left(\frac{\partial_0\delta}{\left(1-\frac{2m}{R}\right)} + \partial_R\delta\right)^2 + \frac{2}{R^2}\delta^2\Bigg] \\ &- 12m\int_R^\infty dr\,\partial_0\delta\frac{f}{r^4} \\ \leqslant &-12m\int_R^\infty dr\,\partial_0\delta\frac{f}{r^4},\end{aligned} \tag{4.5}$$

the inequality following from omission of the nonpositive boundary term. The right-hand side can be bounded further by

$$12m\left[\int_R^\infty dr(\partial_0\delta)^2\right]^{1/2}\left[\int_R^\infty dr\frac{f^2}{r^8}\right]^{1/2}, \tag{4.6}$$

due to the Schwarz inequality. That in turn can be bounded by

$$\frac{12m}{R^{2-\epsilon}\sqrt{\eta_R}} H^{1/2}\left[\int_R^\infty dr\,\frac{f^2}{r^{4+2\epsilon}}\right]^{1/2}. \tag{4.7}$$

The integral in Eq. (4.7) cannot increase along outgoing null directions, and therefore is bounded by initial values $\int_R^\infty dr\, f^2/r^{4+2\epsilon} \leqslant \int_{R_0}^\infty dr\, f^2/r^{4+2\epsilon} \equiv I_{R_{0,\epsilon}}(R)$. Since $I_{R_{0,\epsilon}}(\infty) \leqslant I_{a,\epsilon}(\infty)$, one arrives at

$$(\partial_t + \partial_r^*)H(R,t)^{1/2} \leqslant 6\sqrt{I_{a,\epsilon}(\infty)}\frac{m}{\sqrt{\eta_a}R^{2-\epsilon}}. \tag{4.8}$$

The integration of Eq. (4.8) along $\tilde{\Gamma}_{R_0,(R,t)}$ yields, replacing $I_{a,\epsilon}(\infty)$ by its bound expressed in Eq. (3.4),

$$\begin{aligned}\sqrt{H(R,t)} &\leqslant \frac{6m}{a^\epsilon}\frac{\sqrt{\beta_a(\infty)}}{\eta_a^{3/2}(1-\epsilon)}\sqrt{\frac{1}{\epsilon[(1+2\epsilon)\eta_a^2 - 4m^2/a^2]}} \\ &\quad\times\left(\frac{1}{R_0^{1-\epsilon}} - \frac{1}{R^{1-\epsilon}}\right) \\ &= \frac{mC_1}{\sqrt{2}a^\epsilon}\left(\frac{1}{R_0^{1-\epsilon}} - \frac{1}{R^{1-\epsilon}}\right).\end{aligned} \tag{4.9}$$

Notice that initially δ vanishes and that its propagation is ruled by a hyperbolic equation. Thence at any finite time t the support of δ is bounded. Therefore

$$\begin{aligned}\frac{|\delta(R)|}{R} = \left|\int_\infty^R \partial_r\frac{\delta(r)}{r}\right| &\leqslant \left(\int_\infty^R \frac{1}{r^2}\right)^{1/2}\left[\int_\infty^R\left(\partial_r\delta - \frac{\delta}{r}\right)^2\right]^{1/2} \\ &\leqslant \frac{1}{\sqrt{R-2m}}(2H)^{1/2}(R).\end{aligned} \tag{4.10}$$

Inequalities (4.9) and (4.10) yield the bound of theorem 2 in the case when $b=\infty$.

Let the initial data be of finite support (a, b). Define a region Ω_b consisting of points $(R \geqslant b, t)$ acausal to $(b, t=0)$. The energy $H(b(t),t)$ obviously vanishes for any point $(b(t),t)$ located inside Ω_b. In this case the inequality of theorem 2 can be stated as follows:

$$\frac{|\delta(R)|}{R} \leqslant mC_1\sqrt{\beta_a(b)}\frac{\sqrt{1-(a/b)^{2\epsilon}}}{a^\epsilon\sqrt{R\,\eta_R}}\left(\frac{1}{R_0^{1-\epsilon}} - \frac{1}{R^{1-\epsilon}}\right). \tag{4.11}$$

In what follows it will be assumed that the initial data have compact support located in an annular region (a, b).

V. ESTIMATES OF TWO (NULL) LINE INTEGRALS

In analogy with $\tilde{\Gamma}_{(R,t)}$ defined earlier, let $\Gamma_{(R,t)}$ be a null ingoing geodesic that originates at (R, t). $\Gamma_{(R,t=0)}$ will be shortened to Γ_R. A segment of $\Gamma_{(R_1,t_1)}$ connecting (R_1,t_1) with (R, t) $(t_1<t, R_1>R)$ will be denoted as $\Gamma_{(R_1,t_1),(R,t)}$.

Let a point (R, t) be an intersection of an ingoing null geodesic Γ_{R_1} with an outgoing null geodesic $\tilde{\Gamma}_a$. Let (r,τ), $r \geqslant R$, be a point of $\Gamma_{R_1,(R,t)}$ and define $[R_0(r), t=0]$ as a point of the initial hypersurface such that $\tilde{\Gamma}_{R_0} \cap \Gamma_{R_1} = (r,\tau)$. Fixing a and R_1, one can view R_0 as a function of r; obviously, $R_0(R)=a$, while $R_0(R_1)=R_1$. On the other hand, fixing only a and viewing R_1 as a function of R, one has $R_1(a)=a$; this will be used in the forthcoming proof.

