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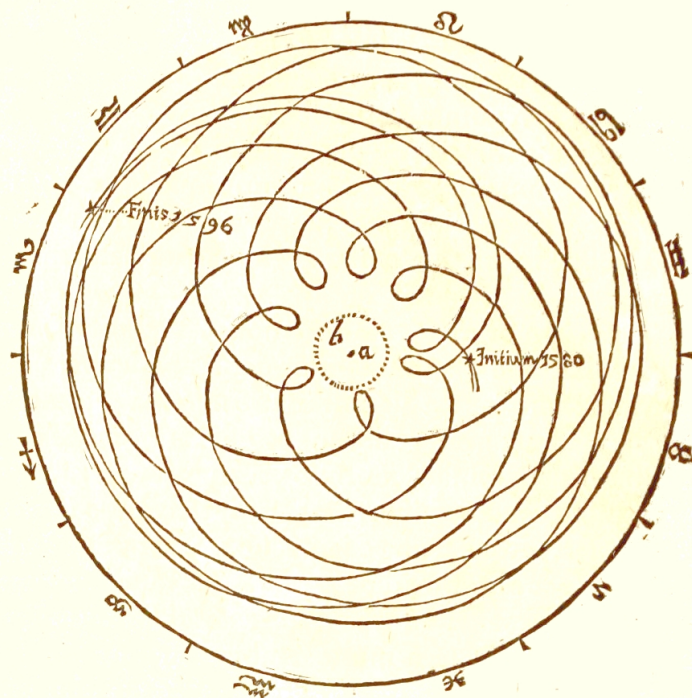
# Lectures on Differential Geometry

April 12, 2022





## DE MOTIB. STELLÆ MARTIS





## Preface

These are lectures on differential geometry written for undergraduate students at University College Cork in 2017. They assume elementary knowledge of metric spaces and topological spaces, including the concepts of homeomorphism, Hausdorff space, quotient space, and covering space. I recommend the reader on a first reading skip any section that says it can be omitted, and skip all problems marked with a \*, although it might be worthwhile to read the hints to the \* problems.

## Contents

|    |                                          |     |
|----|------------------------------------------|-----|
| 1  | Manifolds                                | 1   |
| 2  | Examples of manifolds                    | 17  |
| 3  | Functions on manifolds                   | 23  |
| 4  | Tangent vectors                          | 39  |
| 5  | Embedding the Grassmannian               | 55  |
| 6  | Vector fields                            | 65  |
| 7  | Brackets of vector fields                | 77  |
| 8  | Lie groups                               | 83  |
| 9  | Lie groups under the microscope          | 93  |
| 10 | From small to large in Lie groups        | 99  |
| 11 | Closed subgroups of Lie groups           | 103 |
| 12 | Lie algebras                             | 109 |
| 13 | Foliations                               | 117 |
| 14 | Quotients by Lie group actions           | 127 |
| 15 | Homogeneous spaces                       | 135 |
| 16 | Tangent and cotangent spaces and bundles | 143 |
| 17 | Transverse submanifolds                  | 149 |
| 18 | Sard's theorem                           | 155 |
| 19 | Corners                                  | 165 |
| 20 | Transverse submanifolds with corners     | 177 |
| 21 | Boundaries and corners                   | 183 |
| 22 | Homotopy and degree                      | 187 |
| 23 | Differential forms                       | 197 |
| 24 | Differentiating differential forms       | 203 |
| 25 | Integrating differential forms           | 209 |
| 26 | Stokes's theorem                         | 219 |
| 27 | The Brouwer fixed point theorem          | 223 |
| 28 | Maps and integrals                       | 229 |
| 29 | Integration over chains                  | 235 |
| 30 | Fubini's theorem                         | 239 |

|    |                                      |     |
|----|--------------------------------------|-----|
| 31 | Open covers                          | 247 |
| 32 | Partitions of unity                  | 257 |
| 33 | Star shaped sets                     | 261 |
| 34 | Convexity                            | 265 |
| 35 | Convolution                          | 271 |
| 36 | Smoothing                            | 277 |
| 37 | Integrating along a path             | 287 |
| 38 | de Rham cohomology                   | 297 |
| 39 | Invariant differential forms         | 307 |
| 40 | Homological algebra                  | 323 |
| 41 | Čech–de Rham cohomology              | 341 |
| 42 | Currents                             | 349 |
| 43 | Haar measure                         | 357 |
| 44 | The strong topology                  | 363 |
| 45 | Wiggling into transversality         | 367 |
| 46 | Transversality and homology          | 375 |
| 47 | Morse theory I: the drowning surface | 379 |
| 48 | Morse theory II: attaching handles   | 389 |
| 49 | Homotopy groups                      | 401 |
| 50 | Fiber bundles                        | 407 |
| 51 | Topology of fiber bundles            | 415 |
| 52 | Homotopy groups of fiber bundles     | 425 |
| 53 | Riemannian metrics                   | 439 |
| 54 | The Levi–Civita connection           | 447 |
| 55 | Geodesics                            | 455 |
| 56 | Riemann surfaces                     | 473 |
| 57 | Compact Riemann surfaces             | 485 |
| 58 | Vector bundles                       | 491 |
| 59 | Connections                          | 505 |
| 60 | Characteristic classes               | 513 |
|    | Hints                                | 529 |
|    | Bibliography                         | 629 |
|    | List of notation                     | 631 |
|    | Index                                | 635 |



## Chapter 1

### Manifolds

*With my full philosophical rucksack I can only climb slowly up the mountain of mathematics.*

— Ludwig Wittgenstein  
CULTURE AND VALUE

We define the notion of a manifold, an abstract concept of a “smooth” space.

#### Introduction

Roughly, a *manifold* means a smooth geometric object. A curve is a 1-dimensional manifold, a surface a 2-dimensional manifold, and so on.

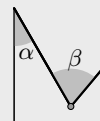
Manifolds arise in many physical problems. Think of the various positions that a complicated robot could maneuver into; the set of all such positions can have many “degrees of freedom”, so is something like a manifold of very high dimension. To follow spacecraft and satellites, or planets and moons, around in the sky, it is natural to approximate their orbits as ellipses. The set of all ellipses is also a manifold in some sense, because we can certainly picture what it would mean to have a smooth family of ellipses.

In a scientific experiment, when measuring large numbers of variables for each of a large number of samples, each sample is represented as a data point in a large dimensional Euclidean space. A theory to try to explain where the data points lie is precisely a high dimensional geometric object roughly predicting their approximate locations, perhaps a large dimensional manifold in that Euclidean space. So any form of scientific investigation leads to differential geometry, i.e. the study of manifolds.

The torus, the surface of a doughnut, is a manifold.



Hold one end of a rigid pendulum at a fixed point and bind another rigid pendulum to the end of it with a ball joint. If the pendula live in the plane, the “configurations” are described by two angles. Hence the “space of configurations” is a surface, indeed a torus: each pendulum swings in a circle, the first around the fixed point, the second around the end of the first.



If we allow the pendula to swing around in 3-dimensional space, the “space of configurations” is a product of two spheres, so a 4-dimensional manifold.

Imagine a rigid body, like a space craft, floating in space. The location of the centre of mass is a point, hence has three coordinates. Once we record the centre of mass, we still need to say how the body is rotated in space, say by angles around three perpendicular axes. So we guess that the “space of configurations” of a rigid body is a 6-dimensional manifold.

### Why we need an abstract concept

Einstein’s general theory of relativity pictures the entirety of space and time being one, perhaps very complicated, manifold. But unlike our intuitive notion of a curve or a surface, as a smooth extended object floating in Euclidean space, it doesn’t make much sense to think about space-time as sitting inside some ambient Euclidean space. If we can measure physical phenomena in an ambient space, it is part of the physical model of space time; if not, it is irrelevant to physics. So we want to think about manifolds in a more abstract manner.

Topological spaces have a notion of continuity, metric spaces a notion of distance, but neither have a notion of smoothness. The concept of manifold lies outside of both topology and metric space theory.

The power in the abstract concept of manifold is that we can construct manifolds implicitly. For example, often the set of solutions of a system of differential equations form a manifold, and we may be able to prove this, and profit from this point of view, even though we can’t solve the equations.

Another example is the construction of a universal covering space. Any connected surface, sitting inside Euclidean space, is a topological space, and has a universal covering space “unwrapping it”. But its universal covering space is an abstract “surface”, not obviously situated in any Euclidean space. Similarly, the quotient of a manifold by a group action is often a manifold. When an abstract manifold arises, we don’t need to figure out how we would realize it as a concrete set of points in Euclidean space; we are freed from the shackles of concrete representation.

### Charts

Even the most abstract manifolds we can imagine must surely all be “smooth spaces”, in some sense, so all locally look like Euclidean space. For example, on the surface of the Earth, we describe where we are using a map, which associates to each point on the curved surface of the Earth a point of a flat plane. As usual let  $\mathbb{R}^n$  denote  $n$ -dimensional Euclidean space. Recall that a *diffeomorphism* is a smooth map between open sets of Euclidean space with a smooth inverse. By *smooth* we mean differentiable with derivatives of all orders, i.e.  $C^\infty$ .

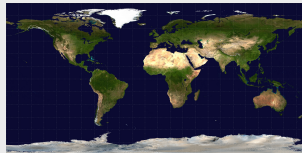
A *chart* on a set  $M$  is a pair  $(U, \varphi)$  where  $U \subset M$  is a subset and  $\varphi: U \rightarrow \varphi(U) \subset \mathbb{R}^n$  is a bijection onto an open subset  $\varphi(U)$  of  $\mathbb{R}^n$ . The map  $\varphi^{-1}$  is a *local parameterization* of  $M$ .



The surface of the planet Earth is a surface.

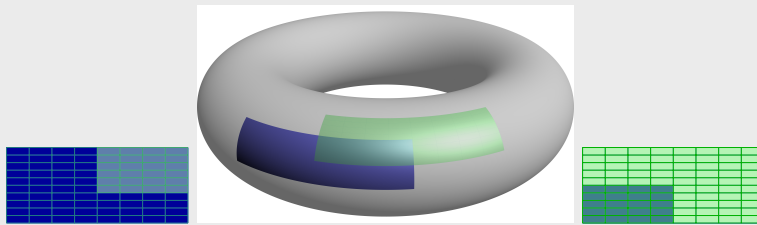


Here are two different charts.



The *transition map* from a chart  $(U, \varphi)$  to another  $(W, \psi)$  is the map  $\psi \circ \varphi^{-1}$ , wherever this is defined.

Two different charts on the torus:



have transition map mapping between the two smaller coloured rectangles.

Two charts are *compatible* if the transition map from each one to the other is either (i) defined nowhere or (ii) is a diffeomorphism. An *atlas* on a set  $M$  is a collection of mutually compatible charts whose domains cover  $M$ .

Let  $M := \mathbb{R}^2$  and define a polar coordinate chart on  $U = \mathbb{R}^2 - \{(x, 0) \mid x > 0\}$ , by setting  $\varphi(x, y) := (r, \theta)$  where  $r := \sqrt{x^2 + y^2}$  and

$$\theta := \begin{cases} \arctan(y/x) & \text{if } x > 0, \\ \operatorname{arccot}(x/y) & \text{if } y > 0, \\ -\operatorname{arccot}(-x/y) & \text{if } y < 0. \end{cases}$$

You construct polar coordinates  $r, \theta$  on the plane with  $0 < \theta < 2\pi$ . I construct polar coordinates  $R, \Theta$  with  $-\pi < \Theta < \pi$ . Each of these polar coordinate systems is defined on a different open set in the plane. The  $r, \theta$  coordinates are defined everywhere but along the points  $(x, 0)$  with  $x \geq 0$ . The  $R, \Theta$  coordinates are defined everywhere but along the points  $(x, 0)$  with  $x \leq 0$ . (Don't allow either coordinates to have  $r = 0$  or  $R = 0$ : we will need some other coordinates near the origin, since neither  $\theta$  nor  $\Theta$  are defined there.) The transition map is  $(r, \theta) \mapsto (R, \Theta)$  where  $R = r$  but

$$\Theta = \begin{cases} \theta, & 0 < \theta < \pi, \\ \theta - 2\pi, & \pi < \theta < 2\pi, \end{cases}$$

and is not defined at  $\theta = \pi$ .

Let  $M$  be the set of all lines in the plane. Intuitively, we know what it means for a line to move smoothly: its slope and intercept should vary smoothly. Associate to the line  $y = mx + b$  the point  $(m, b)$ ; this is a chart, not defined on all of  $M$ , because vertical lines don't have the form  $y = mx + b$ . We can make another chart: to each line  $x = ny + c$  associate the coordinates  $(n, c)$ . The transition map is given by changing the equation  $y = mx + b$  into the form  $x = ny + c$ . If  $y = mx + b$  then  $x = \frac{1}{m}y - \frac{b}{m}$ . So the transition map is  $t(m, b) = (n, c) = (\frac{1}{m}, -\frac{b}{m})$ . The transition map is defined whenever  $m \neq 0$ .

The circle: let  $M := \mathbb{R}/\mathbb{Z}$  be the set of all real numbers but modulo the equivalence relation that two are considered the same if they differ by an integer. Denote the equivalence class of a real number  $s$  by  $[s]$ . Given an equivalence class  $[s]$ , let  $x([s])$  be the real number  $x$  with  $0 < x < 1$  so that  $[s] = [x]$ ;  $x([s])$  is defined for  $[s] \neq [0]$ . Let  $y([s])$  be the real number  $y$  with  $\frac{1}{2} < y < \frac{3}{2}$  and  $[y] = [s]$ ;  $y([s])$  is defined for  $[s] \neq [1/2]$ . These maps are bijective, so charts, and between them their domains cover  $M$ . The transition map

$$y = y(x): \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \rightarrow \left(\frac{1}{2}, 1\right) \cup \left(1, \frac{3}{2}\right)$$

is

$$y(x) = \begin{cases} x + 1, & \text{if } 0 < x < \frac{1}{2}, \\ x, & \text{if } \frac{1}{2} < x < \frac{3}{2}. \end{cases}$$

### Example: the projective plane

Invent a formal symbol  $\infty$  and let  $M = \mathbb{C} \cup \{\infty\}$ , which we call the *Riemann sphere*. Write points of  $M$  as complex numbers  $\zeta$  but allowing  $\zeta = \infty$  among them. Make a complex chart  $z: M - \{\infty\} \rightarrow \mathbb{C}$  by  $z(\zeta) = \zeta$ , and another chart  $w: M - \{0\} \rightarrow \mathbb{C}$  by

$$w(\zeta) = \begin{cases} \frac{1}{\zeta}, & \text{if } \zeta \neq 0, \infty, \\ 0, & \text{if } \zeta = \infty. \end{cases}$$

Check that the transition map is  $w = w(z) = \frac{1}{z}$ , defined on  $\mathbb{C} - \{0\}$ . In terms of real and imaginary parts, this is a smooth map where it is defined.

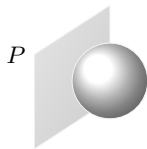
**1.1** If two charts are both compatible with all of the charts in some atlas, prove that they are compatible with one another.

It is convenient to define  $\mathbb{R}^0$ , the *zero dimensional Euclidean space*, to mean just a set consisting of a single point called 0; extend all of our definitions above and below to include dimension zero.

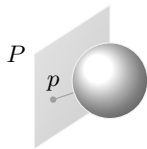
### Example: the projective plane

The *projective plane* is the set of all lines in  $\mathbb{R}^3$  passing through the origin, denoted  $\mathbb{RP}^2$ . To picture  $\mathbb{RP}^2$ : inside each line through the origin, draw a unit length vector. This seems to make  $\mathbb{RP}^2$  look like the unit sphere: the set of all unit vectors coming from the origin. But this is *not* really a picture of  $\mathbb{RP}^2$ : if  $u$  is a unit vector, then  $-u$  and  $u$  both lie in the same line through the origin. So  $\mathbb{RP}^2$  is the sphere, but with opposite points identified.

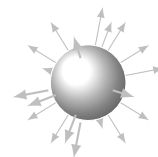
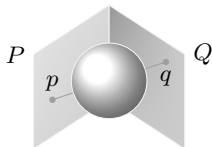
Make charts for  $\mathbb{RP}^2$  as follows. Pick any plane  $P$  *not* passing through the origin.



Somehow draw axes along the plane  $P$ , so that we smoothly identify each point  $p \in P$  with a point  $(x, y)$  of the plane  $\mathbb{R}^2$ . If  $\ell$  is the line  $0p$ , let  $\varphi(\ell) = (x, y)$ . (We need to make sure that 0 is *not* in  $P$ , so that there is a line  $0p$ .)



The transition map is given by following the line from one plane to another.



Write vectors in  $\mathbb{R}^3$  as  $(x, y, z)$ . For example, take  $P$  to be the plane  $x = 1$ . Each point  $p \in P$  has the form  $p = (1, y, z)$ , so is determined by its  $y, z$  coordinates; let  $\varphi(0p) = (y, z)$ . Similarly, we can let  $Q$  be the plane  $y = 1$  and make a chart  $\psi$  by letting  $\psi(0q) = (x, z)$  if  $(x, 1, z)$  is the point where the line  $\ell$  hits the plane  $Q$ . Write each point  $\psi(\ell)$  as  $(\bar{x}, 1, \bar{z})$ .

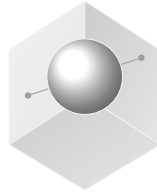
The line  $0p$  through  $p = (1, y, z)$  is the span of  $(1, y, z)$ , so consists in the vectors  $t(1, y, z)$  for any value of a variable  $t$ . To find the point  $(\bar{x}, 1, \bar{z})$  that lies on this line, write out

$$(\bar{x}, 1, \bar{z}) = t(1, y, z) = (t, ty, tz),$$

and solve:  $t = 1/y$ , so that the transition map is

$$(\bar{x}, \bar{z}) = \left( \frac{1}{y}, \frac{z}{y} \right).$$

**1.2** Prove that every line through the origin strikes at least one of the 3 planes  $(x = -1), (y = -1), (z = -1)$ :



So we only need 3 charts to cover the real projective plane.

**1.3** Let  $M$  be the space of all lines in  $\mathbb{R}^3$ ; write down an atlas on  $M$ .

**1.4** Let  $M$  be the space of all pairs  $(p, \ell)$  so that  $p \in \mathbb{R}^n$  and  $\ell$  is a line  $\mathbb{R}^n$  passing through both  $p$  and the origin; write down an atlas on  $M$ .

### Example: the sphere by Ptolemaic projection

Take a vector space  $V$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take a point  $n \in V$  and an affine hyperplane  $P$  not passing through  $n$ . The *projection* of a point  $p$  from the point  $n$  is the point  $q = \pi(p) \in P$  given by drawing the line from  $n$  through  $p$  to the point  $q$  where that line strikes the plane  $P$ . Clearly this is defined for precisely those points  $p \neq n$ :

$$\pi: V - \{n\} \rightarrow P.$$

Write any vector  $p \in \mathbb{R}^{n+1}$  as a pair

$$p = \begin{pmatrix} a \\ b \end{pmatrix}$$

with  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}$ . Denote by  $S^n$  the sphere consisting of the unit vectors in  $\mathbb{R}^{n+1}$ , i.e. the vectors  $p$  so that  $|p| = 1$ . The *north pole* is the point

$$n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S^n.$$

Let  $P$  be the hyperplane consisting of the points

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

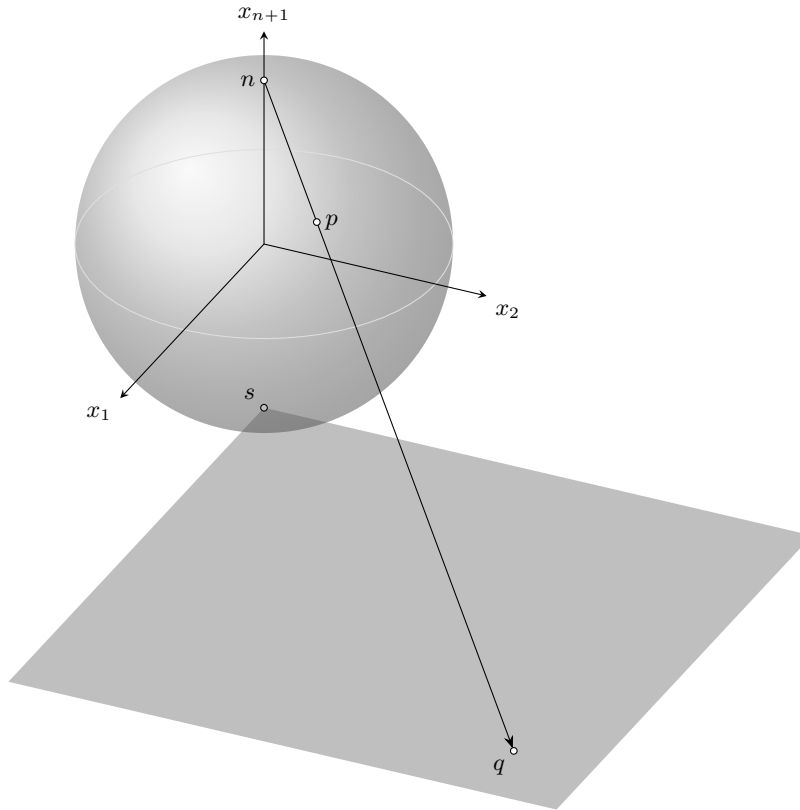
with  $b = -1$ . Take the projection  $\pi$  from  $n$  to the plane  $P$ , writing it as

$$q = \pi(p) = \begin{pmatrix} 2x \\ -1 \end{pmatrix}.$$

(The factor of 2 in front of  $x$  is not important, but will be convenient later on.) Write  $x$  as  $x = \varpi(p)$ . Clearly this is defined for every point  $p \in S^n$  except  $n$ , given a map

$$\varpi: S^n - \{n\} \rightarrow \mathbb{R}^n.$$

This map  $\varpi$  is the *Ptolemaic projection* of the sphere.



**1.5** Compute Ptolemaic projection and its inverse; prove that Ptolemaic projection is a homeomorphism.

**1.6** Try the south pole instead of the north pole.

**1.7** Thinking of Ptolemaic projection from the north pole as a chart on  $S^n - \{n\}$ , and the same for the south pole, compute the transition map between those charts.

## The induced topology

Given a set  $M$  with an atlas, a subset  $V \subset M$  is *open* if, for every chart  $(U, \varphi)$  in the atlas,  $\varphi(U \cap V) \subset \mathbb{R}^n$  is open.

**1.8** To check if a set is open, we don't need to check *all* charts. Take a set  $M$  with an atlas. Prove that a subset  $V \subset M$  is open just when there are some charts  $(U_a, \varphi_a)$  from the atlas so that every point of  $V$  lies in some  $U_a$  and every  $\varphi_a(U_a \cap V)$  is open.

**1.9** Prove that the open sets of a set with an atlas form a topology.

Henceforth, we always use this topology on any set with atlas.

## Smooth structures

If you use one atlas of charts on a sphere, and I use another, we are nonetheless discussing the same sphere, the same manifold. The choice of atlas is not essential. Two atlases are *equivalent* if their union is an atlas.

**1.10** Prove that equivalent atlases give the same topology.

**1.11** Show that the relation of compatibility of charts is *not* transitive.

**1.12** Show that the relation of equivalence of atlases is transitive, symmetric and reflexive, so is in fact an equivalence relation.

An atlas is *maximal* if any chart, compatible with every chart belonging to the atlas, belongs to the atlas.

**1.13** Prove that an atlas is maximal just when any equivalent atlas already lies inside it.

**1.14** Prove that any atlas lies in a unique maximal atlas: the union of all charts compatible with its charts.

A *smooth structure* is a maximal atlas.

**1.15** Prove that every open subset  $U \subset M$  of any set with smooth structure  $M$  has its own smooth structure, with charts being just the restrictions of the charts of  $M$ .

A *diffeomorphism* is a bijection between two sets with smooth structure which, when used to identify the points of the two sets, also identifies the smooth structures. (We will make this more concrete in the next chapter; for now we only need the idea that two smooth structures can be “practically the same”).

## Nasty spaces: different dimensions in different parts

Take  $M$  the union of a line in the plane and a disjoint open disk in the plane:  $\mathbb{R} \cup \circ$ . Then  $M$  has a smooth structure with charts (1) a diffeomorphism  $\varphi$  from the line to the real number line and (2) the identity map  $\psi$  taking the disk to itself. This is strange, because one part of  $M$  is 1-dimensional and another

part is 2-dimensional.

### Nasty spaces: points “on top of each other”

A worse example: take some abstract symbol  $0'$ , not representing any real number, and let  $M := \mathbb{R} \cup \{0'\}$ . Think of  $0'$  as a second zero, somehow sitting “on top of the usual zero”.



To make this more precise, let  $U := \mathbb{R}$  and  $W = \mathbb{R} \cup \{0'\} - \{0\}$ . Take two charts: the chart  $\varphi: U \rightarrow \mathbb{R}$  given by  $\varphi(x) = x$ , and the chart  $\psi: W \rightarrow \mathbb{R}$  given by

$$\psi(x) = \begin{cases} x, & \text{if } x \neq 0', \\ 0, & \text{if } x = 0'. \end{cases}$$

The transition map is  $t: \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{0\}$ ,  $t(x) = x$ . We call  $M$  a *line with doubled point*. The line with doubled point is *not* Hausdorff.

**1.16** What are the continuous functions on  $M$ ? What are the continuous curves in  $M$ ?

### Nasty spaces: too many open sets

Take any set  $M$ . Take as charts just constant maps taking each point to the origin in the zero dimensional Euclidean space  $\mathbb{R}^0 = \{0\}$ . Picture  $M$  as a collection of scattered points, separated from one another. The topology of this smooth structure is the discrete topology.

**1.17** Compute the transition maps between these charts.

**1.18** Prove that every function  $h: M \rightarrow \mathbb{R}$  is continuous.

**1.19** Similarly, prove that a continuous path in  $M$ , i.e. a continuous map  $\mathbb{R} \rightarrow M$ , is constant: you can’t move around in  $M$  at all along continuous paths.

Let  $M$  be the plane  $M = \mathbb{R}^2$ , but picture the plane as being built up as a union of vertical lines, i.e.  $M$  is the product  $M = X \times Y$  where  $X$  is  $\mathbb{R}$  with the discrete topology and  $Y$  is  $\mathbb{R}$  with the Euclidean topology.



For each number  $x$  construct a chart  $\varphi_x: \{x\} \times \mathbb{R} \rightarrow \mathbb{R}$  by  $\varphi_x(x, y) = y$ , drawing out the vertical line along the given  $x$  coordinate.

**1.20** Compute the transition maps and prove that these charts form an atlas for a smooth structure.

**1.21** Prove that the continuous functions on  $M$  are precisely the functions  $h(x, y)$  which are continuous in  $y$  in the usual sense, but can do anything at all in  $x$ .

**1.22** Prove that a continuous path in  $M$  is precisely a map  $t \mapsto (x(t), y(t))$  where  $y(t)$  is a continuous function in the usual sense, and  $x(t)$  is constant.

**1.23** Say that a subset of  $M$  is compact if every open cover has a finite subcover. What are the compact subsets of  $M$ ?

Geometrically, we can think of this  $M$  as being obtained by taking the usual Euclidean plane, but cutting out each vertical line, i.e. the image of each of our charts, and “separating it” from all of the other vertical lines, as if they were spaced apart.

A *basis of open sets* of a topological space is a collection of open sets so that every open set is a union of sets from the basis.

The disks are a basis of open sets in the plane.

The unit disks are *not* a basis of open sets in the plane; no disk of smaller than unit radius is a union of unit disks.

A basis is *countable* if its open sets can be placed into a sequence, say  $U_1, U_2, \dots$

The disks with rational radii centered at points with rational coordinates form a countable basis of open sets in the plane.

**1.24** A topological space is *separable* if it contains a countable dense subset, *metrizable* if it is homeomorphic to a metric space. Prove that every subset of any separable metrizable topological space has a countable basis of open sets. (So a topological space without a countable basis of open sets is “nasty”.)

## Manifolds

For any integer  $n \geq 0$ , an  $n$ -dimensional *topological manifold* is a Hausdorff topological space locally homeomorphic to  $\mathbb{R}^n$  and with a countable basis of open sets.



**1.25** Prove that the plane with the vertical lines “separated apart”, does *not* admit a countable basis of open sets, and so is not a topological manifold.

**1.26** Take two lines in the plane, not parallel. Prove that their union



is not locally homeomorphic to a line, so is not a 1-dimensional topological manifold.

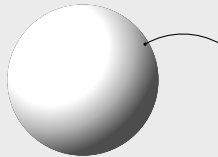
**1.27** Prove that in any Hausdorff space, every compact set is closed.

**1.28** Prove that in any Hausdorff space, every compact set is the intersection of the open sets which contain it.

**1.29\*** Suppose that  $P$  and  $Q$  are compact topological manifolds. Pick points  $p_0 \in P$ ,  $q_0 \in Q$ . Suppose that  $P - \{p_0\}$  is homeomorphic to  $Q - \{q_0\}$ . Prove that  $P$  is homeomorphic to  $Q$ .

We will not pursue the proof, but it turns out that open sets in Euclidean spaces of different dimension are *not* homeomorphic. Hence topological manifolds of different dimensions are *not* homeomorphic, even locally.

The union of a sphere and a curve is not a topological manifold, since it has points near which it is locally homeomorphic to the plane, and others near which it is locally homeomorphic to the line.



**1.30** Prove that for any compact set  $X$  and closed set  $Y$  in a topological manifold  $M$  there is an open set  $U$  containing  $X$  and not intersecting  $Y$  so that  $\bar{U}$  is compact.

**1.31** Prove that any topological manifold is covered by a sequence of open sets, each with compact closure.

**1.32\*** In any topological manifold, prove that every compact set is the intersection of a nested sequence of open sets, each with compact closure.

The definition of topological manifold captures the idea that manifolds should locally look like Euclidean space, and have otherwise reasonable topology. But it still misses the essential idea that manifolds should have a concept of smooth function, so that we can make sense of calculus on manifolds. A *manifold* is a set with smooth structure which is a topological manifold in the topology arising from the charts. In other words, in that topology, it is Hausdorff with a countable basis of open sets, and all of its charts map to a Euclidean space of the same dimension. From now on a *chart* on a manifold means a chart belonging to its smooth structure.

### Trivial examples

Euclidean space  $\mathbb{R}^n$  is a manifold: take the identity map as a chart, and the smooth structure is induced by that chart.

**1.33** Prove that every open subset of any manifold is a manifold of the same dimension, with charts being just the restrictions of the charts of the ambient manifold.

**1.34** Suppose that  $P$  and  $Q$  are manifolds. Explain how to make  $P \times Q$  into a manifold.

If  $X$  and  $Y$  are any sets, the *disjoint union*  $X \sqcup Y$  is “separate copies of  $X$  and  $Y$ ”, i.e.  $X \sqcup Y$  is the set of pairs  $(0, x)$  for  $x \in X$  and  $(1, y)$  for  $y \in Y$ . For example,  $\mathbb{R} \sqcup \mathbb{R}$  is a pair of disjoint lines. Similarly, we can make triple disjoint unions and so on. The disjoint union of two manifolds of equal dimension is clearly a manifold.

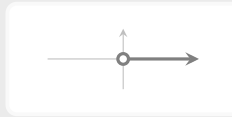
### Submanifolds

A  $d$ -dimensional *embedded submanifold*  $S \subset M$  of a manifold  $M$  is a subset  $S$  so that each point  $x \in S$  lies in an open subset  $U \subset M$ , which is the domain of a chart  $(U, \varphi)$  of  $M$  so that  $\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^d \times \{0\})$ , i.e.  $\varphi$  “flattens out  $S$  locally”.

The unit sphere centered at the origin in  $\mathbb{R}^3$  is locally “flattened out” near the north pole by  $(x, y, z) \mapsto (x, y, z - \sqrt{1 - x^2 - y^2})$ , and similarly near other points, and so is an embedded submanifold, hence a manifold. Any smooth surface we draw, such as a sphere, is an embedded submanifold of 3-dimensional Euclidean space.

Any embedded submanifold  $S \subset M$  has a countable basis of open sets: take the intersections  $S \cap U_i$  with open sets  $U_i$  from any countable basis of open sets of  $M$ . Moreover,  $S$  is Hausdorff because  $M$  is. Hence  $S$  is a  $d$ -dimensional manifold.

In the plane, the positive horizontal axis is an embedded submanifold  $S$ :



Take  $U = (0, \infty) \times \mathbb{R}$  and the identity map as chart. Near the origin,  $S$  is not identified by any chart with an open set in  $\mathbb{R}^1 \times \{0\}$ , but this is ok because the origin does not belong to  $S$ . All the same, an embedded submanifold is much nicer if it is closed.

