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MARKOVIAN COCYCLES ON OPERATOR ALGEBRAS ADAPTED TO A FOCK FILTRATION

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ABSTRACT. The introduction of a Feller-type condition allows the study of Markovian cocycles adapted to a Fock filtration to be extended from von Neumann algebras to C^* -algebras. It is shown that every such cocycle which, along with its adjoint cocycle, is pointwise strongly continuous and whose associated semigroups are norm continuous, weakly satisfies a quantum stochastic differential equation (QSDE). The matrix of coefficients of this equation may thereby be considered as the generator of the cocycle. The QSDE is satisfied strongly in all of the following cases: when the cocycle is completely positive and contractive, or the driving quantum noise is finite dimensional, or the C^* -algebra is finite dimensional and the cocycle generator is bounded. Applying the algebra results to Fock-adapted Markovian cocycles on a Hilbert space we obtain similar characterisations. In particular, contraction cocycles, whose Markov semigroup is norm continuous, strongly satisfy a QSDE. A representation of cocycles in terms of a family of associated semigroups is central to the present analysis, providing the connection with QSDE's through a parallel work ([LW1]).

0. INTRODUCTION

Let $\mathfrak{h}$ and $\mathfrak{k}$ be Hilbert spaces and let $\mathcal{F}_+$ be the symmetric Fock space over $L^2(\mathbb{R}_+; \mathfrak{k})$. Let $(\sigma_t)_{t \geq 0}$ be the one-parameter semigroup of right shifts $\mathcal{B}(\mathcal{F}_+) \rightarrow \mathcal{B}(\mathcal{F}_+)$ induced by the natural time shift on $L^2(\mathbb{R}_+; \mathfrak{k})$. In this paper we undertake a study of the equations

$$V_{s+t} = V_s \sigma_s(V_t) \tag{0.1}$$

$$j_{s+t} = \widehat{j}_s \circ \sigma_s \circ j_t \tag{0.2}$$

where $s, t \geq 0$, extending and refining the work in [HL2], [Jou], [AJL], [Fa2], [Bra] and [AcM]. A solution of (0.1) consists of a family of (bounded) operators on $\mathfrak{h} \otimes \mathcal{F}_+$. Equation (0.2) is studied with reference to an operator algebra $\mathcal{A}$ acting on $\mathfrak{h}$; a solution consists of a family of linear maps of $\mathcal{A}$ into $\mathcal{A}'' \otimes \mathcal{B}(\mathcal{F}_+)$, and $\widehat{j}_s$ is an extension of j_s whose domain includes $\bigcup_{t \geq 0} \sigma_s(j_t(\mathcal{A}))$. In each case solutions are required to satisfy an adaptedness condition with respect to the natural filtration of $\mathfrak{h} \otimes \mathcal{F}_+$ (which induces a filtration of $\mathcal{A}'' \otimes \mathcal{B}(\mathcal{F}_+)$) arising from the continuous tensor product structure of Fock space; they are called *Markovian cocycles*. The natural filtration comes with conditional expectation maps $(\mathbb{E}_s)_{s \geq 0}$ whose basic properties, in particular their behaviour when composed with the shifts $(\sigma_s)_{s \geq 0}$, imply that $(\mathbb{E}_0[V_t])_{t \geq 0}$ and $(\mathbb{E}_0 \circ j_t)_{t \geq 0}$ are one-parameter semigroups — the *Markov semigroups* associated to these cocycles. Equations (0.1) and (0.2) arise in a variety of contexts in probability theory and quantum theory, and written in this form they can be studied from a common view point — see the next section for more discussion of this background.

The equations are related in a number of ways. If V satisfies (0.1) then $j_t(a) = V_t a V_t^*$ defines a Markovian cocycle on $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ that is normal and completely

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positive; if each V_t is isometric (respectively coisometric) then the maps j_t are $*$ -homomorphic (resp. unital). Note also that $j'_t(a) = V_t a$ is a Markovian cocycle, and V can be recovered from j' by evaluation at the identity. To exploit this we work in a wider category than is usual, not restricting to $*$ -homomorphic or even completely positive processes. Results on (0.1) are thereby deduced from the findings of our study of (0.2), which holds our primary focus.

Examples of Markovian cocycles arise by solving quantum stochastic differential equations (QSDE's) of the form

$$dU_t = U_t F_\beta^\alpha d\Lambda_\alpha^\beta(t), \quad dj_t = j_t \circ \theta_\beta^\alpha d\Lambda_\alpha^\beta(t),$$

where $F = [F_\beta^\alpha]$ is a matrix of operators on $\mathfrak{h}$, $\theta = [\theta_\beta^\alpha]$ is a matrix of linear maps on $\mathcal{A}$, $\Lambda = [\Lambda_\beta^\alpha]$ is the matrix of fundamental integrator processes of Hudson-Parthasarathy quantum stochastic calculus; summation over repeated indices being understood ([Me2],[Par]). The main aim of this paper is to work in the opposite direction and find conditions on a cocycle that imply the existence of a *stochastic generator*, that is a matrix F or θ such that the cocycle satisfies the corresponding QSDE for this matrix. In previous work on this question the cocycles were assumed to consist of unitary operators ([HL2]), and injective normal unital $*$ -homomorphisms on a von Neumann algebra ([Bra],[AcM]). The generator of the Markov semigroup is used to construct an associated martingale whose representation as a quantum stochastic integral ([PS1],[PS2]) leads to the existence of the stochastic generator. In these papers crucial use is made of these assumed properties of V_t , j_t and $\mathcal{A}$ in order to apply the results of [PS1] and [PS2], and in the case of cocycles on $\mathcal{A}$, to actually make sense of the defining identity (0.2). The techniques that we develop here (which are closer in spirit to those used in [AJL] and [Fa2]) depend on a new characterisation of those processes V and j satisfying (0.1) and (0.2) respectively, and replace the use of martingale representation by more elementary arguments. Adopting this new characterisation as our *definition* of cocycles (replacing the identities (0.1) and (0.2)) we are now able to deal with a large class of unbounded cocycles. More importantly our definition makes good sense for processes defined on any C^* -algebra provided that the process j satisfies a *generalised Feller condition*.

The plan of the paper is as follows. In the first section we provide some context, by describing the existing work on stochastic cocycles. In Section 2 we set up the language and notation that we shall use throughout. In contrast to the above discussion we shall actually work with two-parameter processes on $\mathfrak{h} \otimes \mathcal{F}$, where $\mathcal{F}$ is the symmetric Fock space over $L^2(\mathbb{R}; \mathfrak{k})$. These are related to their one-parameter counterparts by $X_{r,t} = \sigma_r(V_{t-r})$ and $k_{r,t} = \sigma_r \circ j_{t-r}$, where now $-\infty < r \leq t < \infty$. In Section 3 we give our (new) definition of *flow* on a C^* -algebra that replaces (0.2) (a flow in this paper is the two-parameter version of a cocycle) and explore the basic algebraic consequences of this definition. Working with two-parameter processes we have both *forward* and *backward* flows; they are related by means of a time reversal operation (c.f. [Jou]). We also describe how to pass between the one-parameter picture and the two-parameter picture, and in Section 4 we show that our new definition reduces to (0.2) for normal completely bounded flows on a von Neumann algebra. Section 5 contains our results on the generation of flows as solutions of QSDE's. The most general form of the representation result is given in Theorem 5.6. This achieves a more elegant form in Theorem 5.10 when the maps $k_{r,t}$ that comprise the flow are assumed to be completely positive contractions. Finally, in Section 6, we rapidly develop the corresponding results for left and right evolutions on Hilbert space by exploiting the work of the previous sections.

Notational conventions. In order to distinguish between various types of tensor product we shall use the symbol $\odot$ for the algebraic tensor product of vectors, linear spaces and maps, reserving the symbol $\otimes$ for the Hilbert space tensor product of elements, spaces and bounded operators, and for the von Neumann algebra tensor product of elements, algebras and completely bounded normal maps. For vector spaces V and W , $\mathcal{L}(V; W)$ denotes the space of linear maps from V to W , and for normed spaces X and Y , $\mathcal{B}(X; Y)$ denotes the normed space of bounded operators from X to Y . If V is a dense subspace of X then $\mathcal{L}(V; Y)$ will be viewed as the linear space of (possibly unbounded) operators $X \rightarrow Y$, each with domain V ; $\mathcal{B}(X; Y)$ is thereby identified with the subspace of bounded operators in $\mathcal{L}(V; Y)$ by restriction/continuous linear extension. We denote the indicator function of a set A by $\mathbf{1}_A$.

1. BACKGROUND

Markovian cocycles were introduced into the quantum probability literature by Accardi in [Ac1] as a tool for perturbing quantum Markov processes, thereby obtaining a generalisation of the Feynman-Kac formula. These ideas were elaborated further as part of the axiomatic framework for more general quantum stochastic processes given in [AFL], where a process is defined to be a family $(j_t : \mathcal{A} \rightarrow \mathcal{C})_{t \in T}$ of unital $*$ -homomorphisms between two operator algebras, and a fixed state ω is chosen on the image algebra $\mathcal{C}$ that plays the role of a probability measure. In order to develop the theory further and discuss concepts such as Markovianity it is necessary for $\mathcal{C}$ to be equipped with a filtration, and in [AFL] and early work of Kümmerer ([Küm]) this is done by setting $\mathcal{C}_I$ to be the algebra generated by $\{j_t(a) : a \in \mathcal{A}, t \in I\}$ for each subset $I \subset T$. Thus j is adapted to the filtration by construction. The cocycles considered in [AFL] are used to perturb *stationary* Markov processes, the same class of processes that is treated in greater detail in [Küm]. Here, in contrast to our work, the indexing set T is taken to be the whole line $\mathbb{R}$, and $\mathcal{C}$ is endowed with a one-parameter group of automorphisms $(\theta_t)_{t \in \mathbb{R}}$, compatible with the filtration and leaving the state ω invariant, such that $\theta_t \circ j_0 = j_t$ for all t . Note that j is thus determined by the group θ and the time zero map j_0 , which is often assumed to be injective. A Markovian cocycle in [AFL] is defined to be a family $(m_t)_{t \geq 0} \subset \mathcal{B}(\mathcal{C})$ of unital completely positive maps such that

$$m_{s+t} = m_s \circ \theta_s \circ m_t \circ \theta_{-s} \quad (1.1)$$

and satisfying a number of compatibility conditions with respect to the filtration and the state. An example of such a cocycle is given by setting $m_t(c) = V_t c V_t^*$ for any (adapted) family of coisometries $(V_t)_{t \geq 0} \subset \mathcal{C}$ such that $V_{s+t} = V_s \theta_s(V_t)$, with m being automorphic if V is unitary. Note that this is the same as the first construction used above to relate (0.1) and (0.2).

A different viewpoint in this general setting is taken up in recent work by Hellmich et al. ([HKK]), which starts with a von Neumann algebra $\mathcal{C}$ together with a faithful normal state ω , and equipped with a prescribed filtration. Axioms for a *quantum* or *generalised white noise* θ on $\mathcal{C}$ are given, and unitary cocycles $(u_t)_{t \geq 0}$ with respect to this noise are introduced. Defining $j_t = \text{Ad } u_t \circ \theta_t$ they obtain a stationary Markov process (adapted to the given filtration), which can be considered as a perturbation of the free evolution of the noise. The main result of [HKK] is the establishment of a bijective correspondence between sufficiently well-behaved unitary (multiplicative) cocycles u and *additive* cocycles $(b_t)_{t \geq 0}$, that is families of operators satisfying the equation $b_{s+t} = b_s + \theta_s(b_t)$. Moreover the cocycle u is shown to satisfy a stochastic differential equation in which b is the integrator process.

More concrete realisations of these ideas have been studied on the different varieties of Fock space (boson, fermion, free). In each case the natural time shift on the underlying Hilbert space gives rise to a shift (semi)group σ on the operators of the Fock space $\mathcal{F}$. Unitary or contraction cocycles are then defined as families in $\mathcal{B}(\mathfrak{h} \otimes \mathcal{F})$ satisfying (0.1), where $\mathfrak{h}$ is an additional Hilbert space and σ has been extended to act on this larger algebra, usually by tensoring with the identity of $\mathcal{B}(\mathfrak{h})$. The possibility of using a shift that acts nontrivially on $\mathcal{B}(\mathfrak{h})$ is discussed in [Me1] and [Ac2], and examples of cocycles for such shifts have been constructed in [Fa1].

Stochastic differential equations generate examples of cocycles on a Fock space, and regularity conditions on a cocycle have been found which ensure that it is governed an SDE in the relevant quantum stochastic calculus. In contrast to the general work of Accardi, Kümmerer et al., these representations are achieved without reference to a particular state on $\mathcal{B}(\mathfrak{h} \otimes \mathcal{F})$. The first example is [HL1] for cocycles defined with respect to quantum Brownian motion of non-unit variance; in this case the martingale representation theorem is relatively straightforward. The techniques of this paper were then adapted to the unit variance boson case in [HL2] after the work of Parthasarathy and Sinha, and then to fermionic cocycles in [Str], via the unification of the fermion and boson calculi in [HP2]. In each of these cases the stochastic generator is composed of bounded operators; this follows from the assumption that the associated Markov semigroup is norm continuous. Weaker continuity conditions, and hence unbounded generators, have been studied in [Jou], [AJL] and [Fa2]. In particular in [Jou] unitary cocycles on boson Fock space are considered whose Markov semigroups are only strongly continuous. An alternative notion of generator is introduced, from which the cocycle can be reconstructed via a recursive procedure through the finite particle subspaces. However a counterexample shows that in general such a cocycle will not have a representation as the solution of a QSDE.

