Invariant Polynomials in Harmonic Analysis

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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Abstract

This thesis presents a methodology for analysing equiaffine invariant measures on surfaces, building on a modified version of a comparability lemma from the 2019 work of P. Gressman [10]. The research focuses on simplifying the process of calculating the equiaffine invariant measure for 2-surfaces in $\mathbb{R}^n$ by avoiding the need for a complete set of generators for the algebra of invariant polynomials. Instead, we compute a sufficient set of invariant polynomials whose intersection characterises the set of unstable points under the $SL_2(\mathbb{C})$ action.

Our approach is demonstrated on specific surfaces in $\mathbb{R}^6$, $\mathbb{R}^{10}$, and $\mathbb{R}^{15}$, where we successfully identify the relevant invariant polynomials and use them, along with the modified lemma, to derive the associated density functions. These density functions can then be used to define the equiaffine invariant measure.

The results offer a practical framework for understanding affine invariants in geometric contexts and suggest possible extensions to higher-dimensional surfaces and different group actions. This work aims to provide a foundation for future studies in geometric invariant theory and harmonic analysis, exploring the interplay between algebraic geometry, invariant theory, and analysis.

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Declaration

This is to certify that the work I am submitting is my own and has not been submitted for another degree, either at University College Cork or elsewhere. All external references and sources are clearly acknowledged and identified within the contents. I have read and understood the regulations of University College Cork concerning plagiarism and intellectual property.

Fergal Murphy.

"The general laws of nature are not, for the most part, immediate objects of perception."

— George Boole, *An Investigation of the Laws of Thought*, Chapter 1, 1854.

Contents

Chapter 1

Introduction

Several problems in modern harmonic analysis have sustained interest for many years. These include the Fourier restriction conjecture and the problem of determining the $L^p - L^q$ mapping properties of convolution operators, among many others.

In his 2003 paper [23], D. Oberlin introduced an equiaffine invariant measure to this circle of problems. Although his definition works for manifolds of arbitrary Hausdorff dimension, for a long time, we had no direct description of what the associated density was for the cases of intermediate Hausdorff dimension, i.e. manifolds that are not curves or hypersurfaces. In his seminal 2019 work [10], P. Gressman gave an explicit description of this density for those intermediate dimensions. The calculation of this density however, remains difficult due to an associated infimum in its construction. In order to overcome this issue, Gressman proves the following lemma, which allows the infimum to be bounded by computing polynomials invariant under an $SL_n(\mathbb{R})$ action.

Lemma 1.0.1 (Comparability Lemma). *Suppose that G is a real-reductive algebraic group and that ρ is a $G-$representation on some finite-dimensional real vector space V equipped with a norm $\|\cdot\|$. Let $p_1, \ldots, p_N$ be any collection of homogeneous polynomials on V, with positive degrees $d_1, \ldots, d_N$, which generates the algebra of all G-invariant polynomials. Then there exists constants $0 < C_1 \leqslant C_2 < \infty$ such that for all $v \in V$ we have*

$$C_1 \max_{j=1,\ldots,N} |p_j(v)|^{\frac{1}{d_j}} \leqslant \inf_{M \in G} \|\rho_M v\| \leqslant C_2 \max_{j=1,\ldots,N} |p_j(v)|^{\frac{1}{d_j}}.$$

Lemma 1.0.1 involves several concepts from geometric invariant theory, a subfield of algebraic geometry. The lemma assumes the existence of finitely many polynomials $p_1, \ldots, p_N$ that generate the algebra of homogeneous invariant polynomials. This is a nontrivial fact that requires working with a linearly reductive group[1]. As we shall see, linearly reductive groups are of significant importance because due to the existence of this finite set of generators we can define a "good quotient" of an affine variety X by the reductive group G by examining its ring of regular functions. It

[1] The term "linearly reductive" typically refers to algebraic groups over an algebraically closed field, where every finite-dimensional representation is completely reducible. In contrast, "real reductive" groups are defined over $\mathbb{R}$, and while they exhibit similar reductive properties, the notion of complete reducibility does not always apply in the same way. However, for many practical purposes in geometric invariant theory (GIT), many of our results over $\mathbb{C}$ extend to $\mathbb{R}$ without loss of generality.

turns out that forming a quotient whose regular functions are the invariant functions under the action leads to what is known as the GIT quotient. This GIT quotient, often denoted by $X//G$ comes with a map $\pi : X \to X//G$ that is surjective and satisfies the property that two points get sent to the same point by π if and only if the closure of their orbits have a non-empty intersection. This allows for the classification of points into two important categories: *unstable* and *semistable*. Unstable points are those that include zero in the closure of their orbit, or equivalently, are mapped to the same point as zero when X is a vector space. Semistable points, in contrast, are those whose orbits do not contain zero in their closure. Topologically, this makes the GIT quotient better behaved than the standard set-theoretic quotient, where two points are considered equivalent if they share the same orbit. The standard set-theoretic quotient is typically not Hausdorff, unlike the GIT quotient.

A good understanding of this GIT quotient requires a good understanding of its regular functions. The GIT quotient is defined in terms of its regular functions, which remain invariant under the group action. This requires a fundamental understanding of algebraic geometry, including Hilbert's Nullstellensatz. Indeed, the Nullstellensatz establishes a one-to-one correspondence between the points of an affine variety and the maximal ideals of its coordinate ring. This correspondence allows one to reconstruct the Zariski topology on an affine variety when the Zariski topology on its maximal spectrum is already known, and vice versa.

Using the tools of GIT and algebraic geometry, Lemma 1.0.1 can be employed to calculate Gressman's equiaffine invariant measure, provided one can identify the polynomials that generate the entire algebra of invariants. There is an immediate problem however, and that is that finding generators for the entire algebra of invariant polynomials is a very difficult task. This is despite the fact that we know finitely many such generators always exist for linearly reductive groups. Therefore, we aim to leave the same hypotheses of Lemma 1.0.1 while strengthening the conclusion. That is, we want a variant of Gressman's Comparability Lemma that still allows us to reconstruct the equiaffine invariant measure without having to compute all the generators of the algebra of invariants. In section 4.2, we prove the following result, which is a modified version of 1.0.1.

Lemma 1.0.2 (Modified Comparability Lemma)**.** *Suppose that ρ is a linear action of a finite-dimensional vector space V (over $\mathbb{C}$) on a linearly reductive group G. Furthermore, suppose V is equipped with a norm $\|\cdot\|$. Let $p_1, \ldots, p_n$ be a collection of homogeneous invariant polynomials such that the intersection of their respective vanishing sets equals the set of unstable points for the action. Then there exists constants $0 < C_1 \leqslant C_2 < \infty$ such that for all $v \in V$ we have*

$$C_1 \max_{j=1,\ldots,N} |p_j(v)|^{\frac{1}{d_j}} \leqslant \inf_{M \in G} \|\rho_M v\| \leqslant C_2 \max_{j=1,\ldots,N} |p_j(v)|^{\frac{1}{d_j}},$$

where d_j denotes the degree of p_j.

This modified variation of Gressman's original lemma is presented over $\mathbb{C}$, and it also holds over $\mathbb{R}$ due to a result of Birkes [1]. In section 4.2 we also show how the original lemma can be derived from the modified one in the special case of the $SL_2(\mathbb{R})$ action on a finite dimensional vector space V. An important point is that the number of invariant polynomials appearing in the modified Lemma 1.0.2 is usually significantly smaller than those appearing in Lemma 1.0.1.

The main objectives of this thesis are:

(i) To present the background harmonic analysis and Gressman's results, establishing the connection to geometric invariant theory, and to present and prove the Modified Comparability Lemma.

(ii) To compute sufficient invariant polynomials for the action of $SL_2(\mathbb{C})$ on $Sym^n(\mathbb{C}^2)$ for $3 \leqslant n \leqslant 5$ and demonstrate their application to the Modified Comparability Lemma for $n = 4, 5$.

(iii) To reconstruct Gressman's equiaffine invariant measure for $2-$surfaces in $\mathbb{R}^6, \mathbb{R}^{10}$ and $\mathbb{R}^{15}$.

(iv) To ensure the content is accessible to readers with an undergraduate education in mathematics. Consequently, we present the necessary requisite material from algebraic geometry, representation theory, reductive groups, and geometric invariant theory.

Outline:
The structure of this thesis is as follows. Chapter 2 introduces key problems in harmonic analysis and briefly discusses the role that the equiaffine invariant measure has with respect to these problems. We then introduce Gressman's article [10] and his construction of the equiaffine invariant measure. We take a more in-depth look at the Comparability Lemma and understand why one needs geometric invariant theory.

Section 3.1 outlines the basics of algebraic geometry, including affine varieties, Hilbert's Nullstellensatz, and regular functions, among other concepts. As previously stated, a solid understanding of the basic algebraic geometry is essential for comprehending regular functions and the GIT quotient in later sections. In section 3.2 we discuss representation theory and use it to define and discuss linearly reductive groups. In particular, we show the group $SL_n(\mathbb{C})$ is linearly reductive which is the most important group for our purposes. We will use the Haar measure from real analysis and prove that $SU_n(\mathbb{C})$ is linearly reductive by employing Weyl's Unitarian trick. As we shall see, the same method used for $SU_n(\mathbb{C})$ can demonstrate that any locally compact topological group with finite Haar measure is linearly reductive. In the last part of this section, we prove Hilbert's Finite Generation Theorem.

In section 4.1, we first introduce and discuss the main results of geometric invariant theory (GIT). This includes why one takes a GIT quotient, the fundamental theorem of GIT, and the Hilbert-Mumford criterion among others. We use the Hilbert-Mumford criterion to characterise the unstable points for the action of $SL_2(\mathbb{C})$ on $Sym^n(\mathbb{C}^2)$.

In section 4.2, we compute enough invariant polynomials for the action of $SL_2(\mathbb{C})$ on $Sym^n(\mathbb{C}^2)$ for $n = 4, 5$, so that Lemma 1.0.2 can be applied when computing the equiaffine invariant measure for $2-$surfaces in $\mathbb{R}^{10}$ and $\mathbb{R}^{15}$, which make up most of the contents of section 4.4. In section 4.3 we discuss projective varieties and give some reasons as to why the concept of stability in GIT is very important in other areas of mathematics not necessarily related to this study.

Lastly, there is an appendix of basic results from commutative algebra and tensor products that we require at times but wish not to prove so as not to obfuscate the main goals of this thesis.

Chapter 2

Background in Harmonic Analysis and Overview of Gressman's Paper

Some definitions and results presented in this section also appear in the introduction; however, they are restated here for the reader's convenience.

Key Historical Problems in Harmonic Analysis.
Two principal problems in modern harmonic analysis are the Fourier restriction conjecture and the characterisation of $L^p - L^q$ mapping properties of geometrically constructed convolution operators. We provide a brief introduction to them here.

The Fourier restriction conjecture, introduced by Elias Stein in the 1970s [26], is a fundamental problem in harmonic analysis. It concerns whether the Fourier transform $\hat{f}$ of a function f defined on $\mathbb{R}^d$ can be meaningfully restricted to a lower-dimensional subset $S \subset \mathbb{R}^d$, such as a sphere. This conjecture connects to broader studies in oscillatory integral operators and has applications in geometric measure theory, combinatorics, and partial differential equations (PDEs).

For a function $f \in L^1(\mathbb{R}^d)$, the Fourier transform $\hat{f}$ is classically defined as:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} \, dx,$$

which is a continuous function that decays to zero at infinity (by the Riemann-Lebesgue lemma). However, for $f \in L^p(\mathbb{R}^d)$ with $p > 1$, $\hat{f}$ may not be pointwise defined and lacks the regularity observed for L^1 functions. To extend the Fourier transform in this setting, one often relies on a distributional approach or limiting arguments in function spaces.

The restriction conjecture examines whether, for a Schwartz function f and some subset S of $\mathbb{R}^d$, there exists an inequality of the form:

$$\left(\int_S |\hat{f}(\xi)|^q \, d\sigma(\xi) \right)^{1/q} \leqslant C \|f\|_{L^p(\mathbb{R}^d)},$$

where $d\sigma$ denotes the surface measure on S, such as the unit sphere S^{d-1}, and C is a constant independent of f. The conjecture seeks the range of exponents p and q

for which this inequality holds, specifically in the range $1 \leqslant p \leqslant 2$.

Stein and subsequent mathematicians observed that for certain subsets $S \subset \mathbb{R}^d$, specifically when $S = S^{d-1}$, it is possible to define a restriction of $\hat{f}$ to S if $f \in L^p(\mathbb{R}^d)$ for some values of p in the range $1 \leqslant p < \frac{2d}{d+1}$. The restriction conjecture remains an open problem for general d and has led to significant advances in harmonic analysis, including applications in the study of dispersive PDEs and connections to the Kakeya and Bochner-Riesz problems. Advanced techniques, such as those developed by Jean Bourgain and Terence Tao, use decoupling theory and multilinear methods to approach these challenges, though a complete resolution of the conjecture is still unknown in higher dimensions. For further background on the restriction problem see e.g [4], [14], [22], [27] .

Another important problem of modern harmonic analysis that has remained of sustained interest for a number of years is the computation of the $L^p - L^q$ mapping properties of convolution operators. Given functions f and g, their convolution is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(t)g(x-t)\,dt.$$

A central problem is determining the sharp exponents p and q for which convolution with a measure μ defines a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Specifically, for a measure μ supported on a submanifold $S \subseteq \mathbb{R}^n$, one seeks the optimal exponents p and q such that the convolution operator

$$f \mapsto f * \mu(x) = \int_S f(x-t)\,d\mu(t)$$

is bounded, i.e.,

$$\|f * \mu\|_{L^q(\mathbb{R}^n)} \leqslant C\|f\|_{L^p(\mathbb{R}^n)}.$$

Determining these exponents typically depends on the geometric properties of the support of μ, such as its dimension, smoothness, or curvature. For example, in the case where μ is the surface measure on the unit sphere S^{n-1}, the celebrated *Stein-Tomas theorem* provides sharp $L^p \to L^2$ bounds for the Fourier transform restricted to the sphere, which are closely related to the convolution operator's boundedness properties. The problem of identifying the precise range of exponents is often linked to deep geometric and combinatorial properties of the underlying set S, and continues to be a major area of research with connections to the Kakeya conjecture, Fourier restriction conjecture, and other significant open problems in analysis, see [3], [5], [6], [17].

The Oberlin Condition.
We now speak of the *Oberlin condition*, which has intimate connections with the Fourier restriction and convolution problems and will allow us to formulate in full the main theorem from [10].

Definition 2.0.1. Suppose μ is non-negative measure on a d-dimensional immersed submanifold of $\mathbb{R}^n$. μ satisfies the *Oberlin condition* with exponent $\alpha > 0$ if there exists $C > 0$ such that for every K in the set $\mathcal{K}_n$ of compact convex subsets of $\mathbb{R}^n$, we have that

$$\mu(K) \leqslant C\,|K|^{\alpha}\,, \tag{2.0.1}$$

where $|K|$ denotes the usual Lebesgue measure of K.

This condition was observed to have the following relation to the Fourier restriction and L^p-improving estimation problems. In particular, we have that

(i) If for all Schwartz functions f on $\mathbb{R}^n$ we have that $\left(\int |\hat{f}|^q d\mu\right)^{\frac{1}{q}} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$, then for all $K \in \mathcal{K}_n$ we have

$$\mu(K) \lesssim |K|^{\frac{q}{p'}},$$

that is, the Oberlin condition is satisfied with exponent $\alpha = \frac{q}{p'}$.

(ii) If for all Schwartz functions f on $\mathbb{R}^n$ we have that $\|f * \mu\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$, then for all $K \in \mathcal{K}_n$ we have that

$$\mu(K) \lesssim |K|^{\frac{1}{p}-\frac{1}{q}},$$

that is, the Oberlin condition is satisfied with exponent $\alpha = \frac{1}{p} - \frac{1}{q}$.

We will return shortly to the Oberlin condition and observe that the equiaffine invariant measure constructed by Gressman, discussed in the next subsection, is the largest measure on a manifold satisfying the Oberlin condition.

Affine measure on curves and hypersurfaces.
Given a submanifold $M \subseteq \mathbb{R}^n$, the standard surface measure σ on M is invariant under rigid motions, meaning it is invariant under translation, reflection, or rotation. If M is a curve or hypersurface (i.e., it has dimension 1 or codimension 1), there exists a natural non-zero measure on M which is invariant under the larger group of equiaffine transformations. In the case of curves and hypersurfaces defined by a graph, the measure is given as follows.

Definition 2.0.2. Suppose $\gamma : I \to \mathbb{R}^n$ is a curve. *Affine arclength measure* is defined as

$$\int f \, d\mu_{\mathcal{A}} := \int_I f(\gamma(t)) \left|\det\left(\gamma'(t), \ldots, \gamma^{(n)}(t)\right)\right|^{\frac{2}{n(n+1)}} dt.$$

If $(x, \varphi(x)) \subseteq \mathbb{R}^{n-1} \times \mathbb{R}$ is a graph defined over a subset $U \subseteq \mathbb{R}^{n-1}$, then *affine hypersurface measure* is defined by

$$\int f \, d\mu_{\mathcal{A}} := \int_U f(x, \varphi(x)) \left|\det \nabla^2 \varphi(x)\right|^{\frac{1}{n+1}} dx,$$

where $\nabla^2 \varphi$ is the Hessian matrix of second derivatives of φ.

Since the 1970s it has been known that these measures play a natural role in certain problems in harmonic analysis with respect to manifolds, such as the Fourier restriction conjecture and the problem of determining the $L^p \to L^q$ mapping properties of the convolution operator $f \mapsto f * \sigma$, both of which we discussed earlier. However, no satisfactory theory of equiaffine invariant measures was known for manifolds M of intermediate dimension $1 < d < n - 1$.

Extension of the affine measure to intermediate dimensions.
In [10], P. Gressman constructs an explicit description of the density for the equiaffine invariant measure for the intermediate dimensions of $1 < d < n-1$, the proof of which relies upon elements of geometric invariant theory, convex geometry, and frame theory. We detail the construction of this equiaffine invariant measure now. The construction begins with a lemma that provides a method to obtain a density from any $k-$linear functional on $\mathbb{R}^d$. The second part involves choosing a suitable k-linear functional that measures curvature and is affine invariant. First, we begin with the lemma, for which we introduce the following notation.

Suppose V is a d-dimensional real vector space. Then the special linear group of matrices on V, denoted $SL_d(\mathbb{R}) = SL(V)$ acts on the (topological) dual space of V by

$$M \cdot f(x) := f(M^T x)$$

for $M \in SL(V)$, $f \in V^*$ and $x \in V$. In turn this induces an action of $SL(V)$ on the $k-$fold tensor product $(V^*)^{\otimes k}$, where we have

$$M \cdot \Theta(v_1, \ldots, v_k) := \Theta(M^T v_1, \ldots, M^T v_k)$$

for some $\Theta \in (V^*)^{\otimes k}$, $M \in SL(V)$ and $(v_1, \ldots, v_k) \in V^{\otimes k}$.
Furthermore, if we fix a linearly independent vector $v := (v_1, \ldots, v_d) \in V^d$, we define a norm $\| \cdot \|_v$ on the space of $k-$linear functionals $(V^*)^{\otimes k}$ by

$$\|\mathcal{A}\|_v^2 := \sum_{j_1=1}^{d} \cdots \sum_{j_k=1}^{d} |\mathcal{A}(v_{j_1}, \ldots, v_{j_k})|^2,$$

where $\mathcal{A} \in (V^*)^{\otimes k}$. We then have the following lemma, whose proof we omit but can found in §2.1 of [10].

Lemma 2.0.3. *Let V be an an $\mathbb{R}$-vector space with $\dim(V) = d$ and let $\Theta \in (V^*)^{\otimes Q}$, and let $v = (v_1, \ldots, v_d)$. Then the mapping,*

$$\delta(v_1, \ldots, v_d) := \left(\inf_{M \in SL(V)} \|\rho_M \Theta\|_v \right)^{\frac{d}{Q}}$$

is a density on V, meaning there is a constant $C_\Theta \geqslant 0$ such that

$$\inf_{M \in SL(V)} \|\rho_M \Theta\|_v^{\frac{d}{Q}} = c_\Theta |\det \begin{pmatrix} v_1 & \ldots & v_d \end{pmatrix}|.$$

The second part of the construction of the affine invariant measure is a choice of multilinear functional that depends on a chosen submanifold Σ, of dimension $d \in \mathbb{N}$. Let

$$f : \Sigma \to \mathbb{R}^n$$

be a smooth immersion of Σ into $\mathbb{R}^n$. For any natural number j, let κ_j be the smallest integer such that the dimension of the space $P_d^{\kappa_j}$ of real polynomials of degree κ_j in d variables has dimension at least $j+1$. Then let $\Lambda_{d,n}$ be the index set given by

$$\Lambda_{d,n} := \{(j,k) \mid 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant \kappa_j, \}.$$

Referring to the diagram in Figure 2 of [10], one can note that the set $\Lambda_{d,n}$ is a union of integer points in rectangles D_l, with l such that $\binom{d+l}{l} \leqslant n$ and where $(j,k) \in D_l$ if and only if $1 \leqslant j \leqslant n$ and

$$\binom{d+l-1}{l-1} \leqslant j < \binom{d+l}{l}.$$

For each $\lambda = (j,k) \in D_l \subset \Lambda_{d,n}$, then j can be thought of as indexing $d-$tuples $\alpha_j = ({\alpha_j}^1, \ldots, {\alpha_j}^d) \in \mathbb{N}^d$ with

$$|\alpha_j| := \sum_{i=1}^{d} \alpha_j^i = l.$$

Indeed, note that the cardinality of the set of such α_j with $|\alpha_j| = l$ is exactly

$$\binom{d+l-1}{l} = \binom{d+l}{l} - \binom{d+l-1}{l-1},$$

the length of the rectangle D_l. Hence each $X_{j,k}$ can be chosen equal to some ∂_{x_i}, $1 \leqslant i \leqslant d$, so that

$$\prod_{k=1}^{k_j} X_{j,k} = \partial_{\alpha_j}$$

corresponds to the l-th order derivative with multi-order α_j.

Next define Q as

$$Q = Q_{d,n} := \#\Lambda_{d,n}.$$

Fix a point $p \in \Sigma$ on our manifold. For any finite sequence of vector field X_λ indexed by $\lambda \in \Lambda_{d,n}$, let

$$\begin{aligned}\mathcal{A}_p^f((X_\lambda)_{\lambda\in\Lambda_{d,n}}) := \det\big(&X_{(1,1)}f(p) \wedge \cdots \wedge X_{(j,1)} \ldots X_{(j,k_j)}f(p) \wedge \cdots \\ &\wedge X_{(n,1)} \cdots X_{(n,k_n)}f(p)\big)\,,\end{aligned}$$

where we lexicographically order the elements of $\Lambda_{d,n}$[1]. It is shown in proposition 2, §2 of [10] that $\mathcal{A}_p^f$ depends only on the values of X_λ at p, and it can therefore be identified with a $Q_{d,n}-$linear functional on $T_p\Sigma$. Moreover, $\mathcal{A}_p^f$ is equi-affine invariant. The object $\mathcal{A}_p^f$ is called the *affine curvature tensor at* p, and is a tensor in the usual sense, that is , a multilinear function of the vector fields X_λ.

Combining all of this, the equiaffine invariant measure of our general d-dimensional submanifold is defined as the pushforward via f of the measure on Σ associated to the density generated by the affine curvature tensor. Explicitly, for any nonnegative Borel function F on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} F \, d\mu_{\mathcal{A}} := \int_{\Sigma} (F \circ f) \, d\nu_{\mathcal{A}}, \tag{2.0.2}$$

where $\nu_{\mathcal{A}}$ is the measure uniquely defined from the density.

