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Constant Mean Curvature Surfaces and Heun's Differential Equations

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Contents

This is to certify that the work I am submitting is my own and has not been submitted for another degree, either at University College Cork or elsewhere. All external references and sources are clearly acknowledged and identified within the contents. I have read and understood the regulations of University College Cork concerning plagiarism.

Signature ____________________

Eduardo Mota

Cork, June 2019.

“E sapevo anche un’altra cosa. Che se non fossi andato per il mondo, non avrei capito niente della mia storia, della storia della mia parte. E se non si capisce nulla, a cosa servono le mani fini dei privilegiati? Cosa racconti, cosa scrivi con quelle mani se gli occhi non vedono, se il cuore non desidera e non spera, se lo stomaco non conosce la fame e il fegato la rabbia? Come puoi fare il morso del ciuco alla realtà senza le castagne in tasca?”

- *Alberto Prunetti*

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Finally, thanks to my family, they have been always there supporting me, encouraging me in all of my pursuits and inspiring me to follow my dreams.

Dedicated to *mamá y papá*.

Abstract

This thesis is concerned with the problem of constructing surfaces of constant mean curvature with irregular ends by using the class of Heun's Differential Equations. More specifically, we are interested in obtaining immersion of punctured Riemann spheres into three dimensional Euclidean space with constant mean curvature. These immersions can be described by a Weierstrass representation in terms of holomorphic loop Lie algebra valued 1-forms. We describe how to encode each of the differential equations in Heun's family in the Weierstrass representation. Next, we investigate monodromy problems for each of the cases in order to ensure periodicity of all the resulting immersions. This allows us to find four families of surfaces with constant mean curvature and irregular ends. These families can be described as trinoids, cylinders, perturbed Delaunay surfaces and planes. Finally, we study some symmetry properties of these groups of surfaces.

Introduction

The surfaces with non-zero constant mean curvature (CMC) have been of big interest to mathematicians since the middle of the nineteenth century. An immersion is of constant mean curvature if it is critical for the first variation of area under volume constraint. This volume constraint is mathematically equivalent to a pressure differential across the surface, which in this case is the boundary between two regions. Thus, CMC surfaces appear in fluid problems as soap bubbles since they have the curvature corresponding to a nonzero pressure difference, in the shape of the gas-liquid interface on a superhydrophobic surface, in models for block copolymers where the different components have a nonzero interfacial energy or tension, or in architecture as air-supported structures such as inflatable domes and enclosures. Moreover, CMC surfaces are important mathematical models for the physics of interfaces in the absence of gravity, where they separate two different media.

The first class of CMC surfaces was the one of surfaces of revolution, the Delaunay surfaces. They were found almost 200 years ago by Delaunay [12], and are still of interest, since every properly embedded annular end of a CMC surface is asymptotically a Delaunay surface, proved by Korevaar, Kusner and Solomon in [40].
In the mid twentieth century two important results about the global properties of CMC surfaces were found by Hopf and Alexandrov. In 1955 Hopf [24] showed that the only genus-zero closed CMC surfaces in $\mathbb{R}^3$ are the round spheres. Just a few years later, Alexandrov [1] used the maximum principle to show that the only closed CMC surfaces embedded in $\mathbb{R}^3$ are the round

spheres.

In the 1980's this topic regained new interest after the discovery of the first closed, compact CMC surface different from the round sphere found, by Wente in [60]. Then, integrable systems methods were first used to study CMC surfaces by Pinkall and Sterling [44]. Thanks to them, the moduli space of CMC tori was completely described in terms of hyperelliptic Riemann surfaces (see [44] and also the important work of Bobenko [7]).

It is a well-known fact, proved by Ruh and Vilms in [49], that CMC surfaces have harmonic Gaus map into the sphere $\mathbb{S}^2$, that is, their Gauss map is an extremal value for the energy functional. In [16], Dorfmeister, Pedit and Wu showed that such harmonic maps can be obtained as projections of horizontal holomorphic maps from the universal cover of the surface into a certain loop group. The integrability of the moving frame for CMC surfaces is the celebrated Gauss equation, which can be seen as an infinite dimensional integrable system (see the book of Hélein [22] for details on these ideas).

Regarding higher genus CMC surfaces, the works of Kapouleas [26, 27, 28] gave existence results. He used PDEs machinery and gluing techniques. More recently, Traizet has adapted this approach to the loop groups methods used in this thesis. In [59], he uses the opening nodes technique in the underlying Riemann surface to construct CMC surfaces in $\mathbb{R}^3$ with no restriction on the number of ends and with any genus.

The case of simply connected CMC surfaces was studied by Dorfmeister, Pedit and Wu in [16]. They described them in terms of a meromorphic loop Lie algebra valued 1-form, giving a Weierstrass-type representation analogous to the one for minimal surfaces. Their method can be outlined as follows: for a Riemann surface Σ, take a holomorphic 1-form ξ on its universal cover $\widetilde{\Sigma}$ with values in the loop Lie algebra of $\mathfrak{sl}_2(\mathbb{C})$, a point $z_0 \in \widetilde{\Sigma}$ and an element Φ_0 in the loop Lie group of $\mathrm{SL}_2(\mathbb{C})$. To construct an associated family of (possibly branched) conformal immersions $f_\lambda : \widetilde{\Sigma} \times \mathbb{S}^1 \to \mathfrak{su}_2 \cong \mathbb{R}^3$ with constant mean curvature H,

- Solve the initial value problem
$$\begin{cases} \mathrm{d}\Phi = \Phi\xi, \\ \Phi(z_0) = \Phi_0 \end{cases} \tag{IVP}$$
to obtain a map $\Phi : \widetilde{\Sigma} \to \Lambda\mathrm{SL}_2(\mathbb{C})$, that is, a map into the loop group of $\mathrm{SL}_2(\mathbb{C})$.
- Compute (pointwise) the Iwasawa decomposition of $\Phi = F\,B$, which returns a unique map $F : \widetilde{\Sigma} \to \Lambda\mathrm{SU}_2$. This is the extended frame of a Gauss map of a CMC surface.
- Plug F into the Sym-Bobenko formula
$$f_\lambda = \frac{-1}{2H}\left(i\lambda(\partial_\lambda F)\,F^{-1} + F\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} F^{-1} \right)$$
to obtain the associated family of CMC surfaces. For each $\lambda \in \mathbb{S}^1$, a conformal immersion $f : \widetilde{\Sigma} \to \mathbb{R}^3$ with constant mean curvature $H \neq 0$ is obtained.

Extending this method to non-simply connected domains makes use of the fact that holomorphic bundles are trivial over open Riemann surfaces, ensuring that generally for non-compact CMC surfaces there is a holomorphic potential on the underlying Riemann surfaces, not just on its universal cover.

Since the construction of the immersion involves solving a differential equation which has in general monodromy, the difficulty of the construction is solving period problems in order to close the ends. If λ is the loop parameter and M is the monodromy representation, the following conditions ensure that the resulting immersion $f : \Sigma \to \mathbb{R}^3$ lives on the surface for the value $\lambda = 1$:
$$\begin{aligned} M &\text{ takes values in the unitary loop group,} \\ M(\lambda = 1) &= \pm\mathbb{1}, \\ \partial_\lambda M|_{\lambda=1} &= 0 \end{aligned}$$
These conditions can be ensured by properties on the Weierstrass data, in particular in this thesis this amounts to showing that the monodromy representation is pointwise simultaneously unitarisable on the unit circle via conjugation by a dressing. This approach has been extensively used to construct

CMC surfaces with non-trivial fundamental group. We cite among many others the works of Bobenko [8, 10], Dorfmeister and Haak [15], Kilian, McIntosh and Schmitt [33], Schmitt [52], Schmitt, Kilian, Kobayashi and Rossman [54, 32] and Traizet [58]. Different approaches to this one were used to study CMC surfaces in the works of Korevaar, Kusner, Meeks and Solomon [39] and Grosse-Brauckmann, Kusner and Sullivan [17].

In this thesis we deal with the construction of CMC surfaces arising from the family of Heun's Differential Equations. This is done by identifying the 2×2 system in (IVP) with a second order scalar differential equation and choosing that equation to be a member of that family. This process collects the singularities' behaviour of the scalar ODE in the system (IVP). In this way we have holomorphic data on different punctured Riemann spheres (according to the inherited singularities), and we use it to construct trinoids, cylinders, perturbed Delaunay cylinders and planes with constant mean curvature.

The interesting contribution of Heun's Differential Equations is that four of the five ODEs have irregular singularities of some rank. This yields a pole structure in ξ at the punctures which determines the geometry of the ends of the surface. The irregular singularities produce non-embeded ends similar to those in the Smyth surfaces. However, the irregular ends obtained in our surfaces are not asymptotic to Smyth surfaces because in general they have nonvanishing end weights, while the end weights of Smyth surfaces always vanish. Hence, we offer in this thesis a systematic way of producing new CMC surfaces with a different number of irregular ends arising from Heun's Differential Equations.

The structure of this work goes as follows. In chapter 1 we introduce the topic of CMC surfaces and explain the loop groups methods outlined above in order to produce immersions in $\mathbb{R}^3$. We also give an account of the theory regarding Delaunay surfaces, to exemplify the loop groups methods as well as to establish some fundamental results used later. Finally, we explain how to specifically pass from the system of ODEs to a general second order scalar differential equation and prove results relating the theory of solutions in each of the situations.

Next, in chapter 2, we give a brief discussion about the family of differential equations that we will use later on, Heun's Differential Equations. In there we introduce each of the members of the family, and discuss the relevant concepts for us.

Chapter 3 is the central piece of this thesis, where we construct surfaces with irregular ends, discussing for each of the differential equations what is our approach to the unitarisation of the monodromy representation. In most of the cases, some symmetries need to be imposed in order to unitarise, and we discuss these symmetries in chapter 4.

We explain our conclusions and expose some ideas on how this line work could be continued in chapter 5.

Lastly, the appendices A and B gather some basic results regarding Lie groups, hyperbolic geometry and unitarisation that are used without explanation throughout the thesis.

Chapter 1

Preliminaries. The generalised Weierstrass representation

The first chapter of this work intends to give a mathematical introduction to the process of finding CMC surfaces from Weierstrass holomorphic data. For this, we need to present some basic concepts regarding local surface theory and *conformal immersions* along with setting our notation for the rest of the text. We do this in the coming section. In section 1.2 we formulate the *generalised Weierstrass representation* for CMC surfaces by holomorphic loop $\mathfrak{sl}_2(\mathbb{C})$-valued 1-forms. This Weierstrass type representation is due to Dorfmeister, Pedit and Wu [16], and sometimes is called the DPW method. This representation involves solving a system of ODEs for which the solution takes values in a *loop group*. Thereupon the solution must be split in the loop groups using the *Iwasawa decomposition*, obtaining a *unitary factor* and a *positive factor*. The unitary factor is then used to obtain a CMC immersion by use of the *Sym-Bobenko formula* for $\mathbb{R}^3$, which makes use of the correspondence $\mathfrak{su}_2 \cong \mathbb{R}^3$.

The subsequent sections are intended to outline two important group actions, the *dressing action* and the *gauge action*. Then, we carry on with the description of one of the biggest difficulties when using the generalised Weierstrass representation for CMC surfaces: the *monodromy problem*. Necessary and sufficient conditions for the surfaces to close can be given in terms of the monodromy representation, and in order to prove the existence of our surfaces we need to guarantee those conditions. Afterwards, we introduce one of the most basic examples of CMC surfaces, the *Delaunay surfaces* and we use them to exemplify the steps to follow in this construction of CMC surfaces in $\mathbb{R}^3$. Finally, we conclude describing how the step of solving a system of ODEs in this algorithmic process can be thought instead as solving a 2nd-order scalar ODE. This will be hugely exploited in the coming chapters.

1.1 Basics of local surface theory

Let Σ be a 2-dimensional orientable manifold and $f : \Sigma \to \mathbb{R}^3$ an immersion. This means f is a mapping for which the differential df_p is injective for all $p \in \Sigma$. The Euclidean vector space $\mathbb{R}^3$ is endowed with the standard inner product $\langle \cdot, \cdot \rangle$ and the hereby induced norm $|| \cdot ||$. We can pull this metric back to a metric $\langle \cdot, \cdot \rangle_\Sigma$ on Σ by df, that is,

$$\langle v, w \rangle_\Sigma = \langle df(v), df(w) \rangle \tag{1.1.1}$$

for any two vectors v, w for any tangent space $T_p\Sigma$ of Σ. Let $U \subset \Sigma$ be an open set with chart $\varphi : V \to U$ with $V \subset \mathbb{R}^2$, where $\mathbb{R}^2$ carries the standard Euclidean structure.

Since (x, y) is a coordinate for Σ and f is an immersion, a basis for $T_p\Sigma$ can be chosen as

$$f_x = \frac{\partial}{\partial x} (f \circ \varphi)_p \,, \quad f_y = \frac{\partial}{\partial y} (f \circ \varphi)_p \,, \tag{1.1.2}$$

and then the metric ds^2 can be represented as the matrix

$$g = \begin{pmatrix} \langle f_x, f_x \rangle & \langle f_x, f_y \rangle \\ \langle f_y, f_x \rangle & \langle f_y, f_y \rangle \end{pmatrix}. \tag{1.1.3}$$

We can choose the coordinates on Σ so that ds^2 is a conformal metric. This means that the vectors f_x and f_y are orthogonal and of equal length in $\mathbb{R}^3$ at every point $f(p)$. This implies that $\langle f_x, f_y \rangle = \langle f_y, f_x \rangle = 0$ and that there exists some smooth function $u : U \to \mathbb{R}$, the *conformal factor* of the immersion, so that $4e^{2u} = \langle f_x, f_x\rangle = \langle f_y, f_y\rangle$. Then, the induced metric has the form $ds^2 = 4e^{2u}(dx^2 + dy^2)$.

Now we come to the extrinsic invariants defined for an immersed surface. We can define a unit normal vector to the surface on each coordinate chart by taking the cross product of f_x and f_y and scaling it to have length equal to one:

$$N = \frac{f_x \times f_y}{||f_x \times f_y||}. \tag{1.1.4}$$

This vector is uniquely determined up to sign, which is determined by the orientation of the coordinate chart. We can now define the second fundamental form of f which can be written using symmetric 2-differentials as

$$\mathrm{I\!I} = L\, dx^2 + 2M\, dx\, dy + N\, dy^2. \tag{1.1.5}$$

Its matrix representation in the basis f_x, f_y of $T_p\Sigma$ is

$$h = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} \langle N, f_{xx}\rangle & \langle N, f_{xy}\rangle \\ \langle N, f_{yx}\rangle & \langle N, f_{yy}\rangle \end{pmatrix}. \tag{1.1.6}$$

Since a conformal immersion induces a conformal structure on the surface, Σ can be viewed as a Riemann surface. Let $z = x + iy$ be a holomorphic coordinate and $\langle\cdot,\cdot\rangle$ denote the the bilinear extension of the standard inner product on $\mathbb{R}^3$ to $\mathbb{C}^3$ with induced norm $||\cdot||$ and partial derivatives given by $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$. Then, conformality is given by

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0, \quad \langle f_z, f_{\bar z}\rangle = 2e^{2u}. \tag{1.1.7}$$

The metric is now $ds^2 = 4e^{2u} dz\, d\bar z$. Thus, the second fundamental form can be written as

$$\mathrm{I\!I} = Q\, dz^2 + \widetilde{H}\, dz\, d\bar z + \overline{Q}\, d\bar z^2, \tag{1.1.8}$$

where Q is the complex-valued function

$$Q := \frac{1}{4}(L - N - 2iM), \tag{1.1.9}$$

and $\widetilde{H}$ is the real-valued function

$$\widetilde{H} := \frac{1}{2}(L + N). \tag{1.1.10}$$

The symmetric quadratic differential $Q\,dz^2$ is called the *Hopf differential* of the immersion f. The Hopf differential is globally defined and together with the metric determines f up to rigid motions.

The linear map $S : T_p\Sigma \to T_p\Sigma$ defined by $S := g^{-1}h$ is called the *shape operator* and it maps a vector $v \in T_p\Sigma$ to the vector $-D_vN \in T_p\Sigma$, where D is the directional derivative of $\mathbb{R}^3$. The eigenvalues k_1, k_2 and the corresponding eigenvectors of S are the *principal curvatures* and the *principal curvature directions* of the surface $f(\Sigma)$ at the corresponding point. The half-trace of S, denoted by

$$H = \frac{1}{2}\,\mathrm{tr}(g^{-1}h) = \frac{\widetilde{H}}{4e^{2u}} = \frac{1}{8e^{2u}}\langle f_{xx} + f_{yy}, N\rangle = \frac{1}{2}(k_1 + k_2), \tag{1.1.11}$$

is the *mean curvature* of the surface at the corresponding point. The immersion f is CMC if H is constant. From equation (1.1.11) one concludes that $\langle f_{z\bar{z}}, N\rangle = 2He^{2u}$ and from equation (1.1.9) we have that $Q = \langle N, f_{zz}\rangle$.

A point $p \in \Sigma$ is called *umbilic* if the two principal curvatures coincide $k_1 = k_2$. Umbilic points are precisely the zeroes of the Hopf differential.

Lemma 1.1. *Let Σ be a Riemann surface and $f : \Sigma \to \mathbb{R}^3$ be a conformal immersion. Then a point $p \in \Sigma$ is umbilic if and only if $Q = 0$.*

Proof. The shape operator of f with respect to the basis f_x, f_y is given by

$$S = g^{-1}h = \frac{1}{4e^{2u}}\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{1}{4e^{2u}}\begin{pmatrix} \widetilde{H} + Q + \overline{Q} & i(Q - \overline{Q}) \\ i(Q - \overline{Q}) & \widetilde{H} - Q - \overline{Q} \end{pmatrix}. \tag{1.1.12}$$

The two principal curvatures are the eigenvalues of S and hence are solutions of equating to zero the expression

$$\begin{aligned} \det(S - k\mathbb{1}) &= \frac{1}{4e^{2u}}\left((\widetilde{H} + Q + \overline{Q} - k)(\widetilde{H} - Q - \overline{Q} - k) + (Q - \overline{Q})^2\right) \\ &= \frac{1}{4e^{2u}}\left((\widetilde{H} - k)^2 - (Q + \overline{Q})^2 + (Q - \overline{Q})^2\right) \\ &= \frac{1}{4e^{2u}}\left(\widetilde{H}^2 - 2\widetilde{H}k + k^2\right) - \frac{1}{e^{2u}}|Q|^2 \\ &= 4e^{2u}H^2 - 2Hk + \frac{1}{4e^{2u}}k^2 - \frac{1}{e^{2u}}|Q|^2. \end{aligned} \tag{1.1.13}$$

Thus we obtain

$$k_1 = 2(2e^{2u}H - |Q|), \quad k_2 = 2(2e^{2u}H + |Q|). \tag{1.1.14}$$

Therefore, $k_1 = k_2$ if and only if $Q = 0$. □

The *fundamental theorem of surface theory*, also called Bonnet theorem, states that the first and second fundamental forms determine a surface in $\mathbb{R}^3$ uniquely up to a rigid motion. More in detail, there exists an immersion $f : U \to \mathbb{R}^3$ with first fundamental form I and second fundamental form II with respect to the chosen coordinates if and only if I and II satisfy the *Gauss-Codazzi equations*

$$4u_{z\bar{z}} - |Q|^2 e^{-2u} + 4H^2 e^{2u} = 0, \quad Q_{\bar{z}} = 2e^{2u} H_z. \tag{1.1.15}$$

Note that (1.1.15) are the Gauss and Codazzi equations for the ambient space $\mathbb{R}^3$.

We define $e_1 = e^{-u}(f_z + f_{\bar{z}})/2$ and $e_2 = ie^{-u}(f_z - f_{\bar{z}})/2$ so that the Gauss map N induces a special orthogonal frame $\mathcal{F} : U \to \mathrm{SO}_3$

$$\mathcal{F} = (e_1, e_2, N) \tag{1.1.16}$$

of the surface. Then, $\mathcal{F}^{-1} d\mathcal{F} = \mathcal{F}\mathcal{U} dz + \mathcal{F}\mathcal{V} d\bar{z}$ with $\mathcal{U}, \mathcal{V} : U \to \mathfrak{so}_3$ and the *integrability condition* is $d^2\mathcal{F} = 0$, in other words,

$$\mathcal{V}_z - \mathcal{U}_{\bar{z}} + [\mathcal{U}, \mathcal{V}] = 0. \tag{1.1.17}$$

Rather than working with 3×3 matrices it is more convenient to use the spinor representation identifying $\mathbb{R}^3 \cong \mathfrak{su}_2$ and lifting SO_3 to its double cover SU_2 (see section A.4 for more details on these arguments). We complexify $\mathfrak{su}_2^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ and fix a basis of $\mathfrak{sl}_2(\mathbb{C})$ as

$$\epsilon = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \ \epsilon_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \ \text{ and } \ \epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{1.1.18}$$

for which the following relations hold

$$\begin{gathered} \langle \epsilon_+, \epsilon_+ \rangle = \langle \epsilon_-, \epsilon_- \rangle = 0, \quad \epsilon_-^* = -\epsilon_+, \\ [\epsilon, \epsilon_-] = 2i\epsilon_-, \quad [\epsilon_+, \epsilon] = 2i\epsilon_+, \quad [\epsilon_-, \epsilon_+] = i\epsilon. \end{gathered} \tag{1.1.19}$$

Let us state the fundamental result in this discussion due to Lax [41], for the particular case of unitary frames of dimension 2.

Proposition 1.1. *Let $U \subset \mathbb{C}$ be an open set containing the origin. For $A, B : U \to \mathfrak{sl}_2(\mathbb{C})$, there exists a solution $F = F(z, \bar{z}) : U \to \mathrm{SU}_2(\mathbb{C})$ of the Lax pair*

$$F_z = FA, \quad F_{\bar{z}} = FB \tag{1.1.20}$$

for some initial condition $F(0,0) \in \mathrm{SU}_2$ if and only if $d^2F = 0$, that is, if and only if

$$B_z - A_{\bar{z}} + [A, B] = 0. \tag{1.1.21}$$

The unitary frame $F : U \to \mathrm{SU}_2$ of the surface can be chosen so that

$$f_z = ie^u F\epsilon_- F^{-1}, \quad f_{\bar{z}} = -ie^u F\epsilon_+ F^{-1}, \quad \text{and } N = \frac{1}{2}F\epsilon F^{-1}. \tag{1.1.22}$$

The differential of F can be then written as $dF = F_z dz + F_{\bar{z}} d\bar{z} = FAdz + FBd\bar{z}$ where $A, B : U \to \mathfrak{sl}_2(\mathbb{C})$ work out to be

$$A = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ -2He^u & u_z \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & 2He^u \\ -e^{-u}\overline{Q} & -u_{\bar{z}} \end{pmatrix}. \tag{1.1.23}$$

Note that $A = -\overline{B}^t$. The integrability condition (1.1.21) for the frame splits into a diagonal condition and an off-diagonal condition giving respectively the Gauss and the Codazzi equations (1.1.15).

Identifying $\mathbb{S}^2$ with $\mathbb{CP}^1$ one can see $\mathbb{S}^2 \simeq \mathrm{SU}_2/\mathrm{U}_1$ as a homogeneous space, with U_1 the group of 1×1 complex-valued unitary matrices. Fix a base point $z_0 \in U$ and consider $\varphi : U \to \mathbb{S}^2$ with $\varphi(z_0) = [\mathbb{1}]$. Let $\pi : \mathrm{SU}_2 \to \mathbb{S}^2$ be the canonical projection. Then for any such map there exists a lift $F : U \to \mathrm{SU}_2$ such that $F(z_0) = \mathbb{1}$ with $\varphi = \pi \circ F$. The map F is called the *frame* of φ. The *Maurer-Cartan form* $\alpha := F^{-1}dF$ of F is a 1-form over U taking values in $\mathfrak{su}_2$, and its integrability condition $d^2F = 0$ is called the *Maurer-Cartan equation* and reads

$$d\alpha + \alpha \wedge \alpha = 0. \tag{1.1.24}$$

From equations (1.1.15) three important ideas can be concluded for immersions with CMC :

a) If $H = 1/2$ and $Q = 1$ (note that, away from umbilics, one can choose local coordinates so that this is true), then the Gauss equation in (1.1.15) becomes the *sinh-Gordon* equation

$$2u_{z\bar{z}} + \sinh 2u = 0. \tag{1.1.25}$$

Thus, away from umbilic points of our CMC surface, the Gauss equation can be locally viewed as the *sinh-Gordon* equation and then one can use integrable systems methods (see the works [3, 10, 8, 7]).

b) The Gauss-Codazzi equations are invariant under $Q \mapsto \lambda^{-2}Q$ for $\lambda \in \mathbb{S}^1$. Hence to every solution of equations (1.1.15), the Bonnet theorem assures that there is an $\mathbb{S}^1$-parameter family of CMC surfaces, the so called *associated family.* For $\lambda \in \mathbb{C}^*$, consider the loop of frames F_λ. The map F_λ satisfies that
   - for each $z \in U$, $\lambda \mapsto F_\lambda(z)$ is holomorphic on $\mathbb{C}^*$.
   - $F|_{\lambda=1} \equiv F$ frames a map $\varphi : U \to \mathbb{S}^2$.
   - $\overline{F_{1/\bar{\lambda}}}^t = F_\lambda^{-1}$. In particular, F is SU_2-valued for $\lambda \in \mathbb{S}^1$.

   The differential of this loop of frames is $dF_\lambda = F_\lambda A_\lambda dz + F_\lambda B_\lambda d\bar{z}$ with

$$\begin{aligned} A_\lambda &= \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}\lambda^{-2}Q \\ -2He^u & u_z \end{pmatrix}, \\ B_\lambda &= \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & 2He^u \\ -e^{-u}\lambda^2\overline{Q} & -u_{\bar{z}} \end{pmatrix}. \end{aligned} \tag{1.1.26}$$

   A map F_λ as above is called *extended* or *unitary* frame if its Maurer-Cartan form α_λ satisfies the Maurer-Cartan equation $d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0$ for all $\lambda \in \mathbb{S}^1$. Using $\mathfrak{su}_2^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ we can split $\mathfrak{sl}_2(\mathbb{C})$-valued 1-forms into the $(1,0)$ part α' and the $(0,1)$ part α'' writing $\alpha = \alpha' + \alpha''$. Thus, for α_λ we have that

$$\alpha_\lambda = \alpha'_\lambda + \alpha''_\lambda = (\alpha'_1 + \lambda^2\alpha''_1)\epsilon_- + (\lambda^{-2}\alpha'_2 + \alpha''_2)\epsilon_+ + (\alpha'_3 + \alpha''_3)\epsilon, \tag{1.1.27}$$

   where none of the α'_j, α''_j depends on λ. This decomposition will be used later on.

c) Since $H_z = 0$, the Codazzi equation implies that

$$Q_{\bar{z}} = 0, \tag{1.1.28}$$

that is, that the function Q in the Hopf differential is holomorphic and hence locally it is of the form $Q = az^k$ with a non-negative integer $k \geq 0$ and a complex number $a \in \mathbb{C}$. By a theorem of Ruh and Vilms [49] f is CMC if and only if its Gauss map is *harmonic*.

Putting together a), b) and c), one can prove the following

Theorem 1.1. *Let $f : \Sigma \to \mathbb{R}^3$ be a conformal immersion with metric $ds^2 = 4e^{2u}dzd\bar{z}$, Hopf differential Q, mean curvature H and Gauss map $N : \Sigma \to \mathbb{S}^2$. Then, the following are equivalent:*

(1) H is constant.

(2) Q is holomorphic.

(3) N is harmonic.

(4) There exists an unitary frame F_λ, and its Maurer-Cartan form $\alpha_\lambda = F_\lambda^{-1}dF_\lambda$ satisfies the Maurer-Cartan equation $d\alpha_\lambda + \alpha_\lambda \wedge \alpha_\lambda = 0$ for all $\lambda \in \mathbb{S}^1$.

If $H = 1/2$ and $Q = 1$ then the function u solves the sinh-Gordon *equation*

$$2u_{z\bar{z}} + \sinh 2u = 0. \tag{1.1.29}$$

We have already explained why (1) and (2) are equivalent. That (1) and (3) are equivalent is known as a theorem of Ruh and Vilms [49]. The integrability condition of F_λ is necessary and sufficient for the map $F = F_1$ to be a lift of a harmonic Gauss map $N : U \to \mathbb{S}^2$ of a CMC surface, due to a result of Pohlmeyer [45], proving that (3) and (4) are equivalent. Let us also note that F can be viewed either as the unitary frame of N or the unitary frame of the immersion f, depending on whether one's interest is in harmonic maps or in the CMC immersion. Our objective is to construct from holomorphic data with non-trivial topology unitary frames of harmonic maps that are the Gauss map of a CMC surface.

1.2 Loop groups methods for CMC surfaces

1.2.1 Loop groups

Next we introduce various loop groups and state the Iwasawa and Birkhoff decompositions for this setup. A loop is a smooth map from the unit circle $\mathbb{S}^1$ to a matrix group. The circle variable is denoted λ and called the *spectral parameter*.

For each $0 < r \leq 1$, the circle, open disk (interior) and open annulus are denoted respectively by

$$\begin{aligned} C_r &= \{\lambda \in \mathbb{C} : |\lambda| = r\}, \\ D_r &= \{\lambda \in \mathbb{C} : |\lambda| < r\}, \\ A_r &= \{\lambda \in \mathbb{C} : r < |\lambda| < 1/r\}. \end{aligned} \tag{1.2.1}$$

For $r = 1$, we will omit the subscript. On A_r given a map $F : A_r \to \mathrm{SL}_2(\mathbb{C})$ there is the map $F^* : A_r \to \mathrm{SL}_2(\mathbb{C})$ defined by

$$F^* : \lambda \mapsto \overline{F(1/\bar{\lambda})}^t. \tag{1.2.2}$$

Below we define the (untwisted) loop groups and algebras that we need:

- $\Lambda_r\mathrm{SL}_2(\mathbb{C})$ is the set $\mathcal{C}^\infty(C_r, \mathrm{SL}_2(\mathbb{C}))$, that is, smooth maps $\Phi : C_r \to \mathrm{SL}_2(\mathbb{C})$.
- The Lie algebras of these groups are $\Lambda_r\mathfrak{sl}_2(\mathbb{C}) = \mathcal{C}^\infty(C_r, \mathfrak{sl}_2(\mathbb{C}))$.
- $\Lambda_r^+\mathrm{SL}_2(\mathbb{C}) \subset \Lambda_r\mathrm{SL}_2(\mathbb{C})$ is the set of smooth maps $B \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ which extend analytically to maps $B : D_r \to \mathrm{SL}_2(\mathbb{C})$.
- $\Lambda_r^-\mathrm{SL}_2(\mathbb{C}) \subset \Lambda_r\mathrm{SL}_2(\mathbb{C})$ is the set of smooth maps $B \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ which extend analytically to maps $B : \widehat{\mathbb{C}}\setminus\overline{D_r} \to \mathrm{SL}_2(\mathbb{C})$ and such that $B(\infty) = \mathbb{1}$.
- $\Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) \subset \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$ is the set of smooth maps $B \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ which extend analytically to maps $B : D_r \to \mathrm{SL}_2(\mathbb{C})$ and such that $B(0)$ is upper triangular with positive real elements on the diagonal. We call these *r-positive* loops.

- $\Lambda_r^*\mathrm{SL}_2(\mathbb{C}) \subset \Lambda_r\mathrm{SL}_2(\mathbb{C})$ is the set of smooth maps $F \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ that can be analytically extended to maps $F : A_r \to \mathrm{SL}_2(\mathbb{C})$ and satisfy the *reality condition*
$$F^* = F^{-1}. \tag{1.2.3}$$
Note that since these are maps into some subgroup of $\mathrm{SL}_2(\mathbb{C})$ for each $\lambda \in \mathbb{S}^1$, they have pointwise inverses. Also, note that $F \in \Lambda_r^*\mathrm{SL}_2\mathbb{C}$ implies $F|_{\mathbb{S}^1} \in \mathrm{SU}_2$. We call these r-*unitary* loops.

Sometimes the loop group $\Lambda_r^*\mathrm{SL}_2$ is equivalently defined with the notation $\Lambda_r\mathrm{SU}_2 \subset \Lambda_r\mathrm{SL}_2(\mathbb{C})$ being the set $\mathcal{C}^\infty(C_r, \mathrm{SU}_2)$, that is, smooth maps $F : C_r \to \mathrm{SU}_2$. This notation will be used later on in this work for the sake of simplicity. The loop group $\Lambda\mathrm{SL}_2(\mathbb{C})$ is the infinite-dimensional analogues of $\mathrm{SL}_2(\mathbb{C})$ and is the object to be split by Iwasawa decomposition, analogous to the QR-decomposition of $\mathrm{SL}_2(\mathbb{C})$ explained in lemma A.3 of the appendix. On the other hand, $\Lambda^*\mathrm{SL}_2(\mathbb{C})$ and $\Lambda^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ are the infinite-dimensional analogue of SU_2 and $\mathcal{B}$ that appear in the finite-dimensional splitting. Now we come to the splitting theorems.

Theorem 1.2 (Iwasawa decomposition)**.** *The multiplication* $\Lambda_r^*\mathrm{SL}_2(\mathbb{C}) \times \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ *is a real-analytic diffeomorphic map onto. We call Iwasawa (or r-Iwasawa) decomposition to the unique splitting of an element* $\Phi \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$
$$\Phi = FB, \tag{1.2.4}$$
where $F \in \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ *is called* r-unitary *part of* Φ *and* $B \in \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2\mathbb{C}$ *is called* positive *part of* Φ*.*

Note that since $\mathrm{SU}_2 \cap \mathcal{B} = \{\mathbb{1}\}$ in the finite-dimensional splitting in lemma A.3, also $\Lambda_r^*\mathrm{SL}_2(\mathbb{C}) \cap \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) = \{\mathbb{1}\}$. The condition on $\Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ of having positive real elements on their diagonals is a choice to ensure uniqueness of this factorization. Proofs of theorem 1.2 can be found in the book of Pressley and Segal [46] and in the work of McIntosh [42]. It is also important to say that, in general, this splitting is not explicit.

Theorem 1.3 (Birkhoff decomposition, [6, 5])**.** *The multiplication map* $\Lambda^-\mathrm{SL}_2(\mathbb{C}) \times \Lambda^+\mathrm{SL}_2(\mathbb{C}) \to \mathcal{BC}$ *is a complex-analytic diffeomorphism onto an* *open dense subset, called the* big cell, *of* $\Lambda\mathrm{SL}_2(\mathbb{C})$. *The unique splitting of an element* $\Phi \in \mathcal{BC}$

$$\Phi = B_- B_+, \tag{1.2.5}$$

where $B_- \in \Lambda^-\mathrm{SL}_2(\mathbb{C})$ *and* $B_+ \in \Lambda^+\mathrm{SL}_2(\mathbb{C})$ *is called the Birkhoff decomposition.*

Let us point out that when Φ depends complex-analytically on z, it does not follow that F and B in (1.2.4) will do so too. However, the two elements B_- and B_+ in (1.2.5) will depend complex-analytically on z when Φ does so. On the other hand, the Iwasawa decomposition can be performed on any $\Phi \in \Lambda\mathrm{SL}_2(\mathbb{C})$ whereas Birkhoff is only possible for those Φ lying in $\mathcal{BC}$. Similar arguments are exposed for the analogue finite-dimensional splittings in section A.4.

1.2.2 Holomorphic potentials and unitary frames

The generalised Weierstrass representation for producing CMC surfaces works with the so-called *holomorphic potentials*. Consider $\xi = \xi(z, \lambda)$ a $\mathfrak{sl}_2(\mathbb{C})$-valued holomorphic 1-form on Σ, depending on the spectral parameter $\lambda \in \mathbb{S}^1$. A holomorphic potential ξ can be written in terms of a local coordinate z on Σ as

$$\xi(z, \lambda) = \sum_{j=-1}^{\infty} \xi_j(z)\lambda^j \, dz, \tag{1.2.6}$$

where each $\xi_j(z) \in \mathfrak{sl}_2(\mathbb{C})$ depends holomorphically on z and the elements of $\xi_{-1}(z)$ are all zero except that one in the upper right position.