In [Bra] the identity (0.2) was introduced as an alternative to the definition (1.1), for map-valued cocycles. Note that (1.1) is equivalent to $m_{s+t} \circ \theta_{s+t} = m_s \circ \theta_s \circ m_t \circ \theta_t$, and for this to make sense it is only necessary that the time parameter for θ be the half-line and that m_t be defined on the algebra $\theta_t(\mathcal{C})$, which in [Bra] and [AcM] is a proper subalgebra of $\mathcal{C}$ since θ is composed of nonsurjective $*$ -homomorphisms.

Unitary cocycles can also be generated by randomising one-parameter unitary groups on a Hilbert space, or automorphism groups on a von Neumann algebra using classical noise. For example, given a unitary group $(U_t)_{t \in \mathbb{R}}$ acting on a Hilbert space $\mathfrak{h}$, defining $u_t = U_{B_t}$ ($t \geq 0$), where $(B_t)_{t \geq 0}$ is the standard Brownian motion, gives a family of unitary operators acting on the Hilbert space of vector-valued functions $L^2(\Omega; \mathfrak{h})$ that satisfy the cocycle identity $u_{s+t} = u_s \theta_s(u_t)$, where θ is now the right shift on Brownian paths. A similar construction is possible using the Poisson process, and more generally groups of automorphisms on a von Neumann algebra may be randomised ([LiS]) to obtain cocycles.

Finally, unitary cocycles also appear in the work of Arveson on classifying one-parameter semigroups of unital $*$ -endomorphisms of $\mathcal{B}(\mathcal{H})$ (for a separable Hilbert space $\mathcal{H}$), called E_0 -semigroups ([Ar1]). The study of this structure is motivated by quantum field theory ([AR2]). Given such a semigroup α , if $(U_t)_{t \geq 0}$ is a strongly continuous family of unitaries on $\mathcal{H}$ satisfying $U_{s+t} = U_s \alpha_s(U_t)$, then $X \mapsto U_t \alpha_t(X) U_t^*$ is again an E_0 -semigroup. Two characteristics of an E_0 -semigroup — its type and index — are shown to be invariant under such transformations. These ideas are now being extended to the classification of quantum dynamical semigroups (QDS's) on $\mathcal{B}(\mathcal{H})$, that is, semigroups of unital and completely positive maps on this algebra. Bhat showed that given any QDS τ one can construct a *minimal* dilation of τ to an E_0 -semigroup θ acting on $\mathcal{B}(\mathcal{H}')$, where $\mathcal{H}'$ is a Hilbert

space containing $\mathcal{H}$ as a subspace. That is

$$\tau_t(X) = E^* \theta_t(EXE^*)E \quad \forall X \in \mathcal{B}(\mathcal{H}), \quad (1.2)$$

where E is the inclusion map $\mathcal{H} \rightarrow \mathcal{H}'$, and minimality means that $\mathcal{H}'$ is the closed linear span of $\{\theta_t(EXE^*)Eu : t \geq 0, X \in \mathcal{B}(\mathcal{H}), u \in \mathcal{H}\}$. The index of a QDS is defined to be the index of its minimal dilation ([Bh1]); an equivalent more intrinsic definition appears in [Ar3]. The methods of quantum stochastic calculus are now being introduced into this study ([Bh2]) as a result of the following observation: given any QDS τ with bounded generator it is possible to construct a Markovian cocycle j satisfying (0.2), for some choice of the Fock space $\mathcal{F}$, in which the maps j_t are unital and $*$ -homomorphic, and whose Markov semigroup is τ ([HuS]). Composing the cocycle j with the shift σ then gives an E_0 -semigroup $(\hat{j}_t \circ \sigma_t)_{t \geq 0}$ that dilates τ in the sense that it satisfies (1.2). For a particular class of dilations realised in this manner a “deficiency index” is computed in [Bh2] that measures how far this dilation is from being minimal. This is achieved through a characterisation of all positive contraction operator cocycles in the case when $\mathfrak{h} = \mathbb{C}$.

2. PRELIMINARIES

We fix once and for all a Hilbert space $\mathfrak{h}$, called the *initial space*, and a separable Hilbert space $\mathfrak{k}$, called the *noise dimension space*, for which we choose a fixed orthonormal basis $\eta = (e_i)_{i \geq 1}$, and extend to a basis $(e_\alpha)_{\alpha \geq 0}$ of $\hat{\mathfrak{k}} := \mathbb{C} \oplus \mathfrak{k}$ by putting $e_0 = 1$. Let $D = \dim \hat{\mathfrak{k}}$. Note the convention introduced here which we shall employ throughout the paper: roman indices run from 1 whereas greek indices run from 0. Also, the Einstein summation convention will be used for pairs of repeated indices, subject to this rule.

Given a vector space V , $M_D(V)$ denotes the collection of matrices $v = [v_\beta^\alpha]_{\alpha, \beta \geq 0}$ with entries in V , endowed with a linear structure via pointwise defined operations; the indices range in the set $\{\gamma : 0 \leq \gamma < D\}$. If V has an involution $\dagger$ then this gives rise to an involution $v \mapsto v^\dagger$ on $M_D(V)$, defined by $(v^\dagger)_\beta^\alpha = (v_\alpha^\beta)^\dagger$. Define $c(D)$ to be $\mathbb{C}^{D-1}$ if $D < \infty$, and $c_{00}(\mathbb{N})$ otherwise, where $c_{00}(\mathbb{N})$ is the space of complex sequences whose terms are eventually zero. For $v \in M_D(V)$ and $w, z \in c(D)$, the element $v_{w,z}$ of V is defined by

$$v_{w,z} = w_\alpha v_\beta^\alpha z^\beta \quad (2.1)$$

where $w_\alpha = \overline{w^\alpha}$, and $w^0 = z^0 = 1$. Elements of $M_D(\mathcal{B}(\mathfrak{h}))$ and $M_D(\mathcal{B}(\mathcal{A}))$ are called *operator matrices* and *mapping matrices* respectively, where $\mathcal{A}$ is any operator algebra acting nondegenerately on $\mathfrak{h}$.

Aspects of Fock space. For a Hilbert space $\mathcal{K}$, the (symmetric) Fock space over $\mathcal{K}$ is the Hilbert space

$$\Gamma(\mathcal{K}) := \bigoplus_{n \geq 0} (\otimes_{\text{sym}}^{(n)} \mathcal{K})$$

where $\otimes_{\text{sym}}^{(0)} \mathcal{K} = \mathbb{C}$ and, for $n \geq 1$, $\otimes_{\text{sym}}^{(n)}$ denotes the n -fold symmetric tensor product. For any subset S of $\mathcal{K}$, $\mathcal{E}(S)$ denotes the linear span of the exponential vectors

$$\varepsilon(f) := ((n!)^{-1/2} \otimes^{(n)} f), \quad f \in S.$$

$\mathcal{E}(S)$ is dense in $\Gamma(\mathcal{K})$ whenever S is dense in $\mathcal{K}$. Natural isomorphisms exist between these objects: $\Gamma(\mathcal{K}_1 \oplus \mathcal{K}_2) = \Gamma(\mathcal{K}_1) \otimes \Gamma(\mathcal{K}_2)$, and $\mathcal{E}(S_1 \oplus S_2) = \mathcal{E}(S_1) \odot \mathcal{E}(S_2)$, given by (continuous) linear extension of the map $\varepsilon(f_1, f_2) \mapsto \varepsilon(f_1) \otimes \varepsilon(f_2)$.

Let $\mathcal{B} = \mathcal{B}(\mathcal{F})$ and $\mathcal{H} = \mathfrak{h} \otimes \mathcal{F}$ where $\mathcal{F} = \Gamma(\mathcal{K})$ and $\mathcal{K} = L^2(\mathbb{R}; \mathfrak{k}) = L^2(\mathbb{R}) \otimes \mathfrak{k}$. The *vacuum conditional expectation* of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathfrak{h})$, denoted $\mathbb{E}$, is defined by

$$\mathbb{E}[X] = E^* X E$$

where $E : \mathfrak{h} \rightarrow \mathcal{H}$ is the isometry $u \mapsto u \otimes \varepsilon(0)$. Adaptedness of quantum stochastic processes is defined in terms of the following manifestation of the continuous tensor product structure of the Fock space $\mathcal{F}$:

$$\mathcal{F}_{I \cup J} = \mathcal{F}_I \otimes \mathcal{F}_J$$

for disjoint Borel measurable subsets I and J of $\mathbb{R}$, where $\mathcal{F}_I$ and $\mathcal{F}_J$ are the Fock spaces over $L^2(I; \mathbf{k})$ and $L^2(J; \mathbf{k})$ respectively. The divisibility property also gives the following further identifications and inclusions:

$$\mathcal{H}_{I \cup J} = \mathcal{H}_I \otimes \mathcal{F}_J; \quad \mathcal{B}_{I \cup J} = \mathcal{B}_I \otimes \mathcal{B}_J; \quad \mathcal{H}_I \subset \mathcal{H} \quad (\xi \mapsto \xi \otimes \varepsilon(0))$$

using the same subscript convention, and with I and J disjoint as before.

All of our processes will be defined on the domain $\mathfrak{h} \odot \mathcal{E}$ where $\mathcal{E} := \mathcal{E}(\mathbb{S})$ and

$$\mathbb{S} = \mathbb{S}[\eta] := \text{Lin}\{e_i \mathbf{1}_I : i \geq 1, I \text{ a bounded subinterval of } \mathbb{R}\}.$$

Also define

$$\mathbb{S}_{\text{loc}} := \{f : \mathbb{R} \rightarrow \mathbf{k} \mid f_J \in \mathbb{S} \text{ for each bounded subinterval } J \text{ of } \mathbb{R}\}.$$

where f_J denotes $f \mathbf{1}_J$. We then have another two identifications:

$$\mathbb{S}_{I \cup J} = \mathbb{S}_I \oplus \mathbb{S}_J; \quad \mathcal{E}_{I \cup J} = \mathcal{E}_I \odot \mathcal{E}_J.$$

for I and J disjoint. In the next section, for results concerning the semigroup decomposition of time-homogeneous flows, it is necessary to choose a particular version of a given $f \in \mathbb{S}$, and we shall always work with the right continuous version. Product vectors $\xi \otimes \varepsilon(f)$ in $\mathcal{H}_I \odot \mathcal{E}_J \subset \mathcal{H}$, for disjoint I and J , will be abbreviated to $\xi \varepsilon(f)$.

We make use of the following operators to extend the existing theory of cocycles on an algebra from normal $*$ -homomorphisms on a von Neumann algebra to more general classes of maps on a C^* -algebra. Given $f \in L^2_{\text{loc}}(\mathbb{R}; \mathbf{k})$ and a bounded Borel set $I \subset \mathbb{R}$, define $D_I^f \in \mathcal{B}(\mathfrak{h}; \mathcal{H}_I) \subset \mathcal{B}(\mathfrak{h}; \mathcal{H})$ by

$$D_I^f u = u \varepsilon(f_I). \tag{2.2}$$

Each element c of $\mathcal{B}(\mathcal{H}_I)$ is determined by the family $\{(D_I^f)^* c D_I^g : f, g \in \mathbb{S}_{\text{loc}}\}$ since $\{\varepsilon(f_I) : f \in \mathbb{S}_{\text{loc}}\}$ is total in $\mathcal{F}_I$.

For each $t \in \mathbb{R}$, the right shift s_t on $\mathcal{K}$ is the unitary operator given by

$$(s_t f)(u) = f(u - t), \quad u \in \mathbb{R}.$$

These determine unitary operators $(S_t)_{t \in \mathbb{R}}$ on $\mathcal{H}$ through second quantisation:

$$S_t u \varepsilon(f) = u \varepsilon(s_t f),$$

and these in turn determine shifts on $\mathcal{L}(\mathfrak{h} \odot \mathcal{E}; \mathcal{H})$ by

$$\sigma_t(Z) = S_t Z S_t^*,$$

since s_t leaves $\mathbb{S}$ invariant. Restricted to $\mathcal{M} \otimes \mathcal{B}$, for any von Neumann algebra of operators $\mathcal{M}$ acting on $\mathfrak{h}$, $(\sigma_t)_{t \in \mathbb{R}}$ is an ultraweakly continuous automorphism group.