One can prove the following.

Lemma 3. Under the above conditions and if $R_1 \leqslant b$, the line integral along a null segment geodesic $\Gamma_{R_1,(R,t)}$ is bounded from above:

$$\int_R^{R_1} dr\,\frac{r-R_0}{R_0 r\sqrt{1-2m/R}} \leqslant \frac{1}{2}\left[\ln\frac{R}{b} + \ln\left(\frac{b-2m}{a-2m}\right)\right]. \tag{5.1}$$

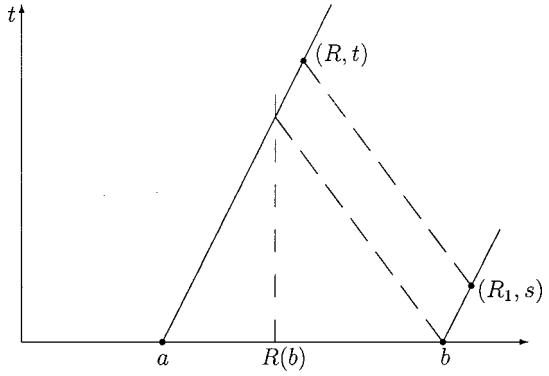


FIG. 1. Solid lines $\tilde{\Gamma}_a, \tilde{\Gamma}_b$ are outgoing null curves. Dashed lines are ingoing null curves $\Gamma_b, \Gamma_{(R_1,s)}$.

Lemma 4. Under the above condition, but with the initial point (R_1,s) $(s>0)$ of the null geodesic segment $\Gamma_{(R_1,s),(R,t)}$ lying on $\tilde{\Gamma}_b$ (Fig. 1), one can prove

$$\int_R^{R_1} dr \frac{r-R_0}{R_0 r\sqrt{1-2m/R}} \leqslant \frac{1}{2}\ln\left(\frac{b-2m}{a-2m}\right). \tag{5.2}$$

Proof of Lemma 3. Let r be a radial coordinate of a point lying on the intersection of $\tilde{\Gamma}_{R_0}$ and Γ_{R_1} (Fig. 2). One finds that the areal distances of three points $(R_0(r),0)$, (r,τ), and $(R_1,0)$ satisfy the following:

$$R_0(r) = 2r - R_1 + 2m\ln\left(\frac{(r-2m)^2}{(R_0(r)-2m)(R_1-2m)}\right). \tag{5.3}$$

That implies

$$dR_0 = 2\frac{1-2m/R_0}{1-2m/r}dr. \tag{5.4}$$

Replacing r by R_0 in the integral of Eq. (5.1), one obtains

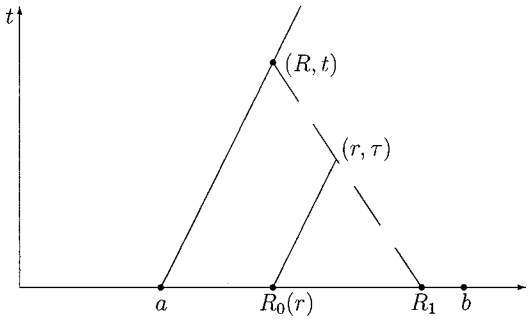


FIG. 2. Solid lines $\tilde{\Gamma}_a, \tilde{\Gamma}_{R_0(r)}$ are outgoing null curves. The dashed line is Γ_{R_1}, an ingoing null curve.

$$\begin{aligned}\int_R^{R_1} dr \frac{r-R_0}{R_0 r\sqrt{1-2m/r}} &= \int_R^{R_1}\frac{dr}{\sqrt{1-2m/r}}\left(\frac{1}{R_0}-\frac{1}{r}\right)\\ &= \frac{1}{2}\int_a^{R_1} dR_0 \frac{\sqrt{1-2m/r}}{R_0-2m}\\ &\quad -\int_R^{R_1}\frac{dr}{r\sqrt{1-2m/r}}\\ &\leqslant \frac{1}{2}\ln\left(\frac{R_1-2m}{a-2m}\right)-\ln\frac{R_1}{R}\\ &= \ln\frac{R}{\sqrt{aR_1}}+\tfrac{1}{2}\ln\frac{1-2m/R_1}{1-2m/a}.\end{aligned} \tag{5.5}$$

Next, one can show that $R_1 \geqslant 2R-a$. Indeed, assuming that a is fixed, one has, from Eq. (5.4), $dR_1/dR \geqslant 2$; since the initial condition is $R_1(a)=a$, the conclusion follows.

Taking into account $R_1 \geqslant 2R-a$, one gets

$$\ln\frac{R}{\sqrt{aR_1}} \leqslant \ln\frac{R}{\sqrt{a(2R-a)}} \leqslant \ln\sqrt{\frac{R}{a}}. \tag{5.6}$$

Replacing R_1 by b in the last term of Eq. (5.5) and inserting Eq. (5.6), one arrives at the first of conjectured inequalities, Eq. (5.1).