[1]Recall the graded lexicographic order $\preceq$ on the multi indices $\mathbb{N}^d \backslash \{0\}$, where $\alpha \prec \beta$ if $|\alpha| < |\beta|$ and $|\alpha| = |(\alpha_1, \ldots, \alpha_d)| = |\alpha_1| + \ldots |\alpha_d|$, and if $|\alpha| = |\beta|$ then we compare them lexicographically, [12, §4].

Next, we formally state the main theorem from Gressman's 2019 paper [10].

Theorem 2.0.4 (Existence and Properties of the Equiaffine Invariant Measure)**.** *Suppose that $\mathcal{M}$ is an immersed d-dimensional submanfiold of $\mathbb{R}^n$. To any such $\mathcal{M}$, one may associate a nonnegative Borel measure $\mu_{\mathcal{A}}$ on $\mathbb{R}^n$, defined by equation 2.0.2, which is supported on $\mathcal{M}$. Then the following statements are true.*

(i) If μ is any nonnegative Borel measure supported on $\mathcal{M}$ which satisfies 2.0.1 with exponent $\alpha > \frac{d}{Q}$, then μ is the zero-measure.

(ii) If μ is any nonnegative Borel measure supported on $\mathcal{M}$ which satisfies 2.0.1 *with exponent $\alpha = \frac{d}{Q}$, then $\mu \lesssim \mu_{\mathcal{A}}$*[2]*.*

(iii) If $\mathcal{M}$ is the image of an immersion $f : \Omega \to \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^d$ is open with compact closure $\overline{\Omega}$ and f extends to be real analytic on a neighbourhood of $\overline{\Omega}$, then the measure $\mu_{\mathcal{A}}$ satisfies 2.0.1 with exponent $\alpha = \frac{d}{Q}$ and is consequently the largest measure, up to a multiplicative constant.

Furthermore, it is also the case that:

(iv) The measure $\mu_{\mathcal{A}}$ agrees up to a normalisation with affine arclength and affine hypersurface measure when $d = 1$ and $n - 1$ respectively.

(v) The correspondence sending $\mathcal{M}$ to $\mu_{\mathcal{A}}$ is equiaffine invariant, meaning that if $\mathcal{M}$ and $\mathcal{M}'$ are immersed submanifolds such that $\mathcal{M}'$ is the image of $\mathcal{M}$ under some equiaffine transformation T of $\mathbb{R}^n$, then the measure $\mu'_{\mathcal{A}}$ corresponding to $\mathcal{M}'$ and the measure $\mu_{\mathcal{A}}$ corresponding to $\mathcal{M}$ satisfy $\mu'_{\mathcal{A}}(T(E)) = \mu_{\mathcal{A}}(E)$ for all Borel sets E.

An Example of the Use of the Comparability Lemma

Example 2.0.5. In section §3.1, Gressman computes the equiaffine invariant measure for a 3-surface in $\mathbb{R}^{10}$ using lemma 1.0.1, an overview of which we give now. For such a 3−surface, one first computes the index set $\Lambda_{3,10}$ and then the associated tensor A_p using the methods outlined previously. We omit the computation for this example because of the large dimensions involved but will explore how this is done later in section 4.4.1. This then boils to computing

$$\inf_{M \in SL_3(\mathbb{R})} \|M \cdot A_p\|.$$

It is known that the algebra of $SL_3(\mathbb{R})-$invariant polynomials of homogeneous degree 3 polynomials in three variables is generated by Aronhold's invariants S and T of degrees 4 and 6 respectively - see [28]. Using lemma 1.0.1, one consequently has that

$$\frac{d\nu_{\mathcal{A}}}{dt} \approx |S(A_p)|^{\frac{1}{4}} + |T(A_p)|^{\frac{1}{6}}.$$

[2]The inequality $A \lesssim B$ means there is a constant $C > 0$, such that $A \leqslant CB$.

A drawback of the comparability lemma, is that in general, computing the generators for the algebra of invariant polynomials is a difficult thing to do, even for relatively simple groups. Indeed, example 2.0.5 was slightly artificial as we knew the generators beforehand. Even for the relatively simple action of $SL_2(\mathbb{C})$ on $Sym^n(\mathbb{C}^2)$, computing the generators is very difficult for $n \geqslant 4$. In fact, it is substantially easier to directly compute the entire of algebra of invariant polynomials than it is to determine the generators. This is mainly due to the nature of the Reynolds operator, see [24].

To overcome this, in section 4.2 we will prove the modified version of the Comparability Lemma, which will allow us to perform our desired analysis without having to compute explicit generators. With this result in place, we will be able to compute the equiaffine invariant measure for $2-$surfaces in $\mathbb{R}^{10}$ and $\mathbb{R}^{15}$ - without computing the generators for the entire algebra of invariant polynomials. We will also compute the equiaffine invariant measure a $2-$surface in $\mathbb{R}^6$, however, this does not explicitly use Lemma 1.0.2 as the algebra of invariant polynomials, in this case, is classically known to be generated by the discriminant polynomial. Instead we use Gressman's original Lemma 1.0.1.

Chapter 3

Preliminaries

3.1 Basic Algebraic Geometry

Throughout this and later sections, K will denote an algebraically closed field.

Definition 3.1.1. If I is an ideal of a ring R, denoted by $I \trianglelefteq R$, then the *radical* of I, denoted by $\sqrt{I}$, is the following set,

$$\sqrt{I} := \{f \in R \mid f^k \in I \text{ for some } k \in \mathbb{N}_0\}.$$

The *affine n-space* over K is the following set,

$$\mathbb{A}^n := \{(a_1, \ldots, a_n) \mid a_i \in K\}.$$

The polynomial ring in n variables $x_1, \ldots, x_n$ is denoted by $K[x_1, \ldots, x_n]$. If $S \subseteq K[x_1, \ldots, x_n]$, we call the following set

$$V(S) := \{x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S\}$$

the *affine zero locus* of S, or simply the *zero locus* when the context is clear. We say a subset $X \subseteq \mathbb{A}^n$ is an *affine variety* if it is the zero locus of a set $S \subseteq K[x_1, \ldots, x_n]$.

Remark 3.1.2. If $S = \{f_1, \ldots f_k\}$ is a finite set, we write $V(f_1, \ldots, f_k)$ to simplify notation. Later on we may change $K[x_1, \ldots, x_n]$ to $\mathbb{C}[x_1, \ldots, x_n]$ to make it obvious we are working over $\mathbb{C}$.

Definition 3.1.3. If S is a subset of the ring R, the ideal generated by S is the smallest ideal in R containing S and is denoted by $\langle S \rangle$.

For commutative rings this coincides with the set of all $R-$linear combinations of elements of S, that is

$$\langle S \rangle = \left\{ \sum_{i=1}^{k} \alpha_i x_i \mid \alpha_i \in R,\, x_i \in S,\, k \in \mathbb{N} \right\}.$$

Consequently, it is a routine check that for commutative rings (which we deal mostly with) we have $V(S) = V(\langle S \rangle)$ whenever $S \subseteq K[x_1, \ldots, x_n]$. From this it follows that an affine variety in $\mathbb{A}^n$ can be written as the zero locus of an ideal in $K[x_1, \ldots, x_n]$. Thus, we may restrict our attention to zero loci of ideals when considering affine varieties.

Definition 3.1.4. If I and J are ideals in $K[x_1, \ldots, x_n]$ and $I \subseteq J$, then $V(I) \supseteq V(J)$. In this case we say that $V(J)$ is a *subvariety* of $V(I)$.

Example 3.1.5. The special linear group of $n \times n$ matrices with determinant 1 and with complex coefficients, $SL_n(\mathbb{C})$, can be identified as the vanishing set

$$SL_n(\mathbb{C}) = \{g \in \mathbb{A}^{n^2} \mid \det(g) - 1 = 0\}.$$

Thus, it is an affine variety. More generally, the set GL_n of $n \times n$ invertible matrices is an affine variety in $\mathbb{A}^{n^2+1}$ as we can view it as

$$\{(x_{1,1}, \ldots, x_{n,n}, y) \in \mathbb{A}^{n^2+1} \mid \det \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{n,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} y - 1 = 0\}.$$

Definition 3.1.6 (Ideal of a subset of $\mathbb{A}^n$)**.** Let $X \subseteq \mathbb{A}^n$. Then the *ideal* of X is the following set,

$$I(X) := \{f \in K[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X\}.$$

Remark 3.1.7. For any $X \subseteq \mathbb{A}^n$, it is a routine check that $I(X)$ is an ideal in the ring-theoretic sense. Moreover, $I(X) = \sqrt{I(X)}$ and so it is a radical ideal. This is a consequence of K being a field.

The following result, known as *Hilbert's Nullstellensatz*, is foundational in algebraic geometry.

Theorem 3.1.8 (Hilbert's Nullstellensatz)**.** *Let K be an algebraically closed field. Suppose $X \subseteq \mathbb{A}^n$ is an affine variety and J is an ideal in the polynomial ring $K[x_1, \ldots, x_n]$. We then have that,*

(i) $V(I(X)) = X$.

(ii) $I(V(J)) = \sqrt{J}$.

Moreover, there is an inclusion reversing bijection

$$\{\textit{affine varieties in } \mathbb{A}^n\} \longleftrightarrow \{\textit{radical ideals in } K[x_1, \ldots, x_n]\}.$$

Example 3.1.9 (Maximum Ideals Correspond to Points)**.** Consider the ideal $J = \langle x_1 - a_1, \ldots, x_n - a_n\rangle \trianglelefteq K[x_1, \ldots, x_n]$. To see it is maximal, consider the map $\theta : K[x_1, \ldots, x_n] \to K$, $\theta(f) := f(a_1, \ldots, a_n)$. It is easy to see that $J \subseteq \ker(\theta)$. For the reverse, suppose $f \in \ker(\theta)$. Then $f(a_1, \ldots a_n) = 0$, and so $(a_1, \ldots, a_n)$ is a root of f. Consequently, there exist polynomials $p_i \in K[x_1, \ldots, x_n]$ for $1 \leqslant i \leqslant n$ such that

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} (x_i - a_i) p_i(x_1, \ldots, x_n).$$

It follows that $f \in J$, whence $J = \ker(\theta)$. It is easy to see that θ is onto, hence the first isomorphism theorem states $K[x_1, \ldots, x_n]/J \cong K$. Since K is a field, it follows that J is maximal.

Conversely, suppose $\mathfrak{m}$ is a maximal ideal in $K[x_1, \ldots, x_n]$. We see that $V(\mathfrak{m}) \neq \varnothing$, otherwise by the Nullstellensatz we would have that $\mathfrak{m} = K[x_1, \ldots, x_n]$ which is precluded by the definition of maximal. Hence, there exists a point $a \in V(\mathfrak{m})$. Taking the ideal of both sides therefore gives

$$I(a) \supseteq I(V(\mathfrak{m})) = \mathfrak{m}$$

because taking the ideal is inclusion reversing.
Now, $I(a) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ and since $\mathfrak{m}$ is maximal, it follows that

$$\langle x_1 - a_1, \ldots, x_n - a_n \rangle = \mathfrak{m}.$$

We therefore have the following one-to-one correspondence between maximal ideals in and points in an affine variety,

$$\text{Points in } \mathbb{A}^n \longleftrightarrow \text{ Maximal Ideals in } K[x_1, \ldots, x_n]$$

Example 3.1.10. Suppose $J \trianglelefteq K[x]$ is a non-zero ideal. By A.0.18 we have that $K[x]$ is a principal ideal domain and so $J = \langle f \rangle$ for an appropriate polynomial $f \in K[x]$. By the fundamental theorem of algebra we can express f in terms of its roots, and so

$$f = c(x - r_1)^{k_1} \ldots (x - r_m)^{k_m}$$

for distinct points $r_1, \ldots, r_m \in \mathbb{A}$, $c \in K$ and $k_1, \ldots, k_m \in \mathbb{N}$. The zero locus of J is therefore the set $V(J) = \{r_1, \ldots, r_m\}$. Hence, by the Nullstellensatz, we see that

$$I(V(J)) = \sqrt{J} = \langle (x - r_1) \ldots (x - r_m) \rangle .$$

This shows the radical of J strictly contains J as it consists all polynomials vanishing at $r_1, \ldots, r_m$, with no restriction on the order.

The assumption that K is algebraically closed for the Nullstellensatz is very important. To see this, consider the ideal $J = \langle x^2 + 1 \rangle \trianglelefteq \mathbb{R}[x]$. Let $\varphi : \mathbb{R}[x] \to \mathbb{C}$, $\varphi(f) = f(i)$. It is easy to see that $\langle 1 + x^2 \rangle \subseteq \ker(\varphi)$. For the reverse, suppose $p \in \ker(\varphi)$. Then $(x + i) \mid p$. Since complex roots come in pairs, we also have $(x - i) \mid p$. Thus, $(x^2 + 1) \mid p$. It follows $p \in \langle 1 + x^2 \rangle$. By the first isomorphism theorem, we get that $\mathbb{R}[x] \backslash \langle 1 + x^2 \rangle \cong \mathbb{C}$. Since $\mathbb{C}$ is a field, it follows by A.0.8.2 that $\langle 1 + x^2 \rangle$ is a maximal ideal in $\mathbb{R}[x]$. Moreover, since $\mathbb{R}[x]$ is a principal ideal domain, by A.0.10, it is prime also. By A.0.11 we therefore have that $\langle 1 + x^2 \rangle = \sqrt{\langle 1 + x^2 \rangle}$. We then compute the following,

$$\begin{aligned} V(\langle 1 + x^2 \rangle) &= \{x \in \mathbb{A} \mid f(x)(1 + x^2) = 0 \text{ for all } f \in R[x]\} \\ &\subseteq \{x \in \mathbb{A} \mid (1 + x^2) = 0 \text{ for all } \} = \varnothing \end{aligned}$$

Using equation (ii) from the Nullstellensatz, we therefore get that $I(V(\langle 1 + x^2 \rangle) = I(\varnothing) = \mathbb{R}[x] = \langle 1 + x^2 \rangle$ which is a contradiction. Thus we see that algebraic closure is essential.

Definition 3.1.11. Let $X = V(I) \subseteq \mathbb{A}^n$ be an affine variety, where $I \trianglelefteq K[x_1, \ldots, x_n]$. The *coordinate ring* of X is the quotient ring,

$$A(X) := K[x_1, \ldots, x_n]/I(X).$$

This may also be considered the collection of polynomials in $K[x_1, \ldots, x_n]$ restricted to X, and may also be called the ring of regular functions on X. Under pointwise addition and multiplication it is also a $K-$algebra.

Morphisms of affine varieties and varieties more generally is a larger topic than is desired here, so we make the following relatively simple definition which shall suit our purposes.

Definition 3.1.12. Suppose X and Y are affine varieties. We say that they are *isomorphic* if their respective rings of regular functions are isomorphic (as rings, not necessarily as $K-$algebras). That is, $A(X) \cong A(Y)$.

Remark 3.1.13. The coordinate ring may also sometimes be called the *ring of regular functions.*

Although a lot more could be said on the basics of algebraic geometry, we will keep it brief so as not to obfuscate the main results later on. In this last part we introduce the *Zariski topology.*

Definition 3.1.14 (Zariski Topology)**.** Let $X \subseteq \mathbb{A}^n$ be an affine variety. The *Zariski topology* on X is the topology whose closed sets are the affine subvarieties of X, that is, sets of the form $V(S)$ for some $S \subseteq A(X)$.

Remark 3.1.15. That the Zariski topology actually is a topology is a clear consequence of basic properties of the zero locus and that $\varnothing = V(1)$ and $X = V(0)$.

Proposition 3.1.16. *Every polynomial $f \in K[x_1, \ldots, x_n]$ when viewed as a function $\mathbb{A}^n \to \mathbb{A}$ is continuous, where both $\mathbb{A}^n$ and $\mathbb{A}$ have the Zariski topology.*

Proof : Let X be a closed set in $\mathbb{A}^1$. In $\mathbb{A}^1$ it is easy to see that an affine variety is either all of $\mathbb{A}^1$ or a finite set. If it is the former, then $f^{-1}(X) = \mathbb{A}^n = V(0)$ is closed.
If $X = \{a_1, \ldots, a_k\}$, then note that

$$\begin{aligned} V\left((f - \alpha_1)(f - \alpha_2)\ldots(f - \alpha_k)\right) &= \{x \in \mathbb{A}^n \mid f(x) \in X\} \\ &= f^{-1}(X). \end{aligned}$$

□

Example 3.1.17. Consider $X = \{x \in \mathbb{A}^1_{\mathbb{C}} \mid |x| \leqslant 1\}$. X is closed in the Euclidean topology but not in the Zariski topology. As a result, the closure of X in the Zariski topology, that is, that smallest affine variety containing X, is $\mathbb{A}$. Thus being closed in the usual sense does not imply Zariski closed. The converse however, is a true statement.

Proposition 3.1.18. *Every Zariski closed set is closed in the Euclidean topology.*

Proof : Suppose $X \subseteq \mathbb{A}^n$ is an affine variety and Y is an affine subvariety of X, that is, Zariski closed. Then $Y = V(S)$ for some $S \subseteq A(X)$. S can be assumed finite and so Y is the vanishing set of some collection of polynomials functions on X, say $f_1, \ldots, f_k$. Note that $Y = \bigcap_{i=1}^k f_i^{-1}(0)$. Moreover, each f_i is a polynomial equation, hence continuous in the Euclidean topology. Furthermore, $\{0\}$ is a singleton, hence closed. It follows that $f_i^{-1}(0)$ is closed, and so Y is too.

□

Suppose X is an affine variety in $\mathbb{A}^n$, and $A(X)$ is its ring of regular functions. We can recover the topology of the affine variety from $A(X)$. For this we require the *maximal spectrum.*

Definition 3.1.19. The *maximal spectrum* of a ring R, denoted by $\mathrm{MaxSpec}(R)$ is the set of all maximal ideals in R.

From example 3.1.9 we know that each point p in the variety X corresponds to a maximal ideal $\mathfrak{m}_p$ of the form

$$\mathfrak{m}_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$$

where $p = (p_1, \ldots, p_n)$. Thus we can write our variety as

$$\begin{aligned} X &= \bigcup_{p \in X} p \\ &= \bigcup_{p \in X} V(I(p)) \\ &= \bigcup_{p \in X} V(\mathfrak{m}_p). \end{aligned}$$

Definition 3.1.20. The Zariski topology on $\mathrm{MaxSpec}(A[X])$ is defined by specifying the closed sets as those sets $Z \subseteq \mathrm{MaxSpec}(A[X])$ such that there exists an ideal $I \subseteq A(X)$ such that

$$Z = V(I) = \{\mathfrak{m} \in \mathrm{MaxSpec}(A[X]) \mid I \subseteq \mathfrak{m}\}.$$

Remark 3.1.21. We see that $V(I)$ consists of all maximal ideals in $K[x_1, \ldots, x_n]$ that contain the ideal I. The Zariski topology on X is the topology where the closed sets are precisely the sets of the form $V(I)$ for some ideal $I \subseteq A(X)$.
Consequently, we can recover the Zariski topology on X from $\mathrm{MaxSpec}(A[X])$ and vice versa. Specifically, a set $Z \subseteq X$ is closed if and only if the set of corresponding maximal ideals $\{\mathfrak{m}_p \mid p \in Z\}$ is closed in the Zariski topology on $\mathrm{MaxSpec}(A[X])$.

Finally, the following result, which we will make use of later, says that for regular maps between affine varieties over $\mathbb{C}$, there is no distinction between the closure of the image in the usual Euclidean topology and the Zariski topology. For a proof of this result, see [25], §5.3, page 63 of Shavarevich's book, *Basic Algebraic Geometry.*

Theorem 3.1.22. *Suppose X and Y are both affine varieties both defined over $\mathbb{C}$. Let $f : X \to Y$ be a regular map between these two affine varieties. Then the closure of $f(X)$ in Y is the same in both the Zariski topology and the usual topology.*

3.2 Representation Theory and Reductive Groups

Throughout this part we assume that $(G, \cdot)$ is a group with identity e.

Definition 3.2.1. Let X be a set and suppose $\alpha : G \times X \to X$ is a map such that

(i) $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$

(ii) $\alpha(e, x) = x$

for all $g_1, g_2 \in G$ and $x \in X$. Then α is called an *action* of G on X.

Remark 3.2.2. To simplify notation we will sometimes use the notation gx or $g \cdot x$ instead of $\alpha(g, x)$ when it is convenient.

Remark 3.2.3. For a fixed element $g \in G$, the map $\alpha(g, \cdot) : X \to X$ is a bijection with inverse map given by $\alpha(g^{-1}, \cdot) : X \to X$.

Definition 3.2.4. For a set X, we denote by $Perm(X)$ the set of bijections on X, that is,

$$Perm(X) := \{f \in X^X \mid f \text{ is bijective }\}.$$

Proposition 3.2.5. *Suppose X is a set. Then $Perm(X)$ is a group under function composition. Moreover, there is a natural bijection between the set of actions $\alpha : G \times X \to X$ and $Hom(G, Perm(X))-$ the set of homomorphisms between G and $Perm(X)$.*

Proof : That $Perm(X)$ is a group under composition is straightforward. For each $\alpha : G \times X \to X$ we define $\alpha : G \to Perm(X)$ as $\alpha(g, \cdot) : X \to X$. The condition $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ is equivalent with $\alpha(g_1 \cdot g_2) = \alpha(g_1) \circ \alpha(g_2)$. □

Definition 3.2.6. Suppose $\alpha_1 : G \times X \to X$ and $\alpha_2 : G \times Y \to Y$ are actions of G on X and Y respectively. We say they are the same if there exists a bijection $\varphi : X \to Y$ such that the following diagram commutes,

$$\begin{tikzcd} G \times X \arrow[r, "\alpha_1"] \arrow[d, "(\mathrm{id},\varphi)"'] & X \arrow[d, "\varphi"] \\ G \times Y \arrow[r, "\alpha_2"'] & Y \end{tikzcd}$$

That is, $\alpha_2(g, \varphi(x)) = \varphi(\alpha_1, (g, x))$, for all $g \in G$ and $x \in X$.

Example 3.2.7. Let H be a subgroup of G and let $G/H = \{gH \mid g \in G\}$. Then the map $\alpha : G \times G/H \to G/H$, $\alpha(g_1, g_2 H) := (g_1 g_2)H$ is an action of G on G/H.

Definition 3.2.8. Let α be an action of G on X and let $x \in X$. The *orbit* of x is the set

$$Gx = Orb(x) := \{gx \mid g \in G\}.$$

The *stabilizer* of x is

$$Stab(x) = \{g \in G \mid gx = x\}.$$

Remark 3.2.9. For an action G on an affine variety X, the closure of the orbit Gx in the Zariski topology is equal to the closure in the standard topology, as this is true in general for all Zariski locally-closed subsets, when the ambient field is $\mathbb{C}$. (see [25, Ch.I, §5.3, Th.6] or [29, §1.3,, Th.1.25]) When working over $\mathbb{R}$ this is not necessarily the case. One can however still say that in this case 0 is in the Zariski closure of an orbit if and only it is in the standard closure.

Proposition 3.2.10. *Suppose G is an action on a set X. Then for each $x \in X$, the action restricts to an action $\alpha|_{Orb(x)} : G \times Orb(x) \to Orb(x)$. Moreover, this action is the same as the action of G on $G/Stab(x)$.*

Proof : Suppose $g_1x \in Orb(x)$. Then $g \cdot (g_1x) = (gg_1)x \in Orb(x)$. It is routine to check that $Stab(x)$ is a subgroup; if $g_1x = x$ and $g_2x = x$ then $(g_1 \cdot g_2)x = g_(g_2x) = g_1x = x$. If $gx = x$ then $g^{-1}gx = g^{-1}x$ which means $g^{-1} \in Stab(x)$.
Let $\varphi : Orb(x) \to G/Stab(x)$ be given by $\varphi(gx) := gStab(x)$. To see this map is a well-defined injection, suppose $gx = g'x$. This happens if and only if $g^{-1}g'x = x \iff g^{-1}g' \in Stab(x)$. This in turn happens if and only if $gStab(x) = g'Stab(x)$ and so $\varphi(gx) = \varphi(g'x)$. Moreover, it is clearly onto, and so a bijection. It follows that both actions are the same, in the sense of definition 3.2.6. □

Remark 3.2.11. The natural actions of G on G/H and G/gHg^{-1} are the same as we can view G/H as the orbit of gH which has $Stab(gH) = gHg^{-1}$, and so its orbit is the same as G/gHg^{-1} as a set with an action G by proposition 3.2.10.