**1.35** The *unit sphere*  $S^n$  is the set of all vectors of unit length in  $\mathbb{R}^{n+1}$ . Prove that the unit sphere is a manifold of dimension  $n$ .

**1.36\*** Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Denote by  $k^{p \times q}$  the set of all  $p \times q$  matrices with entries in  $k$ . The *determinantal manifold*  $M = M_{p,q,r} \subseteq k^{p \times q}$  is the set of these matrices which have rank  $r$ . Prove that the determinantal manifold is an embedded submanifold in  $k^{p \times q}$  of dimension  $dr(p + q - r)$  where  $d = 1, 2$  or  $4$  as  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ .

A *embedded hypersurface* in a manifold  $M$ , if  $M$  has dimension  $n$ , is a embedded submanifold of dimension  $n - 1$ .

**1.37** Suppose that  $S$  is a connected embedded hypersurface in a connected manifold  $M$ . Prove that the complement  $M - S$  has one or two path components. If  $M$  is not compact, prove that the closure of one or both of those path components is not compact.

## Smoothing manifolds

This section can be omitted without loss of continuity.

We might consider manifolds with limited smoothness. For any integer  $k = 0, 1, 2, \dots$ , a  $C^k$  *atlas* on a topological manifold  $M$  is a collection of charts, whose domains cover  $M$ , with all transition maps being  $C^k$ . A  $C^k$  *structure* is a maximal  $C^k$  atlas. A  $C^k$  *manifold* is a topological manifold with  $C^k$  structure. If  $M$  is a  $C^k$  manifold then every open set  $U \subset M$  has an induced  $C^k$  structure generated by the restrictions of the charts of  $M$  to  $U$ .

After this section, we will never think about  $C^k$  manifolds, because of a theorem whose proof we will eventually provide:

**Theorem 1.1** (Smoothing of manifolds). *Every  $C^k$  manifold  $M$ , for any  $k \geq 1$ , admits a smooth structure in its  $C^k$  structure. This smooth structure is unique up to smooth diffeomorphism.*

Henceforth, forget about  $C^k$  manifolds; all manifolds are smooth from here on.

## An inequivalent notion of manifold

This section can be omitted without loss of continuity.

Some authors use a different definition of *manifold*, which is *not* equivalent to the one above, and allows a larger collection of manifolds. A *topological manifold in the broad sense* is a metrizable topological space locally homeomorphic to Euclidean space. This is *not* equivalent to the definition of topological manifold above (which, to draw the distinction, we can call a topological manifold in the *narrow sense*). Recall that a topological space is *metrizable* if it is homeomorphic to a metric space, i.e. its open sets are the unions of open balls of some metric. Repeat the definition of smooth structure above to define a *manifold in the broad sense*. Repeat all of the theorems with the same proofs.

The real number line with the discrete topology is a topological manifold in the broad sense, but does not have a countable basis of open sets, so is not a manifold in the narrow sense.

**1.38** Let  $X$  be a topological space which has no countable basis of open sets. Let  $Y$  be a topological space which has a countable basis of open sets. Prove that there is no subset of  $Y$  homeomorphic to  $X$ .

**1.39\*** Prove that a topological manifold in the broad sense is a topological manifold in the narrow sense just when it has a countable set of components.

Proof that a manifold in the narrow sense is metrizable, and hence is also a manifold in the broad sense, will have to wait for corollary 53.4 on page 443.

## Notes

This section can be omitted without loss of continuity.

- a. We can write down explicitly an atlas of charts on various manifolds. There is *no* manifold on which we can write down explicitly a smooth structure, i.e. a maximal atlas, not even for the real number line  $M = \mathbb{R}$ , since we would have to write down too many arbitrarily complicated charts. We can't even write down explicitly all of the open subsets of the real number line.
- b. We saw that  $\mathbb{RP}^2$  is a surface (i.e. a 2-dimensional manifold). It turns out that  $\mathbb{RP}^2$  is *not* diffeomorphic to any surface in  $\mathbb{R}^3$ . Let  $S^2 \subset \mathbb{R}^3$  be the unit sphere. The map  $f: S^2 \rightarrow \mathbb{R}^6$  given by  $f(x, y, z) = (x^2, y^2, z^2, xy, xz, yz)$  has image diffeomorphic to  $\mathbb{RP}^2$ .
- c. Whitney proved that every manifold of positive dimension, say dimension  $n$ , is diffeomorphic to an embedded submanifold of Euclidean space  $\mathbb{R}^{2n}$ . (What happens in dimension  $n = 0$ ?) So in fact manifolds are not so abstract after all. But we have outlined above the need for an abstract theory of manifolds.
- d. There are topological manifolds, of every dimension  $\geq 4$ , which admit no smooth structure.
- e. Take the map  $x \in \mathbb{R} \mapsto x^3 \in \mathbb{R}$  and compose with all of the smooth functions  $f$  from open sets of  $\mathbb{R}$  to become continuous functions  $f(x^3)$  on open sets of  $\mathbb{R}$ . Note that not all smooth functions arise this way: they have to have two derivatives vanishing at the origin. Our composed functions form a smooth structure on  $\mathbb{R}$ , which is not the usual one near the origin (because the function  $g(x) := x$  is not of this form). This smooth structure is the usual one away from the origin (because the cube root function is smooth away from the origin). Nonetheless,  $\mathbb{R}$  with this strange smooth structure is diffeomorphic to  $\mathbb{R}$  with the usual smooth structure, via the map  $x \mapsto x^3$ .
- f. There is precisely one smooth structure, up to such a diffeomorphism, on any topological manifold of dimension 1, 2 or 3.
- g. But there are manifolds, of every dimension  $\geq 4$ , which admit more than one nondiffeomorphic smooth structure. In particular,  $\mathbb{R}^4$  admits infinitely many nondiffeomorphic smooth structures, in some of which it is nonetheless (surprisingly) diffeomorphic to an open subset of  $\mathbb{R}^4$ . I have no idea how to picture such a thing. Some of these sit as open sets inside the usual  $\mathbb{R}^4$ . There is even a way to slice up  $\mathbb{R}^5$  into submanifolds each of which is homeomorphic to  $\mathbb{R}^4$  with a different smooth structure.

- h.* Perhaps more concrete, but still unimaginable: there is an uncountable collection of contractible open subsets of  $\mathbb{R}^3$  no two of which are diffeomorphic.
- i.* It remains (as of 12 April 2022) unknown whether the 4-dimensional sphere  $S^4$  admits a smooth structure not diffeomorphic to the standard smooth structure.
- j.* It is known that  $S^7$  admits exactly 15 smooth structures, modulo diffeomorphism.
- k.* The proof of smoothing of manifolds fails if applied to a topological manifold, since there are topological manifolds which admit no smooth structure.
- l.* A manifold in the broad sense is a manifold in our more narrow sense just when it is diffeomorphic to an embedded submanifold of Euclidean space.
- m.* Every topological manifold of positive dimension, say dimension  $n$ , admits a continuous embedding into Euclidean space  $\mathbb{R}^{2n+1}$ ; see [14] p. 316. It is not known as of 12 April 2022) if the same holds for  $\mathbb{R}^{2n}$  instead of  $\mathbb{R}^{2n+1}$ , although this is widely believed.
- n.* There is a countable set of compact topological manifolds, up to homeomorphism [3], but an uncountable set of noncompact topological manifolds up to homeomorphism.



## Chapter 2

### Examples of manifolds

*But you must note this: if God exists and if He really did create the world, then, as we all know, He created it according to the geometry of Euclid and the human mind with the conception of only three dimensions in space. Yet there have been and still are geometricians and philosophers, and even some of the most distinguished, who doubt whether the whole universe, or to speak more widely the whole of being, was only created in Euclid's geometry; they even dare to dream that two parallel lines, which according to Euclid can never meet on earth, may meet somewhere in infinity.*

— Fyodor Dostoyevsky  
THE BROTHERS KARAMAZOV

**2.1\*** Prove that a set with smooth structure has a countable basis of open sets just when it has a countable set of charts, sitting inside its smooth structure, whose domains cover the set.

**Lemma 2.1.** *A set with smooth structure is Hausdorff just when any two points of the set either*

- a. *lie in the domain of the same chart or*
- b. *lie in disjoint chart domains,*

*Proof.* Suppose that two points of  $M$  lie in the domain of a chart. The chart identifies an open subset of  $M$  with an open subset of Euclidean space, which is Hausdorff, so they are housed off. Suppose that two points of  $M$  lie in disjoint chart domains. Chart domains are open, so the points are housed off. So  $M$  is Hausdorff.

On the other hand, suppose that  $M$  is Hausdorff. So any two points  $m_0, m_1 \in M$  lie in disjoint open sets. Intersect those open sets with chart domains around those points.  $\square$

The projective plane is a manifold:

- a. We can use finitely many of our charts to cover it. In fact, 3 charts suffice: every line through the origin strikes one of the planes  $(x = 1)$ ,  $(y = 1)$  or  $(z = 1)$ .
- b. Any two lines through the origin in  $\mathbb{R}^3$  pass through some plane  $P \subset \mathbb{R}^3$  not passing through the origin, so lie in the domain of the same chart.

The set of lines in the plane is a manifold:

- a. We can use finitely many of our charts to cover it. In fact, 2 charts suffice: every line in the plane is either of the form  $y = mx + b$  (where the chart taking the line  $y = mx + b$  to the point  $m, b$  is defined), or of the form  $x = ny + c$  (where the chart taking the line  $x = ny + c$  to the point  $n, c$  is defined).
- b. Every pair of lines lies in one or both of our two charts, *except* for a pair of lines  $x = x_0$  and  $y = y_0$ , one horizontal and one vertical line. Construct two open sets: let  $U$  be the set of “nearly horizontal lines”: lines of the form  $y = mx + b$  with  $-1/2 < m < 1/2$  and let  $W$  be the set of “nearly vertical lines”: lines of the form  $x = ny + c$  with  $-1/2 < n < 1/2$ . Clearly these are open sets, since each corresponds in one of the charts to an open subset of the associated plane. These open sets are disjoint, as  $n = \frac{1}{m}$ .

**2.2** Give an atlas of charts on the set of all lines in  $\mathbb{R}^3$ , and prove that it is a manifold with your atlas.

**2.3** Use Ptolemaic projection to ensure that the sphere is a manifold.

**2.4** Let  $M$  be the set of all pairs  $(p, \ell)$  where  $p \in \mathbb{R}^2$  and  $\ell \subset \mathbb{R}^2$  is a line through the origin, and  $p \in \ell$ . Give an atlas of charts on  $M$ , and prove that it is a manifold with your atlas.

**2.5** Prove that any topological manifold is connected just when it is path connected.

**2.6\*** The *Cayley transform* of any square matrix  $A$  is

$$C(A) = (I - A)(I + A)^{-1},$$

which is defined as long as  $-1$  is not an eigenvalue of  $A$ . Prove that the Cayley transform is its own inverse:  $C(C(A)) = A$ . Use this to prove that the Cayley transform is 1-1. Prove that the Cayley transform of a matrix  $A$  is an orthogonal matrix just when  $A$  is antisymmetric, and vice versa. Let  $M$  be the set of all orthogonal matrices. Prove that the Cayley transform gives a chart on  $M$  near the identity matrix, mapping to the space of antisymmetric matrices (which we linearly identify with  $\mathbb{R}^d$  where  $d = \binom{n}{2} = n(n-1)/2$ ). Prove that, for any fixed orthogonal matrix  $g_0$ , the map

$$\varphi(g) = C(g_0^{-1}g)$$

is a chart on  $M$  near  $g_0$ . Explain why the set of orthogonal matrices is a manifold, and find its dimension. Prove that the set of orthogonal matrices with rational number entries is dense in the set of all orthogonal matrices.

## The projective line

Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . The *projective line*, denoted  $\mathbb{P}^1$ , is the set of all lines through the origin in  $k^2$ . Denote the span of a nonzero vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in k^2$$



as

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Each nonhorizontal line has  $y \neq 0$ , so we can rescale the vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

without changing the span, to arrange that  $y = 1$ , giving a chart

$$\begin{bmatrix} x \\ 1 \end{bmatrix} \mapsto x.$$

Think of each point of the projective line as being given by its value of this chart, it looks just like  $k$  itself, but with one point “at infinity”: the unique point

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

for which that chart is not defined. Similarly, we define a chart on the set of nonvertical lines by:

$$\begin{bmatrix} 1 \\ y \end{bmatrix} \mapsto y.$$

The transition map is  $y = 1/x$ , smooth. Every line is either not horizontal or not vertical, so lies in one of the charts. We will soon see that the projective line is a manifold.

Let  $d = 1, 2, 4$  according to whether  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , i.e.  $d = \dim k$  is the dimension of  $k$  as a real vector space. Identify  $k = \mathbb{R}^d$  as usual. Define charts on the spheres  $S^d = S^1, S^2, S^4$  using Ptolemaic projections:  $x = \varpi_n(p)$  and *conjugate*:  $y = \overline{\varpi_s(p)}$ . The transition map (as computed in problem 1.7 on page 7) is  $y = \bar{x}/|x|^2 = \frac{1}{x}$  in  $k$ . This is precisely the transition map of the projective line. In fact, we can just directly write elements of  $\mathbb{R}^{d+1} = k \oplus \mathbb{R}$  as pairs

$$\begin{pmatrix} a \\ b \end{pmatrix}.$$

Map this vector to

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{a}{1-b} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\frac{a}{1-b}} \\ 1 \end{bmatrix}.$$

**2.7** Explain that last equality.

**2.8** Prove that this map is a diffeomorphism.

So projective lines are diffeomorphic to spheres:

$$\mathbb{RP}^1 = S^1, \mathbb{CP}^1 = S^2, \mathbb{HP}^1 = S^4.$$

## Projective spaces

Take  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ,  $V$  a finite dimensional vector space over  $k$ . Let  $\mathbb{P}V$  be the *projectization* of  $V$ , i.e. the set of all lines through the origin of  $V$ , also called the *projective space* of  $V$ . If  $V = k^{n+1}$  is the usual  $k$ -vector space of dimension  $n+1$ , the projective space  $\mathbb{P}V$  is denoted  $\mathbb{P}^n$ .

Map  $\pi: x \in V - \{0\} \rightarrow [x] \in \mathbb{P}V$ , taking each nonzero vector to the line through it and the origin. Suppose that  $V$  has dimension  $n+1$  as a vector space over  $k$ . Take a basis  $e_0, e_1, \dots, e_n$ . Write each vector  $x$  as  $x = \sum_a x_a e_a$ . Let  $V_a$  be the open set of vectors  $x \in V$  with  $x_a \neq 0$ . Note that  $V - \{0\} = V_0 \cup V_1 \cup \dots \cup V_n$ . Identify each affine subspace  $A_a = (x_a = 1) \subset V_a$  with  $k^n$  by writing each vector  $x$  with  $x_a = 1$  as being

$$x = \begin{pmatrix} y_1 \\ \vdots \\ y_{a-1} \\ 1 \\ y_a \\ \vdots \\ y_n \end{pmatrix},$$

a bijection  $\iota_a: y \in k^n \mapsto x \in A_a$ .

Let  $U_a \subset \mathbb{P}V$  be the set of lines  $[x]$  with  $x_a \neq 0$ , i.e

$$U_a = \pi(V_a) = \pi(A_a) = \pi \circ \iota_a(k^n),$$

the image of the bijection  $\pi \circ \iota_a: k^n \rightarrow U_a$ . The inverse of this bijection is the chart  $(U_a, \varphi_a)$  on  $\mathbb{P}V$ :

$$\varphi_a := (\pi \circ \iota_a)^{-1}: U_a \rightarrow k^n.$$

As the reader can check, the transition map  $\varphi_1^{-1} \circ \varphi_0$  is

$$y = \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} x_1^{-1}.$$

Similarly for all other transition maps: the transition maps are smooth. We have covered the projective space by a finite set of charts.

We generalize the construction of the charts somewhat. An *affine map* is a translate  $x \mapsto Mx + b$  of a linear map  $x \mapsto Mx$ . Take an injective affine map  $\iota: k^n \rightarrow V$ , not striking the origin. The set  $U := \pi(\iota(k^n))$  is an *affine cell*. The map  $\pi \circ \iota: k^n \rightarrow U$  is an *affine parameterization*. The inverse of an affine parameterization is an *affine chart*. Elementary linear algebra will convince the reader that every affine chart is a bijection, and that any two affine charts whose images are not equal are, after perhaps composing each with an affine bijection of  $k^n$ , and picking a suitable basis of  $V$ , precisely the charts  $\varphi_0, \varphi_1$  above. Hence affine charts have smooth transition maps.

Any two points  $[x], [y] \in \mathbb{P}V$  have their vectors  $x, y$  lying in the image of the same injective affine map not striking the origin, so lie in the same affine chart. By the same argument as for the projective plane, projective space  $\mathbb{P}V$  is a manifold of dimension  $dn$  where  $V$  has dimension  $n+1$  as a vector space over  $k$ , and  $k$  has dimension  $d$  as a vector space over  $\mathbb{R}$ .

## Grassmannians

Linear systems of equations often contain a parameter. The space of solutions is a linear subspace depending on the parameter, so we need to think about the notion of a smoothly varying linear subspace. Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take a finite dimensional  $k$ -vector space  $V$ . The *Grassmannian*  $\text{Gr}_p V$  is the set of all  $p$ -dimensional linear subspaces of  $V$ . Note that the projective space is  $\mathbb{P}V = \text{Gr}_1 V$ .

### Charts on the Grassmannian

Take a  $k$ -vector space  $V$  and split it into a sum of subspaces  $V = P_0 \oplus Q_0$ , of dimensions  $p$  and  $q$ . Write points of  $V$  as  $(x, y)$  with  $x \in P_0$  and  $y \in Q_0$ . Intuitively, we picture that “most” linear subspaces of dimension  $p$  in  $V$  are graphs  $y = Ax$  for some linear map  $A: P_0 \rightarrow Q_0$ . Define a chart  $(U, \varphi)$  on  $\text{Gr}_p V$  by taking  $U \subset \text{Gr}_p V$  to be the set of all  $p$ -dimensional linear subspaces  $P$  which are graphs of the form  $y = Ax$ , and taking  $\varphi(P) = A$ , or more precisely let  $\varphi(P)$  be the matrix of  $A$  in a basis of  $P_0$  and  $Q_0$ . This  $A$  is (in that basis) a matrix, so we are just writing out its entries.

Let’s make an atlas of charts. Just change the choice of  $P_0$  and  $Q_0$  and use the same chart again. We want to see the transition maps. Take an invertible linear transformation

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and write down variables related by  $g$ :

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Each linear subspace  $\bar{y} = \bar{A}\bar{x}$  becomes  $y = Ax$  where

$$A = (d - \bar{A}b)^{-1}(\bar{A}a - c),$$

and vice versa, as is easy to check. This is the transition map between our two charts  $y = Ax$  and  $\bar{y} = \bar{A}\bar{x}$ , defined on any  $\bar{A}$  for which  $d - \bar{A}b$  is invertible. Since  $g$  is invertible,  $b$  and  $d$  have enough linearly independent rows to ensure that  $d - \bar{A}b$  is invertible for a nonempty open set of matrices  $\bar{A}$ .

**Theorem 2.2.** *The Grassmannian  $\text{Gr}_p V$  of  $p$ -dimensional linear subspaces of a  $(p + q)$ -dimensional vector space  $V$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  is a connected manifold of dimension  $npq$  where  $n$  is the dimension of  $k$  over the real numbers.*

Note that  $k$  has dimension 1, 2 or 4 over the real numbers, if  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ .

*Proof.* Take any basis. Every linear subspace is the graph of some variables as functions of other variables. So we only need to consider  $g$  above to be a permutation matrix, in order to give rise to charts that cover all permutations of the variables, giving an atlas. We only need finitely many charts, for the finitely many permutations of the variables. We computed the transition maps. On the other hand, we could allow arbitrary invertible matrices  $g$ . Any two  $p$ -dimensional linear subspaces both lie in the domain of a single chart, by taking a suitable such matrix  $g$  so that both subspaces become graphs simultaneously.  $\square$

Take  $V = k^4$  with standard basis  $e_0, e_1, e_2, e_3$ . Look at the charts on  $\mathbb{P}^3 = \mathbb{P}V$ . Write the span of any vector  $x \in k^4$  as  $[x]$ . We can take as a 1-dimensional subspace  $P_0 := [e_2]$  and  $Q_0 = \langle e_0, e_1, e_3 \rangle$ . Each 1-dimensional subspace graphed over  $P_0$  is expressed as the graph of a linear map taking  $e_2$  to some vector  $x \in Q_0$ , say  $x_0e_0 + x_1e_1 + x_3e_3$ , so the graph is the span of

$$\begin{pmatrix} x_0 \\ x_1 \\ 1 \\ x_3 \end{pmatrix}.$$

The chart is

$$\begin{bmatrix} x_0 \\ x_1 \\ 1 \\ x_3 \end{bmatrix} \mapsto \begin{pmatrix} x_0 \\ x_1 \\ 1 \\ x_3 \end{pmatrix}.$$

We change chart to the chart with  $P_0 = \langle e_3 \rangle$  by just rescaling by  $x_3$ :

$$\begin{bmatrix} x_0 \\ x_1 \\ 1 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_0x_3^{-1} \\ x_1x_3^{-1} \\ x_3^{-1} \\ 1 \end{bmatrix}.$$

So the transition map between those charts is

$$\begin{pmatrix} x_0 \\ x_1 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_0x_3^{-1} \\ x_1x_3^{-1} \\ x_3^{-1} \end{pmatrix}.$$

**2.9** Work out all of the charts and all of the transition maps for  $\text{Gr}_2\mathbb{R}^4$ , taking as  $P_0$  and as  $Q_0$  any two complementary subspaces spanned each by two standard basis vectors.

## Chapter 3

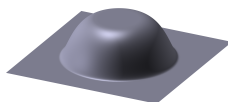
### Functions on manifolds

#### Constructing functions

A  $C^k$  function  $f: M \rightarrow \mathbb{R}$  on a manifold  $M$  is a continuous map so that, for any  $C^k$  chart  $(U, \varphi)$  on  $M$ , the map  $f \circ \varphi^{-1}$ , defined on an open set in Euclidean space, is  $C^k$ ,  $k \geq 0$ . A *smooth* function is one which is  $C^k$  for all  $k$ , also denoted  $C^\infty$ . The *support* of a function  $f$  is the closure of the set of points where  $f \neq 0$ .

Take manifold  $M$ , a point  $m_0 \in M$ . A *bump function* around  $m_0$  is a smooth function  $h: M \rightarrow \mathbb{R}$  with compact support so that

- $0 \leq h \leq 1$  and
- $h = 1$  near  $m_0$ .



**3.1** Prove that, for any point  $x_0 \in U$  and open set  $U \subseteq \mathbb{R}^n$ , there is a bump function around  $x_0$  supported in  $U$ .

**Lemma 3.1.** *Manifolds have bump functions, near any point, supported in any open set.*

*Proof.* Take a chart  $(W, \varphi)$  so that  $m_0 \in W$ . Restricting  $\varphi$  to  $W \cap U$  and replacing  $W$  by  $W \cap U$ , we can assume that  $W \subset U$ . Let  $x_0 := \varphi(m_0)$ . Pick a bump function: a smooth function  $h: \mathbb{R}^n \rightarrow \mathbb{R}$  so that  $0 \leq h \leq 1$  and  $h$  is equal to 1 in some open set containing  $x_0$  and  $h$  is supported on a compact set  $K \subseteq \varphi(W)$ . Let

$$f(m) := \begin{cases} h(\varphi(m)), & \text{if } m \in W, \\ 0, & \text{if } m \notin W. \end{cases}$$

On  $W$ ,  $f$  is smooth because  $h$  and  $\varphi$  are smooth. Each point outside  $W$  lies outside the set  $\varphi^{-1}K$ , which is compact because  $\varphi$  is a homeomorphism. Compact sets are closed, because  $M$  is Hausdorff; see problem 1.27 on page 11. So each point outside  $W$  lies in an open set outside  $\varphi^{-1}K$ , i.e. lies in an open set where  $f = 0$ . So  $f$  is smooth.  $\square$

**3.2** Prove that a set with smooth structure is a manifold just when, for any two points, there is a smooth function taking different values at those points, and there is a sequence of smooth functions, so that for any point, and open set around that point, there is a function in the sequence, positive at that point and vanishing outside that open set.

### Charts versus coordinates

Take a chart  $(U, \varphi)$  on a manifold  $M$ . To each point  $m \in U \subseteq M$ , associate the numbers

$$x_1(m), x_2(m), \dots, x_n(m)$$

by the equation

$$\varphi(m) = (x_1(m), x_2(m), \dots, x_n(m)).$$

These functions  $x_1, x_2, \dots, x_n: U \rightarrow \mathbb{R}$  are called the *coordinates* of the point  $m$  in the chart  $\varphi$ . Mathematicians (and people who use mathematics) prefer not to write about charts, but instead to say “suppose that  $x_1, x_2, \dots, x_n$  are coordinates around a point of  $M$ ”, and this is what they mean.

### Maps

A  $C^k$  map of manifolds  $\Phi: P \rightarrow Q$  is a continuous map so that, for any charts  $(U, \varphi)$  on  $P$  and  $(W, \psi)$  on  $Q$ , the map  $\psi \circ \Phi \circ \varphi^{-1}$ , mapping between open sets of Euclidean space, is  $C^k$ ,  $k \geq 0$ . A *smooth* map is one which is  $C^k$  for all  $k$ , also denoted  $C^\infty$ .

**3.3** Denote by  $S^1$  the unit circle in the plane, centered at the origin. Using only elementary trigonometry, prove that every  $C^k$  map  $\varphi: S^1 \rightarrow S^1$  arises as  $\varphi(e^{i\theta}) = e^{i\Phi(\theta)}$  for a  $C^k$  map  $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ , unique up to adding a constant  $2\pi n$  for an arbitrary integer  $n$ . We will refer to  $\Phi$  as an *angle function* of  $\varphi$ . Prove that the *degree*

$$\frac{\Phi(2\pi) - \Phi(0)}{2\pi}$$

is an integer which depends only on  $\varphi$ , not on the choice of angle function  $\Phi$ .

**3.4** If  $k, K = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , and  $k \subset K$ , suppose that  $V_K$  is a finite dimensional  $K$ -vector space, and let  $V_k$  be the same vector space  $V_K$ , but thought of as a  $k$ -vector space. The *Hopf fibration* is  $h: \mathbb{P}V_k \rightarrow \mathbb{P}V_K$ , defined by taking the  $k$ -span of a vector  $v$  to its  $K$ -span. Prove that the Hopf fibration is smooth.

For an finite dimensional vector space  $V$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , the map

$$\pi: V - \{0\} \rightarrow \mathbb{P}V$$

is expressed in our charts as

$$x \mapsto x_a^{-1}(x_1, \dots, x_{a-1}, x_{a+1}, \dots, x_n),$$

so is smooth. Pick an inner product on  $V$ . The unit sphere  $S \subset V$ , consisting of vectors of unit length, is compact. For any  $[x] \in \mathbb{P}V$ , the vector  $x$  can be rescaled without changing  $[x]$ , to have unit length, i.e. to lie in  $S$ . Hence  $\pi|_S: S \rightarrow \mathbb{P}V$  is a continuous surjective map from a compact set, so  $\mathbb{P}V$  is compact.

**3.5** Suppose that a manifold  $M$  is a product  $M = P \times Q$ . A point  $m_0 \in M$  is precisely a pair  $(p_0, q_0)$  of points  $p_0 \in P$ ,  $q_0 \in Q$ . So  $M$  has two obvious maps  $\pi_P: M \rightarrow P$  and

$\pi_Q: M \rightarrow Q$ , the *projections*, defined by  $\pi_P(p_0, q_0) = p_0$  and  $\pi_Q(p_0, q_0) = q_0$ . For any manifold  $S$ , every map  $\varphi: S \rightarrow M$  is expressed as

$$\varphi(x) = (\pi_P \circ \varphi(x), \pi_Q \circ \varphi(x)),$$

and conversely any two maps  $\psi: S \rightarrow P$  and  $\chi: S \rightarrow Q$  give a unique map

$$\varphi(x) = (\psi(x), \chi(x)),$$

with these *components*. For any manifold  $S$ , prove that a map  $\varphi: S \rightarrow M$  is smooth just when its components are.

**3.6** Prove that every homeomorphism  $\varphi: S \rightarrow S$  of a sphere  $S$  in Euclidean space extends to a homeomorphism  $\varphi: \bar{B} \rightarrow \bar{B}$  of the closed ball that the sphere bounds.

### More examples: covering spaces

A continuous map  $f: X \rightarrow Y$  of topological spaces *evenly covers* an open set  $U_Y \subset Y$  if  $f^{-1}U_Y \subset X$  is a disjoint union of open sets mapped homeomorphically to  $U_Y$  by  $f$ , called *sheets*. The map  $f: X \rightarrow Y$  is a *covering map* if  $Y$  has an open cover by evenly covered open sets. A *covering space* of a topological space  $Y$  is a topological space  $X$  equipped with a covering map  $f: X \rightarrow Y$ .

**3.7** Suppose that  $f: X \rightarrow Y$  is a covering map and that  $Y$  is a manifold. Suppose further that each stalk  $X_y = f^{-1}\{y\}$  of  $f$  is a countable set, for each  $y \in Y$ . Prove that there is a unique smooth structure on  $X$  so that  $f$  is a smooth map of manifolds.

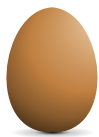
A *local diffeomorphism* is a smooth map  $\varphi: P \rightarrow Q$  of manifolds, so that  $P$  is covered by open sets  $P_a \subset P$  on which every map  $\varphi|_{P_a}: P_a \rightarrow \varphi(P_a) \subseteq Q$  is a diffeomorphism.

**Corollary 3.2.** *Every countable component covering space of any manifold is a manifold, for a unique smooth structure compatible with its topology, so that the covering map is a local diffeomorphism.*

*Proof.* Clearly it is enough to prove the result for each connected component of the covering space, which is then also a covering space. We can assume the target of the covering map is connected. A path connected topological space with a countable basis of simply connected open sets has countable fundamental group [13]. Hence the fundamental group of any connected manifold is countable. The open sets in any countable basis of open sets of the manifold have preimages a countable basis of open sets of the universal covering space: one for each element of the fundamental group. So the universal covering space has a countable basis of open sets. Any connected covering space is a quotient of the universal covering space, so also has a countable basis of open sets. The charts compose with the covering map to give charts for a smooth structure; these must be smooth if the covering map is to be smooth, and so this must be the smooth structure.  $\square$

(For the reader who is still interested in manifolds in the broad sense: the same proof shows that the universal covering space of any connected manifold in the broad sense  $M$  is a manifold in the broad sense  $\tilde{M}$ , so that the covering map  $\tilde{M} \rightarrow M$  is a local diffeomorphism. In this respect, manifolds in the broad sense are superior, but in other respects inferior.)

**3.8** Prove that the fundamental group of any compact connected manifold is finitely generated.



A surface from outside



A Möbius strip

## Orientation

You can travel along a curve either of two ways: . A choice of one of these directions is called an *orientation*. A surface in  $\mathbb{R}^3$  has an inside and an outside. Actually, that isn't quite true: the Möbius strip has only one side. If a surface has two sides, a choice of side is called an *orientation*.

A diffeomorphism  $F$  between open sets of  $\mathbb{R}^n$  is *orientation preserving* if  $\det F' > 0$ . Two charts  $\varphi$  and  $\psi$  on a manifold are *compatibly oriented* if the associated transition map  $\psi \circ \varphi^{-1}$  is orientation preserving where defined. An *orientation* of a positive dimensional manifold  $M$  is a choice of a maximal collection of compatibly oriented charts, whose domains cover  $M$ . A *local orientation* at a point  $m_0 \in M$  is a choice of chart defined near  $m_0$ , but only defined up to being replaced by a chart which is compatibly oriented near  $m_0$ . A 0-dimensional manifold is a collection of isolated points. The notion of orientation doesn't really make sense on a 0-dimensional manifold, so, by definition, an *orientation* of a 0-dimensional manifold is just a choice of  $\pm$  sign for each point.