In order to prove lemma 4 one should start from relation between areal distances of four points (r,τ), $(R_0(r),0)$, (R,t), and $(a,0)$ (see Fig. 2):

$$2\left(r-R+2m\ln\frac{r-2m}{R-2m}\right) = R_0-a+2m\ln\frac{R_0-2m}{a-2m}. \tag{5.7}$$

The variable r ranges from $R_1>b$ to R. Fixing a and R, one again obtains

$$dR_0 = 2\frac{1-2m/R_0}{1-2m/r}dr. \tag{5.8}$$

A straightforward calculation, in which dr is replaced by dR_0, shows that

$$\int_R^{R_1} dr \frac{r-R_0}{R_0 r\sqrt{1-2m/R}} \leqslant \frac{1}{2}\ln\left(\frac{b-2m}{a-2m}\right)-\ln\frac{R_1}{R}. \tag{5.9}$$

Since $R_1 \geqslant R$, one immediately obtains Eq. (5.2).

VI. ESTIMATE OF THE AMPLITUDES BACKSCATTERED INWARD

Define the intensity of the backscattered radiation that is directed inward:

$$h_-(R,t) = \frac{1}{1-2m/R}(\partial_0+\partial_{r*})\delta. \tag{6.1}$$

Equation (4.1) reads now

$$(-\partial_0+\partial_{r*})\left[\left(1-\frac{2m}{R}\right)h_-\right]=\left(1-\frac{2m}{R}\right)\left[\frac{2}{R^2}\,\delta+\frac{6mf}{R^4}\right]. \tag{6.2}$$

The integral form of Eq. (6.2) reads

$$\left(1-\frac{2m}{R}\right)h_-(R,t)=\int_{R_1}^{R}dr\left[\frac{2}{r^2}\,\delta+\frac{6mf}{r^4}\right]+h_-(R_1,s); \tag{6.3}$$

here the integration contour coincides with a null ingoing geodesics $\Gamma_{(R_1,s),(R,t)}$. (R_1,s) lies on the initial hypersurface $(s=0)$ if $R_1\leqslant b$; thus, $h_-(R_1,s=0)=0$, since the initial data are entirely outgoing. If $R_1>b$, then $(R_1,s)\in\tilde{\Gamma}_{b,(R_1,s)}$; also in this case $h_-(R_1,s)=0$, because $\tilde{\Gamma}_{b,R_1}$ constitutes the outer boundary of the outgoing impulse. In either case the radiation amplitude satisfies the integral equation

$$\left(1-\frac{2m}{R}\right)h_-(R,t)=\int_{R_1}^{R}dr\left[\frac{2}{r^2}\,\delta+\frac{6mf}{r^4}\right]. \tag{6.4}$$

The second term is bounded above by

$$\begin{aligned}\int_R^{R_1}dr\,\frac{6m|f|}{r^4}&\leqslant 6m\left(\int_R^{R_1}dr\,\frac{f^2}{r^{4+2\epsilon}}\right)^{1/2}\left(\int_R^{R_1}dr\,\frac{1}{r^{4-2\epsilon}}\right)^{1/2}\\&=\frac{6m}{\sqrt{3-2\epsilon}}\left(\int_R^{R_1}dr\,\frac{f^2}{r^{4+2\epsilon}}\right)^{1/2}\frac{1}{R^{(3/2)-\epsilon}}\\&\quad\times\sqrt{1-\frac{R^{3-2\epsilon}}{R_1^{3-2\epsilon}}}\\&\leqslant\frac{6m}{\sqrt{3-2\epsilon}}\left(\int_R^{R_1}dr\,\frac{f^2}{r^{4+2\epsilon}}\right)^{1/2}\\&\quad\times\frac{1}{R^{(3/2)-\epsilon}}\sqrt{1-\frac{a^{3-2\epsilon}}{b^{3-2\epsilon}}},\end{aligned} \tag{6.5}$$

where the first inequality follows from the Schwartz inequality and the last inequality is due to the fact that $R/R_1\geqslant a/b$ (Appendix A).

In order to find the integral from the last line of Eq. (6.5), it is useful to project it onto the initial data surface, along outgoing null geodesics $\tilde{\Gamma}_{R_0,(r,\tau)}$. Notice that

$$dR_0=2\,\frac{1-2m/R_0(r)}{(1-2m/r)}\,dr;$$

see Eq. (5.4). The $f^2/r^{4+2\epsilon}$ term cannot decrease during this projection. One arrives at

$$\begin{aligned}\int_R^{R_1}dr\,\frac{f^2(r)}{r^{4+2\epsilon}}&\leqslant\int_a^{R_1}dR_0\frac{1-2m/r}{2(1-2m/R_0(r))}\,\frac{f^2(R_0)}{R_0^{4+2\epsilon}}\\&\leqslant\frac{I_{a,\epsilon}(R_1)}{2\,\eta_a}.\end{aligned} \tag{6.6}$$

Inserting the energy estimate of lemma 1 into Eq. (6.6), one gets finally

$$\int_R^{R_1}dr\,\frac{6m|f|}{r^4}\leqslant\sqrt{\beta_a(b)}\frac{mC_2}{a^\epsilon R^{3/2-\epsilon}}\sqrt{1-\left(\frac{a}{b}\right)^{2\epsilon}}. \tag{6.7}$$