When a group G acts linearly on a vector space V, we get a map $\rho : G \to GL(V)$ such that $\rho(e)$ is the identity and $\rho(gh, v) = \rho(g, \rho(h, v))$ for all $g, h \in G$ and $v \in V$. In such a case, we call (V, ρ) a *representation of* G. When the context is clear, we omit ρ and simply say that V is a representation of V. If (V, ρ) and (W, φ) are both representations of G, then a linear map $\alpha : V \to W$ is *equivariant* if for all $g \in G$, the following diagram commutes,

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\rho(g)"'] & W \arrow[d, "\varphi(g)"] \\ V \arrow[r, "\alpha"'] & W \end{tikzcd}$$

That is, for all $x \in V$ and $g \in G$, $\varphi(g)(\alpha(x)) = \alpha(\rho(g)(x))$.

If V is a representation of G and $W \subseteq V$ is a subspace of V, then we say that W is *G-stable* if $\rho(g, W) \subseteq W$ for all $g \in G$. In this case, the corestrction $\rho : G \to GL(W)$ is also a representation, and in this case we say that W is a subrepresentation of V. There will always be at least two subrepresentations, $\{0\}$ and V, and these are called the trivial ones. If only the trivial ones exist, we say that V is irreducible. If however, there is a proper, nontrivial, invariant subspace, then ρ is *reducible.*

Definition 3.2.12. Let $\rho : G \to GL(V)$ be a representation of a group G by a vector space V. We say that ρ is *completely reducible,* or *semisimple*, if there exist subspaces of V, $(W_i)_{i \in I}$, such that $V = \oplus_{i \in I} W_i$, and each W_i is an irreducible subrepresentation of ρ.

The following result is very important in representation theory, a more in-depth survey can be found at [15, §10].

Theorem 3.2.13. *[Schur's Lemma] Suppose V and W are irreducible representations of a group G and $f : V \to W$ is a non-zero equivariant map. Then V and W are isomorphic.*
Furthermore, in the special case that $V = W$ is finite dimensional, $f = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.

Proof : If $x \in \ker(f)$ then $gx \in \ker(f)$ also since f is equivariant. This means $\ker(f)$ is a subrepresentation of G, and because V is irreducible and f is not the zero map, $\ker(f) = \{0\}$. Similarly, $im(f)$ is a subrepresentation of G and since W is irreducible and f is non-zero, $im(f) = W$. Consequently, f is a bijective linear map, i.e. an isomorphism.
For the second part, let λ be an eigenvalue of f (which will always exist because V is finite-dimensional). Consider the map $f' = f - \lambda \mathrm{id}$. It is a trivial task to verify that f' is also an equivariant map from V to V. Consequently, $\ker(f')$ is also a subrepresentation of V, which is irreducible, and is therefore equal to $\{0\}$ or the entire space V. By assumption there is a non-zero eigenvector in the kernel however, and so $\ker(f') = V$. As such, f' is the zero map and we have that $f = \lambda\,\mathrm{id}$, as claimed. □

Corollary 3.2.13.1. *Suppose V is a finite-dimensional irreducible representation of an abelian group G. Then V is one-dimensional.*

Proof : Let $h \in G$, and consider $f_h : V \to V$ given by $f_h(v) = hv$. This is an equivariant map because $gf_h(v) = ghv = hgv = f_h(gv)$ for all $g \in G$. By Schur's lemma 3.2.13, for each $h \in G$ there exists a $t_h \in \mathbb{C}$ such that $f_h = t_h I$. Thus, the span of any non-zero vector v is a non-zero invariant subspace of V.
□

Reductive Groups.
To motivate our discussion of reductive groups, we first discuss their historical importance in the context of invariant theory.
Suppose V is a $K-$vector space. The (algebraic) *dual space* of V, denoted by V^*, is the vector space of all linear maps from V to K, i.e.

$$V^* = \{T \in K^V \mid T \text{ is linear }\}.$$

We define $K[V]$, the $K-$algebra of *polynomial functions on* V to be the space generated by V^*. If V is finite-dimensional, say $\dim(V) = n$, then $dim(V^*) = n$ also and so in this case $K[V] \cong K[x_1, \ldots, x_n]$. If V is an affine variety then we will also sometimes denote by $K[V]$ its ring of regular functions, along with $A(V)$ from earlier. Every vector space is an affine variety and these notions coincide in that case.
When G acts on an affine variety V. Then there is an induced action G on $K[V]$ given by the formula,

$$g \cdot f(x) := f(g^{-1}x)$$

for $g \in G$, $f \in K[V]$ and $x \in V$. One easily verifies this as an action since

$$e \cdot f(x) = f(e^{-1}x) = f(ex) = f(x)$$

and if $g_1, g_2 \in G$ we have that

$$\begin{aligned} g_1 (g_2 f(x)) &= g_2 \left(f(g_1^{-1}x)\right) \\ &= f(g_2^{-1}(g_1^{-1}x)) \\ &= f((g_1^{-1}g_2^{-1})x) \\ &= f((g_1g_2)^{-1}x) \\ &= (g_1g_2)f(x). \end{aligned}$$

A natural question to ask is whether $K[V]^G$ is finitely generated as a K-algebra for an arbitrary group G. This was the 14^{th} problem posed by David Hilbert in the bulletin of the American Mathematical Society in 1902 [13]. The conjecture was eventually resolved in the negative by Japanese mathematician Nagata, who constructed an explicit counterexample [21]. It turns out, however, that $K[V]^G$ is finitely generated for a particular class of groups, namely the linearly reductive ones. In order to formulate these, we first introduce *linear algebraic groups.*

Definition 3.2.14. Suppose G is a subgroup of the invertible group of matrices $GL_n(\mathbb{C})$. Then G is a *linear algebraic group* if it is an affine subvariety of $GL_n(\mathbb{C})$.

Remark 3.2.15. Recall from earlier that $GL_n(\mathbb{C})$ can be viewed as the vanishing set of the polynomial $X \cdot det(M) - 1$ where M is an $n \times n$ matrix, and so it is an affine variety. Furthermore, whenever $\rho : G \times X \to X$ is an action for some affine variety X and linear algebraic group G, we will require that the action itself is *algebraic,* that is, there is an embedding of G and X as affine varieties in affine spaces where ρ is a regular map in the affine coordinates.

Definition 3.2.16. Suppose G is a linear algebraic group. We say that G is *linearly reductive* if whenever V is a representation of G, and $W \subseteq V$ is a subrepresentation, then there exists another subrepresentation $W' \subseteq V$ such that $V = W \oplus W'$.

Remark 3.2.17. Suppose G is linearly reductive. Then the collection

$$V^G = \{v \in V \mid gv = v, \text{ for all } g \in G\}$$

is a subrepresentation of V. We therefore have that $V = V^G \oplus V'$ for some other subrepresentation $V' \subseteq V$. Note that this splitting is not required to irreducible. Indeed, for every $v \in V^G$, the space $span\{v\}$ is another subrepresentation of V.

The literature on linearly reductive groups is immense, and there are many other equivalent definitions of linearly reductive groups. For our purposes, definition 3.2.16 will be the most useful. We will, however, also make use of the following proposition.

Proposition 3.2.18. *Suppose G is a linearly reductive algebraic group. Then, whenever there are representations V and W of G and an epimorphism $\varphi : V \to W$, the induced morphism on invariants $\varphi^G : V^G \to W^G$ is also surjective.*

Proof : Suppose V and W are both representations of G and $\varphi : V \to W$ is an epimorphism. Since G is linearly reductive, there are subrepresentations, V' and W' such that $V = V^G \oplus V'$ and $W = W^G \oplus W'$. We claim that $\varphi(V^G) = W^G$. Suppose $v \in V^G$. Then $g \cdot \varphi(V) = \varphi(vg) = \varphi(v)$, and so $\varphi(V^G) \subseteq W^G$. For the reverse, first split V' into a finite number of irreducible subrepresentations, which we can do inductively by our assumption that G is linearly reductive. This means that for some $n \in \mathbb{N}$ we have that

$$V = V^G \oplus V_1' \oplus \cdots \oplus V_n'.$$

Let $\pi : W \to W^G$ be the projection from W onto the invariants W^G. Next, pick a basis $e_1, \ldots, e_k$ for W^G so that $W^G = \langle e_1, \ldots, e_k \rangle$, and let p_i be the projection onto the i^{th} coordinate, that is,

$$p_i : W^G \to \mathbb{C}e_i.$$

Now, let f_i be the composition of the restriction of φ to V_i' with π and p_i, that is

$$f_i := p_i \circ \pi \circ \varphi \mid_{V_i'}: V_i' \to \mathbb{C}e_i.$$

Since the constituent maps are all equivariant, it follows that f_i is also a morphism of representations. Next note that V_i' is not a trivial representation, meaning $g{\cdot}V_i' \nsubseteq V_i'$, otherwise it would be included in V^G. On the other hand however, $\mathbb{C}e_i$ is a trivial representation of W, since $g \cdot e_i = e_i$ for all $1 \leqslant i \leqslant k$. It follows that the domain and codomain of f_i are not isomorphic. Therefore Schur's lemma 3.2.13 ensures it must be the zero map. As such $f_i(V_i') = 0$, and so $\varphi(V_i') \subseteq W'$ for all $1 \leqslant i \leqslant k$. It follows that $\varphi(V^G) = W^G$, from which the claim holds.

□

A standard reference on the theory of linearly reductive groups can be found at [2, § 1]. Furthermore, when the ambient field is algebraically closed and of zero characterstic, (such is the case over $\mathbb{C}$), the notion of linear reductivity coincides with the classical notion of reductivity, i.e. the radical, which is the maximal connected normal solvable subgroup, is a torus. Since we are primarily working over $\mathbb{C}$, we will henceforth use the two terms interchangeably.

For the purposes of this thesis, the most important linearly reductive group under consideration is $SL_n(\mathbb{C})$, the special linear group of $n \times n$ matrices with complex coefficients. That $SL_n(\mathbb{C})$ is reductive relies upon some interesting theory which we use the rest of this section to detail. We first begin with a study of $SU_n(\mathbb{C})$, the special linear group of unitary matrices.

Definition 3.2.19. The collection of of special unitary matrices $SU_n(\mathbb{C})$ is the following set,

$$SU_n(\mathbb{C}) = \{A \in U_n(\mathbb{C}) \mid \det(A) = 1\},$$

where $U_n(\mathbb{C})$ is the collection of $n \times n$ matrices in $\mathbb{C}$ that are unitary.

Remark 3.2.20. It is routine to check that $SU_n(\mathbb{C})$ is in fact a topological group, that is, a group such that the group operations of multiplication and inversion are continuous maps in the topology. The topology on $SU_n(\mathbb{C})$ is the one induced from the standard Euclidean topology.

Proposition 3.2.21. *The topological group $SU_n(\mathbb{C})$ is compact.*

Proof : First note that $SU_n(\mathbb{C}) = \det^{-1}(\{1\})$, where the determinant function $\det : U_n(\mathbb{C}) \to \mathbb{C}$, given by

$$\det(X) = \det([x_{ij}]) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} x_{\sigma(i),i},$$

is a polynomial in n^2 variables, that is, an element of $\mathbb{C}[x_{11}, \dots, x_{nn}]$. As such, it is continuous, and since the preimage of a closed set by a continuous mapping is closed, it follows that $SU_n(\mathbb{C})$ is closed in $U_n(\mathbb{C})$. Furthermore, if $A \in SU_n(\mathbb{C})$ then $\|A\| = \sup\{\|Ax\| \mid \|x\| = 1\}$. However, we have that

$$\begin{aligned} \|Ax\|^2 &= \langle Ax \mid Ax \rangle \\ &= \langle x \mid x \rangle \\ &= \|x\|^2 = 1. \end{aligned}$$

It follows that $SU_n(\mathbb{C})$ is also bounded, and since it is also closed, it is compact. □

The following theorem, due to Haar, is well known in harmonic analysis, see [7].

Theorem 3.2.22. *Suppose G is a locally compact Hausdorff topological group. Then there is, up to a positive multiplicative constant, a unique, countably additive, nontrivial measure μ on the Borel subsets of G satisfying the following properties:*

(i) μ is left-translation invariant: $\mu(gS) = \mu(S)$ for every $g \in G$ and all Borel sets $S \subseteq G$.

(ii) μ is finite on every compact set: $\mu(K) < \infty$ for all compact $K \subseteq G$.

(iii) μ is outer regular on Borel sets: $\mu(S) = \inf\{\mu(U) \mid S \subseteq U, U \text{ is open }\}$.

(iv) μ is inner regular on open sets: $\mu(U) = \sup\{\mu(K) \mid K \subseteq U, K \text{ is compact }\}$.

Remark 3.2.23. Such a measure is called a *left Haar measure*. In complete analogy, there exists also a *right Haar measure*. The two measures need not coincide, however.

Proposition 3.2.24 (Weyl's Unitarian Trick)**.** *Suppose V is a representation of $SU_n(\mathbb{C})$. Then there is an inner product $\langle \cdot \mid \cdot \rangle$ on V such that $\langle x \mid y \rangle = \langle gx \mid gy \rangle$ for all $g \in G$ and $x, y \in V$.*

Proof : By proposition 3.2.21 $SU_n(\mathbb{C})$ is compact, and so it is also locally compact. Hence, there exists a (left) Haar measure μ on $SU_n(\mathbb{C})$, by theorem 3.2.22. Suppose V is a representation of $SU_n(\mathbb{C})$ and let $\langle \cdot \mid \cdot \rangle$ be an arbitrary inner product on V. Define a new inner product $\langle \cdot \mid \cdot \rangle_G : V \times V \to \mathbb{C}$ as follows,

$$\langle x \mid y \rangle_G := \frac{1}{\mu(SU_n(\mathbb{C}))} \int_{g \in SU_n(\mathbb{C})} \langle gx \mid gy \rangle \, d\mu(g).$$

Note that if $g' \in SU_n(\mathbb{C})$ we have that

$$\begin{aligned} \langle g'x \mid g'y \rangle_G &= \frac{1}{\mu(SU_n(\mathbb{C}))} \int_{g \in SU_n(\mathbb{C})} \langle gg'x \mid gg'y \rangle \, d\mu(g) \\ &= \frac{1}{\mu(SU_n(\mathbb{C}))} \int_{g \in SU_n(\mathbb{C})} \langle gx \mid gy \rangle \, d\mu(g) \\ &= \langle x \mid y \rangle_G \, . \end{aligned}$$

It follows that $\langle \cdot \mid \cdot \rangle_G$ is $SU_n(\mathbb{C})$ invariant, that is, for each $g \in SU_n(\mathbb{C})$, we have that $\langle gx \mid gy \rangle_G = \langle x \mid y \rangle_G$. □

Proposition 3.2.25. *$SU_n(\mathbb{C})$ is linearly reductive.*

Proof : Let V be a representation of $SU_n(\mathbb{C})$, let $\langle \cdot \mid \cdot \rangle_G$ be a $G-$invariant inner product on V from proposition 3.2.24 and suppose that $W \subseteq V$ is a subrepresentation. The orthogonal complement of W with respect to $\langle \cdot \mid \cdot \rangle_G$, denoted by $W^\perp$ is the space $W^\perp = \{v \in V \mid \langle v \mid w \rangle_G = 0 \text{ for all } w \in W\}$. We claim that this is a subrepresentation of V. Indeed, let $g \in SU_n(\mathbb{C})$ and let $v \in W^\perp$. For arbitrary $w \in W$ we then have that

$$\begin{aligned}\langle gv \mid w \rangle_G &= \left\langle g^{-1}gv \mid g^{-1}w \right\rangle_G \\ &= \left\langle v \mid g^{-1}w \right\rangle_G \\ &= 0, \text{ since } W \text{ is a subrepresentation of } V.\end{aligned}$$

It follows that for all $v \in W^\perp$, we have $SU_n(\mathbb{C}) \cdot v \subseteq W^\perp$. Consequently, V splits into $W \oplus W^\perp$, and so $SU_n(\mathbb{C})$ is linearly reductive. □

Corollary 3.2.25.1. *Let G be a locally compact Hausdorff topological group, such that the Haar measure of G is finite. Then G is reductive.*

Proof : Simply follow all the steps above but with G instead of $SU_n(\mathbb{C})$. □

Remark 3.2.26. The assumption that G has finite Haar measure is essential in the last corollary. Indeed, $\mathbb{R}$ is a locally compact topological group without finite Haar measure and it is not linearly reductive. To see this, consider the representation $\rho : \mathbb{R} \to GL_2(\mathbb{R})$ by

$$\rho(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then it is easy to check that $W = span \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is a subrepresentation of $\mathbb{R}^2$, but there is no subspace $W' \subseteq \mathbb{R}^2$ such that

$$\mathbb{R}^2 = W \oplus W'.$$

Theorem 3.2.27. *Any finite group is reductive.*

Proof : Let G be a finite group, and suppose V is a representation of G. We can repeat Weyl's Unitarian trick from proposition 3.2.24 only instead of an integral, we can use a sum, as the group is finite. As such, there is an inner product on V such that $\langle x \mid y \rangle = \langle gx \mid gy \rangle$ for all $g \in G$ and $x, y \in V$. The same argument as used in 3.2.21 is then used to show that G is reductive. □

We temporarily assume the following lemma from finite-dimensional linear algebra, and prove it at the end of this section, so as not to obfuscate the main results.

Lemma 3.2.28 (Singular Value Decomposition)**.** *Suppose $A \in M_n(\mathbb{C})$. Then there exist unitary matrices $U_1, U_2 \in U_n(\mathbb{C})$ and a diagonal matrix D such that*

$$A = U_1 D U_2.$$

In particular,

(i) $GL_n(\mathbb{C}) = U_n(\mathbb{C}) \cdot D_n(\mathbb{C}) \cdot U_n(\mathbb{C})$.

(ii) $SL_n(\mathbb{C}) = SU_n(\mathbb{C}) \cdot D_n(\mathbb{C}) \cdot SU_n(\mathbb{C})$.

Lemma 3.2.29. *The Zariski closure of* $U_n(\mathbb{C})$ *that is, the* $n \times n$ *unitary matrices with complex coefficients, is* $GL_n(\mathbb{C})$.

Proof : First note that $GL_n(\mathbb{C})$ is Zariski-closed because it is given by the vanishing set of the polynomial equation $f = det(A)X - 1$. Thus,

$$\overline{U_n(\mathbb{C})} = \bigcap \{A \in GL_n(\mathbb{C}) \mid U_n(\mathbb{C}) \subseteq A \text{ and } A \text{ is Zariski closed }\}.$$

Next, note that $\overline{S^1} = \mathbb{C}^*$, which follows from the fact that the Zariski-closed sets of $\mathbb{A}^1$ are the finite sets, the empty set, and all of $\mathbb{A}^1$. Also, $\overline{(S^1)^n} \supseteq \overline{S^1} \times \text{id} \times \ldots \times \text{id} = \mathbb{C} \times \text{id} \times \ldots \times \text{id}$. Since this will be true for each copy of $\overline{(S^1)^n}$ we have that $(C^*)^n = D_n(\mathbb{C}) \subseteq \overline{(S^1)^n}$. Next, since $(S^1)^n \subseteq U_n(\mathbb{C})$ we will have that $D_n(\mathbb{C}) \subseteq \overline{U_n(\mathbb{C})}$. In total, we have therefore that

$$U_n(\mathbb{C}) \cdot D_n(\mathbb{C}) \cdot U_n(\mathbb{C}) \subseteq \overline{U_n(\mathbb{C})}.$$

By singular value decomposition 3.2.28, the left-hand side of the above equation is precisely $GL_n(\mathbb{C})$, from which the result follows.

□

Corollary 3.2.29.1. *The Zariski closure of* $SU_n(\mathbb{C})$, *the special unitary matrices with complex coefficients, is* $SL_n(\mathbb{C})$.

Proof : Simply replace $GL_n(\mathbb{C})$ with $SL_n(\mathbb{C})$ in the previous theorem, which only adds the condition that the determinant is 1. □

Theorem 3.2.30. $SL_n(\mathbb{C})$ *is linearly reductive.*

Proof : Denote $SL_n(\mathbb{C})$ by G and let V be a representation of $SL_n(\mathbb{C})$, and suppose $W \subseteq V$ is a subrepresentation of V so that $Gw \subseteq W$ for all $w \in W$ and $g \in G$. Since $SU_n(\mathbb{C})$ is a subgroup of $SL_n(\mathbb{C})$, V is also a representation of $SU_n(\mathbb{C})$, and moreover, W will also be a subrepresentation of $SU_n(\mathbb{C})$. Since $SU_n(\mathbb{C})$ is linearly reductive, there exists a subspace $W' \subseteq V$ such that W' is a subrepresentation of $SU_n(\mathbb{C})$ and $V = W \oplus W'$. We now claim that W' is also a subrepresentation of $SL_n(\mathbb{C})$.

Let $w' \in W'$ and consider the map $f : SL_n(\mathbb{C}) \mapsto SL_n(\mathbb{C}) \cdot w'$. Let $B := f^{-1}\left(\overline{SU_n(\mathbb{C}) \cdot w'}\right)$. Now, f is continuous and therefore B is closed. It follows that $\overline{SU_n(\mathbb{C})} \subseteq B$, which, by corollary 3.2.29.1 means $SL_n(\mathbb{C}) \subseteq f^{-1}((SU_n(\mathbb{C}) \cdot w')$, which in turn means that $SL_n(\mathbb{C}) \cdot w' \subseteq \overline{SU_n(\mathbb{C}) \cdot w'} \subseteq \overline{W'} = W'$. It follows that W' is also an $SL_n(\mathbb{C})$ subrepresentation, and so $SL_n(\mathbb{C})$ is linearly reductive. □

Corollary 3.2.30.1. $\mathbb{S}^1$ *is linearly reductive.*

Proof : $\mathbb{S}^1 = SL_1(\mathbb{C})$ which is linearly reductive by theorem 3.2.30. □

Hilbert's finite generation theorem

We finish this section with the following theorem, which was originally proven by Hilbert in the special case of $G = SL_n(\mathbb{C})$.