For each pair $s \leq t$ the time reversal operator $r_{s,t}$ on $\mathcal{K}$ is the self-adjoint unitary operator given by

$$(r_{s,t} f)(u) = \begin{cases} f(t + s - u), & u \in [s, t] \\ f(u), & u \notin [s, t] \end{cases}.$$

As with the shifts these determine self-adjoint unitaries $(R_{s,t})_{s \leq t}$ on $\mathcal{H}$ through second quantisation, and time reversals $(\rho_{s,t})_{s \leq t}$ on $\mathcal{L}(\mathfrak{h} \odot \mathcal{E}; \mathcal{H})$ by conjugation:

$$R_{s,t} u \varepsilon(f) = u \varepsilon(r_{s,t} f); \quad \rho_{s,t}(Z) = R_{s,t} Z R_{s,t}. \tag{2.3}$$

Completely bounded maps. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^* -algebras acting on Hilbert spaces $\mathfrak{h}_1$ and $\mathfrak{h}_2$ respectively. Let $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ be a linear map, then for each $n \geq 1$ the induced map $\phi^{(n)} : M_n(\mathcal{A}_1) \rightarrow M_n(\mathcal{A}_2)$ is defined by $\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})]$. The map ϕ is *completely bounded* (CB) if

$$\|\phi\|_{\text{cb}} := \sup_n \|\phi^{(n)}\| < \infty.$$

The Wittstock-Paulsen Theorem ([SiS], Theorem 1.3.1) states that for any CB map ϕ there is a representation π of $\mathcal{A}_1$ on some Hilbert space $\mathcal{K}$ and operators $W_i \in \mathcal{B}(\mathfrak{h}_2; \mathcal{K})$, $i = 1, 2$, such that

$$\phi(a) = W_2^* \pi(a) W_1, \quad \|\phi\|_{\text{cb}} = \|W_2\| \|W_1\|.$$

When both $\mathcal{A}_1$ and $\mathcal{A}_2$ are von Neumann algebras and the map ϕ is normal then it is possible to choose π above so that it is normal ([Haa], [SiS] Lemma 1.5.7). Thus, by the standard form for normal representations of von Neumann algebras ([Tak], Theorem IV.5.5), it follows that the map ϕ is CB and normal if and only if there is some Hilbert space $\mathfrak{h}$ and operators $V_i \in \mathcal{B}(\mathfrak{h}_2; \mathfrak{h}_1 \otimes \mathfrak{h})$, $i = 1, 2$, such that

$$\phi(a) = V_2^*(a \otimes 1_{\mathfrak{h}}) V_1, \quad a \in \mathcal{A}_1.$$

In particular, if $\mathcal{C}_1, \mathcal{C}_2$ are two more von Neumann algebras and $\psi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is also CB and normal with decomposition $\psi(c) = U_2^*(c \otimes 1_{\mathfrak{k}}) U_1$, then $\phi \odot \psi$ extends continuously (in the ultraweak topologies) to a map denoted $\phi \otimes \psi : \mathcal{A}_1 \otimes \mathcal{C}_1 \rightarrow \mathcal{A}_2 \otimes \mathcal{C}_2$:

$$(\phi \otimes \psi)(d) = (V_2 \otimes U_2)^*(d \otimes 1_{\mathfrak{h} \otimes \mathfrak{k}})(V_1 \otimes U_1), \quad d \in \mathcal{A}_1 \otimes \mathcal{C}_1.$$

Processes on a Hilbert space. The following definitions are essentially those given in [LW1] except that here we work with the two-parameter index set $\mathbb{R}_{\leq}^2 := \{(r, t) : r, t \in \mathbb{R}, r \leq t\}$; two-parameter processes being more natural and convenient for present algebraic considerations. We identify $\mathbb{R}_{\leq}^2$ with the collection of closed bounded subintervals of $\mathbb{R}$, $\mathcal{I} = \{I \subset \mathbb{R} : I = [r, t], r \leq t\}$. A *quantum stochastic process* on the Hilbert space $\mathfrak{h}$ is a family $(X_{r,t})_{r \leq t} = (X_I)_{I \in \mathcal{I}}$ in $\mathcal{L}(\mathfrak{h} \odot \mathcal{E}; \mathcal{H})$ such that $(r, t) \mapsto X_{r,t} \xi$ is weakly measurable for all $\xi \in \mathfrak{h} \odot \mathcal{E}$, and which satisfies the adaptedness conditions

$$X_I(\mathfrak{h} \odot \mathcal{E}_I) \subset \mathcal{H}_I; \quad X_I u \varepsilon(f) = (X_I u \varepsilon(f_I)) \otimes \varepsilon(f_{I^c}),$$

or, equivalently,

$$\langle u \varepsilon(f), X_I v \varepsilon(g) \rangle = \langle u \varepsilon(f_I), X_I v \varepsilon(g_I) \rangle \langle \varepsilon(f_{I^c}), \varepsilon(g_{I^c}) \rangle, \quad (2.4)$$

for all $u, v \in \mathfrak{h}$, $f, g \in \mathbb{S}$. Processes X and Y will be identified if for all $\zeta, \xi \in \mathfrak{h} \odot \mathcal{E}$, $\langle \zeta, (X_{r,t} - Y_{r,t}) \xi \rangle = 0$ for a.a. (r, t) ; $\mathbb{P}(\mathfrak{h}) = \mathbb{P}(\mathfrak{h}, \mathcal{E})$ denotes the linear space of processes on $\mathfrak{h}$. If a process X on $\mathfrak{h}$ satisfies $\mathfrak{h} \odot \mathcal{E} \subset \text{Dom } X_I^*$ for all $I \in \mathcal{I}$, then

$$X_I^\dagger = X_I^*|_{\mathfrak{h} \odot \mathcal{E}}$$

defines an element of $\mathbb{P}(\mathfrak{h})$ called the *adjoint process* of X . The subspace of processes on $\mathfrak{h}$ having an adjoint is denoted $\mathbb{P}^\dagger(\mathfrak{h}) = \mathbb{P}^\dagger(\mathfrak{h}, \mathcal{E})$.

A process X is *strongly continuous* (respectively *strongly continuous at 0*) if

$$\mathbb{R}_{\leq}^2 \ni (r, t) \mapsto X_{r,t} \xi \text{ is continuous} \quad (\text{resp. } \lim_{t \rightarrow 0} X_{0,t} \xi = X_{0,0} \xi)$$

for all $\xi \in \mathfrak{h} \odot \mathcal{E}$. Weak continuity (at 0) is defined analogously.

We associate to any process X two families of operators $X_I^f = X_{r,t}^f \in \mathcal{L}(\mathfrak{h}; \mathcal{H}_I)$ and $X_I^{f,g} = X_{r,t}^{f,g} \in \mathcal{L}(\mathfrak{h})$ for $I = [r, t]$, defined by

$$X_I^f = X_I D_I^f; \quad X_I^{f,g} = (D_I^f)^* X_I D_I^g$$

for $f, g \in \mathbb{S}_{\text{loc}}, r \leq t$, where D_I^f is defined in (2.2). Note that (2.4) implies that any processes is determined by the family of operators $\{X_I^{f,g}\}$. Note also that since D_I^f depends only on the restriction f_I , the same is true for the operators X_I^f and $X_I^{f,g}$.

We extend the range of regularity conditions on processes that were found useful in [LW1]. The process X is *weakly initial space bounded* (i.s.b.) if each $X_I^{f,g}$ is bounded, and *weakly regular* if also $\sup\{\|X_J^{f,g}\| : J \subset I\} < \infty$ for all $f, g \in \mathbb{S}_{\text{loc}}, I \in \mathcal{I}$. It is *strongly i.s.b.* if each X_I^f is bounded, and *strongly regular* if also $\sup\{\|X_J^f\| : J \subset I\} < \infty$ for all $I \in \mathcal{I}$ and $f \in \mathbb{S}_{\text{loc}}$; and it is a *bounded* (resp. *contraction*) process if each X_I is bounded (resp. contractive).

If X is strongly i.s.b. then

$$\widehat{X}_I \xi \varepsilon(f_I) = (X_I^f \otimes 1_{I^c}) \xi, \quad \xi \in \mathcal{H}_{I^c} \quad (2.5)$$

defines an extension of X_I to an operator $\widehat{X}_I$ in $\mathcal{L}(\mathcal{H}_{I^c} \odot \mathcal{E}_I; \mathcal{H})$. When X is bounded $\widehat{X}_I$ is simply the unique continuous extension of the operator X_I .

Processes on an algebra. Let $\mathcal{A}$ be a unital C^* -algebra, acting on the Hilbert space $\mathfrak{h}$. A *quantum stochastic process on $\mathcal{A}$* is a linear map $k : \mathcal{A} \rightarrow \mathbb{P}(\mathfrak{h})$ for which each operator $k_{s,t}(a) = k_I(a)$ is affiliated to the von Neumann algebra $\mathcal{A}'' \otimes \mathcal{B}$:

$$k_I(a)(a' \otimes 1_{\mathcal{F}})\xi = (a' \otimes 1_{\mathcal{F}})k_I(a)\xi \quad (2.6)$$

for all $a \in \mathcal{A}, a' \in \mathcal{A}', I \in \mathcal{I}$ and $\xi \in \mathfrak{h} \odot \mathcal{E}$. We denote the linear space of quantum stochastic processes on $\mathcal{A}$ by $\mathbb{P}(\mathcal{A})$. A process k is *bounded, completely positive, completely bounded, contractive* or, when $\mathcal{A}$ is a von Neumann algebra, *normal* if each constituent map k_I has that property. It is *pointwise strongly* or *weakly continuous* (at 0) if each process $k(a) \in \mathbb{P}(\mathfrak{h})$ is strongly or weakly continuous (at 0). If a process on $\mathcal{A}$ satisfies $k(a) \in \mathbb{P}^\dagger(\mathfrak{h})$ for each $a \in \mathcal{A}$, then

$$k_I^\dagger(a) = k_I(a^*)^\dagger$$

defines a process $k^\dagger$ called the *adjoint process* to k . The linear space of processes on $\mathcal{A}$ having an adjoint is denoted $\mathbb{P}^\dagger(\mathcal{A})$. As with processes on $\mathfrak{h}$ we associate two families of maps, $k_I^f : \mathcal{A} \rightarrow \mathcal{L}(\mathfrak{h}; \mathcal{H}_I)$ and $k_I^{f,g} : \mathcal{A} \rightarrow \mathcal{L}(\mathfrak{h})$ given by

$$k_I^f(a) = k_I(a)D_I^f; \quad k_I^{f,g}(a) = (D_I^f)^* k_I(a)D_I^g.$$

Since by definition k is adapted it follows that k is uniquely determined by the family $\{k_I^{f,g}\}$.

The following regularity properties are analogues to those for processes on $\mathfrak{h}$, except they also allow for variation in the algebra element a . The process k is *weakly initial algebra bounded* (i.a.b.) if each $k_I^{f,g}(a)$ is bounded (and so belongs to $\mathcal{A}''$) and each operator $k_I^{f,g} : \mathcal{A} \rightarrow \mathcal{A}''$ is bounded. The process is *weakly regular* if also each map $k^{f,g} : \mathbb{R}_{\leq}^2 \rightarrow \mathcal{B}(\mathcal{A}; \mathcal{A}'')$ is locally bounded; it is *strongly i.a.b.* (respectively *strongly regular*) if these hold for the maps $\{k_I^f\}$. The analogue of the extension $\widehat{X}$ is restricted to processes on von Neumann algebras, which are considered in Section 4.

Weak initial algebra boundedness is a minimal condition for defining flows; weak regularity is a natural condition for guaranteeing uniqueness of weak solutions of quantum stochastic differential equations ([LW1], Theorem 3.1); strong regularity often holds for strong solutions of QSDE's ([LW1], Theorem 3.3). These conditions have been introduced in the most convenient way for use in the next section, however

consider the following set of conditions on a given $k \in \mathbb{P}(\mathcal{A})$:

$$\sup\{|\langle u\varepsilon(f), k_I(a)v\varepsilon(g) \rangle| : \|u\| = \|v\| = 1\} < \infty, \quad (2.7a)$$

$$\sup\{|\langle u\varepsilon(f), k_I(a)v\varepsilon(g) \rangle| : \|a\| = 1\} < \infty, \quad (2.7b)$$

$$\sup\{|\langle u\varepsilon(f), k_I(a)v\varepsilon(g) \rangle| : I \subset J\} < \infty. \quad (2.7c)$$

The Banach-Steinhaus Theorem implies that a process k is weakly i.a.b. if and only if (2.7a) and (2.7b) hold, which is true if and only if for all choices of $f, g \in \mathbb{S}_{\text{loc}}$ and $I \in \mathcal{I}$ there is a constant $C_I^{f,g}$ such that

$$|\langle u\varepsilon(f), k_I(a)v\varepsilon(g) \rangle| \leq C_I^{f,g} \|u\| \|a\| \|v\| \quad \forall u, a, v.$$

Similarly, k is weakly regular if and only if (2.7a–c) hold, which is true if and only if for all choices of $f, g \in \mathbb{S}_{\text{loc}}$ and $I \in \mathcal{I}$ there is a constant $K_I^{f,g}$ such that

$$|\langle u\varepsilon(f), k_J(a)v\varepsilon(g) \rangle| \leq K_I^{f,g} \|u\| \|a\| \|v\| \quad \forall u, a, v \text{ and } J \subset I.$$

Also note that if $k \in \mathbb{P}^\dagger(\mathcal{A})$ then $k_I^{f,g}(a) = (k^\dagger)_I^{g,f}(a^*)^*$, and so (2.7a) holds by the Closed Graph Theorem. If, on the other hand, k is pointwise weakly continuous then (2.7c) holds. In particular processes that are both weakly i.a.b. and pointwise weakly continuous are necessarily weakly regular. We shall improve on this observation later in the case when k is a flow (Corollary 3.3).

A similar analysis can be done for strongly i.a.b. and regular processes, replacing $|\langle u\varepsilon(f), k_I(a)v\varepsilon(g) \rangle|$ by $\|k_I(a)u\varepsilon(f)\|$ in (2.7a–c) above.

3. STOCHASTIC FLOWS ON A C^* -ALGEBRA

In this section we give the basic definitions for a flow on a C^* -algebra and develop the purely algebraic part of the theory, postponing until the next section the discussion of how this reformulation extends the framework of the existing literature. Indeed the measurability assumption on processes is irrelevant in this and the following section — it only becomes necessary in Sections 5 and 6 when we discuss stochastic integrals.

Flows on a C^* -algebra. Fix a unital C^* -algebra $\mathcal{A}$ acting on the initial space $\mathfrak{h}$ for this section. A process k on $\mathcal{A}$ is *time-homogeneous* if it satisfies

$$\sigma_p \circ k_I = k_{I+p} \quad (3.1)$$

for all $I \in \mathcal{I}, p \in \mathbb{R}$.