Recall from equation (1.1.27) that replacing $Q \mapsto \lambda^{-2}Q$ in the Maurer-Cartan form $\alpha = F^{-1}dF$ gives a $\Lambda\mathfrak{sl}_2(\mathbb{C})$-valued 1-form with decomposition

$$\alpha_\lambda = (\alpha_1' + \lambda^2\alpha_1'')\epsilon_- + (\lambda^{-2}\alpha_2' + \alpha_2'')\epsilon_+ + (\alpha_3' + \alpha_3'')\epsilon. \tag{1.2.7}$$

The smooth maps $F_\lambda : \Sigma \to \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ for which $\alpha_\lambda = F_\lambda^{-1}dF_\lambda$ is of the form (1.2.7) will be called *r-unitary frames*. Let us recall in lemma 1.2 the method of Dorfmeister, Pedit and Wu [16] to generate r-unitary frames. Define

$$\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C}) = \left\{ \xi \in \Lambda_r \mathfrak{sl}_2(\mathbb{C}) : \xi = \sum_{j \geq -1} \xi_j \lambda^j,\ \xi_{-1} \in \mathbb{C} \otimes \epsilon_+ \right\} \tag{1.2.8}$$

and denote the holomorphic 1-forms on Σ with values in $\Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C})$ by

$$\Lambda_r \Omega(\Sigma) = \Omega^1 \left(\Sigma, \Lambda_r^{-1}\mathfrak{sl}_2(\mathbb{C}) \right). \tag{1.2.9}$$

Lemma 1.2 ([16])**.** *Let Σ be a simply connected Riemann surface, $\xi \in \Lambda_r\Omega(\Sigma)$ and Φ be the solution of $\mathrm{d}\Phi = \Phi\xi$ with initial condition $\Phi_0 \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$ at $z_0 \in \Sigma$. Then the r-unitary part of Φ obtained by r-Iwasawa decomposing $\Phi = FB$ pointwise on Σ is an r-unitary frame.*

1.2.3 The Sym-Bobenko formula

Given an r-unitary frame, an immersion can be obtained by formulas first found by Sym [57] for pseudo-spherical surfaces in $\mathbb{R}^3$ and extended by Bobenko [8] for CMC immersions in the three space forms. Our formula differs from this, since we work in untwisted loop groups. Denote $\partial_\lambda = \partial/\partial\lambda$, and let f be the CMC immersion from section 1.1 with u and Q satisfying the Gauss-Codazzi equations (1.1.15) and A_λ, B_λ as in (1.1.26).

Theorem 1.4. *Let Σ be a simply connected Riemann surface and $H \in \mathbb{R}^*$. Consider the* CMC *immersion f with u and Q satisfying the Gauss-Codazzi equations and F an r-unitary frame for some $r \in (0, 1]$ with A_λ, B_λ its Lax pair. The map $\hat{f}_\lambda$ defined by*

$$\hat{f}_\lambda = \frac{-1}{2H} \left(i\lambda (\partial_\lambda F)\, F^{-1} + F\, \epsilon\, F^{-1} \right) \tag{1.2.10}$$

is a (possibly branched) conformal immersion with metric $4e^{2u}(dx^2 + dy^2)$ and Hopf differential $\lambda^{-2}Q$ that differs from f only by a rigid motion. For each $\lambda \in \mathbb{S}^1$, $\hat{f}_\lambda : \Sigma \to \mathfrak{su}_2 \cong \mathbb{R}^3$ has constant mean curvature H.

Proof. Since F is an r-unitary frame for some $r \in (0, 1]$ its Maurer-Cartan form $\alpha_\lambda = F^{-1}dF$ satisfies (1.1.27). Denoting $(1, 0)$ parts and $(0, 1)$ parts with $'$ and $''$ respectively, and recalling the relations in (1.1.19) we compute

$$
\begin{aligned}
\hat{f}_z = d\hat{f}' &= \tfrac{-1}{2H}\left(i\lambda(\partial_\lambda dF)\,F^{-1} - i\lambda(\partial_\lambda F)\,F^{-1}\,dF\,F^{-1} + dF\,\epsilon\,F^{-1} - F\,\epsilon\,F^{-1}\,dF\,F^{-1}\right)\\
&= \tfrac{-1}{2H}\left(i\lambda(\partial_\lambda F\,\alpha'_\lambda)\,F^{-1} - i\lambda(\partial_\lambda F)\,\alpha'_\lambda\,F^{-1} + F\,\alpha'_\lambda\,\epsilon\,F^{-1} - F\,\epsilon\,\alpha'_\lambda\,F^{-1}\right)\\
&= \tfrac{-1}{2H}\left(i\lambda(\partial_\lambda F)\,\alpha'_\lambda\,F^{-1} + i\lambda\,F(\partial_\lambda\alpha'_\lambda)\,F^{-1} - i\lambda(\partial_\lambda F)\,\alpha'_\lambda\,F^{-1} + F\,[\alpha'_\lambda,\epsilon]\,F^{-1}\right)\\
&= \tfrac{-1}{2H}\left(i\lambda\,F(\partial_\lambda\alpha'_\lambda)\,F^{-1} + F\,[\alpha'_\lambda,\epsilon]\,F^{-1}\right)\\
&= \tfrac{-1}{2H}\left(-2i\lambda^{-2}\,F\,\alpha'_2\,\epsilon_+\,F^{-1} - 2iF\,\alpha'_1\,\epsilon_-\,F^{-1} + 2i\lambda^{-2}\,F\,\alpha'_2\,\epsilon_+\,F^{-1}\right)\\
&= \frac{1}{H}iF\,\alpha'_1\,\epsilon_-\,F^{-1} \qquad (1.2.11)\\
&= \frac{1}{H}i(e^uH)F\,\epsilon_-\,F^{-1}\\
&= ie^uF\,\epsilon_-\,F^{-1}\\
&= f_z.
\end{aligned}
$$

Analogously one can check that $\hat{f}_{\bar{z}} = f_{\bar{z}}$ proving that $\hat{f}$ and f are the same surfaces up to rigid motion. Thus, we can use the identities for f in section 1.1 for $\hat{f}$.

Therefore

$$\hat{f}_x = e^uF\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}F^{-1}, \quad \hat{f}_y = e^uF\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}F^{-1} \qquad (1.2.12)$$

since $\hat{f}_x = \hat{f}_z + \hat{f}_{\bar{z}}$ and $\hat{f}_y = i(\hat{f}_z - \hat{f}_{\bar{z}})$. Then $\widehat{N} = \frac{\hat{f}_x\times\hat{f}_y}{||\hat{f}_x\times\hat{f}_y||} = \frac{1}{2}F\epsilon F^{-1}$. Now using the metric of $\mathbb{R}^3$ in the $\mathfrak{su}_2$ matrix formulation (see section A.4) we obtain

$$\langle\hat{f}_x,\hat{f}_x\rangle = \langle\hat{f}_y,\hat{f}_y\rangle = 0, \quad \langle\hat{f}_x,\hat{f}_y\rangle = 4e^{2u}. \qquad (1.2.13)$$

which gives the conformality for the metric $4e^{2u}(dx^2+dy^2)$. Let $\widehat{Q}$ and $\widehat{H}$ be the Hopf differential and the mean curvature of $\hat{f}$ respectively. Then

$$\widehat{Q} = \langle\hat{f}_{zz},\widehat{N}\rangle = ie^u\,\mathrm{tr}\,[(-u_z\epsilon_- + [\epsilon_-,A_\lambda])\,\epsilon] = \lambda^{-2}Q, \qquad (1.2.14)$$

using the form of A_λ, so $\lambda^{-2}Q$ is the Hopf differential of $\hat{f}$. Finally, using the form of B_λ,

$$\widehat{H} = \frac{1}{2e^{2u}}\langle\hat{f}_{z\bar{z}},\widehat{N}\rangle = ie^{-u}\,\mathrm{tr}\,[(-u_{\bar{z}}\epsilon_- + [\epsilon_-,B_\lambda])\,i\epsilon] = H, \qquad (1.2.15)$$

the mean curvature of f, completing the proof. □

Theorem 1.4 shows how to retrieve the CMC immersion f. Let us remark the following:

- Using different spectral parameters λ in the Sym-Bobenko formula (1.2.10) yield different associated surfaces with the same metric and mean curvature but different Hopf differentials.
- Given the frame F we have the means to find f_z and $f_{\bar{z}}$, so one would expect to need integration in order to know f. Formula (1.2.10) avoids integration using the derivative with respect to λ of F instead.
- When the domain is not simply connected, it is not clear whether the surface f will be well-defined on that domain, as the frame F might not be well-defined there either.

1.2.4 The generalised Weierstrass representation

Summarising the above, by combining lemma 1.2 and theorem 1.4, a conformal CMC immersion from a simply connected Riemann surface Σ can be constructed. The original construction in the work of Dorfmeister, Pedit and Wu [16] was restricted to simply connected domains, but Dorfmeister and Haak in [14] managed to generalised it for arbitrary domains. Let ξ be an holomorphic potential on Σ, z_0 a base point in Σ and Φ_0 an initial condition in $\Lambda_r\mathrm{SL}_2(\mathbb{C})$. Given this data, the construction comprises the following three steps: let $\widetilde{\Sigma}$ be the universal cover of Σ and $\tilde{z}_0 \in \widetilde{\Sigma}$ be an arbitrary element in the fiber of z_0.

Step 1 Solve on $\widetilde{\Sigma}$ the initial value problem with parameter $\lambda \in C_r$

$$\begin{cases} \mathrm{d}\Phi = \Phi\xi, \\ \Phi(z_0) = \Phi_0. \end{cases} \tag{1.2.16}$$

to obtain a solution $\Phi : \widetilde{\Sigma} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$, frequently called the *holomorphic frame* of the surface. Note that in general Φ is only defined on the universal cover $\widetilde{\Sigma}$ of Σ.

Step 2 Compute for each $z \in \widetilde{\Sigma}$, that is, pointwise, the r-Iwasawa decomposition of $\Phi = F\,B$, which returns a unique r-unitary frame F. Both F

and B are real-analytic in z.

Step 3 Plug F into the Sym-Bobenko formula of equation (1.2.10) to obtain the associated family of CMC surfaces. Unless otherwise stated, the induced CMC surface will be the one obtained for $\lambda = 1$.

The theory of these methods (see [54, 16]) states that every conformal immersion in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ can be locally obtained in this way, but note that in general Step 2 is not explicit.

An element $\xi \in \Lambda_r\Omega(\Sigma)$ has an expansion of the form

$$\xi = (\beta + \mathrm{O}(\lambda))\epsilon_- + (\lambda^{-1}\alpha + \mathrm{O}(1))\epsilon_+ + \mathrm{O}(1)\epsilon. \tag{1.2.17}$$

with $\alpha, \beta \in \Omega^1(\Sigma, \mathbb{C})$. The metric of f is a non-vanishing multiple of $|\alpha|^2$ and its Hopf differential is a multiple of $\alpha\beta$. To avoid branch points in our construction, we choose a closed form $\alpha \in \Omega^1(\Sigma, \mathbb{C}^*)$.

We thus have a map $(\xi, \Phi_0, \tilde{z}_0) \mapsto f_\lambda$ and the triple $(\xi, \Phi_0, \tilde{z}_0)$ is called the *Weierstrass data* of a CMC immersion. The correspondence between the Weierstrass data and its resulting immersion is still not well understood, since both the integration in Step 1 and the subsequent Iwasawa decomposition are generally not explicit.

1.2.5 Dressing action

The *dressing action* of $\Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ on $\Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ is the composition of left multiplication and r-Iwasawa decomposition. For a loop $h \in \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ and a solution $\Phi \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{d}\Phi = \Phi\xi$, we define the *dressing* as the multiplication on the left of Φ by h,

$$\widehat{\Phi} := h(\lambda)\,\Phi, \tag{1.2.18}$$

It is clear that $\widehat{\Phi}$ also satisfies $\mathrm{d}\widehat{\Phi} = \widehat{\Phi}\xi$, since

$$\xi = \widehat{\Phi}^{-1}\mathrm{d}\widehat{\Phi} = (h\Phi)^{-1}\,\mathrm{d}(h\Phi) = \Phi^{-1}\,\mathrm{d}\Phi, \tag{1.2.19}$$

and moreover dressing does not change the potential ξ.

By a result due to Burstall and Pedit [11, Proposition 2.9], the dressing action descends to the set of r-unitary frames on Σ, and in this context is the variation of the initial condition in equation (1.2.16) by left multiplication of $\Lambda_r\mathrm{SL}_2(\mathbb{C})$, that is, dressing is equivalent to a modification of the initial condition Φ_0.

Consider the r-Iwasawa decompositions of $\Phi = F\,B$ and $\widehat{\Phi} = \widehat{F}\,\widehat{B}$. To see how the surface f is changed by h one can write

$$\widehat{F}\,\widehat{B} = \widehat{\Phi} = h\Phi = hF\,B = (F_h\,B_h)\,B, \tag{1.2.20}$$

where F_h and B_h are the elements of the Iwasawa decomposition of hF. It is not trivial to understand the change in the frame F to F_h, hence the change in the surface f is also not trivial. The method of dressing has been used to produce bubbletons (one can see the works of Kilian, Schmitt, Sterling and Kobayashi [31, 37, 38]) from cylinders and Delaunay surfaces (see section 1.3). Suppose that we consider the particular case of $h \in \Lambda\mathrm{SU}_2$. Then $\widehat{F} = hF$ and the resulting surface in the Sym-Bobenko formula (1.2.10) evaluated at $\lambda = 1$

$$\hat{f} = h(1)\,f\,h^{-1}(1) - i(\partial_\lambda h(\lambda))|_{\lambda=1}h^{-1}(1) \tag{1.2.21}$$

is a rigid motion of f in $\mathbb{R}^3$ which can be written explicitly in terms of the entries of $h(\lambda)$.

1.2.6 Gauge action

Consider the elements of the group

$$\mathcal{G}_r(\Sigma) = \{g : \Sigma \to \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) \text{ holomorphic}\}, \tag{1.2.22}$$

that is, holomorphic maps g on $z \in \Sigma$ such that $g(z,\cdot) \in \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$. Such maps are called *gauges* and $\mathcal{G}_r(\Sigma)$ is the *gauge group*.

If we define $\Psi := \Phi g$, then Φ and Ψ lead to the same immersion f. This operation is called *gaugeing* and unlike dressing it changes the potential since it depends on z. If Ψ is a solution of $\mathrm{d}\Psi = \Psi\eta$, then

$$\eta = \Psi^{-1}\mathrm{d}\Psi = (\Phi g)^{-1}\mathrm{d}(\Phi g) = g^{-1}\xi g + g^{-1}dg. \tag{1.2.23}$$

Therefore, on the potential level one defines the gauge action $\Lambda_r\Omega(\Sigma) \times \mathcal{G}_r(\Sigma) \to \Lambda_r\Omega(\Sigma)$ by

$$\xi.g := g^{-1}\xi g + g^{-1}dg. \tag{1.2.24}$$

Let us prove that ξ and $\xi.g$ produce the same immersion.

Lemma 1.3. *Let $\xi \in \Lambda_r\Omega(\Sigma)$, Φ be the solution of the initial value problem (1.2.16) with Weierstrass data $(\xi,\, \Phi_0,\, \tilde{z}_0)$ and let $g \in \mathcal{G}_r\big(\widetilde{\Sigma}\big)$ with $g_0 = g(\tilde{z}_0)$. Then the* CMC *surface induced by $(\xi,\, \Phi_0,\, \tilde{z}_0)$ is the same as the one constructed by $(\xi.g,\, \Phi_0 g_0,\, \tilde{z}_0)$.*

Proof. If Φ solves (1.2.16) with data $(\xi,\, \Phi_0,\, \tilde{z}_0)$, then $\Psi = \Phi g$ solves $\mathrm{d}\Psi = \Psi(\xi.g)$ with initial condition $\Psi(\tilde{z}_0) = \Phi(\tilde{z}_0)g(\tilde{z}_0) = \Phi_0 g_0$. This initial condition ensures that the resulting unitary frames are the same, since if the r-Iwasawa splitting of Φ is $\Phi = F\,B$ then Ψ splits $\Psi = \Phi g = (F\,B)g = F(B\,g)$ where $B\,g \in \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$ since both $B, g \in \Lambda_r^{+\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$. □

We therefore conclude that the resulting CMC surface only ‘sees’ $g(\tilde{z}_0)$ of the map $g \in \mathcal{G}_r(\Sigma)$. Note that since $\mathcal{G}_r(\Sigma)$ acts by right multiplication on the fibers of the map $(\xi,\, \Phi_0,\, \tilde{z}_0) \mapsto F$, the map is surjective ([16]).

Gaugeing is a useful tool in reducing potentials to simpler form without changing the immersion and it will be used widely throughout this work. In particular, let us show that a potential can be assumed to be off-diagonal by finding the right gauge.

Lemma 1.4. *For $\xi \in \Lambda_r\Omega(\Sigma)$ there exists a gauge $g \in \mathcal{G}_r(\Sigma)$ such that $\xi.g$ is off-diagonal.*

Proof. Consider some matrix $g = \mathrm{diag}(g_1, g_2)$ and let us write the potential $\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & -\xi_{11} \end{pmatrix}$ with respect to its components. A straightforward calculation gives $\xi.g = \begin{pmatrix} \xi_{11} + g_1^{-1}g_1' & \xi_{12}g_1^{-1}g_2 \\ \xi_{21}g_2^{-1}g_1 & -\xi_{11} + g_2^{-1}g_2' \end{pmatrix}$. The gauged potential $\xi.g$ is off-diagonal if and only if $\xi_{11} + g_1^{-1}g_1' = -\xi_{11} + g_2^{-1}g_2' = 0$, which have solutions $g_1 = \exp(-\int \xi_{11})$ and $g_2 = \exp(\int \xi_{11})$. The form of g guarantees that it is a positive loop, that is, $g \in \mathcal{G}_r(\Sigma)$. □

Therefore, to produce our immersion f we may assume that the holomorphic potential ξ is off-diagonal, since we can find a gauge that changes ξ appropriately.

1.2.7 The monodromy problem

Consider a CMC immersion f constructed from some Weierstrass data defined on a simply connected domain. One can uniquely extend the domain of definition of f to a larger non-simply connected domain, but the extended immersion will not be necessarily well-defined. The extended immersion being well-defined on the larger domain is equivalent to being well-defined on every closed loop γ there. To more directly see this relationship, consider Φ a solution of $\mathrm{d}\Phi = \Phi\xi$ and let us analytically continue the solution along a closed loop. We follow Φ around a closed loop γ in Σ, denoting the result Φ_γ. The analytically continued solution Φ_γ must again be a fundamental matrix for the ODE, however it need not be equal to Φ: given one fundamental solution, we can always multiply it from the left by a constant matrix (with respect to z) to obtain another. This notion is called *monodromy*.
Let Σ be a connected Riemann surface of genus zero with universal cover $\widetilde{\Sigma}$ and let us denote by $\mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$ the group of deck transformations. We identify the fundamental group $\pi_1(\Sigma)$ with $\mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. For the closed loop γ, there is a deck transformation τ associated to γ on $\widetilde{\Sigma}$. Let $\xi \in \Lambda_r\Omega(\Sigma)$ be a holomorphic potential on Σ. The potential being holomorphic imples $\xi(\tau(z), \lambda) = \xi(z, \lambda)$ for all $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. Let Φ be a solution of the ODE $\mathrm{d}\Phi = \Phi\xi$. Note that Φ is only defined on $\widetilde{\Sigma}$. We define the *monodromy matrix* $M_\tau(\lambda) \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ of Φ with respect to τ by

$$M_\tau(\lambda) = \Phi(\tau(z), \lambda)\, \Phi(z, \lambda)^{-1}. \tag{1.2.25}$$

By definition, if our loop γ does not enclose any puncture then $M_\tau = \mathbb{1}$, so M_τ is a measure of the lack of meromorphicity of Φ. In particular, if one can find a gauge for which ξ has no poles enclosed by γ, then $M_\tau = \mathbb{1}$. In this way, poles of ξ that cannot be removed by gauge transformations are associated with non-trivial monodromy matrices. Once a base point z_0 has been picked, the monodromy is independent of the choice of loop within the homotopy class of γ, and this is why sometimes M_τ might be referred to as the monodromy along γ. This has an important implication. Let $z = z_k$, for $k \in \{1, \ldots, n\}$, be the locations of all the poles of ξ, and let M_{τ_k} be the monodromy matrix associated with a loop that encloses only the point z_k. If we follow Φ around a path enclosing all punctures, the other side of the loop encloses no puncture and so the monodromy around that loop must be trivial. In other words,

$$M_{\tau_1} M_{\tau_2} \cdots M_{\tau_n} = \mathbb{1}. \tag{1.2.26}$$

This relation is interesting because when the conjugacy class of each individual M_{τ_k} can be computed from *local* information of the ODE, equation (1.2.26) represents a piece of *global* information.

Moreover, changing the initial condition used to determine Φ results in a conjugation of M_τ. If $\widehat{\Phi}$ is another solution of the Cauchy problem and $\widehat{M_\tau}(\lambda) = \widehat{\Phi}(\tau(z), \lambda)\, \widehat{\Phi}(z, \lambda)^{-1}$, then there exists a constant element $C \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$ such that $\widehat{\Phi} = C\Phi$. Hence $\widehat{M_\tau} = C M_\tau C^{-1}$ and different solutions give rise to mutually conjugate monodromy matrices. A choice of base point $\tilde{z}_0 \in \widetilde{\Sigma}$ and initial condition $\Phi_0 \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$ removes this ambiguity and gives the *monodromy representation* $M : \mathrm{Deck}(\widetilde{\Sigma}/\Sigma) \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ of a holomorphic frame $\Phi \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$. In this way, when we refer to the monodromy representation we are making the assumption that it was induced by a triple $(\xi,\, \Phi_0,\, \tilde{z}_0)$. Note that the invariance $\xi(\tau(z), \lambda) = \xi(z, \lambda)$ for all $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$ is equivalent to

$$\begin{aligned} \mathrm{d}M_\tau &= \mathrm{d}(\Phi(\tau(z))\Phi(z)^{-1}) \\ &= \Phi(\tau(z))\xi(\tau(z))\Phi(z)^{-1} - \Phi(\tau(z))\xi(z)\Phi(z)^{-1} \\ &= 0, \end{aligned} \tag{1.2.27}$$

ensuring that the monodromy is z-independent and thus well-defined. It is shown by Dorfmeister and Haak [15] that CMC immersions of open Riemann surfaces Σ can always be generated by such invariant holomorphic potentials. In order to control the periodicity of the resulting CMC immersion, we need to study the monodromy representation $M_\tau^F = F(\tau(z))F^{-1}$ for the unitary frame F via analytic continuation along generators $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$, but in principle we are not assured that this monodromy is z-independent for all $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. Consider the pointwise r-Iwasawa decomposition $\Phi = F\,B$ of $\Phi : \widetilde{\Sigma} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$ and note that F as well is generally only defined on $\widetilde{\Sigma}$. One way to avoid this problem is ensuring that $M_\tau(\lambda)$ is $\Lambda_r\mathrm{SU}_2$-valued.

Lemma 1.5 ([29])**.** *Let M_τ be the monodromy of a solution Φ of $\mathrm{d}\Phi = \Phi\xi$ with respect to $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. If $M_\tau \in \Lambda_r\mathrm{SU}_2$ then $M_\tau = M_\tau^F$ and so in particular the monodromy of the unitary frame F does not depend on z and is well-defined.*

Proof. From $\Phi = F\,B$ one gets that $M_\tau\,F\,B = F(\tau(z))\,B(\tau(z))$ for all $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. Since we have assumed that M_τ is $\Lambda_r\mathrm{SU}_2$-valued, by uniqueness of the r-Iwasawa decomposition, we have that

$$F(\tau(z))^{-1}\,M_\tau\,F = B(\tau(z))\,B^{-1} = \mathbb{1}. \tag{1.2.28}$$

Therefore, $M_\tau^F = M_\tau$ and is well-defined. □

The converse of lemma 1.5 also holds in the presence of umbilics (see Kilian's PhD thesis [29]). The next result characterises the well known closing conditions for the periods in the monodromy problem.

Theorem 1.5. *Let Φ be a solution of the* ODE $\mathrm{d}\Phi = \Phi\xi$ *with* $\Phi(\tilde{z}_0) = \mathbb{1}$ *and* $\Lambda_r\mathrm{SU}_2$*-valued monodromy M_τ. Let f_λ be as in equation* (1.2.10)*. There is some $\lambda_0 \in C_1$ such that $f_{\lambda_0}(\tau(z)) = f_{\lambda_0}$ for all $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$ if and only if $M_\tau(\lambda_0) = \pm\mathbb{1}$ and $\partial_\lambda M_\tau|_{\lambda_0} = 0$ for all $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$.*

Proof. Assume that $M_\tau(\lambda_0) = \pm\mathbb{1}$ and $\partial_\lambda M_\tau|_{\lambda_0} = 0$ for all $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$. The necessary and sufficient condition for f to be well-defined along the loop γ associated to τ, that is, for $f(\tau(z)) = f(z)$, is that

$$f_{\lambda_0} = \left[-\frac{i\lambda}{2H}(\partial_\lambda M_\tau)M_\tau^{-1} + M_\tau f M_\tau^{-1}\right]_{\lambda=\lambda_0}. \tag{1.2.29}$$

Hence $f_{\lambda_0} : \Sigma \to \mathbb{R}^3$ lives on Σ.

Conversely, suppose that equation (1.2.29) holds. This equation can be thought of as analytic continuation, resulting in a rigid motion of $\mathbb{R}^3$, where the rotational part is given by the adjoint action of the unitary element M_τ on f, and the translation corresponds to the element $-\frac{i\lambda}{2H}(\partial_\lambda M_\tau)M_\tau^{-1}$ evaluated at $\lambda = \lambda_0$. A rigid motion that fixes three linearly independent vectors is a constant multiple of the identity. If three points on the surface form a basis (e_1, e_2, e_3) of $\mathbb{R}^3$ this means

$$e_i = \left[-\frac{i\lambda}{2H}(\partial_\lambda M_\tau)M_\tau^{-1} + M_\tau e_i M_\tau^{-1}\right]_{\lambda=\lambda_0}, \tag{1.2.30}$$

for $i = 1, 2, 3$, forcing $M_\tau(\lambda_0) = \pm c\,\mathbb{1}$ and $\left[-\frac{i\lambda}{2H}(\partial_\lambda M_\tau)M_\tau^{-1}\right]_{\lambda=\lambda_0} = 0$ and the initial condition ensures $c = 1$. Thus, $M_\tau(\lambda_0) = \pm\mathbb{1}$ and $\partial_\lambda M_\tau|_{\lambda_0} = 0$. $\square$

In our construction of CMC surfaces we will work in the Riemann sphere with punctures. The monodromy problem can be now reformulated in the following way: let γ_k be a loop around one of the punctures z_k, and $M_{\tau_k}(\lambda)$ the monodromy of $\Phi(z, \lambda)$ along this loop. The monodromy problem, which guarantees that f_{λ_0} defined by formula (1.2.10) factors through the fundamental group $\pi_1(\Sigma)$ and thus descends to a CMC immersion $f : \Sigma \to \mathbb{R}^3$, amounts to satisfying the following three conditions:

$$M_{\tau_k}(\lambda) \in \Lambda_r \mathrm{SU}_2, \tag{1.2.31a}$$

$$M_{\tau_k}(\lambda_0) = \pm\mathbb{1}, \tag{1.2.31b}$$

$$\partial_\lambda M_{\tau_k}(\lambda)|_{\lambda=\lambda_0} = 0 \tag{1.2.31c}$$

Condition (1.2.31a) is the intrinsic closing condition, while conditions (1.2.31b) and (1.2.31c) are the extrinsic closing conditions, taking care of the rotational and the translational periods respectively.

Note that (1.2.31a) can be ensured if the initial condition Φ_0 in (1.2.16) is unitary and the potential ξ is chosen to be skew hermitian along $\tau \in \pi_1(\Sigma)$ passing through $\tilde{z}_0$. Also, note that forcing $(M_\tau, \partial_\lambda M_\tau)|_{\lambda=\lambda_0} \in \mathrm{SU}_2 \times \mathfrak{su}_2$ to be (up to sign) the identity element is a six dimensional period problem, as the real dimension of $\mathrm{SU}_2 \times \mathfrak{su}_2$ is equal to six. Instead, we will use a different approach: we will ensure (1.2.31b) and (1.2.31c) with our choice of potentials and then condition (1.2.31a) will need to be ensured. The following lemma shows that a certain choice of potentials satisfies automatically the closing conditions for the monodromy.

Lemma 1.6. *Let* $\alpha, \beta \in \Omega^1(\Sigma, \mathbb{C})$ *be holomorphic* 1*-forms on* Σ *and* $t = -1/4\,\lambda^{-1}(\lambda-1)^2$. *Let*

$$\xi = \begin{pmatrix} 0 & \alpha \\ t\beta & 0 \end{pmatrix}. \tag{1.2.32}$$

Let $\tilde{z}_0 \in \widetilde{\Sigma}$ *and* Φ *be the solution to*

$$\begin{cases} \mathrm{d}\Phi = \Phi\xi, \\ \Phi(\tilde{z}_0, t) = \mathbb{1}. \end{cases} \tag{1.2.33}$$

Let $\tau \in \mathrm{Deck}(\widetilde{\Sigma}/\Sigma)$ *and* $\widehat{M}(t) := \Phi(\tau(\tilde{z}_0), t)$. *Suppose that* $\int_{\tilde{z}_0}^{\tau(\tilde{z}_0)} \alpha_{|t=0} = 0$. *Then* $M(\lambda)|_{\lambda=1} = \mathbb{1}$ *and* $\partial_\lambda M(\lambda)|_{\lambda=1} = 0$, *where* $M(\lambda) = \widehat{M}(t)$.

Proof. At $t = 0$ we can explicitly integrate to obtain that

$$\Phi(\tilde{z}_0, 0) = \begin{pmatrix} 1 & \int_{\tilde{z}_0}^{\tau(\tilde{z}_0)} \alpha \\ 0 & 1 \end{pmatrix}. \tag{1.2.34}$$

Hence $\Phi(\tau(\tilde{z}_0), 0) = \mathbb{1}$, and so $M(1) = \mathbb{1}$.
Differentiating $\mathrm{d}\Phi = \Phi\xi$ and $\Phi(\tilde{z}_0, t) = \mathbb{1}$ and evaluating at $t = 0$ yields

$$\mathrm{d}\left(\partial_t \Phi|_{t=0}\right) = \left(\partial_t \Phi|_{t=0}\right) \xi|_{t=0} \tag{1.2.35}$$

with $\partial_t \Phi(\tilde{z}_0, t)|_{t=0} \equiv 0$. Since $\Psi \equiv 0$ is also a solution, by uniqueness of initial value problems, this implies

$$\partial_t \Phi(\tilde{z}, 0) \equiv 0. \tag{1.2.36}$$

Differentiating $\Phi(\tau(\tilde{z}_0), t) = \widehat{M}(t)$ yields $\partial_t \Phi(\tau(\tilde{z}_0), t) = \partial_t \widehat{M}(t)$ which evaluated at $t = 0$ gives $\partial_t \widehat{M}(0) = \partial_t \Phi(\tau(\tilde{z}_0), 0) = 0$ by equation (1.2.36), concluding the proof. ☐

As noted in section 1.2.5, the dressing $h\,\Phi$ generally results in a highly nontrivial relation between the unitary frame of Φ and that of $h\,\Phi$. However, it is easier to understand the relation between their monodromies. Denoting by M_τ and $\widehat{M_\tau}$ the monodromies of Φ and $h\,\Phi$ respectively, we see that

$$\widehat{M_\tau} = h\, M_\tau\, h^{-1}. \tag{1.2.37}$$

Then, encoding conditions (1.2.31b) and (1.2.31c) in the potential, one might be able to solve (1.2.31a) by applying equation (1.2.37). This procedure is called *unitarisation* (see section A.4.1) and it will be our main concern when constructing new families of CMC surfaces.

1.3 Delaunay surfaces

In 1841 Delaunay proved that the only surfaces of revolution with constant mean curvature are the surfaces obtained by rotating the roulettes of the conics. These are the cylinders, spheres, the catenoids ($H = 0$), the unduloids and the nodoids. We will define Delaunay surfaces as CMC surfaces of revolution about a geodesic in the ambient space form. These surfaces have been widely investigated in space forms [40, 39, 56] and in the context of loop groups [29, 30, 35]. Additionally, Kapouleas [26] constructed embedded CMC surfaces by gluing round spheres and pieces of unduloids and nodoids, using PDEs techniques. See also the work of Eells [18]. We use this section to introduce these surfaces, that are central in the study of CMC surfaces and in particular when constructing new examples, and also to exemplify the usage of the methods explained so far.

Let us set $\Sigma = \mathbb{C}^*$, the complex plane with a puncture at 0, and define

$$\xi_D = D\frac{dz}{z}, \qquad \text{with} \qquad D = \begin{pmatrix} c & a\lambda^{-1} + \bar{b} \\ \bar{a}\lambda + b & -c \end{pmatrix}, \tag{1.3.1}$$

with

$$a, b \in \mathbb{C}^*, c \in \mathbb{R} \qquad \text{and} \qquad c^2 + |a + \bar{b}|^2 = \frac{1}{4}. \tag{1.3.2}$$

Sometimes it is convenient to normalise the potential (1.3.1) by gaugeing so that $a, b \in \mathbb{R}^*$ and $c = 0$. The initial value problem

$$\mathrm{d}\Phi = \Phi\xi_D, \quad \Phi(1) = \mathbb{1}, \tag{1.3.3}$$

has an explicit solution

$$\Phi = \exp(D \log z) = z^D, \tag{1.3.4}$$

which allows one to construct explicitly the monodromy matrix. For a loop circling counterclockwise once about $z = 0$ in Σ, the monodromy matrix M of Φ is

$$M(\lambda) = \exp(2\pi i D) = \cos(2\pi\mu)\mathbb{1} + \frac{1}{\mu}\sin(2\pi\mu)D, \tag{1.3.5}$$

where $\pm\mu$ are the eigenvalues of D, given by

$$\mu(\lambda)^2 = -\det D = c^2 + |a|^2 + |b|^2 + ab\lambda^{-1} + \overline{ab}\lambda. \tag{1.3.6}$$

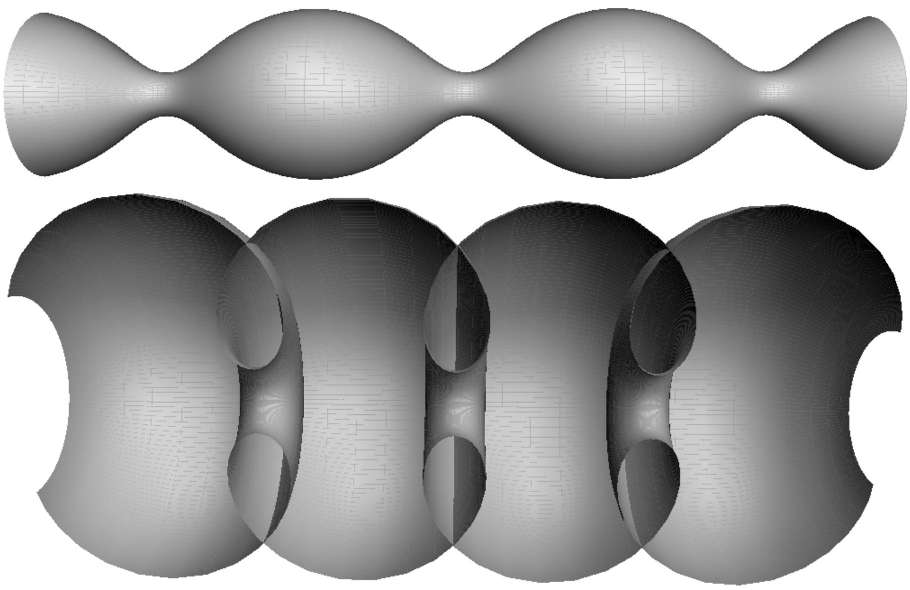

Figure 1.1: The first surface is a Delaunay unduloid and the second surface is a Delaunay nodoid cut away to show the internal structure. The parameters used to construct them are $a = 1/16$, $b = 7/16$ and $a = -3/4$, $b = 1/4$ respectively.