Theorem 3.2.31 (Hilbert)**.** *Suppose G is a reductive linear algebraic group and it acts on an affine variety X. Then the $\mathbb{C}-$algebra of invariant polynomials $\mathbb{C}[V]^G$ is finitely generated.*

Proof : First we reduce to the case when $X = V$, a representation of G. Let $f_1, \ldots, f_\ell$ be generators of the algebra of regular functions $\mathbb{C}[V]$. Let $V^* \subseteq \mathbb{C}[V]$ be a finite-dimensional G-stable vector space containing these generators. This defines a $G-$equivariant surjection $\mathbb{C}[V] \to \mathbb{C}[X]$ that restricts to a surjection $\mathbb{C}[V]^G \to \mathbb{C}[X]^G$ due to the linear reductiveness of G and proposition 3.2.18. As such, we only need to show that $\mathbb{C}[V]^G$ is finitely generated. To this end, let $J = \mathbb{C}[V]\mathbb{C}[V]^G_+$ where $\mathbb{C}[V]^G_+ = \bigoplus_{n>0} \mathbb{C}[V]^G_n$ and $\mathbb{C}[V]^G_n = Sym^n(V^*)$ is the space of all degree n invariant homogeneous polynomials on V. J is an ideal in $\mathbb{C}[V]$ which is Noetherian, and therefore by Hilbert's basis theorem, there exists a finite basis of homogeneous invariant polynomials for J as a $\mathbb{C}$-algebra, let these be $f_1, \ldots, f_k$, so that $\langle f_1, \ldots, f_k \rangle = J$. Suppose $j \in J$, so there are $a_i \in \mathbb{C}[V]$ such that $j = \sum_{i=1}^k a_i f_i$. Suppose j is an invariant polynomial. We wish to show that j can be written as an algebraic expression involving only invariant polynomials.
Now, since G is reductive and acts linearly on $\mathbb{C}[V]$, $\mathbb{C}[V]$ splits into $\mathbb{C}[V]^G \oplus V'$, and there exists a projection $\pi : \mathbb{C}[V] \to \mathbb{C}[V]^G$, which is linear and G-equivariant, meaning $\pi(gv) = g\pi(v) = \pi(v)$, which is a consequence of Schur's lemma 3.2.13. We therefore have that

$$\begin{aligned}\pi(j) &= \pi\left(\sum_{i=1}^k a_i f_i\right)\\ &= \sum_{i=1}^k \pi(a_i f_i)\\ &= \sum_{i=1}^k \pi(a_i) f_i.\end{aligned}$$

Let $b_i = \pi(a_i)$ for each $1 \leqslant i \leqslant k$. Since j is invariant, $\pi(j) = j$ and
$j = b_1 f_1 + \ldots b_k f_k$. It follows that $\mathbb{C}[V]^G$ is finitely generated. □

Example 3.2.32. From theorem 3.2.30, $SL_n(\mathbb{C})$ is linearly reductive. Indeed a minor modification of that theorem also gives that $GL_n(\mathbb{C})$ is linearly reductive, see also [18, §12]. Thus by theorem 3.2.31, whenever V is a representation of either of these groups, the algebra of invariant polynomials is finitely generated.

Proof of Singular Value Decomposition

Here we present a proof of singular value decomposition.

Theorem 3.2.33. *Let V be a finite-dimensional complex inner product space, and let $f : V \to V$ be a linear operator. Then there exist orthonormal bases $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2, \ldots, v_n\}$ of V such that, with respect to these bases, the matrix representation of f is diagonal with non-negative real entries. In other words, there exist unitary operators U, V such that*

$$f = U\Sigma V^*,$$

where Σ is a diagonal matrix with non-negative real entries (the singular values of f), and U and V are unitary matrices.

Proof : Let $f : V \to V$ be a linear operator on a finite-dimensional complex inner product space V. Consider the adjoint operator $f^* : V \to V$ defined by the property

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle, \quad \text{for all } x, y \in V.$$

Define the Hermitian operator $f^*f : V \to V$. Since f^*f is Hermitian, it is also positive semi-definite, meaning that $\langle f^*f(x), x\rangle \geqslant 0$ for all $x \in V$. By the Spectral Theorem for Hermitian operators, there exists an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of V consisting of eigenvectors of f^*f with corresponding real non-negative eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.
Define the singular values $\sigma_i \geqslant 0$ by $\sigma_i = \sqrt{\lambda_i}$ for each $i = 1, 2, \dots, n$. Construct a diagonal matrix $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$, arranging the singular values in non-increasing order.
Next, form the unitary matrix $V \in \mathbb{C}^{n\times n}$ whose columns are the orthonormal eigenvectors $v_1, v_2, \dots, v_n$ of f^*f. For each i such that $\sigma_i > 0$, define the vectors $u_i = \frac{1}{\sigma_i} f(v_i)$. When $\sigma_i = 0$, extend $\{u_1, u_2, \dots, u_r\}$ (where r is the rank of f) to an orthonormal basis $\{u_1, u_2, \dots, u_n\}$ of V.
To see that the vectors u_i for $\sigma_i > 0$ are orthonormal, observe that

$$\langle u_i, u_j\rangle = \left\langle \frac{f(v_i)}{\sigma_i}, \frac{f(v_j)}{\sigma_j}\right\rangle = \frac{1}{\sigma_i\sigma_j}\langle f(v_i), f(v_j)\rangle.$$

Using the definition of the adjoint and the properties of eigenvectors, we have

$$\langle f(v_i), f(v_j)\rangle = \langle v_i, f^*f(v_j)\rangle = \langle v_i, \lambda_j v_j\rangle = \lambda_j\delta_{ij} = \sigma_j^2\delta_{ij}.$$

Thus we have that,

$$\langle u_i, u_j\rangle = \frac{\sigma_j^2}{\sigma_i\sigma_j}\delta_{ij} = \delta_{ij},$$

showing that the vectors u_i are orthonormal. Form the unitary matrix $U = [u_1, u_2, \dots, u_n]$.
Now, observe that for all $i = 1, 2, \dots, n$,

$$f(v_i) = \sigma_i u_i,$$

and hence

$$fV = U\Sigma.$$

Thus, we can express f as

$$f = U\Sigma V^*,$$

where U and V are unitary matrices and Σ is a diagonal matrix with non-negative real entries.

□

Corollary 3.2.33.1. *Let $f : V \to V$ be a linear operator on a finite-dimensional complex inner product space V such that its matrix representation is in the special linear group $SL_n(\mathbb{C})$, i.e.,* $\det(f) = 1$*. Then, in the singular value decomposition*

$$f = U\Sigma V^*,$$

the unitary matrices U and V can be chosen to be in the special unitary group $SU_n(\mathbb{C})$, meaning $\det(U) = 1$ *and* $\det(V) = 1$.

Proof : Since $f : V \to V$ is in $SL_n(\mathbb{C})$, we have

$$\det(f) = 1.$$

Consider the singular value decomposition of f:

$$f = U\Sigma V^*,$$

where $U, V \in U_n(\mathbb{C})$ are unitary matrices and Σ is a diagonal matrix with non-negative real entries (the singular values of f). Taking the determinant, we find

$$\det(f) = \det(U)\det(\Sigma)\det(V^*) = \det(U)\det(\Sigma)\overline{\det(V)}.$$

Given that $\det(f) = 1$ and $\det(V^*) = \overline{\det(V)} = \det(V)^{-1}$, we obtain

$$\det(U)\det(\Sigma)\det(V)^{-1} = 1.$$

Since Σ is a diagonal matrix with non-negative entries,

$$\det(\Sigma) = \sigma_1\sigma_2\cdots\sigma_n \geqslant 0.$$

Since both U and V are unitary matrices, their determinants lie on the unit circle, so $\det(U), \det(V) \in \mathbb{C}$ with $|\det(U)| = 1$ and $|\det(V)| = 1$. We therefore have that

$$\det(\Sigma) = 1,$$

implying $\sigma_1\sigma_2\cdots\sigma_n = 1$. Therefore, we have

$$\det(U)\det(V)^{-1} = 1,$$

which implies that

$$\det(U) = \det(V).$$

Let $\det(U) = \det(V) = \epsilon$ where $|\epsilon| = 1$. Let $U' = \epsilon^{-\frac{1}{n}}U$ so that $\det(U') = \det\left(\epsilon^{-\frac{1}{n}}U\right) = \epsilon^{-1}\det(U) = \epsilon^{-1}\epsilon = 1$. Similarly, let $(V')^* = \epsilon^{\frac{1}{n}}V^*$ so that $\det((V')^*) = \det\left(\epsilon^{\frac{1}{n}}V^*\right) = \epsilon\det(V^*) = \epsilon\cdot\epsilon^{-1} = 1$.
We see that $U', V' \in SU_n(\mathbb{C})$ and we have that

$$f = U\Sigma V^*.$$

□

Chapter 4

GIT and Computing the Equiaffine Invariant Measure

4.1 Geometric Invariant Theory

In this section we introduce the geometric invariant theory quotient of an affine variety X by a linearly reductive group G. In order to motivate the development of a quotient, we first consider a standard set-theoretical quotient.

Example 4.1.1. Suppose our affine variety is $X = \mathbb{C}^2$ and consider the action of $\mathbb{C}^*$ on X by $\lambda \cdot (x, y) := (\lambda x, \lambda^{-1} y)$. We classify the orbits of the resulting set-theoretic quotient as follows,

(i) If $xy \neq 0$ then the orbit of (x, y) is a hyperbola governed by $xy = c$ for some $c \in \mathbb{C}^*$.

(ii) The punctured $x-$axis is an orbit.

(iii) The punctured $y-$axis is an orbit.

(iv) The origin is an orbit.

We see that this object lacks desirable geometric and topological properties - such as being an affine variety. Indeed we can see that the resulting orbit space is not Hausdorff as there are sequences of points in the punctured axes that converge to the origin. Consequently, we introduce the notion of a geometric invariant theory quotient, or GIT quotient for short.

Suppose that G is a linearly reductive group acting on an affine variety X. By theorem 3.2.31, $\mathbb{C}[X]^G$ is finitely generated, and so there are polynomials $f_1, \ldots, f_n$ such that $\mathbb{C}[X]^G = \langle f_1, \ldots, f_n \rangle$. Now, any finitely generated algebra over $\mathbb{C}$ can be written as $\mathbb{C}[x_1, \ldots, x_n]/I$ for some ideal I. Indeed, consider the map $\theta : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[X]^G, x_i \mapsto f_i$, which extends to all $p \in \mathbb{C}[x_1, \ldots, x_n]$ by $p \mapsto p(f_1, \ldots, f_n)$. Then $Ker(\theta) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid \theta(f) = 0\}$ gives $\mathbb{C}[X]^G = \mathbb{C}[x_1, \ldots, x_n]/Ker(\theta)$. With this in mind, we define the *geometric invariant theory quotient*, or *GIT quotient* as follows.

Definition 4.1.2. Suppose G is a linearly reductive group, acting on an affine variety X. Let M be the affine variety whose regular functions are the invariant polynomials on X, that is,

$$\mathbb{C}[M] := \mathbb{C}[X]^G.$$

We call M the *GIT* quotient of X by G, and denote it also by $X//G$.

Remark 4.1.3. The GIT quotient M comes with a map $\pi : X \to M$ that can be described as follows. Let $f_1, \ldots, f_m$ generate M. Then X and M are affine varieties so embed them in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively. Then

$$\pi(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).$$

Theorem 4.1.4. *Let $X//G$ be the GIT quotient of a reductive group G on an affine variety X, and let $\pi : X \to X//M$ be the canonical projection detailed in remark 4.1.3. We then have that*

(i) π is surjective.

(ii) For all $x, y \in X : \pi(x) = \pi(y) \iff \overline{Gx} \cap \overline{Gy} \neq \varnothing$.

Proof : (i) Let $f_1, \ldots, f_m$ generate the ring of invariants $\mathbb{C}[X]^G$. The Zariski closure of the image of the quotient, that is $\overline{\pi(X)}$ is the set of all $(a_1, \ldots, a_n) \in \mathbb{C}^n$ satisfying:

$F(a_1, \ldots, a_n) = 0$ for all polynomial relations $F(f_1, \ldots, f_m) = 0$ among the generators $f_1, \ldots, f_m$ of $\mathbb{C}[X]^G$.

For convenience, denote this set as $Y := \overline{\pi(X)}$. Now, suppose $a = (a_1, \ldots, a_n) \in Y$. Consider the morphism of $\mathbb{C}[X]-$modules,

$$g_a : \bigoplus_{i=1}^{n} \mathbb{C}[X] \to \mathbb{C}[X],$$

$$(b_1, \ldots, b_n) \mapsto \sum_{i=1}^{n} b_i(f_i - a_i).$$

Since each $f_i - a_i \in \mathbb{C}[X]^G$ is $G-$invariant, we see that g_a is a morphism of representations. Also, we see that the induced map g_a^G on invariants is not surjective; its image is the maximal ideal $I_a \subseteq \mathbb{C}[X]^G$ corresponding to the point $a \in Y$. Since G is linearly reductive, this implies that g_a itself cannot be surjective, and therefore its image is contained in some maximal ideal $I \subseteq \mathbb{C}[X]$. It follows that the intersection $I \cap \mathbb{C}[X]^G$ is a maximal ideal in $\mathbb{C}[X]^G$, and therefore, it coincides with the maximal ideal I_a. This shows that $a \in Y$ is the image of the point of X corresponding to I. Lastly, we borrow the fact that $\overline{\pi(X)} = X//G$ from [19] §5.

(ii) $(\Leftarrow)$ First note that π is constant on orbits. Thus if $y \in Gx$, $\pi(x) = \pi(y)$. Moreover, because it is continuous (it is a map of regular functions), it has the same value on the closure of the orbit, and so $\pi(x) = \pi(y)$.

$(\Rightarrow)$ Aiming for contradiction, suppose $\overline{Gx} \cap \overline{Gy} = \varnothing$. Let $I_1 = I(\overline{Gx})$ and $I_2 = I(\overline{Gy})$, Then by hypothesis, $V(I_1 + I_2) = \varnothing$, and so by the Nullstellensatz 3.1.8,

$I_1 + I_2 = \mathbb{C}[X]$. Next, since $\overline{Gx}$ and $\overline{Gy}$ are preserved by the action of G, the ideals I_1, I_2 are subrepresentations of G. This means that the homomorphism of $\mathbb{C}[X]-$modules,

$$I_1 \oplus I_2 \to \mathbb{C}[X], (a, b) \mapsto a + b$$

is also a homorphism of $G-$representations. Since $I_1 + I_2 = \mathbb{C}[X]$, this map is surjective, and so by linear reductivity of the group G, the map

$$(I_1 \cap \mathbb{C}[X]^G) \oplus (I_2 \cap \mathbb{C}[X]^G) \to \mathbb{C}[X]^G$$

is also surjective. In particular, there are invariant functions $f \in I_1 \cap \mathbb{C}[X]^G$ and $f' \in I_2 \cap \mathbb{C}[X]^G$ satisfying $f + f' = 1$. We see that f vanishes on $\overline{Gx}$ and takes the value 1 on $\overline{Gy}$. Consequently, f is zero on $\pi(x)$ but non-zero on $\pi(y)$, yielding the desired contradiction. □

Example 4.1.5. As seen in example 4.1.1, the set-theoretic quotient of $\mathbb{C}^2$ with the action of $G := \mathbb{C}^*$ given by $\lambda \cdot (x, y) := (\lambda x, \lambda^{-1} y)$ is not a Hausdorff space. The GIT quotient is better behaved however, as one sees that $\mathbb{C}(\mathbb{C}^2) \cong \mathbb{C}[X, Y]$ and $\mathbb{C}[\mathbb{C}^2]^G \cong \mathbb{C}[XY] \cong \mathbb{C}[Z]$, and so $\mathbb{C}^2//G \cong \mathbb{C}$ - the affine line. We see that the orbits $(2), (3)$, and (4) cannot be separated by invariant polynomials; the GIT quotient then collapses them to a single point and the resulting object has the structure of an affine variety.

Definition 4.1.6. Suppose a group V is a representation of a group G. We say that a point $x \in V$ is *unstable* for the action if $0 \in \overline{Gx}$, and *semistable* otherwise. If Gx is closed, and the stabilizer of x is finite, we say that x is stable.

Corollary 4.1.6.1. *Let $V//G$ be the GIT quotient of a reductive group G on a vector space V. Then a point $x \in V$ is unstable if and only if $\pi(x) = \pi(0)$.*

Proof : From part (ii) of theorem 4.1.4 we have that $\pi(x) = \pi(0)$ if and only if $\overline{Gx} \cap \overline{G \cdot 0} \neq \emptyset$. However, $\overline{G \cdot 0} = \{0\}$. Thus, $\pi(x) = \pi(0)$ if and only if $0 \in \overline{Gx}$, i.e. if x is unstable. □

The following lemma, which comes from linear programming, will be used in proof of the Hilbert-Mumford criterion. For a proof, see [9].

Lemma 4.1.7 (Gordon's Lemma)**.** *Let a_{ij} be integers for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant n$ that satisfy the following property: Whenever $b_1, \ldots, b_r$ are real numbers that are not all zero that satisfy*

$$b_1 a_{1j} + \ldots b_r a_{rj} = 0 \, for \; all \, j = 1, \ldots, n,$$

then at least two of the b_i have opposite signs.
Then there are real numbers c_i such that

$$a_{i1} c_1 + \cdots + m_{in} c_n > 0 \, for \; all \;\; i = 1, 2, \ldots, r.$$

We provide a proof here of the Hilbert-Mumford criterion when $G = SL_n(\mathbb{C})$ and X is a vector space. For the general case of a reductive group acting an affine variety, see [29, §3.5]. We avoid the general case as it requires speaking of Cartan decompositions and Lie groups. Furthermore, the case of $G = SL_n(\mathbb{C})$ is the one of explicit interest to us due to the paper [10]. To state the theorem we first give the definition of a *one parameter subgroup.*

Definition 4.1.8. Suppose G is a linear algebraic group, and suppose $\lambda : \mathbb{C}^\times \to G$ is a homomorphism. Then $\lambda(\mathbb{C}^*)$ is a *one parameter subgroup* of G.

Theorem 4.1.9 (Hilbert-Mumford Criterion for $G = SL_n(\mathbb{C})$)**.** *Suppose $G = SL_n(\mathbb{C})$ acts on a finite-dimensional vector space V, and x is a point in V. Then the following conditions are equivalent.*

(i) x is unstable.

(ii) There exists a one parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot x = 0$.

Proof : First, suppose there exists such a homomorphism $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot x = 0$. Let $x_n = \lambda(\frac{1}{n})$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n \cdot x = 0$, and therefore by definition, $0 \in \overline{G \cdot x}$.

For the reverse, suppose $x \in V$ is unstable, and so $0 \in \overline{G \cdot x} = \overline{SL_n \cdot x}$. By singular value decomposition 3.2.28, $SL_n = SU_n \cdot D_N \cdot SU_n$, where SU_n are the special unitary matrices and D_n are the diagonal matrices, both square of dimension n with coefficients in $\mathbb{C}$. Thus, $0 \in \overline{SU_n \cdot D_N \cdot SU_n x}$. Next, we may suppose the inner product on V commutes with the action of SU_n, this is essentially Weyl's Unitary trick. In particular, for each $v, w \in V$ and $A \in SU_n$, $\langle Av, Aw\rangle = \langle v, w\rangle$. In turn this inner product induces a metric with the property that for all $y \in V$ and $A \in SU_n$, $d(Ay, 0) = d(y, 0)$. Next, suppose that $0 \notin \overline{D_n SU_n x}$. Then there exists $\varepsilon > 0$ such that for all $y \in \overline{D_n SU_n x}, d(y, 0) \geqslant \varepsilon$. Since $d(y, 0) = d(Ay, 0)$ for all $A \in SU_n$, this implies the existence of a ball $B \subseteq V$ containing 0 such that $B \cap SU_n\overline{D_n SU_n x} = \varnothing$. In particular, this means that $0 \notin \overline{SU_n D_n SU_n x}$. By contraposition, it follows that if $0 \in \overline{SU_n D_n SU_n x}$ then $x \in \overline{D_n SU_n x}$.
In turn, this means that there exists a sequence $y_k = a_k b_k x$ for each $k \in \mathbb{N}$ where $a_k \in D_n$ and $b_k \in SU_n$, and $\lim_{k\to\infty} y_k = 0$. Since SU_n is compact, it follows that $(b_k)_{k\in\mathbb{N}}$ has a convergent subsequence, $(b_{k_j})_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} b_{k_j} = b \in SU_n$. We therefore have that $0 \in \overline{D_n bx}$.

Suppose we find a one parameter subgroup $\lambda_1 : \mathbb{C} \to SL_n$ that works for the vector bx, that is $\lim_{t\to 0} \lambda_1(t) \cdot bx = 0$. Let $\lambda_2(t) = b^{-1}\lambda_1(t)b$. Clearly $\lambda : \mathbb{C}^* \to SL_n$ is a homomorphism, and moreover, $\lim_{t\to 0} \lambda_2(t) \cdot x = \lim_{t\to 0} b^{-1}\lambda_1(t)b \cdot x = b^{-1} \lim_{t\to 0} \lambda_1(t) \cdot bx = 0$. it follows that once a one-subgroup has been found for bx, the job has also been done for x. We therefore tactitly rename bx as x with this understood.
Next, let $\mu_j : D_n \to \mathbb{C}$, $diag(t_1, \ldots, t_n) \mapsto t_j$. Let $\chi_i = \sum_{j=1}^{n} m_{ij}\mu_j$ where the m_{ij} are integers. For such a χ_i let V_{χ_i} be defined by

$$V_{\chi_i} := \{w \in V \mid D \cdot w = \chi_i(D)w, \forall\, D \in D_n\}.$$

Then write our unstable point $x \in V$ as $x = \sum_{i=1}^{r} v_{\chi_i}$. We will then have that for any $D = diag(t_1, \ldots, t_n) \in D_n$,

$$D \cdot v = \sum_{i=1}^{r} t_1^{m_{i1}} \cdots t_n^{m_{in}} v_{\chi_i}.$$

Suppose that $b_1, \ldots, b_n$ are real numbers that are not all zero, such that

$$b_1 m_{1j} + \cdots + b_r m_{rj} = 0$$

for $1 \leqslant j \leqslant n$. Then at least two of the $b_i's$ have opposite signs.
Suppose this were not the case, and that there are b_i, not all zero, and all of the same sign, such that

$$b_1 m_{1j} + \cdots b_r m_{rj} = 0$$

for all $j = 1, \ldots, n$. Let $t^k = diag(t_1^{(k)}, \ldots, t_n^{(k)}) \in D_n$ be a sequence of diagonal matrices for each $k \in \mathbb{N}$ such that $\lim_{k\to\infty} t^{(k)} = 0$. Thus, for all i less than or equal to r we have that

$$\lim_{k\to\infty} \prod_{j=1}^{n} (t_i^{(k)})^{m_{ij}} = \lim_{k\to\infty} (t_1^{(k)})^{m_{i1}} \cdots (t_n^{(k)})^{m_{in}} = 0. \tag{4.1.1}$$

Without loss of generality, suppose that $b_1 \neq 0$. We then have that

$$-m_{1j} = \frac{b_2}{b_1} m_{2j} + \cdots + \frac{b_r}{b_1} m_{rj}$$

for all $j = 1, \ldots, n$. This in turn means that

$$t_1^{-m_{11}} \cdots t_n^{-m_{1n}} = (t_1^{m_{21}} \cdots t_n^{m_{2n}})^{\frac{b_2}{b_1}} \cdots (t_1^{m_{r1}} \cdots t_n^{m_{rn}})^{\frac{b_r}{b_1}}.$$

Since $\frac{b_i}{b_1} \geqslant 0$ for all $i \geqslant 2$ and they are not all zero, the right hand side of the previous equation has limit 0 as $(t_1, \ldots, t_n)$ runs over the sequence $(t_1^{(k)}, \ldots, t_n^{(k)})$. Looking at the left hand side however, this does not tend to 0 and therefore contadicts 4.1.1.
□

Example 4.1.10. Let V be a finite-dimensional $\mathbb{C}$ vector space. Then $Sym^n(V)$ is the subspace of $\bigotimes_{i=1}^n V$ that is invariant under S_n. A general element of this space can be described as $\sum_{v_1,\ldots,v_n} a_{v_1,\ldots,v_n} v_1 \otimes \ldots \otimes v_n$. Now suppose we have a linear action of $SL_2(\mathbb{C})$ on V. Then this action extends to $Sym^n(V)$ in a natural way, by $A \cdot (v_1 \ldots v_n) := Av_1 \cdot Av_2 \cdot \ldots Av_n$. Specialising this to the case $V = \mathbb{C}^2$, then $Sym^n(\mathbb{C}^2)$ is the space of homogeneous polynomials in two variables of degree n. Our action can generally be described as follows, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$A \cdot (x^{n-i} y^i) = (ax + cy)^{n-i}(bx + dy) \in Sym^n(\mathbb{C}^2).$$

We can use this to characterise the unstable points for the action of $SL_2(\mathbb{C})$ on $Sym^n(\mathbb{C}^2)$.