Lemma 3.1. *Let k be a process on $\mathcal{A}$. Then*

$$(D_{I+p}^f)^* \sigma_p \circ k_I(a) D_{I+p}^g = k_I^{s-pf, s-pg}(a)$$

for all $I \in \mathcal{I}, p \in \mathbb{R}, f, g \in \mathbb{S}_{\text{loc}}$ and $a \in \mathcal{A}$.

Proof. This follows from the easily verified identity $S_p^* D_{I+p}^f = D_I^{s-pf}$. $\square$

This result shows that we may easily recognise time-homogeneity in the reduced operators of the process: k is time-homogeneous if and only if it satisfies

$$k_{I+p}^{f,g} = k_I^{s-pf, s-pg}. \quad (3.1)'$$

Flows are defined here in terms of the reduced operators of the process. This allows us to go beyond both previous constraints of being on a von Neumann algebra, and of being bounded.

A (*forward*) *flow on $\mathcal{A}$* (resp. *backward flow on $\mathcal{A}$*) is a weakly i.a.b. process k such that

$$k_{r,t}^{f,g}(\mathcal{A}) \subset \mathcal{A}, \quad k_{r,r}^{f,g} = \text{id}_{\mathcal{A}} \quad \text{and} \quad k_{r,t}^{f,g} = k_{r,s}^{f,g} \circ k_{s,t}^{f,g}, \quad (3.2)$$

(respectively, $k_{r,t}^{f,g} = k_{s,t}^{f,g} \circ k_{r,s}^{f,g}$), for all $r \leq s \leq t$ and $f, g \in \mathbb{S}_{\text{loc}}$. As stated earlier initial algebra boundedness already implies that $k_t^{f,g}(\mathcal{A}) \subset \mathcal{A}''$. When $\mathcal{A}$ is not a von Neumann algebra the condition $k_{r,t}^{f,g}(\mathcal{A}) \subset \mathcal{A}$ should be considered as a *generalised Feller condition*; its immediate role is to guarantee that the maps $k_{r,t}^{f,g}$ can be composed, so that the evolution law may be formulated. When k is a bounded process the maps $k_{r,t}^{f,g}$ can be defined for all $f, g \in L_{\text{loc}}^2(\mathbb{R}; \mathbf{k})$, and moreover $(f, g) \mapsto k_{r,t}^{f,g}$ is continuous $\mathcal{K} \oplus \mathcal{K} \rightarrow \mathcal{B}(\mathcal{A}; \mathcal{A}'')$. Thus if (3.2) holds for all $f, g \in \mathbb{S}_{\text{loc}}$ then it remains valid for all $f, g \in L_{\text{loc}}^2(\mathbb{R}; \mathbf{k})$. In particular, the fact that k is a flow is actually independent of the choice of basis η for $\mathbf{k}$. The existence of an abundance of stochastic flows on a C^* -algebra (as opposed to a von Neumann algebra) is ensured by the results of [LW2] in conjunction with Proposition 5.2 below.

The semigroup decomposition. Identify $c(D)$ with a subspace of $\mathbb{S}_{\text{loc}}$ by associating each $z = (z^i)$ with the constant function $z^i e_i$. The main result of this subsection, which we give next, is an immediate consequence of the definitions since the functions in $c(D) \subset \mathbb{S}_{\text{loc}}$ are left unchanged by the shifts. It gives a useful characterisation of time-homogeneous flows that is an essential ingredient in the proof of the existence of stochastic integral representations in Section 5.

Proposition 3.2. *For $k \in \mathbb{P}(\mathcal{A})$, the following sets of equivalences hold:*

- (i) *k is a time-homogeneous forward flow (resp. backward flow).*
- (ii) *For all $w, z \in c(D)$, $(\mathcal{P}_s^{w,z} := k_{0,s}^{w,z})_{s \geq 0}$ is a one-parameter semigroup in $\mathcal{B}(\mathcal{A})$ and, for all $r \leq t$ and $f, g \in \mathbb{S}_{\text{loc}}$,*

$$k_{r,t}^{f,g} = \mathcal{P}_{t_1-r}^{[0]} \circ \cdots \circ \mathcal{P}_{t-t_n}^{[n]} \quad (3.3)$$

(resp. $k_{r,t}^{f,g} = \mathcal{P}_{t-t_n}^{[n]} \circ \cdots \circ \mathcal{P}_{t_1-r}^{[0]}$), where $\mathcal{P}^{[i]} = \mathcal{P}^{w,z}$ for $w = f(t_i), z = g(t_i)$, whenever $\{r = t_0 \leq \cdots \leq t_{n+1} = t\}$ contains all the discontinuities of $f_{[r,t]}$ and $g_{[r,t]}$.

Thus time-homogeneous flows are determined by their *associated semigroups* $\mathcal{P}^{w,z}, w, z \in c(D)$; $\mathcal{P}^{0,0}$ is called the *Markov semigroup* of the flow and satisfies $\mathcal{P}_{t-s}^{0,0} = \mathbb{E} \circ k_{s,t}$ where $\mathbb{E}$ is the vacuum conditional expectation. A time-homogeneous flow whose Markov semigroup is norm continuous is called *Markov-regular*. The analysis in Section 5 will reveal that if a time-homogeneous flow is both bounded, with locally uniformly bounds, and Markov-regular then all of its associated semigroups are norm continuous.

The decomposition of a flow into its associated semigroups allows us to improve on our observations on weak regularity at the end of Section 2.

Corollary 3.3. *Let k be a time-homogeneous flow. If k is pointwise weakly continuous at 0 then k is weakly regular.*

Proof. Weak continuity of k at 0 implies that for each pair $w, z \in c(D)$,

$$\lim_{t \rightarrow 0} \langle u, (\mathcal{P}_t^{w,z}(a) - a)v \rangle = 0$$

for all $u, v \in \mathfrak{h}$ and $a \in \mathcal{A}$. The Banach-Steinhaus Theorem and the semigroup property imply that $\{\|\mathcal{P}_t^{w,z}\|\}_{t \geq 0}$ is locally bounded, and so the result follows from (3.3). $\square$

Before giving the next result we introduce some more notation: given a process k on $\mathcal{A}$ we denote its *time-reversed process* by $\tilde{k}$, that is

$$\tilde{k}_{r,t} = \rho_{r,t} \circ k_{r,t} \quad (r \leq t), \quad (3.4)$$

where $\rho_{r,t}$ is defined in (2.3). Similarly, given a process X on $\mathfrak{h}$ we denote its time reversed process by $\tilde{X}$. Using this notation we have the following immediate consequence of the reversal of the order of the semigroups in the semigroup decompositions of Proposition 3.2.

Proposition 3.4. *Let $k \in \mathbb{P}(\mathcal{A})$. The following are equivalent:*

- (i) k is a time-homogeneous forward flow.
- (ii) $\tilde{k}$ is a time-homogeneous backward flow.

In view of this result we shall concentrate for the most part on forward flows.

Invertible flows. A forward flow k (respectively backward flow l) on $\mathcal{A}$ is *invertible* if there is a backward flow l (resp. a forward flow k) such that

$$k_I^{f,g} \circ l_I^{f,g} = l_I^{f,g} \circ k_I^{f,g} = \text{id}_{\mathcal{A}} \quad (3.5)$$

for all $I \in \mathcal{I}$, and $f, g \in \mathbb{S}_{\text{loc}}$. Note that if k is invertible then each $l_I^{f,g}$ is uniquely determined, and hence so is l , since l is adapted. Also if k is time-homogeneous then so is l . Such a pair (k, l) is called an *inverse pair*, and given such a pair we can extend the indexing set to all of $\mathbb{R}^2$ in a natural manner as follows: for $x, z \in \mathbb{R}$ define $k_{x,z}$ and $l_{x,z} : \mathcal{A} \rightarrow \mathcal{L}(\mathfrak{h} \odot \mathcal{E}; \mathcal{H})$ by

$$k_{x,z} = \begin{cases} k_{x,z}, & x \leq z \\ l_{z,x}, & x > z \end{cases}; \quad l_{x,z} = \begin{cases} l_{x,z}, & x \leq z \\ k_{z,x}, & x > z \end{cases}.$$

These extended families satisfy the evolution equations

$$k_{x,z}^{f,g} = k_{x,y}^{f,g} \circ k_{y,z}^{f,g}, \quad l_{x,z}^{f,g} = l_{y,z}^{f,g} \circ l_{x,y}^{f,g} \quad (x, y, z \in \mathbb{R}, f, g \in \mathbb{S}_{\text{loc}}),$$

in which $k_{x,z}^{f,g}(\cdot) = (D_{z,x}^f)^* k_{x,z}(\cdot) D_{z,x}^g$ if $z < x$, and similarly for l .

The characterisation of time-homogeneous flows in Proposition 3.2 gives rise to the following characterisation of inverse pairs of time-homogeneous flows in terms of their associated semigroups.

Proposition 3.5. *Let k be a time-homogeneous forward flow and let l be a time-homogeneous backward flow. The following are equivalent:*

- (i) (k, l) is an inverse pair.
- (ii) For all $w, z \in c(D)$ the family of maps $(\mathcal{P}_x^{w,z})_{x \in \mathbb{R}}$ where

$$\mathcal{P}_x^{w,z} = \begin{cases} k_{0,x}^{w,z}, & x \geq 0 \\ l_{0,-x}^{w,z}, & x < 0 \end{cases}$$

is a one-parameter group in $\mathcal{B}(\mathcal{A})$.

Markovian and Feller cocycles. The time-homogeneous part of the theory developed so far may be cast into a form involving processes indexed by only one parameter. Since the majority of the existing work has been presented in this one-parameter form, we use this subsection to relate the two-parameters to one.

A *one-parameter process* on $\mathfrak{h}$ is a family $(W_t)_{t \geq 0}$ in $\mathcal{L}(\mathfrak{h} \odot \mathcal{E}; \mathcal{H})$ such that $t \mapsto W_t \eta$ is weakly measurable and satisfies

$$W_t(\mathfrak{h} \odot \mathcal{E}_{[0,t]}) \subset \mathcal{H}_{[0,t]}; \quad W_t u \varepsilon(f) = (W_t u \varepsilon(f_{[0,t]})) \otimes \varepsilon(f_{[0,t]^c}).$$

A *one-parameter process* on the C^* -algebra $\mathcal{A}$ is a linear map j from $\mathcal{A}$ into the space of one-parameter processes on $\mathfrak{h}$ satisfying the obvious counterpart to the affiliation condition (2.6). The notions of weak and strong initial space (or algebra) boundedness transfer analogously, along with extendability where applicable.

A *Markovian cocycle* on $\mathcal{A}$ (resp. *backward Markovian cocycle* on $\mathcal{A}$) is a weakly i.a.b. one-parameter process j that satisfies

$$j_t^{f,g}(\mathcal{A}) \subset \mathcal{A}, \quad j_0^{f,g} = \text{id}_{\mathcal{A}} \quad \text{and} \quad j_{r+t}^{f,g} = j_r^{f,g} \circ j_t^{s_{-r}f, s_{-r}g},$$

(resp. $j_{r+t}^{f,g} = j_t^{s_{-r}f, s_{-r}g} \circ j_r^{f,g}$), for all $r, t \geq 0$ and $f, g \in \mathbb{S}_{\text{loc}}$. When $\mathcal{A}$ is not a von Neumann algebra we may speak of *Feller cocycles*, in view of the first condition of invariance.

Proposition 3.6. *Time-homogeneous two-parameter processes on $\mathcal{A}$ are in bijective correspondence with one-parameter processes on $\mathcal{A}$ through the relations*

$$j_t = k_{0,t}; \quad k_{r,t} = \sigma_r \circ j_{t-r}. \quad (3.6)$$

Moreover, restricting these relations gives a bijective correspondence between time-homogeneous forward flows (respectively backward flows) and Markovian (resp. backward Markovian) cocycles.

Proof. First note that the time-homogeneity of the two-parameter process constructed from a given one-parameter process is immediate from the group law for σ . This gives the first part.

Suppose now that k is a time-homogeneous forward flow on $\mathcal{A}$. Then

$$k_{0,r}^{f,g} \circ k_{0,t}^{s_{-r}f, s_{-r}g} = k_{0,r}^{f,g} \circ k_{r,r+t}^{f,g} = k_{0,r+t}^{f,g},$$

by (3.1)', in other words j is a Markovian cocycle. Conversely let j be a Markovian cocycle on $\mathcal{A}$ and define k as in the hypothesis. By Lemma 3.1

$$k_{r,t}^{f,g} = j_{t-r}^{s_{-r}f, s_{-r}g},$$

and so $k_{r,t}^{f,g}(\mathcal{A}) \subset \mathcal{A}$. Setting $f' = s_{-r}f, g' = s_{-r}g$ we have for all $r \leq s \leq t$

$$j_{s-r}^{s_{-r}f, s_{-r}g} \circ j_{t-s}^{s_{-s}f, s_{-s}g} = j_{s-r}^{f', g'} \circ j_{t-s}^{s_{r-s}f', s_{r-s}g'} = j_{t-r}^{s_{-r}f, s_{-r}g},$$

so that k is a forward flow. The proof for backward flows and Markovian cocycles is identical. $\square$

Since $j_t(a)$ is a $[0, t]$ -adapted operator it follows that there is some operator $j'_t(a)$ with domain $\mathfrak{h} \odot \mathcal{E}_{[0, \infty[}$ such that $j_t(a) = 1_{]-\infty, 0[} \odot j'_t(a)$. It is the family $(j'_t(a))_{t \geq 0}$ that is studied in [Bra] and [AcM]. An advantage of embedding it into the larger space $\mathcal{H}$ is that the shift operators S_t become unitaries rather than isometries. There are some subtleties that arise in this embedding procedure — see Section 6 for more discussion on this matter.