The above ξ_D and Φ produce Delaunay surfaces via the Sym-Bobenko formula (1.2.10). The conditions (1.3.2) are taken since thus $M(1) = -\mathbb{1}$ and $\partial_\lambda M|_{\lambda=1} = 0$, and so the second (1.2.31b) and third (1.2.31c) parts of the monodromy problem are solved. Also (1.2.31a) is satisfied since $2\pi i D \in \mathfrak{su}_2$ for all $\lambda \in \mathbb{S}^1$ and therefore the monodromy in equation (1.3.5) is in SU_2 for all $\lambda \in \mathbb{S}^1$, that is, the monodromy problem is fulfilled. In this way f closes and becomes homeomorphic to a cylinder. An unduloid is produced when $ab > 0$, a nodoid when $ab < 0$, a twice punctured round sphere when $ab = 0$ and a round cylinder when $|a| = |b|$.

We now describe the *weight* of a Delaunay surface, a quantity that determines the surface up to rigid motion, as defined by Schmitt, Kilian, Kobayashi and Rossman in [54]. Let δ be an oriented loop about an annular end of a CMC surface in $\mathbb{R}^3$ with mean curvature H, and let $\mathcal{D}$ be an immersed disk with boundary δ. Let κ be the unit conormal of the surface along the loop δ and let ν be the unit normal of $\mathcal{D}$, with their signs determined by the orientation of δ. Then the flux of the end with respect to a Killing vector field Y is

$$w(Y) = \frac{2}{\pi}\left(\int_\delta \langle \kappa, Y\rangle - 2H\int_{\mathcal{D}} \langle \nu, Y\rangle\right). \tag{1.3.7}$$

The flux is a homology invariant, proved by Korevaar, Kusner, Meeks III and Solomon [40, 39], that is, it changes sign when the orientation of δ is switched, but otherwise it is independent of the choices of δ and $\mathcal{D}$. In the case that the end is asymptotic to a Delaunay surface with axis ℓ and Y is the Killing vector field associated to unit translation along the direction of ℓ, we abbreviate $w(Y)$ to w and say that w is the weight of the end. In the case of Delaunay surfaces the weight is given by

$$w = \frac{16ab}{|H|}. \tag{1.3.8}$$

The necksize n of the end is the minimum radius of the foliating circles, taken to be negative in the case of nodoids. The weight then can be written $w = 4n(1 - nH)$. In the case $H = 1$, the round cylinder has weight 1 and necksize $1/2$, the unduloids have weights in $(0, 1)$ and necksizes in $(0, 1/2)$, the round sphere has weight and necksize 0, and the nodoids have weights and necksizes in $(-\infty, 0)$.

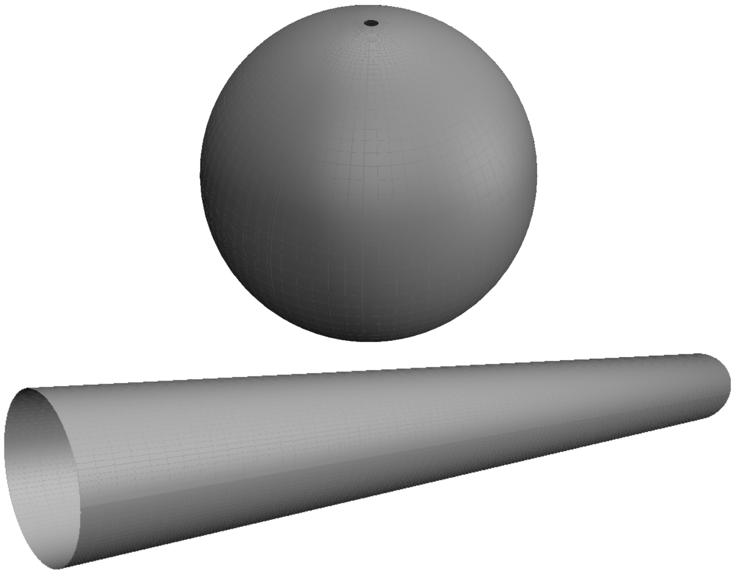

Figure 1.2: Limiting cases of Delaunay surfaces: a twice punctured round sphere and a round cylinder. Here $a = 1/2$, $b = 0$ and $a = 1/4$, $b = 1/4$, respectively.

A *perturbed Delaunay potential* is a potential ξ on $\Sigma = \mathbb{C}^*$ of the form

$$\xi = Az^{-1}\,dz + \mathrm{O}\left(z^0\right) dz. \tag{1.3.9}$$

with A a Delaunay residue, that is, of the same form as D in (1.3.1). Lemma 2.3 due to Kilian, Rossman and Schmitt [35] allows one to find a solution of $\mathrm{d}\Phi = \Phi\xi$ when ξ is a perturbed Delaunay potential. Let $r \in (0, 1)$ and suppose that $\xi = Az^{-1}\,dz + \mathrm{O}\left(z^0\right) dz$ has a simple pole at $z = 0$ and A is a Delaunay residue. A standard result in the theory of ODEs [21, Theorem 10.1] states that under certain conditions on the eigenvalues of A, there exists a solution of the form $\Phi = z^A P = \exp\left(A \log z\right) P$, where P extends holomorphically to $z = 0$. Lemma 1.7 summarizes these ideas for our context.

Lemma 1.7 (The $z^A P$ decomposition)**.** *Consider a potential $\xi = Az^{-1}\,dz + \mathrm{O}\left(z^n\right)\,dz$ with Delaunay residue A for which the eigenvalues are $\pm\mu(\lambda)$. Let $\mathcal{S}_A$ be the discrete set of* resonance points *of A given by $\mathcal{S}_A = \{\lambda \in \mathbb{C}^* :$*

$2\mu(\lambda) \in \mathbb{Z}^*\}$. *Then, for each* $\lambda \in \Sigma \setminus \mathcal{S}_A$, *there exists a map* $P(z, \lambda)$ *in a neighbourhood of* $z = 0$ *such that the* ODE $\mathrm{d}\Phi = \Phi\xi$ *has general solution*

$$\Phi = C(\lambda) \exp\left(A \log z\right) P, \tag{1.3.10}$$

for some analytic map C *and where* P *extends holomorphically to* $z = 0$ *and such that* $P(0, \lambda) = \mathbb{1}$. *Also, since* $\Phi = C \exp\left(A \log z\right) P$, *the monodromy of* Φ *around* $z = 0$ *is* $M = C \exp\left(2\pi i A\right) C^{-1}$.

1.4 The associated second order ODE

Let us then consider a potential ξ of the form

$$\xi = \begin{pmatrix} 0 & \nu(z,\lambda) \\ \rho(z,\lambda) & 0 \end{pmatrix} dz. \tag{1.4.1}$$

Every 2×2 system of ODEs has an associated 2nd-order scalar differential equation. Thus, it will be convenient for us to think of (1.2.16) in terms of its corresponding scalar ODE in order to introduce particular attributes of such equations in the Weierstrass data used to construct CMC surfaces. Denoting by $'$ the derivative with respect to z, the relation between $\mathrm{d}\Phi = \Phi\xi$ and its associated scalar ODE is given by the following straightforward result.

Lemma 1.8. *The solutions* Φ *to* $\mathrm{d}\Phi = \Phi\xi$ *are of the form*

$$\Phi = \begin{pmatrix} y_1'/\nu & y_1 \\ y_2'/\nu & y_2 \end{pmatrix} \tag{1.4.2}$$

where y_1 *and* y_2 *form a set of fundamental solutions of the scalar* ODE

$$y'' - \frac{\nu'}{\nu}\, y' - \rho\, \nu\, y = 0. \tag{1.4.3}$$

Proof. Write a solution to $\mathrm{d}\Phi = \Phi\xi$ as

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}. \tag{1.4.4}$$

Then, $\mathrm{d}\Phi = \Phi\xi$ can be written in matrix form as follows:

$$\begin{pmatrix} \Phi_{11}' & \Phi_{12}' \\ \Phi_{21}' & \Phi_{22}' \end{pmatrix} = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} 0 & \nu \\ \rho & 0 \end{pmatrix}. \tag{1.4.5}$$

The last expression is equivalent to

$$\begin{cases} \Phi_{11}' = \rho\,\Phi_{12}, \\ \Phi_{12}' = \nu\,\Phi_{11}, \\ \Phi_{21}' = \rho\,\Phi_{22}, \\ \Phi_{22}' = \nu\,\Phi_{21}. \end{cases} \tag{1.4.6}$$

Hence, computing second derivatives and using relations in (1.4.6), one obtains

$$\begin{cases} \Phi_{12}'' = \nu'\,\Phi_{11} + \nu\,\Phi_{11}' = \frac{\nu'}{\nu}\,\Phi_{12}' + \rho\,\nu\,\Phi_{12}, \\ \Phi_{22}'' = \nu'\,\Phi_{21} + \nu\,\Phi_{21}' = \frac{\nu'}{\nu}\,\Phi_{22}' + \rho\,\nu\,\Phi_{22}. \end{cases} \tag{1.4.7}$$

Therefore, Φ_{12} and Φ_{22} are solutions of the second order ODE $y'' - \frac{\nu'}{\nu}\,y' - \rho\,\nu\,y = 0$. On the other hand, using again (1.4.6), we see that $\Phi_{11} = \Phi_{12}'/\nu$ and $\Phi_{21} = \Phi_{22}'/\nu$, completing the proof. □

Remark 1.1. The dependence on z and λ has been omitted in the proof above, but note that y_1 and y_2 in lemma 1.8 depend on the parameter λ since (1.4.2) is a solution with values in the loop group $\Lambda\mathrm{SL}_2(\mathbb{C})$. We also omit this dependence in the discussion below.

It seems from lemma 1.8 that fundamental solutions of equation (1.4.3) might be of interest. Specifically, we will need in section 3.2 to find explicit connection relations between the local solutions at two different points z_0 and z_1 of a linear equation of the form (1.4.3). That is what one usually calls a two point *connection problem.* Suppose that an equation of the form (1.4.3), that is,

$$y''(z) + P(z)\,y'(z) + Q(z)\,y(z) = 0, \tag{1.4.8}$$

has the following sets of fundamental solutions fixed by its behaviour in the vicinities of two points z_0 and z_1,

$$\begin{pmatrix} y_{01}(z) \\ y_{02}(z) \end{pmatrix}, \quad \begin{pmatrix} y_{11}(z) \\ y_{12}(z) \end{pmatrix}. \tag{1.4.9}$$

If we perform the analytical continuation of the fundamental pair $\begin{pmatrix} y_{01}(z) \\ y_{02}(z) \end{pmatrix}$ into the vicinity of the point z_1, then we can express the result of this procedure as a linear combination of the fundamental solutions, namely

$$\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = C \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix}. \tag{1.4.10}$$

The corresponding matrix C is called the *connection matrix*. In our setup, this matrix depends only on the spectral parameter λ. To finish this chapter, let us prove that the matrix C in the scalar scheme is the same as the connection matrix for the linear system $\mathrm{d}\Phi = \Phi\xi$ with ξ as in (1.4.1).

Lemma 1.9. *Let C be the connection matrix between two sets of local solutions $(y_{01}(z), y_{02}(z))^t$ and $(y_{11}(z), y_{12}(z))^t$ of*

$$y'' - \frac{\nu'}{\nu} y' - \rho\,\nu\,y = 0 \tag{1.4.11}$$

at two points z_0 and z_1 respectively. Then, the connection matrix $\widehat{C} \in \Lambda\mathrm{SL}_2(\mathbb{C})$ of two local solutions Φ_0 and Φ_1 of $\mathrm{d}\Phi = \Phi\xi$ at z_0 and z_1 with ξ off-diagonal satisfies $\widehat{C} = C$.

Proof. The linear connection (1.4.10) of the scalar solutions is equivalent to

$$\begin{aligned} y_{01}(z) &= c_{11}\, y_{11}(z) + c_{12} y_{12}(z) \\ y_{02}(z) &= c_{21}\, y_{11}(z) + c_{22} y_{12}(z), \end{aligned} \tag{1.4.12}$$

where c_{ij} are the entries of the matrix C. Similarly, using equation (1.4.2) to write the local solutions Φ_0 and Φ_1, we write two of the equations (there is a total of four) relating these solutions:

$$\begin{aligned} y_{01}(z) &= \hat{c}_{11}\, y_{11}(z) + \hat{c}_{12} y_{12}(z) \\ y_{02}(z) &= \hat{c}_{21}\, y_{11}(z) + \hat{c}_{22} y_{12}(z), \end{aligned} \tag{1.4.13}$$

where $\hat{c}_{ij}$ are the entries of the matrix $\widehat{C}$. Subtracting equations in (1.4.12) and those in (1.4.13) one gets

$$\begin{aligned} 0 &= (c_{11} - \hat{c}_{11})\; y_{11}(z) + (c_{12} - \hat{c}_{12})\, y_{12}(z) \\ 0 &= (c_{21} - \hat{c}_{21})\; y_{11}(z) + (c_{22} - \hat{c}_{22})\, y_{12}(z), \end{aligned} \tag{1.4.14}$$

but $(y_{11}(z), y_{12}(z))^t$ is set of fundamental solutions, so by linear independence

$$c_{ij} - \hat{c}_{ij} = 0 \tag{1.4.15}$$

for all $i, j \in \{1, 2\}$. Therefore, $\widehat{C} = C$. $\square$

Chapter 2

Heun's Differential Equations

By virtue of lemma 1.8, second order ODEs can be used to better understand the solutions of (1.2.16) in the general Weierstrass representation for CMC surfaces. In particular, well known facts about solutions of scalar ODEs and their connection problems can be translated to our setup. In that sense, we aim to use the class of ODEs called *Heun's Differential Equations* and its features to construct new families of surfaces with constant mean curvature.

In this chapter we will pay attention to the relevant theory of the different cases of Heun's Differential Equations. The first part is devoted to explain the types of singularities found in these scalar ODEs and what is the effect of them when used to construct CMC surfaces. Then, we introduce the general *Heun's equation* and afterwards explain how the process of *confluence* of two (or more) singularities create new equations. These equations will be used ultimately to construct new families of constant mean curvature surfaces with one, two or three end(s). As stated in chapter 1, the main issue in our construction is the monodromy problem, for which we will employ different approaches depending on the number of singularities.

2.1 Singularities of 2nd-order differential equations

Let us start by giving a brief explanation of some technical terms from the general theory of linear differential equations which will be used later in this chapter and the remaining ones.
We refer to the second order linear homogeneous equation

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0, \tag{2.1.1}$$

where z is regarded as a complex variable, and p, q are rational functions.

1. Ordinary points and singularities.
   A point z_0 is said to be *ordinary* if p and q are both holomorphic at z_0. Any other point is *singular* or a *singularity* of the equation (2.1.1).
2. Regular and irregular singularities.
   Let z_0 be a singularity, then if at z_0 the function p is regular or has a pole of order 1 and q is regular or has a pole of order not exceeding 2, we say that the singularity is *regular*. Otherwise the singularity is *irregular*. Thus, at a regular singularity z_0,

$$\lim_{z\to z_0}(z-z_0)p(z) = P \quad \text{and} \quad \lim_{z\to z_0}(z-z_0)^2 q(z) = Q \tag{2.1.2}$$

   both exist.
3. Indicial equation and characteristic exponents.
   The equation

$$r^2 + (P-1)r + Q = 0 \tag{2.1.3}$$

   is called the *indicial equation* at the regular point z_0. Its roots r_1, r_2 are the *characteristic exponents* at z_0. When the difference between r_1 and r_2 is not an integer, then in a neighbourhood of z_0 equation (2.1.1) has linearly independent solutions of the form

$$(z-z_0)^{r_i}\sum_{n\geq 0} c_n(z-z_0)^n, \quad i \in \{1,2\}. \tag{2.1.4}$$

These solutions are called *Frobenius solutions* and they converge (at least) in a disc centered at z_0 for which the nearest other singularity occurs on its boundary. However, if $r_1 - r_2 \in \mathbb{Z}$, then one of the series is replaced by a solution involving $\log(z - z_0)$.

4. Point at infinity.

   When the substitution $z = 1/w$ is made in equation (2.1.1) it becomes

$$y''(w) + \left(\frac{2}{w} - \frac{1}{w^2}\,p\left(\frac{1}{w}\right)\right) y'(w) + \frac{1}{w^4}\,q\left(\frac{1}{w}\right) y(w) = 0. \qquad (2.1.5)$$

   In the cases of equation (2.1.5) having an ordinary point or a singularity at $w = 0$, we say that equation (2.1.1) has the corresponding feature at $z = \infty$. If $z = \infty$ is a regular singularity of equation (2.1.1), its characteristic exponents are defined as those of equation (2.1.5) at $w = 0$. From now on in this chapter, we stop writing the z-dependence of y in the ODEs to be considered.

5. Rank.

   When dealing with irregular singularities, it is convenient to have an index to measure its degree of irregularity, that is, the extent to which it departs from being regular. In this work we use for this matter the *Poincaré rank* or just *rank*. Suppose that $z = \infty$ is an irregular singularity of equation (2.1.1) and that, as $z \to \infty$,

$$p(z) = \mathrm{O}\left(z^{k_1}\right), \quad q(z) = \mathrm{O}\left(z^{k_2}\right). \qquad (2.1.6)$$

   Since p and q are rational functions, k_1 and k_2 are integers, and in the case of one of them being identically zero, we take the corresponding k_i as $-\infty$. Defining

$$g := \max\left(k_1, \frac{k_2}{2}\right), \qquad (2.1.7)$$

   the Poincaré rank is defined by

$$h := g + 1. \qquad (2.1.8)$$

   Note that if one computed the rank of a regular singularity, it would be 0.

Therefore, when using lemma 1.8, we might consider differential equations with regular or irregular singularities. A very important result was proved by Kilian, Rossman and Schmitt in [35] regarding the asymptotic behaviour of constant mean curvature surfaces that arise locally from an ODE with a regular singularity. They proved that a holomorphic perturbation of an ODE that represents a Delaunay surface generates a CMC surface which has one end that is asymptotically Delaunay.

Theorem 2.1 (Theorem 5.9, [35])**.** *Let A be a Delaunay residue and $\mu : \mathbb{C}^* \to \mathbb{C}$ an eigenvalue of A and suppose that*

$$\max_{\lambda \in C_r} \operatorname{Re} \mu(\lambda) < \frac{n+1}{2}. \tag{2.1.9}$$

On $\mathcal{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, let ξ be a perturbed Delaunay potential of the form (1.3.9). Let f_0 and f be the immersions of $\mathcal{D}^*$ induced by the generalised Weierstrass representation at $\lambda = 1$ by $Az^{-1}\,dz$ and $\xi\,dz$ respectively, so f_0 is a Delaunay immersion. Assume also that f is obtained from a holomorphic r-frame whose monodromy at $z = 0$ is in $\Lambda_r^*\mathrm{SL}_2(\mathbb{C})$. Then f is $\mathcal{C}^\infty$-asymptotic to a Delaunay immersion $R \circ f_0 \circ \psi$, where R is a rotation and ψ is a diffeomorphism.

Henceforth, if the differential equation to be considered has regular singularities and we transform our potential via gauges into a perturbed Delaunay potential at those points, theorem 2.1 guarantees that the ends corresponding to those regular singularities will be half-Delaunay surfaces. Otherwise, when the singularities are irregular, we will obtain surfaces with irregular ends, as those found by Kilian and Schmitt in [36] for the case of two ends.

Thus, to construct surfaces with irregular ends, it is natural to look at Heun's differential equations, since in this family the most general case of second order ODE with four regular singularities is considered and, out of that one, differential equations with different number of irregular singularities are derived.

2.2 Heun's equation

Heun's equation is a natural extension of the Riemann hypergeometric differential equation, which is a Fuchsian differential equation, that is, a differential equation with all singularities being regular. In particular, the hypergeometric equation has three regular singularities located at $z = 0$, $z = 1$ and $z = \infty$ and any second order differential equation with three regular singularities can be reduced to it by suitable transformations of the dependent and independent variables. In this sense, the next second order differential equation of Fuchsian type is Heun's, first written in [23], which has four regular singularities and also any second order Fuchsian differential equation with four singularities can be transformed into the 'canonical' Heun equation. The process is slightly different according as to whether the four singularities all lie in the finite part of the plane or whether one of them is already at infinity. For more on these transformations, see [47, Addendum, Chapter 3].

The occurrence of the extra singularity introduces a qualitative complication: the powerful methods used to investigate the hypergeometric equation no longer work. For example, one can still obtain power series solutions, but they are governed by three-term recursion relations between successive coefficients, making it impossible in general to write down such series explicitly and yielding problems over their convergence.

Heun's equation has, however, a compensating richness because it generates the so-called *confluent equations* with irregular singularities by the *coalescence of singularities* - that is, by making one or more of the singularities coincide with another, to be discussed in the next sections. A very rigorous treatment of confluence processes and their proofs are given in [47, Part A, 2.3].

In this work, the canonical form of Heun's equation will be taken as

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0. \tag{2.2.1}$$

In this, z is regarded as a complex variable and $\alpha, \beta, \gamma, \delta, \epsilon, q, a$ are parameters, generally complex and arbitrary, except that $a \neq 0, 1$ and further restrictions to be taken later. The first five parameters are linked by the relation

$$\gamma + \delta + \epsilon = \alpha + \beta + 1, \tag{2.2.2}$$

and the equation is thus of Fuchsian type, with regular singularities at $z = 0, 1, a, \infty$, the characteristic exponents at these singularities being respectively $\{0, 1-\gamma\}$, $\{0, 1-\delta\}$, $\{0, 1-\epsilon\}$ and $\{\alpha, \beta\}$. According to the general theory of Fuchsian differential equations, the sum of these characteristic exponents must take the value 2, which yields the relationship in equation (2.2.2).
Each singularity is regular, so by the usual theory of linear ODEs, in the neighbourhood of any singularity there exist two linear independent solutions, one corresponding to each of the characteristic exponents there. These solutions are normally valid only in a disc which excludes any other singularity.
There is a set of 24 substitutions each of which produces another equation of Heun type. For later use, we only consider the transformation given by

$$z \mapsto \frac{z}{a}. \tag{2.2.3}$$

Under this transformation, the singularities are mapped as follows:

$$\begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 1/a \\ a &\mapsto 1 \\ \infty &\mapsto \infty. \end{aligned} \tag{2.2.4}$$

If the reader is interested, much more can be found about this family of equations in [47]. Next, we discuss the confluent cases obtained from this equation.

2.3 Confluent Heun's equation

As outlined in the previous section, the confluent Heun equation (CHE) arises as a result of the confluence of two regular singular points in Heun's equation. In this way, one regular singular point (and one parameter) is lost but the point at infinity now becomes irregular. In particular, the parameter a is sent to $a \mapsto \infty$. The CHE in its *non-symmetrical canonical form* is written as

$$y'' + \left(4p + \frac{\gamma}{z} + \frac{\delta}{z-1}\right) y' + \frac{4p\alpha z - \sigma}{z(z-1)} y = 0 \tag{2.3.1}$$

where the parameters $p, \gamma, \delta, \alpha, \sigma$ can be in general complex. In this equation the points $z = 0$ and $z = 1$ are regular singularities while $z = \infty$ is an irregular singular point of rank 1. However, for the purpose of this work we are going to consider the CHE written as in the work of Schäfke and Schmidt [50], that is,

$$\begin{aligned} y'' &+ \left(2a + \tfrac{1-\mu_0}{z} + \tfrac{1-\mu_1}{z-1}\right) y' \\ &+ \left(\tfrac{z[a(2-\mu_0-\mu_1)-(r_0+r_1)] + \frac{1}{2}(\mu_0\mu_1 - 2a(1-\mu_0) - (\mu_0+\mu_1) + 2r_0 + 1)}{z(z-1)}\right) y = 0. \end{aligned} \tag{2.3.2}$$

The parameters $\mu_0, \mu_1, a, r_0, r_1$ are again complex, but we will need to restrict our choices later on in order to unitarise the monodromy.

As is well known, solutions of the hypergeometric equation can be constructed as series with explicit coefficients. In the case of solutions of Heun's equation and its confluent forms, the best results would be three-term recurrence relations for the power series expansion coefficients. We start with power series in a vicinity of the regular singular point $z = 0$. Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < r\}$, the open disk where $1 < r \leq \infty$. No further singularity of equation (2.3.2) can be found within $\mathcal{D}$ apart from 0 and 1. Let

$$y_0(z) = \Gamma(1-\mu_0) \sum_{k=0}^{\infty} c_k z^k \tag{2.3.3}$$

be the series expansion of the unique solution to equation (2.3.2) which is holomorphic in $\mathcal{D} \setminus [1, r)$ and satisfies $y_0(0) = 1$. Here, Γ represents the usual *gamma function*. By standard methods of power series solutions (see [47, Part B, 2.2]), the sequence $\{c_k\}_{k=0}^{\infty}$ of coefficients defined by equation (2.3.3) satisfies the 3-term recurrence

$$U(k)c_{k+1} = V(k)c_k + W(k)c_{k-1}, \quad c_{-1} = 0,\ c_0 = \frac{1}{\Gamma(1-\mu_0)} \tag{2.3.4}$$

where

$$\begin{aligned}
U(k) &:= (1+k)(1+k-\mu_0),\\
V(k) &:= k(k+1-2a-\mu_0-\mu_1)\\
&\quad+\frac{1}{2}(\mu_0-1)(\mu_1-1)+a(\mu_0-1)+r_0,\\
W(k) &:= a(2k-\mu_0-\mu_1)-(r_0+r_1).
\end{aligned} \tag{2.3.5}$$

We deduce from (2.3.4) that in general

$$\frac{c_k}{c_{k+1}} = 1 + \mathrm{O}\left(\frac{1}{k}\right), \tag{2.3.6}$$

and therefore the radius of convergence of the series in (2.3.3) is 1, which is quite natural since that is the distance between $z = 0$ and the other regular singular point at $z = 1$. Other kinds of solutions of the CHE can be found, such as solutions in terms of hypergeometric functions, but for the purpose of our study only power series solutions are considered.

It is shown by Schäfke and Schmidt in [50] that, using the solution in (2.3.3), one can construct two sets of *Floquet solutions* of equation (2.3.2) at $z = 0$ and at $z = 1$, respectively. Basically, these are sets of linearly independent solutions (y_{01}, y_{02}) and (y_{11}, y_{12}) at each of the singularities. An introduction to Floquet theory can be found in, among others, the book of Hartman [21, Part IV, Chapter 6].

The aim of the second part of [50] is to obtain explicit connection relations between the sets of solutions (y_{j1}, y_{j2}), for $j = 0, 1$ (the so-called two-connection problem). They do so in [50, Proposition 2.14], obtaining in particular that for $k = 1, 2$

$$\frac{1}{\Gamma(\mu_1)\Gamma(1-\mu_1)}\, y_{0k} = q(\pm\mu_0, -\mu_1)\, y_{11} - q(\pm\mu_0, \mu_1)\, y_{12}, \tag{2.3.7}$$

with + for $k = 1$ and − for $k = 2$, where q is a function that can be calculated explicitlty by an asymptotic formula in terms of the parameters μ_0, μ_1 and the coefficients c_k of the power series solution in (2.3.3).

The recurrence in (2.3.4), its polynomials (2.3.5), along with the function q for the two-connection problem by Schäfke and Schmidt will be used in section 3.2.4 in order to guarantee unitarisability of the monodromy of the solution of the CHE.

2.4 Double confluent Heun's equation

In this part, we introduce the double confluent Heun equation (DCHE), the linear second order ODE that appears when the two finite singularities in the CHE coalesce. The DCHE therefore has two irregular singularities and it can be written in its *symmetric canonical form* as

$$y'' + \alpha \left(1 + \tfrac{1}{z^2}\right) y' + \left(\left(\beta_1 + \tfrac{1}{2}\right) \tfrac{\alpha}{z} + \left(\tfrac{\alpha^2}{2} - \gamma\right) \tfrac{1}{z^2} + \left(\beta_{-1} - \tfrac{1}{2}\right) \tfrac{\alpha}{z^3}\right) y = 0. \qquad (2.4.1)$$

The independent variable z is regarded as a complex variable and the quantities $\alpha, \beta_1, \beta_{-1}$ and γ are complex parameters, and the solutions y are analytic functions of z. Equation (2.4.1) is a linear ordinary differential equation of second order with meromorphic coefficients. The points 0 and ∞ are irregular singular points of rank 1, and there are no other singularities in $\mathbb{C}^*$. Therefore, all the solutions can be continued analytically within $\mathbb{C}^*$. Since $\mathbb{C}^*$ is not simply connected, the resulting global solutions are in general not single valued on $\mathbb{C}^*$ but on the universal cover $\mathbb{C}$. By the general theory of meromorphic ODEs, there exist uniquely defined fundamental sets of solutions about each singularity. The leading coefficients of the asymptotic series at 0 and at ∞ depend on α, β_1 and β_{-1}, called accordingly 'singular parameters'. Instead, the parameter γ has only secondary influence on the asymptotic behaviour and it is therefore called the 'accessory parameter'.
Although there is a rich theory regarding this equation, such as analytic theory of solutions, asymptotic solutions and two-connection problem, for our application none of this will be needed. We encourage the curious reader to consult [47, Part C] where all these topics are covered.

2.5 Biconfluent Heun's equation

This part deals with the so-called biconfluent Heun's equation (BHE), which is obtained from the CHE when one of the two finite singular points joins the singular point at infinity. Although there exist several forms for the BHE, we choose its *canonical form* according to [47, Part D], which reads

$$y'' + \left(\frac{\alpha+1}{z} - \beta - 2z\right) y' + \left(\gamma - \alpha - 2 - \frac{1}{2z}\left(\beta(\alpha+1)+\delta\right)\right) y = 0. \quad (2.5.1)$$

The linear differential equation (2.5.1) has one regular singularity at the origin and one irregular singularity at infinity of rank 2. One interesting fact is that many Schrödinger equations can be solved using the BHE. In our setup, since equation (2.5.1) counts with one regular singularity, by the general theory of ODEs and theorem 2.1, we expect to construct surfaces with one Delaunay end. The monodromy problem in this case will be automatically solved using the well known theory regarding these surfaces. Despite that, we remark that regarding the theory of equation (2.5.1) a lot is known about fundamental solutions at the origin and at infinity, as well as integral relations between pairs of solutions. We refer the reader to [47, Part D] for a systematic treatment of these subjects.

2.6 Triconfluent Heun's equation

The last possible process of confluence brings together the two finite singular points of the CHE to the existing irregular singularity to generate a higher rank irregular singularity at infinity. In this triconfluent Heun equation (THE), all finite points are ordinary points and only three parameters remain.

Our choice of THE is in *canonical form*, that is,

$$y'' - \left(\gamma + 3z^2\right) y' + \left(\alpha + (\beta - 3)z\right) y = 0. \quad (2.6.1)$$

The irregular singularity is located at $z = \infty$ and has rank 3, the highest we can get out of the process of confluence in the Heun equation. Since no other singularity exists in $\mathbb{C}$, the monodromy is trivial and thus we will not require any condition on the parameters or the solutions in order to construct new CMC planes (surfaces with the topology of the plane).

Chapter 3

CMC surfaces from Heun's Differential Equations

In this chapter we employ all the theory explained in the previous two chapters in order to construct new families of CMC surfaces using the family of ODEs given by Heun (see chapter 2), with special attention on the confluent cases, since those allow us to find new families of surfaces with irregular ends in a systematic way.
The structure of the chapter is as follows: in each of the sections we focus on one of the differential equations, finding the appropriate functions that prescribe it in the initial value problem (1.2.16). Once this is done, our focus will be to solve the monodromy problem. Depending on the number of punctures, this will require different approaches. This step usually means that we need to restrict the range of parameters to consider in Heun's Differential Equations, which in principle are all complex values without restriction. Lastly, the theoretical content is supported by numerical solutions of each of the cases, and images are produced using the software CMCLab [51] developed by Nicholas Schmitt, to whom we are grateful for publishing the software he developed with the GANG group.
Thus, each of the subsequent sections refers to the CMC surfaces that have been found for each of the representatives of the family of Heun's Differential Equations.

3.1 4-noids

Although the main goal of this thesis is to construct surfaces with *irregular ends*, which is done in the subsequent sections of this chapter, we first look in this section at the general Heun equation

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) y' + \frac{\alpha\beta z - q}{z(z-1)(z-a)} y = 0 \qquad (3.1.1)$$

and point out the steps we have followed trying to construct CMC surfaces with 4 Delaunay ends from this equation. This equation has 4 regular singularities at $z = 0, 1, a, \infty$, so as stated in theorem 2.1, one might be able to construct families of surfaces with four Delaunay ends for it. The obstacle is of course the pointwise simultaneously unitarisation of the monodromy matrices.

Constructing CMC surfaces with at least 4 ends is in general more complicated using the classical approaches for unitarisation than the lower ends cases. Some of these approaches involve imposing full symmetries on the surface, such as the work by Rossman and Schmitt [48] with n-noids. A similar but more general approach was used by Schmitt in his work [53], where he reduces the unitarisation problem to that one of a trinoid by constructing a rational map which is an invariant of a finite group of Möbius transformations that lowers the number of poles of the potential ξ to 3. Also, Kapouleas in [26] constructed embedded CMC surfaces with no limitation on the number of ends by gluing round spheres and pieces of Delaunay surfaces, using PDEs techniques. Recently, Traizet [59] has been able to adapt the gluing techniques of Kapouleas to the loop groups methods by *opening nodes* in the underlying Riemann surface, which is a model for Riemann surfaces with *small necks*. In this way Traizet can construct CMC surfaces in $\mathbb{R}^3$ with no restriction on the number of ends and with any genus.

The appearance of the Heun equation as a way to generalise the construction of trinoids with Delaunay ends is already present in the literature. In particular, the problem of unitarising the monodromy was studied for a very particular case by Dorfmeister and Eschenburg in [13]. Their approach is based on the classical *spherical inequalities* for *spherical n-gons*, that is, loops of geodesic segments on a 2-sphere, each of whose side lengths is between 0 and the semicircumference inclusively. It is known that these inequalities do not give a sufficient condition for the unitarisation when $n \geq 4$, but under a sufficient 'simplification' of the equation (3.1.1) the unitarisation problem is asserted to be solved in this paper.

3.1.1 Assumptions for the Heun equation

In what follows, we discuss the approach followed in [13], writing the ODE and the resulting potential for the specific choices that they do and show the obstructions in order to get a proof. The first assumption to be made is to fix the fourth regular singularity at $a = -1$. Next, all the parameters in the differential equation (3.1.1) are considered to be real. Also, we force the parameters δ and ϵ to be equal. These are the characteristic exponents of the singularities at $z = 1$ and $z = -1$. Making them equal, we force a symmetry between the two corresponding ends. We finally eliminate the last parameter in equation (3.1.1), that is, we make $q = 0$. In this way, we work with the ODE

$$y'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\delta}{z+1}\right) y' + \frac{\alpha\beta}{(z-1)(z+1)} y = 0, \tag{3.1.2}$$

for $\gamma, \delta, \alpha, \beta \in \mathbb{R}$, which is invariant under the transformation $z \mapsto \bar{z}$ of the domain, and also under $z \mapsto -z$, which fixes 0 and ∞ and interchanges 1 and -1. Note that this transformation corresponds to the one given in equation (2.2.3) for $a = -1$, which then gives another equation of Heun type. Also, recall the link between the parameters in equation (2.2.2), which now reads

$$\gamma + 2\delta = \alpha + \beta + 1. \tag{3.1.3}$$

3.1.2 A potential for the Heun equation

Let us write an off-diagonal holomorphic potential associated to equation (3.1.2) on $\Sigma = \mathbb{C} \setminus \{0, 1, -1\}$. We proved in lemma 1.4 that, for such an off-diagonal potential, the associated scalar ODE is of the form (1.4.3), where the two functions involved in the ODE are those in the off-diagonal potential. It is easy then to find functions ν, ρ so that the associated second order ODE becomes the Heun equation in (3.1.2). In particular, let us take

$$\begin{aligned}\nu &:= z^{-\gamma}\left(1-z^2\right)^{-\delta}, \quad \text{and} \\ \rho &:= -\frac{\alpha\beta z^{\gamma-1}\left(1-z^2\right)^{\delta}}{(z-1)(z+1)}.\end{aligned} \tag{3.1.4}$$

Note that these two functions seem to depend only on z, but this is just because the dependence on λ is in the parameters of the equation. Therefore, later on, we will write explicitly those relations.