Theorem 4.1.11. *Suppose $p \in Sym^n(\mathbb{C}^2)$. Then p is unstable for the action of $SL_2(\mathbb{C})$ if and only if p is zero or has a linear factor of multiplicity greater than $\frac{n}{2}$.*

Proof : Let $p = a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_0 y^n = \sum_{i=0}^n a_{n-i} x^{n-i} y^i$. p is unstable if and only if there is homomorphism $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot p = 0$. We can make a change of coordinates in $\mathbb{C}^2$ such that

$$\lambda(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix}$$

for some $r > 0$. We therefore have that

$$\begin{aligned}\lambda(t) \cdot p &= \sum_{i=0}^{n} a_{n-i} \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix} \cdot x^{n-i} y^i \\ &= \sum_{i=0}^{n} a_{n-i} (t^r x)^{n-i} (t^{-r} y)^i \\ &= \sum_{i=0}^{n} a_{n-i} x^{n-i} y^i t^{r(n-2i)}.\end{aligned}$$

This goes to 0 in the limit as t goes to 0 in two possible cases. First, all the coefficients a_i are 0, in which case p is the zero polynomial. Otherwise $\lim_{t\to 0} (t^r)^{n-2i} = 0$ if and only if $a_{n-i} = 0$ whenever $n - 2i \leqslant 0$, or $\frac{n}{2} \leqslant i$. In other words, p has a linear factor of multiplicity greater than $\frac{n}{2}$. □

Remark 4.1.12. A standard result from the representation theory of linearly reductive groups says that any representation by a (finite-dimensional) vector space V of the group $SL_2(\mathbb{C})$ can be written as the following direct sum of irreducible subrepresenations,

$$V = \bigoplus_{i=1}^{k} Sym^{n_i}(\mathbb{C}^2),$$

where k and n_i depend on V. A consequence of this result gives the following corollary, which enables us to characterise all the unstable points for the action of $SL_2(\mathbb{C})$ on a vector space.

Corollary 4.1.12.1. *Suppose V is a representation of $SL_2(\mathbb{C})$. A point $p = (p_1, \ldots, p_k) \in V$ is unstable for this action, if each p_i has a linear factor of multiplicity greater than $\frac{n_i}{2}$. This linear factor must be the same for all the see polynomials $p_1, \ldots, p_k$.*

4.2 A Digression on Projective Varieties

In this part of the chapter, we define and study projective varieties, and how one can take GIT quotients of projective varieties once one already has affine quotients. Projective GIT quotients are discussed slightly with respect to their uses in other areas of algebraic geometry. Their construction demonstrates the importance of unstable points from the previous section.

Definition 4.2.1. Let $p_1, \ldots, p_k \in \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous polynomials and let $I = \langle p_1, \ldots, p_k \rangle$. Then we say that $\mathscr{C} = V(I)$ is a *cone*. Notice that if $x \in \mathscr{C}$ then $\lambda x \in \mathscr{C}$ for all $\lambda \in \mathbb{C}^\times$. This can also be viewed as the action $\delta(\lambda, x) := \lambda x$ of $\mathbb{C}^\times$on $\mathscr{C}$. We call the topological quotient

$$(\mathscr{C}\backslash\{0\})/\mathbb{C}^\times$$

the *projective variety* $\mathbb{P}(\mathscr{C})$ associated to the cone $\mathscr{C}$. We say that $\mathscr{C}$ is a cone over $\mathbb{P}(\mathscr{C})$. Notice that if $\mathscr{C}$ is embedded in $\mathbb{C}^n$ then the action extends to the latter and one has the topological quotient

$$\left(\mathbb{C}^n\backslash\{0\}\right)/\mathbb{C}^\times,$$

that can be identified with the projective space $\mathbb{P}\left(\mathbb{C}^n\right)$; then $\mathbb{P}(\mathscr{C})$ is a projective variety in $\mathbb{P}\left(\mathbb{C}^d\right)$ in the usual sense.

We may also consider a slight generalisation of the above construction of projective spaces. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and let $\delta^\alpha : \mathbb{C}^* \times \mathbb{C}^n \times \mathbb{C}^n$ be given by

$$\delta^\alpha(\lambda, x) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n).$$

The topological quotient of $\mathbb{C}^n\backslash\{0\}$ by δ^α is then called the *α-weighted projective space*, $\mathbb{P}_\alpha(\mathbb{C}^n)$. For $\alpha = (1, \ldots, 1)$ we see that this is just the normal projective space. The $\alpha-$weighted projective space is also compact; the map $(X_1, \ldots, X_n) \mapsto (X_1^{\alpha_1}, \ldots, X_n^{\alpha_n})$ induces a continuous surjective mapping $\mathbb{P}(\mathbb{C}^n) \to \mathbb{P}(\mathbb{C})^n$. One may construct $\alpha-$weighted cones, affine varieties that are invariant under δ^α, and also $\alpha-$weighted projective varieties obtained by quotienting by δ^α. The $\alpha-$weighted cones are the vanishing sets of $\alpha-$homogeneous polynomials, that is, polynomials p such that $p(\delta^\alpha(\lambda, x)) = \lambda^k p(x)$ for some $k \in \mathbb{N}$.

Example 4.2.2. Let V be a vector space over $\mathbb{C}$ and let $k \leqslant \dim(V)$. The *cone over the Grassmannian* with parameters k, V is defined as

$$\mathscr{C}Gr(k, V) = \{v_1 \wedge \cdots \wedge v_k \mid v_i \in V, \text{ for all } i = 1, \ldots, k\}.$$

This is an affine variety in the k-fold wedge product $\bigwedge^k V$, which can be shown using Plücker coordinates, [11, §5].

Let $X \subseteq \mathbb{P}^n$ a projective variety and denote by $\hat{X}$ its cone, which is an affine variety. Since G acts on X, there is a natural extension of this to a linear action of G on $\hat{X}$. We don't go into the details of the lifting of this action here, but it can be found in [20].
In order to take the GIT quotient of X by G, we first take the GIT quotient of $\hat{X}$ by G, which we know how to do since $\hat{X}$ is affine. It may be tempting then to simply

take $X//G$ as $\pi(\hat{X})//\mathbb{C}^*$, however, there is a big problem with this definition, as this is in fact the trivial ring. This is because the point $\pi(0)$ is in the $\mathbb{C}^*$ orbit of every point in $\pi(\hat{X})(= \hat{X}//G$ since π is surjective, 4.1.4). To remedy this, we must throw out $\pi(0)$, which motivates the following definition.

Definition 4.2.3. Let $X \subseteq \mathbb{P}^n$ be a projective variety acted on linearly by a linearly reductive group G, and denote by $\hat{X}$ its cone. Then the projective GIT quotient of X by G is the projectivization of the affine GIT quotient of $\hat{X}$ minus the point $\pi(0)$, that is,

$$X//G := \frac{\pi(\hat{X})\backslash\{\pi(0)\}}{\mathbb{C}^*}$$

Remark 4.2.4. Recall that a point is unstable if and only if $\pi(x) = \pi(0)$. Thus, we could first start by throwing out the unstable points of the cone, and *then* taking the affine quotient. This is the same as taking the quotient and then throwing out $\pi(0)$.

Although we are mainly interested in affine GIT quotients, projective GIT quotients are interesting mathematically as they play a large role in the construction of other algebraic objects such as *moduli spaces*. Roughly speaking, a moduli space is a space that parameterizes equivalence classes of certain geometric objects, such as curves, vector bundles, or varieties, up to some notion of equivalence (like isomorphism). The key idea in constructing a moduli space is to capture all possible variations of these objects in a single geometric space. Often, moduli problems can be framed as a problem of classifying objects up to a group action. For instance, classifying vector bundles on a fixed variety or classifying curves of a fixed genus can be approached by identifying equivalent objects under a suitable group action.

One typical method whereby a moduli space is constructed is as a GIT quotient of a space that parameterises all possible objects (like all curves of a fixed degree and genus) by a group that captures the symmetries or equivalences (like isomorphisms of curves). The projective GIT quotient provides a rigorous and geometric way to obtain this moduli space. In particular, the notion of stability in projective GIT quotients is crucial for moduli spaces, as it allows for the exclusion of certain "bad" or "degenerate" objects (unstable points) from the moduli space, ensuring that the quotient represents a well-behaved moduli space.

4.3 Modified Comparability Lemma and Key Results

In this section of the chapter, we prove a modification of the Comparability Lemma from [10] taking place over $\mathbb{C}$. Moreover, we show that the original as it appears in [10] can be derived from the modification in the special case of the $SL_2(\mathbb{R})$ action on $Sym^n(\mathbb{R}^2)$. This avoids implicating the version of the Hilbert-Mumford criterion for real numbers first proven by Birkes in [1].

We then compute enough invariant polynomials such that the intersection of their vanishing sets satisfies the criteria of this modified comparability lemma for the $SL_2(\mathbb{C})$ actions on $Sym^4(\mathbb{C}^2)$ and $Sym^5(\mathbb{C}^2)$. The cases of $Sym^2(\mathbb{C}^2)$ and $Sym^3(\mathbb{C}^2)$ are also mentioned but here it is known that the discriminant polynomial generates the entire algebra of invariants.

In order to do compute these polynomials, we require a study of the resultant, which is the determinant of the Sylvester matrix, all of which we discuss here. A more in-depth analysis of the resultant can be found at [16, Chapter 4, §8].

To begin, we require the following proposition.

Proposition 4.3.1. *Let V be a representation of a linearly reductive group G. Then there exists homogeneous, invariant polynomials $p_1, \ldots, p_k$ such that*

$$\bigcap_{i=1}^{k} V(p_k) = \{ \textit{ unstable points for the action } \}.$$

Proof : By corollary 4.1.6.1 the unstable points in V are the ones sent to the same point that 0 is sent to by the map $\pi : V \to V//G$. That means $x \in V$ is unstable if and only if $x \in \pi^{-1}(\pi(0))$. With this in mind, let

$$J := I(\pi^{-1}(\pi(0))) = \{f \in \mathbb{C}[V]^G \mid f(\pi^{-1}(\pi(0))) = 0.\}.$$

Since $\mathbb{C}[V]^G$ is finitely generated and J is a subalgebra, J must be finitely generated also, and so let $p_1, \ldots, p_k$ be invariant polynomials such that

$$\langle p_1, \ldots p_k \rangle = J.$$

Without loss of generality, we may also assume these p_i are homogeneous, by splitting them into their constituent homogeneous parts if necessary. Now, we claim that $\bigcap_{i=1}^{k} V(p_i)$ is the set of unstable points. Indeed, if x is in the zero set of all the p_i, it is in the zero set of all polynomials they generate, which is precisely J. Consequently,

$$x \in V(I(J)) = \pi^{-1}(\pi(0))$$

where the equality holds because of part (ii) of Hilbert's Nullstellensatz 3.1.8. It follows that $\pi(x) = \pi(0)$ and so x is unstable. Conversely, an essentially similar argument shows that if x is unstable, it vanishes under p_i for $1 \leqslant i \leqslant k$.

□

We now state and prove the Modified Comparability Lemma.

Lemma 4.3.2. *[Modified Comparability Lemma] Suppose that ρ is a linear action of a finite dimensional vector space V (over $\mathbb{C}$) on a linearly reductive group G. Furthermore, suppose V is equipped with a norm $\|\cdot\|$. Let $p_1, \ldots, p_n$ be a collection of a homogeneous invariant polynomials such that $\bigcap_{i=1}^{n} V(p_i)$ equals the set of unstable points for the action. Then there exists constants $0 < C_1 \leqslant C_2 < \infty$ such that for all $v \in V$ we have*

$$C_1 \max_{j=1,\ldots,N} |p_j(v)|^{\frac{1}{d_j}} \leqslant \inf_{M\in G} \|\rho_M v\| \leqslant C_2 \max_{j=1,\ldots,N} |p_j(v)|^{\frac{1}{d_j}},$$

where d_j denoteds the degree of p_j.

Proof : We first show the left inequality. Let $v \in V$ be arbitrary, and then write $v = \|v\|\hat{v}$, where $\hat{v}$ has norm 1. We then have

$$|p_j(v)| = |p_j(\|v\|\hat{v})| = \|v_j\|^d |p_j(\hat{v})|$$

where the last equality holds because p_j is homogeneous with degree d_j. Now, if we allow $\hat{v}$ to vary across the entire unit ball we will have the following inequality, valid for all $v \in V$,

$$|p_j(v)| \leqslant \|v_j\|_j^d \sup_{\hat{v}\in B_1(0)} |p_j(\hat{v})|.$$

If we let $\|p_j\|_\infty = \sup\{|p_j(v)| \mid v \in B_1(0)\}$ we therefore have $|p_j(v)| \leqslant \|v_j\|^{d_j}\|p_j\|_\infty$ which in turn implies that for all $v \in V$

$$\|p_j\|^{-\frac{1}{d_j}} |p_j(v)|^{\frac{1}{d_j}} \leqslant \|v\|.$$

Moreover, because each p_j is invariant under the action of G, we have that

$$\|p_j\|_\infty^{-\frac{1}{d_j}} |p_j(v)| \frac{1}{d_j} = \|p_j\|_\infty |p_j(\rho_M v)|^{\frac{1}{d_j}} \leqslant \|\rho_M v\|.$$

If we therefore take an infimum in $M \in G$ and a supremum in $j \in \{1, \ldots, N\}$ we have that

$$\left[\min_{j=1,\ldots,N} \|p_j\|_\infty^{-\frac{1}{d_j}}\right] \left[\max_{j=1,\ldots,N} |p_j(v)|^{\frac{1}{d_j}}\right] \leqslant \inf_{M\in G} \|\rho_M v\|.$$

Letting $C_1 = \left[\min_{j=1,\ldots,N} \|p_j\|_\infty^{-\frac{1}{d_j}}\right]$ completes the proof of the first inequality.

For the second, suppose otherwise for contradiction, so that the inequality does not hold for all finite constants C_2.

This implies the existence of a sequence $(v_k)_{k\in\mathbb{N}} \in V^{\mathbb{N}}$ such that $\inf_{M\in G} \|\rho_M v_k\| = 1$ for all $k \in \mathbb{N}$ with

$$\max_{j\in\{1,\ldots,N\}} |p_j(v_k)|^{\frac{1}{d_j}} \leqslant \frac{1}{k} \inf_{M\in G} \|\rho_M v_k\| = \frac{1}{k}.$$

We can find elements $M_k \in G$ such that

$$1 \leqslant \|\rho_{M_k}(v_k)\| \leqslant 1 + \frac{1}{k},$$

and by compactness, there exists a subsequence $(u_k)_k := (\rho_{M_k}(v_k))_k$, that converges to a limit u lies inside the unit ball of V (here we are using the topology on V induced by $\|\cdot\|$). By continuity of the polynomials p_j we must also have that $p_j(u) = 0$ for all $j \in \{1, \ldots, n\}$. This means that u is unstable for the action of G, and thus by the Hilbert-Mumford criterion, there exists a one-parameter subgroup $\lambda : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot u = 0$. As a consequence, $\inf_{M\in G} \|\rho_M(u)\| = 0$ as well; but on the other hand, we have by assumption that $\|\rho_M(u_k)\| \geqslant 1$ for all $M \in G$ and $k \in \{1, \ldots, n\}$, and by continuity, $\|\rho_M(u)\| \geqslant 1$, giving the desired contradiction. It follows that there exists some constant C_2 satisfying the criteria.

□

Remark 4.3.3. By invoking the version of the Hilbert-Mumford criterion that takes place over $\mathbb{R}$, proven by Birkes in [1], 4.3.2 is also true with everything taking place over the real numbers instead of the complex numbers. In the next proposition, we show that for the specific action of $SL_2(\mathbb{R})$ on $Sym^n(\mathbb{R}^2)$ however, Gressman's original comparability lemma follows as a corollary of lemma 4.3.2.

Proposition 4.3.4. *Consider the action of $SL_2(\mathbb{R})$ on a finite dimensional real vector space V. Then lemma 4.3.2 holds with everything taking place over $\mathbb{R}$ instead of $\mathbb{C}$.*

Proof : Firstly, the classification of representations of $SL_2(\mathbb{R})$ is essentially the same as representations of $SL_2(\mathbb{C})$. That is, given a representation of $SL_2(\mathbb{R})$ by a real vector space we can write

$$V = \bigoplus_{i=1}^{j} Sym^{n_i}(\mathbb{R}^2),$$

for some $j \in \mathbb{N}$. Pick some $p = (p_1, \ldots, p_j) \in V$. If all of these polynomials have a common root with multiplicity k_i greater than $\frac{n_i}{2}$, where $n_i = deg(p_i)$, then, this root must be real. If this root were not real, since complex roots come in pairs, we would have that both $(x - \overline{\alpha})^{k_i}$ and $(x - \alpha)^{k_i}$ divide p_i for each i. Thus, their product, a degree $2k_i$ polynomial would also divide p_i. Since $k_i > \frac{n_i}{2}$, we have that $n_i > n_i$, a contradiction. Consequently, α must be real.
It follows that if p is unstable we can we can construct a $1-$parameter subgroup $\lambda : \mathbb{R}^* \to SL_2(\mathbb{R})$ such that $\lim_{t\to 0} \lambda(t) \cdot p_i = 0$ for $1 \leqslant i \leqslant k$. Now the same argument over $\mathbb{C}$ can be extended to $\mathbb{R}$.

□

With the modified comparability lemma proven, we are now in the position to compute invariant polynomials and prove that the intersection of these invariants is the unstable points. This is easier than proving they generate the algebra of invariants, which is indeed a very difficult task.
It turns out that one polynomial with variables in $Sym^n(\mathbb{C}^2) \oplus Sym^m(\mathbb{C}^2)$ that will be very important in this analysis is the resultant polynomial. In order to give a full introduction to that we first digress a bit on the procedure of *homogenisation* which will be quite important in this context.

Consider a degree n homogeneous polynomial f which we write as

$$f = \sum_{i=0}^{n} a_i x^{n-i} y^i,$$

If we then let the polynomial f' be defined by

$$f' := \frac{1}{y^n} \cdot f = \sum_{i=0}^{n} a_i \frac{x^{n-i}y^i}{y^{n-i}y^i} = \sum_{i=0}^{n} a_i \left(\frac{x}{y}\right)^{n-i}.$$

we see that $f' \in \mathbb{C}\left[\frac{x}{y}\right]$ and $deg(f) = deg(f') = n$. This process is called *de-homogenization.* Conversely, if $f = \sum_{i=0}^{n} a_i x^i$, then let $f' = y^n \sum_{i=0}^{n} a_i \left(\frac{x}{y}\right)^i = \sum_{i=0}^{n} a_i x^i y^{n-i} \in Sym^n(\mathbb{C}^2)$. This process is called *homogenization.* With this discussed, we now introduce the resultant.

Definition 4.3.5. Suppose p and q are both univariate polynomials over $\mathbb{C}$, such that $p = a_0 + a_1x + \cdots + a_mx^m$ and $q = b_0 + b_1x + \ldots b_nx^n$, and $m, n > 0$. The *Sylvester matrix* of p and q, denoted by $Syl(p,q)$, is the $(m+n) \times (m+n)$ matrix formed as follows.

- The first row is $\begin{pmatrix} a_m & a_{m-1} & \cdots & a_1 & 0 & \cdots & 0 \end{pmatrix}$.
- The second row is the first row but shifted one column to the right; the first element of the row is zero.
- The following $n-2$ rows are obtained the same way by shifting the coefficients one column to the right each time and setting the other entries of the row to zero.
- The $(n+1)^{th}$ row is given by $\begin{pmatrix} b_n & b_{n-1} & \cdots & b_1 & 0 & \cdots & 0 \end{pmatrix}$.
- The final $m-1$ rows are obtained as before.

Example 4.3.6. Suppose $m = 3$ and $n = 4$ from definition 4.3.5. The resulting Sylvester matrix is then

$$Syl(p,q) = \begin{pmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & a_3 & a_2 & a_1 & a_0 \\ b_4 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_4 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_4 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

Definition 4.3.7. Suppose p and q are both univariate polynomials in K. Then their *resultant*, denoted $Res(p,q)$ is the determinant of their Sylvester matrix, that is,

$$Res(p,q) = \det\left(Syl(p,q)\right).$$

Theorem 4.3.8. *Suppose $p, q \in \mathbb{C}[x]$. Then p and q share a common root if and only if $Res(p,q) = 0$.*

Proof : Suppose first that p and q share a common root. Then $p(\alpha) = q(\alpha) = 0$ for some $\alpha \in K$. Let $r \in \mathbb{C}[x]$ be the polynomial defined by $r(x) := \gcd(p(x), q(x))$, which divides both $p(x)$ and $q(x)$. Since α is a root of both polynomials, α must also

be a root of $r(x)$, and so $r(x)$ has positive degree. Consequently, there exist non-zero polynomials $a(x)$ and $b(x)$ such that $p(x) = a(x)r(x)$ and $q(x) = b(x)r(x)$. In particular, this means that $p(x)$ and $q(x)$ are linearly dependent modulo $r(x)$, leading to a non-trivial linear relation among the rows of the sylvester matrix $Syl(p, q)$. Hence $Syl(p, q)$ is singular and $Res(p, q) = 0$.
For the reverse, suppose that $Res(p, q) = 0$. Then $Syl(p, q)$ has linearly dependent rows. Thus, there exists a non-trivial linear combination of the rows of $Syl(p, q)$ that equals zero. This in turn corresponds to a non-trivial relation among the coefficients of $p(x)$ and $q(x)$, which means that there is a common factor of positive degree. Thus $p(x)$ and $q(x)$ have a non-constant greatest common divisor $r(x)$ with $deg(r(x)) > 0$. Hence, $p(x)$ and $q(x)$ share at least one common root.

□

Proposition 4.3.9. *The resultant, when considered as a polynomial in the coefficients p and q, is an invariant polynomial for the action of $SL_2(\mathbb{C})$ on $Sym^n(\mathbb{C}^2) \oplus Sym^m(\mathbb{C}^2)$. That is,*

$$Res(p, q) \in \mathbb{C}[Sym^n(\mathbb{C}^2) \oplus Sym^m(\mathbb{C}^2)]^{SL_2(\mathbb{C})}.$$

Proof : Recall that if $A \in SL_2(\mathbb{C})$ and $(p, q) \in Sym^n(\mathbb{C}^2) \oplus Sym^m(\mathbb{C}^2)$, then

$$A \cdot Res(p, q) = Res(A^{-1}p, A^{-1}q).$$

We must therefore show that for all $A \in SL_2(\mathbb{C})$, $Res(p, q) = Res(A^{-1}p, A^{-1}q)$. Note that an arbitrary $A \in SL_2(\mathbb{C})$ may be viewed as a linear change of coordinates of $\mathbb{C}^2$. This means that if p and q share a common root, so does $A^{-1} \cdot p$ and $A^{-1} \cdot q$. Consequently,

$$V(Res) = V(A \cdot Res).$$

Now, consider the map

$$f : \mathbb{C}^2 \oplus Sym^{n-1}(\mathbb{C}^2) \oplus Sym^{m-1}(\mathbb{C}^2) \to Sym^n(\mathbb{C}^2) \oplus Sym^m(\mathbb{C}^2),$$

$$((ax + by), p_1, p_2) \mapsto ((ax + by)p_1, (ax + by)p_2).$$

The domain of f is irreducible. Indeed, its coordinate ring is the ring of polynomial functions in $n + m$ variables. Note also that $V(Res) \subseteq \mathbb{C}^{n+m} = Sym^n(\mathbb{C}^2) \oplus Sym^m(\mathbb{C}^2) = Im(f)$. Since $f : \mathbb{C}^2 \oplus Sym^{n-1}(\mathbb{C}^2) \oplus Sym^{m-1}(\mathbb{C}^2) \to Im(f)$ is surjective, it follows that $V(Res)$ is irreducible. This means that the resultant is some polynomial to some power. As A acts by a linear change of coordinates, a similar argument shows that $A \cdot Res$ is a constant times the same polynomial to the same power. As such, $Res(p, q) = C_A A \cdot Res(p, q)$ for some constant C_A depending on A. Now note, that the space spanned by by the resultant, that is, $\langle SL_2(\mathbb{C}) \cdot Res \rangle$, is a 1 dimensional representation of $SL_2(\mathbb{C})$. However, from the representation theory of groups, the only such representation is the trivial one, which means that $C_A = 1$. Consequently, $A \cdot Res = Res$, from which the result follows. □

Remark 4.3.10. An essentially similar argument shows that $Res\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)$ is an invariant polynomial for the action of $SL_2(\mathbb{C})$. However, $Res\left(\frac{\partial^2 p}{\partial x \partial y}, \frac{\partial^2 p}{\partial x^2}\right)$ is not. This is because the map $f : Sym^{n-1}(\mathbb{C}^2) \to Sym^{n-2}(\mathbb{C}^2)$ that sends $\frac{\partial p}{\partial x}$ to $\frac{\partial^2 p}{\partial x^2}$ is not $SL_2(\mathbb{C})$ equivariant.