4. FLOWS AND COCYCLES ON A VON NEUMANN ALGEBRA

In this section we show that when we specialise the definitions to normal $*$ -homomorphic processes on a von Neumann algebra we recover the definitions introduced in [Bra].

Flows on a von Neumann algebra. The map-valued process counterpart to the extension $\widehat{X}$ of a process X on $\mathfrak{h}$ given in Section 2 is as follows. A process k on a von Neumann algebra $\mathcal{M}$ is *extendable* if it is strongly i.a.b. and each map $k_I^f : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{B}_I$ is completely bounded and normal. Given such a process

$$\widehat{k}_I(c)\xi\epsilon(f_I) = (k_I^f \otimes \text{id}_{I^c})(c)\xi \quad (4.1)$$

defines an extension of $k_I \odot \text{id}_{I^c}$ to a map $\widehat{k}_I : \mathcal{M} \otimes \mathcal{B}_{I^c} \rightarrow \mathcal{L}(\mathcal{H}_{I^c} \odot \mathcal{E}_I; \mathcal{H})$. If the process itself is completely bounded and normal then clearly k is extendable. On the other hand $k_I \otimes \text{id}_{I^c} : \mathcal{M} \otimes \mathcal{B}_{I^c} \rightarrow \mathcal{M} \otimes \mathcal{B}$ is well-defined, and $k_I \otimes \text{id}_{I^c}(c)$ is the unique continuous extension of $\widehat{k}_I(c)$ to a bounded operator on $\mathcal{H}$.

Lemma 4.1. *Let k be a bounded and extendable process on a von Neumann algebra $\mathcal{M}$, and let I and J be bounded disjoint subintervals of $\mathbb{R}$. Then*

$$(D_{I \cup J}^f)^* \widehat{k}_I(k_J(a)) D_{I \cup J}^g = k_I^{f,g} \circ k_J^{f,g}(a) \quad (4.2)$$

for all $f, g \in \mathbb{S}_{\text{loc}}$ and $a \in \mathcal{M}$.

Proof. Let $c = a_1 \otimes b$ with $a_1 \in \mathcal{M}, b \in \mathcal{B}_{I^c}$, and suppose without loss that f and g are supported by $I \cup J$. Then

$$\begin{aligned} \langle u\varepsilon(f), (k_I^g \otimes \text{id}_{I^c})(c) v\varepsilon(g_J) \rangle &= \langle u\varepsilon(f), [k_I^g(a_1) \otimes b] v\varepsilon(g_J) \rangle \\ &= \langle u, k_I^{f,g}(a_1) v \rangle \langle \varepsilon(f_J), b\varepsilon(g_J) \rangle \\ &= \langle u, k_I^{f,g}((D_J^f)^* c D_J^g) v \rangle. \end{aligned}$$

By linearity and ultraweak continuity this continues to hold for any element c of $\mathcal{M} \otimes \mathcal{B}_{I^c}$. Putting $c = k_J(a)$ gives

$$\langle u\varepsilon(f), \widehat{k}_I(k_J(a)) v\varepsilon(g) \rangle = \langle u, k_I^{f,g}(k_J^{f,g}(a)) \rangle,$$

and (4.2) follows. $\square$

Recall that since any process k is assumed to be adapted, it is determined by the family $\{k_I^{f,g}\}$. Thus if k is a bounded and extendable process, for example a normal *-homomorphic process as in [Bra] and [AcM], then by Lemma 4.1 k is a forward flow (resp. backward flow) if and only if

$$k_{r,r} = \text{id}_{\mathcal{M}}; \quad k_{r,t} = \widehat{k}_{r,s} \circ k_{s,t} \quad (\text{resp. } k_{r,t} = \widehat{k}_{s,t} \circ k_{r,s}) \quad (4.3)$$

for all $r \leq s \leq t$.

The following result gives the relationship between various continuity properties of a time-homogeneous flow and those of its associated semigroups.

Proposition 4.2. *Let k be a bounded (respectively locally uniformly bounded) time-homogeneous flow on a von Neumann algebra $\mathcal{M}$ (resp. on a C^* -algebra $\mathcal{A}$). The following are equivalent:*

- (i) k is normal (resp. pointwise weakly continuous).
- (ii) Each of the associated semigroups $\{\mathcal{P}^{w,z}\}$ is normal (resp. pointwise weakly continuous).

Proof. In both cases the implication (i $\Rightarrow$ ii) is immediate from the definition of $\mathcal{P}^{w,z}$. To prove the normality of k when each $\mathcal{P}^{w,z}$ is normal, note that the weak and ultraweak topologies coincide on bounded sets, and since k is bounded it is sufficient (by [Tak], Corollary III.3.11) to check that $\{k_I(a_\lambda)\}_{\lambda \in \Lambda}$ converges to $k_I(a)$ weakly on $\mathfrak{h} \odot \mathcal{E}$, for any bounded increasing net $\{a_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{M}$ with supremum a . This follows immediately from the semigroup representation in Proposition 3.2.

Suppose instead that each semigroup $\mathcal{P}^{w,z}$ is pointwise weakly continuous. Then given any $(r, t) \in \mathbb{R}_{\leq}^2$, and restricting to test functions $f, g \in \mathbb{S}$ for which r is a point of continuity, it is clear from (3.3) that

$$\lim_{n \rightarrow \infty} \langle u\varepsilon(f), [k_{r_n, t}(a) - k_{r, t}(a)] v\varepsilon(g) \rangle = 0$$

for all $a \in \mathcal{A}, u, v \in \mathfrak{h}$ and any sequence $\{r_n\}$ in $]-\infty, t]$ that converges to r . Thus $r \mapsto k_{r, t}(a)\xi$ is weakly continuous. The result now follows from the strong continuity of the unitary group of shifts and this special case, since for all sequences $\{(r_n, t_n)\} \subset \mathbb{R}_{\leq}^2$ converging (r, t) we have $k_{r_n, t_n}(a) = S_{t_n - t} k_{r_n - t_n + t, t}(a) S_{t_n - t}^*$. $\square$

Markovian cocycles on a von Neumann algebra. Let j be a one-parameter process on a von Neumann algebra $\mathcal{M}$. If j is bounded and extendable it follows (in the same spirit as (4.3)) that j is a forward cocycle if and only if

$$j_0(a) = a \otimes 1_{\mathcal{F}}, \quad j_{r+t} = \widehat{j}_r \circ \sigma_r \circ j_t$$

It is this identity that Bradshaw uses to *define* his cocycles. Conversely a bounded and extendable process j is a backward cocycle if and only if $j_0 = \text{id}_{\mathcal{M}}$ and $j_{r+t} = \widehat{\sigma_r \circ j_t} \circ j_r$, for all $r, t \geq 0$, where $\widehat{\sigma_r \circ j_t} : \mathcal{M} \otimes \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H}_{[r, r+t]^c} \odot \mathcal{E}_{[r, r+t]; \mathcal{H}})$ is the extension of the algebraic tensor product $(\sigma_r \circ j_t) \odot \text{id}_{[r, r+t]^c}$ defined in a manner analogous to (4.1).

The following result was proved in [Bra], for the case where j is unital and $*$ -homomorphic and the noise is one dimensional; its origins can be traced back to [Ac1].

Proposition 4.3. *Let $j \in \mathbb{P}(\mathcal{M})$ be a normal completely bounded process on a von Neumann algebra $\mathcal{M}$, and let J_t be the operator on $\mathcal{M} \otimes \mathcal{B}_{[0, \infty[}$ determined by $\widehat{j}_t \circ \sigma_t$. Then the following are equivalent:*

- (i) j is a forward cocycle.
- (ii) J is a one-parameter semigroup on $\mathcal{M} \otimes \mathcal{B}_{[0, \infty[}$.

Proof. Suppose that j is a cocycle and let $c = a \otimes b$ where $a \in \mathcal{M}, b \in \mathcal{B}_{[0, \infty[}$. Then

$$\begin{aligned} (\widehat{j}_s \circ \sigma_s \circ \widehat{j}_t \circ \sigma_t)(c) &= (\widehat{j}_s \circ \sigma_s)(j_t(a) \otimes \sigma_t(b)) \\ &= \widehat{j}_s(\sigma_s \circ j_t(a) \otimes \sigma_{s+t}(b)) \\ &= (\widehat{j}_s \circ \sigma_s \circ j_t)(a) \otimes \sigma_{s+t}(b) = \widehat{j}_{s+t} \circ \sigma_{s+t}(c), \end{aligned}$$

and so it follows by linearity and ultraweak continuity that J is a one-parameter semigroup. In the other direction note that σ acts trivially on $\mathcal{M}$, so that $\widehat{j}_s \circ \sigma_s(a) = j_s(a)$ for all $a \in \mathcal{M}$. $\square$

5. INFINITESIMAL DESCRIPTION OF FLOWS

In this section we establish the main result of the paper, namely that all sufficiently regular time-homogeneous flows on a C^* -algebra have a simple infinitesimal description as the solution of a quantum stochastic differential equation. The converse was essentially proved in [LW1], Theorem 3.1, except now we are working with two-parameter processes which enables us to study both forward and backward equations.

Quantum stochastic differential equations. Consider the quantum stochastic differential equations (QSDE's)

$$d_t k_{r,t} = k_{r,t} \circ \theta_{\beta}^{\alpha} d\Lambda_{\alpha}^{\beta}(t), \tag{F}$$

$$d_r k_{r,t} = k_{r,t} \circ \theta_{\beta}^{\alpha} d\widetilde{\Lambda}_{\alpha}^{\beta}(r), \tag{B}$$

with initial condition $k_{r,r}(a) = a \otimes 1$. Here Λ is the matrix of fundamental integrators of quantum stochastic calculus, defined with respect to the fixed basis η of the noise dimension space ([Par]), $d\widetilde{\Lambda}_{\beta}^{\alpha}(r) := \Lambda_{\beta}^{\alpha}([r - dr, r])$ are backward pointing differentials, and $\theta \in M_D(\mathcal{B}(\mathcal{A}))$.

Given a mapping matrix θ , a process k on $\mathcal{A}$ is a *weak solution to the forward equation* (F) if it satisfies

$$t \mapsto \langle u\varepsilon(f), k_{r,t}(a)v\varepsilon(g) \rangle \text{ is continuous,}$$

and

$$\langle u\varepsilon(f), [k_{r,t}(a) - a]v\varepsilon(g) \rangle = \int_r^t f_{\alpha}(s)g^{\beta}(s) \langle u\varepsilon(f), k_{r,s}(\theta_{\beta}^{\alpha}(a))v\varepsilon(g) \rangle ds.$$

Note that the sum over α and β in the integrand has only finitely many non-zero terms because of our choice of test functions.

If $F \in \mathbb{P}(\mathfrak{h})$ is such that the $\mathcal{H}$ -valued function $F\xi$ is locally square-integrable in its second variable (in particular the map $t \mapsto F_{r,t}\xi$ must be *strongly* measurable) then a strongly continuous process $(\int_r^t F_{r,s} d\Lambda_\alpha^\beta(s))_{r \leq t}$ is defined for each pair of indices $\alpha, \beta \geq 0$ ([HP1],[Par]). If $L = [L_\beta^\alpha]$ is a matrix of processes satisfying

$$\int_r^t \sum_{\alpha \geq 0} \|L_\beta^\alpha(r, s)\xi\|^2 ds < \infty$$

it is called *f-stochastically integrable*. In this case the operator $\int_r^t L_\beta^\alpha(r, s) d\Lambda_\alpha^\beta(s)$ is defined as the strong limit on $\mathfrak{h} \odot \mathcal{E}$ of the partial sums $\sum_{\alpha, \beta \leq N} \int_r^t L_\beta^\alpha(r, s) d\Lambda_\alpha^\beta(s)$. Note that when $\mathfrak{k}$ is finite dimensional the matrix L is *f-stochastically integrable* if and only if each map $L_\beta^\alpha(r, \cdot)\xi$ is locally square integrable. However when $\mathfrak{k}$ is infinite dimensional there is in addition a square-summability condition that must be fulfilled by each sequence $(L_\beta^\alpha(r, \cdot)\xi)_{\alpha \geq 0}$.

A process k on $\mathcal{A}$ is a *strong solution* of (F) for a given θ if

$$[k(\theta_\beta^\alpha(a))] \text{ is } f\text{-stochastically integrable for all } a \in \mathcal{A},$$

and

$$k_{r,t}(a) = a \otimes 1 + \int_r^t k_{r,s}(\theta_\beta^\alpha(a)) d\Lambda_\alpha^\beta(s).$$

Equivalently, k is a strong solution if it is a weak solution of (F) and the *f-stochastic integrability* condition holds. Backward stochastic integrals, and both weak and strong solutions of (B), are defined similarly, with attention now focused on the first time variable.

Throughout this and the next section we will make repeated use of the following transformations on operator and mapping matrices. Let $\Delta \in M_D(\mathcal{B}(\mathcal{A}))$ be the mapping matrix

$$\Delta_\beta^\alpha = \begin{cases} \text{id}_{\mathcal{A}}, & \alpha = \beta \geq 1 \\ 0, & \text{otherwise} \end{cases},$$

and for $\theta \in M_D(\mathcal{B}(\mathcal{A}))$ and $F \in M_D(\mathcal{B}(\mathfrak{h}))$, define $\hat{\theta}$ and $\hat{F}$ by

$$\hat{\theta} = \theta + \Delta, \quad \hat{F} = F + \Delta(1). \quad (5.1)$$

The next result is Theorem 3.1 of [LW1], trivially extended to two-parameter processes by an identical proof. Recall the notation $\hat{\theta}_{w,z}$, defined for $\hat{\theta} \in M_D(\mathcal{B}(\mathcal{A}))$ and $w, z \in c(D)$ by (2.1).