Hence, our off-diagonal potential that encodes the Heun's equation (3.1.2) is

$$\xi_1 = \begin{pmatrix} 0 & z^{-\gamma}\left(1-z^2\right)^{-\delta} \\ -\frac{\alpha\beta z^{\gamma-1}\left(1-z^2\right)^{\delta}}{(z-1)(z+1)} & 0 \end{pmatrix} dz. \tag{3.1.5}$$

Lastly, let us consider the gauge

$$g_1 = \begin{pmatrix} \left(z^{-\gamma}\left(1-z^2\right)^{-\delta}\right)^{1/2} & 0 \\ \left(\frac{\delta z}{z^2-1}+\frac{\gamma}{2z}\right)\left(z^{-\gamma}\left(1-z^2\right)^{-\delta}\right)^{-1/2} & \left(z^{-\gamma}\left(1-z^2\right)^{-\delta}\right)^{-1/2} \end{pmatrix} \tag{3.1.6}$$

with which we obtain $\xi := \xi_1.g_1$, where

$$\xi = \begin{pmatrix} 0 & 1 \\ \frac{\gamma^2-2\gamma}{4z^2}+\frac{2\alpha\beta-2\gamma\delta-\delta^2}{4(z+1)}+\frac{\delta^2+2\gamma\delta-2\alpha\beta}{4(z-1)}+\frac{\delta^2-2\delta}{4(z-1)^2}+\frac{\delta^2-2\delta}{4(z+1)^2} & 0 \end{pmatrix} dz. \tag{3.1.7}$$

This is the simplest potential in which we can encode the equation (3.1.2). Now we take parameters in order to write a potential that might be used to construct CMC surfaces with four Delaunay ends.

For $t := -\frac{1}{4}\lambda^{-1}(\lambda-1)^2$, let us consider the maps

$$\begin{aligned}\gamma &:= 1-\sqrt{1-rt}, \\ \delta &:= 1-\sqrt{1-st}, \\ \alpha &:= \frac{1}{2}\left(2-\sqrt{1-rt}-2\sqrt{1-st}-\sqrt{1-t(r+2s-4u)}\right), \\ \beta &:= \frac{1}{2}\left(2-\sqrt{1-rt}-2\sqrt{1-st}+\sqrt{1-t(r+2s-4u)}\right).\end{aligned} \tag{3.1.8}$$

Note that, writing $\lambda = e^{i\theta}$ with $\theta \in [0, 2\pi]$, the value of $t = -\frac{1}{4}\lambda^{-1}(\lambda - 1)^2 = \sin^2 \frac{\theta}{2} \in [0, 1]$. In particular, t is real. Thus, as long as $r, s, u \in \mathbb{R}$, all the parameters in equation (3.1.2) are real, as imposed earlier. With the maps in (3.1.8) and using $\Lambda := \mathrm{diag}(\lambda^{1/2}, \lambda^{-1/2})$, the gauged potential

$$\xi_N := \xi.\Lambda = \begin{pmatrix} 0 & 1/\lambda \\ \lambda Q_t & 0 \end{pmatrix} dz, \tag{3.1.9}$$

can be used in the generalised Weierstrass representation for CMC surfaces, where

$$Q_t := t\left(-\frac{r}{4z^2} - \frac{s}{4(z-1)^2} - \frac{s}{4(z+1)^2} + \frac{u}{2(z-1)} - \frac{u}{2(z+1)}\right). \tag{3.1.10}$$

3.1.3 Regular singular points

The function Q_t has double poles at $z = 0$, $z = 1$, $z = -1$ and at $z = \infty$ (the ends of the surface) and no other poles. Note that the potential ξ_N can be gauged to a perturbed Delaunay potential using

$$g_2 = \begin{pmatrix} (z-1)^{1/2} & 0 \\ -\lambda\frac{(z-1)^{-1/2}}{2} & (a + b\lambda)(z-1)^{-1/2} \end{pmatrix}, \tag{3.1.11}$$

with $a = \frac{1}{4}\left(1 + \sqrt{1-s}\right)$ and $b = \frac{1}{4}\left(1 - \sqrt{1-s}\right)$, obtaining

$$\xi_N.g_2 = A_1\frac{dz}{z-1} + \mathrm{O}\left((z-1)^0\right) dz, \tag{3.1.12}$$

where A_1 is a Delaunay residue as in (1.3.1) with $c = 0$.

Thus the ODE $\mathrm{d}\Phi = \Phi\xi_N$, $\Phi(z_0) = \mathbb{1}$ has a solution in a neighbourhood of $z = 1$ containing z_0 which is of the form $\Phi_1 = (z-1)^{A_1}P$ (recall the z^AP lemma 1.7).

A set of generators of the monodromy representation of the potential ξ_N is defined as follows. Choose closed curves γ_0, γ_1, γ_{-1}, γ_∞ based at z_0 which respectively wind around each of the singularities once and not around any other of them, satisfying $\gamma_0\gamma_1\gamma_{-1}\gamma_\infty = \mathbb{1}$. Define $M_0, M_1, M_{-1}, M_\infty : \mathbb{C}^* \to \mathrm{SL}_2(\mathbb{C})$ as the monodromies of the solution Φ to the equation $\mathrm{d}\Phi = \Phi\xi$, $\Phi(z_0) = \mathbb{1}$

along γ_0, γ_1, γ_{-1} and γ_∞ respectively. This choice gives $M_0 M_1 M_{-1} M_\infty = \mathbb{1}$. As a consequence of the remarks above regarding Φ_1, the monodromy around the singularity at $z = 1$ can be written as $M_1 = \exp(2\pi i A_1)$. In particular M_1 is unitary.

3.1.4 Exploring symmetries for the monodromies

Let us define the trnasformation $\kappa(z) = -z$. Thanks to the assumptions made in section 3.1.2, the function Q_t satisfies that $\kappa^* Q = Q$. This gives our potential ξ_N a special symmetry that could be exploited. It holds that

$$\kappa^* \xi_N = h\, \xi_N\, h^{-1}, \tag{3.1.13}$$

where $h = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Naturally, the transformation $\kappa^* \Phi_1$ defines a solution to the differential equation $\mathrm{d}\,(\kappa^* \Phi_1) = (\kappa^* \Phi_1)\,(\kappa^* \xi_N)$, which using (3.1.13) is the same as $\mathrm{d}\,(\kappa^* \Phi_1) = (\kappa^* \Phi_1)\,(h\, \xi_N\, h^{-1})$, that is,

$$\mathrm{d}\,(\kappa^* \Phi_1\, h) = (\kappa^* \Phi_1\, h)\, \xi_N. \tag{3.1.14}$$

It is clear that two solutions of this equation differ by a factor that does not depend on z, in other words, by a matrix $T \in \Lambda\mathrm{SL}_2(\mathbb{C})$. Therefore, the solution Φ_1 has the symmetry

$$T \Phi_1 = \kappa^* \Phi_1\, h. \tag{3.1.15}$$

Suppose that we choose the base point z_0 to be a fixed point of the transformation κ. Then, by evaluating at this point, one gets a useful symmetry for the solution from equation (3.1.15). In particular, if $\kappa(z_0) = z_0$, we would obtain that $T = h$ using $\Phi_1(z_0) = \mathbb{1}$ and then the symmetry of the solution would be

$$\kappa^* \Phi_1 = h\, \Phi_1\, h^{-1}. \tag{3.1.16}$$

With symmetry (3.1.16), the monodromy matrices would satisfy that

$$M_{-1} = h\, M_1\, h^{-1}. \tag{3.1.17}$$

Similar techniques are used to construct n-noids, see for instance the symmetric n-noids by Rossman and Schmitt in [48].
The unitarisation problem consists on the pointwise simultaneously unitarisation of the monodromies M_0, M_1, M_{-1} (recall the relation (1.2.26)) which, in view of equation (3.1.17), amounts to simultaneously unitarise the triple

$$\left(M_0, M_1, h\, M_1\, h^{-1}\right). \tag{3.1.18}$$

Note that the last monodromy to be unitarised is written as a conjugation by a unitary matrix of another monodromy. This means that a simultaneous unitariser for M_0 and M_1 would also unitarise M_{-1} as a consequence of the symmetries. In order to unitarise two monodromies, we have derived criteria that uses the connection coefficients between solutions (this theory is fully explained and used in section 3.2.4). We could expect to solve the unitarisation for M_0 and M_1 using the connection matrix between solutions given in the work of Williams and Batic [61]. For the same 'reduced' equation as in (3.1.2), they worked out these connections coefficients which are given in terms of quotients of the Gamma function $\Gamma(x)$.
The problem, however, is that the only fixed point of the transformation κ is 0, which is one of the singularities, so it can not be chosen, losing thus all the symmetries. We tried overcoming this difficulty by finding a suitable Möbius transformation that fixes our base point and interchanges 1 and -1. This, of course, breaks the symmetry for the potential in (3.1.13), and so it is not a solution.
The above explains why we could not prove the existence of CMC surfaces arising from this sub-class of the Heun equation. One possible explanation is that the usual techniques to unitarise monodromies when constructing CMC surfaces or, in particular, n-noids involve imposing symmetries to reduce the complexity of the problem. It might be that in the case of the Heun equation the freedom to impose symmetries is lost as a consequence of having already fixed three of the singularities.
However, another solution could be found with different techniques. A very similar unitarisation problem for the Heun equation is solved without the

presence of the spectral parameter by Eremenko, Gabrielov and Tarasov in [19], and our experimental graphics suggest that this could be successfully applied to our setup.

3.2 Trinoids with two regular ends and one irregular end

The present section is devoted to prove the existence of a family of *trinoids* with two asymptotically Delaunay ends and one irregular end, constructed using the CHE presented in section 2.3. The content of this section is the core of our joint work with Kilian and Schmitt in [34].
As pointed out in chapter 1, the main difficulty is unitarising the monodromy group. This will be done using the connection matrix for the CHE, found by Schäfke and Schmidt. Our potential for the trinoids will also satisfy the closing conditions (see lemma 1.6), so that the monodromy problem will be fully solved.

3.2.1 Two Regular Singular Points

Let Σ be the thrice-punctured Riemann sphere with punctures $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$. Since our goal is to construct CMC trinoids for which two ends are regular and one end is irregular, we will assume that at the two ends z_0 and z_1, the potential is a holomorphic perturbation of a Delaunay potential, and hence regular singular there. In addition, our potentials will have a singularity of rank 1 at z_∞, making it an irregular end. These choices will determine the form of the associated scalar ODE in (1.4.3).
Note that in this and the following sections, we omit the dependence on the spectral parameter λ. Let $\vartheta_0, \vartheta_1 \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$ be parameters, and a and b functions on Σ such that a is holomorphic at z_0 and z_1, and b is allowed to have

simple poles at z_0 and z_1. Define

$$\begin{aligned} \xi_0 &:= \begin{pmatrix} 0 & 1 \\ Q(z) & 0 \end{pmatrix} dz, \\ Q(z) &:= \frac{\vartheta_0(\vartheta_0-1)}{(z-z_0)^2} + \frac{\vartheta_1(\vartheta_1-1)}{(z-z_1)^2} + b(z). \end{aligned} \tag{3.2.1}$$

The points z_0 and z_1 are regular singular points of ξ_0. These double poles can be gauged to simple poles by

$$g_0 := \begin{pmatrix} 1 & 0 \\ G & 1 \end{pmatrix}, \quad \text{where } G(z) = \frac{\vartheta_0}{z-z_0} + \frac{\vartheta_1}{z-z_1} + a(z), \tag{3.2.2}$$

to obtain

$$\eta := \xi_0.\left(g_0^{-1}\right) = A_0\frac{dz}{z-z_0} + A_1\frac{dz}{z-z_1} + B\,dz \tag{3.2.3}$$

where B is holomorphic at z_0 and z_1 and

$$\begin{aligned} A_k &:= H_k J_k H_k^{-1}, \\ J_k &:= \operatorname{diag}(-\vartheta_k, \vartheta_k), \\ H_k &:= \begin{pmatrix} 1 & 0 \\ h_k & 1 \end{pmatrix}, \end{aligned} \tag{3.2.4}$$

for $k \in \{0,1\}$. Here, $h_0 = a + \frac{\vartheta_1}{z_0-z_1} - \frac{b_0}{2\vartheta_0}$ and $h_1 = a + \frac{\vartheta_0}{z_1-z_0} - \frac{b_1}{2\vartheta_1}$, where b_k is the part of b that has a simple pole at z_k.

Suppose $\mathrm{d}\Phi = \Phi\eta$ for which $\eta = A_k\frac{dz}{z-z_k} + \mathrm{O}\left(z^0\right)dz$ has a simple pole at $z = z_k$ and Delaunay residue A_k. Let us prove in lemma 3.1 a version of the $z^A P$ lemma for this context.

Lemma 3.1. *For $k \in \{0,1\}$, the* ODE $\mathrm{d}\Phi = \Phi\xi_0$ *has solutions*

$$\Phi_k = \exp\left(J_k \log\left((-1)^k(z-z_k)\right)\right) P_k(z) g_0(z), \tag{3.2.5}$$

at z_k, where $P_k(z)$ is holomorphic at $z = z_k$ and $P_k(z_k) = H_k^{-1}$.

Proof. The potential η has a simple pole at z_k with residue A_k. Since $\vartheta_k \notin \frac{1}{2}\mathbb{Z} \setminus \{0\}$, by the theory of regular singular points, there exists a solution to the ODE $\mathrm{d}\Psi = \Psi\eta$ of the form

$$\Psi_k(z) = \exp\left(A_k \log\left(z-z_k\right)\right) R_k(z), \tag{3.2.6}$$

where $R_k(z)$ is holomorphic at $z = z_k$ and $R_k(z_k) = \mathbb{1}$. Since $\xi_0 = \eta . g_0$, then

$$\widehat{\Phi}_k(z) = H_k^{-1} \exp\left(A_k \log\left(z - z_k\right)\right) R_k(z) g_0(z) \tag{3.2.7}$$

is a solution of the ODE $\mathrm{d}\Phi = \Phi\xi_0$. Since $A_k H_k = H_k J_k$, then

$$\widehat{\Phi}_k(z) = \exp\left(J_k \log\left(z - z_k\right)\right) H_k^{-1} R_k(z) g_0(z). \tag{3.2.8}$$

The result follows with $P_k := H_k^{-1} R_k$, $\Phi_0 := \widehat{\Phi}_0$ and
$\Phi_1 := \operatorname{diag}\left((-1)^{-\vartheta_1}, (-1)^{\vartheta_1}\right) \widehat{\Phi}_1$. $\square$

3.2.2 Prescribing the CHE

In this part we find a potential associated to the CHE of the form

$$\begin{aligned} y'' &+ \left(2a + \tfrac{1-\mu_0}{z} + \tfrac{1-\mu_1}{z-1}\right) y' \\ &+ \left(\tfrac{z[a(2-\mu_0-\mu_1)-(r_0+r_1)] + \frac{1}{2}(\mu_0\mu_1 - 2a(1-\mu_0) - (\mu_0+\mu_1) + 2r_0 + 1)}{z(z-1)}\right) y = 0. \end{aligned} \tag{3.2.9}$$

This is our choice of ODE with two regular singular points at $z = 0$ and $z = 1$, and one irregular singularity at $z = \infty$, so the correspondence with the parameters and functions used in section 3.2.1 is as follows: set $z_0 = 0$ and $z_1 = 1$ and for $k \in \{0, 1\}$ let

$$\begin{aligned} \vartheta_k &:= \frac{1 - \mu_k}{2}, \\ a(z) &:= a, \\ b(z) &:= \frac{r_0}{z} + \frac{r_1}{z-1} + a^2. \end{aligned} \tag{3.2.10}$$

Consider an off-diagonal holomorphic potential ξ_1 on $\Sigma = \mathbb{C} \setminus \{0, 1\}$. We just need to find functions ν, ρ so that the associated second order ODE becomes equation (3.2.9). This is done by taking

$$\begin{aligned} \nu :=& z^{\mu_0 - 1}(z-1)^{\mu_1 - 1} e^{-2az}, \qquad \text{and} \\ \rho :=& -e^{2az} z^{-\mu_0}(z-1)^{-\mu_1}[z(a(2 - \mu_0 - \mu_1) - (r_0 + r_1)) \\ &+ \frac{1}{2}(\mu_0\mu_1 - 2a(1 - \mu_0) - (\mu_0 + \mu_1) + 2r_0 + 1)]. \end{aligned} \tag{3.2.11}$$

Note that these functions depend on λ, which is encoded in the parameters of the equation. In this way, an off-diagonal potential prescribing the CHE is given by $\xi_1 = \begin{pmatrix} 0 & \nu \\ \rho & 0 \end{pmatrix}$.

In order to simplify the potential that will be used to construct trinoids, we gauge by

$$g_1 = \begin{pmatrix} \left(z^{\mu_0-1}(z-1)^{\mu_1-1}\, e^{-2az}\right)^{1/2} & 0 \\ \frac{2az^2 - z(2a+\mu_0+\mu_1-2)+\mu_0-1}{2z(z-1)\left(z^{\mu_0-1}(z-1)^{\mu_1-1}\, e^{-2az}\right)^{1/2}} & \left(z^{\mu_0-1}(z-1)^{\mu_1-1}\, e^{-2az}\right)^{-1/2} \end{pmatrix}. \quad (3.2.12)$$

This gives a very simplified potential,

$$\xi = \xi_1.g_1 = \begin{pmatrix} 0 & 1 \\ \frac{\mu_0^2-1}{4z^2} + \frac{\mu_1^2-1}{4(z-1)^2} + \frac{r_0}{z} + \frac{r_1}{z-1} + a^2 & 0 \end{pmatrix} dz. \quad (3.2.13)$$

Before writing the constructing potential, we proceed with the unitarisation of the monodromies using the connection matrix for two solutions at regular singular points.

3.2.3 Connection matrix

Schäfke and Schmidt give in [50] an asymptotic formula for the connection coefficients between a set of two solutions of equation (3.2.9) around $z = 0$ and another set of two solutions of equation (3.2.9) around $z = 1$, in terms of their power series expansion coefficients. The main results to be used from [50] are Proposition 2.14 and Theorem 2.15. In the first of them, the connection coefficients between a set of fundamental solutions (y_{01}, y_{02}) at $z = 0$ and a set of fundamental solutions (y_{11}, y_{12}) at $z = 1$ are given, in terms of a function q. Then, in the second result, they provide an asymptotic expression for this function. Thus, computing this function one can obtain the connection coefficients between two sets of fundamental solutions of the CHE. With lemma 1.9, it is straightforward to obtain the *connection matrix* for our 2×2 setup. Our connection matrix C is a λ-dependent matrix such that two local solutions Φ_0, Φ_1 of $\mathrm{d}\Phi = \Phi\xi$ at $z = 0$ and at $z = 1$ respectively, are related by $\Phi_0 = C\Phi_1$.

From the two equations appearing in [50, Proposition 2.14], we only need the first one in order to write down our matrix relationship $C = \Phi_0\Phi_1^{-1}$, as in what follows we will only consider the unique solution at $z = 0$ introduced in equation (2.3.3). These connection coefficients can be written in a matrix form allowing us to define the connection matrix between the two local solutions Φ_0 and Φ_1.

Theorem 3.1 ([50]). *The connection matrix $C := \Phi_0\Phi_1^{-1}$ is*

$$C = \begin{pmatrix} \frac{\Gamma(\mu_0)}{\Gamma(1-\mu_0)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q(\mu_0, -\mu_1) & q(\mu_0, \mu_1) \\ q(-\mu_0, -\mu_1) & q(-\mu_0, \mu_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\Gamma(\mu_1)}{\Gamma(1-\mu_1)} \end{pmatrix}, \tag{3.2.14}$$

where the asymptotic formula from which q can be calculated explicitly is

$$q(\mu_0, \mu_1) = \Gamma(1 - \mu_0)\Gamma(1 - \mu_1) \lim_{k\to\infty} \frac{\Gamma(k+1)}{\Gamma(k - \mu_1)} c_k. \tag{3.2.15}$$

Proof. The proof amounts to converting the notation of Schäfke-Schmidt to our notation. For $k \in \{0, 1\}$, define

$$\begin{aligned} D_k &:= \operatorname{diag}\left(\Gamma(\mu_k), (-1)^k\Gamma(1 - \mu_k)\right), \\ w(z) &:= z^{(\mu_0-1)/2}(1 - z)^{(\mu_1-1)/2}e^{-a(z-1)}. \end{aligned} \tag{3.2.16}$$

Then, the relation between our solutions and those in [50] is

$$\widehat{\Phi}_k = w\, D_k^{-1}\, \Phi_k\, g_0^{-1}. \tag{3.2.17}$$

By [50, Proposition 2.14, Theorem 2.15], the connection matrix $\widehat{C} = \widehat{\Phi}_0\widehat{\Phi}_1^{-1} \in \mathrm{GL}_2(\mathbb{C})$ is

$$\widehat{C} = \Gamma(\mu_1)\Gamma(1 - \mu_1) \begin{pmatrix} \hat{q}(\mu_0, -\mu_1) & -\hat{q}(\mu_0, \mu_1) \\ \hat{q}(-\mu_0, -\mu_1) & -\hat{q}(-\mu_0, \mu_1) \end{pmatrix}, \tag{3.2.18}$$

where

$$\hat{q}(\mu_0, \mu_1) = \lim_{k\to\infty} \frac{\Gamma(k+1)}{\Gamma(k - \mu_1)} c_k. \tag{3.2.19}$$

The theorem follows by the relations between our notation and that of [50]:

$$q(\mu_0, \mu_1) = \Gamma(1 - \mu_0)\Gamma(1 - \mu_1)\hat{q}(\mu_0, \mu_1) \quad \text{and} \quad C = D_0\widehat{C}D_1^{-1}. \tag{3.2.20}$$

□

The function q will be used in the next sections to find conditions for the unitarisability of two monodromies.

3.2.4 Unitarisability of the monodromy

In this part, we assume that all the definitions and results explained in section A.4.1 regarding the unitarisability of matrices in $\mathrm{SL}_2(\mathbb{C})$ are known. In our context, the two matrices to be unitarised are of the form $\widehat{M_0}$ and $C\widehat{M_1}C^{-1}$, where $\widehat{M_0}$ and $\widehat{M_1}$ can be chosen to be diagonal as they are the local monodromies of two solutions at regular singular points, and C is the connection matrix between those two local solutions.
The proof of the next proposition is deferred to the appendix, in B.10.

Proposition 3.1. *Let $M_0, M_1 \in \mathrm{SL}_2(\mathbb{C})\backslash\{\pm\mathbb{1}\}$ be irreducible and individually unitarisable. Let $\varphi, \varphi' \in \mathbb{CP}^1$ and $\psi, \psi' \in \mathbb{CP}^1$ be the respective eigenlines of M_0 and M_1. Then M_0 and M_1 are simultaneously unitarisable if and only if the cross-ratio*

$$[\varphi, \psi, \varphi', \psi'] \in \mathbb{R}_-. \tag{3.2.21}$$

The unitarisability criterion in proposition 3.1 for this case can be expressed as follows.

Proposition 3.2. *Consider two matrices $M_0 := \widehat{M_0}$ and $M_1 := C\widehat{M_1}C^{-1}$, where $\widehat{M_0}, \widehat{M_1} \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$ are diagonal matrices and*

$$C =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}). \tag{3.2.22}$$

(i) M_0 and M_1 are irreducible if and only if a, b, c and d are non-zero.

(ii) If M_0 and M_1 are irreducible and individually unitarisable, then M_0 and M_1 are simultaneously unitarisable if and only if the ratio $\frac{bc}{ad} \in \mathbb{R}_-$.

Proof. For (i), since M_0 is diagonal and not $\pm\mathbb{1}$, then M_0 and M_1 are reducible if and only if M_1 is upper or lower triangular. Writing $\widehat{M_1} = \mathrm{diag}(\beta, \beta^{-1})$, compute

$$M_1 = C\widehat{M_1}C^{-1} = \begin{pmatrix} ad\beta - bc\beta^{-1} & -ab(\beta - \beta^{-1}) \\ cd(\beta - \beta^{-1}) & ad\beta^{-1} - bc\beta \end{pmatrix}. \tag{3.2.23}$$

Then M_1 is upper or lower triangular if and only if at least one of a, b, c or d vanishes.

We now prove (ii). We note first that a diagonal matrix is unitarisable if and only if it is already unitary, because the eigenvalues of a unitary matrix are unitary complex numbers. Since M_0 is diagonal, its eigenlines in $\mathbb{CP}^1$ are $\varphi_1 = 0$ and $\varphi_2 = \infty$. Since $M_1 C = C\widehat{M_1}$, the eigenlines of M_1 in $\mathbb{C}^2$ are the columns of C, so its eigenlines in $\mathbb{CP}^1$ are $\psi_1 = a/c$ and $\psi_2 = b/d$. Then

$$[\varphi_1, \psi_1, \varphi_2, \psi_2] = \frac{\psi_2}{\psi_1} = \frac{bc}{ad}. \tag{3.2.24}$$

By proposition 3.1, M_0 and M_1 are simultaneously unitarisable if and only if this cross ratio is in $\mathbb{R}_-$. □

We will apply the above criterion to the monodromy of an ODE as follows. Consider a potential ξ with singularities at z_0 and z_1. Let τ_0 and τ_1 be the deck transformations corresponding to closed paths around z_0 and z_1 respectively, not enclosing the other singularity. Let Φ_0 and Φ_1 be local solutions to the ODE $\mathrm{d}\Phi = \Phi\xi$ chosen so that their respective monodromies

$$\widehat{M_0} := \Phi_0(\tau_0(z))\Phi_0^{-1} \quad \text{and} \quad \widehat{M_1} := \Phi_1(\tau_1(z))\Phi_1^{-1} \tag{3.2.25}$$

are diagonal. Let $C = \Phi_0\Phi_1^{-1}$ be the connection matrix between these two solutions with entries as in (3.2.22). The monodromies of Φ_0 at z_0 and z_1 are respectively

$$\begin{aligned} &M_0 = \widehat{M_0} \quad \text{and} \\ &M_1 = \Phi_0(\tau_1(z))\Phi_0^{-1} = C\Phi_1(\tau_1(z))\Phi_1^{-1}C^{-1} = C\widehat{M_1}C^{-1}. \end{aligned} \tag{3.2.26}$$

Suppose M_0 and M_1 are irreducible and individually unitarisable. By proposition 3.2, M_0 and M_1 are simultaneously unitarisable if and only if

$$\frac{bc}{ad} \in \mathbb{R}_-. \tag{3.2.27}$$

For the remainder of the paper we choose all parameters in the CHE to be real. Also, we assume that the CHE parameter a is positive. It is easy to check that the coefficients $U(k)$, $V(k)$ and $W(k)$ of the CHE recurrence (2.3.4) are positive for all sufficient large k. Under this assumption, the signs of the entries of the connection matrix C in (3.2.14) can be computed as follows.

Proposition 3.3. *Suppose $\mu_j < 1$ for $j \in \{0,1\}$ and that there exists $k_0 \in \mathbb{N}$ such that $U(k) > 0$, $V(k) > 0$ and $W(k) > 0$ for all $k \geq k_0$. If $c_{k_0-1} > 0$ and $c_{k_0} > 0$, then q defined in equation* (3.2.15) *satisfies $q \geq 0$. Similarly, if $c_{k_0-1} < 0$ and $c_{k_0} < 0$, then $q \leq 0$.*

Proof. Since $\mu_j < 1$ for $j \in \{0,1\}$ and the function $\Gamma(x) > 0$ for all $x > 0$, we have that $\Gamma(1-\mu_j) > 0$ for $j \in \{0,1\}$, $\Gamma(k+1) > 0$ and $\Gamma(k-\mu_1) > 0$ for all $k \geq k_0$.

By hypothesis $U(k)$, $V(k)$ and $W(k)$ are positive for all $k \geq k_0$. Hence, in the case of $c_{k_0-1} > 0$ and $c_{k_0} > 0$, the third term c_{k_0+1} must be positive as well. Then, by induction, $\{c_k\}_{k=k_0-1}^{\infty}$ are all positive coefficients. This implies that $q \geq 0$.

Similarly, if $c_{k_0-1} < 0$ and $c_{k_0} < 0$ the coefficients $\{c_k\}_{k=k_0-1}^{\infty}$ are negative and therefore $q \leq 0$. □

The criterion for unitarisability in proposition 3.2, the asymptotic formula for the connection matrix in theorem 3.1, and the recurrence relation for the CHE solution in equation (2.3.4) yield the following sufficient condition for the unitarisability of the monodromy.

Let us write the parameters for the CHE as a 5-tuple $\chi := (\mu_0, \mu_1, r_0, r_1, a) \in \mathbb{R}^5$. Define the finite integer

$$m(\chi) := \min_{k_0 \in \mathbb{N}} \{U(k,\chi) > 0, V(k,\chi) > 0 \text{ and } W(k,\chi) > 0 \; \forall\, k \geq k_0\}, \quad (3.2.28)$$

and the sets

$$\begin{aligned} \mathcal{S}_+ &:= \{\chi \in \mathbb{R}^5 \mid \exists\, \ell \geq m(\chi) \text{ such that } c_{\ell-1} > 0 \text{ and } c_\ell > 0\}, \\ \mathcal{S}_- &:= \{\chi \in \mathbb{R}^5 \mid \exists\, \ell \geq m(\chi) \text{ such that } c_{\ell-1} < 0 \text{ and } c_\ell < 0\}. \end{aligned} \quad (3.2.29)$$

Proposition 3.4. *If each of the four 5-tuples $(\pm\mu_0, \pm\mu_1, r_0, r_1, a) \in \mathbb{R}^5$ lies in $\mathcal{S}_+ \cup \mathcal{S}_-$, then the monodromy is unitarisable if and only if an odd number of these tuples lie in $\mathcal{S}_+$.*

Proof. By Proposition 3.3, if $\chi \in \mathcal{S}_+$ then $q(\chi) \geq 0$, and if $\chi \in \mathcal{S}_-$ then $q(\chi) \leq 0$. An odd number of the tuples lie in $\mathcal{S}_+$ if and only if

$$\frac{q(\mu_0,\mu_1)q(-\mu_0,-\mu_1)}{q(\mu_0,-\mu_1)q(-\mu_0,\mu_1)} \in \mathbb{R}_-, \quad (3.2.30)$$

that is, by Proposition 3.2 and the remarks thereafter, if and only if the monodromy is unitarisable. □

With this criteria we are ready to construct new trinoids.

3.2.5 Construction of new trinoids

The trinoid potential

To construct trinoids we choose the potential

$$\xi_T = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda Q_t & 0 \end{pmatrix} dz, \tag{3.2.31}$$

where

$$\begin{aligned} Q_t &:= t\left(-\frac{w_0}{4z^2} - \frac{w_1}{4(z-1)^2} + \frac{\hat{r}_0}{z} + \frac{\hat{r}_1}{z-1} + p^2\right), \\ t &:= -\frac{1}{4}\lambda^{-1}(\lambda-1)^2, \end{aligned} \tag{3.2.32}$$

and $w_0, w_1, \hat{r}_0, \hat{r}_1, p \in \mathbb{R}$ are free parameters. The parameters w_0 and w_1 will be the asymptotic end weights of the Delaunay ends at 0 and 1. The parameters $\hat{r}_0$ and $\hat{r}_1$ affect the weight of the irregular end, and p the shape of the trinoid.
With $\Lambda := \mathrm{diag}(\lambda^{1/2}, \lambda^{-1/2})$, the gauged potential

$$\xi_T.(\Lambda^{-1}) = \begin{pmatrix} 0 & 1 \\ Q_t & 0 \end{pmatrix} dz \tag{3.2.33}$$

has the form of the CHE potential defined in (3.2.13), where the coefficients $(\mu_0, \mu_1, r_0, r_1, a)$ in the CHE equation are related to the parameters $(w_0, w_1, \hat{r}_0, \hat{r}_1, p)$ in the potential ξ_T by

$$\mu_k = \sqrt{1 - w_k t}, \quad r_k = \hat{r}_k t, \quad a^2 = p^2 t, \quad k \in \{0, 1\}. \tag{3.2.34}$$

The monodromy of the trinoid potential is unitarisable along $\mathbb{S}^1$ if and only if that of the gauged potential $\xi_T.(\Lambda^{-1})$ is.

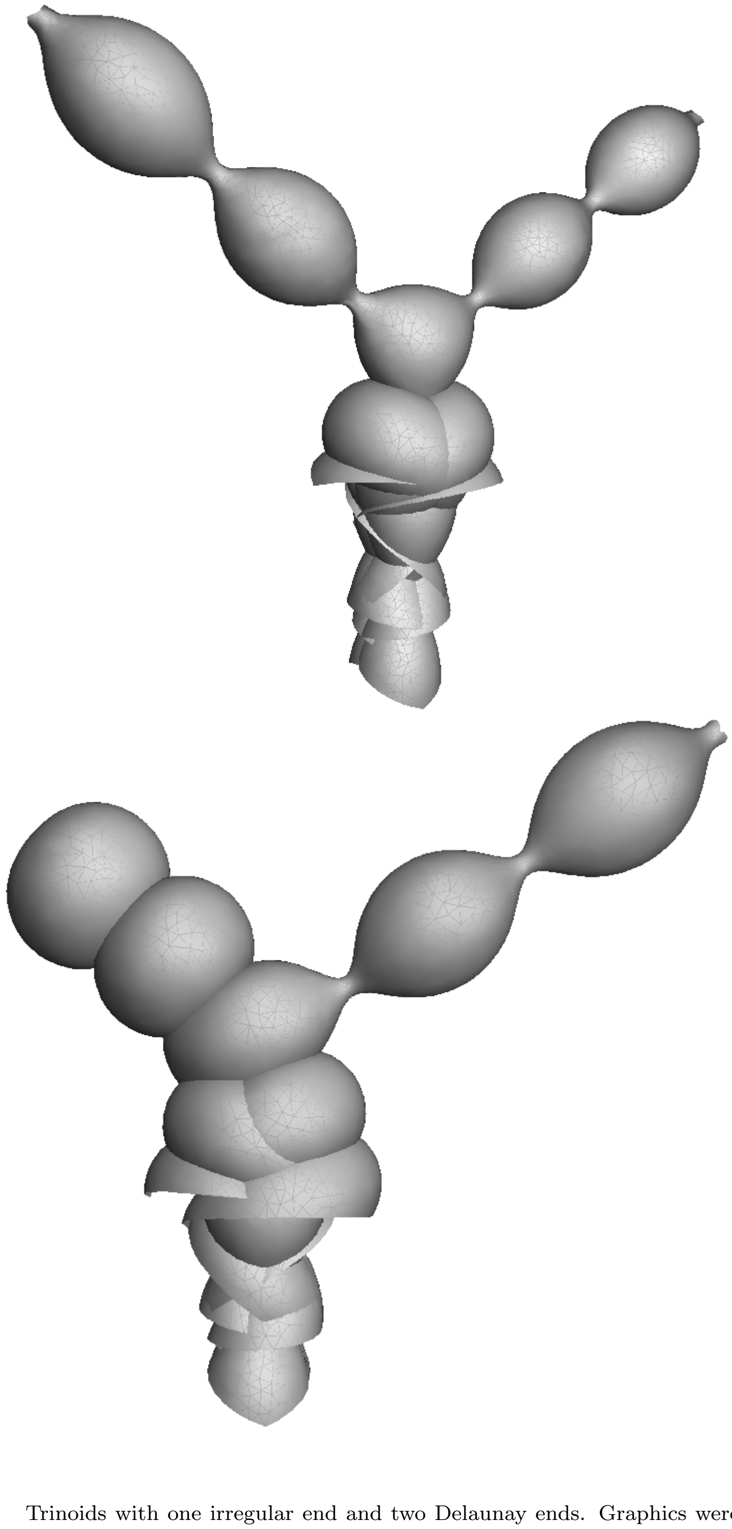

Figure 3.1: Trinoids with one irregular end and two Delaunay ends. Graphics were produced with CMCLab [51]. The regular end weights are either $\frac{1}{2}$ or $-\frac{1}{2}$ while the irregular end weights vary. The parameters $(w_0, w_1, \hat{r}_0, \hat{r}_1, p)$ used to construct each of them are $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$, and $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$.