Here the constant C_2 is given by

$$C_2=\frac{3\sqrt{2(1-a^{3-2\epsilon}/b^{3-2\epsilon})}}{\eta_a^{3/2}\sqrt{\epsilon\left[(1+2\epsilon)\,\eta_a^2-\dfrac{4m^2}{a^2}\right](3-2\epsilon)}}. \tag{6.8}$$

The δ-related term of Eq. (6.4) is bounded, due to Eq. (4.2), by

$$\frac{2mC_1}{a^\epsilon}\sqrt{\beta_a(b)}\sqrt{1-\left(\frac{a}{b}\right)^{2\epsilon}}\int_R^{R_1}dr\,\frac{1}{\eta_R r^{3/2}}\left(\frac{1}{R_0^{1-\epsilon}}-\frac{1}{r^{1-\epsilon}}\right). \tag{6.9}$$

Here $r\geqslant R_0$ and r, $R_0\in\tilde{\Gamma}_{R_0,(r,\tau)}$. Thus $1/(r^\epsilon R_0^{1-\epsilon})\leqslant 1/R_0$. Therefore expression (6.9) is bounded above by

$$\frac{2mC_1}{a_\epsilon}\sqrt{\beta_a(b)}\frac{\sqrt{1-(a/b)^{2\epsilon}}}{R^{3/2-\epsilon}}\int_R^{R_1}dr\,\frac{1}{\eta_R}\left(\frac{1}{R_0}-\frac{1}{r}\right). \tag{6.10}$$

The results of lemma 3 and 4 lead now to a pair of estimates. If $R_1\leqslant b$, then

$$2\int_R^{R_1}dr\,\frac{|\delta|}{r^2}\leqslant mC_1\sqrt{\beta_a(b)}\frac{\sqrt{1-(a/b)^{2\epsilon}}}{a^\epsilon R^{3/2-\epsilon}}\left(\ln\frac{R}{a}+\ln\frac{\eta_b}{\eta_a}\right), \tag{6.11}$$

and if $R_1>b$ [in which case $(R,t)\in\Gamma_{(R_1,s)}$], then

$$2\int_R^{R_1}dr\,\frac{|\delta|}{r^2}\leqslant mC_1\sqrt{\beta_a(b)}\frac{\sqrt{1-(a/b)^{2\epsilon}}}{a^\epsilon R^{3/2-\epsilon}}\left(\ln\frac{b}{a}+\ln\frac{\eta_b}{\eta_a}\right). \tag{6.12}$$

In summary, the radiation amplitude is bounded above by

$$\left(1-\frac{2m}{R}\right)|h_-(R,t)|\leqslant\frac{C_3}{a^\epsilon R^{3/2-\epsilon}}\left[C_4+C_1\ln\frac{b\Theta(R_1(R)-b)+R\Theta(-R_1(R)+b)}{a}\right], \tag{6.13}$$

where $\Theta(b-R_1)=0$ if $b-R_1<0$ and $\Theta(b-R_1)=1$ if $b-R_1\geqslant 0$. The constants C_3 and C_4 are defined by

$$C_3\equiv m\sqrt{\beta_a(b)}\sqrt{1-\left(\frac{a}{b}\right)^{2\epsilon}},$$

$$C_4\equiv C_2+C_1\ln\frac{\eta_b}{\eta_a}. \tag{6.14}$$

VII. REFINING THE BOUND ON δ

Equation (6.1) can be written in the integral form

$$\delta(R,t)=\int_{R_0}^{(R,t)} dr\, h_- +\delta(R_0), \tag{7.1}$$

where the integration contour coincides with $\tilde{\Gamma}_{(R_0),(R,t)}$ and R_0 is a point of the initial Cauchy slice defined earlier. Since initially δ vanishes, one has $\delta(R,t)=\int_{R_0}^{R} dr\, h_-$. It becomes clear in Sec. VIII that one needs to bound $\delta(R,t)$ only along $\tilde{\Gamma}_{a,\infty}$; in what follows, this situation is always meant. Define $R(b)\equiv[R{:}R_1(R)=b]$ (see Fig. 1). Inserting the bound of Eq. (6.13) [but notice that Eq. (6.13) bounds $\eta_r|h(r)|$, not $|h(r)|$ itself], one obtains

$$\begin{aligned}|\delta(R)|\leqslant\frac{C_3}{\eta_a a^{\epsilon}}\Bigg[&C_4\int_a^R\frac{dr}{r^{3/2-\epsilon}}+C_1\Theta(R(b)-R)\int_a^R dr\,\frac{\ln(r/a)}{r^{3/2-\epsilon}}\\&+C_1\Theta(R-R(b))\Bigg(\int_a^{R(b)} dr\,\frac{\ln(r/a)}{r^{3/2-\epsilon}}\\&+\ln\frac{b}{a}\int_{R(b)}^R\frac{dr}{r^{3/2-\epsilon}}\Bigg)\Bigg].\end{aligned} \tag{7.2}$$

The integrand of the second integral is non-negative: therefore, extending the integration up to $R(b)$ can give only a bigger quantity. Thus one gets, after elementary integration,

$$\begin{aligned}|\delta(R)|\leqslant\frac{2C_3}{\eta_a(1-2\epsilon)a^{1/2}}\Bigg(&C_4\left[1-\left(\frac{a}{R}\right)^{1/2-\epsilon}\right]+\frac{2C_1}{1-2\epsilon}\\&\times\left[1-\left(\frac{a}{R(b)}\right)^{1/2-\epsilon}\right]+C_1\left(\frac{a}{R(b)}\right)^{1/2-\epsilon}\\&\times\Bigg\{-\ln\frac{R(b)}{a}+\Theta(R-R(b))\\&\times\ln\frac{b}{a}\left[1-\left(\frac{R(b)}{R}\right)^{1/2-\epsilon}\right]\Bigg\}\Bigg).\end{aligned} \tag{7.3}$$