Theorem 5.1. *Let $\theta \in M_D(\mathcal{B}(\mathcal{A}))$ and let $k \in \mathbb{P}(\mathcal{A})$ be weakly regular process on $\mathcal{A}$. The following sets of equivalences hold:*

- (i) *k is a weak solution of (F) (resp. (B)).*
- (ii) *For all $f, g \in \mathbb{S}_{\text{loc}}$ and $r \leq t$,*

$$k_{r,t}^{f,g} = \mathcal{P}_{t_1-r}^{[0]} \circ \dots \circ \mathcal{P}_{t-t_n}^{[n]} \quad (5.2)$$

(resp. $k_{r,t}^{f,g} = \mathcal{P}_{t-t_n}^{[n]} \circ \dots \circ \mathcal{P}_{t_1-r}^{[0]}$), where $\mathcal{P}^{[k]}$ is the one-parameter semigroup on $\mathcal{A}$ with generator $\hat{\theta}_{w,z}$ with $w = f(t_k)$ and $z = g(t_k)$, whenever $\{r = t_0 \leq \dots \leq t_{n+1} = t\}$ contains all the points of discontinuity of $f_{[r,t]}$ and $g_{[r,t]}$.

Caution. The semigroups above differ from those of [LW1] in that here the generators are $\hat{\theta}_{w,z}$ rather than $\theta_{w,z}$, yielding the simpler form of representation (5.2).

The next result is an immediate consequence of the above *semigroup representation* for weakly regular weak solutions to the backward and forward equations. Recall the definitions of the time reversed process (3.4), and inverse pairs (3.5).

Proposition 5.2. *Let $\theta, \psi \in M_D(\mathcal{B}(\mathcal{A}))$, and let $k, l \in \mathbb{P}(\mathcal{A})$ be weakly regular.*

- (a) *If k is a weak solution of (F) then k is a time-homogeneous forward flow, and moreover each of the associated semigroups $\mathcal{P}^{w,z}$ is norm continuous with generator $\hat{\theta}_{w,z}$. The same is true with forward replaced by backward, and (F) by (B).*
- (b) *The following are equivalent:*
 - (i) *k is a weak solution to (F).*
 - (ii) *$\tilde{k}$ is a weak solution to (B).*
- (c) *If k is a weak solution to (F) for θ , and l is a weak solution to (B) for ψ , then the following are equivalent:*
 - (i) *(k, l) is an inverse pair.*
 - (ii) *$\theta = -\psi$.*

Generators for flows. Our aim now is to give a converse of Proposition 5.2(a). That is, given a time-homogeneous flow k , determine conditions that ensure that k is the solution of a QSDE. We concentrate on the case when the associated semigroups $\{\mathcal{P}^{w,z}\}$ are norm continuous, and denote its generator by $\phi_{(w,z)}$. Let $\{h_\alpha\}_{\alpha \geq 0}$ be the sequence in $c(D)$ defined by letting $h_0 = 0$, and for each $i \geq 1$ letting h_i be the vector whose only non-zero entry is a 1 in the i th position. The coefficients of the QSDE will (eventually) be specified in terms of the mapping matrix $\phi \in M_D(\mathcal{B}(\mathcal{A}))$, where

$$\left. \begin{aligned} \phi_0^0 &:= \phi_{(h_0, h_0)} \\ \phi_j^0 &:= \phi_{(h_0, h_j)} - \phi_{(h_0, h_0)}; \quad \phi_0^i := \phi_{(h_i, h_0)} - \phi_{(h_0, h_0)} \\ \phi_j^i &:= \phi_{(h_i, h_j)} - \phi_{(h_0, h_j)} - \phi_{(h_i, h_0)} + \phi_{(h_0, h_0)} \end{aligned} \right\} \quad (5.3)$$

We now seek conditions on the flow k that guarantee that the $\{\mathcal{P}^{w,z}\}$ are norm-continuous, that their generators are the correct linear combinations of the $\{\phi_\beta^\alpha\}$ in (5.3), and that the flow does indeed satisfy the appropriate QSDE for $\theta = \phi - \Delta \in M_D(\mathcal{B}(\mathcal{A}))$. To do this we proceed via a number of lemmas and auxiliary results, in which we shall use both l^1 and l^2 norms on $c(D)$:

$$\|z\|_1 := \sum_i |z^i|, \quad \|z\|_2 := (\sum_i |z^i|^2)^{1/2} \quad (z \in c(D)).$$

Recall also that we view $c(D)$ as a subspace of $\mathbb{S}_{\text{loc}}$ by identifying z with the constant function $z^i e_i$. Thus $\{h_\alpha \mathbf{1}_{[0,t]}\}$ is the sequence $\{0, e_1 \mathbf{1}_{[0,t]}, e_2 \mathbf{1}_{[0,t]}, \dots\}$ in $L^2(\mathbb{R}; \mathbf{k})$.

Lemma 5.3. *For $u \in \mathfrak{h}$, $z \in c(D)$ and $t > 0$,*

- (a) $\|u\varepsilon(z \mathbf{1}_{[0,t]}) - u\| \leq \sqrt{t} \|u\| \|z\|_2 \exp(t\|z\|_2^2/2),$
- (b) $\|u\varepsilon(z \mathbf{1}_{[0,t]}) - u - uz \mathbf{1}_{[0,t]}\| \leq t \|u\| \|z\|_2^2 \exp(t\|z\|_2^2/2),$
- (c) $\|\gamma_{u,z,t}\| \leq t \|u\| \left\{ \|z\|_2^2 \exp(t\|z\|_2^2) + \|z\|_1 \exp(t/2) \right\},$ where

$$\gamma_{u,z,t} = u\varepsilon(z \mathbf{1}_{[0,t]}) - z^\alpha [u\varepsilon(h_\alpha \mathbf{1}_{[0,t]}) - \nu_\alpha u]$$

$$\text{and } \nu_\alpha = (1 - \delta_\alpha^0).$$

Proof. (a) and (b) are trivial to prove, and (c) follows from (b) since $\gamma_{u,z,t}$ is equal to

$$u\varepsilon(z \mathbf{1}_{[0,t]}) - u - uz \mathbf{1}_{[0,t]} - z^i [u\varepsilon(h_i \mathbf{1}_{[0,t]}) - u - uh_i \mathbf{1}_{[0,t]}]. \quad \square$$

Recall the definitions of pointwise weak and strong continuity of a process k from Section 2. Note that if k is bounded with locally uniform bounds then it is pointwise

strongly continuous at 0 if and only if $\lim_{t \rightarrow 0} k_{0,t}(a)u = au$ for all $a \in \mathcal{A}, u \in \mathfrak{h}$, since it is assumed to be adapted.

Proposition 5.4. *Let k be a bounded time-homogeneous flow on $\mathcal{A}$ with locally uniform bounds.*

- (a) *Either all or none of the associated semigroups $\{\mathcal{P}^{w,z}\}$ are norm continuous (respectively, pointwise strongly continuous or pointwise weakly continuous).*
- (b) *If k is Markov-regular then it is pointwise weakly continuous.*

Proof. (a) Since k is bounded Lemma 5.3 implies that

$$\begin{aligned} |\langle u, (\mathcal{P}_t^{w,z}(a) - \mathcal{P}_t^{0,0}(a))v \rangle| &\leq \sqrt{t} \|u\| \|a\| \|v\| \|k_{0,t}\| \exp(t\|z\|_2^2/2) \\ &\quad \times (\|w\|_2 \exp(t\|w\|_2^2/2) + \|z\|_2), \end{aligned}$$

from which the result follows.

- (b) This now follows from Proposition 4.2. $\square$

Since we also wish to consider unbounded flows we must refine the estimates used in the above proposition, and also establish the relation between $\phi_{(w,z)}$ and the maps ϕ_β^α .

Proposition 5.5. *Let $k \in \mathbb{P}^\dagger(\mathcal{A})$ be a time-homogeneous flow for which each of the semigroups $\mathcal{P}^{h_\alpha, h_\beta}$ is norm continuous.*

- (a) *If k and $k^\dagger$ are strongly regular then all of the associated semigroups $\mathcal{P}^{w,z}$ are norm continuous.*
- (b) *If k and $k^\dagger$ are pointwise strongly continuous at 0 then all of the associated semigroups $\mathcal{P}^{w,z}$ are norm continuous. Moreover their generators are related to those of $\mathcal{P}^{h_\alpha, h_\beta}$ through*

$$\phi_{(w,z)} = \phi_{w,z} \tag{5.4}$$

where the matrix $\phi = [\phi_\beta^\alpha]$ is given by (5.3).

Proof. For each $\alpha, \beta \geq 0$ define $\mathcal{R}_t^{(\alpha, \beta)} \in \mathcal{B}(\mathcal{A})$ to be

$$(\mathcal{P}_t^{h_\alpha, h_\beta} - \text{id}) - \nu_\alpha(\mathcal{P}_t^{h_0, h_\beta} - \text{id}) - \nu_\beta(\mathcal{P}_t^{h_\alpha, h_0} - \text{id}) + \nu_\alpha \nu_\beta(\mathcal{P}_t^{h_0, h_0} - \text{id}).$$

where ν is defined as in Lemma 5.3. Fix $w, z \in c(D), u, v \in \mathfrak{h}$ and $a \in \mathcal{A}$, then algebraic manipulation reveals the identities

$$\begin{aligned} (1) &:= \langle u, (\mathcal{P}_t^{w,z}(a) - a)v \rangle - (e^{t\langle w, z \rangle} - 1)\langle u, av \rangle \\ &= \langle u\varepsilon(w\mathbf{1}_{[0,t]}), [k_t(a) - a]v\varepsilon(z\mathbf{1}_{[0,t]}) \rangle, \end{aligned}$$

and

$$\begin{aligned} (2) &:= w_\alpha z^\beta \langle u, \mathcal{R}_t^{(\alpha, \beta)}(a)v \rangle + \langle w, z \rangle (1 - e^t) \langle u, av \rangle \\ &= w_\alpha z^\beta \langle [u\varepsilon(h_\alpha \mathbf{1}_{[0,t]}) - \nu_\alpha u], [k_t(a) - a][v\varepsilon(h_\beta \mathbf{1}_{[0,t]}) - \nu_\beta v] \rangle. \end{aligned}$$

Therefore

$$(1) - (2) = \langle \gamma_{u,w,t}, [k_t(a) - a]v\varepsilon(z\mathbf{1}_{[0,t]}) \rangle + \langle w^\alpha [u\varepsilon(h_\alpha \mathbf{1}_{[0,t]}) - \nu_\alpha u], [k_t(a) - a]\gamma_{v,z,t} \rangle,$$

where γ is defined as in Lemma 5.3, and hence

$$\begin{aligned} |\langle u, [(\mathcal{P}_t^{w,z}(a) - a) - w_\alpha z^\beta \mathcal{R}_t^{(\alpha, \beta)}(a)]v \rangle| &\leq \|\gamma_{u,w,t}\| \| [k_{0,t}(a) - a]v\varepsilon(z\mathbf{1}_{[0,t]}) \| \\ &+ \| [k_{0,t}^\dagger(a) - a]w^\alpha [u\varepsilon(h_\alpha \mathbf{1}_{[0,t]}) - \nu_\alpha u] \| \|\gamma_{v,z,t}\| + |\{e^{t\langle w, z \rangle} - 1 + \langle w, z \rangle(1 - e^t)\} \langle u, av \rangle|. \end{aligned}$$

(a) follows immediately from the estimates in Lemma 5.3. For (b) note that the pointwise strong continuity at 0 of k and $k^\dagger$ ensures that

$$\lim_{t \rightarrow 0} \langle u, t^{-1}(\mathcal{P}_t^{w,z}(a) - a)v \rangle = \langle u, w_\alpha z^\beta \phi_\beta^\alpha(a)v \rangle$$

for all $u, v \in \mathfrak{h}$ and $a \in \mathcal{A}$. Repeated application of the Banach-Steinhaus Theorem as in Corollary 3.3 shows that the family $\{\|t^{-1}(\mathcal{P}_t^{w,z} - \text{id})\|\}_{t>0}$ is bounded in each bounded neighbourhood of 0, and the result follows. $\square$

We now combine the above results to give the most general form of the stochastic integral representation result for time-homogeneous flows.

Theorem 5.6. *Let $k \in \mathbb{P}^\dagger(\mathcal{A})$ be a time-homogeneous forward flow, and assume that k and $k^\dagger$ are pointwise strongly continuous at 0, and each of the semigroups $\mathcal{P}^{h_\alpha, h_\beta}$ is norm continuous. Then k is the unique weakly regular weak solution of a forward QSDE of the form (F) for some $\theta \in M_D(\mathcal{B}(\mathcal{A}))$.*

The above remains true if forward is replaced by backwards and (F) by (B) throughout.

Proof. By Proposition 3.2,

$$k_{r,t}^{f,g} = \mathcal{P}_{t_1-r}^{[0]} \circ \dots \circ \mathcal{P}_{t-t_n}^{[n]}$$

where $\mathcal{P}^{[i]} = \mathcal{P}^{w,z}$ with $w = f(t_i)$ and $z = g(t_i)$, and $\{r = t_0 \leq t_1 \leq \dots \leq t_{n+1} = t\}$ containing all of the discontinuities of $f_{[r,t]}$ and $g_{[r,t]}$. By Proposition 5.5 the generator of $\mathcal{P}^{w,z}$ is given by (5.4), moreover k is weakly regular by Corollary 3.3. Putting $\theta = \phi - \Delta$, the result now follows from the semigroup representation of weakly regular weak solutions of QSDE's (Theorem 5.1). $\square$

We next dispose of two cases where k strongly satisfies its QSDE.