Consider the unit sphere  $S^2 \subset \mathbb{R}^3$ , the set of vectors  $(x, y, z)$  so that

$$x^2 + y^2 + z^2 = 1.$$

Define a chart, on the open set where  $z > 0$ , by projecting linearly to the point  $(x, y)$  given by dropping the  $z$  coordinate: the shadow on the horizontal plane. We can do the same where  $z < 0$ , giving another chart, but now keeping  $(y, x)$  instead of  $(x, y)$ . We smoothly recover the point  $p$  by the equation

$$z = \sqrt{1 - x^2 - y^2},$$

for  $z > 0$ , and

$$z = -\sqrt{1 - x^2 - y^2},$$

for  $z < 0$ . Similarly, on the set where  $x > 0$  we have a chart given by projection to  $(y, z)$ . The transition map to this chart from the chart for  $z > 0$  is

$$(x, y) \mapsto (y, z) = (y, \sqrt{1 - x^2 - y^2}).$$

Check that this is orientation preserving by differentiating. Similarly, by cyclic permutation of  $(x, y, z)$  in these equations (but not just any permutation: only cyclic ones), we can produce a complicated system of charts, with orientation preserving transition maps. The same ideas work in all dimensions: the unit sphere in Euclidean space is oriented.

Return to Ptolemaic projection on the sphere  $S^n$ , which the reader worked



out: take any point

$$\begin{pmatrix} a \\ b \end{pmatrix} \in S^n \subset \mathbb{R}^{n+1}$$

with  $a \in \mathbb{R}^n, b \in \mathbb{R}$  with  $b \neq 1$ , and map

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto x = \frac{a}{1-b}.$$

Recall that

$$\begin{pmatrix} 2x \\ -1 \end{pmatrix}$$

is the Ptolemaic projection to the plane ( $b = -1$ ) tangent to sphere at the south pole. When we use the same picture to carry out Ptolemaic projection “upside down” to the plane ( $b = 1$ ) tangent to the sphere at the north pole, that plane is “upside down”, so we will always prefer to reflect that plane in the last coordinate, giving a chart

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto y = \frac{\bar{a}}{1+b}$$

where if

$$a = (a_1, \dots, a_{n-1}, a_n)$$

then  $\bar{a}$  means

$$\bar{a} = (a_1, \dots, a_{n-1}, -a_n)$$

or any other linear reflection mapping.

**3.9** Check that the transition map is now

$$y = \bar{x}/|x|^2$$

and that these charts are compatibly oriented.

If  $P$  and  $Q$  are oriented manifolds then  $P \times Q$  is oriented, by taking as positively oriented charts the products of maps, i.e. if  $(U, \varphi)$  is a positively oriented chart on  $P$  and  $(W, \psi)$  is one on  $Q$ , then

$$(\varphi, \psi): (p, q) \in U \times W \mapsto (\varphi(p), \psi(q)) \in \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q}$$

is a positively oriented chart on  $P \times Q$ , where  $p, q$  are the dimensions of  $P, Q$ . N.B. the orientation of  $Q \times P$  does *not* match that of  $P \times Q$  unless one of  $P, Q$  has even dimension. To see this, note that we could permute the coordinates of  $\mathbb{R}^{p+q}$  one at a time, to get from one chart to the other. The determinant of the matrix of such a permutation is  $-1$ , so to permute all variables is  $pq$  permutations.

**3.10** Suppose that  $M$  is a manifold covering by oriented open sets  $M_a \subset M$ . Suppose that the overlaps  $M_a \cap M_b$  are covered by some path connected open sets  $M_\alpha \subset M$ . Imagine a graph  $\Gamma$ , with one vertex  $x_a$  for each open set  $M_a$ , and one edge  $e_{a\alpha b} = e_{b\alpha a}$

from  $x_a$  to  $x_b$  if  $M_\alpha$  contains a point of  $M_a \cap M_b$ . On that edge, write the label  $+1$  if  $M_a$  and  $M_b$  have the same orientation near some point of  $M_\alpha$ ,  $-1$  otherwise.

- a. Prove that this sign is independent of which point of  $M_\alpha$  we choose.
- b. Give examples to (not rigorously) show that every finite graph  $\Gamma$ , with any choice of  $\pm 1$  edge labels, arises in this way for some compact surface  $M$ .
- c. Pick some examples of manifolds, each with a covering by some oriented open sets, and compute their associated graphs for various orientations.
- d. Explain why, if we change the orientation of some  $M_a$ , we change all signs of the edges coming out of  $x_a$ .
- e. Suppose that the graph is countable. Prove that  $M$  is orientable if and only if we can pick vertices  $x_1, x_2, \dots$ , and change the signs of all edges coming out of  $x_1$ , and then of  $x_2$ , and so on, until all signs turn to 1.
- f. Prove that if  $M$  is orientable, then the product of the  $\pm 1$  on any loop in  $\Gamma$  is 1. Hence the group morphism  $\pi_1(\Gamma) \rightarrow \{\pm 1\}$ , giving by multiplying around a loop, maps to  $\{1\}$ .
- g. Prove that, if there are precisely two open sets  $M_a, M_b$  and any finite number of  $M_\alpha$ , then  $M$  is orientable just when all edge signs are the same.
- h. Prove that, if there are precisely three open sets  $M_a, M_b, M_c$  and three open sets  $M_\alpha, M_\beta, M_\gamma$ , then  $M$  is orientable just when: (a) the graph is a line, or (b) the graph is a triangle and there are zero or two  $-1$  signs.
- i. Prove that  $M$  is orientable just when that group morphism  $\pi_1(\Gamma) \rightarrow \{\pm 1\}$  maps to  $\{1\}$ .

This last problem shows that simply connected manifolds are orientable. The unit sphere  $S^n \subset \mathbb{R}^{n+1}$  in Euclidean space is simply connected, so is orientable.

**3.11\*** Suppose that  $M$  is a manifold. Let  $\hat{M}$  be the set of all choices of pairs  $(m, o)$  where  $m \in M$  and  $o$  is a local orientation at  $m$ . Give  $\hat{M}$  the structure of a manifold so that  $\hat{M} \rightarrow M$  is a 2-to-1 local diffeomorphism and covering map. Explain why  $\hat{M}$  is oriented. Suppose that  $M$  is connected. Prove that  $\hat{M}$  is connected if and only if  $M$  is not orientable.

**Lemma 3.3.** *Suppose that  $M$  is an unorientable manifold. Then there is an orientable manifold  $\hat{M}$  and a two-to-one covering map  $p: \hat{M} \rightarrow M$  so that, near each point of  $m$ , the two sheets map to  $M$  with opposite orientations. If  $M$  is connected, then  $\hat{M}$  is connected. Any local diffeomorphism  $\varphi: N \rightarrow M$  from an oriented manifold  $N$  factors  $\varphi = p \circ \hat{\varphi}$  for a unique orientation preserving local diffeomorphism  $\hat{\varphi}: N \rightarrow \hat{M}$ . In particular, if there are two two-to-one covering maps  $p_1: M_1 \rightarrow M$  and  $p_2: M_2 \rightarrow M$  by oriented manifolds then there is a unique orientation preserving diffeomorphism  $\varphi: M_1 \rightarrow M_2$  so that  $p_1 = p_2 \circ \varphi$ .*

*Proof.* For most of the proof, see problem 3.11. We take a point  $n_0 \in N$  and map it to the point  $\hat{m}_0 = \hat{\varphi}(n_0)$  with the same image in  $M$  and the same nearby orientation. In local coordinates pulled back from  $M$ , this is the identity map, so smooth.  $\square$

## More examples: quotients

An *action* of a group  $\Gamma$  on a manifold  $M$  is a map associating to each  $g \in \Gamma$  a smooth map  $M \rightarrow M$  denoted  $x \mapsto gx$  or sometimes denoted  $x \mapsto g \cdot x$ , so that  $g(hx) = (gh)x$  for any  $g, h \in \Gamma$  and  $x \in M$  and so that  $1x = x$  for any  $x \in M$ . The action is *free* if for any  $x \in M$ , the only element  $g \in \Gamma$  for which  $gx = x$  is  $g = 1$ . The *orbit* of a point  $x \in M$  is the set  $\Gamma x := \{gx \mid g \in \Gamma\}$ . The *quotient space*  $M/\Gamma$  of the group action is the set of orbits, with the *quotient map*  $x \in M \mapsto \Gamma x \in M/\Gamma$ . A group action is *proper* when any two points of  $M$  from different orbits lie in disjoint  $\Gamma$ -invariant open sets.

**Lemma 3.4.** *If a group  $\Gamma$  acts freely and properly on a manifold  $M$ , then the quotient space  $\bar{M} := \Gamma \backslash M$  has a unique manifold structure so that the quotient map  $M \rightarrow \bar{M}$  is a local diffeomorphism and a covering map.*

*Proof.* The reader experienced in topology will already know that  $\bar{M}$  is a Hausdorff topological space [13]. We want to prove that the action is a covering action. By properness, any point lies in an open set containing only finitely many points of its orbit. We can make a smaller open set containing none of the other points of the orbit. Since the action is free, the action is a covering action and  $M \rightarrow \bar{M}$  is a covering map.

For each point  $\bar{m} \in \bar{M}$ , pick an open set  $\bar{U} \subset \bar{M}$  around it, evenly covered by sheets which are translates of one another. Shrink  $\bar{U}$  if needed so one of the sheets  $U$  is the domain of a chart  $(U, \varphi)$ . Compose  $\varphi$  with the inverse of  $U \rightarrow \bar{U}$  as a chart on  $\bar{M}$ ; take all such charts to give an atlas on  $\bar{M}$ . An overlap of two charts in  $\bar{M}$  corresponds to an overlap of two of the sheets in  $M$ , on which the charts are smooth charts for  $M$ , so overlap smoothly. Each open subset of  $M$  is a union of subsets  $U$  which arise as sheets over evenly covered open sets  $\bar{U} \subset \bar{M}$ , making a basis of open sets. These give open sets  $\bar{U}_m \subset \bar{M}$ . Hence the image of any open set in  $M$  is open in  $\bar{M}$ . In particular, any countable basis of open sets for  $M$  induces one for  $\bar{M}$ . Thus  $\bar{M}$  is a manifold and the quotient map  $M \rightarrow \bar{M}$  is a smooth local diffeomorphism, via the agreement of the charts.  $\square$

**3.12** Explain why the set of triangles in the plane is a 6-dimensional manifold.

**3.13** Suppose that  $M$  is a manifold and a metric space, sharing the same topology, and that a group  $\Gamma$  acts freely on  $M$  by isometries of the metric. Prove that  $\Gamma$  acts on  $M$  properly if and only if all of the orbits are discrete. Prove that if this occurs, then there is a unique smooth structure on  $M/\Gamma$  and a unique metric on  $M/\Gamma$  for which the quotient map  $M \rightarrow M/\Gamma$  is a locally isometric covering map and a local diffeomorphism.

**3.14** Suppose that a finite group  $\Gamma$  acts smoothly and freely on an oriented connected manifold  $M$ . Prove that  $\Gamma \backslash M$  is orientable just when every  $\gamma \in \Gamma$  preserves orientation.

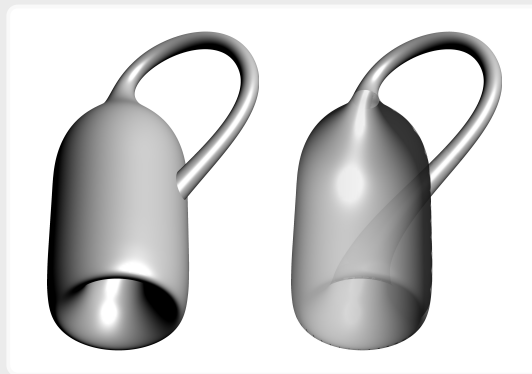
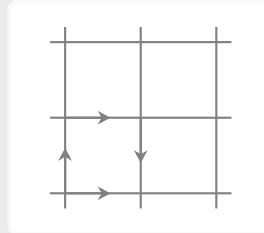
The *Klein bottle* is the quotient  $\mathbb{R}^2/\Gamma$  by the rigid motions  $\Gamma$  of the plane generated by the two transformations

$$(x, y) \mapsto (x, y + 1)$$

and

$$(x, y) \mapsto (x + 1, -y).$$

gluing together opposite sides of unit squares like:



The Klein bottle is a surface (a manifold of dimension 2), with locally isometric and locally diffeomorphic universal covering map from the plane, as we can see immediately from problem 3.13 on page 29.

**3.15** Roughly, where are the lines and arrows of that square on the two pictures of the Klein bottle?

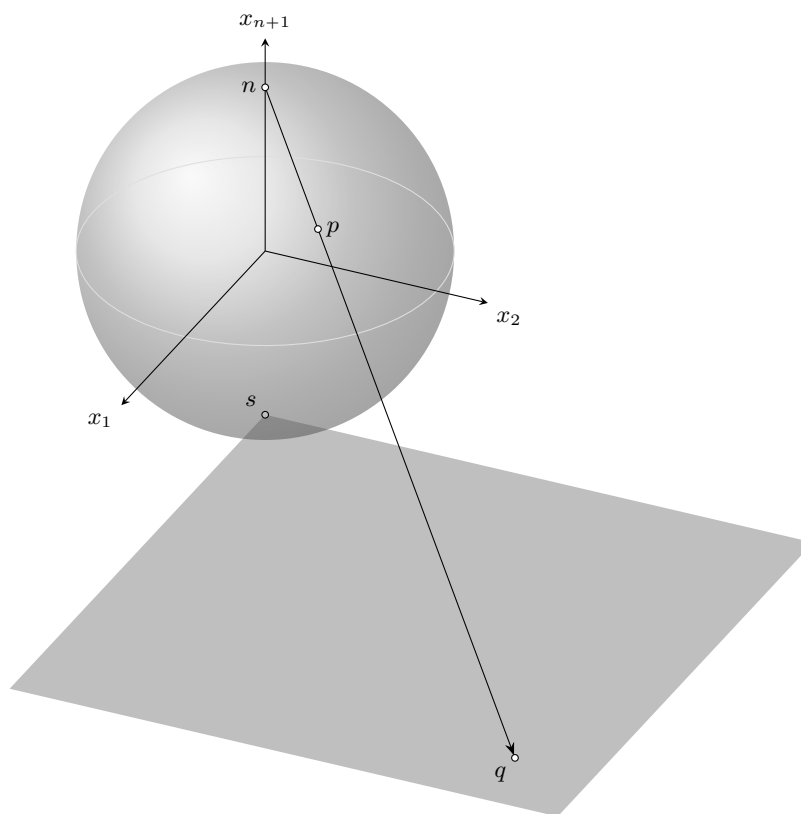
The torus  $T^n = \mathbb{R}^n / \mathbb{Z}^n$  is the quotient of the  $\mathbb{R}^n$  by the free action of  $\mathbb{Z}^n$ . The quotient map  $\mathbb{R}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$  is covering map.

The real projective plane  $\mathbb{RP}^2$  is the quotient of the unit sphere  $S^2 \subset \mathbb{R}^3$  by  $\{\pm 1\}$ . The quotient map  $S^2 \rightarrow \mathbb{RP}^2$  is a covering map.

Let  $M := \mathbb{C}$ , let  $w := e^{2\pi i/3}$  and let  $\Gamma$  be the set of complex numbers of the form  $2a + 2bw + 2cw^2$ , for any integers  $a, b, c$ , acting by addition on  $M$ . The quotient binds together the sides of the regular hexagon bounded by the equations on real parts:  $-1 < \operatorname{Re} z < 1$ ,  $-1 < \operatorname{Re}(wz) < 1$  and  $-1 < \operatorname{Re}(w^2z) < 1$ . The quotient is a flat torus, the *hexagonal torus* but with symmetry by the group of rotations  $\Gamma_1 := \{1, w, w^2\}$  acting by multiplication.

**3.16\*** Let  $M \subset \mathbb{R}^6$  be the set of pairs  $(u, v)$  so that  $u, v \in \mathbb{R}^3$  and  $|u| = 1$  and  $\langle u, v \rangle = 0$ . Prove that  $M$  is an embedded submanifold of  $\mathbb{R}^6$ . To each point  $(u, v) \in M$  associate the line  $\ell \subset \mathbb{R}^3$  which is the set of points  $v + tu$  for  $t \in \mathbb{R}$ . This line is oriented, pointing in the direction  $u$ . Prove that the point  $v$  is the unique point of  $\ell$  closest to the origin. Prove that the map  $(u, v) \mapsto \ell$  is a bijection to the space of oriented lines. Explain how to quotient  $M$  by a group action to get the space of (unoriented) lines to be a manifold.

**3.17** Suppose that some finite group acts on some Hausdorff space by homeomorphisms. Prove that it acts properly. Use this to prove that any smooth free action of a finite group on a manifold has a manifold quotient and the quotient map is a smooth covering map and a local diffeomorphism



Returning to Ptolemaic projection in  $\mathbb{R}^{n+1}$ , the  $n$ -dimensional sphere punctured at north and south pole is identified with  $\mathbb{R}^n - \{0\}$ . Identify opposite points of the twice punctured sphere to get the once punctured real projective space. In our Ptolemaic projection, the punctured real projective space is the punctured Euclidean space but with “opposite” points identified, i.e.  $x \sim y$  where  $y = -x/|x|^2$ , hence a quotient of the twice punctured sphere by a smooth free action of the group with two elements. For  $n = 2$  (as in the picture), taking polar coordinates  $r, \theta$  on the plane, and then writing  $r$  as  $r = e^u$ , we have coordinates  $u, \theta$ . The group action takes  $u, \theta$  to  $-u, \theta + \pi$ . The quotient is the *Möbius strip*. So the Möbius strip is the real projective plane punctured at one point, but we can also draw it as something like:



Take  $n$  vectors in  $\mathbb{R}^n$ . Put each vector into a column of a matrix. The resulting matrix is orthogonal if and only if the vectors form an orthonormal basis. Hence the set of all ordered orthonormal bases of  $\mathbb{R}^n$  is identified with the set of  $n \times n$  orthogonal matrices, which we have seen is a manifold (see problem 2.6 on page 18). The unordered orthonormal bases form the quotient by the permutation group, which clearly acts smoothly on  $n \times n$  matrices, permuting their columns, freely because the vectors in a basis are all different, and properly because the set of orthogonal matrices is compact. So the space of orthonormal bases (unordered) is a manifold of dimension  $\binom{n}{2} = n(n-1)/2$ .

**3.18** Denote by  $\text{SO}(n)$  the set of all  $n \times n$  rotation matrices. Prove that  $\text{SO}(n)$  is a manifold.

**3.19** Find a finite group  $\Gamma$  acting smoothly and transitively on  $\text{SO}(n) \times \Lambda$  without fixed points, so that the quotient manifold is the set  $M$  of all symmetric matrices without multiple eigenvalues. Use this to prove the *smooth spectral theorem*: if  $g: P \rightarrow M$  is a smooth map of a connected and simply connected smooth manifold  $P$ , associating to each point  $p \in P$  a symmetric matrix  $g(p)$  without multiple eigenvalues, then there are smooth functions on  $P$  giving the eigenvalues of  $g(p)$  and smooth vector-valued functions giving a basis of unit length eigenvectors.

## Commutative diagrams

A *commutative diagram* is a collection of maps, drawn with arrows, like

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow h \\ Z & \xrightarrow{k} & W \end{array}$$

so that, if you travel along one path, following the direction of the arrows, and I travel along another, and we start and end at the same place, then the compositions of maps we encounter are the same. In our example, this says that  $h \circ f = k \circ g$ , for instance.

## Gluing manifolds

This section can be omitted without loss of continuity.

Picture again the line with doubled point on page 9.



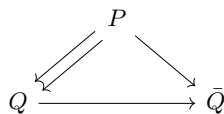
We glue together two manifolds (both copies of the real number line) along open subsets: the nonzero real numbers. The resulting space has a double

point: points on each manifold, in the boundary of the open sets, which “should” be glued into one point.

Picture the sphere  $M$ . Let  $M_0, M_1, M_{01}$  be  $M$  punctured at the north pole, the south pole, and both poles. Gluing together  $M_0$  and  $M_1$  along  $M_{01}$ , we don’t have such a double point, and we recover  $M$ . The north pole is glued on far away from the south pole.

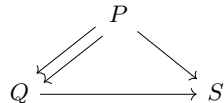
A *gluing plan* is a pair of manifolds  $P, Q$  and two maps of  $P$  to  $Q$ , which we denote by  $P \rightrightarrows Q$ . A *gluing* of that gluing plan is a manifold  $\bar{Q}$  with smooth maps  $P \rightarrow \bar{Q}$  and  $Q \rightarrow \bar{Q}$  so that

•

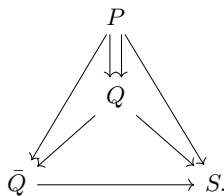


commutes (in other words, if we start at any point of  $P$ , we can follow either map from  $P$  to  $Q$ , and then the one from  $Q$  to  $\bar{Q}$ , or the one from  $P$  to  $\bar{Q}$ , and arrive at the same point of  $\bar{Q}$ ) and

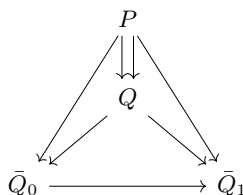
- for any manifold  $S$  and smooth maps  $P \rightarrow S$  and  $Q \rightarrow S$ , if



commutes then there is a unique smooth map  $\bar{Q} \rightarrow S$  making commute



**3.20** Prove that any two gluings  $\bar{Q}_0, \bar{Q}_1$  of a gluing plan  $P \rightrightarrows Q$  agree up to a unique diffeomorphism making commutative



The *equivalence relation* of a gluing plan is the one generated by asking each point of  $P$  to be equivalent to its two images under the two maps. An open subset  $U \subseteq Q$



is *invariant* for the gluing plan if whenever  $U$  contains some point, it contains all equivalent points. Let  $\bar{Q}$  be the quotient of  $P \sqcup Q$  by the equivalence relation, with quotient map

$$\pi: P \sqcup Q \rightarrow \bar{Q},$$

with the quotient topology.

**3.21** Prove that the invariant open sets in  $Q$  are precisely the preimages in  $Q$  of the open sets in  $\bar{Q}$  and the open sets in  $\bar{Q}$  are precisely the images of the invariant open sets in  $Q$ .

**3.22** Suppose that the maps  $P \rightrightarrows Q$  are open maps. Prove that the quotient topology on  $\bar{Q}$  has a countable basis of open sets.

A *double point* of the gluing plan is a pair of points  $q_0, q_1 \in Q$ , one on the boundary of the image of one of the the maps  $P \rightrightarrows Q$  and the other on the boundary of the image of the other, so that for any disjoint open sets  $U_0, U_1 \subseteq Q$  with  $q_0 \in U_0$  and  $q_1 \in U_1$ , the preimages in  $P$  by the corresponding maps overlap.

**3.23** Suppose that a gluing plan has a double point. Prove that the quotient space  $\bar{Q}$  is not Hausdorff.

**Theorem 3.5.** *Take two manifolds  $P, Q$  of equal dimension and gluing plan  $P \rightrightarrows Q$  consisting of two embeddings with disjoint images. The following are equivalent:*

- *There are no double points.*
- *The quotient of the gluing plan, identifying points of  $Q$  if they are images of the same point of  $P$  under the two maps, is Hausdorff.*
- *The quotient of the gluing plan is a gluing.*

*Proof.* As above,  $\bar{Q}$  has a countable basis of open sets, but fails to be Hausdorff just when there are distinct points  $q_0, q_1 \in Q$  so that any invariant open sets  $U_0, U_1 \subseteq Q$  containing them overlap. Denote the maps  $P \rightrightarrows Q$  by  $\varphi_0, \varphi_1$ . Denote by  $P_0$  the image of  $\varphi_0$ , and by  $P_1$  the image of  $\varphi_1$ . If  $q_0 \notin \bar{P}_0 \cup \bar{P}_1$ , take an open set  $U_0$  around  $q_0$  with compact closure disjoint from  $\bar{P}_0 \cup \bar{P}_1$ , and not containing  $q_1$ . Then  $U_0$  and the complement of  $\bar{U}_0$  house off and are invariant. So we can assume that both  $q_0, q_1$  lie in the closure of the images of the maps.

Suppose they both lie in the image of a map, but of different maps. They are equivalent to points in the image of the same map, so we can replace them by these, without changing the invariant open sets that contain them, so they are still not housed off by invariant open sets. Suppose they both lie in the image of the same map  $P \rightarrow Q$ . The map is an embedding, so we can house their preimages in  $P$ , and take the images by both maps of the houses to get an invariant open set. Since the map images are disjoint, these invariant open sets are disjoint.

Suppose that they both lie on  $\partial P_0$  and not on  $\partial P_1$ . House them off by some disjoint open sets  $U_0, U_1$  in  $Q$ . Make those open sets smaller to avoid  $P_1$ , smaller still to avoid  $\bar{P}_1$ . Adding the equivalent points in  $P_1$ , we create disjoint invariant open sets. So we can suppose that  $q_0 \in \partial P_0$  and  $q_1 \in \partial P_1$ .

**3.24** Prove that  $q_0, q_1$  is a double point.

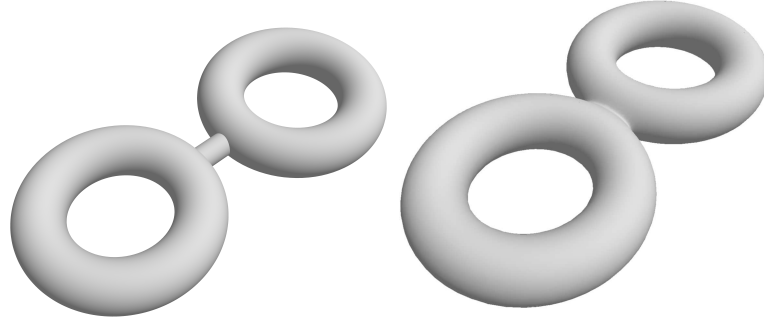
**3.25** Suppose that there are no double points. Prove that  $\bar{Q}$  is covered by open sets on which  $\pi$  is a local homeomorphism; construct charts on  $\bar{Q}$ .

**3.26** Suppose that there are no double points. Prove that  $\bar{Q}$  is a gluing. □

### Connected sums

This section can be omitted without loss of continuity.

Joining manifolds by a tube, then smoothing out, we form their connected sum.



Let us make this picture into rigorous mathematics. Take two oriented manifolds  $M_0, M_1$  of the same dimension. Let  $B$  be a ball of the same dimension as  $M_0$  and  $M_1$ . *Connected sum data* is a pair of orientation preserving embeddings

$$M_0 \leftarrow B \rightarrow M_1.$$

Let  $m_0 \in M_0$  and  $m_1 \in M_1$  be the images in  $M_0, M_1$  of the center of the ball  $0 \in B$ . Let  $B^\circ := B - \{0\}$ ,  $M_0^\circ := M_0 - \{m_0\}$  and  $M_1^\circ := M_1 - \{m_1\}$  the punctured manifolds. Denote by  $x \mapsto \bar{x}$  the reflection in one of the coordinates, an orientation reversing diffeomorphism mapping vectors near the unit sphere to other such vectors, and vectors near the origin to other such vectors. On the punctured ball let

$$\varphi(x) = (1 - |x|) \frac{\bar{x}}{|x|}.$$

So  $\varphi$  takes any vector with length near 1 to a vector of the reflected direction with length near zero, and vice versa. It would reverse orientation if we just scaled  $x$ , so we put in  $\bar{x}$  to reverse orientation again:  $\varphi$  preserves orientation. (More generally, we could take  $\varphi$  to be any orientation preserving proper diffeomorphism of the punctured ball which moves points near the unit sphere to points near the origin and vice versa; our gluing will still give us a manifold.) Composing our given map  $B \rightarrow M_1$  with the map  $\varphi: B^\circ \rightarrow B^\circ$  gives a map  $B^\circ \rightarrow M_1$ ; we only use this map as our map  $B^\circ \rightarrow M_1$  from here on. A *connected sum*  $M$  of  $M_0$  and  $M_1$  over this data is an oriented manifold  $M$  of the same dimension as  $M_0$  and  $M_1$  with smooth orientation preserving embeddings  $M_0^\circ \rightarrow M \leftarrow M_1^\circ$  so that

a. compositions agree on the punctured ball, i.e. the diagram

$$\begin{array}{ccc} B^\circ & \longrightarrow & M_0^\circ \\ \downarrow & & \downarrow \\ M_1^\circ & \longrightarrow & M \end{array}$$

commutes and

- b. for any manifold  $N$  and smooth maps for which

$$\begin{array}{ccc} B^\circ & \longrightarrow & M_0^\circ \\ \downarrow & & \downarrow \\ M_1^\circ & \longrightarrow & N \end{array}$$

commutes, there is a unique smooth map  $M \rightarrow N$  so that

$$\begin{array}{ccccc} B^\circ & \longrightarrow & M_0^\circ & & \\ \downarrow & & \downarrow & \searrow & \\ M_1^\circ & \longrightarrow & M & \searrow & \\ & & & \searrow & \\ & & & & N \end{array}$$

commutes.

**Theorem 3.6.** *Every connected sum data has a connected sum, unique up to diffeomorphism.*

*Proof.* Let

$$M := M_0^\circ \sqcup_{B^\circ} M_1^\circ$$

be the quotient with gluing data  $M_0^\circ, M_1^\circ, B^\circ$  where we map  $B^\circ \rightarrow M_0$  in the obvious way, but  $B^\circ \rightarrow M_1$  by first applying  $\varphi$  and then applying our given embedding of the ball to  $M_1$ . We have to check for double points. To simplify notation, we write  $B^\circ$  to denote our punctured ball and also its images in  $M_0$  and  $M_1$ . To prevent double points it suffices to find open sets  $U_0 \subseteq M_0^\circ$  and  $U_1 \subseteq M_1^\circ$  so that their intersections with  $B^\circ$  are disjoint. We take some compact set  $K \subset B$  containing a neighborhood  $U$  of the origin. Take  $U_0$  to be the complement of  $K$  in  $M_0^\circ$ . Take the complement  $L \subset B$  of  $\alpha(U)$  in the ball; so  $L$  is also compact. Then take  $U_1$  to be the complement of  $L^\circ$  in  $M_1^\circ$ . So there are no double points.  $\square$

**3.27** “Connected sum” a manifold with itself: take two orientation preserving embeddings  $M \leftarrow B \rightarrow M$  to the same manifold  $M$ , whose images lie inside disjoint compact sets in  $M$ . Suppose that  $B$  is the ball in Euclidean space of the same dimension as  $M$ . Prove that the quotient  $\bar{M}$ , known as  $M$  with a handle attached, is a smooth manifold, and any map  $\Phi: M \sqcup B \rightarrow N$  of manifolds arises as composition  $\Phi = \varphi \circ \pi$  from a unique smooth map  $\varphi: \bar{M} \rightarrow N$  if and only if  $\Phi$  takes each point of  $B$  to its two matched points in  $M$ .

## Notes

As we shall see, by Whitney’s theorem every smooth manifold admits an embedding as an embedded smooth manifold of Euclidean space. The Akubulut–King–Nash–Tognoli theorem tells us that every compact embedded submanifold  $M \subset \mathbb{R}^n$  is diffeomorphic to the set of solutions of a finite collection of polynomial equations in  $\mathbb{R}^{n+1}$ . For example, every smooth surface embeds in  $\mathbb{R}^5$ , and so every compact smooth surface embeds as the solution of polynomial equations in  $\mathbb{R}^6$ .



## Chapter 4

### Tangent vectors

We define the tangent vectors of a manifold, i.e. the velocities of particles moving on the manifold. We differentiate smooth maps between manifolds. We generalize some theorems of several variable calculus to manifolds.

#### Background material from several variable calculus

We recall some terminology on maps in Euclidean space [12]. Take a map  $f: A \rightarrow B$  between subsets  $A \subseteq \mathbb{R}^p$  and  $B \subseteq \mathbb{R}^q$ . Write  $f(x) = O(x)$  to mean that  $f(x) \rightarrow 0$  as  $x \rightarrow 0$  while  $x$  stays in  $A$ , and  $f(x) = o(x)$  to mean that  $f(x)/|x| \rightarrow 0$  as  $x \rightarrow 0$  while  $x$  stays in  $A$ . Recall that  $f$  is continuous at  $x$  if  $f(x+h) - f(x) = O(h)$ , and continuous, or  $C^0$ , if continuous everywhere in  $A$ . At any interior point  $x \in A$ , we try to define a linear map  $f'(x): \mathbb{R}^p \rightarrow \mathbb{R}^q$  by asking that

$$f(x+h) - f(x) = f'(x)h + o(h).$$

The reader will know (and can prove) that this uniquely determines  $f'(x)$ , if any such  $f'(x)$  exists, i.e. if  $f$  is *differentiable* at  $x$ . A map  $f$  is  $C^k$  if  $x \mapsto f'(x)$  is  $C^{k-1}$ ,  $C^\infty$  if  $C^k$  for all  $k$ .