Lemma 4.3.11. *Let $g : \mathbb{C}^n \to \mathbb{C}^n$ be given by $g(t_1, \ldots, t_n) = (s_0, \ldots, s_{n-1})$ where s_i is the coefficient of x^i in the polynomial $(x - t_1)(x - t_2) \cdots (x - t_n)$. Then the algebra of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ invariant under the action of S_n is the algebra of polynomials in the variables $s_0, \ldots, s_{n-1}$, that is, $\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[s_0, \ldots, s_{n-1}]$.*

Proof : This is a standard proof and can be found in many algebra textbooks, for instance consider [8] §1. Notice that the orbits of S^n are exactly the fibres of g. This together, with the fact that $\mathbb{C}^n$ is smooth implies the result, (any $f : X \to Y$ that is regular and Y smooth implies f is an isomorphism). $\square$

Remark 4.3.12. By expanding the polynomial $(x - t_1) \cdots (x - t_n)$ we see that the coefficient s_{n-k} is given by

$$s_{n-k} = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \prod_{j \in \{i_1, \ldots, i_k\}} (-t_j)$$

for $0 \leqslant k < n$.

Definition 4.3.13. Let $f : \mathbb{C}^{2n} \to Sym^n(\mathbb{C}^2)$ be given by

$$f(x_1, y_1, \ldots, x_n, y_n) = (b_0, \ldots, b_n)$$

where b_i are the coefficients of $\prod_{i=1}^n (xy_i - yx_i) = b_n x^n + b_{n-1} x^{n-1} y + \cdots + b_0 y^n$. Enumeration gives the following formula for the coefficients b_j. We see that for $1 \leqslant k \leqslant n$ we have

$$b_{n-k} = \sum_{1 \leqslant i_1 < \ldots i_k \leqslant n} (-x_{i_1}) \cdots (-x_{i_k}) \prod_{j \notin \{i_1, \ldots, i_k\}} y_j.$$

Note: If we let $\frac{x_i}{y_i} := t_i$, then $b_j = y_1 \cdots y_n \cdot s_j$.

Definition 4.3.14. We say that a polynomial in the variables $(x_1, y_1, \ldots, x_n, y_n)$ is multihomogeneous of degree $\underbrace{(d, d, \ldots, d)}_{n \text{ times}}$ if, for all $1 \leqslant i \leqslant n$, when viewed as a polynomial in the variables x_i and y_i, it is homogeneous of degree d.

Theorem 4.3.15. *Every polynomial in the variables $b_0, \ldots, b_n$ which is homogeneous of degree d can be written uniquely as a multihomogeneous polynomial of degree $\underbrace{(d, \ldots, d)}_{n}$ in the variables (x_i, y_i), which is S_n invariant, that is,*

$$\mathbb{C}[b_0, \ldots, b_n] =$$
$$\{p \in \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{S_n} \mid p \text{ is multihomogeneous of degree } \underbrace{(d, \ldots, d)}_{n}\}.$$

Proof : The b_i are multilinear of degree $(1, \cdots, 1)$. A monomial of degree d in the b_i's is therefore multilinear of degree $(d, \cdots, d)$. Moreover, all of the b_i's are trivially S_n invariant. The forward inclusion therefore holds. For the reverse, suppose $p \in \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ is S_n invariant and multihomgenous of degree $(d, \cdots, d)$. Let $p' = \frac{p}{y_1^d \cdots y_n^d}$. Then $p' \in \mathbb{C}[t_1, \ldots, t_n]$ where $t_i = \frac{x_i}{y_i}$. By lemma 4.3.11, we have that $p' \in \mathbb{C}[s_0, \ldots, s_{n-1}]$. Multiplying back by $y_1^d \cdots y_n^d$ gives

$$p \in \mathbb{C}[s_0, \ldots, s_{n-1}] \cdot y_1^d \cdots y_n^d.$$

Since $y_1 \cdots y_n \cdot s_j = b_j$ we have that $p \in \mathbb{C}[b_0, \ldots, b_n]$, from which the claim holds. □

We assume the following theorem which is quite classical and known as the first fundamental theorem of invariant theory. A full proof can be found in the textbook "Lectures on Invariant Theory" by Dolgachev, see [8] §2.2.

Theorem 4.3.16. *When viewed as an algebra, the set of invariant polynomials on* $(\mathbb{C}^2)^n (= (Sym^1(\mathbb{C}^2))^n)$ *for the action of* $SL_2(\mathbb{C})$ *is generated by the polynomials* $(x_i y_j - x_j y_i)$ *for* $1 \leqslant i < j \leqslant n$*. That is,*

$$\mathbb{C}[(Sym^1\mathbb{C}^2)^n]^{SL_2(\mathbb{C})} = \langle x_i y_j - x_j y_i \mid 1 \leqslant i \leqslant j \leqslant n \rangle .$$

Lastly, we will also require the following theorem.

Theorem 4.3.17. *Let* A *be the set of all polynomials* p *in the variable* $x_i y_j - x_j y_i$ *for* $1 \leqslant i, j \leqslant n$ *that are multihomogeneous of degree* $(d, \ldots, d)$*. Then* A *is the set of all invariant polynomials for the action of* $SL_2(\mathbb{C})$ *on* $Sym^n(\mathbb{C}^2)$*, that is,*

$$A = \mathbb{C}[Sym^n(\mathbb{C}^2)]^{SL_2(\mathbb{C})}.$$

Proof : The forward inclusion is a direct result of the previous theorem 4.3.16. For the reverse inclusion, suppose $p \in \mathbb{C}[Sym^n(\mathbb{C}^2)]^{SL_2(\mathbb{C})}$. First note that we can split up these invariants into their homogeneous parts,

$$\mathbb{C}[Sym^n(\mathbb{C}^2)]^{SL_2(\mathbb{C})} = \bigoplus_{i=0}^{\infty} \mathbb{C}[Sym^n(\mathbb{C}^2)]_i.$$

This fact, that taking invariants commutes with the sum, is a result of Schur's lemma. Then let $p = \sum_{i=0}^k p_i$ be written in its homogeneous decomposition. By theorem 4.3.15, each p_i a polynomial in the variables (x_i, y_i) for $1 \leqslant i \leqslant n$ and is multihomogeneous of degree $(i, \ldots, i)$, and is S_n invariant (in addition to being $SL_2(\mathbb{C})$ invariant). Consequently, by theorem 4.3.16, p_i is a polynomial in the variables $x_i y_j - x_j y_i$ for $1 \leqslant i \leqslant j \leqslant n$ and is multihomogeneous of degree $(i, \ldots, i)$. Putting all the p_i back together, we get that p is multihomogeneous of degree $(d, \ldots, d)$ in the variables $x_i y_j - x_j y_i$ for $1 \leqslant i \leqslant j \leqslant n$. □

Computing the Invariants.

$Sym^2(\mathbb{C}^2)$ and $Sym^3(\mathbb{C}^2)$

In both cases, the resultant of $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ generates the invariant algebra, that is,

$$\mathbb{C}[Sym^n(\mathbb{C}^2)]^{SL_2(\mathbb{C})} = \left\langle Res\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)\right\rangle = \langle \Delta(p) \rangle ,$$

when $n = 2, 3$, and where $\Delta(p)$ denote the discriminant of the polynomial p. This is a classical result, see §3.6 [28].

• $Sym^4(\mathbb{C}^2)$

We find invariant polynomials p_1 and p_2 such that $V(p_1) \cap V(p_2)$ equals the set of unstable points for the action of $SL_2(\mathbb{C})$ on $Sym^4(\mathbb{C}^2)$. For any $p \in Sym^4(\mathbb{C}^2)$, $Res\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial x}\right)$ is such a polynomial. If $p(x) = \prod_{i=1}^4 (xy_i - yx_i)$, then let P_1 be

$$P_1 := Res\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) = \left[\prod_{1 \leqslant i < j \leqslant 4} (x_i y_j - x_j y_i)\right]^2.$$

P_1 is indeed an example of a multihomogeneous degree $(2, 2, 2, 2)$ polynomial in $(x_i y_j - x_j y_i)$ which is S_4 invariant, which is zero if and only if p has a multiple root. Next, let P_2 be defined by

$$\begin{aligned} P_2 = &(x_1y_2 - x_2y_1)^2(x_3y_4 - x_4y_3)^2 \\ &+ (x_1y_3 - x_3y_1)^2(x_2y_4 - x_4y_2)^2 \\ &+ (x_1y_4 - x_4y_1)^2(x_2y_3 - x_3y_2)^2. \end{aligned}$$

We now check that $V(P_1) \cap V(P_2)$ is indeed the collection of unstable points. As $P_1 = 0$ either $(x_1y_2 - x_2y_1) = 0$ or $(x_1y_3 - x_3y_1) = 0$ or $(x_1y_4 - x_4y_1) = 0$. Because of the symmetry we can assume that $(x_1y_2 - x_2y_1) = 0$. We check that

$$V(x_1y_2 - x_2y_1) \cap V(P_2) \subseteq \{ \text{ unstable points } \}.$$

Since $x_1y_2 - x_2y_1 = 0$ this means $(x_2, y_2) = c(x_1, y_1)$ for some constant c which we can insert into P_2 and get

$$c^2(x_1y_3 - x_3y_1)^2(x_1y_4 - x_4y_1)^2 + c^2(x_1y_4 - x_4y_1)^2(x_1y_3 - x_3y_1)^2 = 0.$$

Consequently, either $x_1y_3 - x_3y_1 = 0$ or $x_1y_4 - x_4y_1 = 0$. This means that either $(x_1, y_1) = c'(x_3, y_3)$ or $(x_1, y_1) = c''(x_3, y_3)$, so $V(P_1) \cap V(P_2)$ consists of polynomials which have at least 3 common roots, which is exactly the set of unstable points by theorem 4.1.11.

• $Sym^5(\mathbb{C}^2)$

We now describe how to get enough invariant polynomials p_i with variables in $Sym^5(\mathbb{C}^2)$ such that $\bigcap_i v(p_i)$ is the set of unstable points in $Sym^5(\mathbb{C}^2)$. This example will suggest how one could proceed in the more general case of $Sym^n(\mathbb{C}^2)$ for $n > 5$.

First, we denote $e_{ij} := x_i y_j - x_j y_i$. As in the case of $Sym^4(\mathbb{C}^2)$, we start with $p_1 := Res\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) = \prod_{1 \leqslant i < j \leqslant 5} e_{ij}^2$.

We consider now

$$p_2 := \sum_{\sigma \in S_5} \left(\prod_{i \neq 1,2} e^4_{\sigma(1)\sigma(i)} \prod_{i \neq 1,2} e^4_{\sigma(2)\sigma(i)} \prod_{i,j \in \{3,4,5\}, i<j} e^2_{\sigma(i)\sigma(j)} \right).$$

All powers are taken to be even such that the summation doesn't cancel the terms. The power for the first 2 terms of the product is twice the power for the third to

make sure that the multi homogeneity condition is satisfied.
Lastly, for $k = 1, 2, 3$ we define the polynomials p_{3k} by

$$p_{3k} = \sum_{\sigma \in S_5} e^{2k}_{\sigma(1)\sigma(2)} e^{2k}_{\sigma(2)\sigma(3)} e^{2k}_{\sigma(3)\sigma(4)} e^{2k}_{\sigma(4)\sigma(5)} e^{2k}_{\sigma(5)\sigma(1)}.$$

The set of unstable points in $Sym^5(\mathbb{C}^2)$ for the action of $SL_2(\mathbb{C})$ is then given by the vanishing sets of all the defined polynomials, that is,

$$V(p_1) \cap V(p_2) \cap \left(\bigcap_{k=1}^{3} V(p_{3k}) \right).$$

Any point satisfying $Res\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)$ satisfies one of the $e_{ij} = 0$. Because of the symmetry, without loss of generality we may assume $e_{12} = 0$. When restricted at the set where $e_{12} = 0$, and using the fact that $e_{1k} = ce_{2k}$, we get that

$$p_2 = c \prod_{i \neq 1,2} e^4_{\sigma(2)\sigma(i)} \prod_{i,j \in \{3,4,5\}} e_{\sigma(i)\sigma(j)^2}.$$

Because of the S_5 symmetry without loss of generally we can assume that $e_{12} = e_{34} = 0$ and then modulo these relations we have that $p_{3k} = c\, e^{2k}_{13} e^{2k}_{54} e^{2k}_{15} \left(e^{4k}_{13} + e^{4k}_{54} + e^{4k}_{15}\right)$. If the last bracket is 0 is for all $1 \leqslant k \leqslant 3$ then $e_{13} = e_{54} = e_{15} = 0$ due to the Newton identities. As the products are all zero, consequently, at least one of e_{13}, e_{54}, e_{51} has to be 0.

4.4 Construction of the Equiaffine Invariant Measure for Specific Surfaces

4.4.1 A 2-Surface in $\mathbb{R}^6$

In this section we construct the affine invariant tensor and the associated equiaffine invariant measure of Gressman for a two surface $\mathcal{M}$ in $\mathbb{R}^6$. For completeness we make some formal definitions now.

Definition 4.4.1. For a point p on a smooth manifold $\mathcal{M}$, the *tangent space* $T_p\mathcal{M}$ is the vector space consisting of all tangent vectors to $\mathcal{M}$ at p. Formally, it is the space of all derivations at p, which are linear maps $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ satisfying the Leibniz rule:

$$v(fg) = f(p)\, v(g) + g(p)\, v(f),$$

for all smooth functions $f, g \in C^\infty(\mathcal{M})$.

Definition 4.4.2. An *immersion* of a 2-manifold M into $\mathbb{R}^6$ is a smooth map $f : M \to \mathbb{R}^6$ such that the differential $df_p : T_pM \to T_{f(p)}\mathbb{R}^6$ is injective at every point $p \in M$. This implies that locally, f maps M smoothly into $\mathbb{R}^6$ without collapsing dimensions, though it may have self-intersections globally.

Now, let $f : \mathcal{M} \to \mathbb{R}^6$ be an immersion of $\mathcal{M}$.

Determination of the Index Set $\Lambda_{2,6}$

To construct the affine curvature tensor, we first define the index set $\Lambda_{d,n}$, which helps identify the necessary derivatives involved. For a 2-dimensional surface ($d = 2$) in a 6-dimensional space ($n = 6$), the index set $\Lambda_{2,6}$ consists of pairs (j, k), where $j \in \{1, 2, \ldots, 6\}$ and $k \in \{1, \ldots, \kappa_j\}$. Here, κ_j is the smallest integer such that the space of polynomials of degree κ_j in 2 variables has dimension at least $j + 1$. Now, a polynomial of degree κ_j in the variables x and y consists of terms of the form $x^a y^b$ where a, b are non-negative integers such that $a + b \leqslant \kappa_j$. The number of distinct terms is therefore equivalent to the number of solutions to the equation $a + b \leqslant \kappa_j$, which is a well-known problem from combinatorics and the number of such terms is given by $\frac{(\kappa_j+2)(\kappa_j+1)}{2}$. This formula comes from the "stars and bars" theorem in combinatorics, which is a way to count the number of ways to distribute indistinguishable objects (the total degree of the polynomial) into distinguishable bins (the exponents a and b in $x^a y^b$).

j	Required Dimension $j+1$	Minimum κ_j
1	2	1
2	3	1
3	4	2
4	5	2
5	6	2
6	7	3

Table 4.1: Calculation of the minimum κ_j for the index set $\Lambda_{2,6}$.

From table 4.4.1 we see that the index set is as follows:

$$\Lambda_{2,6} = \{(1,1),(2,1),(3,1),(3,2),(4,1),(4,2),(5,1),(5,2),(6,1),(6,2),(6,3)\}.$$

Now, at any point $p \in M$, for any finite sequence of vector fields X_λ indexed by $\lambda \in \Lambda_{2,6}$ the affine curvature tensor A_p is defined as as follows,

$$\begin{aligned} A_p((X_\lambda)_{\lambda\in\Lambda_{2,6}}) = \det\Big(& X_{(1,1)}f(p) \wedge X_{(2,1)}f(p) \wedge \\ & \big(X_{(3,1)}X_{(3,2)}f(p)\big) \wedge \\ & \big(X_{(4,1)}X_{(4,2)}f(p)\big) \wedge \\ & \big(X_{(5,1)}X_{(5,2)}f(p)\big) \wedge \\ & \big(X_{(6,1)}X_{(6,2)}X_{(6,3)}f(p)\big)\Big). \end{aligned} \tag{4.4.1}$$

Now, we can view A_p as an $11-$multilinear map

$$A_p : \bigoplus_{i=1}^{11} \mathbb{R}^2 \to \mathbb{R}.$$

By the universal property of tensor products [B.0.1], this consequently corresponds uniquely to an element $\tilde{A}_p$ in the following dual space

$$\tilde{A}_p \in \left(\bigotimes_{i=1}^{11} \mathbb{R}^2\right)^*.$$

Let $\{x, y\}$ be a basis for space $(\mathbb{R}^2)^*$. Consider the following,

(i) The first two terms of 4.4.1 are elements of $(\mathbb{R}^2)^*$ and interchanging them produces a sign change.

(ii) The next three terms of 4.4.1 are elements of $Sym^2(\mathbb{R}^2)^*$, and a interchanging them again produces a sign change.

(iii) The final term is an element of $Sym^3(\mathbb{R}^2)^*$.

We therefore have that

$$\tilde{A}_p \in \left(\bigwedge^2 (\mathbb{R}^2)^*\right) \otimes \bigwedge^3 Sym^2((\mathbb{R}^2)^*) \otimes Sym^3((\mathbb{R}^2)^*).$$

Now, the space $\left(\bigwedge^2(\mathbb{R}^2)^*\right)$ is generated by the vector $x \wedge y$ and consequently is one dimensional. Similarly, the space $\bigwedge^3 Sym^2((\mathbb{R}^2)^*)$ is generated by the vector $x^2 \wedge xy \wedge y^2$ and is therefore also one dimensional. Finally, the space $Sym^3((\mathbb{R}^2)^*)$ has basis vectors x^3, x^2y, xy^2, and y^3 and is therefore 4 dimensional. If we denote the space $\left(\bigwedge^2(\mathbb{R}^2)^*\right) \otimes \bigwedge^3 Sym^2((\mathbb{R}^2)^*) \otimes Sym^3((\mathbb{R}^2)^*)$ by V, we then see that

$$\begin{aligned} V &= \left\langle x \wedge y \otimes x^2 \wedge xy \wedge y^2 \otimes \xi \mid \xi \in \{x^3, x^2y, xy^2, y^3\}\right\rangle \\ &\cong Sym^3(\mathbb{R}^2). \end{aligned}$$

It follows that V is 4-dimensional. Consequently, we can describe $\tilde{A}_p$, and therefore the equiaffine invariant tensor A_p, completely by its coordinates in terms of these 4 basis vectors. To do this, we form the following matrix made up of partial derivatives of the immersion f, as detailed in the construction by Gressman.

$$A(f) := \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_1}{\partial y^2} & \frac{\partial^3 f_1}{\partial x^3} & \frac{\partial^3 f_1}{\partial x^2 \partial y} & \frac{\partial^3 f_1}{\partial x \partial y^2} & \frac{\partial^3 f_1}{\partial y^3} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial^2 f_2}{\partial x^2} & \frac{\partial^2 f_2}{\partial x \partial y} & \frac{\partial^2 f_2}{\partial y^2} & \frac{\partial^3 f_2}{\partial x^3} & \frac{\partial^3 f_2}{\partial x^2 \partial y} & \frac{\partial^3 f_2}{\partial x \partial y^2} & \frac{\partial^3 f_2}{\partial y^3} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial^2 f_3}{\partial x^2} & \frac{\partial^2 f_3}{\partial x \partial y} & \frac{\partial^2 f_3}{\partial y^2} & \frac{\partial^3 f_3}{\partial x^3} & \frac{\partial^3 f_3}{\partial x^2 \partial y} & \frac{\partial^3 f_3}{\partial x \partial y^2} & \frac{\partial^3 f_3}{\partial y^3} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial^2 f_4}{\partial x^2} & \frac{\partial^2 f_4}{\partial x \partial y} & \frac{\partial^2 f_4}{\partial y^2} & \frac{\partial^3 f_4}{\partial x^3} & \frac{\partial^3 f_4}{\partial x^2 \partial y} & \frac{\partial^3 f_4}{\partial x \partial y^2} & \frac{\partial^3 f_4}{\partial y^3} \\ \frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial y} & \frac{\partial^2 f_5}{\partial x^2} & \frac{\partial^2 f_5}{\partial x \partial y} & \frac{\partial^2 f_5}{\partial y^2} & \frac{\partial^3 f_5}{\partial x^3} & \frac{\partial^3 f_5}{\partial x^2 \partial y} & \frac{\partial^3 f_5}{\partial x \partial y^2} & \frac{\partial^3 f_5}{\partial y^3} \\ \frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial y} & \frac{\partial^2 f_6}{\partial x^2} & \frac{\partial^2 f_6}{\partial x \partial y} & \frac{\partial^2 f_6}{\partial y^2} & \frac{\partial^3 f_6}{\partial x^3} & \frac{\partial^3 f_6}{\partial x^2 \partial y} & \frac{\partial^3 f_6}{\partial x \partial y^2} & \frac{\partial^3 f_6}{\partial y^3} \end{pmatrix}$$

$A(f)$ has 6 rows and 9 columns. We remove the last four columns of $A(f)$, resulting in a truncated 6×5 matrix $A'(f)$, given by

$$A'(f) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial^2 f_1}{\partial x^2} & \frac{\partial^2 f_1}{\partial x \partial y} & \frac{\partial^2 f_1}{\partial y^2} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial^2 f_2}{\partial x^2} & \frac{\partial^2 f_2}{\partial x \partial y} & \frac{\partial^2 f_2}{\partial y^2} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial^2 f_3}{\partial x^2} & \frac{\partial^2 f_3}{\partial x \partial y} & \frac{\partial^2 f_3}{\partial y^2} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial^2 f_4}{\partial x^2} & \frac{\partial^2 f_4}{\partial x \partial y} & \frac{\partial^2 f_4}{\partial y^2} \\ \frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial y} & \frac{\partial^2 f_5}{\partial x^2} & \frac{\partial^2 f_5}{\partial x \partial y} & \frac{\partial^2 f_5}{\partial y^2} \\ \frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial y} & \frac{\partial^2 f_6}{\partial x^2} & \frac{\partial^2 f_6}{\partial x \partial y} & \frac{\partial^2 f_6}{\partial y^2} \end{pmatrix}.$$

Next, for each of the removed columns, we adjoin them back to $A'(f)$ to form four different 6×6 matrices:

$$A_i(f) = [A'(f) \mid C_i], \quad \text{for } i = 1, 2, 3, 4,$$

where C_i represents each of the columns:

Column	Expression
C_1	$\left(\frac{\partial^3 f_k}{\partial x^3}\right)_{k=1}^{6}$
C_2	$\left(\frac{\partial^3 f_k}{\partial x^2 \partial y}\right)_{k=1}^{6}$
C_3	$\left(\frac{\partial^3 f_k}{\partial x \partial y^2}\right)_{k=1}^{6}$
C_4	$\left(\frac{\partial^3 f_k}{\partial y^3}\right)_{k=1}^{6}$

We then compute the determinant of each matrix $A_i(f)$. Denote by γ_i these determinants, that is,

$$\gamma_i := \det(A_i(f)) = \det([A'(f) \mid C_i]), \quad \text{for } i = 1,2,3,4.$$

Now, we will have that

$$\tilde{A}_p = x \wedge y \otimes x^2 \wedge xy \wedge y^2 \otimes (\gamma_1 x^3 + \gamma_2 x^2 y + \gamma_3 x y^2 + \gamma_4 y^3).$$

We have therefore fully described our equiaffine invariant tensor A_p.
Next, we want to pass to the density function, which will then give the equiaffine invariant measure. Again, following Gressman's construction, let

$$\delta(v_1, v_2) = \inf_{M \in SL_2(\mathbb{R})} \|M \cdot \tilde{A}_p\|^{\frac{2}{11}}.$$

Then the equiaffine measure $\nu_{\mathcal{A}}$ can be described by the relation

$$\nu_{\mathcal{A}}(Y) = \int_Y \delta(v_1, v_2)\, d\mu,$$

where $Y \subseteq \mathcal{M}$ is an open set on the manifold $\mathcal{M}$ and μ is the usual surface measure. However, we can also describe the measure as follows. We know from section 4.3 that the polynomials invariant under the $SL_2(\mathbb{R})$ action on $Sym^3(\mathbb{R}^2)$ are generated by the discriminant polynomial $\Delta(p)$. We can therefore use this function to define the density, in conjunction with Gressman's Comparability Lemma 1.0.1. We will have that

$$\inf_{M \in SL_2(\mathbb{R})} \|M\tilde{A}_p\| \approx \left| Res\left(\frac{\partial \tilde{A}_p}{\partial x}, \frac{\partial \tilde{A}_p}{\partial y}\right)\right|^{\frac{1}{4}}$$

where $p \in Sym^3(\mathbb{R}^{2*})$. We will therefore have that

$$\delta(v_1, v_2) \approx \left(\left| Res\left(\frac{\partial \tilde{A}_p}{\partial x}, \frac{\partial \tilde{A}_p}{\partial y}\right)\right|^{\frac{1}{4}}\right)^{\frac{2}{11}}$$

and have again therefore described our equiaffine invariant measure $\nu_{\mathcal{A}}$.