Dropping out the negative term

$$-\Theta(R-R(b))\ln\frac{b}{a}\left(\frac{R(b)}{R}\right)^{1/2-\epsilon}$$

and taking into account that

$$-\ln\frac{R(b)}{a}+\Theta(R-R(b))\ln\frac{b}{a}\leqslant\ln\frac{b}{R(b)},$$

one arrives at

$$\begin{aligned}|\delta(R)|\leqslant\frac{2C_3}{\eta_a(1-2\epsilon)a^{1/2}}\Bigg\{&C_4\left[1-\left(\frac{a}{R}\right)^{1/2-\epsilon}\right]\\&+\frac{2C_1}{1-2\epsilon}\left[1-\left(\frac{a}{R(b)}\right)^{1/2-\epsilon}\right]+C_1\left(\frac{a}{R(b)}\right)^{1/2-\epsilon}\\&\times\ln\frac{b}{R(b)}\Bigg\}.\end{aligned} \tag{7.4}$$

Define

$$\kappa\equiv(b-a)/a. \tag{7.5}$$

One can show (see Appendix B) that

$$\frac{a+b}{2}-m\kappa\leqslant R(b)\leqslant\frac{a+b}{2}; \tag{7.6}$$

the equality is achieved in Minkowski space-time ($m=0$). Since $b=a+a\kappa$, one has $R(b)\geqslant a+\eta_a\kappa/2$ or, defining

$$\alpha\equiv\frac{\eta_a}{2}, \tag{7.7}$$

$R(b)\geqslant a+\alpha\kappa$. The insertion of the above into Eq. (7.4) yields

$$|\delta(R)|\leqslant\frac{C_3}{\eta_a(1-2\epsilon)a^{1/2}}\left\{C_4\left[1-\left(\frac{a}{R}\right)^{1/2-\epsilon}\right]+C_5\right\}, \tag{7.8}$$

where

$$\begin{aligned}C_5\equiv C_1\frac{\ln[(1+\kappa)/(1+\alpha\kappa)]}{(1+\alpha\kappa)^{1/2-\epsilon}}\\+\frac{2C_1}{1-2\epsilon}\left(1-\frac{1}{(1+\kappa/2)^{1/2-\epsilon}}\right).\end{aligned} \tag{7.9}$$

This estimate gives a better control over the asymptotic behavior of δ than the former one, Eq. (4.10), by a factor $1/\sqrt{R}$. In particular, now δ^2/R^2 is known to be integrable. This integrability will be exploited in the next section.

VIII. BOUNDING THE RADIATION ENERGY LOSS

The energy $E_R(t)$ of the electromagnetic field Ψ contained in the exterior of a sphere of a radius R reads

$$E_R(t)=2\pi\int_R^\infty dr\left[\frac{(\partial_0\Psi)^2}{1-2m/r}+\left(1-\frac{2m}{r}\right)(\partial_r\Psi)^2+\frac{2(\Psi)^2}{r^2}\right]. \tag{8.1}$$

Let the initial data be as specified hitherto, $\Psi(t=0)=\tilde{\Psi}$ and $\partial_0\Psi(t=0)=\partial_0\tilde{\Psi}$ for some $\tilde{\Psi}$; thus, they vanish outside an

annular region (a, b). Define $E_a^b = E_a(0)$ as the energy of the initial pulse. If initial configuration is purely outgoing, then $\partial_t \Psi = -\partial_{r*}\tilde{\Psi} + f\partial_{r*}(1/R)$.

Let an outgoing null cone C_a originate from $(a,0)$. In Minkowski space-time the outgoing radiation contained outside C_a does not leak inward and its energy remains constant. In a curved spacetime, however, some energy will be lost from the main stream due to the diffusion of the radiation h_- through C_a. Most of the backscattered radiation will fall onto the center of the gravitational attraction. The forthcoming theorem gives a bound on the amount of diffused energy.

Theorem 3. Under the above assumptions, the fraction of the diffused energy $\delta E_a / E_a^b$ satisfies the inequality

$$\frac{\delta E_a}{E_a^b} \leqslant \left(\frac{2m}{a}\right)^2 \frac{1-1/(1+\kappa)^{2\epsilon}}{\eta_a^2} \times \left[\frac{C_4^2}{(1-\epsilon)}\left(\frac{\eta_a}{16} + \frac{2\epsilon^2}{(1-2\epsilon)^2(3-2\epsilon)}\right) + \frac{C_5^2}{(1-2\epsilon)^2} + C_6 + \frac{2C_4C_5}{(1-2\epsilon)(3-2\epsilon)}\right], \quad (8.2)$$

where $C_1 - C_5$ have been defined earlier and

$$C_6 = \frac{\eta_a C_1^2}{16(1-\epsilon)}\left(\frac{\ln^2[(1+\kappa)/(1+\alpha\kappa)]}{(1+\alpha\kappa)^{2-2\epsilon}} + \frac{1-1/(1+\kappa/2)^{2-2\epsilon}}{(1-\epsilon)^2}\right) + \frac{C_1C_4\eta_a}{16(1-\epsilon)}\left(\frac{2\ln[(1+\kappa)/(1+\alpha\kappa)]}{(1+\alpha\kappa)^{2-2\epsilon}} \times \left(1 + \frac{C_1\ln(1+\kappa/2)}{2C_4(1-\epsilon)}\right) + \frac{1-1/(1+\kappa/2)^{2-2\epsilon}}{(1-\epsilon)}\right). \quad (8.3)$$