A  $C^k$  (or  $C^\infty$ ) *diffeomorphism* is a  $C^k$  (or  $C^\infty$ ) map with a  $C^k$  (or  $C^\infty$ ) inverse. If  $f$  is  $C^1$  at  $x$ , then  $f'(x)$  is the matrix of partial derivatives

$$f'(x) = \left( \frac{\partial f_i}{\partial x_j} \right).$$

The chain rule:  $(f \circ g)'(x) = f'(g(x))g'(x)$  if  $f$  is differentiable at  $g(x)$  and  $g$  is differentiable at  $x$ . If the linear approximation  $\varphi'(x_0)$  is invertible, then locally  $\varphi$  is invertible; to be precise:

**Theorem 4.1** (The inverse function theorem). *Suppose that  $U \subseteq \mathbb{R}^n$  is an open set,  $x_0$  is a point of  $U$ , and  $\varphi: U \rightarrow \mathbb{R}^n$  is a  $C^k$  map, some  $k \geq 1$ . The matrix  $\varphi'(x_0)$  is invertible just when, after perhaps replacing the open set  $U$  by a smaller open set containing  $x_0$ ,  $\varphi$  has a  $C^k$  inverse defined on some open set around  $\varphi(x_0)$  and, if  $y = \varphi(x)$ , then*

$$(\varphi^{-1})'(y) = \varphi'(x)^{-1}.$$

**4.1** If a smooth map has a  $C^1$  inverse, then prove its inverse is smooth.

**Theorem 4.2** (Implicit function theorem). *Suppose that  $U \subseteq \mathbb{R}^{p+q}$  is an open set and  $\varphi: U \rightarrow \mathbb{R}^q$  is a  $C^k$  map, some  $k \geq 1$ . Write each point of  $U$  as a pair  $(x, y)$*

with  $x \in \mathbb{R}^p$  and  $y \in \mathbb{R}^q$  and write  $\varphi$  as  $z = \varphi(x, y)$ . Write  $\frac{\partial z}{\partial y}$  to mean the matrix

$$\frac{\partial z}{\partial y} = \begin{pmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} & \cdots & \frac{\partial z_1}{\partial y_q} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2} & \cdots & \frac{\partial z_2}{\partial y_q} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial z_q}{\partial y_1} & \frac{\partial z_q}{\partial y_2} & \cdots & \frac{\partial z_q}{\partial y_q} \end{pmatrix}.$$

Suppose that  $\frac{\partial z}{\partial y}$  is an invertible matrix at a point  $(x, y) = (x_0, y_0)$ . Let  $z_0 := \varphi(x_0, y_0)$ . For any  $x$  close enough to  $x_0$  and  $z$  close enough to  $z_0$ , there is a unique solution  $y$  to the equation  $z = \varphi(x, y)$  close enough to  $y_0$ . This solution is a  $C^k$  map  $y = y(x, z)$ .

**4.2** Consider the curve in  $\mathbb{R}^3$  consisting of the intersection of the paraboloid  $z = x^2 + y^2$  and the cylinder  $x^2 + z^2 = 1$ . Near which points of this curve does the implicit function theorem say that we can write the curve as the graph of smooth functions  $y = y(x), z = z(x)$ ?

**4.3** Prove that there is no 1-1 continuously differentiable map  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

**4.4\*** Prove that, in any open set of the plane containing the origin, none of  $x^2 + y^2, x^2 - y^2, x^2 + y^3$  is the composition  $f \circ g$  of a smooth function  $g$  which has some partial derivative not zero, and a smooth function  $f = f(x)$  of one variable.

A  $C^1$  map  $\varphi$  between open sets in Euclidean spaces (possibly of different dimensions) has *constant rank* if the linear map  $\varphi'(x)$  has the same rank at every point  $x$ .

**Theorem 4.3** (The rank theorem). *Constant rank maps are locally linear up to changing variables in domain and range. To be precise: suppose that  $U \subseteq \mathbb{R}^p$  is an open set and  $\varphi: U \rightarrow \mathbb{R}^q$  is a  $C^k$  map of constant rank. Then  $\varphi$  is locally identified by  $C^k$  diffeomorphisms with a linear map. In other words, for each point  $x_0 \in U$ , there is a  $C^k$  diffeomorphism  $\psi$  near  $x_0$  and a  $C^k$  diffeomorphism  $\chi$  near  $\varphi(x_0)$ , so that  $\chi \circ \varphi \circ \psi^{-1}$  is a linear map.*

### Definition

Take a continuously differentiable path  $m(t)$  on a manifold  $M$ , and a time  $t = t_0$ , and suppose that  $m_0 := m(t_0)$ . How do we measure the velocity of  $m(t)$  at  $t = t_0$ ? Choose a chart  $\varphi$  defined on an open set containing  $m_0$ . Let  $x(t) := \varphi(m(t))$  and  $x_0 := \varphi(m_0)$ . Then  $x(t)$  is a path in Euclidean space  $\mathbb{R}^n$ , with velocity vector  $x'(t)$ .

If I choose some other chart  $\psi$ , let  $y(t) := \psi(m(t))$  and  $y_0 := \psi(m_0)$ , then that changes the velocity measurement. How? Write the transition map between the charts as  $y = y(x) = \psi \circ \varphi^{-1}(x)$ , so

$$\begin{aligned} y(t) &= \psi(m(t)), \\ &= \psi(\varphi^{-1}(x(t))), \\ &= y(x(t)). \end{aligned}$$

The path changes by composing with the transition map. Differentiate to get the velocity:

$$\frac{dy}{dt} = y'(x(t)) \frac{dx}{dt}.$$

Compute at the time  $t = t_0$ :

$$\left. \frac{dy}{dt} \right|_{t=t_0} = y'(x_0) \left. \frac{dx}{dt} \right|_{t=t_0}.$$

So we convert *every* velocity vector, of *any* path going through  $m_0$  at time  $t = t_0$ , from your measurements to mine, by multiplying by the matrix  $y'(x_0)$ , the derivative of the transition map, a linear transformation.

Take an  $n$ -dimensional manifold  $M$  and a point  $m_0 \in M$ . A *tangent vector representative* at  $m_0$  is a pair  $(\varphi, v)$ , where  $\varphi$  is a chart defined near  $m_0$  and  $v \in \mathbb{R}^n$ . Two tangent vector representatives  $(\varphi, v)$  and  $(\psi, w)$  at  $m_0$  are *equivalent* just when the transition map  $y = y(x) = \psi(\varphi^{-1}(x))$  gives  $w = y'(x_0)v$  where  $x_0 = \varphi_1(m_0)$ . A *tangent vector* at  $m_0$  is an equivalence class of tangent vector representatives. The set of tangent vectors at  $m_0$  is the *tangent space*  $T_{m_0}M$ . The tangent space is a vector space, using addition and scaling of representatives in any chart; addition and scaling are independent of choice of chart, by the chain rule. The *velocity*, denoted  $m'(t_0)$ , of a path  $m(t)$  at the time  $t = t_0$  is the tangent vector with representative  $(\varphi, v)$  where  $v = (\varphi \circ m)'(t_0)$ .

Given a tangent vector at some point  $m_0$ , say the equivalence class of some representative  $(\varphi, v)$ , and a chart  $(W, \psi)$  with  $m_0 \in W$ , there is a vector  $w \in \mathbb{R}^n$  arising in a representative  $(\psi, w)$  for that tangent vector, and the choice of  $w$  is uniquely determined: we must have  $w = (\psi \circ \varphi^{-1})'(x_0)v$  where  $x_0 = \varphi(m_0)$ .

If  $M = \mathbb{R}^n$ , we can use the “identity chart”  $\varphi(x) = x$  to identify each tangent vector representative  $(\varphi, v)$  with  $v$  itself. The same works if  $M$  is any open set in  $\mathbb{R}^n$ . Along the same lines, there is no danger just thinking of tangent vectors as ordinary vectors, once we fix a single chart to work in.

Given a continuously differentiable map  $\Phi: P \rightarrow Q$  of manifolds, and point  $p_0 \in P$ , let  $q_0 := \varphi(p_0)$  and define  $\Phi'(p_0): T_{p_0}P \rightarrow T_{q_0}Q$  by  $\Phi'(p_0)p'(t_0) = (\Phi \circ p)'(t_0)$ , for any continuously differentiable path  $p(t)$  with  $p(t_0) = p_0$ .

**4.5** Use the chain rule to prove that  $\Phi'(p_0)$  is well defined.

Suppose that  $U \subseteq \mathbb{R}^n$  is an open set and  $\text{id}: U \rightarrow U$  is the identity map. At each point  $x_0 \in U$ , identify each vector  $v \in \mathbb{R}^n$  with the tangent vector representative  $(\text{id}, v)$ . Clearly a bijection:  $T_{x_0}U = \mathbb{R}^n$ .

If  $f: P \rightarrow \mathbb{R}$ ,  $f'(p_0)v \in T_{f(p_0)}\mathbb{R}$ , and we have identified  $T_{f(p_0)}\mathbb{R} = \mathbb{R}$  above, so  $f'(p_0)v \in \mathbb{R}$ . We also write  $f'(p_0)v$  as  $vf$ .

**4.6** Use the chain rule for maps of open sets of Euclidean space to prove the chain rule for maps of manifolds: if  $\varphi: P \rightarrow Q$  and  $\psi: Q \rightarrow R$  are continuously differentiable maps, then

$$(\psi \circ \varphi)'(p_0) = \psi'(\varphi(p_0))\varphi'(p_0).$$

**4.7** Suppose that  $M \subset \mathbb{R}^n$  is a smooth embedded submanifold. For any point  $m_0 \in M$ , we let  $T'_{m_0}M$  be the set of all vectors  $v \in \mathbb{R}^n$  so that  $v = m'(0)$  for some smooth path  $m(t) \in M$  with  $m(0) = m_0$ . Explain how to identify  $T_{m_0}M$  with this simpler object  $T'_{m_0}M$  as vector spaces. (Henceforth we will not distinguish the two.)

The unit sphere  $S^n \subset \mathbb{R}^{n+1}$  around the origin is the set of points  $x$  of length 1, i.e.  $\langle x, x \rangle = 1$  in the usual dot product. Any path  $x(t) \in S^n$  has velocity  $x'(t_0)$  at some time  $t = t_0$ . Let  $x_0 = x(t_0)$  and  $v_0 = x'(t_0)$ . Differentiate  $\langle x, x \rangle = 1$  to find  $\langle v_0, x_0 \rangle = 0$ , i.e. velocity is perpendicular to position. Conversely, any vector  $v_0$  perpendicular to a point  $x_0 \in S^n$  is the velocity of that path  $x(t) = \cos(t)x_0 + \sin(t)v_0$ .



**4.8** If  $\varphi: P \rightarrow Q$  is a  $C^1$  map of manifolds, prove that  $\varphi$  is constant on each connected component of  $P$  just when  $\varphi'(p_0)v = 0$  for any point  $p_0 \in P$  and tangent vector  $v \in T_{p_0}P$ .

A *centre* of a set  $A \subseteq \mathbb{R}^n$  is a point  $x_0 \in A$  so that, for any point  $x \in A$ , the line segment  $x_0x$  lies in  $A$ . A set is *star-shaped* if it has a centre, *convex* if every point is a centre.



**4.9** Suppose that  $f: U \rightarrow \mathbb{R}$  is a smooth function on an open set  $U \subset \mathbb{R}^{p+q}$ . (It might help to think of the case  $p = q = 1$  to draw a picture.) Write points of  $U$  as  $(x, y)$  with  $x \in \mathbb{R}^p$ ,  $y \in \mathbb{R}^q$ . Suppose that  $f(0, y) = 0$  for all  $y$ . Suppose that each “vertical slice”  $U \cap (y = y_0) \subset \mathbb{R}^p$  is either empty or star shaped, with centre at the origin, for all  $y_0 \in \mathbb{R}^q$ . Prove that there are smooth functions  $f_1, \dots, f_p$  so that

$$f(x, y) = x_1 f_1(x, y) + \dots + x_p f_p(x, y).$$

## The inverse function theorem

**Theorem 4.4** (The inverse function theorem). *If the linear approximation to a map is invertible, the map is locally invertible. To be precise: suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  map of manifolds,  $k \geq 1$ . Take a point  $p_0 \in P$  and let  $q_0 := \varphi(p_0)$ . Then  $\varphi'(p_0): T_{p_0}P \rightarrow T_{q_0}Q$  is an invertible linear transformation if and only if, after perhaps replacing  $P$  and  $Q$  by open subsets containing  $p_0$  and  $q_0$ ,  $\varphi$  has a  $C^k$  inverse.*

The proof: just take charts, and the result turns into the one above in Euclidean space.



## Tubular neighborhoods in Euclidean space

This section can be omitted without loss of continuity.

The *normal space*  $\nu_{S,x}$  of an embedded submanifold in Euclidean space  $S \subseteq \mathbb{R}^n$  at a point  $x \in S$  is the set of vectors  $v \in \mathbb{R}^n$  so that  $v$  is perpendicular to  $T_x S$ .

**4.10** What is the normal space of the unit sphere at the north pole? The south pole?

The *normal bundle*  $\nu_S$  of an embedded submanifold in Euclidean space  $S \subseteq \mathbb{R}^n$  is the set of pairs  $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$  so that  $x \in S$  and  $v \in \nu_{S,x}$ .

**4.11** What is the normal bundle of the unit circle in the plane? The unit sphere in  $\mathbb{R}^n$ ?

**4.12** Prove that the normal bundle of a  $C^k$  embedded submanifold  $S \subseteq \mathbb{R}^n$ ,  $k \geq 2$ , is a  $C^{k-1}$  embedded submanifold of  $\mathbb{R}^{2n}$ .

The *normal map* of an embedded submanifold  $S \subset \mathbb{R}^n$  is the map

$$\epsilon: (x, v) \in \nu_S \rightarrow x + v \in \mathbb{R}^n.$$

A *tubular neighborhood*  $U \subset \mathbb{R}^n$  of  $S$  is an open subset so that there is a Lipschitz continuous function  $\rho > 0$  on  $S$  for which  $\epsilon$  takes the set

$$U_\rho := \{ (x, v) \in \nu_S \mid |v| < \rho(x) \}$$

diffeomorphically to  $U$ .

**Lemma 4.5.** *Every  $C^k$  embedded submanifold of Euclidean space,  $2 \leq k \leq \infty$ , has a tubular neighborhood.*

*Proof.* Note that  $\epsilon$  is the restriction to  $\nu_S \subset \mathbb{R}^{2n}$  of the linear map  $\epsilon(x, v) = x + v$  on  $\mathbb{R}^{2n}$ , so on  $\mathbb{R}^{2n}$ ,

$$\epsilon'(x, v)(\dot{x}, \dot{v}) = \dot{x} + \dot{v}.$$

Pick a point  $(x, 0) \in \nu_S$ . The restriction of  $\epsilon'(x, 0)$  to vectors of the form  $(\dot{x}, 0) \in T_x S$  is the identity map  $T_x S \rightarrow T_x S$ . Its restriction to vectors of the form  $(0, \dot{v})$  is the identity map  $\nu_{S,x} \rightarrow \nu_{S,x}$ . So  $\epsilon'(x, 0)$  is onto for every  $x \in \nu_S$ . Counting dimension, by the inverse function theorem,  $\epsilon$  is a local diffeomorphism near each point  $(x, 0) \in \nu_S$ . For each point  $x \in S$  and number  $\delta > 0$ , let  $B_{\delta,x}$  be the “bidisk” of pairs  $(y, v) \in \nu_S$  so that  $|x - y| < \delta$  and  $|v| < \delta$ . Each point  $x$  has a largest positive number  $r = r(x)$  so that the normal map  $\epsilon$  is a diffeomorphism on the bidisk  $B_{r,x}$ ; if this number is  $r(x)$  is infinite, redefine it to be 1 instead.

If  $x, y \in S$  and  $|x - y| < r(x)$  then by the triangle inequality

$$B_{\delta,y} \subseteq B_{r(x),x},$$

for any  $\delta$  with  $\delta + |x - y| \leq r(x)$ . Therefore  $r(y) + |x - y| \geq r(x)$ . Reversing roles of  $x, y$ :

$$\frac{|r(x) - r(y)|}{|x - y|} \leq 1.$$

So  $r$  is a Lipschitz function.

Let  $\rho(x) := r(x)/2$ . By definition of  $U_\rho$ , every point of  $U_\rho$  lies in a bidisk on which  $\epsilon$  is a diffeomorphism. So  $\epsilon'(x, v)$  is a linear isomorphism at every  $(x, v) \in U_\rho$ . So

$\epsilon$  is a local diffeomorphism  $U_\rho \rightarrow \mathbb{R}^n$ . Suppose that  $(x, v), (y, w) \in U_\rho$ . Swapping names if needed, we can suppose that  $r(x) \geq r(y)$ . So  $|w| < r(y)/2 \leq r(x)/2$ , and  $x + v = y + w$ , so

$$\begin{aligned} |x - y| &= |v - w|, \\ &\leq |v| + |w|, \\ &< \frac{r(x)}{2} + \frac{r(y)}{2}, \\ &\leq r(x). \end{aligned}$$

So both  $(x, v), (y, w)$  lie in the bidisk of  $x$ :  $B_{\rho(x), x}$ , on which  $\epsilon$  is injective. So  $(x, v) = (y, w)$ , i.e.  $\epsilon$  is injective on  $U_\rho$ .  $\square$

A *smooth deformation retraction* of a manifold  $M$  to a subset  $S \subseteq M$  is a smooth map

$$\varphi: M \times [0, 1] \rightarrow M$$

so that

- $\varphi(m, 0) = m$  for any  $m \in M$ ,
- $\varphi(m, 1) \in S$  for any  $m \in M$ ,
- $\varphi(s, t) = s$  for any  $s \in S$ .

We say that  $S$  is a *smooth retract* of  $M$ .

**4.13** Prove that every embedded submanifold of Euclidean space is a smooth retract of one of its open neighborhoods.

## Regular and critical points

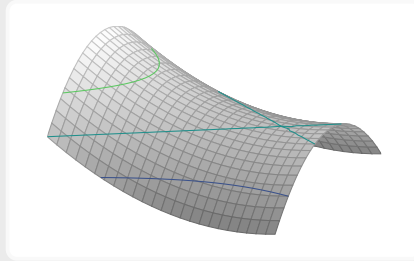
A *regular point* of a continuously differentiable map  $\varphi: P \rightarrow Q$  between manifolds is a point  $p \in P$  where  $\varphi'(p)$  is a surjective linear map; any other point is called a *critical point*. A map for which every point is regular is a *submersion*. A point  $q \in Q$  is called a *critical value* of a continuously differentiable map  $\varphi: P \rightarrow Q$  if  $q = \varphi(p)$  for some critical point  $p \in P$ . A point  $q \in Q$  which is not a critical value is called a *regular value*. In other words, a point  $q \in Q$  is a regular value if there is no critical point  $p \in P$  so that  $\varphi(p) = q$ .

**4.14** Which are the regular points, critical points, regular values and critical values of

- a.  $y = x^2$ ,
- b.  $y = x^3 - x^2$ ,
- c.  $y = 1$ ,
- d.  $y = \sin x$ ,
- e.  $y = \arctan x$ ?

The function  $z = f(x, y) = x^2 - y^2$ ,  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  has derivatives

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = -2y.$$



Critical points are where both vanish, so the origin  $(x, y) = (0, 0)$  is the only critical point.

The value  $f(0, 0) = 0$  is the only critical value. So any value  $z_0 \neq 0$  is a regular value. If  $z_0 \neq 0$  the level set  $(f = z_0)$  is a hyperbola  $x^2 - y^2 = z_0$ , so a smooth curve. If  $z_0 = 0$ , the critical value, the level set  $(f = 0)$  is a pair of lines:

$$0 = x^2 - y^2 = (x - y)(x + y)$$

so not a smooth curve.

Near the points where  $y \neq 0$ ,  $y$  is locally a function of  $x, z$  by the implicit function theorem, so we can change variables by the diffeomorphism  $(x, y) \mapsto (x, z) = (x, f(x, y))$ , so that our function is then, in those new variables, just  $f(x, z) = z$ . Similarly where  $x \neq 0$ , we can change variables by  $(x, y) \mapsto (z, y) = (f(x, y), y)$ . No change of variables can fix the problem that the zero level set of  $f$  is not smooth at the origin.

**Theorem 4.6** (The implicit function theorem). *Near any regular point, any smooth map is a linear projection, in some charts. To be precise: suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  map of manifolds,  $k \geq 1$ . If  $p_0 \in P$  is a regular point of  $\varphi$ , then there are  $C^k$  charts defined in  $P$  near  $p_0$  and in  $Q$  near  $q_0$  identifying  $\varphi$  with the map  $(x, y) \mapsto x$ .*

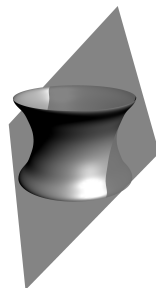
**4.15** Under the same hypotheses as theorem 4.6, prove that, for each regular value  $q_0 \in Q$ , the level set  $M := \varphi^{-1}\{q_0\}$  is a  $C^k$  embedded submanifold of  $P$ , of dimension  $\dim P - \dim Q$ , with tangent space

$$T_p M = \ker \varphi'(p),$$

for every point  $p \in M$ .

**4.16** Use the implicit function theorem to prove that the intersection of the paraboloid  $z = x^2 + y^2$  with the paraboloid  $y = x^2 + z^2$  is a manifold in  $\mathbb{R}^3$ . What is the dimension of this manifold?

**4.17** Use the implicit function theorem to prove that the intersection of the hyperboloid  $x^2 + y^2 = z^2 + 1$  with the plane  $x + y + z = 0$  is a manifold in  $\mathbb{R}^3$ . What is the dimension of this manifold?



**4.18** Consider the sphere  $x^2 + y^2 + z^2 = 1$  and the cylinder  $(x - a)^2 + y^2 = r^2$  where  $r > 0$  and  $a \geq 0$ . Let  $M$  be the intersection of the sphere and cylinder. For what values of  $a$  and  $r$  and near what points  $(x, y, z) \in M$  does the implicit function theorem ensure that  $M$  is a manifold? Draw pictures of what the sphere and cylinder look like at all these  $a$  and  $r$  values, near each of these  $(x, y, z)$  points, where the implicit function theorem doesn't apply? Hint: let

$$\varphi(x, y, z) := \begin{pmatrix} x^2 + y^2 + z^2 - 1 \\ (x - a)^2 + y^2 - r^2 \end{pmatrix}.$$

**4.19** Pick positive constants  $r_1 > r_0$ . The *torus* is the image of the map

$$\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^3, (\alpha, \beta) \mapsto ((r_1 + r_0 \cos \alpha) \cos \beta, (r_1 + r_0 \cos \alpha) \sin \beta, r_0 \sin \alpha).$$

Prove that the torus is a surface. Find an equation of the torus in  $x, y, z$ .

**4.20** Use the implicit function theorem to prove that the intersection of the paraboloid  $z = x^2 + y^2$  with the cylinder  $x^2 + z^2 = 4$  is a manifold in  $\mathbb{R}^3$ . What is the dimension of the manifold?

**Corollary 4.7.** *suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  map of manifolds,  $k \geq 1$ . If  $\varphi'(p)$  is a surjective linear map, then every submanifold  $S_Q \subseteq Q$ , say of codimension  $k$ , has preimage  $S_P := \varphi^{-1}S_Q \subseteq P$  a submanifold, also of codimension  $k$ , with tangent space*

$$T_p S_P = \varphi'(p)^{-1} T_q S_Q,$$

where  $q = \varphi(p)$ .

**4.21** Prove that the graph  $y = \varphi(x)$  of a map  $\varphi: \text{open } \subset \mathbb{R}^p \rightarrow \mathbb{R}^q$  is a  $p$ -dimensional  $C^k$  manifold in  $\mathbb{R}^{p+q}$  just when  $\varphi$  is  $C^k$ .

**4.22** Suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  map of manifolds and  $q_0 \in Q$  is a regular value of  $\varphi$ . Suppose that  $P$  and  $Q$  are oriented. Prove that there is a unique orientation on  $M := \varphi^{-1}\{q_0\}$ , so that, near each point of  $M$ , in some positively oriented charts on  $P$  and  $Q$ ,  $\varphi$  is a linear projection

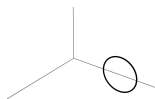
$$(x^1, \dots, x^p, y^1, \dots, y^q) \mapsto (x^1, \dots, x^p),$$

with  $y = (y^1, \dots, y^q)$  a positively oriented chart on  $M$ .

Take  $P = \mathbb{R}^{n+1}$ ,  $Q = \mathbb{R}$ , with usual orientations, and  $\varphi(x) = |x|^2$ . Then  $1 \in Q$  is a regular value and the unit sphere  $M = \varphi^{-1}\{1\}$  is a smooth oriented manifold.

**4.23\*** Continuity of eigenvalues and eigenvectors: suppose that  $A(t)$  is an  $n \times n$  matrix depending as a continuously differentiable function on a real variable  $t$ . Suppose that at  $t = 0$ , the matrix  $A_0 := A(0)$  has an eigenvalue  $\lambda_0$  and that the generalized  $\lambda_0$ -eigenspace is 1-dimensional containing a  $\lambda_0$ -eigenvector  $x$  with  $x_1 = 1$ . Prove that, for all  $t$  close to 0, there is a unique continuously differentiable choice of eigenvalue  $\lambda(t)$  of  $A(t)$  so that  $\lambda(0) = \lambda_0$ , and there is a unique continuously differentiable eigenvector  $x(t)$  with eigenvalue  $\lambda(t)$  and with  $x_1(t) = 1$ .

**4.24** Denote points of  $\mathbb{R}^3$  as  $(x, y, z)$ . Suppose that  $C$  is a curve in  $\mathbb{R}^3$  which lies inside the plane where  $x = 0$ , in the half of that plane in which  $y > 0$ .



Let  $S$  be the set of points of  $\mathbb{R}^3$  of the form  $(y \sin \theta, y \cos \theta, z)$  for  $(0, y, z) \in C$ .



Prove that  $S$  is a surface. Hint: you could consider whether the map

$$F(\theta, y, z) = (y \sin \theta, y \cos \theta, z)$$

is a local diffeomorphism on the set where  $y > 0$

**4.25** Suppose that  $f(x, y)$  is a smooth function so that at every point  $(x, y)$  at which  $f(x, y) = 0$ ,  $f'(x, y) \neq 0$ . Prove that the set

$$S := \{ (x, y, z) \mid f(x, y) + z^2 = 0 \}$$

is a surface, containing two points  $(x, y, z)$  above each point  $(x, y)$  on which  $f(x, y) < 0$ , one point above each point on which  $f(x, y) = 0$ , and no points above each point  $(x, y)$  on which  $f(x, y) > 0$ .

**4.26** Pick an integer  $g \geq 0$  and  $g$  nonoverlapping closed disks  $D_i$  in the plane, say with centre  $(x_i, y_i)$  and radius  $r_i > 0$ ,  $i = 1, 2, \dots, g$ . Take a closed disk  $D_0$  containing all of these  $D_i$  in its interior. Let

$$f_i(x, y) := (x - x_i)^2 + (y - y_i)^2 - r_i^2,$$

and let  $f := f_0 f_1 f_2 \dots f_g$ . Use problem 4.25 to prove that

$$S := \{ (x, y, z) \mid f(x, y) + z^2 = 0 \}$$

is a compact smooth surface, the *compact oriented surface of genus g*. What does it look like?

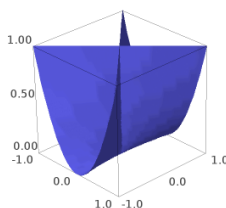
Arrange the “holes”, i.e. the disks  $D_1, \dots, D_g$ , to sit at points on a circle around the center of  $D_0$ , with equal radii, centered at the vertices of a regular polygon, so the dihedral group acts as diffeomorphisms. Note that  $z \mapsto -z$  is also a diffeomorphism of these surfaces. So the compact oriented surface of genus  $g$  has a group of order  $4g$  acting on it by diffeomorphisms.

**4.27** Pick  $2g$  disjoint disks in the plane, carried to one another, each to a different one, under rotation of the plane by  $180^\circ$  around the origin. Pick a disk containing all of them, invariant under the rotation. Act on the compact oriented surface of genus  $2g$  by

$$(x, y, z) \mapsto (-x, -y, -z).$$

Prove that this is a diffeomorphism, reversing orientation, without fixed points. Prove that the quotient is a smooth compact unorientable surface, the *compact unorientable surface of genus  $g$* .

**4.28** The *Whitney umbrella* is the set of points  $(x, y, z) \in \mathbb{R}^3$  for which  $x^2 = y^2z$ .



Near which points of the umbrella is it a surface? Near which points is it a curve?

**4.29\*** Prove that the union of two lines in the plane is a manifold if and only if the lines are parallel. Hint: a manifold of dimension 1 would locally cut into two pieces if you cut out a point.

**4.30** Prove that  $[0, \infty)$  is not a manifold.

**4.31\*** Let  $O(n)$  be the set of orthogonal  $n \times n$  matrices. Prove that  $M \subset \mathbb{R}^{n^2}$  is a manifold of dimension  $n(n-1)/2$ .

**4.32** Find an example of a system of algebraic equations, say  $F(x) = 0$ , whose set of solutions  $M$  is a manifold, but so that  $F'(x)$  does not have full rank at any point of  $M$ .

**4.33\*** Suppose that  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}$  is a surjective submersion. Are the fibers all diffeomorphic to  $\mathbb{R}$ ? Are they at least all diffeomorphic to one another?

**4.34\*** Denote by  $SL(n, \mathbb{R})$  the set of  $n \times n$  matrices of determinant 1. Prove that  $SL(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$  is a manifold of dimension  $n^2 - 1$ .

**4.35\*** Denote by  $SO(n)$  the set of  $n \times n$  rotation matrices (in other words: orthogonal matrices of determinant 1). Prove that  $SO(n) \subset \mathbb{R}^{n^2}$  is a manifold of dimension  $n(n-1)/2$ .

**4.36** Splitting quaternions  $a = a_0 + ia_1 + ja_2 + ka_3$  in real and imaginary (or vector) parts  $a_0$  and  $ia_1 + ja_2 + ka_3$ , explain how to write quaternion multiplication in terms of dot and cross product.

**4.37\*** Extending injectivity from compact sets: suppose that  $\varphi: P \rightarrow Q$  is a smooth map, injective on some compact set  $X \subseteq P$ , and with injective differential  $\varphi'(x)$  at every point  $x \in X$ . Prove that  $X$  lies in an open set on which  $\varphi$  is an immersion. If  $P$  and  $Q$  have the same dimension, prove that  $\varphi$  is a diffeomorphism from that open set to its image.

### Descent

Suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  map of manifolds, some  $k \geq 1$ . The *fiber* of  $\varphi$  over a point  $q_0 \in Q$  is the set  $\varphi^{-1}\{q_0\}$ , denoted  $P_{q_0}$ . A map  $\Psi: Q \rightarrow M$  of manifolds is a *descent* by  $\varphi$  of a map  $\psi: P \rightarrow M$  if  $\psi = \Psi \circ \varphi$ , which we denote

$$\begin{array}{ccc} P & & \\ \varphi \downarrow & \searrow \psi & \\ Q & & M \end{array}$$

or, if the various names of the maps are just understood, as

$$\begin{array}{ccc} P & & \\ \downarrow & \searrow & \\ Q & & M. \end{array}$$

Take  $P = Q = M = \mathbb{R}$  and the smooth maps  $\varphi: x \in P \mapsto x^3 \in Q$ , and  $\psi: x \in P \mapsto x^2 \in M$ . The map descends to the  $C^0$  map  $y \in Q \mapsto y^{2/3} \in M$ , not smooth. Similarly, the smooth map  $x^k$  descends to  $y^{k/3}$ , which is  $C^\infty$  if  $k$  is a multiple of 3, but otherwise only  $C^\ell$  where

$$\ell = \left\lfloor \frac{k}{3} \right\rfloor.$$

**4.38** Give an example of two different descents of the same map.

**4.39\*** Given a  $C^k$  map  $\varphi: P \rightarrow Q$ , prove:

- Every map which descends is constant on the fibers.
- If every  $C^k$  map constant on the fibers descends to a continuous map, then  $\varphi$  has closed image.
- Every continuous map constant on the fibers has at most one descent if and only if  $\varphi$  has dense image.
- If  $\varphi$  is a  $C^k$  surjective submersion then every  $C^k$  map constant on the fibers descends uniquely to a  $C^k$  map.