4.4.2 A 2-Surface in $\mathbb{R}^{10}$.

Following on from the previous section, we now construct an equiaffine invariant tensor and the associated equiaffine invariant measure for a 2-surface in $\mathbb{R}^{10}$. The approach parallels our earlier construction for a 2-surface in $\mathbb{R}^6$, with necessary adaptations to account for the higher-dimensional setting. Let $\mathcal{M}$ be our 2-surface and let $f : \mathcal{M} \to \mathbb{R}^{10}$ be an immersion of $\mathcal{M}$. We will begin by defining the appropriate index set for this context and proceed to compute the affine curvature tensor, following the general method outlined previously.

In this case, $d = 2$ and $k = 10$. We therefore compute the index set $\Lambda_{2,10}$ as follows.

j	Required Dimension $j+1$	Minimum κ_j
1	2	1
2	3	1
3	4	2
4	5	2
5	6	2
6	7	3
7	8	3
8	9	3
9	10	3
10	11	4

Table 4.2: Calculation of the minimum κ_j for the index set $\Lambda_{2,10}$.

From Table 4.4.2, we see that the index set is as follows:

$$\Lambda_{2,10} = \begin{Bmatrix} (1,1),(2,1), \\ (3,1),(3,2),(4,1),(4,2),(5,1),(5,2), \\ (6,1),(6,2),(6,3),(7,1),(7,2),(7,3), \\ (8,1),(8,2),(8,3),(9,1),(9,2),(9,3), \\ (10,1),(10,2),(10,3),(10,4) \end{Bmatrix}.$$

For any finite sequence of vector fields X_λ indexed by $\lambda \in \Lambda_{2,10}$, the affine curvature tensor A_p is defined as as follows,

$$\begin{aligned} A_p((X_\lambda)_{\lambda\in\Lambda_{2,10}}) = \det\big(&X_{(1,1)}f(p) \wedge X_{(2,1)}f(p) \wedge \\ &X_{(3,1)}X_{(3,2)}f(p) \wedge X_{(4,1)}X_{(4,2)}f(p) \wedge X_{(5,1)}X_{(5,2)}f(p) \wedge \\ &X_{(6,1)}X_{(6,2)}X_{(6,3)}f(p) \wedge \\ &X_{(7,1)}X_{(7,2)}X_{(7,3)}f(p) \wedge \\ &X_{(8,1)}X_{(8,2)}X_{(8,3)}f(p) \wedge \\ &X_{(9,1)}X_{(9,2)}X_{(9,3)}f(p) \wedge \\ &X_{(10,1)}X_{(10,2)}X_{(10,3)}X_{(10,4)}f(p)\big). \end{aligned} \tag{4.4.2}$$

Now, we can also view A_p as a 21-multilinear map,

$$A_p : \bigoplus_{i=1}^{21} \mathbb{R}^2 \to \mathbb{R}.$$

By the universal property of tensor products B.0.1 this corresponds to an element $\tilde{A}_p$ in the dual space,

$$\tilde{A}_p \in \left(\bigotimes_{i=1}^{21} \mathbb{R}^2\right)^*$$

Using the properties of the determinant we can see that

$$\tilde{A}_p \in \left(\bigwedge^2 (\mathbb{R}^2)^*\right) \otimes \bigwedge^3 Sym^2((\mathbb{R}^2)^*) \otimes \bigwedge^4 Sym^3((\mathbb{R}^2)^*) \otimes Sym^4((\mathbb{R}^2)^*).$$

Writing down the standard basis vectors for these spaces we also have that

$$\begin{aligned}\tilde{A}_p &\in \langle x \wedge y \rangle \otimes \left\langle x^2 \wedge xy \wedge y^2 \right\rangle \\ &\otimes \left\langle x^3 \wedge x^2y \wedge xy^2 \wedge y^3 \right\rangle \otimes \left\langle x^4, x^3y, x^2y^2, xy^3, y^4 \right\rangle \\ &\cong Sym^4(\mathbb{R}^2).\end{aligned}$$

By the multilinearity of the tensor product, $\tilde{A}_p$ is completely determined by its coefficients on the basis vectors $x^4, x^3y, x^2y^2, xy^3, y^4$. In order to calculate these, we proceed as follows, similarly to the last section. Denote by

$$\frac{\partial^p f}{\partial x^a y^b} = \left(\frac{\partial^p f_i}{\partial x^a y^b}\right)_{i=1}^{10}$$

where $a, b, p \in \mathbb{N}_0$ and $a + b = p$.
Let $A'(f)$ be the 10×9 matrix defined as follows,

$$A'(f) := \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^3 f}{\partial x^3} & \frac{\partial^3 f}{\partial y^3} & \frac{\partial^3 f}{\partial x^2 \partial y} & \frac{\partial^3 f}{\partial x \partial y^2} \end{pmatrix}$$

For $1 \leqslant i \leqslant 5$ let C_i be columns defined as in the following table.

Column	Expression
C_1	$\left(\frac{\partial^4 f_i}{\partial x^4}\right)_{i=1}^{10}$
C_2	$\left(\frac{\partial^4 f_i}{\partial x^3 \partial y}\right)_{i=1}^{10}$
C_3	$\left(\frac{\partial^4 f_i}{\partial x^2 \partial y^2}\right)_{i=1}^{10}$
C_4	$\left(\frac{\partial^4 f_i}{\partial x \partial y^3}\right)_{i=1}^{10}$
C_5	$\left(\frac{\partial^4 f_i}{\partial y^4}\right)_{i=1}^{10}$

For each of the columns C_i, we adjoin them to $A'(f)$ to form five different 10×10 matrices:

$$A_i(f) = [A'(f) \mid C_i], \quad \text{for } i = 1, 2, 3, 4, 5.$$

For $1 \leqslant i \leqslant 5$, let γ_i be the determinant of $A_i(f)$, that is

$$\gamma_i := \det(A_i(f)) = \det([A'(f) \mid C_i]), \quad \text{for } i = 1, 2, 3, 4, 5.$$

We will then have that

$$\begin{aligned}\tilde{A}_p =& x \wedge y \otimes x^2 \wedge xy \wedge y^2 \otimes x^3 \wedge x^2y \wedge xy^2 \wedge y^3 \\ &\otimes \gamma_1 x^4 + \gamma_2 x^3 y + \gamma_3 x^2 y^2 + \gamma_4 x y^3 + \gamma_5 y^4.\end{aligned}$$

We have therefore fully described our equiaffine invariant tensor A_p. To pass to the density function, let

$$\delta(v_1, v_2) = \inf_{M \in SL_2(\mathbb{R})} \|M \cdot \tilde{A}_p\|^{\frac{2}{21}}.$$

Computing this infimum can be quite difficult in practice. Therefore, we can use our modified comparability lemma and the invariant polynomials we computed in section 4.3 for the $SL_2(\mathbb{R})$ action on $Sym^4(\mathbb{R}^2)$. Let P_1 be the polynomial defined by

$$P_1 := Res\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) = \left[\prod_{1 \leqslant i < j \leqslant 4} (x_i y_j - x_j y_i)\right]^2.$$

and let P_2 be defined by

$$\begin{aligned}P_2(x, y) := &(x_1 y_2 - x_2 y_1)^2 (x_3 y_4 - x_4 y_3)^2 \\ &+ (x_1 y_3 - x_3 y_1)^2 (x_2 y_4 - x_4 y_2)^2 \\ &+ (x_1 y_4 - x_4 y_1)^2 (x_2 y_3 - x_3 y_2)^2.\end{aligned}$$

Then by the modified comparability lemma 4.3.2 we have that

$$\inf_{M \in SL_2(\mathbb{R})} \|M \cdot \tilde{A}_p\| \approx \max\left(|P_1(\tilde{A}_p)|^{\frac{1}{6}}, |P_2(\tilde{A}_p)|^{\frac{1}{2}}\right),$$

which in turn means

$$\delta(v_1, v_2) \approx \max\left(|P_1(\tilde{A}_p)|^{\frac{1}{6}}, |P_2(\tilde{A}_p)|^{\frac{1}{2}}\right)^{\frac{2}{21}}.$$

4.4.3 A 2-Surface in $\mathbb{R}^{15}$.

Let $\mathcal{M} \subseteq \mathbb{R}^{15}$ be a 2-surface, and let $f : \mathcal{M} \to \mathbb{R}^{15}$ be an immersion of $\mathcal{M}$. In the context of Gressman's construction of the equiaffine invariant tensor, we will have $k = 2$ and $d = 15$. We therefore compute our index set $\Lambda_{2,15}$ as follows,

j	Required Dimension $j+1$	Minimum κ_j
1	2	1
2	3	1
3	4	2
4	5	2
5	6	2
6	7	3
7	8	3
8	9	3
9	10	3
10	11	4
11	12	4
12	13	4
13	14	4
14	15	4
15	16	5

Table 4.3: Calculation of the minimum κ_j for the index set $\Lambda_{2,15}$.

From Table 4.4.3 we see that the index set is as follows:

$$\Lambda_{2,15} = \left\{ \begin{array}{l} (1,1),(2,1), \\ (3,1),(3,2),(4,1),(4,2),(5,1),(5,2), \\ (6,1),(6,2),(6,3),(7,1),(7,2),(7,3), \\ (8,1),(8,2),(8,3),(9,1),(9,2),(9,3), \\ (10,1),(10,2),(10,3),(10,4), \\ (11,1),(11,2),(11,3),(11,4), \\ (12,1),(12,2),(12,3),(12,4), \\ (13,1),(13,2),(13,3),(13,4), \\ (14,1),(14,2),(14,3),(14,4), \\ (15,1),(15,2),(15,3),(15,4),(15,5) \end{array} \right\}.$$

For any finite sequence of vector fields X_λ indexed by $\lambda \in \Lambda_{2,15}$, the affine curvature tensor A_p is defined as follows:

$$\begin{aligned} A_p((X_\lambda)_{\lambda\in\Lambda_{2,15}}) = \det\big(&X_{(1,1)}f(p)\wedge X_{(2,1)}f(p)\wedge \\ &X_{(3,1)}X_{(3,2)}f(p)\wedge X_{(4,1)}X_{(4,2)}f(p)\wedge X_{(5,1)}X_{(5,2)}f(p)\wedge \\ &X_{(6,1)}X_{(6,2)}X_{(6,3)}f(p)\wedge \\ &X_{(7,1)}X_{(7,2)}X_{(7,3)}f(p)\wedge \\ &X_{(8,1)}X_{(8,2)}X_{(8,3)}f(p)\wedge \\ &X_{(9,1)}X_{(9,2)}X_{(9,3)}f(p)\wedge \\ &X_{(10,1)}X_{(10,2)}X_{(10,3)}X_{(10,4)}f(p)\wedge \\ &X_{(11,1)}X_{(11,2)}X_{(11,3)}X_{(11,4)}f(p)\wedge \\ &X_{(12,1)}X_{(12,2)}X_{(12,3)}X_{(12,4)}f(p)\wedge \\ &X_{(13,1)}X_{(13,2)}X_{(13,3)}X_{(13,4)}f(p)\wedge \\ &X_{(14,1)}X_{(14,2)}X_{(14,3)}X_{(14,4)}f(p)\wedge \\ &X_{(15,1)}X_{(15,2)}X_{(15,3)}X_{(15,4)}X_{(15,5)}f(p)\big). \end{aligned} \tag{4.4.3}$$

Now, $\#\Lambda_{2,15} = 45$ and so we view A_p as a $45-$multilinear map

$$A_p : \bigoplus_{i=1}^{45} \mathbb{R}^2 \to \mathbb{R}.$$

By the universal property of tensor products, this corresponds to an element $\tilde{A}_p$ in the dual space,

$$\tilde{A}_p \in \left(\bigotimes_{i=1}^{45} \mathbb{R}^2\right)^* .$$

Using the properties of the determinant, we can see that

$$\begin{aligned} \tilde{A}_p \in &\left(\bigwedge^2 (\mathbb{R}^2)^*\right) \otimes \bigwedge^3 Sym^2((\mathbb{R}^2)^*) \otimes \bigwedge^4 Sym^3((\mathbb{R}^2)^*) \\ &\otimes \bigwedge^5 Sym^4((\mathbb{R}^2)^*) \otimes Sym^5((\mathbb{R}^2)^*). \end{aligned}$$

Writing down the standard basis vectors for these spaces, we also have that

$$\begin{aligned} \tilde{A}_p \in &\langle x\wedge y\rangle \otimes \left\langle x^2\wedge xy\wedge y^2\right\rangle \otimes \left\langle x^3\wedge x^2y\wedge xy^2\wedge y^3\right\rangle \\ &\otimes \left\langle x^4\wedge x^3y\wedge x^2y^2\wedge xy^3\wedge y^4\right\rangle \otimes \left\langle x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5\right\rangle \\ &\cong Sym^5(\mathbb{R}^2). \end{aligned}$$

By the multilinearity of the tensor product, $\tilde{A}_p$ is completely determined by its coefficients on the basis vectors $x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5$. In order to calculate these, we proceed as follows, similarly to the two sections. Once again, denote by

$$\frac{\partial^p f}{\partial x^a y^b} = \left(\frac{\partial^p f_i}{\partial x^a y^b}\right)_{i=1}^{15}$$

where $a, b, p \in \mathbb{N}_0$ and $a + b = p$.

Let $A'(f)$ be the 15×14 matrix with columns given by the following ordered partial derivatives up to degree 4,

Order	Partial Derivatives
1	$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$
2	$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$
3	$\frac{\partial^3 f}{\partial x^3}, \frac{\partial^3 f}{\partial x^2 \partial y}, \frac{\partial^3 f}{\partial x \partial y^2}, \frac{\partial^3 f}{\partial y^3}$
4	$\frac{\partial^4 f}{\partial x^4}, \frac{\partial^4 f}{\partial x^3 \partial y}, \frac{\partial^4 f}{\partial x^2 \partial y^2}, \frac{\partial^4 f}{\partial x \partial y^3}, \frac{\partial^4 f}{\partial y^4}$

For $1 \leqslant i \leqslant 6$, let C_i be columns defined as in the following table.

Column	Expression
C_1	$\left(\frac{\partial^5 f_i}{\partial x^5}\right)_{i=1}^{15}$
C_2	$\left(\frac{\partial^5 f_i}{\partial x^4 \partial y}\right)_{i=1}^{15}$
C_3	$\left(\frac{\partial^5 f_i}{\partial x^3 \partial y^2}\right)_{i=1}^{15}$
C_4	$\left(\frac{\partial^5 f_i}{\partial x^2 \partial y^3}\right)_{i=1}^{15}$
C_5	$\left(\frac{\partial^5 f_i}{\partial x \partial y^4}\right)_{i=1}^{15}$
C_6	$\left(\frac{\partial^5 f_i}{\partial y^5}\right)_{i=1}^{15}$

For each of the columns C_i, we adjoin them to $A'(f)$ to form six different 15×15 matrices:

$$A_i(f) = [A'(f) \mid C_i], \quad \text{for } i = 1, 2, 3, 4, 5, 6.$$

For $1 \leqslant i \leqslant 6$, let γ_i be the determinant of $A_i(f)$, that is

$$\gamma_i := \det(A_i(f)) = \det([A'(f) \mid C_i]), \quad \text{for } i = 1, 2, 3, 4, 5, 6.$$

We then have that

$$\begin{aligned}\tilde{A}_p =& x \wedge y \otimes x^2 \wedge xy \wedge y^2 \otimes x^3 \wedge x^2y \wedge xy^2 \wedge y^3 \\ & \otimes x^4 \wedge x^3y \wedge x^2y^2 \wedge xy^3 \wedge y^4 \\ & \otimes \gamma_1 x^5 + \gamma_2 x^4 y + \gamma_3 x^3 y^2 + \gamma_4 x^2 y^3 + \gamma_5 x y^4 + \gamma_6 y^5.\end{aligned}$$

This fully describes the equiaffine invariant tensor A_p. To pass to the density function, let

$$\delta(v_1, v_2) = \inf_{M \in SL_2(\mathbb{R})} \|M \cdot \tilde{A}_p\|^{\frac{2}{45}}.$$

Recall from section 4.3 where we computed enough invariant polynomials to generate the unstable points for the $SL_2(\mathbb{C})$ action on $Sym^5(\mathbb{C}^2)$. These polynomials are given in the following table.

Polynomial	Expression
p_1	$\mathrm{Res}\left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right)$
p_2	$\sum_{\sigma\in S_5}\left(\prod_{i\neq 1,2} e^4_{\sigma(1)\sigma(i)}\right)\left(\prod_{i,j\in\{3,4,5\},i<j} e^2_{\sigma(i)\sigma(j)}\right)$
p_{3k}	$\sum_{\sigma\in S_5} e^{2k}_{\sigma(1)\sigma(2)} e^{2k}_{\sigma(2)\sigma(3)} e^{2k}_{\sigma(3)\sigma(4)} e^{2k}_{\sigma(4)\sigma(5)} e^{2k}_{\sigma(5)\sigma(1)} \quad$ for $k = 1, 2, 3$

The polynomials have the following degrees,

Polynomial	Degree
p_1	8
p_2	12
p_{3k}	$4k \quad$ for $k = 1, 2, 3$

Then by the modified comparability lemma 4.3.2 we have that

$$\inf_{M\in SL_2(\mathbb{R})} \|M \cdot \tilde{A}_p\| \approx \max\left(|p_1(\tilde{A}_p)|^{\frac{1}{8}}, |p_2(\tilde{A}_p)|^{\frac{1}{12}}, |p_3(\tilde{A}_p)|^{\frac{1}{4}}, |p_6(\tilde{A}_p)|^{\frac{1}{8}}, |p_9(\tilde{A}_p)|^{\frac{1}{12}}\right)$$

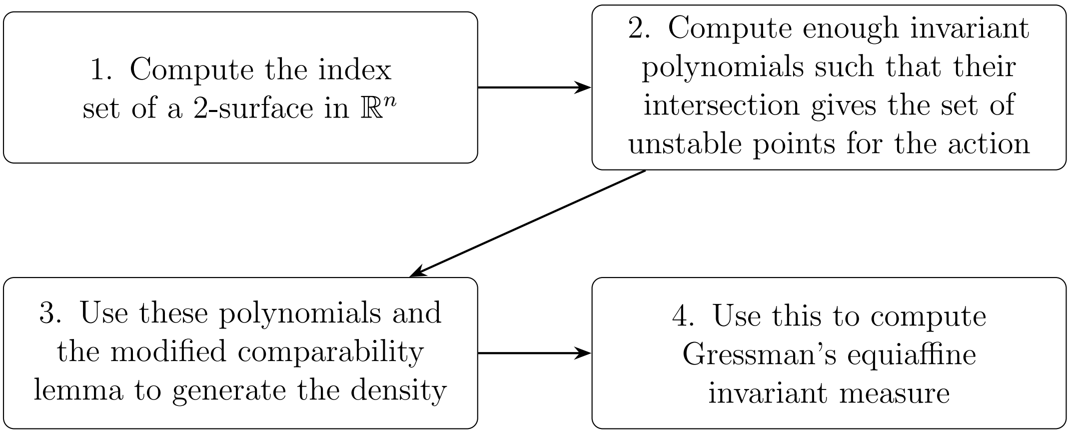
which in turn means

$$\delta(v_1, v_2) \approx \max\left(|p_1(\tilde{A}_p)|^{\frac{1}{8}}, |p_2(\tilde{A}_p)|^{\frac{1}{12}}, |p_3(\tilde{A}_p)|^{\frac{1}{4}}, |p_6(\tilde{A}_p)|^{\frac{1}{8}}, |p_9(\tilde{A}_p)|^{\frac{1}{12}}\right)^{\frac{2}{45}}.$$

4.4.4 Further Remarks about this Construction

In this last chapter, we have developed an explicit method for analysing the equiaffine invariant measures on surfaces, drawing upon our modified version of Gressman's comparability lemma. However, it was no accident that the surfaces we analysed were 2−surfaces in $\mathbb{R}^n$ for $n = 6, 10$ and 15, as these corresponded nicely to the analysis we performed in section 4.3. For $n \in \{7, 8, 9, 11, 12, 13, 14\}$, a more detailed analysis is required. However, by exploiting known identities involving the wedge product, computing the equiaffine measure for some of these surfaces could be made a less challenging task.

This presents a new method for the computation of Gressman's measure and generally promises better computational utility. This methodology is summarised in the following diagram.



Chapter 5

Conclusion

In this thesis, we have developed an explicit methodology for analysing the equiaffine invariant measures on specific surfaces. By modifying Gressman's comparability lemma, we have created a more computationally feasible approach to determining the equiaffine invariant measure for specific manifolds of intermediate dimensions, bypassing the need to compute a complete set of generators for the algebra of invariant polynomials.