Proof. The rate of the energy change along C_a is given by

$$(\partial_0 + \partial_{r*})E_a = -2\pi\left(1-\frac{2m}{R}\right)\left[\left(1-\frac{2m}{R}\right)\left(\frac{\partial_0\Psi}{1-2m/R} + \partial_R\Psi\right)^2 + \frac{2}{R^2}\Psi^2\right] = -2\pi\left(1-\frac{2m}{R}\right)\left[\left(1-\frac{2m}{R}\right)\left(h_- - \frac{f}{R^2}\right)^2 + \frac{2}{R^2}(\tilde{\Psi}+\delta)^2\right]. \quad (8.4)$$

The functions f and $\tilde{\Psi}$ are assumed to vanish on the null cone C_a. Therefore $\Psi = \delta$, $\partial_R\Psi = \partial_R\delta$, and $\partial_t\Psi = \partial_t\delta$ on C_a. In such a case the rate of the energy change reads

$$(\partial_0 + \partial_{r*})E_a = -2\pi\left(1-\frac{2m}{R}\right)\left[\left(1-\frac{2m}{R}\right)h_-^2 + \frac{2\delta^2}{R^2}\right]. \quad (8.5)$$

The energy loss is equal to a line integral along $\tilde{\Gamma}_a$:

$$\delta E_a \equiv E_a - E_\infty = 2\pi\int_a^\infty dr\left[\left(1-\frac{2m}{r}\right)h_-^2 + \frac{2\delta^2}{r^2}\right]. \quad (8.6)$$

The derivation of Eq. (8.2) requires the use of estimates (6.13) and (7.8). The calculation of the δ-related part of the right-hand side of Eq. (8.6) is straightforward and it yields

$$4\pi\int_a^\infty dr\,\frac{\delta^2}{r^2} \leqslant 4\pi\beta_a(b) \times \left(\frac{2m}{a}\right)^2 \frac{1-1/(1+\kappa)^{2\epsilon}}{\eta_a^2(1-2\epsilon)^2}\left[C_4^2\frac{2\epsilon^2}{(3-2\epsilon)(1-\epsilon)} + C_5^2 + 2C_4C_5\frac{1-2\epsilon}{3-2\epsilon}\right]. \quad (8.7)$$

In order to bound the contribution coming from the backscattered radiation amplitude h_-, one needs the estimate (6.13). A straightforward calculation shows that

$$2\pi\int_a^\infty dr\left(1-\frac{2m}{r}\right)h_-^2 \leqslant \pi\beta_a(b)\left(\frac{2m}{a}\right)^2 \frac{1-1/(1+\kappa)^{2\epsilon}}{2\eta_a^2(1-\epsilon)} \times \left[\frac{C_4^2}{2} + C_1C_4\left(y\ln\frac{b}{R(b)} + \frac{1}{2(1-\epsilon)}(1-y)\right) + C_1^2\left(\frac{y}{2}\ln^2\frac{b}{R(b)} - \frac{y}{2(1-\epsilon)}\ln\frac{R(b)}{a}\left(1-2\ln\frac{b}{R(b)}\right) + \frac{1}{4(1-\epsilon)^2}(1-y)\right)\right], \quad (8.8)$$

where $y \equiv [a/R(b)]^{2-2\epsilon}$. Neglecting the negative term with ln... and, using the bounds of Appendix B on $b/R(b)$, one arrives at a bound that in conjunction with Eq. (8.7) proves theorem 3.

Remark. The above estimate depends on the parameter ϵ, which should be chosen in such a way as to optimize the bound. The exact value of the optimal ϵ depends on a and κ, but the value $\epsilon = 1/8$ is proved to yield satisfactory estimates.

IX. DEPENDENCE OF BACKSCATTER ON THE FREQUENCY OF WAVES

The coefficients $C_4 - C_6$ appearing in theorem 3 change with κ, but remain finite in the whole $(0, \infty)$ range of possible values of $\kappa = (b-a)/a$. In the case when the support of the initial radiation is very narrow, i.e., $\kappa \ll 1$, then the coefficient

$$\frac{1-1/(1+\kappa)^{2\epsilon}}{\eta_a^2} \sim \kappa.$$

In such a case one obtains that

$$\frac{\delta E_a}{E_a^b} \leqslant C\left(\frac{2m}{a}\right)^2 \kappa, \tag{9.1}$$

where C is a constant. In the limit $\kappa \to 0$ the ratio $\delta E_a / E_a^b$ becomes 0; the backscattering is negligible in the case of initial pulses of electromagnetic energy that are very narrow. And conversely, the bound becomes bigger with an increase of the width of the radiation pulse. The physical meaning of that can be deduced with the help of the Fourier transform theory. The *similarity theorem* [15] states that compression of the support of a function corresponds to expansion of the frequency scale. If a support of initial data is made narrow, then the wavelength scale of the pulse extends in the direction of short lengths. Therefore the message behind the obtained results must be that high-frequency radiation is essentially unhindered by the effect of backscattering and that long waves can be backscattered.