Let  $Q$  be the set of triangles in the plane, and  $P$  the set of triangles in the plane with a choice of ordering of the vertices (a first vertex, a second, and a

third). Clearly  $P \subset \mathbb{R}^6$  is an open subset, given by writing out the Cartesian coordinates of the vertices. But  $Q$  is a more abstract manifold. Perhaps it is not entirely clear to the reader that  $Q$  is a manifold; it will become clear eventually. The length of the first side is a smooth function on  $P$ . But it doesn't descend to  $Q$ , because the sides have no ordering in an abstractly given triangle. The perimeter and area functions descend to smooth functions on  $Q$ , for the obvious map  $\varphi: P \rightarrow Q$  which forgets the ordering of the vertices.

Variations on the notion of descent are important in differential geometry. As in this simple example, typically  $P$  is more concrete than  $Q$ , but  $Q$  is the manifold we are interested in, so we prefer to work in  $P$  and somehow derive important conclusions about  $Q$ .

### The rank theorem

A  $C^k$  map  $\varphi: P \rightarrow Q$  has *constant rank* if  $\varphi'(p)$  has the same rank at each point  $p \in P$ .

**Theorem 4.8** (The rank theorem). *A  $C^k$  map of constant rank is locally identified by  $C^k$  charts with a linear map.*

In other words, for each point  $p_0 \in P$ , there are coordinates  $x^1, x^2, \dots, x^p$  on  $P$  defined near  $p_0$  and there are coordinates  $y^1, y^2, \dots, y^q$  on  $Q$  defined near  $q_0 = \varphi(p_0)$ , so that in the coordinates  $\varphi$  is expressed as a linear map.

The proof: take any coordinates, and apply the rank theorem 4.3 on page 40.

**Corollary 4.9.** *Every fiber  $S = \varphi^{-1}\{q_0\} \subset P$  of an constant rank  $C^k$  map  $\varphi: P \rightarrow Q$ ,  $k \geq 1$ , is an embedded  $C^k$  submanifold of  $P$  with tangent spaces  $T_p S$  the kernel of  $\varphi'(p)$  for every  $p \in S$ .*

*Proof.* If  $\varphi$  is a linear map, we can arrange that  $q_0$  is the origin and then  $S$  is the kernel of the linear map, a linear subspace, hence a submanifold.  $\square$

Denote the transpose of a matrix  $A$  as  $A^t$ . An  $n \times n$  real matrix  $A$  is *orthogonal* if  $AA^t = I$ . The orthogonal group  $O(n)$  is the collection of orthogonal  $n \times n$  matrices. Think of the space of  $n \times n$  matrices as  $\mathbb{R}^{n^2}$ . We want to see that the orthogonal group is an embedded submanifold. Consider the map  $\varphi(A) = AA^t$ . If we can calculate  $\varphi'$  and show that  $\varphi'$  has constant rank, we can apply the corollary of the rank theorem, as the orthogonal group is

$$O(n) = \varphi^{-1}\{I\}.$$

Since the  $n \times n$  matrices form  $\mathbb{R}^{n^2}$ , not some abstract manifold, we can differentiate  $\varphi$  directly, using

$$\varphi(A + \varepsilon B) = (A + \varepsilon B)(A + \varepsilon B)^t = AA^t + \varepsilon(BA^t + AB^t) + \dots,$$

so that the linear map  $\varphi'(A)$  is the map on matrices

$$B \mapsto BA^t + AB^t.$$



The orthogonal group is not an open subset of the  $n \times n$  matrices, so to apply the rank theorem, we have to allow  $A$  to vary among some open set of matrices, not just the orthogonal ones. Since the determinant is a continuous function, the invertible matrices (i.e. with nonzero determinant) form an open set, so we can apply the rank theorem to  $\varphi$  on that open set. So we suppose that  $A$  is invertible.

Every  $n \times n$  matrix  $B$  can be written as  $B = C(A^t)^{-1}$  for a unique  $n \times n$  matrix  $C = BA^t$ , and the map

$$B \mapsto C = BA^t,$$

is a linear isomorphism on matrices, with inverse

$$C \mapsto B = C(A^t)^{-1}.$$

Hence, wherever  $A$  is invertible, we can compute the kernel of the linear map  $\varphi'(A)$  by asking when

$$0 = BA^t + AB^t = C + C^t.$$

So these matrices  $C$  are the antisymmetric matrices, always the same dimension, for any  $A$ , and so the dimension of matrices  $B$  solving  $0 = BA^t + AB^t$  is constant, equal to the dimension of the antisymmetric matrices.

A subset  $S \subset M$  of a manifold has *codimension at least  $k$*  if every point  $m_0 \in S$  lies in an open subset  $U \subseteq M$  in which there is a submanifold  $X \subset U$  of codimension  $k$  so that  $S \cap U \subset X$ .

**4.40** Find a smooth map  $\varphi: P \rightarrow Q$  between open sets  $P \subset \mathbb{R}^3$  and  $Q \subset \mathbb{R}^2$  and a sequence of values  $y_1, y_2, \dots \in Q$  converging to a value  $y \in Q$ , so that the fibers

$$P_{y_1} := \varphi^{-1}\{y_1\}, P_{y_2}, \dots,$$

are connected, but so that  $P_y$  is disconnected. Hint: a tire swing on a sunny day at noon casts a shadow under it.

**4.41\*** Take a collection of functions, say  $A$ , on a manifold  $M$ . Let  $Z \subseteq M$  be the points where these functions all vanish. A *probe* of  $A$  is a map

$$(f_1, \dots, f_\ell): M \rightarrow \mathbb{R}^\ell$$

for functions  $f_1, \dots, f_\ell$  from  $A$ . Say that  $A$  has *rank  $k$*  at a point of  $M$  if some probe has rank  $k$  at that point, and no probe has rank  $k + 1$ . Let  $M_k \subseteq M$  be the points where  $A$  has rank  $k$ . Prove that  $M_k \cap Z \subset M$  is a subset of codimension at least  $k$ . (Give an example for which  $M_k \cap Z$  is *not* itself a submanifold.) Pick a point  $m_0 \in M_k \cap Z$ . Prove that the following are equivalent

- $M_k \cap Z$  is a submanifold of codimension  $k$  near  $m_0$
- $Z$  is a submanifold of codimension  $k$  near  $m_0$



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- c. There is a probe  $f_1, \dots, f_k$  of rank  $k$  near  $m_0$ . For any such probe, and any smooth function  $f$  defined near  $m_0$ ,  $f$  vanishes on  $Z$  near  $m_0$  just when there are smooth functions  $g_1, \dots, g_k$  defined near  $m_0$  so that

$$f = g_1 f_1 + \dots + g_k f_k$$

near  $m_0$ . In particular, all functions  $f$  in  $A$  have this form near  $m_0$ .

**4.42\*** A *retraction* to a subset  $A \subseteq M$  of a manifold  $M$  is a map  $\varphi: M \rightarrow M$  so that  $\varphi(m) \in A$  for all  $m \in M$  and  $\varphi(a) = a$  for any  $a \in A$ . If  $\varphi$  is smooth, prove that every path component of  $A$  is an embedded submanifold.

## Cones

This section can be omitted without loss of continuity.

It can be helpful to see that some subset of a manifold is *not* an embedded submanifold. A subset  $C \subset \mathbb{R}^n$  is a *cone* if  $0 \in C$  and  $ax \in C$  for any  $x \in C$  and  $a \geq 0$ . The *tangent cone* of a subset  $S \subset M$  of a manifold at a point  $x_0 \in S$  is the set of velocities  $v = x'(0)$  of continuously differentiable curves  $x(t) \in S$  with  $x(0) = x_0$ , defined on some interval  $0 \leq t < \varepsilon$ .

**4.43** Prove that the tangent cone is a cone.

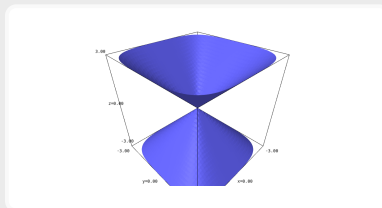
**4.44** Prove that the tangent cone at any point of any submanifold of any manifold is its tangent space.

So if the tangent cone of a set at a point is not a linear subspace, the set is not an embedded submanifold near that point.

**4.45** Prove that every closed cone in Euclidean space is its own tangent cone.

**4.46** Prove that a cone is an embedded submanifold of Euclidean space just when it is a linear subspace.

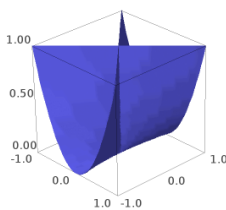
The cone  $C = \{(x, y, z) \mid x^4 + y^4 - z^4 = 0\}$  is not a linear subspace, so is not an embedded submanifold near the origin.



Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . The *determinantal variety*  $V = V_{p,q,r}$  is the set of all  $p \times q$  matrices with entries in  $k$  of rank at most  $r$ , a cone inside the set  $k^{p \times q}$  of all  $p \times q$  matrices with entries in  $k$ . We can increase rank by adding suitable matrices together, so it is not difficult to see that  $V$  is not closed

under addition, so not a real linear subspace, and hence is not an embedded submanifold of  $k^{p \times q}$ .

**4.47\*** The *Whitney umbrella* is the set of points  $(x, y, z) \in \mathbb{R}^3$  for which  $x^2 = y^2z$ . Find the tangent cone of the Whitney umbrella at the origin. Prove that the Whitney umbrella is not a smooth surface near the origin.



## Orientation and the implicit function theorem

This section can be omitted without loss of continuity.

A *positively oriented basis* of  $\mathbb{R}^n$  is an ordered basis so that the matrix formed by taking those vectors as its columns has positive determinant.

**4.48** Prove that a basis of  $\mathbb{R}^n$  is positively oriented just when you can continuously move its vectors through a family of bases to the standard basis.

Suppose that  $M$  is an oriented manifold and  $m \in M$  a point. A *positively oriented basis* of  $T_m M$  is a basis which is identified by a positively oriented chart with a positively oriented basis of  $\mathbb{R}^n$ . An orientation of a positive dimensional connected manifold is clearly uniquely identified by a single positively oriented basis at a single point. (A zero dimensional connected manifold is a point, on which an orientation is just a choice of a + or - sign.)

Take a  $C^k$  map  $\varphi: P \rightarrow Q$ ,  $k \geq 1$  and a regular value  $q_0 \in Q$ . The fiber  $S := \varphi^{-1}\{q_0\}$  is a  $C^k$  submanifold with tangent spaces  $T_p S$  the kernel of  $\varphi'(p)$ , for any  $p \in S$ . Suppose  $P$  and  $Q$  are oriented. Pick a positively oriented basis of  $T_p P$  so that the first so many basis vectors lie in the kernel. The rest then map to a basis of  $T_q Q$ .

Suppose that the kernel and image have positive dimension. We can always change signs of two basis vectors to get that basis in  $T_q Q$  to be positively oriented. Our basis is uniquely determined up to changing by linear combinations of basis vectors forming a block matrix

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

because we need to keep the first so many basis vectors inside that kernel. Keeping the basis in  $T_q Q$  positively oriented is precisely  $\det D > 0$ . Keeping the basis in  $T_p P$  positively oriented is precisely  $\det A \det D > 0$ . Hence we have an orientation of  $T_p S$ : the positively oriented bases are those for which the basis extends to a positively oriented basis of  $T_p P$  which maps to a positive oriented basis of  $T_q Q$ . By continuity, this orientation is defined throughout  $S$ .

If the kernel has dimension zero, i.e.  $\varphi$  is a local diffeomorphism, there is no choice: our positively oriented basis of  $P$  maps to some basis of  $Q$ , and we can't control its

orientation. The set  $S$  is a discrete set of points of  $P$ . Define  $T_p S$  to have orientation  $+$  if  $\varphi'(p)$  takes positive oriented bases of  $T_p P$  to those of  $T_q Q$ , and  $-$  otherwise.

If  $Q$  has dimension zero, i.e.  $\varphi$  is a locally constant map to a discrete set of points, we have  $S$  a union of components of  $P$  and we orient each component of  $S$  as the component of  $P$ .

## Chapter 5

### Embedding the Grassmannian

We prove a criterion to test when a map has image an embedded submanifold. Finally, as a long example, we embed the Grassmannian into Euclidean space.

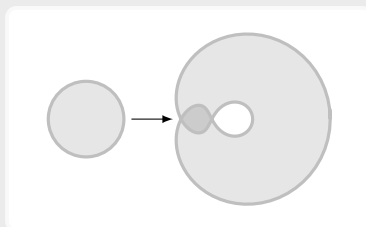
When is the image of a smooth map an embedded submanifold?

The map  $t \mapsto (x(t), y(t)) = (t^2, t^3)$  has image  $y = \pm x^{3/2}$ , not smooth.

<

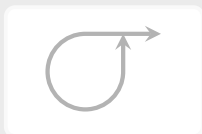
An *immersion* is a smooth map  $\varphi: P \rightarrow Q$  of manifolds for which  $\varphi'(p)$  is injective for every  $p \in P$ . By the rank theorem, for any immersion, every point of  $P$  lies in an open subset in which the immersion is an embedding. But this is *not* enough to ensure an embedding.

An immersion might not be injective:



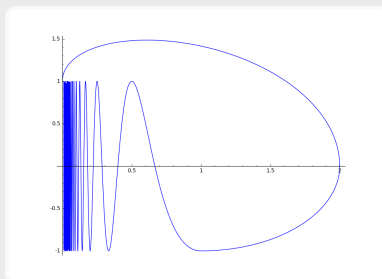
An *immersed submanifold* is an immersion  $\varphi: P \rightarrow Q$ , but only defined up to diffeomorphism of the domain.

A more sophisticated example: map the real number line to the plane, i.e. imagine a particle moving in the plane, so that the particle approaches a point where it has been before, but never reaches that point, because the particle slows down.



As the particle never crosses its path, the map from the real number line is injective. But since it approaches its path, that point where it approaches does not lie in an open set where we can “straighten out” the path.

A more sophisticated example: the *topologists’ sine curve*.



Take the graph of the function  $y = \sin(1/x)$ , for  $x > 0$ . For small  $x$ , this function oscillates with explosive frequency. Cut off the curve at some point off to the right, and attach a smooth “tail” that curves over to the left, smoothly joining the tail to the line segment  $\{0\} \times (-1, 1]$ . Since  $y = \sin(1/x)$  takes on all values between  $-1 \leq y \leq 1$ , all points of that line segment are in the limit of the graph of that function. So the topologist’s sine curve “approaches itself” along a line segment. The topologist’s sine curve is an immersion of the real number line to the plane, which fails to be an embedded submanifold near any point of that line segment.

Take a smooth map  $\varphi: P \rightarrow Q$  and let  $S := \varphi(P)$ . The map is an *embedding* if  $S$  is an embedded submanifold of  $Q$  and  $\varphi$  a diffeomorphism of  $P$  to  $S$ . A map is *proper* if the preimage of any compact set is compact.

**Lemma 5.1.** *Every proper injective immersion is an embedding.*

*Proof.* From topology, a proper injection to a locally compact Hausdorff space is a homeomorphism to its image [13] p.20, Corollary 2.6. So  $\varphi: P \rightarrow S$  is a homeomorphism. Take a point  $s_0 \in S$  and  $p_0 := \varphi^{-1}(s_0)$ . The rank theorem picks out an open set  $U_P \subset P$  and produces local coordinates on an open set  $U_Q \subset Q$  which make each  $U_S := \varphi(U_P)$  into the zero locus of equations  $0 = x^{p+1} = \dots = x^n$ . Hence every point  $s_0 \in S$  then lies in such an open set  $U_S$ . The local coordinates we construct on  $S$  are also local coordinates on  $P$ , so that  $\varphi$  is a diffeomorphism. Every point  $x_0$  of the closure of  $S$  lies in an open set with compact closure in  $M$ . This compact closure has compact preimage in  $P$ , and so intersects  $S$  in a compact set containing any sequence of points of  $S$  approaching  $x_0$ . A convergent subsequence converges in  $S$ , so  $x_0 \in S$ .  $\square$

**5.1\*** Prove that any bijective immersion of manifolds of equal dimension is a diffeomorphism.

**5.2** Give an example of a bijective immersion of manifolds in the broad sense which is not a diffeomorphism.

**5.3** Suppose that  $\varphi: P \rightarrow Q$  is a surjective submersion. Prove that a map  $\psi: Q \rightarrow M$  of manifolds is smooth just when  $\psi \circ \varphi$  is smooth.

**5.4** Suppose that  $\varphi: P \rightarrow Q$  is a surjective continuous map of topological spaces, and that  $P$  is a smooth manifold. Prove that there is at most one smooth structure on  $Q$  for which  $\varphi$  is a submersion.

**5.5** Let  $\varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^5$  be defined by

$$\varphi(x, y, z) = (x^2 - y^2, x^2 - z^2, xy, xz, yz).$$

Prove that  $\varphi$  is an immersion away from the origin, and that  $\varphi(-x, -y, -z) = \varphi(x, y, z)$ . Prove that  $\varphi$  restricts to the unit sphere to an immersion to  $\mathbb{R}^5$ . Explain how this immersion identifies  $\mathbb{RP}^2$  with a smooth surface in  $\mathbb{R}^5$ .

An immersed submanifold  $S \subseteq M$  is *weakly embedded* (also called *initial*) if every smooth map  $P \rightarrow M$  with image inside  $S$ , from any manifold  $P$ , is a smooth map to  $S$ .

The identity map taking one figure 8 to the other is the identity map of the plane, so a smooth map to the plane, but a discontinuous map of the figure 8s: the figure 8s are immersed but not weakly embedded.



**5.6\*** Suppose that  $S \subseteq M$  is a smooth immersed submanifold and  $\varphi: P \rightarrow M$  a smooth map with image in  $S$ . Prove that  $\varphi$  is smooth as a map to  $S$  if and only if it is continuous to  $S$ . In particular, if  $S$  is embedded, smooth maps to  $S$  are precisely smooth maps to  $M$  with image in  $S$ .

## Quaternions

This section can be omitted without loss of continuity.

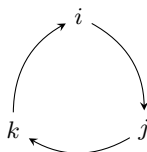
The *quaternions* are the quadruples  $a = (a_0, a_1, a_2, a_3)$  of real numbers, each written as

$$a = a_0 + a_1i + a_2j + a_3k.$$

Add component by component. Multiply by distributive law and the rules

$$i^2 = j^2 = k^2 = -1,$$

and so that when we follow around this loop,



starting anywhere, then the first two elements we hit multiply to give the third, but with a minus sign if we go backwards. The *conjugate* of  $a$  is

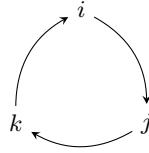
$$\bar{a} := a_0 - a_1i - a_2j - a_3k.$$

Denote by  $\mathbb{H}$  the set of all quaternions, i.e.  $\mathbb{H} = \mathbb{R}^4$ .

**5.7** Prove that  $\mathbb{H}$  is associative.

**5.8** Prove that  $\overline{ab} = \bar{b}\bar{a}$  for any quaternions  $a, b$ .

It is immediate from the definitions that we can replace  $i, j, k$  by any three abstract symbols with ordering defined only up to even permutations, so for example



describes an automorphism of the quaternions.

**5.9** Check that

$$a\bar{a} = \bar{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Prove that every quaternion  $a \neq 0$  has a unique inverse: a quaternion denoted  $a^{-1}$ , so that  $aa^{-1} = a^{-1}a = 1$ .

Every quaternion can be thought of as a  $4 \times 4$  real matrix:

$$a_0 + a_1i + a_2j + a_3k \mapsto \begin{pmatrix} a_0 & -a_1 & -a_2 & a_3 \\ a_1 & a_0 & -a_3 & -a_2 \\ a_2 & a_3 & a_0 & a_1 \\ -a_3 & a_2 & -a_1 & a_0 \end{pmatrix},$$

or as a  $2 \times 2$  complex matrix:

$$a_0 + a_1i + a_2j + a_3k = z + wj \mapsto \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix}$$

so that quaternion addition, subtraction and multiplication correspond to matrix addition, subtraction and multiplication.

### Quaternionic vector spaces

This section can be omitted without loss of continuity.

Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Add vectors  $x \in k^n$  as usual component by component, but multiply by elements  $a \in k$  on the right:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} a = \begin{pmatrix} x_1 a \\ x_2 a \\ \vdots \\ x_n a \end{pmatrix}.$$

A *vector space*  $V$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  is a set  $V$  equipped with an operation  $u + v$  and an operation  $va$  so that



*Addition laws:*

1.  $u + v$  is in  $V$
2.  $(u + v) + w = u + (v + w)$
3.  $u + v = v + u$

for any vectors  $u, v, w$  in  $V$ ,

*Zero laws:*

1. There is a vector  $0$  in  $V$  so that  $0 + v = v$  for any vector  $v$  in  $V$ .
2. For each vector  $v$  in  $V$ , there is a vector  $w$  in  $V$ , for which  $v + w = 0$ .

*Multiplication laws:*

1.  $va$  is in  $V$
2.  $v \cdot 1 = v$
3.  $v(ab) = (va)b$
4.  $v(a + b) = va + vb$
5.  $(u + v)a = ua + va$

for any numbers  $a, b \in k$ , and any vectors  $u$  and  $v$  in  $V$ .

For example,  $V = k^n$  is a vector space, with obvious componentwise addition and scaling. A *subspace* of a vector space  $V$  is a nonempty subset  $U$  of  $V$  so that when we apply the same scaling and addition operations to elements of  $U$  we always get elements of  $U$ . In particular,  $U$  is also a vector space.

A *linear relation* between vectors  $v_1, v_2, \dots, v_p \in V$  is an equation

$$0 = v_1 a_1 + v_2 a_2 + \dots + v_p a_p$$

satisfied by these vectors, in which not all of the coefficients  $a_1, a_2, \dots, a_p$  vanish. Vectors with no linear relation are *linearly independent*. A *linear combination* of vectors  $v_1, v_2, \dots, v_p \in V$  is vector

$$v_1 a_1 + v_2 a_2 + \dots + v_p a_p$$

for some elements  $a_1, a_2, \dots, a_p$  from  $k$ . One weird special case: we consider the vector  $0$  in  $V$  to be a linear combination of an *empty* set of vectors. The *span* of a set of vectors is the set of all of their linear combinations. A *basis* for  $V$  is a set of vectors so that every vector in  $V$  is a unique linear combination of these vectors. Note that some vector spaces might not have any basis. Also, if  $V = \{0\}$  then the empty set is a basis of  $V$ .

**Theorem 5.2.** *Any two bases of a vector space have the same number of elements.*

*Proof.* If  $V = \{0\}$ , the reader can check this special case and see that the empty set is the only basis. Take two bases  $u_1, u_2, \dots, u_p$  and  $v_1, v_2, \dots, v_q$  for  $V$ . If needed, swap bases to arrange that  $p \leq q$ . Write the vector  $v_1$  as a linear combination

$$v_1 = u_1 a_1 + u_2 a_2 + \dots + u_p a_p.$$

If all of the elements  $a_1, a_2, \dots, a_p$  vanish then  $v_1 = 0$ , and then zero can be written as more than one linear combination, a contradiction. By perhaps reordering our basis vectors, we can suppose that  $a_1 \neq 0$ . Solve for  $u_1$ :

$$u_1 = (v_1 + u_2(-a_2) + \dots + u_p(-a_p))a_1^{-1}.$$

We can write every basis vector  $u_1, u_2, \dots, u_p$  as a linear combination of  $v_1, u_2, \dots, u_p$ , and so the span of  $v_1, u_2, \dots, u_p$  is  $V$ . If there is more than one way to write a vector in  $V$  as such a linear combination, say

$$v_1b_1 + u_2b_2 + \dots + u_pb_p = v_1c_1 + u_2c_2 + \dots + u_pc_p$$

then the difference between the left and right hand sides is a linear relation

$$0 = v_1d_1 + u_2d_2 + \dots + u_pd_p.$$

Solve for  $v_1$  in terms of  $u_1$  and plug in:

$$0 = u_1a_1d_1 + u_2(d_2 + a_2d_1) + \dots + u_p(d_p + a_pd_1).$$

Since these  $u_1, u_2, \dots, u_p$  form a basis,

$$0 = a_1d_1 = d_2 + a_2d_1 = \dots = d_p + a_pd_1.$$

Multiplying the first equation by  $a_1^{-1}$  gives  $d_1 = 0$ , and then that gives  $0 = d_2 = \dots = d_p$ , a contradiction. Therefore  $v_1, u_2, \dots, u_p$  is a basis. By induction, repeating this process, we can get more and more of the vectors  $v_1, v_2, \dots, v_q$  to replace the vectors  $u_1, u_2, \dots, u_p$  and still have a basis. Finally, we will find a basis  $u_1, u_2, \dots, u_p$  consisting of a subset of  $p$  of the vectors  $v_1, v_2, \dots, v_q$ . But then every vector is a unique linear combination of some subset of these vectors, a contradiction unless that subset is all of them, i.e.  $p = q$ .  $\square$

The *dimension* of a vector space is the number of elements in a basis, or  $\infty$  if there is no basis. A *linear map*  $\varphi: V \rightarrow W$  between vector spaces  $V, W$  is a map so that  $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$  for any  $v_1, v_2$  in  $V$  and  $\varphi(va) = \varphi(v)a$  for any  $v$  in  $V$  and  $a$  in  $k$ .

**5.10** Prove that for any linear map  $\varphi: V \rightarrow W$  between vector spaces  $V = k^q$  and  $W = k^p$ , there is a unique  $p \times q$  matrix  $A$  with entries from  $k$  so that  $\varphi(v) = Av$  for any  $v$  in  $V$ .

The *kernel* of a linear map  $\varphi: V \rightarrow W$  is the set of all vectors  $v$  in  $V$  so that  $\varphi(v) = 0$ . The *image* of a linear map  $\varphi: V \rightarrow W$  is the set of all vectors  $w$  in  $W$  of the form  $w = \varphi(v)$  for some  $v$  in  $V$ .

Multiply matrices times vectors as usual. A matrix  $A$  is invertible if there is another such matrix  $B$  so that  $AB = BA = I$ .

**5.11** Prove that the kernel and image of a linear map  $\varphi: V \rightarrow W$  are subspaces of  $V$  and of  $W$  and that their dimensions add up to the dimension of  $V$ .

**5.12** Prove that for any basis  $v_1, v_2, \dots, v_n$  for a vector space  $V$ , there is a unique linear map  $\varphi: V \rightarrow W$  with has any given values  $\varphi(v_i) = w_i$ .

**5.13** Suppose that  $V$  and  $W$  are vector spaces of equal finite dimension, and that  $\varphi: V \rightarrow W$  and  $\psi: W \rightarrow V$  are linear maps with  $\varphi\psi = I$  the identity map. Prove that  $\psi\varphi = I$ .

**5.14** Suppose that  $V$  is a complex vector space with a real linear map  $J: V \rightarrow V$  which is complex conjugate linear, i.e.  $J(cv) = \bar{c}Jv$  for  $c \in \mathbb{C}$  and  $v \in V$ , and satisfies  $J^2 = -I$ . Explain how to turn  $V$  into a quaternionic vector space. Conversely, prove that every quaternionic vector space arises uniquely in this manner.

**5.15** For  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , take  $k$ -vector spaces  $V, W$  of equal dimension and a  $k$ -linear map  $\varphi: V \rightarrow W$ . Prove that the following are equivalent:

- a.  $\varphi$  is a bijection,
- b.  $\varphi$  has zero kernel,
- c.  $\varphi$  has a  $k$ -linear inverse.

By turning each quaternion into a  $4 \times 4$  real matrix, we can turn any  $n \times n$  quaternion matrix into a  $(4n) \times (4n)$  real matrix. There is no reasonable notion of quaternionic determinant, but:

**5.16\*** Prove that a quaternionic  $n \times n$  matrix has a quaternionic matrix inverse just when its determinant as a real  $4n \times 4n$  matrix is nonzero, and this occurs just when its determinant as a complex  $2n \times 2n$  matrix is nonzero.

## Inner products

A function  $f: V \rightarrow k$  on a  $k$ -vector space is *conjugate linear* if it is additive, i.e.  $f(x+y) = f(x) + f(y)$  for any  $x, y \in V$ , and  $f(xc) = \bar{c}f(x)$  for any  $x \in V$  and  $c \in k$ . An *inner product* on a  $k$ -vector space is a map

$$x, y \in V \mapsto \langle x, y \rangle,$$

- linear in  $y$  and conjugate linear in  $x$  and
- positive definite:  $\langle x, x \rangle > 0$  for any  $x \neq 0$ .

A  $k$ -vector space with an inner product is an *inner product space*.

**Lemma 5.3.** *Any two finite dimensional inner product spaces of the same dimension over the same skew field  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  are isometrically isomorphic, i.e. admit an isomorphism preserving inner products.*

*Proof.* Apply the Gram–Schmidt orthogonalization process to any basis exactly as in a complex vector space; the resulting orthonormal basis matches the standard basis of  $k^n$  by such a unique isomorphism, which identifies the inner products.  $\square$

## Idempotent isometries

An *idempotent linear transformation* on a vector space  $V$  is a linear map  $g: V \rightarrow V$  so that  $g^2 = I$ .

**5.17** Prove that every idempotent linear map is invertible.

**5.18** Prove that every idempotent linear map has only  $\pm 1$  as eigenvalues.

Take a vector space  $V$  of dimension  $n$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Equip  $V$  with an inner product; we say that  $V$  is an *inner product space*. A *linear isometry* is a  $k$ -linear isomorphism preserving given inner products. Let  $G$  be the group of linear isometries  $V \rightarrow V$ . For  $k = \mathbb{R}$ ,  $G = O(n)$  is the *orthogonal group*, while for  $k = \mathbb{C}$ ,  $G = U(n)$  is the *unitary group*, and for  $k = \mathbb{H}$  the group  $G$  is the *compact symplectic group*, denoted  $Sp(n)$ .

**5.19** Use the Cayley transform or the implicit function theorem to prove that  $G \subset GL(V)$  is a subgroup and a smooth compact submanifold.

To each linear subspace  $P \subseteq V$  of dimension  $p$ , associate the idempotent linear operator  $g = g_P$  so that  $gv = v$  for  $v \in P$  and so that  $gv = -v$  for  $v \in P^\perp$ .

**5.20** Let  $G$  be the group of linear isometries of  $V$  in its inner product. Prove that the map  $P \in \text{Gr}_p V \mapsto g = g_P \in G$  is an injection.

**5.21** Prove that the image of  $P \mapsto g_P$  is precisely the set of all isometric idempotent linear maps of with  $p$ -dimensional eigenspace of eigenvalue 1 and  $q$ -dimensional eigenspace of eigenvalue  $-1$ .

**Lemma 5.4.** *On any finite dimensional inner product space  $V$ , the isometric idempotent linear maps form a compact subset of the set of invertible linear maps  $GL(V)$ .*

*Proof.* Pick a linear subspace  $P \subset V$ . Since the linear isometry group  $G$  acts transitively on  $p$ -dimensional subspaces, it also acts transitively on isometric idempotent linear maps, by conjugation, and  $hgh^{-1} = g_{hP}$ , for any  $h \in G$ , a smooth map  $h \in G \mapsto hg_P h^{-1}$ . But  $G \subset k^{n \times n}$  is compact, so the image of this map is compact.  $\square$

**5.22** If  $g$  is an orthogonal idempotent linear map on a vector space  $V$ , with eigenspaces  $P, Q \subset V$  of eigenvalues 1,  $-1$  respectively, prove that  $A := \frac{1}{2}(I - g)$  is the orthogonal projection to  $Q$ , i.e.  $A = I$  of  $Q$  and  $A = 0$  on  $P$ .

**5.23** Prove that for all square real or complex matrices  $A$  whose complex eigenvalues  $\lambda$  all have  $|\lambda| < 1$ ,

$$(I + A)^{-1} = I - A + A^2 - A^3 + A^4 - \dots$$

Careful: prove absolute convergence of the right hand side. Prove the result for real matrices and then use it to prove that the same equation holds for any complex or quaternionic square matrix  $A$  if  $A$  is close enough to the identity matrix.