Trinoids construction

Theorem 3.2. *Let ξ_T be a potential with unitarisable monodromy on $\mathbb{S}^1$ minus a finite set. Let Φ be a solution of $\mathrm{d}\Phi = \Phi\xi_T$. Then there exists a positive dressing h such that the* CMC *immersion induced by $h\Phi$ via the generalized Weierstrass representation on the universal cover descends to the three-punctured sphere. The ends at $z = 0$ and $z = 1$ are asymptotic to Delaunay surfaces.*

Proof. By [54], there exists a positive loop $h : \mathcal{D}_1 \to \mathrm{GL}_2(\mathbb{C})$ such that the monodromy of $h\Phi$ is unitary. The local unitary monodromies M_0 and M_1 satisfy the closing conditions $M_k(0) = \pm\mathbb{1}$ and $M_k'(0) = 0$, for $k \in \{0, 1\}$. Hence $h\Phi$ induces an immersion of the three-punctured sphere via the $\mathrm{GL}_2(\mathbb{C})$ version of the generalised Weierstrass representation. The ends at $z = 0$ and $z = 1$ are asymptotic to Delaunay cylinders with respective weights w_0 and w_1 by [35, Theorem 5.9]. □

Remark 3.1. Due to the structure of ξ_T, any trinoid T constructed from ξ_T in fact lies in a one-parameter family of trinoids T_κ with monotonically varying Delaunay end weights. If $(\mu_0, \mu_1, r_0, r_1, a)$ are the parameters for T, then the parameters for the family of trinoids T_κ are $(\kappa w_0, \kappa w_1, \kappa\hat{r}_0, \kappa\hat{r}_1, \sqrt{\kappa}p)$ with κ ranging over the interval $(0, 1]$.

Unitarisability conditions for the parameters

It remains to find values of the 5 parameters in ξ_T so that the monodromy is unitarisable. An algorithm to test the hypotheses of proposition 3.4 is as follows. For a 5-tuple $\chi = (\mu_0, \mu_1, r_0, r_1, a)$, consider the $(k+1)$-coefficient of the recurrence in (2.3.4), which is given by

$$c_{k+1}(\chi) = \frac{V(k,\chi)c_k(\chi) + W(k,\chi)c_{k-1}(\chi)}{U(k,\chi)}. \tag{3.2.35}$$

The radicals appearing in $c_{\ell+1}(\chi)$ can be eliminated, reducing the problem to showing that a polynomial is positive in an interval. Let $\Theta = (w_0, w_1, \hat{r}_0, \hat{r}_1, p) \in$

$\mathbb{R}^5$ be a choice of parameters for ξ_T. Note that $c_{\ell+1}(y_0, y_1, \hat{r}_0x^2, \hat{r}_1x^2, px)$ defines a rational function

$$\frac{\mathcal{P}_{\ell+1}(y_0, y_1, \hat{r}_0x^2, \hat{r}_1x^2, px)}{\mathcal{Q}_{\ell+1}(y_0, y_1, \hat{r}_0x^2, \hat{r}_1x^2, px)} \tag{3.2.36}$$

for some polynomials $\mathcal{P}_{\ell+1}, \mathcal{Q}_{\ell+1} \in \mathbb{R}[x, y_0, y_1]$ depending on ℓ and Θ. Define the polynomial functions $F_k, G_k \in \mathbb{R}[x, y_0, y_1]$

$$\begin{aligned} F_k(x, y_0, y_1) &:= \mathcal{P}_k(y_0, y_1, \hat{r}_0x^2, \hat{r}_1x^2, px), \qquad (3.2.37)\\ G_k(x, y_0, y_1) &:= F_k(x, y_0, y_1)F_k(x, y_0, -y_1)F_k(x, -y_0, y_1)F_k(x, -y_0, -y_1). \end{aligned}$$

Since G_k is even in y_0 and in y_1, then the function f_k depending on k and Θ

$$f_k(x) := G_k\left(x, \sqrt{1 - w_0x^2}, \sqrt{1 - w_1x^2}\right) \tag{3.2.38}$$

is in $\mathbb{R}[x]$.

Proposition 3.5. *Let $\Theta := (w_0, w_1, \hat{r}_0, \hat{r}_1, p) \in \mathbb{R}^5$ be a choice of parameters for the trinoid potential ξ_T and let*

$$\begin{aligned} \chi_{\pm\pm} &:= (\pm\mu_0, \pm\mu_1, r_0, r_1, a)\\ &= \left(\pm\sqrt{1 - w_0t}, \pm\sqrt{1 - w_1t}, \hat{r}_0t, \hat{r}_1t, p\sqrt{t}\right). \end{aligned} \tag{3.2.39}$$

Let $k_0 \in \mathbb{N}$ be such that for each of the four choices of signs, $U(k, \chi_{\pm\pm}) > 0$, $V(k, \chi_{\pm\pm}) > 0$ and $W(k, \chi_{\pm\pm}) > 0$ for all $k \geq k_0$. Suppose

(i) $f_{k_0-1}(x) \neq 0$ and $f_{k_0}(x) \neq 0$ along $x \in (0, 1)$,

(ii) for each of the four choices $\chi_{\pm\pm}$, and some $t_0 \in (0, 1)$,

$$\operatorname{sign} c_{k_0-1}(\chi_{\pm\pm}(t_0)) = \operatorname{sign} c_{k_0}(\chi_{\pm\pm}(t_0)), \tag{3.2.40}$$

(iii) of the four signs in (ii) an odd number are $+$ and an odd number are $-$.

Then, the monodromy with parameters Θ is unitarisable.

Proof. By its definition, $f_{\ell+1}(x)$ has a zero along $x \in (0, 1)$ if and only if at least one of the four functions $c_{\ell+1}(\chi_{\pm\pm}(t))$ has a zero along $t \in (0, 1)$. Thus by (i), none of the eight functions $c_{k_0-1}(\chi_{\pm\pm}(t))$ and $c_{k_0}(\chi_{\pm\pm}(t))$ has a zero along $t \in (0, 1)$. By (ii) and continuity, all $\chi_{\pm\pm} \in \mathcal{S}_+ \cup \mathcal{S}_-$, where $\mathcal{S}_+$ and $\mathcal{S}_-$ are the sets defined in (3.2.29). The monodromy is unitarisable by proposition 3.5(iii) and proposition 3.4. $\square$

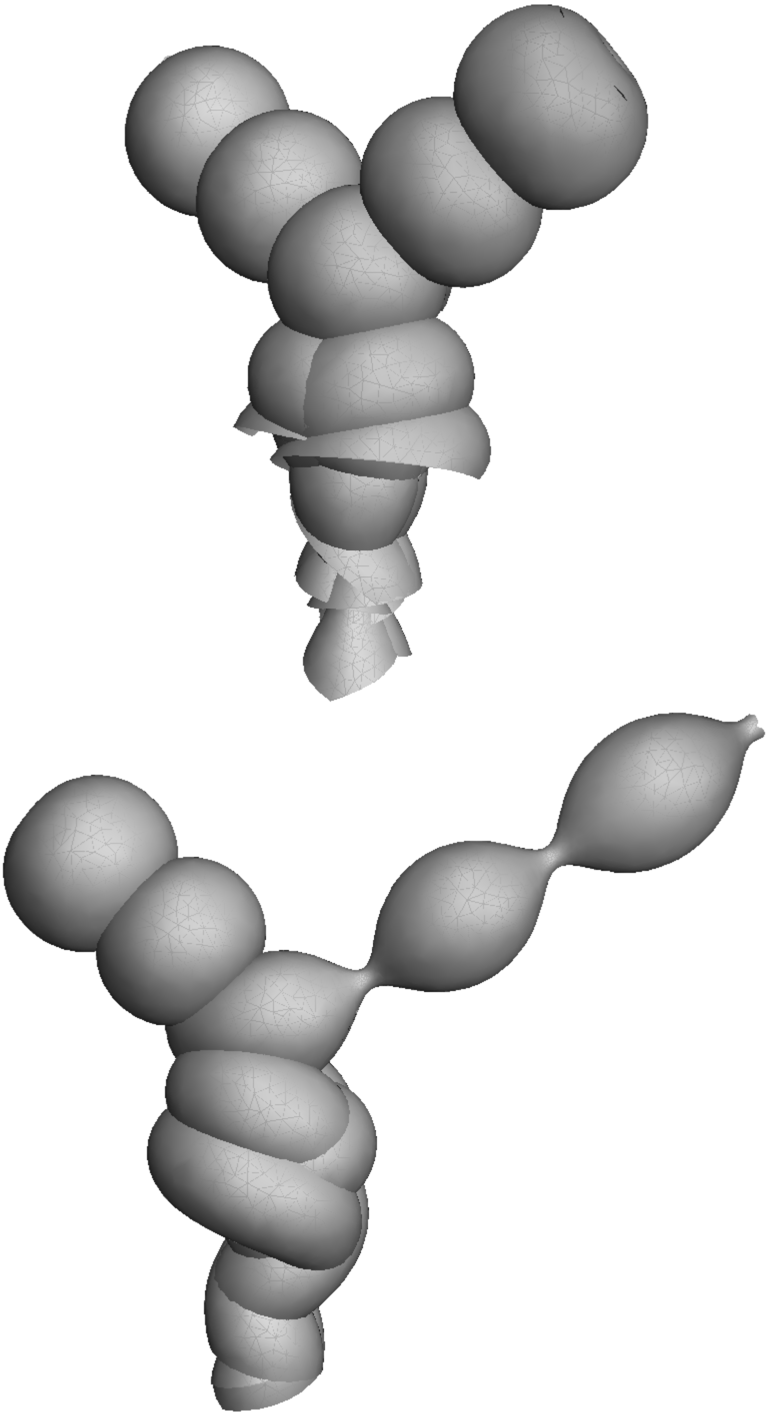

Figure 3.2: Trinoids with one irregular end and two Delaunay ends. The regular end weights are either $\frac{1}{2}$ or $-\frac{1}{2}$ while the irregular end weights vary. The parameters $(w_0, w_1, \hat{r}_0, \hat{r}_1, p)$ used to construct each of them are $\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{8}, -\frac{1}{8}, \frac{1}{8}\right)$, and $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right)$.

The examples in figures 3.1, 3.2 and 3.3 were computed by theorem 3.2 and the unitarisability criterion in proposition 3.5.

Proposition 3.6. *The conditions of proposition 3.5 are satisfied for the choice of parameters*

$$\Theta = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right). \tag{3.2.41}$$

Proof. For this choice of Θ, we compute

$$\begin{aligned} f_1(x) = x^4 &\left(393216 - 245760x + 243712x^2 - 57856x^3 + \right. \\ &\left. 26368x^4 - 5120x^5 + 1008x^6 - 252x^7 + 27x^8\right) \\ &\times \left(-1179648 + 1032192x - 2260992x^2 - 61440x^3 + \right. \\ &\left. 1267968x^4 - 241088x^5 + 189328x^6 - 41116x^7 + 6859x^8\right) \end{aligned} \tag{3.2.42}$$

$$\begin{aligned} f_2(x) = x^4 &\left(21743271936 - 6794772480x + 6455033856x^2 - 2021916672x^3 + \right. \\ &762642432x^4 - 174735360x^5 + 41717760x^6 - \\ &\left. 5871616x^7 + 1031424x^8 - 109248x^9 + 12240x^{10} - 1512x^{11} + 81x^{12}\right) \\ &\times \left(-79725330432 + 37144756224x - 24310185984x^2 + \right. \\ &358612992x^3 + 24553881600x^4 - 9189408768x^5 + \\ &7316135936x^6 - 1422535168x^7 + 570771456x^8 - \\ &\left. 78774080x^9 + 16092368x^{10} - 1562408x^{11} + 130321x^{12}\right) \end{aligned} \tag{3.2.43}$$

The *Sturm chain* of a polynomial $P(x)$ with real coefficients is the sequence of polynomials $P_0, P_1, \dots,$ such that

$$\begin{aligned} P_0 &= P, \\ P_1 &= P', \\ P_{i+1} &= -\operatorname{rem}(P_{i-1}, P_i), \end{aligned} \tag{3.2.44}$$

for $i \geq 1$, where P' is the derivative of P, and $\operatorname{rem}(P_{i-1}, P_i)$ is the remainder of the Euclidean division of polynomials of P_{i-1} by P_i. *Sturm's theorem* expresses the number of distinct real roots of P located in an interval in terms of the number of changes of signs of the values of the Sturm sequence at the bounds of the interval. This result can be applied to f_1 and f_2 to show that these two polynomials have no zero on the interval $(0, 1]$. The computations are easy but lengthy, so we omit them. This verifies proposition 3.5(i). The conditions (ii) and (iii) are verified by computing at $t_0 = \frac{4}{5}$, for $k \in \{1, 2\}$, yielding $c_k(\chi_{++}(t_0)) < 0$, $c_k(\chi_{+-}(t_0)) > 0$, $c_k(\chi_{-+}(t_0)) > 0$ and $c_k(\chi_{--}(t_0)) > 0$. In particular, the numerical results of the computation of these coefficients are

$$\begin{aligned}
&c_0(\chi_{++}(4/5)) = -0.016346 < 0, && c_1(\chi_{++}(4/5)) = -0.016346 < 0\\
&c_0(\chi_{+-}(4/5)) = 0.755228 > 0, && c_1(\chi_{+-}(4/5)) = 2.07486 > 0\\
&c_0(\chi_{-+}(4/5)) = 0.0202225 > 0, && c_1(\chi_{-+}(4/5)) = 0.0454555 > 0\\
&c_0(\chi_{--}(4/5)) = 3.79543 > 0, && c_1(\chi_{--}(4/5)) = 7.49907 > 0.
\end{aligned} \tag{3.2.45}$$

□

Theorem 3.3. *There exists a five parameter family of* CMC *trinoids with two Delaunay ends and one irregular end.*

Proof. The coefficients $U(k)$, $V(k)$ and $W(k)$ of the recurrence in (2.3.4) depend holomorphically on the parameters of the choice Θ. Each term of the sequence $c_{k+1}(\chi)$ is a rational function $\frac{\mathcal{P}_k(\chi)}{\mathcal{Q}_k(\chi)}$, where $\mathcal{P}_k$ and $\mathcal{Q}_k$ are polynomials in $W(0), \ldots, W(k)$ and in $V(0), \ldots, V(k)$ and $U(0), \ldots, U(k)$ respectively. Thus they also depend holomorphically on the parameters. Note that under the assumptions made for the CHE parameters, in particular $\mu_0 < 1$, the polynomial $U(k)$ is never zero. Therefore, the function f_k defined in (3.2.38) also depends holomorphically on the parameters. Let Θ be as in proposition 3.6, for which we have checked that the conditions of proposition 3.5 are satisfied. The polynomials f_1 and f_2 have a zero of order 4 at $x = 0$. A calculation shows that this order is preserved under a small perturbation of $\Theta \in \mathbb{R}^5$. For $i \in \{1, 2\}$, let us denote $f := f_i$. We have that $f(x) = x^4 g(x)$, where g has no zeros on $[0, 1]$, that is, $\varepsilon := \inf_{x\in[0,1]} |g(x)| > 0$. We can make a small perturbation of g by choosing $\tilde{g}$ such that $|g - \tilde{g}|_{[0,1]} < \varepsilon$. If we consider

$\tilde{f}(x) := x^4\tilde{g}(x)$, then we obtain that

$$\begin{aligned} |\tilde{f}(x)| &= x^4|\tilde{g}(x)| = x^4|g(x) - (g(x) - \tilde{g}(x))| \\ &\geq x^4\,||g(x)| - |g(x) - \tilde{g}(x)|| \\ &> x^4|\varepsilon - \varepsilon| = 0. \end{aligned} \tag{3.2.46}$$

Hence, $\tilde{f}$ has no zeros on $(0, 1]$. It is also easy to check that $x = 0$ is not a zero of the perturbation $\tilde{g}$:

$$|\tilde{g}(0)| = |g(0) - (g(0) - \tilde{g}(0))| \geq ||g(0)| - |g(0) - \tilde{g}(0)|| > |\varepsilon - \varepsilon| = 0. \tag{3.2.47}$$

It follows that the condition in proposition 3.5(i) is preserved under small perturbations of Θ. Also conditions (ii) and (iii) are trivially preserved under such perturbations. Hence by proposition 3.5, the monodromy of ξ_T is unitarisable in a small neighborhood of $\Theta \in \mathbb{R}^5$. Theorem 3.2 constructs a five parameter family of trinoids with one irregular end. $\square$

3.2.6 Formal computation of the end weight at ∞

We conclude by formally computing the weight at the irregular end of our trinoids. For our computations now, and also to use it in the next section, it is convenient to compute the series expansion of the monodromy at $\lambda = 1$. We do this in a general way in the next result. Let Φ be the solution of $\mathrm{d}\Phi = \Phi\xi$, $\Phi(z_0) = \mathbb{1}$ and M its monodromy with respect to the curve $\gamma(s) = z_0\, e^{is}$, with $s \in [0, 2\pi]$.

Proposition 3.7. *Suppose there exists a gauge g with monodromy $\pm\mathbb{1}$ for which*

$$\xi_g = \xi.g = \xi_0 + t\xi_1, \quad \xi_0 = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad \xi_1 = \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{3.2.48}$$

where

$$\alpha = dz, \quad \beta = h(z)\, dz \quad t = -\frac{1}{4}\lambda^{-1}(\lambda - 1)^2 = \sin^2\frac{\theta}{2} \tag{3.2.49}$$

and the matrix in ξ_1 is constant. Then, the series expansion of M with respect to λ at 1 is of the form

$$M = \pm \mathbb{1} + M_2(\lambda - 1)^2 + \mathrm{O}\left((\lambda - 1)^3\right), \tag{3.2.50}$$

where M_2 can be explicitly computed in terms of the coefficients of the series expansion of h.

Proof. In order to find the monodromy series of M, we first consider $\Psi = \Psi_0 + \Psi_1 t + \mathrm{O}\left(t^2\right)$, the solution to the initial value problem

$$\begin{cases} \mathrm{d}\Psi = \Psi\xi_g, \\ \Psi(z_0) = \mathbb{1}\ , \end{cases} \tag{3.2.51}$$

and let $P = P_0 + P_1 t + \mathrm{O}\left(t^2\right)$ be the monodromy of Ψ. From equation (3.2.51) we obtain the expression

$$\mathrm{d}\Psi_0 + \mathrm{d}\Psi_1 t + \mathrm{O}\left(t^2\right) = \left(\Psi_0 + \Psi_1 t + \mathrm{O}\left(t^2\right)\right)\left(\xi_0 + \xi_1 t\right), \tag{3.2.52}$$

which allows us to find P_0 and P_1 by comparing the coefficients of t. Equation (3.2.52) yields

$$\mathrm{d}\Psi_0 = \Psi_0\xi_0, \qquad \Psi_0(z_0) = \mathbb{1}, \tag{3.2.53a}$$

$$\mathrm{d}\left(\Psi_1\Psi_0^{-1}\right) = \Psi_0\xi_1\Psi_0^{-1}, \quad \Psi_1(z_0) = 0, \tag{3.2.53b}$$

where equation (3.2.53b) is found using equation (3.2.53a) in $\mathrm{d}\Psi_1 = \Psi_0\xi_1 + \Psi_1\xi_0$. The solution to (3.2.53a) is given by

$$\Psi_0 = \begin{pmatrix} 1 & \int\alpha \\ 0 & 1 \end{pmatrix}, \tag{3.2.54}$$

where the integral goes along a path based at z_0. Since $\int\alpha = z - z_0$, then $\int_\gamma \alpha = 0$ for $\gamma(s) = z_0\, e^{is}$, $s \in [0, 2\pi]$. Hence we find that $P_0 = \mathbb{1}$. Equation (3.2.53b) can be solved by integrating

$$\begin{aligned} \Psi_1\Psi_0^{-1} &= \int \Psi_0\xi_1\Psi_0^{-1} \\ &= \int \beta\, \Psi_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Psi_0^{-1} \\ &= \int \beta \begin{pmatrix} a + c(z - z_0) & b - (z - z_0)(a - d + c(z - z_0)) \\ c & d - c(z - z_0) \end{pmatrix}. \end{aligned} \tag{3.2.55}$$

Suppose that the curve γ encloses a finite number of punctures $z_1, \ldots, z_n$ of Σ. Then, by the Residue theorem,

$$P_1 = \left(\int_\gamma \left(\Psi_0 \xi_1 \Psi_0^{-1}\right)\right) P_0 \tag{3.2.56}$$
$$= 2\pi i \begin{pmatrix} \sum \operatorname{Res}(\beta(a + c(z - z_0)), z_k) & \sum \operatorname{Res}(\beta(b - (z - z_0)(a - d + c(z - z_0))), z_k) \\ \sum \operatorname{Res}(\beta c, z_k) & \sum \operatorname{Res}(\beta(d - c(z - z_0)), z_k) \end{pmatrix}.$$

The series for M in equation (3.2.50) follows from the monodromy of g being $\pm\mathbb{1}$ and thus the relation between M and P given by $M = \pm g(z_0) P g^{-1}(z_0)$. The last equation guarantees that $\Psi(z_0) = \mathbb{1}$. Since the series expression for P is in terms of t, one needs to use the relation between t and λ in order to obtain the specific form of M_2. $\square$

Let

$$\xi = \begin{pmatrix} 0 & 1 \\ tQ & 0 \end{pmatrix} dz \tag{3.2.57}$$

be a potential where Q is holomorphic on the circle $|z| = 2$ with Laurent series $Q = \sum_{k=-\infty}^{\infty} a_k z^k$. The first few terms of the series of the monodromy M of the solution of $\mathrm{d}\Phi = \Phi\xi$, $\Phi(2) = \mathbb{1}$ along the circle $|z| = 2$ are as follows.

Corollary 3.1. *Let M be the monodromy of Φ with respect to the curve $\gamma(s) = 2\,e^{is}$, $s \in [0, 2\pi]$ for the trinoid potential ξ_T. Define $\lambda = e^{i\theta}$ and let $\sum_{k=0}^{\infty} M_k \theta^k$ be the series expansion of M along $|z| = 2$. Then $M_0 = \mathbb{1}$, $M_1 = 0$ and*

$$M_2 = 2\pi i \begin{pmatrix} \frac{a_{-2}}{4} - \frac{a_{-1}}{2} & a_{-2} - a_{-1} - \frac{a_{-3}}{4} \\ \frac{a_{-1}}{4} & \frac{a_{-1}}{2} - \frac{a_{-2}}{4} \end{pmatrix}. \tag{3.2.58}$$

Proof. By the gauge $\Lambda := \operatorname{diag}(\lambda^{1/2}, \lambda^{-1/2})$ we obtain

$$\xi = \xi_T.(\Lambda^{-1}) = \xi_0 + t\xi_1, \quad \xi_0 = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad \xi_1 = \beta \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.2.59}$$

where

$$\alpha = dz, \quad \beta = Q_t\, dz \;\text{ and }\; t = -\frac{1}{4}\lambda^{-1}(\lambda - 1)^2 = \sin^2\frac{\theta}{2}. \tag{3.2.60}$$

The proof follows by applying proposition 3.7 with $z_0 = 2$ and the 1-forms ξ_0, ξ_1 above. To obtain the explicit P_1 from proposition 3.7, we integrate along the path γ enclosing the two regular singular points so, by the residue theorem, we obtain

$$\begin{aligned} P_1 &= \left(\int_\gamma \left(\Psi_0 \xi_1 \Psi_0^{-1}\right)\right) P_0 \\ &= 2\pi i \begin{pmatrix} a_{-2} - 2a_{-1} & 4a_{-2} - 4a_{-1} - a_{-3} \\ a_{-1} & 2a_{-1} - a_{-2} \end{pmatrix}. \end{aligned} \tag{3.2.61}$$

The series for the monodromy M of Φ follows from Λ not having monodromy and expressing the series in terms of θ as defined above. $\square$

The *force* associated to an element in the fundamental group [40, 9, 17] is the matrix $A \in \mathfrak{su}_2$ in the series expansion of the monodromy

$$M = \mathbb{1} + A\theta^2 + \mathrm{O}\left(\theta^3\right) \tag{3.2.62}$$

where $\lambda = e^{i\theta}$. The force is a homomorphism from the fundamental group to $\mathfrak{su}_2 \cong \mathbb{R}^3$. Its length $|A| = \sqrt{\det A}$ is the weight of the end.

Proposition 3.8. *The weights of a trinoid constructed from ξ_T with parameters $(w_0, w_1, \hat{r}_0, \hat{r}_1, p)$ at $z = 0, 1, \infty$ are respectively w_0, w_1, w_∞ where*

$$w_\infty = \frac{\pi}{8}\sqrt{(w_0 + w_1)^2 + 8\hat{r}_0(w_1 - 2\hat{r}_1) - 8\hat{r}_1 w_0}. \tag{3.2.63}$$

Proof. By corollary 3.1, the weight of the irregular end of a trinoid constructed using its potential (3.2.31) is given by

$$\frac{\pi}{2}\sqrt{a_{-2}^2 - a_{-1}a_{-3}} \tag{3.2.64}$$

where a_k are the Laurent coefficients of Q_t as before. The result follows by a computation of these coefficients. $\square$

Remark 3.2. The three weights (lengths of the weight vectors) determine the three weight vectors. In the case of all three ends being regular, a necessary condition for the unitarisability of the monodromy comes from the balancing formula [17, 54]: if the monodromy is unitary, the weight vectors $W_0, W_1, W_\infty \in \mathbb{R}^3$ satisfy $W_0 + W_1 + W_\infty = 0$. The weights are $w_k = \pm|W_k|$. It follows that $|w_i| \leq |w_j| + |w_k|$ for all permutations of $(0, 1, \infty)$. A counterexample can be found in the presence of one irregular end: for the parameters

$(w_0, w_1, \hat{r}_0, \hat{r}_1, p) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{17}{128}, \frac{1}{8}\right)$ the resulting monodromy is unitarisable, but the balancing formula does not hold for all permutations of $(0, 1, \infty)$. This counterexample means that the relation between the weight and the monodromy only holds for regular ends and so equation (3.2.63) just gives a formal way of computing the end weight at ∞.

However, Smyth and Tinaglia showed in [55] a very general computation of the force and torque 1-forms on an immersed CMC surface. If ω is the 1-form defined on Σ by

$$\omega = (H\, f + N) \times df, \tag{3.2.65}$$

where H is the mean curvature, f is the immersion with differential df and N its oriented normal, then one can check that ω is a closed 1-form on Σ. By Stokes' theorem one obtains that

$$W_i = \int_{\gamma_i} \omega \tag{3.2.66}$$

with γ_i the oriented loop around one of the punctures. This is the force of the component γ_i defined in [55] that corresponds to the general definition of an end weight of a CMC surface. One can check also by Stokes' theorem that with this definition the balancing $\sum W_i = 0$ holds.

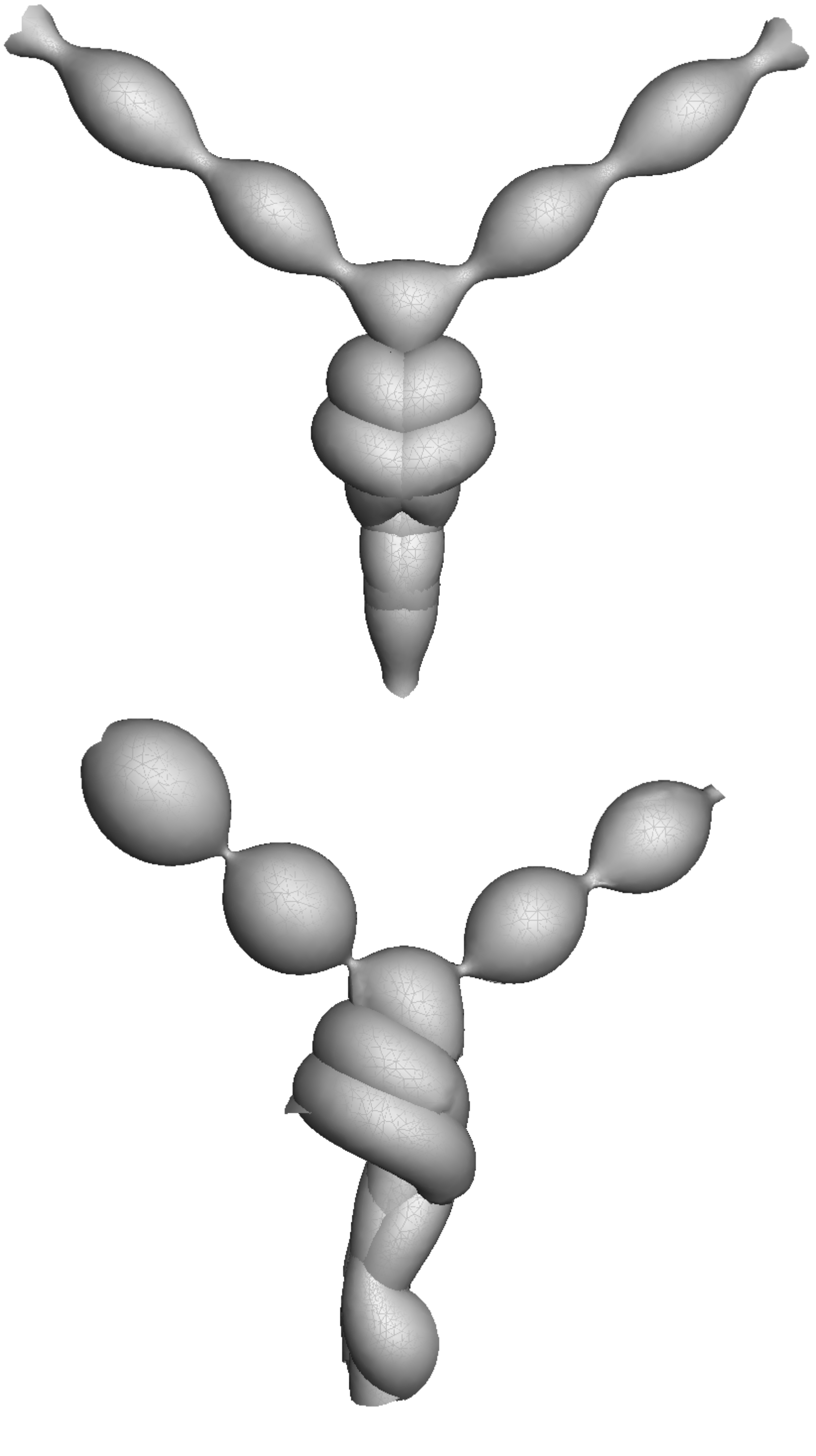

Figure 3.3: Trinoids with one irregular end and two Delaunay ends. The first surface has equal regular ends, while the second one has two different regular end weights. The parameters $(w_0, w_1, \hat{r}_0, \hat{r}_1, p)$ used to construct each of them are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$, and $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{17}{128}, \frac{1}{8}\right)$.

3.3 Cylinders with two irregular ends

In the present section we show how the DCHE can be used to construct CMC cylinders. In order to unitarise the monodromy matrix, we identify one symmetry in our potential and we impose a second symmetry by eliminating one of the parameters inherited from the DCHE. These symmetries yield symmetries in the monodromy that happens to have equal real elements on its diagonal. This allows us to define a diagonal unitariser on all of $\mathbb{S}^1$, by a theorem in the paper of Schmitt, Kilian, Kobayashi, and Rossman [54]. In the work of Kilian and Schmitt [36], similar techniques are used to construct CMC cylinders with two irregular ends parametrized by a holomorphic function. Our cylinders are the subclass of these that emerge from the DCHE.
Let us recall the form of the DCHE,

$$y'' + \alpha\left(1+\tfrac{1}{z^2}\right)y' + \left(\left(\beta_1+\tfrac{1}{2}\right)\tfrac{\alpha}{z} + \left(\tfrac{\alpha^2}{2}-\gamma\right)\tfrac{1}{z^2} + \left(\beta_{-1}-\tfrac{1}{2}\right)\tfrac{\alpha}{z^3}\right)y = 0. \tag{3.3.1}$$

First, let us find a suitable potential for this ODE which has irregular singularities at $z=0$ and $z=\infty$ and no other singularities. Setting $\Sigma=\mathbb{C}^*$, we can choose functions ν and ρ for an off-diagonal potential so that the DCHE becomes the associated ODE for the initial value problem (1.2.16). In particular, taking the functions

$$\begin{aligned} \nu &:= e^{\frac{\alpha}{z}-\alpha z} \quad \text{and} \\ \rho &:= \frac{e^{\alpha z-\frac{\alpha}{z}}\left(2\gamma z-\alpha(2\beta_{-1}+z(\alpha+2\beta_1 z+z)-1)\right)}{2z^3}, \end{aligned} \tag{3.3.2}$$

and plugging them into equation (1.4.3), one obtains the DCHE and thus prescribes this ODE in the algorithm for CMC surfaces seen in section 1.2.
In particular, the potential we are going to work with is

$$\xi_1 = \begin{pmatrix} 0 & e^{\frac{\alpha}{z}-\alpha z} \\ \frac{e^{\alpha z-\frac{\alpha}{z}}(2\gamma z-\alpha(2\beta_{-1}+z(\alpha+2\beta_1 z+z)-1))}{2z^3} & 0 \end{pmatrix} dz. \tag{3.3.3}$$

3.3.1 The potential for cylinders

Next let us introduce our choice of holomorphic potential to construct CMC cylinders from the DCHE. The form of the potential guarantees that the extrinsic closing conditions equation (1.2.31b) and equation (1.2.31c) for the immersion will be satisfied, as we will show in the next part of this chapter. The unitarisation of the monodromy that solves equation (1.2.31a) is deferred to section 3.3.3.

The potential for our construction on $\mathbb{C}^*$ is of the form

$$\xi_C = \begin{pmatrix} 0 & \lambda^{-1} \\ \frac{1}{4}\lambda + (\lambda-1)^2 Q(z) & 0 \end{pmatrix} \frac{dz}{z}, \tag{3.3.4}$$

where

$$Q(z) = \left(\frac{rz^2}{4} + \frac{s}{2z} + u + \frac{wz}{2} + \frac{r}{4z^2} \right). \tag{3.3.5}$$

The 4 umbilic points of the resulting surfaces will be located at the zeroes of $Q(z)$.

Next we show how to obtain the potential in (3.3.4) from the potential ξ_1 above.

