Dispersive Time-Delay Dynamical Systems
Supplemental Material

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CONTENTS

I. MODULATIONAL INSTABILITY IN THE DELAYED MODEL OF A RING LASER

We consider stability of CW solution $A(t) = A_0 e^{i\nu t}$, $G(t) = G_0$, where

$$|A_0|^2 = \frac{g_0 - G_0}{e^{G_0} - 1} e^{\frac{2d\Gamma}{\Gamma^2 + (\Omega+\nu)^2}}, \tag{1}$$

$$G_0 = \ln\left[\frac{\gamma^2 + (w-\nu)^2}{\gamma^2 \kappa} e^{\frac{2d\Gamma}{\Gamma^2+(\Omega+\nu)^2}}\right], \tag{2}$$

$d = \sigma L$, and the frequencies ν of CW solutions satisfy a transcendental equation

$$\tan\left[\frac{\alpha G_0}{2} + \nu T - \frac{d(\nu+\Omega)}{\Gamma^2 + (\Omega+\nu)^2}\right] = \frac{w-\nu}{\gamma}. \tag{3}$$

Then we can obtain the following characteristic equation for the eigenvalues λ describing the stability of CW solutions:

$$a(\lambda) Y(\lambda T)^2 + b(\lambda) Y(\lambda T) + c(\lambda) = 0, \tag{4}$$

where $Y(\lambda) = e^{-\lambda T}$,

$$a(\lambda) = -\gamma^2 \kappa e^{G_0 - \frac{2d(\Gamma+\lambda)}{(\Gamma+\lambda)^2 + (\nu+\Omega)^2}} \left[\lambda + \gamma_g \left(1 + |A_0|^2 e^{-\frac{2d\Gamma}{\Gamma^2+(\nu+\Omega)^2}}\right)\right],$$

$$c(\lambda) = -\left[(\gamma+\lambda)^2 + (w-\nu)^2\right]\left[\lambda + \gamma_g\left(1 + |A_0|^2 e^{G_0 - \frac{2d\Gamma}{\Gamma^2+(\nu+\Omega)^2}}\right)\right],$$

$$b(\lambda) = e^{-\frac{d(\Gamma+\lambda)}{(\Gamma+\lambda)^2+(\Omega+\nu)^2}} \left\{p(\lambda)\cos\left[\Psi - \Theta(\lambda)\right] + q(\lambda)\sin\left[\Psi - \Theta(\lambda)\right]\right\},$$

$$p(\lambda) = \gamma e^{G_0/2}\sqrt{\kappa}\left\{2(\gamma_g+\lambda)(\gamma+\lambda) + \gamma_g |A_0|^2 e^{-\frac{2d\Gamma}{\Gamma^2+(\nu+\Omega)^2}}\left[\left(e^{G_0}+1\right)(\lambda+\gamma) + \alpha\left(e^{G_0}-1\right)(\nu - w)\right]\right\},$$

$$q(\lambda) = \gamma e^{G_0/2}\sqrt{\kappa}\left\{2(\gamma_g+\lambda)(w-\nu) + \gamma_g |A_0|^2 e^{-\frac{2d\Gamma}{\Gamma^2+(\nu+\Omega)^2}}\left[\left(e^{G_0}+1\right)(w-\nu) + \alpha\left(e^{G_0}-1\right)(\gamma+\lambda)\right]\right\},$$

and

$$\Psi = \frac{\alpha G_0}{2} + \nu T, \quad \Theta(\lambda) = \frac{d(\nu+\Omega)}{(\Gamma+\lambda)^2 + (\nu+\Omega)^2}.$$

Finally, in the limit of large delay time $T \to \infty$ we can represent the eigenvalues belonging to the pseudo-continuous spectrum in the form $\lambda = i\mu + \frac{\Lambda}{T} + \mathcal{O}(1/T^2)$ with real μ [18]. Then, keeping only the single leading term $i\mu$ in $a(\lambda)$,$b(\lambda)$, $c(\lambda)$ and two terms $i\mu + \frac{\Lambda}{T}$ in $Y(\lambda T)$, we obtain two branches of pseudo-continuous spectrum $\Lambda_{\pm}(\mu)$ given by

$$\Lambda_{\pm}(\mu) + i\mu T = -\ln\left[\frac{-b(i\mu) \pm \sqrt{b(i\mu)^2 - 4a(i\mu)c(i\mu)}}{2a(i\mu)}\right]. \qquad (5)$$