## Compactness

**Proposition 5.5.** *Every Grassmannian is compact.*

*Proof.* Take an inner product space  $V$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , and a linear subspace  $P_0 \subset V$  of dimension  $p$ . Elementary linear algebra: every linear subspace  $P \subset V$  of that same dimension is  $P = gP_0$  for some isometry  $g$ . Let  $G$  be the group of isometric linear isomorphisms of  $V$ , a smooth compact manifold (as we saw in problem 2.6 on page 18 and problem 5.19). Map  $\psi: g \in G \mapsto gP_0 \in \text{Gr}_p V$ , a surjective map. This

map is smooth: our charts on the Grassmannian are constructed precisely by applying linear transformations  $g$  to linear subspaces  $P_0 = (y = Ax)$ , so our expressions for transition maps are also expressions for how the entries of  $g$  alter  $P_0$  as expressed in the charts. So the map is continuous, and surjective, mapping from a compact manifold, so having compact image.  $\square$

**Lemma 5.6.** *Every Grassmannian is connected.*

*Proof.* Any two of our charts have overlapping domains, and connected images.  $\square$

### The Cartan embedding

Take an inner product space  $V$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , and a linear subspace  $P_0 \subset V$  of dimension  $p$ , say with associated idempotent isometry  $g_{P_0}$ . The *Cartan embedding* is the map that associates to every linear subspace  $P \subset V$  of dimension  $p$  the isometry  $g_P g_{P_0}$ . By definition, the Cartan embedding takes  $g_{P_0} \mapsto I$ , and so takes  $\text{Gr}_p V$  into the identity component of the isometry group.

**Lemma 5.7.** *The Cartan embedding is a smooth embedding of the Grassmannian as a compact submanifold of the group of linear isometries.*

*Proof.* Take a linear subspace  $P \subset k^{p+q}$  which is represented by a linear map  $y = Ax$ . Write  $A^*$  to mean the transpose conjugate of a matrix, i.e. transpose with all entries conjugated. (If  $k = \mathbb{R}$ ,  $A^*$  will mean transpose.) Let

$$h = (I + A^* A)^{-1},$$

and let

$$g = 2 \begin{pmatrix} h & hA^* \\ Ah & AhA^* \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Any element of  $P$  has the form

$$v = \begin{pmatrix} x \\ Ax \end{pmatrix}.$$

Plug in to see that  $gv = v$ . Any element of  $P^\perp$  has the form

$$v = \begin{pmatrix} -A^* y \\ y \end{pmatrix}.$$

Plug in and expand to see that  $gv = -v$ . So  $g = g_P$  is the idempotent associated to  $P$  and  $P^\perp$ . It follows that  $g$  is a linear isometry, but the reader can also check that  $gg^* = g^*g = I$ . For the open set of matrices  $A$  for which  $A^*A$  does not have  $-1$  as an eigenvalue, we let  $P$  be the graph of  $y = Ax$  and then the value of  $g = g_P$  depends smoothly on the value of  $P$ , and so does the value of  $g_P g_{P_0}$ , the Cartan embedding. By changing the choice of basis, we cover all of the Grassmannian with such charts. So the Cartan embedding is smooth in those charts. The Cartan embedding is clearly injective. We differentiate  $g$  at  $A = 0$ , as a function of  $A$ . First

$$\begin{aligned} h'(0)A &= \left. \frac{d}{dt} \right|_{t=0} (I + tA^*tA)^{-1}, \\ &= \left. \frac{d}{dt} \right|_{t=0} (I - t^2 A^* A + \dots), \\ &= 0. \end{aligned}$$

Hence

$$g'(0)A = 2 \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

Hence  $g'(0)$  is injective, so  $A \mapsto gg_{P_0}$  is an immersion near  $A = 0$ . For any given plane  $P$ , pick linear coordinates  $(x, y)$  in which  $P$  is represented by the graph  $y = 0$ . Then near  $g_P g_{P_0}$ , the Grassmannian is immersed by the above calculation. The Cartan embedding is an injective immersion of a compact manifold, and hence proper, so an embedding by lemma 5.1 on page 56.  $\square$

**Corollary 5.8.** *The Grassmannian  $\text{Gr}_p V$  of an inner product space  $V$  is canonically diffeomorphic to the set of all orthogonal projections of rank  $p$  on  $V$ , by taking each  $p$ -plane to the orthogonal projection with that  $p$ -plane as image.*

*Proof.* As in problem 5.22 on page 62, we can take any orthogonal idempotent linear map  $g$  of  $V$  with eigenspaces  $P, Q \subset V$  of eigenvalues  $1, -1$  respectively, to  $A := \frac{1}{2}(I + g)$ : the orthogonal projection to  $Q$ , i.e.  $A = I$  on  $P$  and  $A = 0$  on  $Q$ . Then  $g = 2A - I$ . These maps  $A \mapsto (2A - I)g_0$  and  $gg_0 \mapsto \frac{1}{2}(I + g)$  are clearly smooth.  $\square$

## Chapter 6

### Vector fields

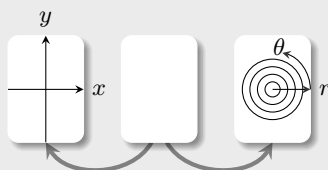
We define the notion of vector field, and use results from the theory of ordinary differential equations to prove that they flow.

#### Einstein notation

From here on, we adopt *Einstein notation*: write coordinates not as  $x_1, \dots, x_n$ , but as  $x^1, \dots, x^n$ . So  $x^2$  does *not* mean the square of something called  $x$ , when we are using Einstein notation. Similarly Einstein asks us to write each vector  $v$  in  $\mathbb{R}^n$  as having components  $v^i$ .

#### Functions

Think about how we work in rectangular and polar coordinates. Picture that there is a plane, and we draw two copies of it: one with  $x, y$  axes drawn, and another with  $r, \theta$  variables drawn:



These are two charts on that same plane. The function  $h = x^4 + y^4 + 2x^2y^2$  in rectangular coordinates is the function  $h = r^4$  in polar coordinates. We use the same letter  $h$  for these functions, each defined on its own copy of the plane, its own chart image.

If  $h: M \rightarrow \mathbb{R}$  is a function, and  $(U, \varphi)$  is a chart on  $M$ , consider the function  $h \circ \varphi^{-1}: \varphi(U) \rightarrow \mathbb{R}$ . It is traditional to refer to this function as “ $h$  in the coordinates  $x^1, x^2, \dots, x^n$ ”, and write it in the (bizarrel!) notation

$$h(x^1, x^2, \dots, x^n).$$

Strictly speaking, this is meaningless, since  $h$  is defined on  $M$ , not on  $\varphi(U)$ . But it is so useful a notational convention that we cannot resist the urge to follow this tradition. We were *already* doing this in polar coordinates: we still called the function  $h$ .

Similarly, given a smooth map  $F: P \rightarrow Q$  between manifolds, we write “ $y = F(x)$ ” in the coordinates  $x^1, x^2, \dots, x^p$  and  $y^1, y^2, \dots, y^q$  to mean the map  $\psi \circ F \circ \varphi^{-1}$  where  $(U, \varphi)$  is a chart on  $P$  with coordinates  $x^1, x^2, \dots, x^p$  and  $(W, \psi)$  is a chart on  $Q$  with coordinates  $y^1, y^2, \dots, y^q$ , and  $W$  contains  $F(U)$ . Never mention charts if we can avoid it, and just say “in the coordinates ...”.

Similarly, if you pick coordinates

$$x^1, x^2, \dots, x^n$$

and I pick coordinates

$$y^1, y^2, \dots, y^n,$$

on some manifold  $M$ , we mean that we each pick a chart on  $M$ . The transition map is then denoted  $y = y(x)$ .

### Einstein summation notation

From here on, we adopt *Einstein summation notation*: if we write an expression such as  $a_i b^i$ , where some index  $i$  arises twice in a single term, once up and once down, then we mean  $\sum_i a_i b^i$ . This saves writing a great many summation signs.

### Vector fields



Leonardo da Vinci, *Study of water falling into still water*, c. 1508-9

Roughly speaking, a vector field  $X$  on a manifold  $M$  is an representation of the velocity  $X(p)$  at each point  $p \in M$  of some fluid that fills the manifold. A *vector field* on a manifold  $M$  is a map  $X$  associating to each point  $m \in M$  a tangent vector  $X(m) \in T_m M$ . If we pick a chart  $(U, \varphi)$ , then on the domain of the chart, each vector  $X(m)$  has a tangent vector representative  $(\varphi, v)$ , which we write as  $(\varphi, X(x))$  where  $x = \varphi(m)$ . Just as above, this notation is a bit sneaky because we are writing out a vector  $X(x)$  in Euclidean space, as a function of a point  $x \in \varphi(U)$  in an open set in Euclidean space; so we use the same name  $X$  to mean the vector field on  $M$  and



on Euclidean space. We say that a vector field  $X$  on  $M$  is  $C^k$  to mean that the corresponding  $X(x)$  is  $C^k$  as a function of  $x \in \varphi(U)$  in every chart.

If  $\varphi: P \rightarrow Q$  is a  $C^1$  diffeomorphism of manifolds, and  $X$  is a vector field on  $P$ , we define a vector field  $\varphi_*X$  on  $Q$  by  $\varphi_*X(q) = \varphi'(p)X(p)$  whenever  $q = \varphi(p)$ . In particular, if  $\varphi$  is a chart, then  $X$  is identified with the vector field  $\varphi_*X$  in  $\mathbb{R}^n$ , which is just exactly what we just called  $X(x)$ , with  $x = \varphi(m)$ .

Write out the vector as a column:

$$X(x) = \begin{pmatrix} X^1(x) \\ X^2(x) \\ \vdots \\ X^n(x) \end{pmatrix},$$

for various functions  $X^i(x)$  defined on  $\varphi(U)$ .

Now the notation gets even more dangerous: if we change charts to  $(W, \psi)$ , written as usual as  $y = \psi(m)$ , then we have a different function  $X(y)$ . By the chain rule, if we change chart, with transition map  $y = y(x)$ , then the components change to some functions

$$\begin{pmatrix} X^1(y) \\ X^2(y) \\ \vdots \\ X^n(y) \end{pmatrix} = y'(x) \begin{pmatrix} X^1(x) \\ X^2(x) \\ \vdots \\ X^n(x) \end{pmatrix}.$$

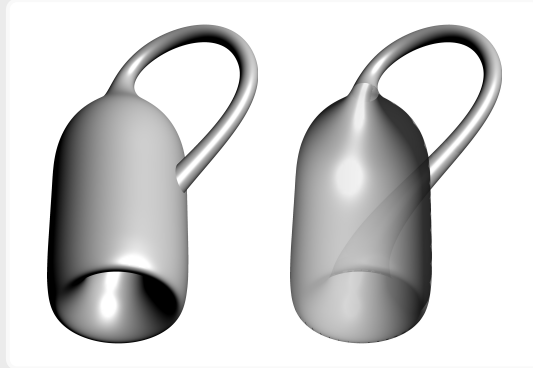
Count derivatives: on a  $C^\ell$  manifold, each transition map  $y(x)$  is  $C^\ell$ , but then  $y'(x)$  is only  $C^{\ell-1}$ , so there is no definition of a  $C^k$  vector field on a  $C^\ell$  manifold unless either  $k \leq \ell - 1$  or  $\ell = \infty$ .

The *Klein bottle* is the quotient  $\mathbb{R}^2/\Gamma$  by the rigid motions  $\Gamma$  of the plane generated by the two transformations

$$(x, y) \mapsto (x, y + 1)$$

and

$$(x, y) \mapsto (x + 1, -y).$$



Small enough open sets of the plane project locally diffeomorphically to the Klein bottle. The vector field  $X(x, y) = (1, 0)$  is invariant under these transfor-

mations, so projected by these local diffeomorphisms to a vector field, unambiguously defined on the Klein bottle. The vector field  $Y(x, y) = (0, 1)$  is not invariant under the second of these transformations, so it does *not* determine a vector field on the Klein bottle. But  $Z(x, y) = (0, \sin \pi x)$  does, for example, as does  $W(x, y) = (\sin(2\pi x), 0)$ .

The torus  $T^n = \mathbb{R}^n / \mathbb{Z}^n$  is the quotient of the  $\mathbb{R}^n$  by integer vector translations  $\mathbb{Z}^n$ . The constant vector fields on  $\mathbb{R}^n$  are invariant under  $\mathbb{Z}^n$ , so descend:



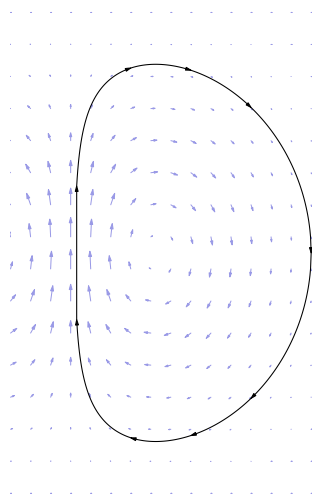
So do any vector fields whose components are functions expressible in terms of

$$\cos 2\pi x_1, \sin 2\pi x_1, \dots, \cos 2\pi x_n, \sin 2\pi x_n.$$

The real projective plane  $\mathbb{RP}^2$  is the quotient of the unit sphere  $S^2 \subset \mathbb{R}^3$  by  $\{\pm 1\}$ . A vector field  $X = X(x, y, z)$  on the sphere descends to  $\mathbb{RP}^2$  just when it satisfies  $X(-x, -y, -z) = -X(x, y, z)$ , so that it “pushes” a particle at  $-(x, y, z)$  with exactly the negative velocity to which it applies to  $(x, y, z)$ , i.e. the particles stay at opposite poles. For example, the vector field  $X(x, y, z) = (y, -x, 0)$  on the sphere descends.

**6.1** Is there a smooth vector field on the sphere  $S^2 \subset \mathbb{R}^3$  that vanishes at exactly one point?

Flow lines



Take a vector field  $X$  on a manifold  $M$ . Picture each vector  $X(p)$  as the velocity at the point  $p$  of some fluid that fills the manifold. If a small particle sat in a fluid flowing this way, how would the particle move? The *flow line* of  $X$  through a point  $p \in U$  is the path  $x(t)$  so that  $x(0) = p$  (start at  $p$ ), and so that

$$\frac{dx}{dt} = X(x(t))$$

defined for  $t$  in a maximal open interval for which such a path exists. From prior experience in ordinary differential equations, you know the existence and uniqueness theorem of ordinary differential equations, which states that for  $M$  an open subset of Euclidean space:

**Theorem 6.1** (Picard). *Every  $C^k$  vector field has a unique  $C^{k+1}$  flow line through each point.*

The same is true on any manifold; the proof is straightforward, using charts to identify open sets of  $M$  with open sets of Euclidean space, and thereby construct flow lines on each. The notation gets messy when we “glue” those flow lines together; [2] chapter 1 gives a proof. A vector field is *complete* if all of its flow lines are defined for all time.

**6.2** Find the flow lines of the vector field  $X(x, y) = (y, -x)$  on  $\mathbb{R}^2$ .

**6.3** Find the flow lines of the vector field  $X(x) = x^k$  on  $\mathbb{R}$  for  $k$  a positive integer.

Besides the existence and uniqueness of solutions of ordinary differential equations, you also know about the continuous dependence on initial conditions:

**Theorem 6.2** (Picard). *Pick a  $C^k$  vector field  $X$  on a manifold  $M$ . There is a unique  $C^k$  map  $\Phi: \text{open} \subset \mathbb{R} \times M \rightarrow M$ , called the flow of the vector field  $X$ , written  $\Phi_t p$  to mean  $\Phi(t, p)$ , so that, for each  $p \in M$ ,  $\Phi_t p$  is the flow line of  $X$  through  $p$ .*

See [2] p. 224 for a proof. From the definition of flow lines,

a.

$$\frac{d}{dt} \Phi_t(p) = X(\Phi_t(p))$$

and

b.  $\Phi_0 p = p$  for any  $p \in M$  and

c.  $\Phi_{t_2} \Phi_{t_1} p = \Phi_{t_1+t_2} p$  where defined, and if the left hand side is defined, so is the right, and

d. if  $\Phi_t p$  is defined, so is  $\Phi_{-t} q$  where  $q = \Phi_t p$ .

**6.4** Suppose that  $X$  is a smooth vector field on a manifold  $M$  and  $K \subset M$  is a compact set. Prove that there is a real number  $\varepsilon > 0$  so that every flow line through any point  $x \in K$  is defined for  $-\varepsilon \leq t \leq \varepsilon$ .

**6.5** Suppose that some flow line of a vector field stays inside some compact set. Prove that it is defined for all time.

**6.6** Prove that any vector field with compact support is complete.

**6.7** Take a constant  $c$  and a vector field  $X$  with flow  $\Phi$ . Let  $\Psi_t p := \Phi_{ct} p$ . Prove that  $\Psi$  is the flow of  $cX$ .

**6.8** Gronwall's inequality: A smooth proper function  $f: M \rightarrow \mathbb{R}$  *compresses* a vector field  $X$  if there are constants  $a_-, b_-, c_-$  and  $a_+, b_+, c_+$  so that for any  $x \in M$ , if  $f(x) > c_+$  then  $Xf < a_+|f| + b_+$ , and if  $f(x) < c_-$  then  $-Xf < a_-|f| + b_-$ . Prove that  $X$  is complete. Give an example where  $X$  is not obviously complete.

**6.9** For any points  $x_0, x_1$  in the interior of a closed ball  $\bar{B} \subset \mathbb{R}^n$ , prove that there is an orientation preserving smooth diffeomorphism  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  so that  $\varphi(x_0) = x_1$  and  $\varphi(x) = x$  for all points  $x$  outside of some smaller ball.

### Flow boxes

**Theorem 6.3** (The flow box theorem). *Pick a  $C^k$  vector field  $X$  on a manifold  $M$  and a point  $m_0 \in M$  at which  $X(m_0) \neq 0$ . There is a  $C^k$  chart in which  $X$  becomes*

$$X(x) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

*Proof.* Taking a chart around  $m_0$ , we can assume that  $M$  is an open subset of  $\mathbb{R}^n$ . Translate to arrange that  $m_0 = 0$ . Rotate to arrange that  $X(0)$  is not tangent to the set  $H = \{x_1 = 0\}$  on which the first coordinate vanishes. In other words, the  $x^1$ -component of  $X(0)$  is not zero. Or in other words, in the standard basis  $e_1, e_2, \dots, e_n$  of  $\mathbb{R}^n$ , the vectors  $X(0), e_2, e_3, \dots, e_n$  are linearly independent.

Let  $\Phi$  be the flow of  $X$ . Each point  $x$  in  $H$  has the form

$$x = (0, x_2, x_3, \dots, x_n).$$

Let

$$\varphi(s_1, s_2, s_3, \dots, s_n) := \Phi_{s_1}(0, s_2, s_3, \dots, s_n).$$

At the point  $s = 0$ , note that

$$\frac{\partial \varphi}{\partial s_1} = X, \frac{\partial \varphi}{\partial s_2} = e_2, \dots, \frac{\partial \varphi}{\partial s_n} = e_n,$$

linearly independent, so  $\varphi'(0)$  is an invertible matrix. Hence  $\varphi$  is a local diffeomorphism near the origin, by the inverse function theorem. The flow line of the vector field  $e_1$  through each point  $s = (s_1, \dots, s_n)$  is the path  $s + te_1$ . The flow lines of  $\varphi_* e_1$  are the paths

$$\begin{aligned} \varphi(s + te_1) &= \Phi_{s_1+t}(0, s_2, s_3, \dots, s_n), \\ &= \Phi_t \Phi_{s_1}(0, s_2, \dots, s_n), \\ &= \Phi_t \varphi(s), \end{aligned}$$

the flow lines of  $X$  through  $\varphi(s)$ . The velocity of each of these flow lines at  $t = 0$  is  $X(\varphi(s))$ , so  $\varphi_* e_1 = X$ .  $\square$

## Notation

Take a continuously differentiable vector field  $X$  on an open set in  $\mathbb{R}^n$ . We associated a flow  $\Phi_t$  to each continuously differentiable vector field, so that  $\Phi_0(x) = x$  and  $\frac{d}{dt}\Phi_t(x) = X(\Phi_t(x))$ . To each continuously differentiable function  $f$ , denote by  $Xf$  the rate at which  $f$  changes along the flow of  $X$ , in other words

$$Xf = \left. \frac{d}{dt} \right|_{t=0} f(\Phi_t(x)).$$

We can write  $X$  uniquely as

$$X = (X^1, X^2, \dots, X^n),$$

for some functions  $X^1, X^2, \dots, X^n$ . By the chain rule,

$$Xf(x) = X^1 \frac{\partial f}{\partial x^1} + X^2 \frac{\partial f}{\partial x^2} + \dots + X^n \frac{\partial f}{\partial x^n}.$$

For this reason, it is traditional to write vectors fields as

$$X = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2} + \dots + X^n \frac{\partial}{\partial x^n}$$

and think of them as operations eating functions. Using Einstein notation

$$X = X^i \frac{\partial}{\partial x^i}.$$

**6.10** Prove the *Leibnitz rule*:  $X(fg) = (Xf)g + f(Xg)$  for any two continuously differentiable functions  $f$  and  $g$ .

Henceforth, if we pick a chart  $x = \varphi(m)$ , we write our vector field  $\varphi_*X$  as

$$X^i \frac{\partial}{\partial x^i}$$

where the  $X^i$  are the components. We will be deliberately sloppy with notation, and write this expression as

$$X = X^i \frac{\partial}{\partial x^i},$$

and say that this is “ $X$  expressed in the coordinates  $x^1, \dots, x^n$ ”. If a vector field  $X$  is expressed in coordinates  $x^1, x^2, \dots, x^n$  as

$$X = X^i \frac{\partial}{\partial x^i}$$

then it is expressed in in coordinates  $y^1, y^2, \dots, y^n$  as

$$X = X^i(y) \frac{\partial}{\partial y^i} = \frac{\partial y^i}{\partial x^j} X^j(x) \frac{\partial}{\partial y^i}.$$

We begin to see the convenience of Einstein’s summation convention.

The real projective plane  $\mathbb{RP}^2$  is the quotient of the unit sphere  $S^2 \subset \mathbb{R}^3$  by  $\{\pm 1\}$ . A vector field  $X = X(x, y, z)$  on the sphere descends to  $\mathbb{RP}^2$  just when it satisfies  $X(-x, -y, -z) = -X(x, y, z)$ , so that it “pushes” a particle at  $-(x, y, z)$  with exactly the negative velocity with which it “pushes”  $(x, y, z)$ , i.e. the particles stay at opposite poles. For example, the vector field  $X(x, y, z) = (y, -x, 0)$  on the sphere descends. In our current notation, this is

$$X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

Take a chart on  $\mathbb{RP}^2$ , given as in the example on page 5, setting  $x = 1$ . So to each line, written as the span of a vector  $(x, y, z)$ , we let  $v = y/x$ ,  $w = z/x$  and then  $(v, w)$  is our chart. Along the flow of  $X$ ,  $\dot{x} = y$ ,  $\dot{y} = -x$  and  $\dot{z} = 0$ . So

$$\begin{aligned} \dot{v} &= \frac{x\dot{y} - y\dot{x}}{x^2}, \\ &= -\frac{x^2 + y^2}{x^2}, \\ &= -1 - v^2. \end{aligned}$$

Similarly,  $\dot{w} = -vw$ . So the vector field  $X$ , in our chart, is

$$X = -(1 + v^2) \frac{\partial}{\partial v} - vw \frac{\partial}{\partial w}.$$

A continuously differentiable map  $\psi: P \rightarrow Q$  is a *local diffeomorphism* if  $P$  can be covered by open sets  $U_a$  so that  $\psi$  restricts to each open set to be a diffeomorphism to some open subset of  $Q$ .

**6.11** If  $\psi: P \rightarrow Q$  is a local diffeomorphism and  $X$  is a vector field on  $Q$ , then there is a unique vector field  $\psi^*X$  on  $P$ , the *pullback*, so that  $\psi'(p)(\psi^*X)(p) = X(\psi(p))$  for every point  $p \in P$ .

## Cylinders

This section can be omitted without loss of continuity.

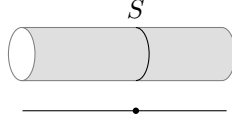
How do we see that a manifold “looks like a cylinder”?

**Theorem 6.4.** Suppose that  $f: M \rightarrow I$  is a smooth proper submersion to an interval  $(a, b) \subset \mathbb{R}$ .



Pick some  $c$  with  $a < c < b$ , and let  $S := (f = c)$ , i.e.

$$S := \{m \in M \mid f(m) = c\}.$$



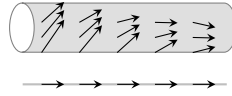
Then  $S \subset M$  is a smooth compact submanifold and there is a diffeomorphism

$$\varphi: S \times (a, b) \rightarrow M,$$

so that if  $m = \varphi(s, t)$  then  $t = f(m)$ .

*Proof.*

**6.12** Prove that there is a vector field  $X$  on  $M$  so that  $Xf = 1$ .



Since  $Xf = 1$ ,  $f(\Phi_t(x)) = t + f(m)$ .

**6.13\*** Prove that the flow  $\Phi$  of  $X$  is defined on the set

$$W := \{ (m, t) \in M \times \mathbb{R} \mid a < f(m) + t < b \}.$$

Let  $\varphi(s, t) := \Phi_{t-c}(s)$ , so  $f(\varphi(s, t)) = t$ .

**6.14\*** Prove that  $\varphi$  is a diffeomorphism.

□

**6.15** Suppose that  $\psi: M \rightarrow C$  is a proper submersion to a circle  $C$ . Prove that all fibers are diffeomorphic.

**6.16\*** Suppose that  $\varphi: P \rightarrow Q$  is a smooth proper submersion to a connected manifold  $Q$ . Prove that all fibers  $P_q := \varphi^{-1}\{q\}$  are diffeomorphic.

**6.17** We constructed many compact surfaces of any given genus  $g$  in problem 4.26 on page 47; why are they all diffeomorphic for any given  $g$ ?

## Gluing

This section can be omitted without loss of continuity.

Our current description of gluing on page 33 is inconvenient, because we like to describe a manifold as being glued together along a hypersurface.

**Theorem 6.5.** Take a smooth function  $f: M \rightarrow \mathbb{R}$  on a manifold  $M$ . Suppose that the level set  $(f = 0)$  is compact, and  $f' \neq 0$  at every point of that level set. Then, for any sufficiently small  $\varepsilon > 0$ ,  $M_{01} := (-\varepsilon < f < \varepsilon)$  is diffeomorphic to  $(f = 0) \times (-\varepsilon, \varepsilon)$ . Moreover  $M$  is obtained by gluing  $M_0 := (f > 0)$  to  $M_1 := (f < 0)$  along  $M_{01}$ .

Speaking informally, we say that  $M$  is  $(f \geq 0)$  glued to  $(f \leq 0)$  along  $(f = 0)$ , although this is not strictly correct from our definition of gluing.

*Proof.* Using a partition of unity, we put together a vector field  $X$  defined near  $(f = 0)$  along which  $Xf = 1$ . Its flow is defined on some interval  $-\varepsilon < t < \varepsilon$  by compactness of  $(f = 0)$ . So it flows the compact  $(f = 0)$  to various  $(f = t)$ , which are therefore also compact. So  $f$  is proper where  $-\varepsilon < f < \varepsilon$ . Apply theorem 6.4 on page 72.  $\square$

The compact oriented surface of genus  $g$  in  $\mathbb{R}^3$  from problem 4.26 on page 47 sits symmetrically around the horizontal plane, intersecting the plane in  $g + 1$  circles. Apply theorem 6.5 on the preceding page to the function  $z$ : the surface is obtained by gluing  $(z > 0)$  and  $(z < 0)$  to  $(z = 0) \times (-\varepsilon, \varepsilon)$ . Note that  $(z = 0)$  is a collection of  $g + 1$  disjoint circles. Both  $(z > 0)$  and  $(z < 0)$  are mapped by  $(x, y, z) \mapsto (x, y)$  diffeomorphically to the interior of a large disk with  $g$  smaller disjoint closed disks cut out from it. So informally we can say that the surface of genus  $g$  consists of one disk with  $g$  holes poked in it, glued to another along the  $g + 1$  circle boundaries.

**6.18\*** Prove that every compact oriented surface of genus  $g$  is diffeomorphic to the connected sum of  $g$  compact oriented surfaces of genus 1.

### The classification of compact curves

A *curve* means a 1-dimensional manifold.

**Theorem 6.6.** *Every compact  $C^k$  curve is  $C^k$  diffeomorphic to the disjoint union of finitely many circles, for any  $k \geq 1$  or  $k = \infty$ .*

This is difficult to prove. We need some lemmata.

**Lemma 6.7.** *Every oriented  $C^k$  curve bears a nowhere vanishing  $C^{k-1}$  vector field.*

*Proof.* On a curve, every vector field is locally

$$f(x) \frac{\partial}{\partial x}$$

for some function  $f(x)$ . On an oriented curve, in a positively oriented chart, the sign of  $f(x)$  is unchanged by change of chart. A vector field is *positive* if  $f(x) > 0$ . Each point of the curve lies in the domain of a positively oriented chart  $(U, x)$ . There is a bump function  $f \geq 0$  vanishing outside the chart domain and positive at the point. The vector field

$$f(x) \frac{\partial}{\partial x}$$

is positive at the given point, and defined on the whole curve. Each point lies in an open set on which some vector field is positive, and the vector field is defined on the whole curve and nowhere negative. Because the curve is compact, we can take a finite subcover by those open sets. Add up the vector fields.  $\square$

**Lemma 6.8.** *Every oriented compact  $C^k$  curve is  $C^k$  diffeomorphic to the disjoint union of finitely many circles,  $k \geq 1$  or  $k = \infty$ .*



*Proof.* Call the curve  $C$ . We can assume  $C$  is connected and bears a nowhere zero vector field, flowing in the positively oriented direction. Since  $C$  is compact, the vector field is complete. Each flow line is a local diffeomorphism  $x: \mathbb{R} \rightarrow C$ . The image is open. Every point lies in the image of a flow line, and these images are all open, so they cover  $C$  in open sets. If two flow lines pass through the same point of  $C$ , say at some times  $t_0$  and  $t_1$  respectively, then the first is the same as the second but with time  $t$  shifted by  $t_1 - t_0$ . So their image is the same open subset of  $C$ . So the open sets that are the images of flow lines are either equal or disjoint. Since  $C$  is connected, there can be only one such open set, i.e. each flow line is surjective. If injective, it is a bijective local diffeomorphism, so a diffeomorphism. But  $\mathbb{R}$  is not compact, so no flow line can be a diffeomorphism. So the flow line is not injective, say  $x(t_0 + h) = x(t_0)$  for some  $t_0$  and some  $h > 0$ . Because  $x$  is a local diffeomorphism, once we fix  $t_0$ , there is a least such  $h > 0$ . It follows that  $t \mapsto x(t + h)$  is also a flow line through the same point, so  $x(t + h) = x(t)$  for all  $t$ . Hence  $x$  descends to a bijective local diffeomorphism from the circle of perimeter  $h$ , hence a diffeomorphism.  $\square$

Finally, we prove theorem 6.6 on the preceding page:

**Lemma 6.9.** *Every compact  $C^1$  curve is orientable.*

*Proof.* Its two-to-one oriented covering space  $\hat{C} \rightarrow C$  (as in problem 3.11 on page 28) is a compact oriented curve, so a circle. The quotient  $\hat{C} \rightarrow C$  quotients out 2 points to 1 by a local diffeomorphism, so a unique diffeomorphism  $\hat{C} \rightarrow \hat{C}$  swaps these two points. This diffeomorphism is not orientation preserving; otherwise that orientation would survive on  $C$ . This diffeomorphism has no fixed points, because it swaps each point of  $\hat{C}$  with the unique other point of  $\hat{C}$  which maps to the same point of  $C$ . Since  $\hat{C}$  is a circle, we can write each point of  $\hat{C}$  as  $e^{i\theta}$ , and the diffeomorphism  $\hat{C} \rightarrow \hat{C}$  as  $e^{i\theta} \mapsto e^{i\alpha(\theta)}$ , for an angle function, i.e. diffeomorphism  $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ , reversing orientation, i.e. decreasing. But then  $\alpha$  must take large positive  $\theta$  to large negative  $\alpha(\theta)$  and vice versa. Therefore  $\alpha(\theta) = \theta$  has a unique solution  $\theta = \theta_0$ , a fixed point of  $\hat{C} \rightarrow \hat{C}$ . But then  $\hat{C} \rightarrow \hat{C}$  is not swapping two points there.  $\square$

**6.19** Suppose that  $M$  is a 1-dimensional manifold on which there is a nowhere zero complete vector field. Prove that every component of  $M$  is diffeomorphic to either a circle or the real number line. (Also see problem 3.2.2 on page 257.)

**6.20** Prove that every compact  $C^1$  curve admits a unique  $C^\infty$  smooth structure up to diffeomorphism.

## Tubular neighborhoods

This section can be omitted without loss of continuity.