Proposition 5.7. *Let $k \in \mathbb{P}^\dagger(\mathcal{A})$ be as in the above Theorem. Then, in both of the following cases, k strongly satisfies its forward QSDE:*

- (a) $\dim \mathfrak{k} < \infty$;
- (b) $\dim \mathcal{A} < \infty$ and θ is bounded.

Note. A mapping matrix $\theta \in M_D(\mathcal{B}(\mathcal{A}))$ is *bounded* if there is an operator $\tilde{\theta} : \mathcal{A} \rightarrow \mathcal{B}(\mathfrak{h} \otimes \hat{\mathfrak{k}})$ such that $\theta_\beta^\alpha(a) = E^\alpha \tilde{\theta}(a) E_\beta$ where $E_\gamma : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \hat{\mathfrak{k}}$ is the sequence of operators defined by $E_\gamma u = u \otimes e_\gamma$, and $E^\gamma = (E_\gamma)^*$. We will identify $\tilde{\theta}$ with θ in what follows.

Proof. In case (a) the existence and uniqueness results of [LiP] ensure that $k = k^\theta$, a solution of the forward QSDE obtained by Picard iteration. In case (b) θ is (completely) regular, by Corollary 1.5 of [LW1], and the existence and uniqueness results of [LW1] again ensure that $k = k^\theta$. In both cases k^θ is a strong solution, so the result follows. $\square$

Corollary 5.8. *Let k be a CP contraction process on $\mathcal{A}$. If k is a Markov-regular time-homogeneous flow then k weakly satisfies a forward QSDE (F) for some $\theta \in M_D(\mathcal{B}(\mathcal{A}))$.*

Proof. Since Markov-regularity means that its Markov semigroup is norm continuous, Proposition 5.4 ensures that all of the associated semigroups are norm continuous, and also that k is pointwise weakly continuous. Since k is contractive and completely positive, the operator Schwarz inequality implies that

$$\|(k_{0,t}(a) - a)\xi\|^2 \leq \langle \xi, k_{0,t}(a^*a)\xi \rangle - 2\text{Re}\langle a\xi, k_{0,t}(a)\xi \rangle + \langle \xi, a^*a\xi \rangle,$$

and so k is pointwise strongly continuous at 0; since $k = k^\dagger$ the result follows. $\square$

Our goal now is to show that such flows satisfy their forward QSDE strongly, that is we must show that the matrix of processes $[k(\theta_\beta^\alpha(a))]$ is f -stochastically integrable. When the noise is finite dimensional this is an immediate consequence of general existence and uniqueness results such as those in [LiP], as noted in Proposition 5.7. We therefore now show how to overcome infinite dimensionality of k , by first proving a general lemma on the measurability of weak solutions, and then making use of the positivity of k in Theorem 5.10, which includes a refinement of Section 5 of [LW1].

Lemma 5.9. *Let $\theta \in M_D(\mathcal{B}(\mathcal{A}))$ and suppose that $k \in \mathbb{P}(\mathcal{A})$ is a weakly regular weak solution of (F) for this θ . Then the map $t \mapsto k_{r,t}(a)\xi$ is strongly measurable $[r, \infty[\rightarrow \mathcal{H}$ for all $r \in \mathbb{R}, a \in \mathcal{A}$ and $\xi \in \mathfrak{h} \odot \mathcal{E}$.*

Weakly regular weak solutions of (B) satisfy the analogous strong measurability condition in the first time variable.

Proof. When k is finite dimensional this follows from the general existence and uniqueness results of [LiP]. So assume k is infinite dimensional, and for each $d \geq 1$ let J_d be the isometry $\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}; \mathbb{C}^d)) \rightarrow \mathcal{H}$ defined by continuous linear extension of $u\varepsilon(\mathbf{f}) \mapsto u\varepsilon(\sum_{i=1}^d f^i e_i)$. Let $\theta^{[d]} \in M_{d+1}(\mathcal{B}(\mathcal{A}))$ be the $(d+1) \times (d+1)$ submatrix of bounded maps on $\mathcal{A}$ taken from the top left corner of the infinite matrix θ . For each d there is a strongly regular strong solution ${}^d k$ to the equation (F) for $\theta^{[d]}$ on $\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}; \mathbb{C}^d))$ generated by a Picard iteration method, such that $t \mapsto {}^d k_{r,t}(a)u\varepsilon(\mathbf{f})$ is continuous, hence strongly measurable. By uniqueness of weakly regular weak solutions to (F) we have that $J_d^* k_{r,t}(a) J_d = {}^d k_{r,t}(a)$; the result now follows since the sequence $\{J_d J_d^*\}$ of projections on $\mathcal{H}$ is strongly convergent to the identity. $\square$

Theorem 5.10. *Let k be a CP contraction process on $\mathcal{A}$. Then the following are equivalent:*

- (i) *k is a weak solution to the forward QSDE (F) for some $\theta \in M_D(\mathcal{B}(\mathcal{A}))$.*
- (ii) *k is a Markov-regular time-homogeneous forward flow.*
- (iii) *k a strong solution to the forward QSDE (F) for some $\theta \in M_D(\mathcal{B}(\mathcal{A}))$.*

The same is true if forward is replaced by backward and (F) by (B).

Proof. Note that k is weakly regular since it is a contraction process. Thus (i $\Rightarrow$ ii) by Proposition 5.2. If (ii) holds then, by Corollary 5.8, k weakly satisfies the forward QSDE (F) for some $\theta \in M_D(\mathcal{B}(\mathcal{A}))$, and so Lemma 5.9 can be applied to deduce the required pointwise strong measurability of k . Now, by Theorem 5.2 of [LW1], the mapping matrix θ is *bounded*, so combining the operator bound

$$\begin{aligned} \sum_{\alpha=0}^N \theta_\beta^\alpha(a)^* \theta_\beta^\alpha(a) &= \sum_{\alpha=0}^N E^\beta \theta(a)^* E_\alpha E^\alpha \theta(a) E_\beta \\ &\leq E^\beta \theta(a)^* \theta(a) E_\beta \leq \|\theta(a) E_\beta\|^2 1, \end{aligned}$$

with the operator Schwarz inequality, applied to $k_{r,t}^{(N+1)}$, we obtain: for all $t \geq r$, $\beta \geq 0$, $a \in \mathcal{A}$, $\xi \in \mathfrak{h} \odot \mathcal{E}$ and $N \geq 1$,

$$\begin{aligned}
& \sum_{\alpha=0}^N \|k_{r,t}(\theta_\beta^\alpha(a))\xi\|^2 \\
&= \left\| k_{r,t}^{(N+1)} \left(\begin{bmatrix} \theta_\beta^0(a) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \theta_\beta^N(a) & 0 & \cdots & 0 \end{bmatrix} \right) \begin{bmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|^2 \\
&\leq \|k_{r,t}^{(N+1)}(1)\| \left\langle \begin{bmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{bmatrix}, k_{r,t}^{(N+1)} \left(\begin{bmatrix} \sum_{\alpha=0}^N \theta_\beta^\alpha(a)^* \theta_\beta^\alpha(a) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right) \begin{bmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\rangle \\
&\leq \langle \xi, k_{r,t}(\sum_{\alpha=0}^N \theta_\beta^\alpha(a)^* \theta_\beta^\alpha(a))\xi \rangle \\
&\leq \|\theta(a)E_\beta\|^2 \|\xi\|^2.
\end{aligned}$$

Since this holds for all N , we have

$$\int_r^t \sum_{\alpha \geq 0} \|k_{r,s}(\theta_\beta^\alpha(a))\xi\|^2 ds < \infty,$$

so that the matrix of processes $[k(\theta_\beta^\alpha(a))]$ is f -stochastically integrable, therefore k satisfies (F) strongly, and (iii) holds. The above argument is equally valid for backward equations and flows. $\square$

Remark. The form of the bounded mapping matrix θ in the above case is given by [LW1], Theorem 5.2:

$$\widehat{\theta}(a) = \psi(a) + J^* a E_0 + E_0 a J$$

where $\psi : \mathcal{A} \rightarrow \mathcal{A}'' \otimes \mathcal{B}$ is completely positive, $J \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathfrak{k}}; \mathfrak{h})$ satisfies $J E_\gamma \in \mathcal{A}''$ for all $\gamma \geq 0$, and $E_\gamma : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \widehat{\mathfrak{k}}$ are the isometric embeddings defined above.

6. STOCHASTIC EVOLUTIONS ON A HILBERT SPACE

We now turn to the study of the corresponding evolution laws for processes on $\mathfrak{h}$, as studied in [HL2], [Jou], [AJL] and [Fa2], extending those results to evolutions composed of not necessarily unitary (or even bounded) operators, and also considering evolutions affiliated to a particular von Neumann subalgebra of $\mathcal{B}(\mathfrak{h})$. As in Section 7 of [LW1] the philosophy is to deduce these results from the above analysis of flows on an algebra.

Definitions. Let X be a process on the Hilbert space $\mathfrak{h}$. It is *time-homogeneous* if

$$X_{I+p} = \sigma_p(X_I).$$

and is a *left evolution* (resp. *right evolution*) if it is weakly i.s.b. and satisfies

$$X_{r,r}^{f,g} = 1, \quad X_{r,t}^{f,g} = X_{r,s}^{f,g} X_{s,t}^{f,g} \quad (\text{resp. } X_{r,t}^{f,g} = X_{s,t}^{f,g} X_{r,s}^{f,g}),$$

for all $r \leq s \leq t$. An evolution X is *affiliated* to a given von Neumann algebra $\mathcal{M}$ acting on $\mathfrak{h}$ if

$$X_{r,t}(a' \otimes 1) = (a' \otimes 1)X_{r,t}$$

for all $a' \in \mathcal{M}'$, $r \leq t$, equivalently $X_{r,t}^{f,g} \in \mathcal{M}$ for all $f, g \in \mathbb{S}_{\text{loc}}$.

Note that as with flows on an algebra, time-homogeneity may be characterised in terms of the reduced operators, since X is time-homogeneous if and only if

$$X_{I+p}^{f,g} = X_I^{s-pf, s-pg}.$$

Also if X is strongly i.s.b. then we can form the extension $\widehat{X}$ through (2.5), and it follows that X is a left (resp. right) evolution if and only if

$$X_{r,r} = 1, \quad X_{r,t} = \widehat{X}_{r,s} X_{s,t} \quad (\text{resp. } X_{r,t} = \widehat{X}_{s,t} X_{r,s}) \quad (6.1)$$

for all $r \leq s \leq t$. We emphasise that X need not necessarily be bounded for this alternative characterisation. The theory of invertible evolutions can be developed in the same way as we did for flows on an algebra.

In the corresponding one-parameter theory a weakly i.s.b. process W is a *left* (resp. *right*) *Markovian cocycle* on $\mathfrak{h}$ if

$$W_0^{f,g} = 1, \quad W_{r+t}^{f,g} = W_r^{f,g} W_t^{s-rf, s-rg} \quad (\text{resp. } W_{r+t}^{f,g} = W_t^{s-rf, s-rg} W_r^{f,g}),$$

for all $r, t \geq 0$ and $f, g \in \mathbb{S}_{\text{loc}}$. Imitating (3.6), the identities

$$W_t = X_{0,t} \quad (t \geq 0); \quad X_{r,t} = \sigma_r(W_{t-r}) \quad (r \leq t), \quad (6.2)$$

define a bijective correspondence between time-homogeneous two parameter processes, and one-parameter processes, which restricts to a bijective correspondence between time-homogeneous left (resp. right) evolutions and left (resp. right) Markovian cocycles on $\mathfrak{h}$. The one-parameter version of (6.1) is as follows: a strongly i.s.b. process W is a left (resp. right) cocycle if and only if it satisfies

$$W_0 = 1, \quad W_{r+t} = \widehat{W}_r \sigma_r(W_t) \quad (\text{resp. } W_{r+t} = \sigma_r(\widehat{W}_t) W_r). \quad (6.3)$$

If W is in fact bounded, and we identify each W_t (and hence also $\widehat{W}_t$) with the unique continuous extension to a bounded operator on $\mathcal{H}$, then the above identities clearly hold without the hats. Note that in the previous literature on Markovian cocycles ([Ac1],[HL2],[Jou],[Fa2]) it is more common to work with $\mathcal{H}_{[0,\infty[}$ (rather than $\mathcal{H}$) where the right shift S'_t is no longer coisometric. As a result the shift σ'_t defined in terms of S'_t is no longer surjective, however it is related to σ_t through the identity

$$\sigma_t(1 \otimes X) = 1 \otimes \sigma'_t(X) \quad (t \geq 0, X \in \mathcal{B}(\mathcal{H}_{[0,\infty[}))$$

where 1 is the identity on $\mathcal{F}_{]-\infty,0]}$. Thus the identities (6.3) are effectively the defining relations used in these earlier works, since our processes are adapted.