Our approach has been demonstrated through specific cases, including surfaces in $\mathbb{R}^{10}$ and $\mathbb{R}^{15}$, where we computed enough invariant polynomials to generate the set of unstable points. We also provided a clearer path to computing the equiaffine invariant measure using these polynomials and the modified lemma, illustrating its potential utility in the broader context of harmonic analysis. Future research could build on this foundation by applying the Modified Comparability Lemma to other settings, such as higher-dimensional surfaces or different types of group actions, to further understand its broader implications and potential applications.

Appendix A

Facts from Commutative Algebra

Definition A.0.1 (Ideal)**.** Let R be a commutative ring and let $S \subseteq R$. We say that S is an *ideal* if for all $r \in R$ and $x, y \in S$, we have that

(i) $x + y \in S$.

(ii) $rx \in S$.

Definition A.0.2. Let $S \subseteq R$. The *ideal generated by* S, denoted by $\langle S \rangle$ is the smallest ideal containing S, that is

$$\langle S \rangle = \bigcap \{F \trianglelefteq R \mid S \subseteq F\}.$$

Lemma A.0.3. *Let R be a commutative ring and let $S \subseteq R$. Then $\langle S \rangle$ coincides with the collection of all $R-$linear combinations of elements of S.*

Proof : For brevity let $X = \left\{\sum_{i=1}^{k} \alpha_i x_i \mid \alpha_i \in R,\, x_i \in S,\, k \in \mathbb{N}\right\}$. Clearly, X is an ideal containing S, and so $\langle S \rangle \subseteq X$. For the reverse, suppose $p \in X$, so $p = \alpha_1 x_1 + \ldots \alpha_k x_k$ some choice of coefficients α_i. If F is an ideal containing S, it is obvious that p is in F also, thus p is in all such ideals. It follows $p \in \langle S \rangle$, whence $X = \langle S \rangle$. □

Definition A.0.4. We say $I \trianglelefteq R$ is *maximal* if it is proper and for all $F \trianglelefteq R$, if $I \subseteq F$, then either $I = F$ or $F = R$.

Definition A.0.5. Suppose $P \trianglelefteq R$. We say P is *prime* if it is proper and for all $a, b \in R$, if $ab \in P$, then either $a \in P$ or $b \in P$.

Definition A.0.6. Let $p \in R$. We say p is *prime* if it is a non-zero non unit and for every $a, b \in R$, if $p \mid ab$ then either $p \mid a$ or $p \mid b$.

Proposition A.0.7. *Suppose p is an element in the ring R. Then $\langle p \rangle$ is a prime ideal if and only if p is prime.*

Theorem A.0.8. *Let $I \trianglelefteq R$. Then the collection of ideals of R/I is in one-to-one correspondence with those of R containing I.*

Proof : Let $\mathfrak{C}$ be the collection of ideals in R containing I and let $\mathfrak{D}$ be the collection of ideals in R/I. Let $\pi : \mathfrak{C} \to R/I$ be the restriction of the canonical surjection to $\mathfrak{C}$. We show π is injective and that $\mathrm{Im}(\pi) = \mathfrak{D}$, proving the correspondence.

Let $\mathfrak{m} \in \mathfrak{C}$ and suppose $\mathfrak{m} \in \ker(\pi)$, that is, $\mathfrak{m} + I = I$. It follows $\mathfrak{m} \subseteq J$. Moreover, since $\mathfrak{m} \in \mathfrak{C}$ it follows $I \subseteq \mathfrak{m}$, from which we have $\mathfrak{m} = I$. Since the kernel is trivial, it follows π is injective.
Once again take $\mathfrak{m} \in \mathfrak{C}$. That $\mathfrak{m} + I$ is an ideal in R/I is straightforward and will not be shown here. We must therefore show that if $J \in \mathfrak{D}$, then there exists $\mathfrak{n} \in \mathfrak{C}$ such that $\mathfrak{n} + I = J$. Given such a $J \in \mathfrak{D}$, let $\mathfrak{n} = \{x \in R \mid x + I \in J\}$. Then clearly $\mathfrak{n}$ is an ideal in R since J is an ideal in R/I. Moreover, it is easily seen that $I \subseteq \mathfrak{n}$ since for each $p \in I$, $p + I = I \in J$. Lastly, $\pi(\mathfrak{n}) = J$ by construction. Surjectivity follows, and therefore the bijection also.

□

Corollary A.0.8.1. *Let A be a proper ideal of a ring R. The following statements are equivalent.*

(i) A is a maximal ideal of R.

(ii) The factor ring R/A has no non-zero proper ideals.

Proof : By Theorem A.0.8, the ideals of R/A are in one-to-one correspondence with those of R containing A. There are only two of the latter, and so there are only two of the former. □

Corollary A.0.8.2. *Let A be an ideal in a ring R. Then A is a maximal ideal of R if and only if the factor ring $R\backslash A$ is a field.*

Theorem A.0.9. *Suppose $I \trianglelefteq R$ is a maximal ideal. Then it is prime.*

Proof : Let $a, b \in R$ and suppose $ab \in I$. If $a \in I$ we are done, so suppose $a \notin I$. Consider the ideal $I + \langle a \rangle$. This strictly contains I, and hence by maximality must be the full ring. Thus there exists $x \in I$ and $r \in R$ such that $x + ra = 1$. Multiplying across by b we see that $bx + rab = b$. Since bx and rab are both members of I, it follows b is too. □

We can upgrade the previous result in the case that R is a principal ideal domain.

Theorem A.0.10. *Suppose R is a principal ideal domain. Then an ideal P is prime if and only if it is maximal.*

Proof : The if part follows from the previous theorem. For the only if part, suppose P is prime. Since R is a PID, there is $p \in R$ prime such that $\langle p \rangle = P \neq R$. Suppose there is an ideal $\langle m \rangle = M \subseteq R$ such that $P \subseteq M \iff \langle p \rangle = \langle m \rangle$. Thus, there exists $x \in R$ such that $p = x \cdot m$. Since p is prime, either x or m is a unit. If x is a unit, then $P = M$. Otherwise $M = R$. It follows that P is maximal. □

Theorem A.0.11. *Suppose $P \trianglelefteq R$ is a prime ideal. Then it is also radical.*

Proof : We must show that if $x^k \in P$, then $x \in P$ too. Suppose k is the minimal natural number that does the job. If it is 1 we are done. Otherwise, $k \geqslant 2$. This means $x \cdot x^{k-1} \in P$, and by minimality of k this forces $x \in P$, so we are done. □

Lemma A.0.12. *Let I_1 and I_2 be ideals in a ring R. Then $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$.*

Proof : First we show $\sqrt{I_1 I_2} \subseteq \sqrt{I_1 \cap I_2}$. Suppose $f \in \sqrt{I_1 I_2}$. Then $f^k \in I_1 I_2$ for some $k \in \mathbb{N}$. It follows that $f^k = g \cdot h$ for some $g \in I_1$ and $h \in I_2$. Thus $f^k \in I_1 \cap I_2$ and so $f \in \sqrt{I_1 \cap I_2}$.
For the reverse suppose $f \in \sqrt{I_1 \cap I_2}$. Then, as before $f^k \in I_1 \cap I_2$ for some natural number k. It follows that $g = f^k \cdot f^k = f^{2k} \in I_1 I_2$ and so $f \in \sqrt{I_1 I_2}$. □

Lemma A.0.13. *Let I_1 and I_2 be ideals in a ring R. Then $\langle I_1 \cup I_2 \rangle = I_1 + I_2$.*

Proof : Recall that $(I_1 \cup I_2)$ is the smallest ideal containing $I_1 \cup I_2$. $I_1 + I_2$ being an ideal follows trivially from the fact that I_1 and I_2 are both ideals. To see that $(I_1 \cup I_2) \subseteq I_1 + I_2$, suppose $f \in (I_1 \cup I_2)$. Then $f = g \cdot h$ for some $g \in K[x_1, \ldots, x_n]$ and some $h \in I_1 \cup I_2$. Without loss of generality assume $h \in I_1$. Then $f = g \cdot h + 0 \in I_1 + I_2$ and so $(I_1 \cup I_2) \subseteq I_1 + I_2$.
Lastly, suppose J is another ideal containing $I_1 \cup I_2$. Let $f \in I_1 + I_2$ so $f = g + h$ for some $g \in I_1$ and $h \in I_2$. Then $f \in (I_1 \cup I_2) \subseteq J$. It follows $I_1 + I_2$ is the smallest ideal containing $I_1 \cup I_2$. □

Proposition A.0.14. *Let R be a commutative unital ring. Then the following conditions on R are equivalent.*

(i) Every ideal in R is finitely generated.

(ii) Every ascending chain of ideals in R terminates.

(iii) Every non-empty collection of ideals in R contains a maximal element.

Remark A.0.15. A ring satisfying any of the three equivalent conditions in the previous proposition is called *Noetherian.*

Lemma A.0.16. *[Hilbert's Basis Theorem] If R is Noetherian, then $R[x]$ is Noetherian.*

Example A.0.17. (i) Principal ideal domains are Noetherian, as every ideal is generated by a single element.
In particular, every field K only has two ideals, namely $\langle 0 \rangle$ and $\langle 1 \rangle = K$, hence every field is Noetherian.

(ii) The polynomial ring $K[x_1, x_2, \ldots]$ in infinitely many variables is not Noetherian. An example of an aschending chain of ideals that doesn't terminate is given as follows,

$$\langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \langle x_1, x_2, x_3 \rangle \subsetneq \ldots$$

Conversely, the polynomial ring in finitely many variables $K[x_1, \ldots, x_n]$ is Noetherian. We prove this by induction on the number of variables.
In the base case, the field K is Noetherian as shown in the previous example. Now suppose the polynomial ring $K[x_1, \ldots, x_{n-1}]$ is Noetherian. The crucial observation is that we can view the polynomial ring in n variables with coefficients in K as the polynomial ring in 1 variable with coefficients from $K[x_1, \ldots, x_{n-1}]$. Since the latter ring is Noetherian, using Lemma A.0.16, we conclude that $K[x_1, \ldots, x_n]$ is Noetherian.

Theorem A.0.18. *$K[x]$ is a principal ideal domain.*

Proof : Suppose $J \trianglelefteq K[x]$. If $J = K[x]$ then $J = \langle 1 \rangle$, and similarly, if $J = \{0\}$ then $J = \langle 0 \rangle$. Suppose $\langle 0 \rangle \subsetneq J \subsetneq K[x]$. Since J must contain a non-zero polynomial, let $m = \min\{\deg(p) \mid p \in K[x]\}$, which exists by the well-ordering principle of $\mathbb{N}$. If $m = 0$ then J contains a constant, call it c. Since J is an ideal, $c\dot{c}^{-1} = 1 \in J$ which is impossible, hence $m = 0$ is impossible and so $m \geqslant 1$.
Now, let $p \in K[x]$ have degree m. We show that $\langle p \rangle = J$. The forward direction is clearly true. For the reverse, suppose $f \in J$. By the division algorithm for polynomials, we have that $f = q \cdot p + r$ where $q, r \in K[x]$ and either $r = 0$ or $0 \leqslant \deg(r) < m$. But $r = f - q \cdot p \in J$ which forces $r = 0$. Hence, $f = q \cdot p$ and so $f \in \langle p \rangle$. □

Proposition A.0.19. *If R is Noetherian and I is an ideal of R, then R/I is also Noetherian.*

Proof : Suppose we have an ascending chain of ideals in R/I as follows,

$$J_0 \subseteq J_1 \subseteq J_2 \subseteq J_3 \subseteq \dots$$

By A.0.8 these are in bijection with an escending chain of ideals in R, which must terminate by the Noetherian proeprty. As a result, the ascending chain in R/I must terminate also.

□

Theorem A.0.20 (Chinese Remainder Theorem)**.** *Suppose I and J are ideals in a commutative ring R such that $I + J = R$. Then the rings $R/(I \cap J)$ and $R/I \times R/J$ are isomorphic via the assignment*

$$\varphi : x + (I \cap J) \mapsto (x + I, x + J).$$

The inverse of this isomorphism admist the following description. Write $1 = a + b$ with $a \in I$ and $b \in J$. Then $\varphi^{-1} : (x + I, y + J) \mapsto (bx + ay) + (I \cap J)$.

Proof : Consider the ring homomorphism $\theta : R \to R/I \times R/J$, $x \mapsto (x + I, x + J)$. Since $\ker(\theta) = I \cap J$, we get an induced injective ring homomorphism

$$\varphi : R/(I \cap J) \to R/I \times R/J,\ x + (I \cap J) \mapsto (x + I, x + J).$$

Write $1 = a + b$ for appropriate $a \in I$ and $b \in J$, using $R = I + J$. Suppose $(x, y) \in R/I \times R/J$, we then calculate

$$\varphi(bx + ay) = (bx + I, ay + J) = (ax + bx + I, ay + by + J) = (x + I, y + J),$$

and so φ is onto, and φ^{-1} has the desired form. □

Appendix B

Tensor Products and their Universal Property

Definition B.0.1. Let R be a commutative ring with unity, and let M and N be two R-modules. The *tensor product* of M and N, denoted by $M \otimes_R N$, is an R-module that is defined together with a bilinear map

$$\phi : M \times N \to M \otimes_R N,$$

such that for any other R-module P and any bilinear map $f : M \times N \to P$, there exists a unique R-linear map $\tilde{f} : M \otimes_R N \to P$ such that:

$$f = \tilde{f} \circ \phi.$$

This property is known as the *universal property of the tensor product.*

Construction of the Tensor Product

The tensor product $M \otimes_R N$ can be constructed explicitly as follows:

(i) Consider the free R-module generated by the Cartesian product $M \times N$, denoted by $F = R^{M \times N}$.

(ii) Define a submodule $S \subseteq F$ generated by the relations of bilinearity:

$$\begin{aligned}
&(m_1 + m_2, n) - (m_1, n) - (m_2, n),\\
&(m, n_1 + n_2) - (m, n_1) - (m, n_2),\\
&(rm, n) - r(m, n),\\
&(m, rn) - r(m, n),
\end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, and $r \in R$.

(iii) The tensor product $M \otimes_R N$ is then defined as the quotient module F/S.

The Tensor Product is Unique

The uniqueness of the tensor product up to isomorphism follows directly from its universal property. To see this, let $M \otimes_R N$ and $M' \otimes_R N'$ be two tensor products of

M and N satisfying the universal property. Then, by the universal property, there exists a unique R-linear map $f : M \otimes_R N \to M' \otimes_R N'$ and a unique R-linear map $g : M' \otimes_R N' \to M \otimes_R N$ such that the compositions $g \circ f$ and $f \circ g$ are the identity maps on their respective modules. Thus, $M \otimes_R N \cong M' \otimes_R N'$.

The Dimension Formula

Let M and N be finite-dimensional R-modules with dimensions $\dim(M) = m$ and $\dim(N) = n$. The tensor product $M \otimes_R N$ has dimension:

$$\dim(M \otimes_R N) = \dim(M) \cdot \dim(N).$$

Proof:

(i) Choose a basis $\{m_1, m_2, \ldots, m_m\}$ for M and a basis $\{n_1, n_2, \ldots, n_n\}$ for N.

(ii) Consider the set of elements $\{m_i \otimes n_j \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ in $M \otimes_R N$.

(iii) To show that this set forms a basis, we prove that it is linearly independent and spans $M \otimes_R N$.

  (a) **Linear Independence**: Suppose

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}(m_i \otimes n_j) = 0,$$

  for some $c_{ij} \in R$. Then, applying the linearity in both arguments, we deduce that all $c_{ij} = 0$.

  (b) **Spanning Set**: Any element in $M \otimes_R N$ can be expressed as a finite sum $\sum c_{ij}(m_i \otimes n_j)$, proving that the set spans $M \otimes_R N$.

Thus, the set $\{m_i \otimes n_j\}$ forms a basis of $M \otimes_R N$, giving $\dim(M \otimes_R N) = m \cdot n$.

Tensor Algebras

Let V be a vector space over a field K. The *tensor algebra* of V, denoted $T(V)$, is an associative algebra that contains V and is generated by it in a universal way.

Formally, the tensor algebra is defined as:

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

where:

- $V^{\otimes n}$ denotes the n-th tensor power of V, with the convention that $V^{\otimes 0} = K$ (the base field).
- The direct sum $\bigoplus_{n=0}^{\infty} V^{\otimes n}$ means that the tensor algebra is the vector space formed by the infinite sum of all tensor powers of V.

Construction of the Tensor Algebra

To construct the tensor algebra $T(V)$:

(i) **Direct Sum of Tensor Powers**: Start with the vector space V. Form the tensor powers $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$ (n times). Take the direct sum of all these spaces:

$$T(V) = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

(ii) **Algebra Structure**: Define multiplication in $T(V)$ using the tensor product. For $x \in V^{\otimes m}$ and $y \in V^{\otimes n}$, define the product $x \cdot y = x \otimes y \in V^{\otimes (m+n)}$.

(iii) **Universal Property**: The tensor algebra $T(V)$ is characterised by the following universal property: For any associative algebra A and any linear map $f : V \to A$, there exists a unique algebra homomorphism $\tilde{f} : T(V) \to A$ such that $\tilde{f}|_V = f$.

Properties of the Tensor Algebra

(i) **Graded Algebra**: $T(V)$ is a graded algebra with respect to the natural grading given by the degrees of tensors:

$$T(V) = \bigoplus_{n=0}^{\infty} T^n(V),$$

where $T^n(V) = V^{\otimes n}$.

(ii) **Associativity**: The tensor algebra is associative, meaning for any elements $x, y, z \in T(V)$, we have:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

(iii) **Algebra Generated by V**: $T(V)$ is the free associative algebra generated by the vector space V. This means any algebraic relation among elements of V arises from the relations imposed by the tensor product.

Examples of Tensor Algebras

(i) **Trivial Vector Space**: If $V = 0$ (the trivial vector space), then $T(V) = K$, the base field.

(ii) **One-Dimensional Vector Space**: If $V = K$, a one-dimensional vector space over itself, then $T(V)$ is isomorphic to the polynomial ring $K[x]$, where x is a basis element of V.

(iii) **Higher-Dimensional Vector Space**: If $V = K^n$ is an n-dimensional vector space with basis $\{e_1, e_2, \ldots, e_n\}$, then the tensor algebra $T(V)$ contains all possible tensors formed by the elements of the basis, with no additional relations except those implied by the tensor product.

The Quotient of the Tensor Algebra: Symmetric and Exterior Algebras

The tensor algebra $T(V)$ can be further specialized to construct other important algebras by taking suitable quotients:

(i) **Symmetric Algebra**: The *symmetric algebra* $S(V)$ is obtained by taking the quotient of $T(V)$ by the ideal generated by all elements of the form $v \otimes w - w \otimes v$ for $v, w \in V$. This algebra captures the commutative polynomials over V.

(ii) **Exterior Algebra**: The *exterior algebra* $\Lambda(V)$ is obtained by taking the quotient of $T(V)$ by the ideal generated by all elements of the form $v \otimes v$ for $v \in V$. This algebra plays a crucial role in differential geometry and representation theory, especially for alternating multilinear maps.

Bibliography

[1] David Birkes. "Orbits of Linear Algebraic Groups". *Annals of Mathematics* 93.3 (1971), pp. 459–475.

[2] Roger W Carter and Meinolf Geck. *Representations of Reductive Groups*. 16. Cambridge University Press, 1998.

[3] Youngwoo Choi. "The $L^p - L^p$ Mapping Properties of Convolution Operators with the Affine Arclength Measure on Space Curves". *Journal of the Australian Mathematical Society* 75.2 (2003), pp. 247–262.

[4] Ciprian Demeter. "Bourgain's Work in Fourier Restriction". *Bulletin of the American Mathematical Society* 58.2 (2021), pp. 191–204.

[5] Spyridon Dendrinos and Betsy Stovall. "Uniform Bounds for Convolution and Restricted X-Ray Transforms along Degenerate Curves". *Journal of Functional Analysis* 268.3 (2015), pp. 585–633.

[6] Spyridon Dendrinos and Eugen Zimmermann. "On L^p-Improving for Averages Associated to Mixed Homogeneous Polynomial Hypersurfaces in $\mathbb{R}^3$". *Journal d'Analyse Mathématique* 138.2 (2019), pp. 563–595.

[7] Joe Diestel and Angela Spalsbury. *The Joys of Haar Measure*. American Mathematical Soc., 2014.

[8] Igor Dolgachev. *Lectures on Invariant Theory*. 296. Cambridge University Press, 2003.

[9] Paul Gordan. "Über die Auflösung Linearer Gleichungen mit Reellen Coefficienten". *Mathematische Annalen* 6.1 (1873), pp. 23–28.

[10] Philip T. Gressman. "On the Oberlin Affine Curvature Condition". *Duke Mathematical Journal* 168.11 (2019).

[11] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, 2014.

[12] Egbert Harzheim. *Ordered sets*. Vol. 7. Springer Science & Business Media, 2005.

[13] Michiel Hazewinkel. "David Hilbert, Paper on 'Mathematical Problems'(1901)". In: *Landmark Writings in Western Mathematics 1640-1940*. Elsevier, 2005, pp. 732–747.

[14] Isroil A Ikromov and Detlef Müller. *Fourier Restriction for Hypersurfaces in Three Dimensions and Newton Polyhedra (AM-194)*. Vol. 194. Princeton University Press, 2016.

[15] Gordon Douglas James and Martin W Liebeck. *Representations and Characters of Groups*. Cambridge university press, 2001.

[16] Serge Lang. *Algebra*. Vol. 211. Springer Science & Business Media, 2012.

[17] Conor Meade. *Uniform Convolution and Fourier Restriction Estimates for Complex Polynomial Curves in* $\mathbb{C}^3$. 2020. arXiv: `2012.09651 [math.CV]`. URL: `https://arxiv.org/abs/2012.09651`.

[18] James S Milne. *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Vol. 170. Cambridge University Press, 2017.

[19] Shigeru Mukai. *An Introduction to Invariants and Moduli*. Vol. 81. Cambridge University Press, 2003.

[20] David Mumford, John Fogarty, and Frances Kirwan. *Geometric invariant theory*. Vol. 34. Springer Science & Business Media, 1994.

[21] Masayoshi Nagata. "On the 14^{th} Problem of Hilbert". *American Journal of Mathematics* 81.3 (1959), pp. 766–772.

[22] Fabio Nicola. "Slicing surfaces and the Fourier restriction conjecture". *Proceedings of the Edinburgh Mathematical Society* 52.2 (2009), pp. 515–527.

[23] Daniel M Oberlin. "Affine Dimension: Measuring the Vestiges of Curvature". *Michigan Mathematical Journal* 51.1 (2003), pp. 13–26.

[24] Gian-Carlo Rota. "Reynolds Operators". In: *Proceedings of Symposia in Applied Mathematics*. Vol. 16. American Mathematical Society Providence, RI. 1964, pp. 70–83.

[25] Igor Rostislavovich Shafarevich and Miles Reid. *Basic Algebraic Geometry*. Vol. 2. Springer, 1994.

[26] Elias M Stein. "Some Problems in Harmonic Analysis". *Harmonic analysis in Euclidean spaces, Proceedings of the Symposium in Pure Mathematics of the Amer. Math. Soc., Williams College, Mass, Proc. Sympos. Pure Math., XXXV Part I, 1979* (1979), pp. 3–20.

[27] Betsy Stovall. "Waves, Spheres, and Tubes: A Selection of Fourier Restriction Problems, Methods, and Applications". *Notices of the American Mathematical Society* 66.7 (2019).

[28] Bernd Sturmfels. *Algorithms in Invariant Theory*. Springer Science & Business Media, 2008.

[29] Nolan R Wallach. *Geometric invariant theory*. Springer, 2017.

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