The *normal space*  $\nu_{S,x}$  at a point  $x \in S$  of an immersed submanifold  $S \subseteq M$  of a manifold  $M$  is the quotient space  $\nu_{S,x} := T_x M / T_x S$ . The *normal bundle*  $\nu_S$  is the set of pairs  $(x, v)$  so that  $x \in S$  and  $v \in \nu_{S,x}$ . Let  $\pi: (x, v) \in \nu_S \rightarrow x \in S$ . A *local section* of the normal bundle over an open set  $U \subseteq S$  is a choice of element  $v(x) \in \nu_{S,x}$  for so that, at least locally, there is some vector field  $X$  on an open subset of  $M$  with  $v(x) = X(x) + T_x S$ .

**6.21** If  $S$  is a  $C^k$  immersed submanifold,  $k \geq 2$ , prove that  $\nu_S$  has a  $C^{k-1}$  structure so that

- $\pi$  is  $C^{k-1}$  and
- every  $C^{k-1}$  local section is a  $C^{k-1}$  map to  $\nu_S$ .

A *tubular neighborhood* of  $S$  is a smooth  $C^{k-1}$  embedding  $\varphi: U \rightarrow M$  so that:

- The set  $U \subseteq \nu_S$  is open.
- Each normal space intersects  $U$  in a convex set containing the origin.
- For every point  $x \in S$ ,  $\varphi(x, 0) = x$ .
- For every point  $(x, v) \in U$ ,  $\varphi'(x, 0)(0, v) = v$ .
- The topological boundary of  $U \subset \nu_S$  is Lipschitz in the associated charts we have described above on  $\nu_S$ .

**6.22** Suppose that  $S \subseteq M$  is an embedded submanifold and that  $M \subseteq \mathbb{R}^n$  is an embedded submanifold. Prove that  $S$  has a tubular neighborhood in  $M$ . Hint: smooth deformation retraction, problem 4.13 on page 44.

## Chapter 7

### Brackets of vector fields

#### The flow of a vector field

Recall that, on any manifold, every  $C^k$  vector field  $X$ ,  $k \geq 1$ , has a unique *flow*, a  $C^k$  map  $\Phi: \text{open} \subset \mathbb{R} \times M \rightarrow M$ , denoted  $\Phi_t m$  to mean  $\Phi(t, m)$ , so that

a.  $\Phi_0 m = m$  for any  $m$  and

b.

$$\frac{d}{dt} \Phi_t m = X(\Phi_t m)$$

for any  $t, m$  for which  $\Phi_t m$  is defined and,

c. for each  $m$ ,  $\Phi_t m$  is defined on a connected open interval of real numbers  $t$  and

d.  $\Phi_{t_2} \Phi_{t_1} m = \Phi_{t_1+t_2} m$  whenever this is defined.

e.  $\Phi$  is defined on the largest open set in  $\mathbb{R} \times M$  for which all of these properties hold.

**7.1** Take an  $n \times n$  matrix  $A$  and consider the vector field  $X(x) = Ax$  on  $\mathbb{R}^n$ . Prove that the flow of  $X$  is  $\Phi_t x = e^{tA} x$ , where  $e^{tA}$  means

$$e^A = I + A + \frac{A^2}{2} + \cdots = \sum_n \frac{A^n}{n!}.$$

Therefore we denote the flow of *any* vector field  $X$  as  $e^{tX} m$  instead of  $\Phi_t m$ . *Danger:* the flow of a vector field  $X$  is *not* usually given in coordinates by anything as simple as just some exponential function; the notation  $e^{tX}$  is dangerously misleading. By problem 6.7 on page 70, the flow  $\Phi_t$  of a vector field  $X$  at time  $t$  depends only on the product  $tX$ , so the notation is not ambiguous. The notation fits neatly with the fact that  $e^{sX} e^{tX} = e^{(s+t)X}$ , and

$$\frac{d}{dt} e^{tX} = X \circ e^{tX},$$

similar to the exponential function.

**7.2** Show that the flow of  $X(x) = x^2 \frac{\partial}{\partial x}$  on the real number line  $x \in \mathbb{R}$  is

$$e^{tX} x_0 = \frac{x_0}{1 - x_0 t}.$$

(In particular, you can see that there is really no exponential function involved in this flow. You can also see that the flow goes “to infinity” in finite time.)

### Complete vector fields

**7.3** A *symmetry* of a vector field  $X$  on a manifold  $M$  is a diffeomorphism  $\phi: M \rightarrow M$  so that  $\phi_*X = X$ . Prove that a vector field is complete if and only if its group of symmetries acts transitively on every one of its flow lines.

**7.4** Suppose that  $X$  and  $Y$  are vector fields on a manifold  $M$  and that  $X = Y$  outside of some compact subset of  $M$ . Prove that  $X$  is complete if and only if  $Y$  is complete.

**7.5** Give an example of two complete vector fields  $X$  and  $Y$  on the same manifold  $M$  so that  $X + Y$  is incomplete.

### Definition of the Lie bracket

Recall that  $Xf$  means the rate at which the function  $f$  increases along the flow of the vector field  $X$ .

**7.6** If  $X, Y$  are two vector fields, prove that the operation  $f \mapsto X(Yf) - Y(Xf)$  is also  $Zf$  for a unique vector field  $Z$ .

Denote this vector field  $[XY]$  (also denoted  $[X, Y]$ ), and call it the *Lie bracket* of  $X$  and  $Y$ . Clearly  $[XY]$  is linear in  $X$  and in  $Y$  and  $[YX] = -[XY]$ .

**7.7** If  $X, Y$  are two vector fields, then in any local coordinates  $x^1, x^2, \dots, x^n$ , write out

$$X(x) = X^i(x) \frac{\partial}{\partial x^i}, Y(x) = Y^i(x) \frac{\partial}{\partial x^i},$$

and prove that

$$[XY] = \left( X^i(x) \frac{\partial Y^j}{\partial x^i} - Y^i(x) \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

**7.8** Suppose that  $A$  and  $B$  are  $n \times n$  matrices, and that  $X$  and  $Y$  are the vector fields on  $\mathbb{R}^n$  given by  $X(x) = Ax$  and  $Y(x) = Bx$ . Calculate  $[XY]$ .

**7.9** Check the tedious but elementary *Jacobi identity*:

$$0 = [X[YZ]] + [Z[XY]] + [Y[ZX]]$$

for any  $C^2$  vector fields  $X, Y, Z$ .

### The meaning of the Lie bracket

**Lemma 7.1.** On a manifold  $M$ , if  $f: M \rightarrow \mathbb{R}$  is a  $C^2$  function and  $X$  and  $Y$  are  $C^2$  vector fields on  $M$  then

$$\frac{\partial^2}{\partial s \partial t} \Big|_{s=t=0} \left( f \circ e^{sX} \circ e^{tY} - f \circ e^{tY} \circ e^{sX} \right) = -[XY]f.$$

*Proof.* Clearly

$$Xf = \frac{d}{ds} \Big|_{s=0} f \circ e^{sX}$$

so

$$YXf = \frac{d}{dt} \Big|_{t=0} \frac{d}{ds} \Big|_{s=0} f \circ e^{sX} \circ e^{tY},$$

and if we switch  $X$  and  $Y$  the result is clear.  $\square$

**Corollary 7.2.**

$$f \circ e^{sX} \circ e^{tY} - f \circ e^{tY} \circ e^{sX} = -st[XY]f + \text{cubic terms in } s, t$$

*Proof.* The left hand side vanishes if  $s = 0$  or  $t = 0$  so its Taylor series has no powers purely of  $s$  or of  $t$ .  $\square$

**Theorem 7.3.** Two  $C^2$  vector fields  $X$  and  $Y$  commute, i.e.  $[XY] = 0$ , just when their flows commute, i.e.  $e^{sX} e^{tY} m = e^{tY} e^{sX} m$  for all  $s, t, m$  for which this is defined.

*Proof.* By lemma 7.1 on the facing page, if the flows commute, then

$$f \circ e^{sX} \circ e^{tY} - f \circ e^{tY} \circ e^{sX}$$

vanishes, and so its derivatives vanish, for any choice of smooth function  $f$ , and so  $[XY] = 0$ .

Suppose that  $[XY] = 0$ , and we will show that the flows commute. The function

$$f \circ e^{sX} \circ e^{tY} - f \circ e^{tY} \circ e^{sX}$$

is bounded by a cubic polynomial in  $s, t$  for small  $s, t$ . Note that all bounds in this proof are uniform on compact sets. Let us first prove the result in some chart, so assume our manifold is an open set in Euclidean space. Since we can take  $f$  to be any smooth function, we can in particular take  $f$  to be any of the coordinate functions  $f(x) = x^1$  or  $f(x) = x^2$ , etc., so  $e^{sX} e^{tY} - e^{tY} e^{sX}$  is bounded by a cubic polynomial in  $s, t$  for small  $s, t$ .

Take a large integer  $N$ , let  $\varepsilon := 1/N$  and look at the expression  $e^{sX} = e^{\varepsilon sX} \dots e^{\varepsilon sX}$ , with  $N$  factors. Write out

$$e^{sX} e^{tY} = e^{\varepsilon sX} \dots e^{\varepsilon sX} e^{\varepsilon tY} \dots e^{\varepsilon tY}.$$

Swap the middle two factors:

$$e^{sX} e^{tY} = e^{\varepsilon sX} \dots \underbrace{e^{\varepsilon sX} e^{\varepsilon tY}} \dots e^{\varepsilon tY}.$$

You alter the result by an “error” proportional to  $\varepsilon^3$ . Swap *all* of the  $e^{\varepsilon sX}$  factors over *all* of the  $e^{\varepsilon tY}$  factors; we make  $N(N+1)/2 \sim N^2$  swaps, so total error  $\varepsilon^3 N^2 \sim \varepsilon$ :

$$e^{sX} e^{tY} - e^{tY} e^{sX} \sim \varepsilon$$

As  $N \rightarrow \infty$ ,  $\varepsilon \rightarrow 0$ , while the left hand side of this expression is independent of  $N$ , so

$$e^{sX} e^{tY} = e^{tY} e^{sX}.$$

Our proof so far only works if we force  $s$  and  $t$  to lie inside some small enough compact set near  $s = t = 0$ , so that all of the expressions we compute are defined and we stay inside some chart. But once we have this result near a point, for small enough  $s$  and  $t$ , we can then split up a large time flow into a product of smaller time flows.  $\square$

**7.10** For any vector field  $X$  and diffeomorphism  $F$ , prove that  $e^{t(F_*X)} \circ F = F \circ e^{tX}$ .

**7.11** For any vector fields  $X, Y$ , prove that  $e^{t(e_*^s X Y)} \circ e^{sX} = e^{sX} \circ e^{tY}$ .

**7.12\*** Prove that, for any smooth function  $f$  and smooth vector fields  $X, Y$  on a manifold  $M$ ,

$$\left. \frac{d}{dt} \right|_{t=0} f \circ e^{-\sqrt{t}Y} e^{-\sqrt{t}X} e^{\sqrt{t}Y} e^{\sqrt{t}X} = [XY]f.$$

In other words, you can move as close as you like to any point along the flow of  $[XY]$ , by rapidly switching from flowing along  $X$  and  $Y$ .

Example: skating



Skate on ice. Your skate sits at a point  $(x, y)$  of the plane, pointing in the direction of an angle  $\theta$ . You can only slide forward on a skate:

$$\begin{aligned}\dot{x} &= u \cos \theta, \\ \dot{y} &= u \sin \theta,\end{aligned}$$

where  $u$  is your speed. You control your speed by pushing backward with the other skate (on the other foot). You lean to one side or to other to control how your angle changes:  $\dot{\theta} = w$ , as you move forward, so that  $w$  is actually the curvature of the path you follow along the ice. On the manifold  $\mathbb{R}_{x,y}^2 \times S_\theta^1$ , where  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$  is parameterized by  $\theta$ , we have vector fields

$$\begin{aligned}X(x, y, \theta) &= \cos \theta \partial_x + \sin \theta \partial_y, \\ Y(x, y, \theta) &= \cos \theta \partial_x + \sin \theta \partial_y + \partial_\theta,\end{aligned}$$

writing  $\partial_x$  to mean  $\frac{\partial}{\partial x}$ . The vector field  $X$  represents skating without leaning over, while  $Y$  has you leaning to the left. There is some choice here about how much curvature you generate in  $Y$  when you lean over; we set the coefficient of  $\partial_\theta$  to 1 for simplicity of notation. Skate by taking numbers  $u, w$  and following the flow of  $uX + wY$ : we have 2 degrees of freedom  $u, w$ .

**7.13** Compute the flows of  $X$  and  $Y$ . Draw pictures of skates in the plane to explain  $X$  and  $Y$ .

The bracket is

$$[XY] = \sin \theta \frac{\partial}{\partial x} - \cos \theta \frac{\partial}{\partial y},$$

a third, linearly independent, vector field. Geometrically,  $[XY]$  fixes the angle of the skate, and slides the skate along the ice, in the direction perpendicular to the blade. This is not physically possible motion for a skate. By problem 7.12 on the preceding page, we can approximate the flow of  $[XY]$  by moving back and forth along the flows of  $X$  and  $Y$  small amounts. Indeed, if we draw out the motion  $e^{-\sqrt{i}Y}e^{-\sqrt{i}X}e^{\sqrt{i}Y}e^{\sqrt{i}X}$  explicitly, we see the skate “parallel parking”. We can thereby move between any points of  $M$ : we can skate from anywhere to anywhere, with any starting and ending angles.

**7.14** Describe the steering of a car, in a similar fashion to the description of skating above. Explain how to parallel park, in terms of the flow of the “steering without moving” vector field and the “driving in a fixed direction” vector field.

**7.15** Suppose that  $S \subset M$  is an immersed submanifold, and that  $X, Y$  are vector fields on  $M$ , with both  $X(s), Y(s) \in T_s S$  for every point  $s \in S$ . Prove that  $[XY](s) \in T_s S$  for every point  $s \in S$ .

**Lemma 7.4.** For any vector fields  $X, Y$  on a manifold  $M$ ,

$$[XY] = \left. \frac{d}{dt} \right|_{t=0} e_*^{-tX} Y.$$

*Proof.* Pick a smooth function  $f$ . Fix a number  $t$  and let  $\phi = e^{-tX}$ . The definition of  $\phi_*$  is

$$(\phi_* Y)f = (Y(f \circ \phi)) \circ \phi^{-1}$$

for any smooth function  $f: M \rightarrow \mathbb{R}$ . Replace  $f$  by  $f \circ \phi^{-1}$ :

$$(\phi_* Y)(f \circ \phi^{-1}) = (Yf) \circ \phi^{-1},$$

for any smooth function  $f: M \rightarrow \mathbb{R}$ , i.e.

$$(e_*^{-tX} Y)(f \circ e^{tX}) = (Yf) \circ e^{tX}.$$

Differentiate both sides:

$$\left. \frac{d}{dt} \right|_{t=0} (e_*^{-tX} Y)(f \circ e^{tX}) = \left. \frac{d}{dt} \right|_{t=0} (Yf) \circ e^{tX}$$

to find

$$\left. \frac{d}{dt} \right|_{t=0} e_*^{-tX} Yf + Y(Xf) = X(Yf).$$

□

### The simultaneous flow box theorem

**Theorem 7.5** (The simultaneous flow box theorem). *Suppose that  $X_1, \dots, X_p$  are linearly independent commuting vector fields on a manifold  $M$ . Near each point of  $M$ , there are coordinates  $x^1, \dots, x^n$  in which these vector fields are*

$$X_1 = \partial_{x^1}, \dots, X_p = \partial_{x^p}.$$

*Proof.* Take any coordinates  $x^1, \dots, x^p, y^1, \dots, y^q$  with  $p + q = n$ , so that  $X_1, \dots, X_p$  have linearly independent components in the  $x$ -variables. Define a map

$$\varphi(t^1, \dots, t^p, y) = e^{t^1 X_1} \dots e^{t^p X_p}(0, y)$$

defined in some open set in  $\mathbb{R}^{p+q}$ . At the origin,

$$\varphi'(0, 0) \partial_{t^i} = X_i(0, 0),$$

while

$$\varphi'(0, 0) \partial_{y^j} = \partial_{y^j}.$$

By the inverse function theorem,  $\varphi$  is a diffeomorphism near the origin. At any point,

$$\varphi_* \frac{\partial}{\partial t^1} = \frac{\partial \varphi}{\partial t^1}(t, y) = X_1.$$

But all of the vector fields commute, so we can reorder the exponentials and so

$$\varphi_* \frac{\partial}{\partial t^j} = \frac{\partial \varphi}{\partial t^j}(t, y) = X_j$$

for every  $j = 1, 2, \dots, p$ . □

### Maps of manifolds and correspondence of vector fields

If  $\varphi: P \rightarrow Q$  is a smooth map, a pair of vector fields  $X$  on  $P$  and  $Y$  on  $Q$  *correspond* if  $\varphi'(p)X(p) = Y(q)$  when  $q = \varphi(p)$ .

The map  $(x, y) = \varphi(s, t) = (s, 0)$  makes the vector field  $X = s\partial_s + st\partial_t$  correspond to *both* of the vector fields  $Y = x\partial_x$  and  $Z = (x + y)\partial_x + y^3\partial_y$ .

**7.16** Draw a picture of a vector field  $X$  on  $\mathbb{R}^2$  so that, under the map  $\varphi(s, t) = s$ , there is *no* corresponding vector field  $Y$  on  $\mathbb{R}$ . What do vector fields  $X$  on  $\mathbb{R}^2$  have to look like to have corresponding vector fields on  $\mathbb{R}$  under this particular map?

**7.17** If  $\varphi: P \rightarrow Q$  is a submersion, prove that each vector field  $X$  on  $P$  has at *most* one corresponding vector field  $Y$  on  $Q$ .

**7.18** Prove that the flows of corresponding vector fields correspond:  $\varphi \circ e^{tX} = e^{tY} \circ \varphi$ , wherever  $e^{tX}$  is defined.

**7.19** Prove that if  $X, Y$  have corresponding vector fields  $\bar{X}, \bar{Y}$ , then  $[XY]$  has corresponding vector field  $[\bar{X}\bar{Y}]$ .



## Chapter 8

### Lie groups

#### Definition

A *Lie group* is a group  $G$  which is also a manifold so that the multiplication operation  $(x, y) \in G \times G \mapsto xy \in G$  is smooth. Denote the identity element as  $1 \in G$ .

**8.1** Prove that the inverse operation is then also smooth by the implicit function theorem.

#### Examples

Most differential equations arising in science and engineering have symmetry groups which are Lie groups. Lie thought of Lie groups as symmetry groups of differential equations, but all of the easy examples, and all of the important ones, come from linear algebra.

Let  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$  be the real numbers, complex numbers or quaternions. If  $V$  is a finite dimensional  $k$ -vector space, the group  $G = \text{GL}(V)$  of  $k$ -linear automorphisms of  $V$  is a Lie group, since (in any basis) it is an open subset of  $k^{n \times n}$  and the group operation is given by multiplying matrix entries and adding up. If  $V = k^n$ , denote  $\text{GL}(V)$  by  $\text{GL}(n, k)$ .

If  $G$  is a Lie group, and  $H \subseteq G$  is a submanifold and a subgroup then  $H$  is a *Lie subgroup*.

The group of lower triangular invertible matrices, and the group of strictly lower triangular invertible matrices are Lie groups.

The quaternions under addition form a Lie group, as do the nonzero quaternions under multiplication, and the unit length quaternions under multiplication.

We will say that a quaternionic  $n \times n$  matrix  $A$  belongs to the special quaternionic linear group  $\text{SL}(n, \mathbb{H})$  if its determinant as a real  $4n \times 4n$  matrix is 1.

An *affine transformation* of  $k^n$  is a transformation  $Tx = Ax + b$  where  $A \in \text{GL}(n, k)$  and  $b \in k^n$ , i.e. a linear transformation plus a translation. The collection of all affine transformations forms a group, the *affine group* of  $k^n$ , a Lie group by taking entries of  $A$  and  $b$  as a chart.

We will see (theorem 11.4 on page 105) that any subgroup of a Lie group which is also a closed subset of that Lie group is an embedded submanifold, and so an embedded Lie subgroup. From this fact (or using the implicit function theorem and problem 9.8), one easily sees that all of the following are embedded Lie subgroups of  $\text{GL}(n, k)$ , so they are Lie groups.

|                             |                                                                                                                                                                             |
|-----------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\text{GL}(n, k)$           | general linear group: invertible $n \times n$ matrices in $k^{n \times n}$                                                                                                  |
| $\text{SL}(n, \mathbb{R})$  | special linear group: $\det = 1$                                                                                                                                            |
| $\text{SL}(n, \mathbb{C})$  | complex special linear group: $\det = 1$                                                                                                                                    |
| $\text{SL}(n, \mathbb{H})$  | quaternionic special linear group: $\text{SL}(4n, \mathbb{R}) \cap \text{GL}(n, \mathbb{H})$                                                                                |
| $\text{O}(n)$               | orthogonal group: stabilizer of $dx_1^2 + dx_2^2 + \cdots + dx_n^2$                                                                                                         |
| $\text{SO}(n)$              | special orthogonal group: $\text{SL}(n, \mathbb{R}) \cap \text{O}(n)$                                                                                                       |
| $\text{SO}(p, q)$           | special pseudoorthogonal group: stabilizer of orientation and $\sum_p dx_i^2 - \sum_q dy_i^2$                                                                               |
| $\text{SO}(n, \mathbb{C})$  | complex special orthogonal group: complex linear stabilizer of $dz_1^2 + dz_2^2 + \cdots + dz_n^2$                                                                          |
| $\text{SO}(n, \mathbb{H})$  | quaternionic orthogonal group: stabilizer of $d\bar{q}_1 idq_1 + \cdots + d\bar{q}_n idq_n$ in $\text{GL}(n, \mathbb{H})$                                                   |
| $\text{U}(n)$               | unitary group: $\text{O}(2n) \cap \text{GL}(n, \mathbb{C})$                                                                                                                 |
| $\text{SU}(n)$              | special unitary group: $\text{SL}(n, \mathbb{C}) \cap \text{U}(n)$                                                                                                          |
| $\text{SU}(p, q)$           | special pseudounitary group: stabilizer in $\text{SL}(n, \mathbb{C})$ of a Hermitian $(p, q)$ -form $ dz_1 ^2 + \cdots +  dz_p ^2 - ( dz_{p+1} ^2 + \cdots +  dz_{p+q} ^2)$ |
| $\text{Sp}(2n, \mathbb{R})$ | symplectic group: stabilizer of $\sum dx_j \wedge dy_j$                                                                                                                     |
| $\text{Sp}(2n, \mathbb{C})$ | complex symplectic group: complex stabilizer of $\sum dz_j \wedge dw_j$                                                                                                     |
| $\text{Sp}(n)$              | quaternionic unitary group: $\text{SU}(2n) \cap \text{GL}(n, \mathbb{H})$                                                                                                   |
| $\text{Sp}(p, q)$           | quaternionic pseudounitary group: $\text{SU}(2p, 2q) \cap \text{GL}(n, \mathbb{H})$                                                                                         |

**8.2** Prove that  $\text{SU}(2)$  is diffeomorphic to the 3-sphere  $S^3 \subset \mathbb{R}^4$ .

**8.3** Denote by  $\text{GL}(n, \mathbb{R})$  the set of invertible  $n \times n$  matrices. Prove that  $\text{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$  is an open set.

An *isometry* is a distance preserving transformation. A *rotation* about a point is an isometry fixing that point, which can be continuously deformed, to the identity transformation, through a family of isometries fixing that same point. An orthogonal matrix  $A$  is a square matrix which satisfies  $A^t A = I$ , where  $A^t$  is the transpose of  $A$ ; if in addition  $\det A = 1$ ,  $A$  is a *rotation matrix*.

**8.4\*** Prove:

- a. A linear transformation of  $\mathbb{R}^n$  is represented by an orthogonal matrix just when it is an isometry.
- b. Every isometry of  $\mathbb{R}^n$  fixing the origin is a linear transformation, expressed by an orthogonal matrix.
- c. Every isometry of  $\mathbb{R}^n$  is expressed uniquely as a composition of a translation with an orthogonal linear transformation, i.e.  $x \mapsto Ax + b$ , where  $b \in \mathbb{R}^n$  and  $A$  is an orthogonal matrix.
- d. A linear transformation of  $\mathbb{R}^n$  is represented by a unique rotation matrix just when it is a rotation fixing the origin.
- e. Every rotation of  $\mathbb{R}^n$  fixing the origin is a linear map, expressed by a rotation matrix.

**8.5\*** Find  $\det'$ : let  $f: \mathbb{R}^{n^2} \rightarrow \mathbb{R}$  be the map  $f(M) = \det(M)$ . Prove that if  $M$  is an invertible  $n \times n$  matrix then

$$f'(M)N = \det(M) \operatorname{tr}(M^{-1}N).$$

**8.6\***

- a. Prove that the product of any two upper triangular square matrices is upper triangular. Hint: induction, and write any upper triangular matrix  $A$  as

$$A = \begin{pmatrix} B & c \\ 0 & d \end{pmatrix}$$

where  $B$  is upper triangular and  $d$  is a number.

- b. Use the same trick to prove that the inverse of any invertible upper triangular matrix is upper triangular.
- c. Why are the invertible upper triangular matrices a Lie subgroup of the Lie group of invertible matrices?

**8.7** Let  $\operatorname{GL}^+(n, \mathbb{R})$  be the set of all  $n \times n$  real matrices with positive determinant. Use Gram–Schmidt orthogonalization to construct retractions

$$\begin{aligned} \operatorname{GL}^+(n, \mathbb{R}) &\rightarrow \operatorname{SO}(n), \\ \operatorname{GL}(n, \mathbb{R}) &\rightarrow \operatorname{O}(n), \\ \operatorname{GL}(n, \mathbb{C}) &\rightarrow \operatorname{U}(n), \\ \operatorname{SL}(n, \mathbb{C}) &\rightarrow \operatorname{SU}(n), \\ \operatorname{GL}(n, \mathbb{H}) &\rightarrow \operatorname{Sp}(n). \end{aligned}$$

By problem 4.42 on page 52  $\operatorname{SO}(n)$ ,  $\operatorname{O}(n)$ ,  $\operatorname{SU}(n)$ ,  $\operatorname{U}(n)$ ,  $\operatorname{Sp}(n)$  are submanifolds of  $\operatorname{GL}^+(n, \mathbb{R})$ ,  $\operatorname{GL}(n, \mathbb{R})$ ,  $\operatorname{SL}(n, \mathbb{C})$ ,  $\operatorname{GL}(n, \mathbb{C})$ ,  $\operatorname{GL}(n, \mathbb{H})$ .

**8.8** Suppose that  $K(t)$  is an orthogonal matrix, for each time  $t$ , with entries smooth functions of  $t$ . Differentiate to prove that  $K^{-1}K'$  is an antisymmetric matrix.

**8.9** Suppose that  $P$  is a symmetric matrix with only positive eigenvalues. Use linear algebra to prove that there is a unique “square root”  $\sqrt{P}$  of  $P$ , i.e. a symmetric matrix  $Q$  with positive eigenvalues so that  $P = Q^2$ . Prove that  $P \mapsto \sqrt{P}$  is a smooth function.

**8.10\*** It is well known that every invertible matrix  $A$  decomposes uniquely into a product  $A = QR$  where  $Q$  is an orthogonal matrix and  $R$  is an upper triangular matrix whose diagonal entries are positive. Assume that this is true and then use the inverse function theorem to prove that  $Q$  and  $R$  are smooth functions of  $A$ .

**8.11\*** It is well known that the set of invertible matrices  $A$  that can be written as  $A = LU$  where  $L$  is a lower triangular matrix with 1's down the diagonal is an open set of matrices. The matrix  $U$  is an invertible upper triangular matrix, and  $L$  and  $U$  are unique. Prove that  $L$  and  $U$  are smooth functions of  $A$ .

**8.12\*** The polar decomposition: it is well known that every invertible matrix  $A$  can be written uniquely as  $A = KP$  where  $K$  is an orthogonal matrix and  $P$  is an invertible symmetric matrix all of whose eigenvalues are positive. Prove that  $K$  and  $P$  are smooth functions of  $A$ .

The unit circle  $S^1 \subset \mathbb{C}$  in the complex numbers is a Lie group, under usual multiplication of complex numbers.

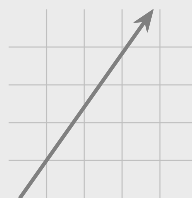
Products  $G \times H$  of Lie groups are Lie groups.

The  $n$ -dimensional torus  $T^n = S^1 \times S^1 \times \cdots \times S^1$  is a Lie group.

A Lie group *morphism* is a group morphism  $G \rightarrow H$  which is smooth.

The map  $x \in \mathbb{R}^n \mapsto (e^{2\pi i x_1}, e^{2\pi i x_2}, \dots, e^{2\pi i x_n}) \in T^n$  is a Lie group morphism.

Take the line in the plane consisting of the points  $(x, y)$  with  $y = \sqrt{2}x$ . It is a Lie subgroup of the plane under addition. Its image in the 2-dimensional torus  $T^2$  is an immersed curve, but comes arbitrarily close to every point of the torus. So Lie subgroups can be quite badly behaved.



A Lie group *isomorphism* is a Lie group morphism which has an inverse Lie group morphism.

**8.13** Consider the groups  $G_1, G_2, G_3$  of  $2 \times 2$  matrices of the respective forms

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}$$

for  $a, b \in \mathbb{R}$  with  $a \neq 0$ . Are these isomorphic Lie groups?

**8.14** Denote by  $\{\pm 1\}$  the group of integers  $1, -1$  under multiplication. For which integers  $n \geq 2$  is the orthogonal group  $O(n)$  diffeomorphic to  $SO(n) \times \{\pm 1\}$  as a manifold? For which integers  $n \geq 2$  is the orthogonal group  $O(n)$  isomorphic to  $SO(n) \times \{\pm 1\}$  as a Lie group?

It is difficult to give an example of a subgroup of a Lie group which is *not* a Lie subgroup; there are theorems to prove that they exist, even subgroups of the real number line, but they are not of interest to us.

**8.15** Write any quaternion  $q$  as  $q = q_0 + iq_1 + jq_2 + kq_3 = (q_0 + iq_1) + (q_2 + iq_3)j = z + jw$  where  $z, w \in \mathbb{C}$ . Associate to  $q$  the complex matrix

$$\begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix}.$$

Prove that this identifies  $\mathbb{H}$  with matrices in  $\mathbb{C}^{2 \times 2}$ , taking sums to sums, differences to differences, products to products. Use this entry by entry to map  $n \times n$   $\mathbb{H}$ -matrices to  $2n \times 2n$   $\mathbb{C}$ -matrices; prove again that this takes sums to sums, differences to differences, products to products.

**8.16** Give a Lie group isomorphism  $\mathrm{Sp}(1) \rightarrow \mathrm{SU}(2)$ .

**8.17\*** Is  $\mathrm{GL}(n, \mathbb{R})$  isomorphic to  $\mathrm{SL}(n, \mathbb{R}) \times \mathbb{R}^\times$ ?

**8.18\*** For various of the Lie groups we have described above (all of which are groups of matrices), use the spectral theorem for normal linear maps to find the degree of the map  $g \mapsto g^d$  for integer  $d \neq 0$ .

## Actions

An *action* of a Lie group  $G$  on a smooth manifold  $M$  is a smooth group action, i.e. a smooth map  $G \times M \rightarrow M$ , denoted  $g, m \mapsto gm$ , so that  $1m = m$  for all  $m \in M$  and  $(gh)m = g(hm)$  for all  $g, h \in G$  and  $m \in M$ . The *orbit* of the action through a point  $m \in M$  is the set

$$Gm = \{ gm \mid g \in G \}.$$

If a Lie group  $G$  acts on manifolds  $P$  and  $Q$ , a smooth map  $f: P \rightarrow Q$  is  $G$ -equivariant if  $f(gp) = gf(p)$  for any  $g \in G$  and  $p \in P$ . A *homogeneous space* of a Lie group  $G$  is a manifold  $M$  equipped with a transitive Lie group action, i.e. all points of  $M$  lie in the same orbit.

The unit sphere is a homogeneous space of the group of rotations of Euclidean space.

For  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , if  $V$  is a finite vector space over  $k$ ,  $\mathrm{Gr}_p V$  denotes the set of all  $p$ -dimensional subspaces of  $V$ , called the *Grassmannian*. We will see that the Grassmannian is a homogeneous space for the action of  $\mathrm{GL}(n, k)$ . If  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  respectively, then let  $K = \mathrm{O}(n), \mathrm{U}(n)$  or  $\mathrm{Sp}(n)$  respectively. The Grassmannian is also a homogeneous space of  $K$ , so the same manifold, but two different group actions.