Note that for

$$g_1 = \begin{pmatrix} \left(e^{\frac{\alpha}{z}-\alpha z}\right)^{1/2} & 0 \\ \frac{\alpha\left(z^2+1\right)}{2z^2\left(e^{\frac{\alpha}{z}-\alpha z}\right)^{1/2}} & \left(e^{\frac{\alpha}{z}-\alpha z}\right)^{-1/2} \end{pmatrix}, \quad g_2 = \begin{pmatrix} z^{1/2} & 0 \\ -\frac{1}{2}z^{-1/2} & z^{-1/2} \end{pmatrix}. \tag{3.3.6}$$

the gauge $\xi_1.(g_1 g_2)$ gives

$$\xi = \begin{pmatrix} 0 & 1 \\ \frac{\alpha^2+\alpha^2 z^4 - 2z^3(2\alpha\beta_1+\alpha)+(4\gamma+1)z^2-2z(2\alpha\beta_{-1}+\alpha)}{4z^2} & 0 \end{pmatrix} \frac{dz}{z}. \tag{3.3.7}$$

Setting $\alpha := \sqrt{r}\sqrt{\tau}$, $\beta_1 := -\frac{\sqrt{r}+w\sqrt{\tau}}{2\sqrt{r}}$, $\beta_{-1} := -\frac{\sqrt{r}+s\sqrt{\tau}}{2\sqrt{r}}$ and $\gamma := u\tau$ for $\tau = \lambda^{-1}(\lambda-1)^2$ (note that the parameters must depend on λ) and using the gauge $\Lambda = \mathrm{diag}(\lambda^{1/2}, \lambda^{-1/2})$ we obtain the potential ξ_C in (3.3.4).

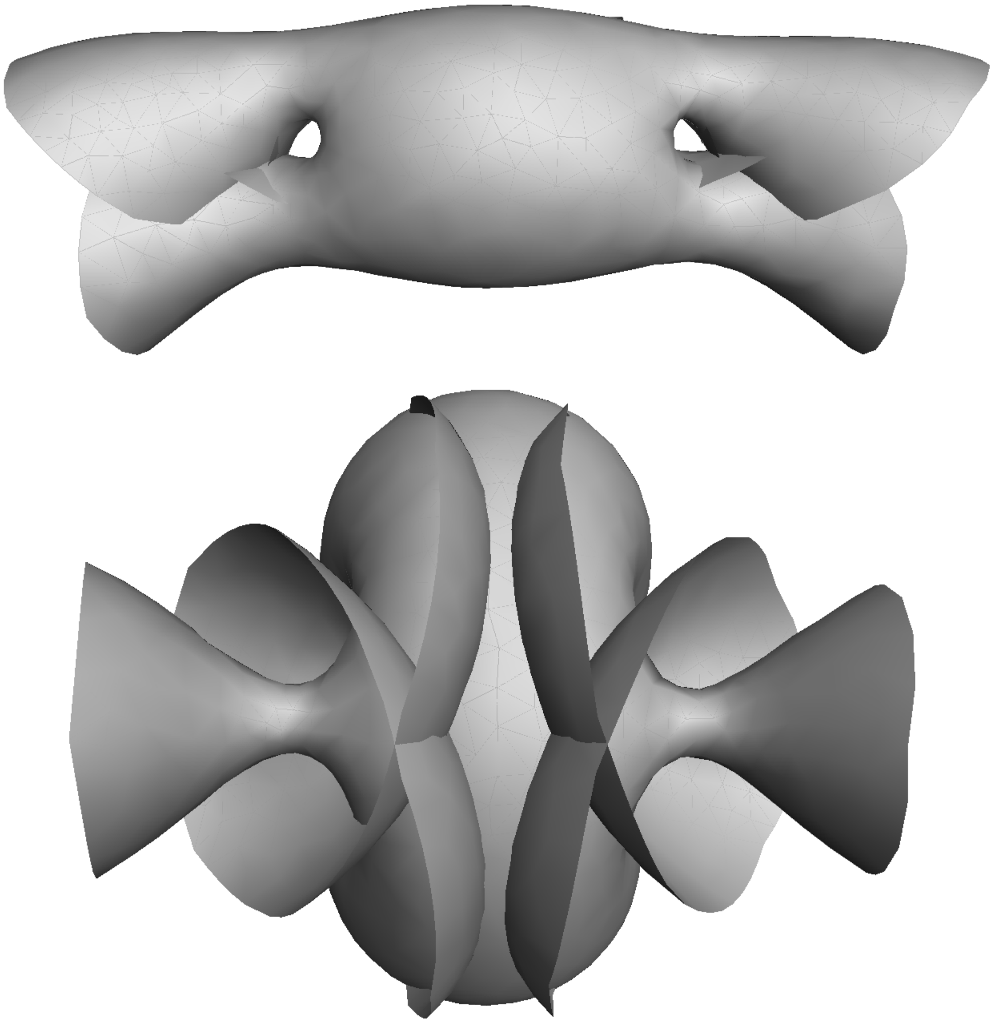

Figure 3.4: Cylinders with two irregular ends. The parameters r, s, u used to construct each of them are $\left(\frac{1}{25}, 0, \frac{1}{16}\right)$, and $\left(\frac{1}{32}, \frac{1}{12}, -\frac{1}{8}\right)$.

3.3.2 Series expansion of the monodromy and closing conditions

Let Φ be the solution of $\mathrm{d}\Phi = \Phi\xi$, $\Phi(1) = \mathbb{1}$ and M its monodromy with respect to the curve $\gamma(s) = e^{is}$, with $s \in [0, 2\pi]$. The first terms of the series expansion of M at $\lambda = 1$ can be computed in terms of the parameters r, s, u, w of the potential (3.3.4) as follows.

Corollary 3.2. *The series expansion of M with respect to λ at 1 is given by*

$$M = -\mathbb{1} + M_2(\lambda-1)^2 + \mathrm{O}\left((\lambda-1)^4\right), \tag{3.3.8}$$

with

$$M_2 = \frac{\pi i}{2}\begin{pmatrix}1 & -1\\ 1/2 & 1/2\end{pmatrix}\begin{pmatrix}-4u & 2s\\ -2w & 4u\end{pmatrix}\begin{pmatrix}1 & -1\\ 1/2 & 1/2\end{pmatrix}^{-1}. \tag{3.3.9}$$

Proof. Consider the gauge

$$g := \begin{pmatrix}\lambda^{-1/2} & 0\\ 0 & \lambda^{1/2}\end{pmatrix}\begin{pmatrix}z^{-1/2} & 0\\ 0 & z^{1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 1/2z & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}. \tag{3.3.10}$$

This gauge transforms the potential ξ_C into

$$\xi_g = \xi_C.g = \xi_0 + t\xi_1, \quad \xi_0 = \begin{pmatrix}0 & \alpha\\ 0 & 0\end{pmatrix}, \quad \xi_1 = \beta\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}, \tag{3.3.11}$$

where

$$\alpha = dz, \quad \beta = -\frac{4Q(z)}{z^2}\,dz \ \text{ and } \ t = -\frac{1}{4}\lambda^{-1}(\lambda-1)^2 = \sin^2\frac{\theta}{2}. \tag{3.3.12}$$

The proof follows by applying proposition 3.7 with $z_0 = 1$ and the 1-forms ξ_0, ξ_1 above. To obtain the explicit P_1 from proposition 3.7, we integrate along the path γ enclosing 0 so, by the residue theorem, we conclude that

$$P_1 = \left(\int_\gamma \left(\Psi_0\xi_1\Psi_0^{-1}\right)\right)P_0 = 2\pi i\begin{pmatrix}-4u & 2s\\ -2w & 4u\end{pmatrix}. \tag{3.3.13}$$

The series for M in equation (3.3.8) follows from g having monodromy $-\mathbb{1}$, the relation $M = -g(1)Pg^{-1}(1)$ and using the relation between t and λ. $\square$

Corollary 3.2 solves the closing conditions in equation (1.2.31b) and equation (1.2.31c). It turns out that in order to unitarise, we will need a Taylor series expansion for the trace of the monodromy. We compute it in the next result.

Theorem 3.4. *The trace of the monodromy has a Taylor expansion about $\lambda = 1$ of the form*

$$\operatorname{tr} M = -2 - \zeta(\lambda-1)^4 + \mathrm{O}\left((\lambda-1)^6\right). \tag{3.3.14}$$

Proof. For the potential ξ_C, let us replace Q with $x\,Q$ for $x > 0$ and consider the gauge

$$\xi = \xi_C.g = \begin{pmatrix} 0 & 1 \\ \lambda^{-1}(\lambda-1)^2\, x\,\frac{Q}{z^2} & 0 \end{pmatrix} dz \tag{3.3.15}$$

where

$$g := \begin{pmatrix} \lambda^{-1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix} \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/2z & 1 \end{pmatrix}. \tag{3.3.16}$$

Consider $\widehat{M}$ the monodromy associated to $\mathrm{d}\Phi = \Phi\xi$, and note that $M = -\widehat{M}$ due to the monodromy of g being $-\mathbb{1}$. To obtain the Taylor series of $\widehat{M}$ at $\lambda = 1$ we differentiate $\mathrm{d}\Phi = \Phi\xi$ with respect to λ and arrive at the inhomogeneous equation $\mathrm{d}\Psi = \Psi\xi + \Phi\partial_\lambda\xi$ for $\Psi := \partial_\lambda\Phi$. The ansatz $\Psi = C\,\Phi$ and variation of parameters gives

$$C(\tilde{z},\lambda) = \int_0^{\tilde{z}} \Phi\,\partial_\lambda\xi\,\Phi^{-1} = x\,(1-\lambda^{-2}) \int_0^{\tilde{z}} \Phi \begin{pmatrix} 0 & 0 \\ Q/z^2 & 0 \end{pmatrix} dz\,\Phi^{-1}. \tag{3.3.17}$$

Consequently $\partial_\lambda\Phi(\tilde{z},\lambda) = C(\tilde{z},\lambda)\,\Phi(\tilde{z},\lambda)$. Since $\widehat{M} = \Phi(2\pi,\lambda)$ we obtain

$$\partial_\lambda\widehat{M} = C(2\pi,\lambda)\,\widehat{M}. \tag{3.3.18}$$

Clearly $C(\tilde{z},1) \equiv 0$ and from $\operatorname{tr} C(\tilde{z},\lambda) = 0$ it follows that $\operatorname{tr}\partial_\lambda^n C(\tilde{z},\lambda) = 0$ for all $n \in \mathbb{N}$. The coefficients of the quadratic and cubic terms in the Taylor series have no trace since

$$\partial_\lambda^2\widehat{M}\Big|_{\lambda=1} = \partial_\lambda C(2\pi,\lambda)|_{\lambda=1} \quad \text{and} \quad \partial_\lambda^3\widehat{M}\Big|_{\lambda=1} = \partial_\lambda^2 C(2\pi,\lambda)\big|_{\lambda=1}. \tag{3.3.19}$$

Computing the fourth derivative gives

$$\partial_\lambda^4\widehat{M}\Big|_{\lambda=1} = \partial_\lambda^3 C(2\pi,\lambda)\big|_{\lambda=1} + 3\left(\partial_\lambda C(2\pi,\lambda)|_{\lambda=1}\right)^2 \tag{3.3.20}$$

Hence the trace in $\partial_\lambda^4\widehat{M}\Big|_{\lambda=1}$ comes from $\left(\partial_\lambda C(2\pi,\lambda)|_{\lambda=1}\right)^2$, which we compute next. The n-th derivative with respect to λ for $n \geq 2$ of the potential is

$$\partial_\lambda^n\xi = (-1)^n x\,n!\,\lambda^{-n-1} \begin{pmatrix} 0 & 0 \\ Q/z^2 & 0 \end{pmatrix} dz \tag{3.3.21}$$

and

$$\partial_\lambda C(2\pi,\lambda) = \int_0^{2\pi} \partial_\lambda \Phi\, \partial_\lambda \xi\, \Phi^{-1} + \Phi\, \partial_\lambda^2 \xi\, \Phi^{-1} - \Phi\, \partial_\lambda \xi\, \Phi^{-1} \partial_\lambda \Phi\, \Phi^{-1} \tag{3.3.22}$$

Using $\partial_\lambda \xi|_{\lambda=1} = 0$ and $\Phi(\tilde{z},1) = \begin{pmatrix} 1 & \int_0^{\tilde{z}} dz \\ 0 & 1 \end{pmatrix}$ gives

$$\begin{aligned} \partial_\lambda C(2\pi,\lambda)|_{\lambda=1} &= \int_0^{2\pi} \Phi\, \partial_\lambda^2 \xi\, \Phi^{-1}|_{\lambda=1} \\ &= 2\,x \int_0^{2\pi} \begin{pmatrix} \frac{Q}{z^2} \int_0^{\tilde{z}} dz & -\frac{Q}{z^2} \left(\int_0^{\tilde{z}} dz\right)^2 \\ \frac{Q}{z^2} & -\frac{Q}{z^2} \int_0^{\tilde{z}} dz \end{pmatrix} dz \end{aligned} \tag{3.3.23}$$

and consequently $\zeta := \operatorname{tr} \partial_\lambda^4 \widehat{M}\Big|_{\lambda=1} = \operatorname{tr}\left(\partial_\lambda C(2\pi,\,\lambda)|_{\lambda=1}\right)^2$, using the Cayley-Hamilton theorem, is given by

$$\begin{aligned} \zeta = 8x^2 &\left[\left(\int_0^{2\pi} [\frac{Q}{z^2} \int_0^{\tilde{z}} dz] dz \right)^2 \right. \\ &\left. - \left(\int_0^{2\pi} \frac{Q}{z^2} dz \right) \left(\int_0^{2\pi} [\frac{Q}{z^2} (\int_0^{\tilde{z}} dz)^2] dz \right) \right]. \end{aligned} \tag{3.3.24}$$

Thus, using equation (3.3.24), corollary 3.2 and the fact that $M = -\widehat{M}$, the trace of the monodromy M has a Taylor series expansion about $\lambda = 1$ of the form $\operatorname{tr} M = -2 - \zeta(\lambda-1)^4 + \mathrm{O}\left((\lambda-1)^6\right)$, completing the proof. □

A simple calculation gives the value of ζ in terms of the parameters in Q, obtaining that

$$\operatorname{tr} M = -2 + \left(8x^2\pi^2(4u^2 - sw)\right)(\lambda-1)^4 + \mathrm{O}\left((\lambda-1)^6\right). \tag{3.3.25}$$

A condition on the parameters appearing in equation (3.3.25) will be imposed later in order to guarantee unitarisability.

It is straightforward to check that the closing conditions are preserved by dressing, so they are still satisfied after unitarisation. Thus, it remains only to solve the unitarisation in equation (1.2.31a), which we do in the next sections.

3.3.3 Unitarisation of the monodromy

For the CMC immersion to close, it is left to show how to solve the unitarisation problem in equation (1.2.31a). Employing similar techniques as those in [36, 54] we construct a diagonal unitariser for M smooth on all $\mathbb{S}^1$. Once the monodromy is pointwise unitarisable on $\mathbb{S}^1$ then our unitariser can be constructed. In order to use this unitarisation criteria, we will need to impose a condition on our potential, which will give us an extra symmetry in the resulting surface.

We will use without proof (see section A.4.1 for a proof) the well-known fact that a matrix $M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$ is unitarisable if and only if $\operatorname{tr} M \in (-2, 2)$. We give now conditions for an element $M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$ being unitarisable by a diagonal element of $\mathrm{SL}_2(\mathbb{C})$ (see also lemma A.8 for a non-loop version of the result).

Proposition 3.9. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{C}^\infty(A_r, \mathrm{SL}_2(\mathbb{C}))$ *is unitarisable for some* $r \in (0, 1]$ *by a diagonal matrix if and only if* $d = \bar{a}$ *and* $|a|^2 \leq 1$.

Proof. Let $D = \operatorname{diag}(\rho, \rho^{-1})$ such that $DMD^{-1} = \begin{pmatrix} a & \rho^2 b \\ \rho^{-2} c & d \end{pmatrix}$ is unitary. Then $d = \bar{a}$ and $\rho^{-2}c = -\overline{\rho^2 b}$. As both summands in $\det M = a\bar{a} + (\rho\bar{\rho})^2 b\bar{b} = 1$ are non-negative and real, we conclude $a\bar{a} = |a|^2 \leq 1$.

Conversely, let $d = \bar{a}$ and $a\bar{a} \leq 1$. Then $\det M = 1$ implies that $bc \in [-1, 0]$ and hence $\sqrt{bc} \in i[0, 1]$. Thus $\overline{\sqrt{bc}} = -\sqrt{bc}$ or equivalently $\sqrt{\bar{c}/b} = -\sqrt{c/\bar{b}}$. Setting $\rho = \left(-c/\bar{b}\right)^{1/4}$, then $\bar{\rho} = -\rho$ and $D = \operatorname{diag}(\rho, \rho^{-1})$ unitarises M away from the set $\mathcal{I} = \{\lambda \in A_r : M(\lambda) = \pm\mathbb{1}\}$. □

If under certain conditions a monodromy representation of an ODE is unitarisable pointwise on $\mathbb{S}^1$, then by [54, Theorem 4] the monodromy is unitarisable by a dressing matrix on an r-circle, which is analytic in λ. Hence, using proposition 3.9, we deduce the following result.

Proposition 3.10. *If* $M : \mathbb{S}^1 \to \mathrm{SL}_2(\mathbb{C})$ *is pointwise unitarisable by a diagonal element except at finitely many points on* $\mathbb{S}^1$*, then there exists a holomorphic map* D *defined in the open disk* D_1 *with values in the subgroup of* *diagonal elements of* $\mathrm{SL}_2(\mathbb{C})$*, which* r*-unitarises* M *for every* $r \in (0,1)$*.*

3.3.4 Main theorem. Construction of cylinders

With this section we conclude the construction of cylinders with irregular ends from the DCHE. To do so, we will apply the unitarisation theory shown in section 3.3.3, and for this we need to impose the condition $s = w$ in the parameters of the potential (3.3.4). Under this assumption, for $\sigma(z) = \bar{z}$ and $\chi(z) = 1/\bar{z}$, we have that

$$Q = \overline{\sigma^* Q} \quad \text{and} \quad Q = \overline{\chi^* Q}. \tag{3.3.26}$$

Note that the second symmetry in equation (3.3.26) would not hold without the assumption of $s = w$, so henceforth we only consider these two parameters being equal. With these symmetries, we will show that the monodrmy has real trace along $\mathbb{S}^1$ and furthermore, under the condition $4u^2 - s^2 > 0$, the trace of M along $\mathbb{S}^1$ will be shown to be increasing from -2 at the point $\lambda = 1$. Note that this condition comes from our computation of the trace of M in equation (3.3.25) and that the parameters in figures 3.4 and 3.5 satisfy it. Then, a technique used in [32] and [36], that consists of re-scaling the potential

$$\xi_x = \begin{pmatrix} 0 & \lambda^{-1} \\ \frac{1}{4}\lambda + (\lambda - 1)^2\, x\, Q(z) & 0 \end{pmatrix} \frac{dz}{z} \tag{3.3.27}$$

by some small enough $x > 0$, ensures that the trace of M remains in $[-2, 2]$ for all $\lambda \in \mathbb{S}^1$. In this way, M is pointwise unitarisable by a diagonal matrix and proposition 3.10 gives a smooth unitariser for M.

Theorem 3.5. *Assume that* $4u^2 - s^2 > 0$*. For a small enough* $x > 0$*,* ξ_x *constructs via the generalised Weierstrass representation a family of* CMC *cylinders with irregular ends arising from the DCHE with four umbilic points located at the roots of* Q*.*

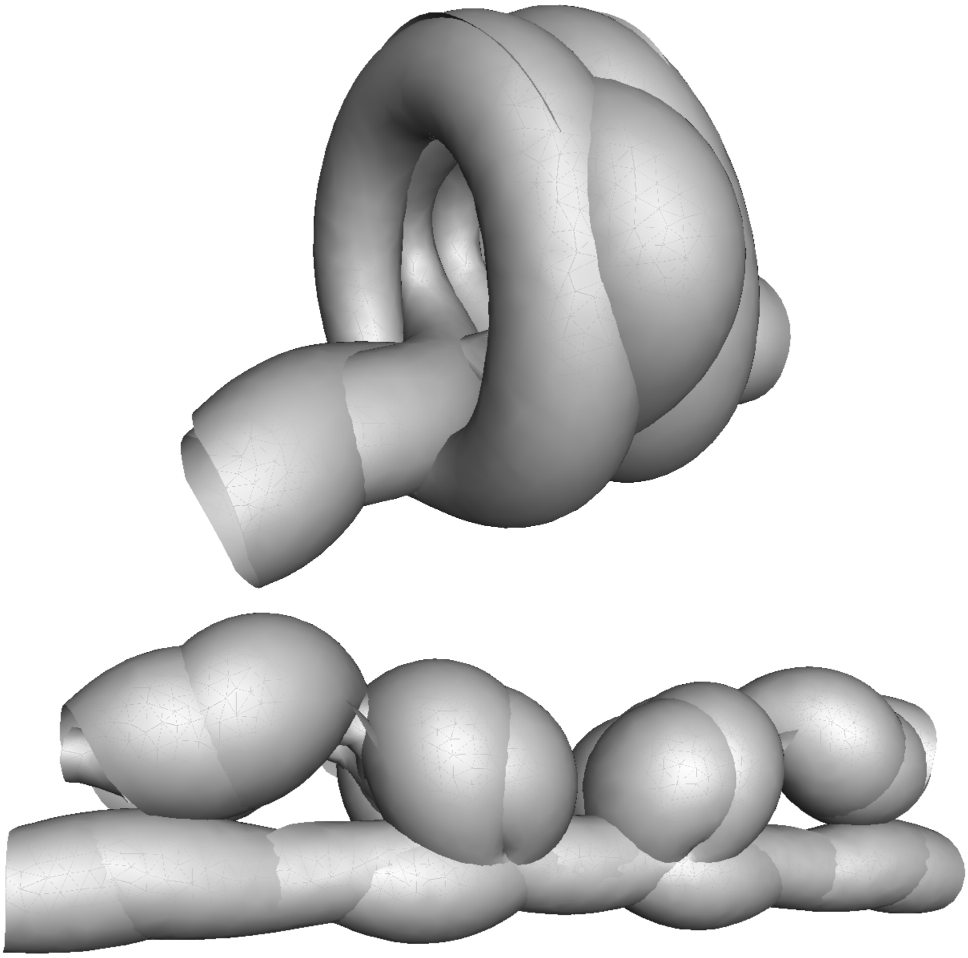

Figure 3.5: Cylinders with two irregular ends. The parameters r, s, u used to construct each of them are $\left(-\frac{1}{64}, -\frac{1}{2}, -\frac{1}{2}\right)$, and $\left(\frac{1}{64}, -\frac{1}{12}, \frac{1}{2}\right)$.

Proof. Consider Φ_x a solution of $\mathrm{d}\Phi_x = \Phi_x \xi_x$, $\Phi_x(1, \lambda) = \mathbb{1}$, and let M_x be its monodromy matrix along the circle $|z| = 1$. By corollary 3.2 the conditions $M_x|_{\lambda=1} = \pm\mathbb{1}$ and $\frac{d}{d\lambda} M_x\big|_{\lambda=1} = 0$ for the resulting surfaces to close are satisfied for all $x > 0$.

Let $\Lambda = \operatorname{diag}(\lambda, 1/\lambda)$ and $h = (i, -i)$. The symmetries on Q given in equation (3.3.26) imply that our scaled potential has the symmetries

$$\xi_x = \Lambda^{-1} \overline{\sigma^* \xi_x(1/\bar{\lambda})} \Lambda \quad \text{and} \quad \xi_x = \Lambda^{-1} h \overline{\chi^* \xi_x(1/\bar{\lambda})} h^{-1} \Lambda. \tag{3.3.28}$$

Naturally, the transformation $\overline{\sigma^* \Phi_x(1/\bar{\lambda})} =: \Psi$ defines a solution to the differential equation $\mathrm{d}\Psi = \Psi\left(\overline{\sigma^*\xi_x(1/\bar{\lambda})}\right)$, which in view of the first symmetry in (3.3.28) reads $\mathrm{d}\Psi = \Psi\,(\Lambda\,\xi_x\,\Lambda^{-1})$, that is,

$$\mathrm{d}\,(\Psi\Lambda) = (\Psi\Lambda)\,\xi_x. \tag{3.3.29}$$

Since any two solutions of this equation differ by a factor that is constant in z, that is by a matrix R in the loop group of $\mathrm{SL}_2(\mathbb{C})$ we conclude that Φ_x has the symmetry

$$R\Phi_x = \overline{\sigma^*\Phi_x(1/\bar{\lambda})}\Lambda, \tag{3.3.30}$$

where R does not depend on z. Similarly, using the second symmetry in (3.3.28), we obtain that

$$S\Phi_x = \overline{\chi^*\Phi_x(1/\bar{\lambda})}h^{-1}\Lambda, \tag{3.3.31}$$

for some z-independent S. Evaluation at the fixed point $z = 1$ of σ and χ gives $R = \Lambda$ and $S = h^{-1}\Lambda$. These symmetries induce symmetries on the monodromy, namely

$$M_x(\lambda) = \Lambda^{-1}\overline{M_x(1/\bar{\lambda})}^{-1}\Lambda, \tag{3.3.32a}$$
$$M_x(\lambda) = \Lambda^{-1}h\overline{M_x(1/\bar{\lambda})}h^{-1}\Lambda, \tag{3.3.32b}$$

where equation (3.3.32a) is deduced using that, for a deck transformation on the universal cover $\mathbb{C}$, its composition with σ is equal to the composition of σ with the opposite deck transformation and that monodromies with respect to opposite deck transformations are inverses of each other.

Putting together the symmetries in equations (3.3.32) one gets that M_x has real diagonal elements on $\mathbb{S}^1$ which are equal to each other, and thus it has real trace on $\mathbb{S}^1$.

Let $M = M_1$ and $t = -1/4\,\lambda^{-1}(\lambda-1)^2$ as before. The assumption $4u^2 - s^2 > 0$ together with theorem 3.4 implies that $t = 0$ is a local minimum of $\operatorname{tr} M$. Hence, there exists $x_0 \in (0, 1]$ such that $\operatorname{tr} M \leq 2$ for $t \in [0, x_0)$. It follows that $\operatorname{tr} M_{x_0} \in (-2, 2)$ for $\lambda \in \mathbb{S}^1 \setminus \{1\}$ and is therefore unitarisable there.

Note that, since the diagonal elements of M are equal and real on $\mathbb{S}^1$ and $\operatorname{tr} M \in (-2, 2)$ for $\lambda \in \mathbb{S}^1 \setminus \{1\}$, then their product is less than 1. Hence by proposition 3.9 and proposition 3.10 there exists a map D such that DMD^{-1} is unitary in all $\mathbb{S}^1$. The closing conditions are still satisfied after unitarising. $\square$

Very similar computations as those in section 3.2.6, allow one to find the weight at ∞ for the cylinders constructed in theorem 3.5, which is given by

$$w_\infty = 2\pi\sqrt{4u^2 - s^2}. \tag{3.3.33}$$

As already discussed in section 3.2.6, these are just formal computations that are valid only in the case of regular ends.

3.4 Perturbed Delaunay cylinders with one irregular end

In the spirit of the recent work [43], our goal is to employ a second order ODE, the BHE, to find a new family of perturbed Delaunay cylinders with constant mean curvature. In [43], the differential equation considered is the Bessel equation, which shares with the BHE the singularities' behaviour: they both have one regular singularity at $z = 0$ and one irregular singularity at $z = \infty$ of rank 2. Thus, we aim to find a family of CMC surfaces with one asymptotically Delaunay end and one irregular end, corresponding respectively to the regular and irregular singularities in the BHE. The first step in the construction, as in the previous sections, is to write down a suitable potential ξ. The fact that one of the ends of the surfaces is asymptotic to half a Delaunay surface follows because at this end ξ is a perturbation of the potential of a Delaunay surface, which is proved in the work of Kilian, Rossman and Schmitt [35].
Since our Riemann surface Σ will be now the twice-punctured Riemann sphere, it will be enough to guarantee closing conditions at the Delaunay end in order to solve the period problem.
Let us first show how the BHE can be encoded in our potential. Let $\Sigma = \mathbb{C}^*$ be the punctured complex plane. We want to find functions ν and ρ so that

the BHE

$$y'' + \left(\frac{\alpha+1}{z} - \beta - 2z\right) y' + \left(\gamma - \alpha - 2 - \frac{1}{2z}\left(\beta(\alpha+1)+\delta\right)\right) y = 0 \quad (3.4.1)$$

appears in a holomorphic off-diagonal potential in Σ. With this purpose we pick

$$\begin{aligned} \nu &:= z^{-\alpha-1}e^{z(z+\beta)} \quad \text{and} \\ \rho &:= \frac{1}{2}z^{\alpha}e^{-z(z+\beta)}\left(2z(\alpha-\gamma+2)+\alpha\beta+\beta+\delta\right). \end{aligned} \quad (3.4.2)$$

Plugging ν and ρ into equation (1.4.3), one obtains the BHE and, in particular, prescribes this ODE in the initial value problem of the generalised Weierstrass representation for CMC surfaces. For later use, we consider an off-diagonal potential with ν and ρ the entries of the secondary diagonal,

$$\xi_0 = \begin{pmatrix} 0 & z^{-\alpha-1}e^{z(z+\beta)} \\ \frac{1}{2}z^{\alpha}e^{-z(z+\beta)}\left(2z(\alpha-\gamma+2)+\alpha\beta+\beta+\delta\right) & 0 \end{pmatrix} dz. \quad (3.4.3)$$

3.4.1 Perturbed Delaunay potential

Constructing CMC surfaces with two ends via the BHE is done in the following two steps:

- Write down a potential on $\mathbb{C}^*$ which prescribes the BHE as associated ODE and which is locally gauge-equivalent to a perturbation of a Delaunay potential at $z=0$.
- Solve the period problem for the monodromy representation M (to be done in section 3.4.2).

Let us begin by writing down the potential which will be used to produce cylinders with one irregular end and one asymptotic Delaunay end. Near the puncture $z=0$ the potential is a local perturbation of a Delaunay potential via gauge equivalence.

Let $\Sigma = \mathbb{C}^*$ and let $r \in (-\infty, 1) \setminus \{0\}$, and $s, u, w \in \mathbb{R}$. Define the $\mathfrak{sl}_2(\mathbb{C})$-valued potential by

$$\xi_{PD} = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda z^2 + \lambda Q_t & 0 \end{pmatrix} dz, \quad (3.4.4)$$

where

$$Q_t = t\left(-\frac{r}{4z^2} + \frac{s}{2z} + \frac{w}{4} + \frac{uz}{2}\right) \tag{3.4.5}$$

and $t := -1/4\,\lambda^{-1}(\lambda-1)^2$, for all $\lambda \in \mathbb{S}^1$.

Let us show now that the potential in (3.4.3) with associated second order ODE the BHE, is gauge-equivalent to the potential ξ_{PD} defined above. Then, we will also prove that ξ_{PD} is a perturbed Delaunay potential.

Note that for

$$g_0 = \begin{pmatrix} \left(z^{-\alpha-1}e^{z(z+\beta)}\right)^{1/2} & 0 \\ \frac{1+\alpha-z(2z+\beta)}{2z\left(z^{-\alpha-1}e^{z(z+\beta)}\right)^{1/2}} & \left(z^{-\alpha-1}e^{z(z+\beta)}\right)^{-1/2} \end{pmatrix}. \tag{3.4.6}$$

the gauge $\xi_0.g_0$ gives the simple potential

$$\xi = \begin{pmatrix} 0 & 1 \\ \frac{\alpha^2-1}{4z^2} + \frac{\delta}{2z} + \frac{1}{4}\left(\beta^2 - 4\gamma\right) + z\beta + z^2 & 0 \end{pmatrix} dz. \tag{3.4.7}$$

Defining

$$\alpha := \sqrt{1-rt}, \delta := st, \beta := \frac{u}{2}t \quad \text{and} \quad \gamma := \frac{1}{16}\left(u^2t - 4w\right)t, \tag{3.4.8}$$

and using the gauge $\Lambda = \operatorname{diag}(\lambda^{1/2}, \lambda^{-1/2})$ we obtain the constructing potential ξ_{PD} in (3.4.4).

Lemma 3.2. *Let ξ_{PD} be the potential defined above. Then there exists a neighbourhood U of $z=0$ and a positive gauge g such that the expansion of $\xi_{PD}.g$ is*

$$A\frac{dz}{z} + \mathrm{O}\left(z^0\right)dz, \tag{3.4.9}$$

for A a Delaunay residue. That is, ξ_{PD} is gauge-equivalent to a perturbed Delaunay potential.

Proof. Let

$$g_z = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}, \quad g_r = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\lambda & a+b\lambda \end{pmatrix}. \tag{3.4.10}$$

For the real values $a = \frac{1}{4}\left(1+\sqrt{1-r}\right)$ and $b = \frac{1}{4}\left(1-\sqrt{1-r}\right)$, one gets that the gauged potential $\xi_{PD}.(g_z g_r)$ has a simple pole at $z=0$ and is of the form

$$\begin{pmatrix} 1/2 & 1/\lambda \\ -\frac{\lambda}{4}(a\lambda+b)(a+b\lambda) & -1/2 \end{pmatrix}\frac{dz}{z} + \mathrm{O}\left(z^0\right)dz. \tag{3.4.11}$$

Note that, since $r \in (-\infty, 1) \setminus \{0\}$, both $a, b \in \mathbb{R}$ and also $a + b = 1/2$. Thus, the last potential can be easily gauged to obtain a Delaunay residue by considering $g_{ab} = \begin{pmatrix} 1 & 0 \\ -\lambda/2 & a + b\lambda \end{pmatrix}$. Therefore, ξ_{PD} is gauge-equivalent to the perturbed Delaunay potential

$$\begin{pmatrix} 0 & \frac{1}{4}\left(1+\sqrt{1-r}\right)\lambda^{-1} + \frac{1}{4}\left(1-\sqrt{1-r}\right) \\ \frac{1}{4}\left(1+\sqrt{1-r}\right)\lambda + \frac{1}{4}\left(1-\sqrt{1-r}\right) & 0 \end{pmatrix} \frac{dz}{z} + \mathrm{O}\left(z^0\right) dz, \quad (3.4.12)$$

for the gauge $g = g_z g_r g_{ab}$. $\square$

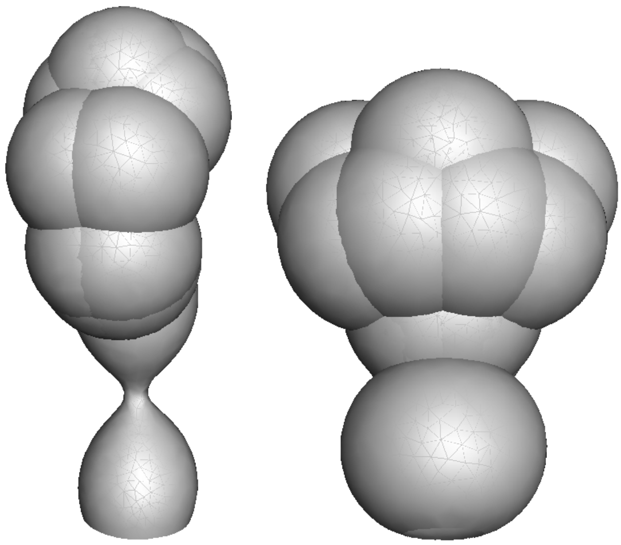

Figure 3.6: Perturbed Delaunay surfaces with one Delaunay end and one irregular end. The parameters r, s, u, w used to construct each of them are $\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{8}, \frac{1}{8}\right)$, and $\left(-\frac{1}{4}, -\frac{1}{8}, \frac{1}{2}, \frac{1}{5}\right)$.