Since the shift S_t on $\mathcal{H}$ is unitary it follows that a bounded process W is a left (resp. right) cocycle if and only if $(W_t S_t)_{t \geq 0}$ (resp. $(S_t^* W_t)_{t \geq 0}$) is a one-parameter semigroup (c.f. Proposition 4.3). This correspondence breaks down if one adopts the half-line point of view: whereas $(W'_t S'_t)_{t \geq 0}$ defines a one-parameter semigroup on $\mathcal{H}_{[0,\infty[}$ for any left cocycle W' on $\mathcal{H}_{[0,\infty[}$, putting

$$W'_t = 1_{\mathfrak{h}} \otimes |\varepsilon(0|_{[0,t])}\rangle \langle \varepsilon(f|_{[0,t]})| \otimes 1_{[t,\infty[}$$

for a nonzero f in $L^2(\mathbb{R}_+; \mathfrak{k})$ defines a process which is not a cocycle, but satisfies $W'_t S'_t = S'_t$, and so $(W'_t S'_t)_{t \geq 0}$ is nevertheless a semigroup.

Representation results. We now give the results for processes on $\mathfrak{h}$ corresponding to those for processes on $\mathcal{A}$. While they could be proved directly, by adapting the proofs given in Sections 3 and 5, they can be viewed as simple corollaries of these earlier results via the following construction: set $\mathcal{A} = \mathcal{B}(\mathfrak{h})$, the full algebra of bounded operators on $\mathfrak{h}$, and for each process X on $\mathfrak{h}$, define processes ${}^1k$ and ${}^2k$ on $\mathcal{B}(\mathfrak{h})$ by

$${}^1k_I(a) = aX_I; \quad {}^2k_I(a) = X_I a. \quad (6.4)$$

Applying the earlier results to either of these two map-valued process and evaluating at $a = 1$ we obtain the required results for X , noting that each of the various properties or continuity assumptions on ${}^1k$ and ${}^2k$ follow from the corresponding hypotheses on X as detailed in the following:

Lemma 6.1. *Let $X \in \mathbb{P}(\mathfrak{h})$ and let ${}^1k$ and ${}^2k$ be the processes on $\mathcal{B}(\mathfrak{h})$ defined by (6.4).*

- (a) *If X is weakly i.s.b. then ${}^1k$ and ${}^2k$ are weakly i.a.b. with*

$${}^1k_I^{f,g}(a) = aX_I^{f,g}, \quad {}^2k_I^{f,g}(a) = X_I^{f,g}a.$$

- (b) *If X is strongly i.s.b. then ${}^1k$ and ${}^2k$ are extendable, with*

$${}^1k_I^f(a) = aX_I^f, \quad {}^2k_I^f(a) = X_I^f a, \quad \text{and, } \widehat{{}^1k_I}(c) = c\widehat{X_I}, \quad \widehat{{}^2k_I}(c) = \widehat{X_I}c$$

for $c \in \mathcal{B}(\mathfrak{h}) \otimes \mathcal{B}_{I^c}$.

- (c) *If X is time-homogeneous then ${}^1k$ and ${}^2k$ are time-homogeneous.*
 (d) *If X is a left evolution then ${}^1k$ is a backward flow and ${}^2k$ is a forward flow. On the other hand if X is a right evolution then ${}^1k$ is a forward flow and ${}^2k$ is a backward flow.*
 (e) *If $X \in \mathbb{P}^\dagger(\mathfrak{h})$ then ${}^1k, {}^2k \in \mathbb{P}^\dagger(\mathcal{A})$, with*

$$({}^1k_I)^\dagger(a) = X_I^\dagger a \quad \text{and} \quad ({}^2k_I)^\dagger(a) = aX_I^\dagger.$$

Proof. The proofs of these statements are all largely a matter of chasing definitions. For (b) note that if $c = a \otimes b$, where $a \in \mathcal{B}(\mathfrak{h})$ and $b \in \mathcal{B}_{I^c}$, then the required identity is immediate, and so the result follows by linearity and continuity. $\square$

Proposition 6.2. *Let X be a weakly i.s.b. process on $\mathfrak{h}$. The following sets of equivalences hold:*

- (i) *X is a time-homogeneous left evolution (resp. right evolution).*
 (ii) *For all $w, z \in c(D)$, $(P_s^{w,z} := X_{0,s}^{w,z})_{s \geq 0}$ is a one-parameter semigroup on $\mathfrak{h}$, and for all $r \leq t$ and $f, g \in \mathbb{S}_{\text{loc}}$,*

$$X_{r,t}^{f,g} = P_{t_1-r}^{[0]} \cdots P_{t-t_n}^{[n]} \quad (\text{resp. } X_{r,t}^{f,g} = P_{t-t_n}^{[n]} \cdots P_{t_1-r}^{[0]}),$$

where $P^{[i]} = P^{w,z}$ for $w = f(t_i), z = g(t_i)$, and $\{r = t_0 \leq \dots \leq t_{n+1} = t\}$ contains the discontinuities of $f_{[r,t]}$ and $g_{[r,t]}$.

The one-parameter semigroups $\{P^{w,z}\}$ defined by a time-homogeneous evolution are called its *associated semigroups*; $P^{0,0}$ is called the *Markov semigroup*. An evolution is *Markov-regular* if $P^{0,0}$ is norm continuous. Note that X is affiliated to a given von Neumann algebra $\mathcal{M}$ acting on $\mathfrak{h}$ if and only if $P_t^{w,z} \in \mathcal{M}$ for all w, z and t . Recall the notation $\tilde{X} = \rho(X)$ for the time reversed process associated to $X \in \mathbb{P}(\mathfrak{h})$ (c.f. 3.4).

Corollary 6.3. *Let $X \in \mathbb{P}(\mathfrak{h})$. The following are equivalent:*

- (i) *X is a time-homogeneous left evolution.*
 (ii) *$\tilde{X}$ is a time-homogeneous right evolution.*

Since we can multiply a process on either side by the coefficients there is a total of four different QSDE's to consider, the forward and backward, left and right equations:

$$\text{(FR)} \quad d_t X_{r,t} = F_\beta^\alpha X_{r,t} d\Lambda_\alpha^\beta(t) \quad \quad d_t X_{r,t} = X_{r,t} F_\beta^\alpha d\Lambda_\alpha^\beta(t) \quad \quad \text{(FL)}$$

$$\text{(BR)} \quad d_r X_{r,t} = F_\beta^\alpha X_{r,t} d\tilde{\Lambda}_\alpha^\beta(t) \quad \quad d_r X_{r,t} = X_{r,t} F_\beta^\alpha d\tilde{\Lambda}_\alpha^\beta(t) \quad \quad \text{(BL)}$$

each with initial condition $X_{r,r} = 1$. Weak and strong solutions are defined as for processes on an algebra. The following result extends the semigroup representation

of solutions obtained in Theorem 7.1 of [LW1] to two-parameter processes and backward equations. Recall the transform $F \mapsto \widehat{F}$ on $M_D(\mathcal{B}(\mathfrak{h}))$ (5.1).

Theorem 6.4. *Let $F \in M_D(\mathcal{B}(\mathfrak{h}))$ and let $X \in \mathbb{P}(\mathfrak{h})$ be weakly regular. Then the following sets of equivalences hold:*

- (i) X is a weak solution of (FL) (resp. (FR)) for F .
- (ii) X is a weak solution of (BR) (resp. (BL)) for F .
- (iii) For $f, g \in \mathbb{S}_{\text{loc}}$,

$$X_{r,t}^{f,g} = P_{t_1-r}^{[0]} \cdots P_{t-t_n}^{[n]}, \quad (\text{resp. } X_{r,t}^{f,g} = P_{t-t_n}^{[n]} \cdots P_{t_1-r}^{[0]}),$$

where $\{r = t_0 \leq \dots \leq t_{n+1} = t\}$ contains all the points of discontinuity of $f_{[r,t]}$ and $g_{[r,t]}$, and where $P^{[k]}$ is the one-parameter semigroup on $\mathfrak{h}$ with generator $\widehat{F}_{w,z}$ with $w = f(t_k)$ and $z = g(t_k)$.

Corollary 6.5. *Let $F \in M_D(\mathcal{B}(\mathfrak{h}))$ and let $X \in \mathbb{P}(\mathfrak{h})$ be weakly regular.*

- (a) *If X is a weak solution of (FL) (resp. (FR)) for F , then X is a time-homogeneous left (resp. right) evolution. Moreover if $\mathcal{M}$ is a von Neumann algebra acting on $\mathfrak{h}$, then X is affiliated to $\mathcal{M}$ if and only if $F_\beta^\alpha \in \mathcal{M}$ for all α, β .*
- (b) *The following are equivalent:*
 - (i) X is a weak solution of (FL) for F .
 - (ii) $\tilde{X}$ is a weak solution of (BL) for F .

As with processes on an algebra our goal is to specify conditions on a time-homogeneous evolution on $\mathfrak{h}$ that allows us to represent it as a solution of some QSDE. Once again we restrict our attention to the case when the associated semigroups $\{P^{w,z}\}$ are norm continuous, and when this is the case we denote the generator of $P^{w,z}$ by $G_{(w,z)}$. The coefficients of the QSDE will be given in terms of

$$\left. \begin{aligned} G_0^0 &:= G_{(h_0, h_0)} \\ G_j^0 &:= G_{(h_0, h_j)} - G_{(h_0, h_0)}; \quad G_0^i := G_{(h_i, h_0)} - G_{(h_0, h_0)} \\ G_j^i &:= G_{(h_i, h_j)} - G_{(h_0, h_j)} - G_{(h_i, h_0)} + G_{(h_0, h_0)} \end{aligned} \right\} \quad (6.5)$$

where $\{h_\alpha\}$ is the sequence in $c(D) \subset \mathbb{S}_{\text{loc}}$ used in the previous section. Given a time-homogeneous evolution X , consider the corresponding flows ${}^1k$ and ${}^2k$ on $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ defined by (6.4), and their associated semigroups $\{{}^1\mathcal{P}^{w,z}\}$ and $\{{}^2\mathcal{P}^{w,z}\}$. Since ${}^1\mathcal{P}_t^{w,z}(a) = aP_t^{w,z}$ and ${}^2\mathcal{P}_t^{w,z}(a) = P_t^{w,z}a$, it follows that $P^{w,z}$ is norm continuous if and only if ${}^1\mathcal{P}^{w,z}$ and ${}^2\mathcal{P}^{w,z}$ are norm continuous.

Theorem 6.6. *Let $X \in \mathbb{P}^\dagger(\mathfrak{h})$ be a time-homogeneous left evolution.*

- (a) *If X is Markov-regular and bounded, with locally uniformly bounds, then all of its associated semigroups are norm continuous, and X is weakly continuous.*
- (b) *If X and $X^\dagger$ are strongly continuous at 0, and each of the semigroups P^{h_α, h_β} is norm continuous, then X is a (weakly regular) weak solution of a forward left QSDE of the form (FL) for some $F \in M_D(\mathcal{B}(\mathfrak{h}))$. Moreover $G := \widehat{F}$ is determined by the associated semigroup generators through (6.5).*
- (c) *If in (b) either the noise is finite dimensional then X satisfies (FL) strongly.*

The above remains true with left changed to right and (FL) to (FR) throughout.

As with CP contraction flows on a C^* -algebra $\mathcal{A}$, a stronger version of the above theorem holds for contraction evolutions on $\mathfrak{h}$. A proof may be constructed along the lines used in [HL2] for unitary evolutions, using the Parthasarathy-Sinha martingale representation theorem ([PS1,2]) and the independence of quantum stochastic

integrals ([LW1], Proposition 2.2) (c.f. [W]). However, a more direct approach is available due to the analysis of QSDE's in Section 7 of [LW1], moreover in this approach no restriction of separability of the initial space is necessary.

Theorem 6.7. *Let X be a contraction process on $\mathfrak{h}$. The following are equivalent:*

- (i) *X is a weak solution of the forward left QSDE (FL) for some F in $M_D(\mathcal{B}(\mathfrak{h}))$.*
- (ii) *X is a Markov-regular time-homogeneous left evolution on $\mathfrak{h}$.*
- (iii) *X is a strong solution of the QSDE (FL) for some $F \in M_D(\mathcal{B}(\mathfrak{h}))$.*

The same is true with left changed to right and (FL) to (FR) throughout.

Proof. Suppose that (ii) holds. Then by Theorem 6.6(a) X is weakly continuous. Since, for any contraction C on $\mathfrak{h}$,

$$\|(C - 1)\xi\|^2 \leq 2(\|\xi\|^2 - \operatorname{Re}\langle \xi, C\xi \rangle),$$

both X and X^* are strongly continuous at 0. Thus, by Theorem 6.6(b), (i) holds. If (i) holds then by [LW1], Proposition 7.6, the matrix $F = [F_\beta^\alpha]$ defines a bounded operator on $\mathfrak{h} \otimes \hat{\mathfrak{k}}$ and X satisfies the equation strongly ([LW1], Theorem 7.1), so (iii) holds. The implication (iii) $\Rightarrow$ (i) is trivial. $\square$

Remark. By Theorem 7.5 and Proposition 7.6 of [LW1] the operator matrix in the above is bounded and satisfies $F + F^* + F^*\Delta(1)F \leq 0$, with equality if and only if X is isometric; the identity $F + F^* + F\Delta(I)F^* = 0$ holds if and only if X is coisometric ([LW1],[Fa2],[Moh]).

7. CONCLUSION

The results here extend all previous work on *regular* Markovian cocycles adapted to a Fock filtration, both on a Hilbert space and on a C^* -algebra. Accardi and Mohari have identified a natural domain condition for a normal $*$ -homomorphic Markovian cocycle on a von Neumann algebra, whose Markov semigroup is *ultra weakly continuous*, to be generated by a QSDE ([AcM]). Following [HL2] and [Bra] they use the martingale representation theorem of Parthasarathy and Sinha ([PS1,2]).

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