As it can be seen from Fig. 3(b) in the letter, appearance of the modulational instability is associated with the change of the sign of the curvature of one of the two eigenvalue branches at the origin $\mu = 0$ [16]. Therefore, we assume $y(\mu) = -\operatorname{Re}\ln\hat{Y}(\mu)$, $\hat{Y}(\mu) = Y(i\mu)$, hence the modulational instability threshold can be found using the condition $y''(0) = 0$. Since

$$y''(0) = \operatorname{Re}\left[\frac{\hat{Y}'(0)^2}{\hat{Y}(0)^2} - \frac{\hat{Y}''(0)}{Y(0)}\right],$$

we can find $\hat{Y}(0)$, $\hat{Y}'(0)$, $\hat{Y}''(0)$ from (4), first and second derivative of (4) at $\mu = 0$. One can see that for the corresponding branch of eigenvalues $\hat{Y}(0) = 1$,

$$\hat{Y}'(0) = i\left[\frac{\gamma - \alpha(w-\nu)}{\gamma^2 + (w-\nu)^2} + \frac{d}{\Gamma^2 + (\Omega+\nu)^2}\left(1 - \frac{2\Gamma\big(\Gamma - \alpha(\Omega+\nu)\big)}{\Gamma^2 + (\Omega+\nu)^2}\right)\right],$$

and, finally, we obtain the condition for modulational instability above the lasing threshold

$$-\left(\frac{\gamma - \alpha(w-\nu)}{\gamma^2 + (w-\nu)^2}\right)^2 - \alpha D_2 + F(w,\nu,d,\Omega) > 0, \qquad (6)$$

where $F = -r_s + r_u$ is almost independent of the chromatic dispersion above the lasing threshold for $|\Omega| \gg 1$: $F(w,\nu,d,\Omega) \approx F(w,\nu,0,\Omega)$, and in the absence of dispersion this term is responsible for the appearance of modulational instability at negative detunings $w < \nu$. Moreover,

$$D_2 = \frac{2d(\Omega+\nu)\big(-3\Gamma^2 + (\Omega+\nu)^2\big)}{\big(\Gamma^2 + (\Omega+\nu)^2\big)^3},$$

$$r_s = \frac{2d\Gamma\big(3(\Omega+\nu)^2 - \Gamma^2\big)}{\big(\Gamma^2 + (\Omega+\nu)^2\big)^3} > 0,$$

$$r_u = \left(1+\alpha^2\right)\left[\left(\frac{w-\nu}{\gamma^2 + (w-\nu)^2}\right)^2 + \left(-1 + \frac{2}{r_1} + \frac{2}{r_2}\right)\left(\frac{w-\nu}{\gamma^2 + (w-\nu)^2} - \frac{2d\Gamma(\Omega+\nu)}{(\Gamma^2 + (\Omega+\nu)^2)^2}\right)^2\right].$$

Here, r_1 and r_2 are the parts of the CW field $A_0 = \sqrt{\frac{r_1\kappa}{r_2}}$, where

$$r_2 = \frac{\gamma^2 + (w-\nu)^2}{\gamma^2} - \kappa\exp\frac{-2d\Gamma}{\Gamma^2 + (\Omega+w)^2},$$

$$r_1 = g_0 + \ln\kappa + \ln\frac{\gamma^2}{\gamma^2 + (w-\nu)^2} - \frac{2d\Gamma}{\Gamma^2 + (\Omega+\nu)^2}.$$

One can see that the term $r_s > 0$ increases MI threshold, however for $\Omega \gg 1$ this term is small ($r_s \ll 1$). The term r_u destabilizes the CW regime at $w = \nu$ for any type of dispersion (normal or anomalous) in a small vicinity of the lasing threshold where $0 < g_0 + \ln\kappa - 2d\Gamma/\big(\Gamma^2 + (\Omega+\nu)^2\big) \ll \min(\Gamma, \frac{1}{\Gamma})$ and $y''(0) > 0$ (for $\Omega \gg 1$ we have to assume $\Gamma \ll 1$ in order to avoid unrealistically large losses at $\nu = w$, hence this vicinity is negligibly small). For larger g_0 the term r_u becomes sufficiently small and the CW regime gains stability ($y''(0) < 0$) till another modulational instability threshold is reached ($y''(0) > 0$) and the CW regime is destabilized once again.

We note that the examination of first two terms in (6) implies that for $w > \nu$ the modulational instability threshold value of the dispersion strength is lower, and for some $0 < \nu - w < \alpha\gamma$ it is higher than for $\nu = w$. This explains the asymmetry of the black curve in Fig. 3(c) in the main paper with respect to $w - \nu = 0$.