This dependence of the backscattering on the wavelength has been in fact observed in the numerical investigation of the propagation of pulses of scalar massless fields [16]. In this case halving of the length has led to a similar decrease of the fraction of the diffused energy.

In the case of a black hole or a neutron star, the scale is set essentially by the Schwarzschild radius $R_S = 2m$; waves with lengths much shorter than R_S are not backscattered, while waves of lengths $\sim R_S$ can reveal quite a strong effect. Moreover, one can show that the $(2m/R)^2$ dependence of the effect implies that most of the energy diffusion occurs in regions that are not very far (as compared to the Schwarzschild radius) from the center.

In order to exemplify the above remarks, consider the diffusion effect in following two cases. Assume the same location $a = 4R_S$, of both radiative dipoles and (i) $\kappa = 1/8$ (i.e., the fundamental wavelength R_S) for a pulse I, and (ii) $\kappa = 1/128$ (i.e., the fundamental wavelength $R_S/16$) for the pulse II.

In the calculation that is reported below, ϵ is chosen to be 1/8, in accordance with the remark ending the preceding section. Then in case I one obtains $\delta E_a / E_a^b < 0.37$, while in case II (of shorter waves) one gets $\delta E_a / E_a^b < 0.004$.

The evolution equation (6.2) can be written in another form as

$$(\partial_0 + \partial_{r*})\left[\left(1 - \frac{2m}{R}\right) h_+\right] = \left(1 - \frac{2m}{R}\right)\left[\frac{2}{R^2}\,\delta + \frac{6mf}{R^4}\right], \tag{9.2}$$

where

$$h_+(R,t) = \frac{1}{1 - 2m/R}(-\partial_0 + \partial_{r*})(\delta + f) \tag{9.3}$$

is the intensity of the outgoing part of the radiation. The inequality (3.7) can be written as follows, applying lemma 1 and the remark following it:

$$\frac{|f(R)|}{R^{3/2}} \leqslant C\sqrt{E_a^b}\left(1 - \left[\frac{a}{b}\right]^{2\epsilon}\right]; \tag{9.4}$$

here, C is some constant. The integration of Eq. (9.2) along a null geodesic $\tilde{\Gamma}_a$ yields now

$$\left(1 - \frac{2m}{R}\right) h_+(R(t),t) - \left(1 - \frac{2m}{a}\right) h_+(R(0), t=0) \propto \frac{1}{a^{3/2}}\left[1 - \left(\frac{a}{b}\right)^{2\epsilon}\right], \tag{9.5}$$

where the proportionality constant depends only on ϵ, $2m/a$, and the initial energy E_a^b. Fixing the energy E_a^b, one notices that in the regime $(b-a)/a \ll 1$ the right-hand side of Eq. (9.5) is essentially zero. Thus the product $(1 - 2m/R) h_+$ is constant. In this case one clearly sees the manifestation of the redshift—the rescaling of the amplitude h_+:

$$h_+(\infty) = \eta_a h_+(a, t=0). \tag{9.6}$$

See also a discussion of that fact in a massless scalar field theory [11].

X. DISTANCE DEPENDENCE OF ENERGY DIFFUSION AND SHARPNESS OF THE ESTIMATES

The bound of theorem 3 depends on the source location—it contains, among other factors, a square of the factor $2m/a$. Thus the bounds in question decrease with the increase of a. The dependence on the distance can actually be much stronger. In order to see this, consider the dipole radiation II of (ii), described in the preceding section, but located at $a = 4m$ (instead of $a = 8m$, as assumed formerly). One obtains that now $\delta E_a / E_a^b < 0.77$, instead of the former bound 0.004. Numerical results concerning the propagation of massless scalar fields also show that the backscattered energy decreases rapidly with the increasing distance [11].

It is of interest to establish how accurate the final estimate is. Most of the inequalities derived in this paper are sharp, in the sense that one can find examples that saturate them. Thus, for instance, the two null-line integrals of Sec. V are estimated sharply (the inequalities saturate in Minkowski space-time). Similarly results of Appendixes A and B are also exact; again, the inequalities become equalities in Minkowski space-time. The energy estimate of Sec. III is not sharp, but the ''loss of sharpness,'' to say, can be less than 25% (see the final remark in Sec. III). The main source of unsharpness is the omission of negative terms in a bound on δ (Sec. VII) and in bounds of diffused energy in Sec. VIII, but that becomes insignificant with the decrease of κ, i.e., when the width of the pulse becomes small in comparison to the Schwarzschild radius. On the other hand, the combination of two exact steps can be associated with some loss in the accuracy.

Taking this into account, it is quite likely that in the case of sources characterized by $\kappa < 1$ the bound in question gives an order of the diffused energy. On the other hand, if $\kappa \gg 1$, then the bound of theorem 3 becomes very inaccurate, with

$$\frac{\delta E_a}{E_a^b} \propto C\left(\frac{2m}{a}\right)^2, \tag{10.1}$$

where $C \approx 10^4$. As mentioned before, the main source of unsharpness is the omission of negative terms in a bound on δ (Sec. VII) and in bounds of Sec. VIII. A more accurate treatment would significantly improve the estimates in the long-wave regime.