3.4.2 Closing periods and main result

Since the fundamental group of the twice-punctured Riemann sphere has only 1 generator, we just need to solve the monodromy problem at $z = 0$. Consider the differential equation $\mathrm{d}\Phi = \Phi\xi_{PD}$. Since ξ_{PD} is a perturbed Delaunay potential, the $z^A P$ lemma 1.7 assures that under certain conditions on the eigenvalues of A, there exists a solution in a neighbourhood of $z = 0$ of

the form $\Phi_0 = z^A P$ with A the Delaunay residue of the perturbed potential, such that $P(0,\lambda) = \mathbb{1}$. It is easy to check that, restricting our choice of r to $r \in (-\infty, 1) \setminus \mathbb{Z}_{\leq 0}$, we avoid the resonance points of A (see lemma 1.7).
Then, $\Phi_0 = z^A P$ and taking as initial value $\Phi(1) = P(1)$, the monodromy of Φ_0 around $z = 0$ is given by

$$M(\lambda) = \exp(2\pi i A) = \cos(2\pi\mu)\mathbb{1} + \frac{1}{\mu}\sin(2\pi\mu)A, \tag{3.4.13}$$

where $\pm\mu$ are the eigenvalues of A, whose squares are given by

$$\mu(\lambda)^2 = -\det A = a^2 + b^2 + ab\lambda^{-1} + ab\lambda = \frac{r}{16}\lambda + \frac{2-r}{8} + \frac{r}{16}\lambda^{-1}. \tag{3.4.14}$$

Thus, since $a + b = \frac{1}{4}\left(1 + \sqrt{1-r}\right) + \frac{1}{4}\left(1 - \sqrt{1-r}\right) = 1/2$, M is unitary for all $\lambda \in \mathbb{S}^1$ and (1.2.31a) is solved. As a consequence, also the closing conditions (1.2.31b) and (1.2.31c) hold, so the immersion will factor through the fundamental group.

Theorem 3.6. *Let $\Sigma = \mathbb{C}^*$ and let $r \in (-\infty, 1) \setminus \mathbb{Z}_{\leq 0}$. Then, there exists a conformal* CMC *immersion $f : \Sigma \to \mathbb{R}^3$ with one end asymptotic to half a Delaunay surface and one irregular end.*

Proof. Let ξ_{PD} be a potential as in (3.4.4). A local solution Φ in a neighbourhood of $z = 0$ of the initial value problem $\mathrm{d}\Phi = \Phi\xi_{PD}$, $\Phi(1) = P(1)$ can be found with the $z^A P$ lemma. Let M be the monodromy of Φ around the puncture $z = 0$. By equation (3.4.13) and the subsequent remarks, M is unitary for all $\lambda \in \mathbb{S}^1$, $M(1) = -\mathbb{1}$ and $\partial_\lambda M|_{\lambda=1} = 0$, so the monodromy problem is solved.
Then, the general Weierstrass representation constructs a CMC immersion f in $\mathbb{R}^3$ which has two ends corresponding to the singularities from the BHE. By lemma 3.2, the potential is locally gauge-equivalent to a Delaunay potential and thus, by the asymptotics theorem of Kilian, Rossman and Schmitt, [35, Theorem 5.9], the end at $z = 0$ is asymptotic to half a Delaunay surface. $\square$

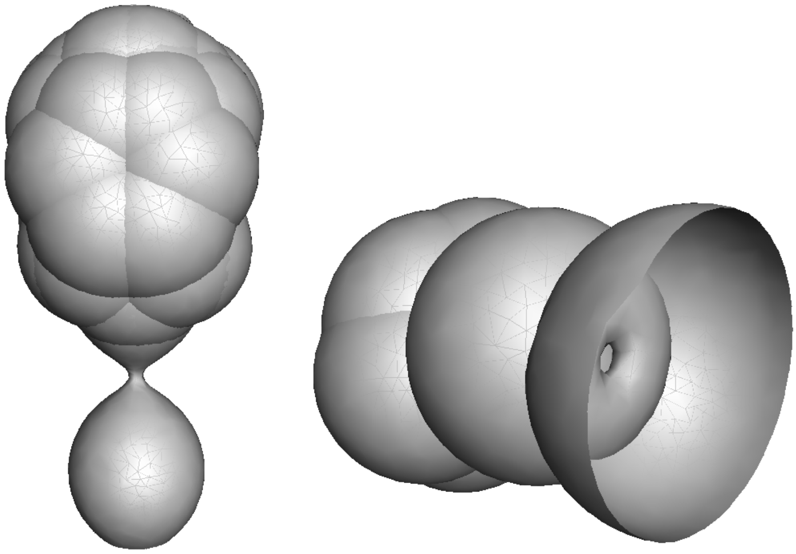

Figure 3.7: Perturbed Delaunay surfaces with one Delaunay end and one irregular end. The parameters r, s, u, w used to construct each of them are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$, and $\left(-\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}\right)$.

3.5 CMC surfaces with the topology of the plane

We finish this chapter constructing the last family of CMC surfaces for Heun's Differential Equations. The last equation to consider in our programme is the THE. Recall that this equation is defined in $\Sigma = \mathbb{C}$ and is of the form

$$y'' - \left(\gamma + 3z^2\right) y' + (\alpha + (\beta - 3)z)\, y = 0. \tag{3.5.1}$$

This equation can be encoded in a potential by finding the appropriate functions ν and ρ such that equation (1.4.3) turns into the THE. By easy computations we get a potential that encodes this second order differential equation. Pick

$$\begin{aligned} &\nu := e^{z^3+\gamma z} \quad \text{and} \\ &\rho := -e^{-z\left(z^2+\gamma\right)}\left(\alpha + (\beta - 3)z\right), \end{aligned} \tag{3.5.2}$$

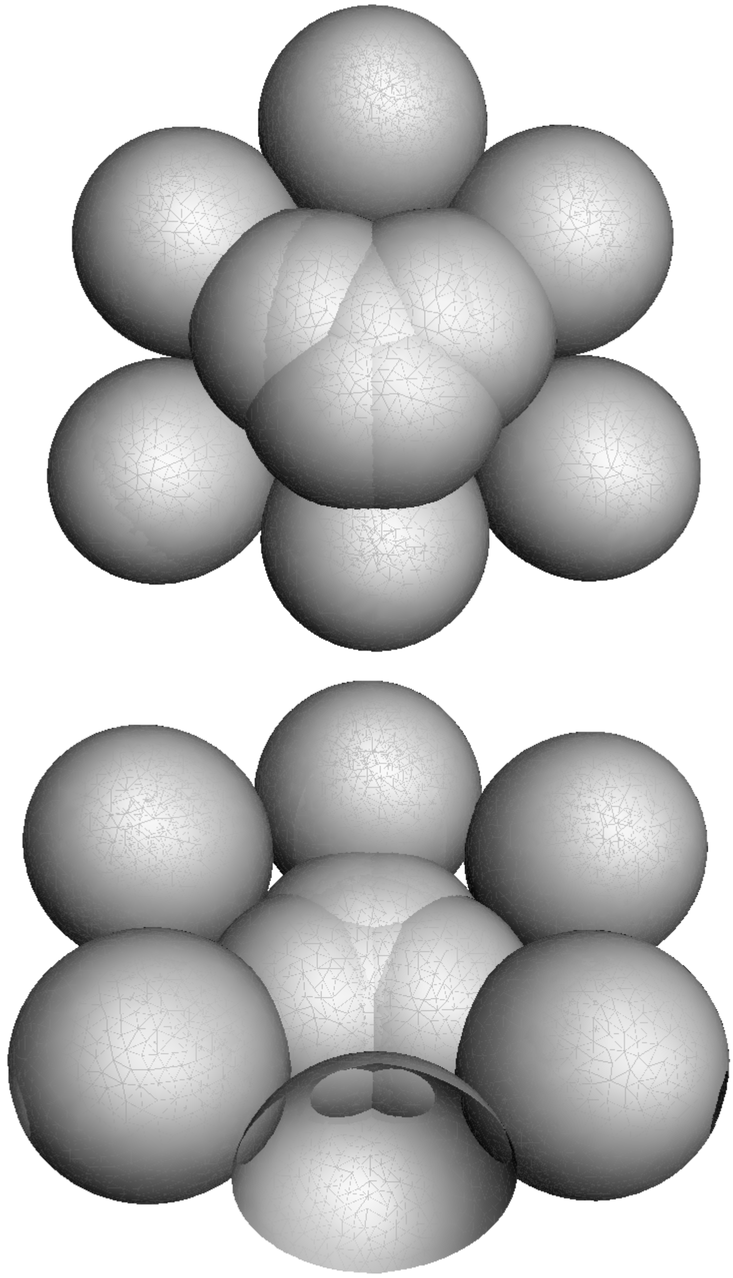

Figure 3.8: CMC surfaces with the topology of the plane. The parameters r, s, u used to construct each of them are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, and $\left(-\frac{1}{3}, \frac{1}{3}, \frac{1}{8}\right)$.

so that

$$\xi_0 = \begin{pmatrix} 0 & e^{z^3+\gamma z} \\ -e^{-z\left(z^2+\gamma\right)}\left(\alpha+(\beta-3)z\right) & 0 \end{pmatrix} dz \tag{3.5.3}$$

is a potential that carries the THE.
Consider now the loop

$$g_0 = \begin{pmatrix} \left(e^{z^3+z\gamma}\right)^{1/2} & 0 \\ -\frac{3z^2+\gamma}{2\left(e^{z^3+z\gamma}\right)^{1/2}} & \left(e^{z^3+z\gamma}\right)^{-1/2} \end{pmatrix}. \tag{3.5.4}$$

We can simplify ξ_0 by employing this gauge, obtaining

$$\xi = \xi_0.g_0 = \begin{pmatrix} 0 & 1 \\ \frac{1}{4}\left(9z^4+6\gamma z^2-4\beta z+\gamma^2-4\alpha\right) & 0 \end{pmatrix} dz. \tag{3.5.5}$$

Let us pick now the following correspondences for the parameters in the THE, in order to have a constructing potential:

$$\alpha := \frac{1}{36}\left(u^2t^2-9st\right), \beta := -rt, \quad \text{and} \quad \gamma := \frac{u}{3}t, \tag{3.5.6}$$

where $t := -1/4\,\lambda^{-1}(\lambda-1)^2$ for $\lambda \in \mathbb{S}^1$.
With these choices, and transforming ξ by the gauge $\Lambda = \mathrm{diag}(\lambda^{1/2}, \lambda^{-1/2})$, a potential for the THE is obtained which can be used in the generalised Weierstrass representation. The potential is

$$\xi_P = \begin{pmatrix} 0 & \lambda^{-1} \\ \frac{9}{4}z^4\lambda+\lambda Q_t & 0 \end{pmatrix} dz, \tag{3.5.7}$$

where

$$Q_t = t\left(\frac{u}{2}z^2+rz+\frac{s}{4}\right). \tag{3.5.8}$$

Consider a solution on $\Sigma = \mathbb{C}$ to $\mathrm{d}\Phi = \Phi\xi_P$, with $\Phi(0) = \mathbb{1}$. Since $\mathbb{C}$ is contractible, any loop can be contracted into a point, so the monodromy is trivial. In other words, the only singularity of the system lives at ∞, so every solution Φ can be analytically continued along any path on Σ, and the immersion f induced by it will be well-defined on every closed loop in Σ.

Theorem 3.7. *Let $\Sigma = \mathbb{C}$ and let $r, s, u \in \mathbb{C}$ not all of them zero. Then, the potential ξ_P constructs via the generalised Weierstrass representation a* CMC *plane arising from the Triconfluent Heun Equation.*

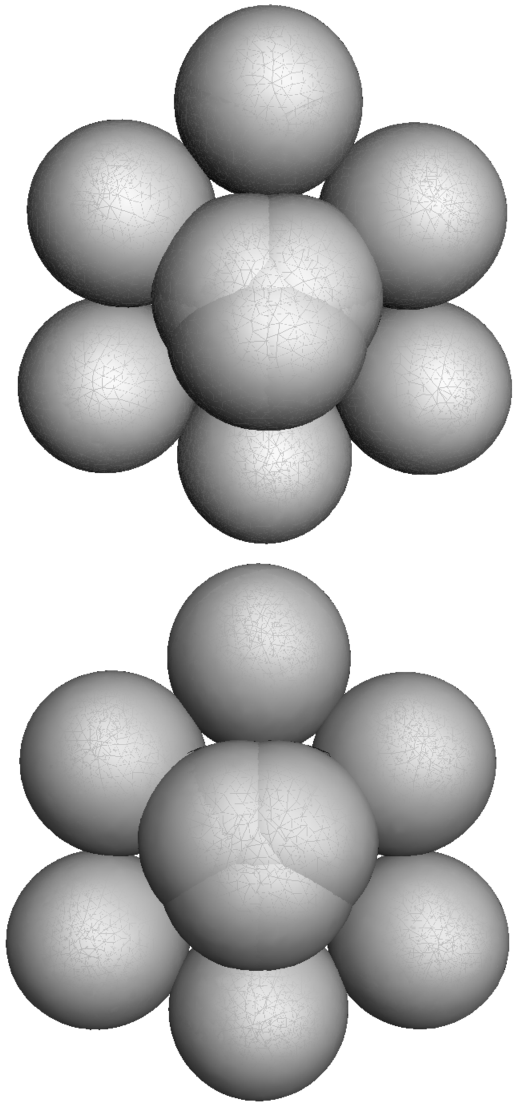

Figure 3.9: CMC surfaces with the topology of the plane. The parameters r, s, u used to construct each of them are $\left(\frac{1}{4}, -\frac{1}{3}, \frac{1}{8}\right)$, and $\left(\frac{1}{4}, 0, \frac{1}{32}\right)$.

Chapter 4

Constant mean curvature surfaces with symmetries

Throughout this chapter we explore some symmetries appearing in the surfaces constructed in chapter 3. We prove that these symmetries in the resultant surfaces can be tracked to the level of the potentials, which are transformed under automorphisms on the domain.
First we prove results that allow us to identify these symmetries in the $\mathfrak{sl}_2(\mathbb{C})$ potentials, and then we investigate which of our potentials from chapter 3 that construct CMC surfaces satisfy them.
Let $\mathrm{Iso}_3(\mathbb{R})$ denote the *isometry group* of $\mathbb{R}^3$, that is, the group of all bijective, distance-preserving maps of $\mathbb{R}^3$ with respect to the standard Euclidean metric. The elements of this group, also referred to as *Euclidean motions* on $\mathbb{R}^3$, are of the form

$$x \mapsto \phi(x) := Ax + b, \tag{4.0.1}$$

where A denotes a real orthogonal 3×3 matrix, and b denotes a vector in $\mathbb{R}^3$. Thus, the elements of $\mathrm{Iso}_3(\mathbb{R})$ are composed of an orthogonal transformation (which describe rotations in the space, see chapter A), and a translation. Moreover, the isometries preserve orientation on $\mathbb{R}^3$ if and only if the matrix A has $\det A = 1$, that is, when $A \in \mathrm{SO}_3$. On the other hand, an isometry ϕ reverses orientation if and only if $\det A = -1$, i.e., if and only if $A \in \mathrm{O}_3 \setminus \mathrm{SO}_3$. Given a conformal CMC immersion $f : \Sigma \to \mathbb{R}^3$ we define the *symmetry group* of f by

$$\mathcal{S}(f) := \{\phi \in \mathrm{Iso}_3(\mathbb{R}) \ | \ \phi(f(\Sigma)) = f(\Sigma)\}. \tag{4.0.2}$$

The immersion f is said to be *symmetric* with respect to the Euclidean motion $\phi \in \mathrm{Iso}_3(\mathbb{R})$ if and only if $\phi \in \mathcal{S}(f)$. In other words, we say that ϕ is a *symmetry* of the CMC immersion f.

A transformation ϕ is an involution if $\phi^2 = \mathrm{id}$ but $\phi \neq \mathrm{id}$. The following proposition holds for elements of $\mathrm{Iso}_3(\mathbb{R})$ that are involutions.

Proposition 4.1. *The involutory isometries are the reflections and the half-turns (rotations by π) and the central symmetry with respect to the origin.*

In what follows, to lighten notation, we may denote the dependence on λ at the different levels of the generalised Weierstrass representatoin with a subscript, that is,

$$\begin{aligned} \xi_\lambda &= \xi(z,\lambda), \\ \Phi_\lambda &= \Phi(z,\lambda), \\ F_\lambda &= F(z,\lambda). \end{aligned} \tag{4.0.3}$$

Also, in order to simplify computations, let us consider the Sym-Bobenko formula of a immersion induced by a frame F given by

$$\mathrm{Sym}[F_\lambda] = \frac{-1}{2H} i\lambda(\partial_\lambda F)F^{-1}. \tag{4.0.4}$$

We are omitting the normal term of the formula, which gives the parallel CMC surface.

4.1 Surfaces with a reflectional symmetry

Consider the orientation reversing automorphism of Σ given by $\sigma(z) = \bar{z}$. It defines a symmetry that reflects the domain across the real axis. We prove the following

Theorem 4.1. *Consider a $\Lambda^{-1}\mathfrak{sl}_2(\mathbb{C})$-potential ξ that generates via the Weierstrass representation a* CMC *family of immersions f_λ. Suppose that ξ satisfies the symmetry*

$$\overline{\sigma^*\xi_{\bar{\lambda}}} = \xi_\lambda. \tag{4.1.1}$$

Then, the induced immersion $\overline{\sigma^* f_{\bar\lambda}}$ *possesses a reflective symmetry by a plane.*

Proof. Let Φ_λ be the solution of the initial value problem $\mathrm{d}\Phi_\lambda = \Phi_\lambda \xi_\lambda$, with $\Phi_\lambda(z_0) = \Phi_0$ and let us assume that z_0 is a fixed point of σ and that $\Phi_0 \in \Lambda^*\mathrm{SL}_2(\mathbb{C})$. The transformation $\overline{\sigma^*\Phi_{\bar\lambda}} =: \Psi$ defines a solution to the differential equation $\mathrm{d}\Psi = \Psi\left(\overline{\sigma^*\xi_{\bar\lambda}}\right)$, which in view of the symmetry in (4.1.1) reads

$$\mathrm{d}\Psi = \Psi\xi_\lambda. \tag{4.1.2}$$

Since any two solutions of this equation differ by a factor that is constant in z, that is by a matrix $R_\lambda \in \Lambda\mathrm{SL}_2(\mathbb{C})$, we see that Φ_λ has the symmetry

$$R_\lambda \Phi_\lambda = \overline{\sigma^*\Phi_{\bar\lambda}}, \tag{4.1.3}$$

for some z-idependent R_λ.

Evaluation at the fixed point z_0 of σ, using equation (4.1.3), yields

$$R_\lambda = \overline{\Phi_0(\bar\lambda)}\,\Phi_0(\lambda)^{-1}. \tag{4.1.4}$$

Since $\Phi_0 \in \Lambda^*\mathrm{SL}_2(\mathbb{C})$, then one gets that R_λ is unitary for all $\lambda \in \mathbb{S}^1$.

Let us write the Iwasawa splittings $\Phi_\lambda = F_\lambda B_\lambda$ and

$$\overline{\sigma^* F_{\bar\lambda}}\,\overline{\sigma^* B_{\bar\lambda}} = \overline{\sigma^*\Phi_{\bar\lambda}} = R_\lambda\Phi_\lambda = R_\lambda F B. \tag{4.1.5}$$

The uniqueness of this splitting allows us to identify unitary and positive parts respectively, giving

$$\overline{\sigma^* F_{\bar\lambda}} = R_\lambda F_\lambda. \tag{4.1.6}$$

This implies that, using the generalised Weierstrass representation, $\overline{\sigma^*\xi_{\bar\lambda}}$ produces on the one hand the family of immersions given by plugging $\overline{\sigma^* F_{\bar\lambda}}$ in the Sym-Bobenko formula (4.0.4) and on the other hand the one obtained using $R_\lambda F_\lambda$ in equation (4.0.4). Consequently, these two surfaces coincide.

At the level of the immersion $f := \mathrm{Sym}[F_\lambda] = \frac{-1}{2H} i\lambda(\partial_\lambda F)F^{-1}$, the symmetry

(4.1.6) of the unitary frame appears in the Sym-Bobenko formula as follows:

$$\begin{aligned}
\mathrm{Sym}\left[\overline{\sigma^* F_{\bar\lambda}}\right] &= \frac{-1}{2H} i\lambda \left(\partial_\lambda \overline{\sigma^* F_{\bar\lambda}}\right) \left(\overline{\sigma^* F_{\bar\lambda}}\right)^{-1} \\
&= \frac{-1}{2H} i\lambda \left(\partial_\lambda (RF)\right) (RF)^{-1} \\
&= \frac{-1}{2H} i\lambda \left(R' R^{-1} + R F' F^{-1} R^{-1}\right) \\
&= \frac{-1}{2H} i\lambda \left(R' R^{-1} + R\, f\, R^{-1}\right),
\end{aligned} \tag{4.1.7}$$

where in the last steps of (4.1.7) we introduce the prime notation for the derivative ∂_λ.

The right hand side of equation (4.1.7) has the form of a rigid motion of f. It is left to prove that this symmetry is a reflection. To do so, we show that the transformation is an involution. That is, if we started again from the level of the potential applying all the above symmetries at each step, we would recover again the immersion f. This follows from the fact that all three transformations $\lambda \mapsto \bar\lambda$, σ and conjugation are involutions. Therefore reapplying them again at the level of the potential, that is in equation (4.1.1), gives the same relation, meaning that we recover the surface produced by the potential ξ_λ, which is f. That is, this symmetry is an involution. Since it is orientation reversing, by proposition 4.1, this symmetry must be a reflection which fixes the ends at $z = 0$ and $z = \infty$. □

In the light of theorem 4.1, we obtain as direct consequence symmetries in surfaces constructed in chapter 3. Consider a solution Φ of the initial value problem $\mathrm{d}\Phi = \Phi\xi$, with $\Phi_\lambda(z_0) = \Phi_0$ and $\Phi_0 \in \Lambda^*\mathrm{SL}_2(\mathbb{C})$ diagonal. Under this assumption, we conclude the following

Corollary 4.1. *For $\xi = \xi_T$, the surfaces constructed in theorem 3.2 have a reflection plane that fixes the ends at* 0 *and* ∞*.*

Corollary 4.2. *For $\xi = \xi_C$, the surfaces constructed in theorem 3.5 have a reflection plane that fixes the ends at* 0 *and* ∞*.*

Corollary 4.3. *For $\xi = \xi_{PD}$, the surfaces constructed in theorem 3.6 have a reflection plane that fixes the ends at* 0 *and* ∞*.*

Corollary 4.4. *For $\xi = \xi_P$ with parameters $r, s, u \in \mathbb{R}$, the surfaces constructed in theorem 3.7 have a reflection plane that fixes the ends at* 0 *and* ∞.

The proofs follow straightaway by checking for each potential that symmetry in (4.1.1) holds.

4.2 Surfaces with a rotational symmetry

In this part we look at the automorphism of Σ given by $\mu = 1/z$, which is an inversion followed by a reflection through the real axis. In a similar way as with the other symmetry in theorem 4.1, we prove the following

Theorem 4.2. *Consider a $\Lambda^{-1}\mathfrak{sl}_2(\mathbb{C})$-potential ξ that generates via the Weierstrass representation a* CMC *family of immersions f_λ. Suppose that ξ satisfies the symmetry*

$$h\xi\, h^{-1} = \mu^*\xi, \tag{4.2.1}$$

where $h = \mathrm{diag}(i, -i)$. Then, the induced immersion $\mu^* f_\lambda$ has a rotational symmetry.

Proof. Let Φ be the solution of the initial value problem $\mathrm{d}\Phi = \Phi\xi$, with $\Phi(z_0) = \Phi_0$ and let us assume that z_0 is a fixed point of μ and that Φ_0 is diagonal. Since the potential ξ has the symmetry $h\xi\, h^{-1} = \mu^*\xi$, then the solution Φ has the symmetry

$$S\Phi = \mu^*\Phi h, \tag{4.2.2}$$

for some z-independent S. Evaluating at the fixed point z_0, we find that $S = \Phi_0\, h\, \Phi_0^{-1} = h$, which is a unitary matrix. Then, Iwasawa decomposition gives

$$\mu^* F\, \mu^* B = \mu^*\Phi = h\Phi h^{-1} = hFh^{-1}\, hBh^{-1} \tag{4.2.3}$$

Identifying unitary parts and translating to the immersion $\mu^* f := \mathrm{Sym}[\mu^* F]$

one gets

$$\begin{aligned}\mathrm{Sym}[\mu^* F] &= \frac{-1}{2H} i\lambda (\partial_\lambda h F h^{-1})(hFh^{-1})^{-1} \\ &= \frac{-1}{2H} i\lambda (h \partial_\lambda F F^{-1} h^{-1}) \\ &= h\, f\, h^{-1}.\end{aligned} \tag{4.2.4}$$

Equation (4.2.4) represents a rotation by π which interchanges the ends at $z = 0$ and $z = \infty$. □

Thanks to theorem 4.2 we obtain as direct consequence that

Corollary 4.5. *The surfaces constructed in theorem 3.5 have a rotation that interchanges the ends at* 0 *and* ∞*.*

As with the other symmetry, the proof follows easily just by checking the symmetry in the potential.

Chapter 5

Conclusions and future work

The appearance of the class of Heun's Differential Equations in the construction of CMC surfaces with different topology has been proved to be interesting. It gives an structured way of constructing the surfaces with the presence of at least one irregular end.
The fact that we are dealing with a (relatively) well known class of linear differential equations, allows us to employ the theory that has been developed for those equations regarding their sets of local solutions, power series expansion solutions and asymptotics or connection problems between solutions. All this has been already fruitful in our work, but could be also extended to other cases.
In the same way as the hypergeometric equation is generalised by the Heun equation to obtain a Fuchsian ODE with four regular singularities, it might be interesting to increase the number of regular singular points to five, obtaining the next Fuchsian differential equation in terms of number of singularities. Presumably, this would lead to similar processes of confluence of singularities as those seen for Heun's equation in chapter 2, which may give the possibility to obtain CMC surfaces in a very similar way as in this work, but with more irregular ends than those obtained here.

Another conceptually simple extension of this work leads to constructing CMC surfaces in other spaces such as space forms or product spaces.

Although we have not mentioned this yet in the thesis, the generalised Weierstrass representation for CMC surfaces also works in the three-sphere $\mathbb{S}^3$ and in the hyperbolic space $\mathbb{H}^3$. Of course if one wants to obtain conformal immersions in $\mathbb{S}^3$ or $\mathbb{H}^3$, it is necessary to adapt some of what is done in chapter 1, such as the expression of the mean curvature, the Sym-Bobenko formula, the monodromy problem conditions, etc. However, it is common to find papers where the theory that is developed is applied to several space forms at the same time, such as in [54]. In this sense, it should be relatively easy (and maybe natural) to adapt the constructions done in this thesis to those in $\mathbb{S}^3$ and $\mathbb{H}^3$. To the best knowledge of the author, nothing has been said yet about irregular ends in those spaces.

Another interesting idea would be using the generalised Weierstrass representation to analyse the asymptotics of the irregular ends of the CMC surfaces construcuted in this thesis, that is, ends of surfaces that arise locally from ODEs with an irregular singularity. Initially, it could be thought that these ends are asymptotic to Smyth surfaces but, as pointed out in the introduction, in general our surfaces have nonvanishing end weights at the irregular ends, while the end weights of Smyth surfaces vanish. Therefore, there might be room to study if there is some common asymptotic behaviour between these ends, probably depending in some way on the rank of the singularity.

Another possible line of continuity for this work would be generating CMC surfaces with bubbletons and irregular ends. The bubbletons are surfaces of constant mean curvature made from Bäcklund transformations of round cylinders. They are shaped like cylinders with attached ‘bubbles’. Therefore, if for instance we are able to construct a trinoid with one irregular end and (at least) one round cylinder in the regular ends, we might use the known techniques to add bubbles there. In particular the case of our trinoids would be probably more difficult because the monodromy representation is only known asymptotically.

Lastly, another problem to continue our work was suggested recently by Martin Traizet, for what we are very grateful. He argued that, since irregular singularities appear from a process of confluence of two regular singularities in Heun's equation, one could study the limiting cases in the surfaces. That is, study the process of variation of the singularity that is merged and use this to deduce how the two ends are combined together and, from being Delaunay ends, how they become irregular. This might be somewhat related with the study of the asymptotics of the irregular ends suggested above or, in other words, could give a great insight into it.

Appendix A

Matrix Lie groups and unitarisability

In this section we present some matrix Lie groups and summarise very well-known facts about them. Of particular interest are the Lie groups $\mathrm{SL}_2(\mathbb{C})$ and SU_2, and their Lie algebras $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{su}_2$, as the matrices we deal with in the construction of CMC surfaces belong to their loop groups. We also prove some technical results regarding the unitarisability of matrices in $\mathrm{SL}_2(\mathbb{C})$. For the most basic material on Lie theory, its notation and some proofs that we do not include, we refer the reader to [2] and [20].

A.1 Affine group $\mathrm{Aff}_3(\mathbb{R})$

The 3-dimensional *affine group* over $\mathbb{R}$ is defined as

$$\mathrm{Aff}_3(\mathbb{R}) = \left\{ \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} \mid A \in \mathrm{GL}_3(\mathbb{R}),\ t \in \mathbb{R}^3 \right\}. \tag{A.1.1}$$

This is a subgroup of $\mathrm{GL}_4(\mathbb{R})$. If we identify $v \in \mathbb{R}^3$ with $\begin{pmatrix} v \\ 1 \end{pmatrix} \in \mathbb{R}^4$, then we obtain an action of $\mathrm{Aff}_3(\mathbb{R})$ on $\mathbb{R}^3$ as a consequence of the formula

$$\begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = \begin{pmatrix} Av + t \\ 1 \end{pmatrix}. \tag{A.1.2}$$

Transformations of $\mathbb{R}^3$ with the form $v \mapsto Av + t$ with A invertible are called *affine transformations* and they preserve lines, i.e., translates of 1-dimensional subspaces of $\mathbb{R}^3$. The associated geometry is *affine geometry* and it has $\mathrm{Aff}_3(\mathbb{R})$ as its symmetry group. The vector space $\mathbb{R}^3$ can be viewed as the *translation subgroup* of $\mathrm{Aff}_3(\mathbb{R})$,

$$\mathrm{Trans}_3(\mathbb{R}) = \left\{ \begin{pmatrix} I_3 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R}^3 \right\}. \tag{A.1.3}$$

There is also the subgroup

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in \mathrm{GL}_3(\mathbb{R}) \right\}, \tag{A.1.4}$$

which we will identify with $\mathrm{GL}_3(\mathbb{R})$. We state the following

Proposition A.1. $\mathrm{Aff}_3(\mathbb{R})$ *can be expressed as the semi-direct product of* $\mathrm{Trans}_3(\mathbb{R})$ *and* $\mathrm{GL}_3(\mathbb{R})$*,*

$$\mathrm{Aff}_3(\mathbb{R}) = \mathrm{GL}_3(\mathbb{R}) \ltimes \mathrm{Trans}_3(\mathbb{R}) = \{AT \mid A \in \mathrm{GL}_3(\mathbb{R}),\ T \in \mathrm{Trans}_3(\mathbb{R})\}, \tag{A.1.5}$$

with $\mathrm{GL}_3(\mathbb{R}) \cap \mathrm{Trans}_3(\mathbb{R}) = \{\mathbb{1}\}$.

A.2 Orthogonal and Isometry groups

A 3×3 real matrix M for which $M^t M = \mathbb{1}$ is called an *orthogonal* matrix. The set

$$\mathrm{O}_3(\mathbb{R}) = \{O \in \mathrm{M}_3(\mathbb{R}) \mid O^t O = O O^t = \mathbb{1}\} \tag{A.2.1}$$

is a subgroup of $\mathrm{GL}_3(\mathbb{R})$ and is called the 3×3 (real) *orthogonal group.* Consider the determinant function restricted to $\mathrm{O}_3(\mathbb{R})$, $\det : \mathrm{O}_3(\mathbb{R}) \to \mathbb{R}^*$. For $O \in \mathrm{O}_3(\mathbb{R})$,

$$(\det O)^2 = \det O^t \det O = \det(O^t O) = \det \mathbb{1} = 1, \tag{A.2.2}$$

which implies that $\det O = \pm 1$. Thus we have

$$\mathrm{O}_3(\mathbb{R}) = \mathrm{O}_3(\mathbb{R})^+ \cup \mathrm{O}_3(\mathbb{R})^-, \tag{A.2.3}$$

where

$$\mathrm{O}_3(\mathbb{R})^+ = \{O \in \mathrm{O}_3(\mathbb{R}) \mid \det O = 1\}, \quad \mathrm{O}_3(\mathbb{R})^- = \{O \in \mathrm{O}_3(\mathbb{R}) \mid \det O = -1\}. \tag{A.2.4}$$

A very important subgroup is

$$\mathrm{SO}_3(\mathbb{R}) = \mathrm{O}_3(\mathbb{R})^+ \lhd \mathrm{O}_3(\mathbb{R}), \tag{A.2.5}$$

that is,

$$\mathrm{SO}_3(\mathbb{R}) = \{O \in \mathrm{M}_3(\mathbb{R}) \mid O^tO = OO^t = \mathbb{1},\ \det O = 1\} \tag{A.2.6}$$

the 3×3 (real) *special orthogonal group.*

One of the reasons for the study of the orthogonal groups is their relationship with *isometries* of $\mathbb{R}^3$, that is, distance-preserving bijections. If such f fixes the origin then it is actually a linear isometry, and so with respect to the standard basis it corresponds to a matrix $A \in \mathrm{GL}_3(\mathbb{R})$.

Proposition A.2. *If $A \in \mathrm{GL}_3(\mathbb{R})$, then the following are equivalent.*

- *A is a linear isometry.*
- *$Av \cdot Aw = v \cdot w$ for all vectors $v, w \in \mathbb{R}^3$.*
- *$A^tA = \mathbb{1}$, that is, A is orthogonal.*

Elements of $\mathrm{SO}_3(\mathbb{R})$ are called *direct isometries* or *rotations*, while those in $\mathrm{O}_3(\mathbb{R})^-$ are called *indirect isometries.* We also define the *isometry group* of $\mathbb{R}^3$,

$$\mathrm{Iso}_3(\mathbb{R}) = \{f : \mathbb{R}^3 \to \mathbb{R}^3 \mid f \text{ is an isometry}\}, \tag{A.2.7}$$

which contains the subgroup of translations. In fact, $\mathrm{Iso}_3(\mathbb{R}) \leq \mathrm{Aff}_3(\mathbb{R})$ and it is also a matrix subgroup. We have the semi-direct product decomposition

$$\mathrm{Iso}_3(\mathbb{R}) = \left\{ \begin{pmatrix} O & t \\ 0 & 1 \end{pmatrix} \mid O \in \mathrm{O}_3(\mathbb{R}), t \in \mathbb{R}^3 \right\}. \tag{A.2.8}$$

In other words, we have

Proposition A.3. $\mathrm{Iso}_3(\mathbb{R})$ *can be expressed as the semi-direct product of* $\mathrm{O}_3(\mathbb{R})$ *and* $\mathrm{Trans}_3(\mathbb{R})$,

$$\mathrm{Iso}_3(\mathbb{R}) = \mathrm{O}_3(\mathbb{R}) \ltimes \mathrm{Trans}_3(\mathbb{R}) = \{OT \mid O \in \mathrm{O}_3(\mathbb{R}), T \in \mathrm{Trans}_3(\mathbb{R})\}. \tag{A.2.9}$$

Lemma A.1. *For a plane $\Pi \subset \mathbb{R}^3$, the reflection across Π is an indirect isometry of $\mathbb{R}^3$, i.e., is a linear transformation that belongs to $\mathrm{O}_3(\mathbb{R})^-$.*

Proposition A.4. *Every element $O \in \mathrm{O}_3(\mathbb{R})$ is a product of plane reflections. The number of these is even if $O \in \mathrm{SO}_3(\mathbb{R})$ and odd if $O \in \mathrm{O}_3(\mathbb{R})^-$.*

A.3 Special linear group $\mathrm{SL}_2(\mathbb{C})$

The matrix group $\mathrm{SL}_2(\mathbb{C})$ is called *Complex Special Linear Group* of order 2. The word 'special' in this context means that the matrices in this group have unit determinant. Hence, this group can be written as

$$\mathrm{SL}_2(\mathbb{C}) = \{A \in \mathrm{M}_2(\mathbb{C}) \ \mid \ \det A = 1\}. \tag{A.3.1}$$

For later use, let us also define $\Delta \subset \mathrm{SL}_2(\mathbb{C})$ the subgroup of diagonal elements, which sometimes we denote as $\mathrm{diag}(\rho, 1/\rho)$.