XI. DISCUSSION

The main result of this paper, theorem 3, states that the dipole energy diffusion due to the backscattering depends on the square of $2m/a$, where m is the mass of the gravitational source and a is a location of the pulse of radiation. Sections IX and X show that the high-frequency radiation essentially is not backscattered, but that the low-frequency radiation can manifest a significant diffusion effect. The last statement is best described in terms of the dimensionless parameter $\tilde{\kappa} \equiv R_S/\lambda$, where λ is the fundamental radiation length. If $\tilde{\kappa} \gg 1$, then the backscattering is negligible, but if $\tilde{\kappa} \approx 1$, then it can be significant. The above results demonstrate that the effect becomes negligible at distances much bigger than the Schwarzschild radius of a central mass. That rules out most stars as objects that can induce observable backscattering effects. For a star of a solar type and $\lambda \sim R_S$, for instance, the ratio $\delta E_a/E_a^b$ can be at most 10^{-20}. In the case of white dwarves and $\lambda \sim R_S$, the bound (10.1) gives $\delta E_a/E_a^b < 10^{-8}$. For long-wave radiation the bounds are bigger—the effect even looks as marginally relevant, for white dwarves, when $\delta E_a/E_a^b \sim 10^{-2}$. However, a sharper estimate would lower that by several orders.

On the other hand, two astrophysical compact objects, neutron stars and (most likely) black holes, are not excluded as objects of interest.

The backscattering would damp the total luminosity produced in accretion disks that exist in vicinities of compact objects, but since the most efficient regions of the disks are located at a distance of (at least) several Schwarzschild radii, the effect would be probably weak. More relevant can be ''echoes''—aftermaths of violent flashy eruptions, produced by a part of radiation reflected from a close vicinity of a horizon of a black hole. Numerical calculations done in the massless scalar fields propagation suggest that the amplitude of the reflected radiation can constitute up to 20% of the incident one, assuming that the length of the wave is comparable to the Schwarzschild radius of a black hole.

The results of this section can be in principle generalized into the case of higher-order multipoles. The key point would consist in showing analogues of the energy estimates of Sec. III that would bound the higher multipole moments. That should lead to a variant of theorem 3 valid under reservations similar to those expressed earlier.

An analysis similar to that of the present paper can be repeated also in the case of a weak gravitational radiation produced in disks rotating around Schwarzschildean black holes. The conclusions concerning the fraction of the diffused energy can be similar.

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APPENDIX A

Lemma A. Let (R,t), $R_1 \in \Gamma_{R_1,(R,t)}$, $R_1 \geqslant R$ and $R > 4m$. Then,

$$\frac{R}{R_1} \geqslant \frac{a}{b}. \tag{A1}$$

Proof. There are two separate cases that need to be considered.

(i) If R_1 lies on the initial hypersurface, then $R_1 \leqslant b$ and $R \geqslant a$ and the inequality follows immediately.

(ii) If $(R_1,s) \in \tilde{\Gamma}_{b,(R_1,s)}$. In this case one has

$$2\left(R_1 - R + 2m \ln\frac{R_1 - 2m}{R - 2m}\right) = b - a + 2m \ln\frac{b-2m}{a-2m}. \tag{A2}$$

Define $X \equiv R_1 - R$. One finds from Eq. (A2) that

$$\frac{d}{dR}\ln\frac{X}{R} = \frac{2m/R}{1-2m/R}\frac{1}{R+X} - \frac{1}{R} \leqslant 0 \tag{A3}$$

provided that $R \leqslant 4m$. Thus X/R is a nonincreasing function which means that R/R_1 is a nondecreasing function and $R/R_1 \geqslant R(b)/b$. Since $R(b) \geqslant a$, one arrives at the postulated inequality.

APPENDIX B

Lemma B. Define $\kappa \equiv (b-a)/a \leqslant 0$. Define $(R(b),t)$ as the intersection point of Γ_b and $\tilde{\Gamma}_a$. Then,

$$\frac{a+b}{2} - m\kappa \leqslant R(b) \leqslant \frac{a+b}{2}. \tag{B1}$$

Proof. The relation (5.3) (see the main text), with $R_1 = b$, $r = R(b)$ and $R_0 = a$, can be written as

$$a = 2R(b) - b + 2m \ln\left(\frac{(R(b)-2m)^2}{(a-2m)(b-2m)}\right). \tag{B2}$$

We will treat Eq. (B2) as a relation between b and $R(b)$, with fixed a. Obviously $R(b) = b = a$ when $b = a$. One easily finds that

$$\partial_b R(b) = \frac{1}{2}\frac{\eta_{R(b)}}{\eta_b}. \tag{B3}$$

Notice that $R(b) \leqslant b$. Thus $\partial_b R(b) \leqslant 1/2$. On the other hand, $R(b) \geqslant a$. Thus $\partial_b R(b) \geqslant (1/2)\,\eta_{R(b)} \geqslant (1/2)\,\eta_a$. The use of those two bounds on $\partial_b R(b)$ and the initial condition $R(a) = a$ immediately imply the lemma.

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