Take a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.3.2}$$

in $\mathrm{SL}_2(\mathbb{C})$, which means that $ad - bc = 1$. Its characteristic polynomial is given by

$$\det \begin{pmatrix} a-\mu & b \\ c & d-\mu \end{pmatrix} = \mu^2 - (a+d)\mu + 1. \tag{A.3.3}$$

Then, the eigenvalues of A are

$$\mu_\pm = \frac{\mathrm{tr}\, A \pm \sqrt{(\mathrm{tr}\, A)^2 - 4}}{2}, \tag{A.3.4}$$

and it holds that $\mu_+ = 1/\mu_-$. We also have that

$$\mathrm{tr}\, A = \mu_+ + \mu_-. \tag{A.3.5}$$

Note that if $\mu_+ \neq \mu_-$, then A is diagonalisable, while if $\mu_+ = \mu_-$ (then ± 1 is the unique eigenvalue of A) then A is diagonalisable if and only if

$$A = SDS^{-1} = S(\mu_+ \mathbb{1})S^{-1} = \mu_+ SS^{-1} = \pm \mathbb{1}. \tag{A.3.6}$$

In relation with the diagonalisability, the elements of $\mathrm{SL}_2(\mathbb{C})$ can be classified up to conjugation. An element $A \in \mathrm{SL}_2(\mathbb{C})$ may be associated with a Möbuis transformation, that is, a rational function of the form

$$h_A(z) = \frac{az+b}{cz+d}, \tag{A.3.7}$$

of one complex variable z; here the coefficients a, b, c, d are complex numbers satisfying $ad-bc = 1$ (in general, Möbius transformations satisfy that $ad-bc \neq 0$). Consider the surjective map $A \in \mathrm{SL}_2(\mathbb{C}) \mapsto h_A \in \mathcal{M}(2)$. One can check that this map has kernel $\{\pm\mathbb{1}\}$ so that $\mathrm{SL}_2(\mathbb{C})/\{\pm\mathbb{1}\}$ is isomorphic to the subgroup of the Möbius group $\mathcal{M}(2)$ of transformations satisfying $ad-bc = 1$. Non-trivial elements of $\mathrm{SL}_2(\mathbb{C})$ can be classified into three types depending on their Jordan normal form. Note that these types can be distinguished by looking at the trace $\operatorname{tr} A = a + d$, which is invariant under conjugation, that is, $\operatorname{tr} GAG^{-1} = \operatorname{tr} A$.

(i) *Elliptic elements.* They have two distinct eigenvalues $\mu_\pm \neq \pm 1$ (that is, these matrices are diagonalisable) with modulus 1, so that these elements are conjugate to

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \tag{A.3.8}$$

with $\alpha \neq 0$ and $|\alpha| = 1$. It holds that $\frac{1}{2}\operatorname{tr} A \in (-1, 1)$ for every elliptic element.

(ii) *Loxodromic elements.* Just like in the previous case, these have two distinct eigenvalues $\mu_\pm \neq \pm 1$ (diagonalisable) but the modulus is not equal to 1. The subclass of real trace with absolute value greater than 2 is called *hyperbolic.*

(iii) *Parabolic elements.* In this case diagonalisation is not possible. These elements are conjugate to

$$\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}. \tag{A.3.9}$$

For a parabolic element A the quantity $\frac{1}{2}\operatorname{tr} A$ is equal to ± 1.

Following these ideas, we prove the following

Lemma A.2. *Let* $M_1, M_2 \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm \mathbb{1}\}$. *Then,*

$$\operatorname{tr} M_1 = \operatorname{tr} M_2 \iff M_1 \textit{ and } M_2 \textit{ are conjugate.}$$

Proof. Suppose that M_1 and M_2 are conjugate, then there exist $T \in \mathrm{SL}_2(\mathbb{C})$ such that $TM_1T^{-1} = M_2$, and therefore $\operatorname{tr} M_2 = \operatorname{tr}(TM_1T^{-1}) = \operatorname{tr}(M_1T^{-1}T) = \operatorname{tr} M_1$.

Suppose now that the traces are equal. Assume that M_1 and M_2 are diagonal, then they are conjugate if and only if $M_1 = M_2$ or $M_1 = M_2^{-1}$. In either case, $\operatorname{tr} M_1 = \operatorname{tr} M_2$. Now we distinguish two cases:

If M_1 and M_2 are diagonalisable, then both are conjugate to a diagonal matrix and $\operatorname{tr} M_1 = \operatorname{tr} M_2$ using the previous assumption. Hence, M_1 and M_2 are conjugate. If one of them is not diagonalisable, say M_1, then its eigenvalues satisfy $\mu_+ = \mu_- = \pm 1$ and therefore, since $\operatorname{tr} M_1 = \mu_+ + \mu_-$, we have that $\frac{1}{2} \operatorname{tr} M_1 = \pm 1$ and also $\frac{1}{2} \operatorname{tr} M_2 = \pm 1$ by hypothesis. Therefore, both M_1 and M_2 are triangular. And also in this case they are conjugate, finishing the proof. $\square$

Summarizing, the conjugacy classes in $\mathrm{SL}_2(\mathbb{C})$ are represented by the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \ \alpha \neq 0, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{A.3.10}$$

It will be clear later why we are interested in matrices of $\mathrm{SL}_2(\mathbb{C})$ associated to elliptic elements of $\mathcal{M}$. They represent rotations - writing $\alpha = e^{i\theta}$ for $\theta \in [0, 2\pi)$, they are conjugate to

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{A.3.11}$$

We shall give a brief definition of the Lie algebra of $\mathrm{SL}_2(\mathbb{C})$, which is a complex vector space made up of traceless complex matrices of order 2, that is,

$$\mathfrak{sl}_2(\mathbb{C}) = \{X \in \mathrm{M}_2(\mathbb{C}) \ \mid \ \operatorname{tr} X = 0\}. \tag{A.3.12}$$

The exponential map $\exp : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ is not surjective, but misses only one conjugacy class in $\mathrm{SL}_2(\mathbb{C})$, namely the one represented by

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}. \tag{A.3.13}$$

A.4 Special unitary group SU_2

The *special unitary group* denoted SU_2 is the Lie group of matrices U with determinant 1 that are unitary, that is, such that $U^*U = UU^* = \mathbb{1}$. A representation of this group is

$$\mathrm{SU}_2 = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\; |a|^2 + |b|^2 = 1 \right\}. \tag{A.4.1}$$

A diffeomorphism with $\mathbb{S}^3$ can be established. If we write $a = x_1 + iy_1$ and $b = x_2 + iy_2$ then $|a|^2 + |b|^2 = 1$ is equivalent to $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$, the equation of the 3-sphere. The map $\varphi : \mathbb{C}^2 \to \mathrm{M}_2(\mathbb{C})$ given by

$$(a, b) \mapsto \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \tag{A.4.2}$$

is an injective real map (by considering $\mathbb{C}^2$ diffeomorphic to $\mathbb{R}^4$ and $\mathrm{M}_2(\mathbb{C})$ to $\mathbb{R}^8$). Hence, the restriction of φ to the 3-sphere (since the modulus is 1) is an embedding of $\mathbb{S}^3$ onto a compact submanifold of $\mathrm{M}_2(\mathbb{C})$, namely $\varphi(\mathbb{S}^3) = \mathrm{SU}_2$. The discussion done in section A.3 gives a picture of the decomposition of $\mathrm{SL}_2(\mathbb{C})$ into its conjugacy classes, but there is another point of view in the decompositions of this group that is interesting for us.

Lemma A.3 (QR decomposition)**.** *Let $K = \mathrm{SU}_2$ and let $\mathcal{B}$ be the subgroup of $\mathrm{SL}_2(\mathbb{C})$ of upper triangular matrices with determinant 1 and positive real diagonal entries, that is, the subgroup of matrices of the form*

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}, \; \alpha \in \mathbb{R}_{>0},\; \beta \in \mathbb{C}. \tag{A.4.3}$$

Then, $\mathrm{SL}_2(\mathbb{C})$ admits a decomposition $\mathrm{SL}_2(\mathbb{C}) = K\mathcal{B}$ in the sense that every element $A \in \mathrm{SL}_2(\mathbb{C})$ can be uniquely written as $A = UB$ with $U \in \mathrm{SU}_2$ and

$B \in \mathcal{B}$. *In particular, the unique splitting can be written*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{a\bar{a}+c\bar{c}}} & \frac{-\bar{c}}{\sqrt{a\bar{a}+c\bar{c}}} \\ \frac{c}{\sqrt{a\bar{a}+c\bar{c}}} & \frac{\bar{a}}{\sqrt{a\bar{a}+c\bar{c}}} \end{pmatrix} \begin{pmatrix} \sqrt{a\bar{a}+c\bar{c}} & \frac{\bar{a}b+\bar{c}d}{\sqrt{a\bar{a}+c\bar{c}}} \\ 0 & \frac{1}{\sqrt{a\bar{a}+c\bar{c}}} \end{pmatrix}. \tag{A.4.4}$$

Thus, topologically, $\mathrm{SL}_2(\mathbb{C})$ looks like $\mathbb{S}^3 \times \mathbb{R} \times \mathbb{C}$. The Iwasawa decomposition presented in section 1.2.1 is similar in spirit to the above splitting, but is done on infinite-dimensional loop groups that depend on a spectral parameter. We also consider Birkhoff splitting on loop groups in section 1.2.1, so let us describe its finite-dimensional analog here, which is the LU decomposition. Consider for all $z \in \mathbb{C}$, $|z| < 1$, a holomorphic $\mathrm{SL}_2(\mathbb{C})$ matrix $\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$ with $a(z)$ having only isolated zeroes. Then, one LU decomposition is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}. \tag{A.4.5}$$

This splitting has the advantage over the QR decomposition that the two resulting matrices are holomorphic in z (away from $a = 0$). However, it has the drawback of not being globally defined, unlike (A.4.4). Let us also set

$$\mathfrak{su}_2 = \{X \in \mathrm{M}_2(\mathbb{C}) \mid \operatorname{tr} X = 0, X^* = -X\}. \tag{A.4.6}$$

This is a *real* vector space of matrices (*not* a complex space) closed under the bracket operation. The real Lie algebra $\mathfrak{su}_2$ has a basis given by

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad u_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{A.4.7}$$

which satisfy

$$[u_1, u_2] = u_3, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = u_2. \tag{A.4.8}$$

In other words, every element of $\mathfrak{su}_2$ may be written in the form

$$X = \frac{-i}{2}\begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix}. \tag{A.4.9}$$

With such a matrix we associate the vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. In this way we identify $\mathbb{R}^3 \cong \mathfrak{su}_2$ via $x \longleftrightarrow x_1u_1 + x_2u_2 + x_3u_3 = X$. For $X, Y \in \mathfrak{su}_2$, the inner product, norm and cross-product in this Lie algebra are given by

$$\begin{aligned} \langle X, Y \rangle &= -2\,\mathrm{tr}(XY), \\ ||X|| &= \sqrt{4 \det X}, \\ X \times Y &= [X, Y], \end{aligned} \tag{A.4.10}$$

where $[\cdot, \cdot]$ represents the Lie bracket on $\mathfrak{su}_2$. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is the complexified Lie algebra $\mathfrak{su}_2$, denoted $\mathfrak{su}_2^{\mathbb{C}} = \mathfrak{su}_2 + i\mathfrak{su}_2 \cong \mathfrak{sl}_2(\mathbb{C})$. As long as one is working with representations over $\mathbb{C}$ this passage from real to complexified Lie algebra is harmless. A reason for passing to the complexification is that it allows one to construct a basis of a type that does not exist in the real Lie algebra $\mathfrak{su}_2$. The complexified Lie algebra is spanned by three elements ϵ_-, ϵ_+, and ϵ. Thus we fix the following basis of $\mathfrak{sl}_2(\mathbb{C})$ as

$$\begin{aligned} \epsilon &= \frac{1}{i}(-2iu_3) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \\ \epsilon_- &= \frac{1}{i}(u_1 + iu_2) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \\ \epsilon_+ &= \frac{1}{i}(-u_1 + iu_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{A.4.11}$$

which satisfy the relations

$$\begin{aligned} &\langle \epsilon_+, \epsilon_+ \rangle = \langle \epsilon_-, \epsilon_- \rangle = 0, \quad \epsilon_-^* = -\epsilon_+, \\ &[\epsilon, \epsilon_-] = 2i\epsilon_-, \quad [\epsilon_+, \epsilon] = 2i\epsilon_+, \quad [\epsilon_-, \epsilon_+] = i\epsilon. \end{aligned} \tag{A.4.12}$$

SU_2 operates on $\mathfrak{su}_2$ through the adjoint action $\mathrm{Ad}_U X = UXU^{-1}$. For $U \in \mathrm{SU}_2$, consider the linear transformation $p(U)$ of $\mathbb{R}^3$ corresponding to the linear transformation Ad_U of $\mathfrak{su}_2$.

Lemma A.4. *The homomorphism p that maps SU_2 onto SO_3 with kernel $\{\pm\mathbb{1}\}$ is a covering.*

In particular, the Lie group SU_2 is a double cover of SO_3, and also $\mathrm{Ad}_{\mathrm{SU}_2} \cong \mathrm{SO}_3$: the adjoint actions of SU_2 become the usual rotation action of SO_3 on

$\mathbb{R}^3$. A rigid motion in $\mathbb{R}^3$ given by $x \mapsto Ox + b$ with $O \in \mathrm{SO}_3$ and $b \in \mathbb{R}^3$ corresponds to $X \mapsto \mathrm{Ad}_U X + B$ where $U \in \mathrm{SU}_2$ is a lift of O and $B \in \mathfrak{su}_2$ corresponds to $b \in \mathbb{R}^3$ under the above isomorphism.

SU_2 may also be realised as the group of quaternions. Recall that quaternions $\mathbb{H}$ are expressions of the form

$$q = a\mathbf{1} + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}, \quad a, b, c, d \in \mathbb{R}, \tag{A.4.13}$$

added and multiplied in the obvious way subject to

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{A.4.14}$$

Identify $\mathbb{C}$ with quaternions of the form $a + ib$. Any quaternion of norm 1 can be uniquely written as $\alpha + \beta j$ with $\alpha, \beta \in \mathbb{C}$. The map

$$\alpha + \beta j \longleftrightarrow \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \tag{A.4.15}$$

sets up a bijection between the subgroup of $\mathbb{H}$ with norm 1 and SU_2.

A.4.1 Unitarisation of $\mathrm{SL}_2(\mathbb{C})$

In what follows, we explore the relevant definitions and results about unitarisation, that are used without further explanation in the different chapters of this thesis.

Definition A.1. A matrix $A \in \mathrm{SL}_2(\mathbb{C})$ is *unitarisable* if there exists $T \in \mathrm{SL}_2(\mathbb{C})$ such that $TAT^{-1} \in \mathrm{SU}_2$. The matrix T is called *unitariser*. On the other hand, matrices $A_1, \ldots, A_n \in \mathrm{SL}_2(\mathbb{C})$ are *individually unitarisable* iff A_k is unitarisable for $k = 1, \ldots, n$. We say that they are *simultaneously unitarisable* iff there exists $T \in \mathrm{SL}_2(\mathbb{C})$ such that $TA_kT^{-1} \in \mathrm{SU}_2$ for all $k = 1, \ldots, n$.

Definition A.2. A matrix $A \in \mathrm{SL}_2(\mathbb{C})$ is *irreducible* if it cannot be conjugated to upper triangular matrices.

Let us state a lemma that allows one to find a unitariser.

Lemma A.5. *Let $A_1, \ldots, A_n \in \mathrm{SL}_2(\mathbb{C})$. The following are equivalent:*

(i.) $T \in \mathrm{SL}_2(\mathbb{C})$ *simultaneously unitarises* $A_1, \ldots, A_n \in \mathrm{SL}_2(\mathbb{C})$.

(ii.) T^*T *is in the kernel of the linear operators defined by*

$$X \mapsto XA_k - A_k^{*-1}X, \forall k. \tag{A.4.16}$$

Remark A.1. Thus to construct the simultaneous unitariser of $A_1, \ldots, A_n \in \mathrm{SL}_2(\mathbb{C})$, let X be a Hermitian positive-definite element in the kernel. Then X factors into $X = T^*T$, and T is a simultaneous unitariser.

We conclude this exposition by giving two results that allow us to characterise matrices that are unitarisable.

Lemma A.6. *For a matrix $U \in \mathrm{SU}_2$ it holds that $\left|\frac{1}{2}\operatorname{tr} U\right| \leq 1$ and if $\frac{1}{2}\operatorname{tr} U = \pm 1$ then $U = \pm\mathbb{1}$ respectively.*

Proof. Let us write $U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathrm{SU}_2$. We have that $\operatorname{tr} U = 2\operatorname{Re}(a)$. On the other hand, $|a|^2 + |b|^2 = 1$, which is equivalent to

$$\operatorname{Re}(a)^2 + \operatorname{Im}(a)^2 + \operatorname{Re}(b)^2 + \operatorname{Im}(b)^2 = 1. \tag{A.4.17}$$

This implies for the real part of a that $-1 \leq \operatorname{Re}(a) \leq 1$ and therefore,

$$-1 \leq \frac{1}{2}\operatorname{tr} U \leq 1. \tag{A.4.18}$$

If $\frac{1}{2}\operatorname{tr} U = \pm 1$, we equivalently have that $\operatorname{Re}(a) = \pm 1$ and then, using equation (A.4.17), one gets

$$\operatorname{Re}(b)^2 + \operatorname{Im}(b)^2 = -\operatorname{Im}(a)^2. \tag{A.4.19}$$

The left hand side of equation (A.4.19) is non-negative while the right hand side is not positive, therefore both are equal to 0. Hence, $\operatorname{Im}(a) = 0$ and $b = 0$ which implies that $U = \pm\mathbb{1}$. □

Lemma A.7. *A matrix $A \in \mathrm{SL}_2(\mathbb{C})$ is unitarisable if and only if $\frac{1}{2}\operatorname{tr} A \in (-1, 1)$ or $A = \pm\mathbb{1}$.*

Proof. Suppose that A is unitarisable. Then, there exists $T \in \mathrm{SL}_2(\mathbb{C})$ such that $TAT^{-1} =: U \in \mathrm{SU}_2$. Then, by lemma A.6, either

$$(-1,1) \ni \frac{1}{2}\operatorname{tr} U = \frac{1}{2}\operatorname{tr} A, \tag{A.4.20}$$

or

$$\pm\,\mathbb{1} = U = TAT^{-1} \iff A = \pm\mathbb{1}. \tag{A.4.21}$$

If $A = \pm\mathbb{1}$, A is already unitary.
On the other hand, suppose that $\frac{1}{2}\operatorname{tr} A \in (-1,1)$ and $A \neq \pm\mathbb{1}$. By lemma A.6, every matrix in SU_2 has trace in $(-2,2)$ and, since all possible values are attained, we can pick $U \in \mathrm{SU}_2$ with equal trace as A. We may use now lemma A.2 to argue that A and U are conjugate, completing thus the proof. □

Lemma A.8. *An element $M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$ is unitarisable by an element of Δ if and only if its diagonal elements are complex conjugates of each other, and their product is less than* 1.

Proof. Assume first that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitarisable by $T = \operatorname{diag}(\rho, 1/\rho)$. Then,

$$TMT^{-1} = U \in \mathrm{SU}_2. \tag{A.4.22}$$

Write $U = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$, then $M = T^{-1}UT = \begin{pmatrix} z & -\bar{w}/\rho^2 \\ w\rho^2 & \bar{z} \end{pmatrix}$. Therefore one has that $a = z$ and $d = \bar{z}$, so the diagonal elements of M are complex conjugates of each other. Also, $b = -\bar{w}/\rho^2$ and $c = w\rho^2$. Then, $z\bar{z} = ad = 1 + bc = 1 - w\bar{w} = 1 - |w|^2$, and it follows that $ad < 1$, since $|w|^2 > 0$. Note that $|w|^2 \neq 0$ for otherwise $M = \pm\mathbb{1}$.
Assume now that $|a| < 1$. For any $\rho \neq 0$,

$$\begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix}\begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix}\begin{pmatrix} 1/\rho & 0 \\ 0 & \rho \end{pmatrix} = \begin{pmatrix} a & b\rho^2 \\ c/\rho^2 & \bar{a} \end{pmatrix}. \tag{A.4.23}$$

Since we want this matrix to be unitary, we multiply it by its conjugate transpose, obtaining

$$\begin{pmatrix} a & b\rho^2 \\ c/\rho^2 & \bar{a} \end{pmatrix}^* \begin{pmatrix} a & b\rho^2 \\ c/\rho^2 & \bar{a} \end{pmatrix} = \begin{pmatrix} |a|^2 + |c|^2/|\rho|^4 & \bar{a}(b\rho^2 + \bar{c}/\bar{\rho}^2) \\ a(\bar{b}\bar{\rho}^2 + c/\rho^2) & |a|^2 + |b|^2|\rho|^4 \end{pmatrix}. \tag{A.4.24}$$

For the latter matrix to be the identity, we need $b\rho^2 + \bar{c}/\bar{\rho}^2 = 0$. Multiplying by $\bar{\rho}^2$, we obtain that $0 = b|\rho|^4 + \bar{c}$, so

$$|\rho|^4 = -\bar{c}/b = -\frac{(|a|^2 - 1)/\bar{b}}{b} = \frac{1 - |a|^2}{|b|^2}. \tag{A.4.25}$$

Also,

$$|a|^2 + |c|^2/|\rho|^4 = |a|^2 + |c|^2/(-\bar{c}/b) = |a|^2 - bc = 1 \tag{A.4.26}$$

and

$$|a|^2 + |b|^2|\rho|^4 = |a|^2 + |b|^2(-\bar{c}/b) = |a|^2 - \overline{bc} = \overline{|a|^2 - bc} = 1. \tag{A.4.27}$$

So any $\rho \in \mathbb{C}$ with

$$|\rho| = \left(\frac{1 - |a|^2}{|b|^2}\right)^{1/4} \tag{A.4.28}$$

allows us to define an element $T \in \Delta$ which makes TMT^{-1} in SU_2. $\square$

Appendix B

Geometry of unitarisability

We expose in what follows some results regarding the hyperbolic space and cross ratios that allow us to give a criterion for the simultaneous unitarisability of two matrices in $\mathrm{SL}_2(\mathbb{C})$ in terms of their eigenlines. This section is inspired by an unpublished work of Nicholas Schmitt, to whom we are grateful.

B.1 Hyperbolic 3-space

Hyperbolic 3-space $\mathbb{H}^3$ can be identified with the quotient $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2$. For $X \in \mathrm{SL}_2(\mathbb{C})$, let $[X] \in \mathbb{H}^3$ denote the left coset

$$[[X]] := \{XU \ \mid \ U \in \mathrm{SU}_2\}. \tag{B.1.1}$$

$M \in \mathrm{SL}_2(\mathbb{C})$ acts isometrically on the hyperbolic 3-space $\mathbb{H}^3$ by

$$[[X]] \mapsto [[MX]]\,. \tag{B.1.2}$$

The fixed point set of this action is

$$\mathrm{fix}(M) = \{[[X]] \in \mathbb{H}^3 \ \mid \ X^{-1}MX \in \mathrm{SU}_2\}. \tag{B.1.3}$$

The fixed point set $\mathrm{fix}(M)$, if non-empty, is the *axis* of M.
Hyperbolic 3-space can be extended to include the sphere at infinity as follows. Let

$$\mathrm{GU}_2 := \{X \in \mathrm{M}_2(\mathbb{C}) \ \mid \ XX^* = x\mathbb{1},\, x \neq 0\} \tag{B.1.4}$$

be the group of unitary similitudes. Let $\mathrm{N} := \mathrm{M}_2(\mathbb{C}^*)$ and $\Xi := \mathrm{N}/\mathrm{GU}_2$. Then $\Xi = A \sqcup B$ where

$$A := \{[[X]] \in \Xi \mid \det X \neq 0\}, \quad B := \{[[X]] \in \Xi \mid \det X = 0\} \tag{B.1.5}$$

Then $A = \mathrm{GL}_2(\mathbb{C})/\mathrm{GU}_2 = \mathrm{SL}_2(\mathbb{C})/\mathrm{SU}_2 = \mathbb{H}^3$.
To show $B = \mathbb{CP}^1$, note that any $X \in \mathrm{N}$ with $\det X = 0$ can be written in the form

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} \tag{B.1.6}$$

with $(x,y)^t$, $(a,b) \in \mathbb{C}^2 \setminus \{0\}$. The first factor $(x,y)^t$ is unique up to multiplication by an element of $\mathbb{C}^*$, so the map $\phi : B \to \mathbb{CP}^1$ given by

$$\left[\left[\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix}\right]\right] \mapsto [x, y] \tag{B.1.7}$$

is well defined.

Proposition B.1. GU_2 *acts transitively on* $\mathbb{C}^2 \setminus \{0\}$.

Proof.

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{GU}_2 \tag{B.1.8}$$

satisfies $(a,b) = (1,0)\,U$. Hence $(1,0)$ can be mapped to any element of $\mathbb{C}^2 \setminus \{0\}$ via an element of GU_2. Likewise, $(1,0) = (a,b)\,U^{-1}$, so any element $(a,b) \in \mathbb{C}^2 \setminus \{0\}$ can be mapped to $(1,0)$ via an element of GU_2. Hence, any element of $\mathbb{C}^2 \setminus \{0\}$ can be mapped to any other element via an element of GU_2. ☐

Proposition B.2. *The map* $\phi : B \to \mathbb{CP}^1$ *is a bijection.*

Proof. The map is clearly surjective. To show injectivity, suppose $\phi([[X]]) = \phi([[Y]])$. Then

$$X = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix}, \quad Y = \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix} \tag{B.1.9}$$

for some $(x,y)^t$, (a,b) $(c,d) \in \mathbb{C}^2 \setminus \{0\}$. Since by proposition B.1 GU_2 acts transitively on $\mathbb{C}^2 \setminus \{0\}$, there exists $U \in \mathrm{GU}_2$ such that

$$\begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} c & d \end{pmatrix} U. \tag{B.1.10}$$

Hence $X = YU$, so $[[X]] = [[Y]]$. □

B.2 Unitarisability

For the final proof of this section, we will need some characterizations regarding the unitarisability of matrices in $\mathrm{SL}_2(\mathbb{C})$.

Proposition B.3. *$M \in \mathrm{SL}_2(\mathbb{C})$ is unitarisable if and only if it has an axis.*

Proof. This is immediate from equation (B.1.3). □

Proposition B.4. *Individually unitarisable matrices $M_1, \dots, M_n \in \mathrm{SL}_2(\mathbb{C})$ are simultaneously unitarisable if and only if their axes intersect in a common point.*

Proof. $M_1, \dots, M_n \in \mathrm{SL}_2(\mathbb{C})$ are simultaneously unitarisable if and only if there exists $X \in \mathrm{SL}_2(\mathbb{C})$ such that

$$X^{-1} M_k X \in \mathrm{SU}_2, \quad k \in \{1, \dots, n\}. \tag{B.2.1}$$

This is if and only if

$$X \in \mathrm{fix}(M_k), \quad k \in \{1, \dots, n\}. \tag{B.2.2}$$

□

B.3 Eigenlines

The fixed point set $\mathrm{fix}(M)$ of $M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm \mathbb{1}\}$ is a disjoint union of a (possibly empty) component in $\mathbb{H}^3$ and a component on the sphere at infinity:

$$\mathrm{fix}(M) = (\mathrm{fix}(M) \cap A) \sqcup (\mathrm{fix}(M) \cap B). \tag{B.3.1}$$

The part in $\mathbb{H}^3$, if non-empty, is the *axis* of M. The part on the sphere at infinity is the set of *eigenlines* of M, as the following proposition shows. Note that this part consists of exactly one or two points, since M has one or two eigenlines.

Proposition B.5. *For $M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$, the set $\phi(\mathrm{fix}(M) \cap B) \subset \mathbb{CP}^1$ is the set of eigenlines of M.*

Proof. Note that $\mathrm{fix}(M) \cap B$ is the set of elements $X \in \mathrm{M}_2(\mathbb{C})$ with $\det X = 0$ such that $[[MX]] = [[X]]$. Since $\det X = 0$, X is of the form equation (B.1.6) with $(x, y)^t$, $(a, b) \in \mathbb{C}^2 \setminus \{0\}$. By proposition B.2, $[[MX]] = [[X]]$ if and only if $\phi([[MX]]) = \phi([[X]])$. That is, if and only if in $\mathbb{CP}^1$

$$\left[M \begin{pmatrix} x \\ y \end{pmatrix} \right] = \left[\begin{pmatrix} x \\ y \end{pmatrix} \right]. \tag{B.3.2}$$

That is, if and only if $(x, y)^t$ is an eigenline of M. □

B.4 The Klein model of $\mathbb{H}^3$

The Klein model of $\mathbb{H}^3$ is the unit ball in $\mathbb{R}^3$

$$\mathbb{B} = \{x \in \mathbb{R}^3 \mid ||x|| \leq 1\}. \tag{B.4.1}$$

The sphere at infinity is its boundary

$$\partial\mathbb{B} = \{x \in \mathbb{R}^3 \mid ||x|| = 1\}. \tag{B.4.2}$$

The map $K : \mathbb{H}^3 \to \overline{\mathbb{B}}$ is defined as the map $[[X]] \mapsto XX^*$ followed by the map

$$\begin{pmatrix} a+b & c+id \\ c-id & a-b \end{pmatrix} = \frac{1}{a}(b, c, d). \tag{B.4.3}$$

We need some well-known results about this model. For details, see [25, Sections II.5, VIII] and [4, Sections A.4, A.5].

Non-trivial isometries in $\mathbb{H}^3$ are identified with elements of $\mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$. In particular, we have the following

Proposition B.6. *$M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$ is unitarisable if and only if M is elliptic as isometry of $\mathbb{H}^3$.*

Proposition B.7. *The geodesics in the Klein model of $\mathbb{H}^3$ are the Euclidean straight line segments in $\mathbb{B}$.*

Proposition B.8. *If $M \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$ is unitarizable, then the axis of M in the Klein model of $\mathbb{H}^3$ is a Euclidean straight line segment in $\overline{\mathbb{B}}$ with two distinct endpoints on $\partial\mathbb{B}$.*

Proof. Since $M \in \mathrm{SL}_2(\mathbb{C})$ is unitarisable, by proposition B.3, $\mathrm{fix}(M) \neq \emptyset$. Following [25, Section VIII.11], M has two fixed endpoints on $\partial\mathbb{B}$ and the axis through these is a geodesic. It is a well-known result proposition B.7 that geodesics are straight line segments. Hence the axis of M is a straight line segment. □

B.5 The cross ratio

The cross ratio of four distinct points in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, taking values in $\mathbb{C}^*$, is

$$[a, b, c, d] := \frac{(b-c)(d-a)}{(b-a)(d-c)}. \tag{B.5.1}$$

The cross ratio is chosen so that $[0, 1, \infty, x] = x$. The cross ratio is invariant under Möbius transformations. The cross ratio $[a, b, c, d]$ of four distinct points is in $\mathbb{R} \setminus \{0\}$ if and only if the points lie on a circle $\mathcal{C}$.

Proposition B.9. *Let a, b, c and d be distinct points in $\mathbb{CP}^1$, and suppose $[a, b, c, d] \in \mathbb{R} \setminus \{0\}$, so a, b, c, d lie on a circle $\mathcal{C}$.*

1. *$[a, b, c, d] \in \mathbb{R}_+$ if and only if b and d lie in the same connected component of $\mathcal{C} \setminus \{a, c\}$.*
2. *$[a, b, c, d] \in \mathbb{R}_-$ if and only if b and d lie on different connected components of $\mathcal{C} \setminus \{a, c\}$.*

Proof. There exists a unique Möbius transformation taking a, b and c to 0, 1 and ∞ respectively, taking circles to circles, and preserving or reversing the order of a, b, c and d on the circle. Thus we may assume $a = 0$, $b = 1$, $c = \infty$. Then

$$[a, b, c, d] = [0, 1, \infty, d] = d \in \mathbb{R} \setminus \{0, 1\}. \tag{B.5.2}$$

Then a, b, c and d lie on the circle $\mathcal{C} = \mathbb{R} \cup \{\infty\}$, and the two connected components of $\mathcal{C} \setminus \{a, c\}$ are $\mathbb{R}_-$ and $\mathbb{R}_+$. The theorem follows by an examination of the two cases $d \in \mathbb{R}_+ \setminus \{1\}$ and $d \in \mathbb{R}_-$. □

B.6 Unitarisability of two matrices

We finish this part giving a characterization for the unitarisability of two matrices in $\mathrm{SL}_2(\mathbb{C})$.

Proposition B.10. *Let $M_0, M_1 \in \mathrm{SL}_2(\mathbb{C}) \setminus \{\pm\mathbb{1}\}$ be irreducible and individually unitarisable. Let $\varphi, \varphi' \in \mathbb{CP}^1$ and $\psi, \psi' \in \mathbb{CP}^1$ be the respective eigenlines of M_0 and M_1. Then M_0 and M_1 are simultaneuosly unitarisable if and only if*

$$[\varphi, \psi, \varphi', \psi'] \in \mathbb{R}_-. \tag{B.6.1}$$

Proof. Since M_0 and M_1 are individually unitarisable then, by proposition B.3, $\mathrm{fix}(M_0) \neq \emptyset$ and $\mathrm{fix}(M_1) \neq \emptyset$. By proposition B.4, M_0 and M_1 are simultaneuosly unitarisable if and only if $\mathrm{fix}(M_0) \cap \mathrm{fix}(M_1) \neq \emptyset$.
First suppose that $\mathrm{fix}(M_0) \cap \mathrm{fix}(M_1) \neq \emptyset$. By proposition B.8, $\mathrm{fix}(M_0)$ and $\mathrm{fix}(M_1)$ are straight line segments. And since they intersect, they lie in a unique Euclidean plane $\mathcal{P} \subset \mathbb{R}^3$. Then $\mathcal{C} := \mathcal{P} \cap \partial\mathbb{B}$ is a circle. By proposition B.5, the endpoints of $\mathrm{fix}(M_0)$ and $\mathrm{fix}(M_1)$ are φ, φ' and ψ, ψ' respectively. Since $\mathrm{fix}(M_0)$ and $\mathrm{fix}(M_1)$ intersect at a point inside the disk $\mathcal{P} \cap \mathbb{B}$ bounded by $\mathcal{C}$, it follows that ψ and ψ' lie in different connected components of $\mathcal{C} \setminus \{\varphi, \varphi'\}$. By proposition B.9, $[\varphi, \psi, \varphi', \psi'] \in \mathbb{R}_-$.
Conversely, suppose $[\varphi, \psi, \varphi', \psi'] \in \mathbb{R}_-$. By proposition B.9 φ, ψ, φ' and ψ' lie on some circle $\mathcal{C} \subset \partial\mathbb{B}$, and ψ and ψ' lie on different connected components of $\mathcal{C} \setminus \{\varphi, \varphi'\}$. Let $\mathcal{P}$ be the unique Euclidean plane containing $\mathcal{C}$. By proposition B.5 and proposition B.8, $\mathrm{fix}(M_0)$ is the straight line segment with endpoints φ and φ', and $\mathrm{fix}(M_1)$ is the straight line segment with endpoints ψ and ψ'. Hence $\mathrm{fix}(M_0)$ and $\mathrm{fix}(M_1)$ lie on $\mathcal{P}$ and intersect in the disk $\mathcal{P} \cap \mathbb{B}$ bounded by $\mathcal{C}$. Hence $\mathrm{fix}(M_0) \cap \mathrm{fix}(M_1) \neq \varnothing$. $\square$

Remark B.1. Similar criteria exist for the simultaneous unitarisability of n matrices in $\mathrm{SL}_2(\mathbb{C})$ in terms of cross ratios of their eigenlines, generalizing proposition B.10. Other criteria exist in terms of the eigenvalues of the matrices and certain of the products.

List of Figures

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