**8.19** Let  $M$  be the set of all homogeneous polynomials of degree 3 in two variables:

$$p(x, y) = ax^3 + bx^2y + cxy^2 + dy^3.$$

Let  $G = \text{GL}(2, \mathbb{R})$  be the group of linear changes of variable, acting in the obvious way: we define  $(gp)(x, y)$  to mean change variables by  $g^{-1}$ :

$$(gp)(x, y) = p(g^{-1}(x, y)).$$

Find the orbits.

**8.20** Suppose that a Lie group  $G$  has a Lie group action on a manifold  $X$ , and acts, perhaps not smoothly, but as a group action, on a smooth manifold  $Y$ , and that  $p: X \rightarrow Y$  is a  $G$ -equivariant submersion. Prove that the action of  $G$  on  $Y$  is smooth.

A *representation* of a Lie group  $G$  on a vector space  $V$  is an action in which each element  $g \in G$  acts as a linear map on  $V$ .

All of our examples are groups of linear maps on vector spaces, so have obvious representations.

**8.21** Each representation of a Lie group  $G$  on a vector space  $V$  associates to each element  $g \in G$  some linear map  $\varphi(g) \in \text{GL}(V)$ . Prove that this map is a morphism of Lie groups. Conversely, prove that every morphism  $G \rightarrow \text{GL}(V)$  arises from a unique representation; hence we also call a morphism  $G \rightarrow \text{GL}(V)$  a *representation*.

### Rotations and quaternions

The group  $\text{Sp}(1)$  is the unit 3-sphere inside the quaternions  $\mathbb{H} \cong \mathbb{R}^4$ , i.e. the set of quaternions

$$q = q_0 + q_1i + q_2j + q_3k$$

for which  $1 = q_0^2 + q_1^2 + q_2^2 + q_3^2$ . One can also write this as  $q\bar{q} = 1$ , i.e.  $\bar{q} = q^{-1}$ .

**8.22** Prove that  $|ab| = |a||b|$  for any quaternions  $a, b \in \mathbb{H}$ .

The *imaginary quaternions* are those of the form  $q_1i + q_2j + q_3k$ , i.e. the quaternions  $q$  with  $q + \bar{q} = 0$ . We denote the set of imaginary quaternions just by  $\mathbb{R}^3$ .

**8.23** Prove that, if  $q \neq 0$  is a quaternion and  $r$  is imaginary quaternion, then  $qr\bar{q}$  is also imaginary.

Moreover,

$$|qr\bar{q}|^2 = |r|^2.$$

So  $qr\bar{q}$  is the same length as  $r$ . Associate to each  $q \in \text{Sp}(1)$  the linear transformation  $\rho(q)$  which maps  $r \mapsto qr\bar{q}$ . So  $\rho(q)$  is a length preserving linear transformation of  $\mathbb{R}^3$ , i.e. an orthogonal linear transformation. Our aim in this section is to prove:

**Theorem 8.1.** *The map  $\rho: \text{Sp}(1) \rightarrow \text{SO}(3)$  is a surjective Lie group morphism with kernel  $\pm 1$ . So  $\text{SO}(3) \cong \text{Sp}(1)/\pm 1$ .*

Note again that  $\text{Sp}(1)$  is the unit sphere in the quaternions, so this theorem shows that  $\text{SO}(3) = \mathbb{RP}^3$  as a manifold. Since  $\text{Sp}(1)$  is a 3-dimensional sphere,  $\text{Sp}(1)$  is connected, so the orthogonal linear transformations  $\rho(q)$  that can arise from various  $q$  all have positive determinant, and so are rotations:

$$\rho: q \in \text{Sp}(1) \mapsto \rho(q) \in \text{SO}(3)$$

is a representation. The kernel of  $\rho$  consists of the  $q$  so that  $qr\bar{q} = r$  for all  $r \in \mathbb{R}^3$ , i.e.  $qr = rq$  for all  $r \in \mathbb{R}^3$ . Check with  $r = i, j, k$  that this forces  $q = \pm 1$ . So  $\rho$  has kernel  $\pm 1$ .

**8.24** Prove that the inner product of any two quaternions  $q, r$  is  $(q\bar{r} + r\bar{q})/2$ .

**8.25** Prove that every imaginary quaternion  $u$  of unit length satisfies  $u^2 = -1$ .

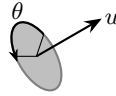
**8.26** Prove that imaginary quaternions *anticommute* (i.e.  $vu = -uv$ ) just when they are perpendicular.

**8.27** Prove that, for any perpendicular imaginary quaternions  $u, v$ , the triple  $u, v, uv$  is an orthonormal basis of  $\mathbb{R}^3$ .

**8.28** Take a unit length imaginary quaternion  $u$ . Let  $q := e^{\theta u}$  for real  $\theta$ . Prove that  $\rho(q)$  is the rotation about the axis spanned by  $u$ , by angle  $2\theta$ .

**8.29** Prove that  $\rho(q) = 1$  if and only if  $q = \pm 1$ .

Picture rotating around an axis, spanned by a unit vector  $u$ , by an angle  $\theta$ :



For each  $u \in S^2$  and angle  $e^{i\theta} \in S^1$ , we obtain a rotation in  $\text{SO}(3)$ . We get the identity rotation just when  $e^{i\theta} = 1$ , i.e. turn by  $\theta = 0$ , no matter what  $u$  we choose. If we change the sign of  $u$  and  $\theta$ , we get the same rotation, so we have a surjective smooth map

$$\phi: M := (S^2 \times S^1)/\pm 1 \rightarrow \text{SO}(3).$$

Any rotation matrix in  $\text{SO}(3)$  is a rotation about some axis; the axis is unique if the rotation is *not* the identity. For each rotation angle  $e^{i\theta}$ , the 2-sphere  $S^2 \times \{e^{i\theta}\} \subset S^2 \times S^1$  maps to a surface  $M_\theta \subset M$ , diffeomorphic to  $\mathbb{RP}^2$  for  $\theta = 0$  and for  $\theta = \pi$ ; otherwise  $M_\theta$  is diffeomorphic  $S^2$ . The map  $\phi$  is a diffeomorphism from  $M - M_0$  to  $\text{SO}(3) - \{I\}$ , but “collapses”  $M_0 \cong \mathbb{RP}^2$  to  $\{I\} \in \text{SO}(3)$ .

**8.30\*** Suppose that  $s \neq \pm 1$  is a unit quaternion. Prove that, for any integer  $n \neq 0$ , the equation  $q^n = s$  has precisely  $n$  solutions, all of which are unit quaternions. Find the degree of the map  $q \in S^3 \mapsto q^n \in S^3$ .

**8.31\*** Prove that a map  $\varphi: \mathbb{H} \rightarrow \mathbb{H}$  is an isomorphism of the ring of quaternions if and only if there is a quaternion  $q$  of unit length, i.e.  $q\bar{q} = 1$ , so that  $\varphi(a) = qa\bar{q}$  for any quaternion  $a$ . Identify the group of automorphisms with the group of rotations of the imaginary quaternions, i.e.  $\text{SO}(3)$ .

## Rotations in four dimensions

Given two unit quaternions  $q, r$ , define a real linear map on quaternions by

$$\rho(q, r): s \in \mathbb{H} \mapsto qs\bar{r} \in \mathbb{H}$$

**8.32** Prove that  $\rho$  is a morphism of groups.

**8.33** Prove that  $|qs\bar{r}| = |s|$  for all  $s \in \mathbb{H}$ .

So the map  $\rho$  takes values in the orthogonal group in 4 dimensions. Orthogonal matrices have determinant  $\pm 1$ ; rotations are those orthogonal transformations which have determinant 1. Since  $q, r$  both belong to the unit sphere in the quaternions, which is connected, the map  $\rho$  takes values in the rotation group in 4 dimensions:

$$\rho: \mathrm{Sp}(1) \times \mathrm{Sp}(1) \rightarrow \mathrm{SO}(4).$$

**8.34** Prove that the kernel of  $\rho$  is  $\pm(1, 1)$ .

**Theorem 8.2.** *The map*

$$\rho: \mathrm{Sp}(1) \times \mathrm{Sp}(1) \rightarrow \mathrm{SO}(4)$$

*is a surjective Lie group morphism, a representation of Lie groups, with kernel  $\pm(1, 1)$ .*

*Proof.* We want to achieve some rotation matrix  $g \in \mathrm{SO}(4)$ . First let  $t := g^{-1}1$ , and let  $q = t^{-1}$  as a quaternion. Then apply  $\rho(q, 1)$ , to get 1 to go to  $t$ . This rotation  $\rho(q, 1)$  might not be exactly equal to  $g$ , but it agrees with  $g$  on the quaternion  $t \in \mathbb{H}$ , taking it to 1. So after this, we have to achieve a rotation which fixes  $1 \in \mathbb{H}$ . By theorem 8.1 on page 88, every rotation of imaginary quaternions (so fixing  $1 \in \mathbb{H}$ ) arises as  $s \mapsto qs\bar{q}$  for some  $q$ .  $\square$

In particular, as a manifold,  $\mathrm{SO}(4) = (S^3 \times S^3)/\pm 1$ .

### The Hopf fibration on the projective line

Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ; let  $d$  be the real dimension of  $k$ . Consider the map

$$\varphi: \begin{pmatrix} a \\ b \end{pmatrix} \in k^2 \rightarrow \begin{pmatrix} 2a\bar{b} \\ |a|^2 - |b|^2 \end{pmatrix} \in k \oplus \mathbb{R}.$$

**8.35** Check that this takes the unit sphere  $S^{2d-1} \subset k^2$  to the unit sphere  $S^d$ .

Write the unit elements of  $k$  as  $S^{d-1} \subset k$ . Our map is invariant under the action

$$g \in S^{d-1}, \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2d-1} \mapsto \begin{pmatrix} a\bar{g} \\ b\bar{g} \end{pmatrix},$$

and equivariant under the action

$$g \in S^{d-1}, \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2d-1} \mapsto \begin{pmatrix} ga \\ gb \end{pmatrix}.$$

**8.36** Prove that the orbits of that first action are precisely the fibers of  $\varphi$ .

### Notes

Under fairly mild hypotheses, the symmetries of a geometric structure form a Lie group. We can't prove any of these results yet, but we state them now to provide some motivating examples. Don't worry if you are not familiar with the terminology used in these examples below; we won't need them.



A *biholomorphism* of a complex manifold means a complex-analytic diffeomorphism. The biholomorphisms of  $\mathbb{C}$  form a Lie group.

The biholomorphisms of any bounded open set in  $\mathbb{C}^n$  form a Lie group.

The biholomorphisms of any compact complex manifold form a Lie group.

The diffeomorphisms of a manifold preserving an affine connection form a Lie group.

Therefore the diffeomorphisms of a manifold preserving a Riemannian, or pseudo-Riemannian, metric form a Lie group: a closed subgroup of the diffeomorphisms preserving the Levi-Civita connection.

If a manifold contains infinitely many points, then its diffeomorphisms do *not* form a Lie group. If the manifold has positive dimension, they form an “infinite dimensional group”. If the manifold has infinitely many components, the group has uncountably many components.

If a manifold contains infinitely many points and has a smooth measure, then its volume preserving diffeomorphisms manifold do *not* form a Lie group; another “infinite dimensional group”.

The biholomorphisms of  $\mathbb{C}^2$  do *not* form a Lie group. They contain all transformations of the form

$$(z, w) \mapsto (z, w + f(z))$$

for any complex analytic function  $f: \mathbb{C} \rightarrow \mathbb{C}$ ; another “infinite dimensional group”.

Typically, the symmetries of any sort of geometric structure on a manifold are usually either “infinite dimensional” or form a Lie group.



## Chapter 9

### Lie groups under the microscope

*Sütterlin script crawled across the blackboard with spiky malevolence, its curves falsely padded, chalking the inscription that marked the beginning of a new stage in life. And in fact the Sütterlin script is well suited for items of marked importance, for succinctly striking statements, for slogans of the day. And there are certain documents that, though I admit I've never seen them, I nonetheless envision in Sütterlin script. I have in mind immunization certificates, sports records, and handwritten death sentences. Even back then, when I could see through the Sütterlin script but couldn't read it, the double loop of the Sütterlin  $\infty$  with which the inscription began, malicious and smelling of hemp, reminded me of the scaffold.*

— Günther Grass  
THE TIN DRUM

#### Connectivity

**9.1** Take a Lie group  $G$ . Prove that the path component  $G^0 \subseteq G$  that contains the identity element 1 is a normal subgroup, and so  $G/G^0$  is a discrete group, i.e. a zero dimensional Lie group, so that  $G \rightarrow G/G^0$  is a Lie group morphism, giving morphisms

$$1 \rightarrow G^0 \rightarrow G \rightarrow G/G^0 \rightarrow 1$$

each having image the kernel of the next.

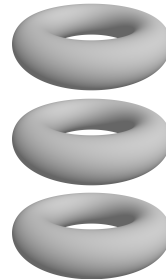
To classify Lie groups:

- classify connected Lie groups,
- classify discrete Lie groups,
- classify extensions of discrete groups by connected groups.

Every step is impossible in general, but most interesting Lie groups only have a few components, so in practice it is often worth while to focus on connected Lie groups.

**9.2** Use elementary linear algebra to prove that  $\mathrm{SL}(n, \mathbb{C})$ ,  $\mathrm{GL}(n, \mathbb{C})$ ,  $\mathrm{GL}(n, \mathbb{H})$ ,  $\mathrm{SO}(n)$ ,  $\mathrm{U}(n)$ ,  $\mathrm{Sp}(n)$  are connected and that  $\mathrm{O}(n)$  and  $\mathrm{GL}(n, \mathbb{R})$  each have two components.

**Lemma 9.1.** *Any open subset of any Lie group generates a subgroup which is a union of components of the Lie group.*



*Proof.* Take an open set  $U \subseteq G$  of a Lie group and let  $H \subseteq G$  be the subgroup generated by  $U$ . Pick a point  $g \in U$  and a point  $h \in H$ . Because  $H$  is a group,  $hg^{-1} \in H$  and so  $hg^{-1}U \subseteq H$ , i.e.  $h$  lies in an open set inside  $H$ . Hence  $H$  is an open subset of  $G$ . If  $g \in G$  is outside  $H$ , then  $gH$  is outside  $H$ , so the complement of  $H$  is open, i.e.  $H$  is a union of components of  $G$ .  $\square$

**Corollary 9.2.** *Every finite dimensional field extension of the real numbers is isomorphic as a field to the real numbers or to the complex numbers.*

*Proof.* Suppose that  $\mathbb{R} \subset k$  is a finite dimensional extension. We can assume that  $k \neq \mathbb{R}$ , so  $k$  has dimension 2 or more as an  $\mathbb{R}$ -vector space. Let  $k^\times := k - \{0\}$  be the set of nonzero elements, an abelian Lie group under multiplication. The squaring map  $s: x \in k^\times \mapsto x^2 \in k^\times$  is smooth with  $s'(x)h = 2xh$ , so  $s'(x)$  is a linear isomorphism for all  $x \in k^\times$ . Hence the image of  $s$  is an open subgroup of  $k^\times$ . Since  $k^\times$  is connected,  $s(k^\times) = k^\times$ . In particular,  $k^\times$  contains a square root of  $-1$ .

Check that  $s(x) = 1$  just when  $x^2 = 1$ , so just when  $(x+1)(x-1) = 0$ , so  $x = \pm 1$  since  $k$  is a field. Let  $\mathbb{R}^\times = \mathbb{R} - \{0\}$ ; usual squaring of real numbers:  $s^{-1}\mathbb{R}^+ = \mathbb{R}^\times$ . So  $s: k^\times/\mathbb{R}^\times \rightarrow k^\times/\mathbb{R}^+$ , a 2-to-1 local diffeomorphism from the projective space to the sphere, hence a covering map.

If  $k$  has dimension 2, this is fine, as the real projective space and sphere are both circles, and can wind around one another any number of times. In topology, we see that real projective space of dimension 2 or more, cannot cover the sphere, as the real projective spaces are not simply connected, while the sphere is. So  $\dim k = 2$  and  $k$  contains some  $i$  with  $i^2 = -1$ , so one easily checks that  $k$  is isomorphic to  $\mathbb{C}$ .  $\square$

**Theorem 9.3** (Fundamental theorem of algebra). *Every nonconstant polynomial function of one complex variable has a complex root.*

*Proof.* If  $p(z)$  has no root then  $k = \mathbb{C}[z]/(p(z))$  is a finite dimensional extension field of  $\mathbb{R}$ . Apply corollary 9.2.  $\square$

## Lie algebras

Any Lie group  $G$  has a distinguished point  $1 \in G$ , and so a distinguished tangent space  $T_1G$ , traditionally printed in lowercase Fraktur font:  $\mathfrak{g}$ . When we write these by hand, traditionally we use lowercase Sütterlin font  $\mathfrak{g}$ , our ancestors used the more complicated Kurrentschrift.

|           |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |                |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Roman     | A              | B              | C              | D              | E              | F              | G              | H              | I              | J              | K              | L              | M              | N              | O              | P              | Q              | R              | S              | T              | U              | V              | W              | X              | Y              | Z              |
| Fraktur   | $\mathfrak{a}$ | $\mathfrak{b}$ | $\mathfrak{c}$ | $\mathfrak{d}$ | $\mathfrak{e}$ | $\mathfrak{f}$ | $\mathfrak{g}$ | $\mathfrak{h}$ | $\mathfrak{i}$ | $\mathfrak{j}$ | $\mathfrak{k}$ | $\mathfrak{l}$ | $\mathfrak{m}$ | $\mathfrak{n}$ | $\mathfrak{o}$ | $\mathfrak{p}$ | $\mathfrak{q}$ | $\mathfrak{r}$ | $\mathfrak{s}$ | $\mathfrak{t}$ | $\mathfrak{u}$ | $\mathfrak{v}$ | $\mathfrak{w}$ | $\mathfrak{x}$ | $\mathfrak{y}$ | $\mathfrak{z}$ |
| Sütterlin | $\mathfrak{a}$ | $\mathfrak{b}$ | $\mathfrak{c}$ | $\mathfrak{d}$ | $\mathfrak{e}$ | $\mathfrak{f}$ | $\mathfrak{g}$ | $\mathfrak{h}$ | $\mathfrak{i}$ | $\mathfrak{j}$ | $\mathfrak{k}$ | $\mathfrak{l}$ | $\mathfrak{m}$ | $\mathfrak{n}$ | $\mathfrak{o}$ | $\mathfrak{p}$ | $\mathfrak{q}$ | $\mathfrak{r}$ | $\mathfrak{s}$ | $\mathfrak{t}$ | $\mathfrak{u}$ | $\mathfrak{v}$ | $\mathfrak{w}$ | $\mathfrak{x}$ | $\mathfrak{y}$ | $\mathfrak{z}$ |

Being a tangent space,  $\mathfrak{g}$  is a vector space of dimension equal to the dimension of  $G$ .

Try  $G = O(n)$  cut out by the equations

$$g^t g = I,$$

which we differentiate along a curve  $g(t) = I + tA + \dots$  at  $t = 0$ , to find the equation

$$A^t + A = 0$$

on tangent vectors: the velocity  $A$  of  $g(t)$  at  $t = 0$  is an antisymmetric matrix. Conversely if  $A$  is antisymmetric, then  $g(t) = e^{tA}$  is an orthogonal matrix

$$g(t)^t = e^{tA^t} = e^{-tA} = g(t)^{-1},$$

and note that  $g'(0) = A$ . So  $\mathfrak{g} = \mathfrak{o}(n)$  is the set of antisymmetric  $n \times n$  matrices.

Similarly, for  $G = U(n)$ ,  $\mathfrak{g} = \mathfrak{u}(n)$  is the set of skew-adjoint  $n \times n$  matrices

**9.3** If  $G = \mathrm{SL}(n, \mathbb{R})$ , prove that  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$  is the set of trace zero  $n \times n$  matrices.

**9.4** From the above, we see that every  $A \in \mathfrak{sp}(n)$  has  $A^* + A = 0$ , where the *adjoint*  $A^*$  is the transpose and quaternionic conjugate matrix. Prove that if  $A^* + A = 0$ , then  $e^{tA} \in \mathrm{Sp}(n)$  for all  $t$ , and differentiate this to prove that  $\mathfrak{sp}(n)$  is precisely the set of quaternionic matrices  $A$  with  $A^* + A = 0$ .

**9.5** Use the above to prove the following dimensions:

| $G$                          | $\mathfrak{g}$                 | equation of $\mathfrak{g}$                    | $\dim_{\mathbb{R}} G$ |
|------------------------------|--------------------------------|-----------------------------------------------|-----------------------|
| $\mathrm{GL}(n, \mathbb{R})$ | $\mathfrak{gl}(n, \mathbb{R})$ | $\mathbb{R}^{n \times n}$                     | $n^2$                 |
| $\mathrm{GL}(n, \mathbb{C})$ | $\mathfrak{gl}(n, \mathbb{C})$ | $\mathbb{C}^{n \times n}$                     | $2n^2$                |
| $\mathrm{GL}(n, \mathbb{H})$ | $\mathfrak{gl}(n, \mathbb{H})$ | $\mathbb{H}^{n \times n}$                     | $4n^2$                |
| $\mathrm{SL}(n, \mathbb{R})$ | $\mathfrak{sl}(n, \mathbb{R})$ | $\mathrm{tr} A = 0$                           | $n^2 - 1$             |
| $\mathrm{SL}(n, \mathbb{C})$ | $\mathfrak{sl}(n, \mathbb{C})$ | $\mathrm{tr}_{\mathbb{C}} A = 0$              | $2(n^2 - 1)$          |
| $\mathrm{SL}(n, \mathbb{H})$ | $\mathfrak{sl}(n, \mathbb{H})$ | $\mathrm{tr}_{\mathbb{R}} A = 0$              | $4n^2 - 1$            |
| $\mathrm{O}(n)$              | $\mathfrak{so}(n)$             | $A^t + A = 0$                                 | $n(n-1)/2$            |
| $\mathrm{SO}(n)$             | $\mathfrak{so}(n)$             | $A^t + A = 0$                                 | $n(n-1)/2$            |
| $\mathrm{SO}(n, \mathbb{C})$ | $\mathfrak{so}(n, \mathbb{C})$ | $A^t + A = 0$                                 | $n(n-1)$              |
| $\mathrm{U}(n)$              | $\mathfrak{u}(n)$              | $A^* + A = 0$                                 | $n^2$                 |
| $\mathrm{SU}(n)$             | $\mathfrak{su}(n)$             | $A^* + A = 0, \mathrm{tr}_{\mathbb{C}} A = 0$ | $n^2 - 1$             |
| $\mathrm{Sp}(n)$             | $\mathfrak{sp}(n)$             | $A^* + A = 0$                                 | $n(2n+1)$             |

**9.6** Prove that the exponential maps  $\mathfrak{g} \rightarrow G$  are onto for

| $\mathfrak{g}$     | $G$              |
|--------------------|------------------|
| $\mathfrak{so}(n)$ | $\mathrm{SO}(n)$ |
| $\mathfrak{u}(n)$  | $\mathrm{U}(n)$  |
| $\mathfrak{su}(n)$ | $\mathrm{SU}(n)$ |
| $\mathfrak{sp}(n)$ | $\mathrm{Sp}(n)$ |

What singles out these Lie groups from the others on the previous list?

**9.7\*** What is the image of the exponential map  $\mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathrm{SL}(2, \mathbb{R})$ ?

### Vector fields on Lie groups

In a Lie group  $G$ ,  $L_x: y \in G \mapsto xy \in G$  is *left multiplication* while  $R_x: y \in G \mapsto yx \in G$  is *right multiplication*. For each  $x \in G$ ,  $L_x$  is a diffeomorphism, as is  $R_x$ , both taking 1 to  $x$ .

**9.8** Suppose that  $G$  is a Lie group and that  $H \subseteq G$  is a subgroup. Prove that  $H$  is an embedded submanifold of  $G$  if and only if there is an open subset  $U \subseteq G$  containing 1 so that  $H \cap U$  is an embedded submanifold of  $G$ .

If we take a vector  $A \in \mathfrak{g} = T_1G$ , there is a unique way to “left translate” it to sit in  $T_xG$ , i.e. by  $L_{x*}A \in T_xG$ . So we can define a smooth vector field  $\vec{A}$  everywhere on  $G$  by  $\vec{A}(x) = L_{x*}A$ ; such a vector field is *left invariant*, i.e. invariant by all left translations, and conversely every left invariant vector field has this form, as its value at  $x$  is just the left translate of its value at 1. Similarly, define a smooth vector field  $\tilde{A}$  everywhere on  $G$  by  $\tilde{A}(x) = R_{x*}A$ ; such a vector field is *right invariant*, i.e. invariant by all right translations, and conversely every right invariant vector field has this form.

In this way, we identify  $\mathfrak{g}$  with the set of all left invariant vector fields on  $G$ . The Lie bracket is diffeomorphism invariant, so the brackets of left invariant vector fields are left invariant. Hence the left invariant vector fields form an algebra under Lie bracket, the *Lie algebra* of  $G$ . (We could have used the right invariant vector fields for this definition, but it is standard to use the left invariant ones; the resulting Lie algebra structure on  $\mathfrak{g}$  is not identical, but is isomorphic.)

If  $G = \mathrm{GL}(n, k)$ , then  $G$  is an open subset of the  $n \times n$  matrices over  $k$ , so  $T_1G = k^{n \times n}$  is the set of all  $n \times n$  matrices, and so  $\mathfrak{g} \cong T_1G = k^{n \times n}$ . It is traditional to write  $k^{n \times n}$  as  $\mathfrak{gl}(n, k)$  when we want to emphasize its role as Lie algebra. The left invariant vector fields are  $\vec{A}(g) = gA$ , and the right invariant vector fields are  $\tilde{A}(g) = Ag$ . The Lie brackets of these vector fields are

$$[\vec{A}\vec{B}] = \overline{[A, B]},$$

and

$$[\tilde{A}\tilde{B}] = -\overline{[A, B]},$$

where  $[AB] := AB - BA$ : the Lie bracket is just the usual commutator bracket of matrices. Since all of our examples are subgroups of  $\mathrm{GL}(n, k)$ , all of our brackets look like this: for each immersed Lie subgroup  $G \subseteq \mathrm{GL}(n, k)$ , its Lie algebra  $\mathfrak{g} \subset \mathfrak{gl}(n, k)$  is its tangent space  $T_1G$ , with Lie bracket given by matrix commutator bracket.

The Lie algebra  $\mathfrak{so}(3)$  of  $\mathrm{SO}(3)$  consists of the  $3 \times 3$  skew-symmetric matrices

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

Identifying each such matrix with a vector  $(a, b, c) \in \mathbb{R}^3$  turns the bracket into the cross-product of vectors; in classical mechanics, the vector  $(a, b, c)$  is the *angular velocity*.

The *Heisenberg group* is the group  $N_3$  of  $3 \times 3$  strictly upper triangular real matrices, i.e. matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Its Lie algebra  $\mathfrak{n}_3$  is the collection of matrices of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie groups  $\mathrm{SL}(2, \mathbb{R})$ ,  $\mathrm{SO}(3)$  and  $N_3$  are 3-dimensional. Are they perhaps isomorphic? The Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$  contains the matrices

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix},$$

which form the Lie subalgebra corresponding to the Lie subgroup  $G \subset \mathrm{SL}(2, \mathbb{R})$  leaving the span of  $e_1$  invariant. But  $\mathfrak{so}(3)$  has no 2-dimensional subalgebras, because the cross-product of any two vectors is perpendicular to both. Hence  $\mathrm{SO}(3)$  is not isomorphic as a Lie group to  $\mathrm{SL}(2, \mathbb{R})$ . On the other hand, the bracket in  $\mathfrak{n}_3$  is

$$\left[ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 & az - xc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with image in a 1-dimensional subspace of  $\mathfrak{n}_3$ . Every vector in either of  $\mathfrak{sl}(2, \mathbb{R})$  or  $\mathfrak{so}(3)$  arises as a bracket of two vectors (easy to check), so none of the Lie groups  $\mathrm{SL}(2, \mathbb{R})$ ,  $\mathrm{SO}(3)$  or  $N_3$  are isomorphic.

**9.9** Find all subalgebras of  $\mathfrak{so}(3)$  up to  $\mathrm{SO}(3)$ -conjugacy.

**9.10** Prove that the time  $t$  flow of  $\vec{A}$  through a point  $g_0 \in \mathrm{GL}(n, k)$  is  $g_0 e^{tA}$ , where  $e^{tA}$  here is the usual matrix exponential function

$$e^A = I + A + \frac{A^2}{2} + \cdots + \frac{A^k}{k!} + \cdots$$

Prove that this exponential converges for all  $A$ .

**9.11** Prove that the brackets are therefore

$$[\vec{A}\vec{B}] = [\overline{AB}],$$

where  $[AB] = AB - BA$ .

Each affine transformation  $T$  has an associated matrix

$$g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

**9.12** Prove that

$$T \mapsto g = g(T) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

is a Lie group morphism.

Hence the Lie algebra of the affine group of  $\mathbb{R}^n$  is the Lie algebra of matrices

$$\begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix}.$$

**9.13** Compute the Lie bracket.



## Chapter 10

### From small to large in Lie groups

*The world only exists in your eyes—your conception of it. You can make it as big or as small as you want to.*

— F. Scott Fitzgerald  
THE CRACK UP

We will see that any morphism of connected Lie groups is determined uniquely by its derivative at the identity.

#### The exponential map

**Lemma 10.1.** *Every left invariant vector field on any Lie group has complete flow.*

*Proof.* Take any flow line, defined on some time interval. Its left translates are also flow lines, defined on the same time interval. So the flow line through any point is defined for just as long a duration as the flow line through any other point. If some flow line is only defined for a finite amount of time, then that amount of time gets smaller as you flow along the flow line. But the duration of each flow line is constant, the same at all points.  $\square$

This encourages us to define a smooth map, the *exponential map*: for any  $A \in \mathfrak{g}$ ,  $e^A$  is the time 1 flow of  $\vec{A}$  through the point  $1 \in G$ . By definition, the exponential map is smooth. By left invariance, the flow of  $\vec{A}$  through each point  $g_0 \in G$  is then just  $g_0 e^A$ .

**10.1** Prove that  $\exp'(0) = I$ .

So the map  $\exp$  is a local diffeomorphism from some open set around  $0 \in T_1 G \cong \mathfrak{g}$  to some open set around  $1 \in G$ . Denote the local inverse as  $\log$ .

**10.2** Compute the exponential map of the Heisenberg group.

**10.3** Prove an element of a Lie group lies in the identity component if and only if it is a finite product of exponentials.

**10.4** Compute the exponential map of the affine group.

**10.5** The intersection of any two embedded Lie subgroups of a Lie group is an embedded Lie subgroup, with Lie algebra the intersection of the Lie algebras. (Careful: connected Lie subgroups may have disconnected intersection.)

## Morphisms of Lie algebras and of Lie groups

For each  $A \in \mathfrak{g}$ , the path  $t \in \mathbb{R} \mapsto e^{tA} \in G$  is a *one parameter subgroup*.

**Theorem 10.2.** *Any Lie group morphism  $\varphi: G \rightarrow H$  on a connected Lie group is uniquely determined by its differential at the identity  $\varphi'(1): \mathfrak{g} \rightarrow \mathfrak{h}$ , which is a Lie algebra morphism, i.e. takes Lie brackets to Lie brackets. One parameter subgroups map by  $\varphi(e^{tA}) = e^{t\varphi'(1)A}$ .*

Consequently,  $\varphi'(1)$  is also routinely denoted  $\varphi$ .

*Proof.* The map  $\varphi$  satisfies  $\varphi(g_0 g_1) = \varphi(g_0) \varphi(g_1)$ , so preserves left translation. Therefore if  $h_0 = \varphi(g_0)$  then  $\varphi(g_0 e^{tA}) = h_0 \varphi(e^{tA})$ . Differentiate to get

$$\varphi'(g_0) \vec{A}(g_0) = L'_{h_0}(1) \vec{A}(1).$$

Let  $B := \varphi'(1)A$ , so  $\varphi_* \vec{A} = \vec{B}$ , i.e. the left invariant vector fields correspond. Corresponding vector fields have corresponding Lie brackets:  $\varphi'(1)$  is a Lie algebra morphism. The flows of the left invariant vector fields through the identity are the one parameter subgroups  $e^{tA}$ .

Two morphisms with the same differential at 1 will give the same correspondence between left invariant vector fields, matching up their flows. Since the exponential map is a local diffeomorphism near  $e^0 = 1$ , the flows  $A \mapsto e^A$  reach a neighborhood of 1, where the morphisms agree; apply lemma 9.1 on page 93.  $\square$

**Corollary 10.3.** *Take Lie groups  $G, H$  with Lie algebras  $\mathfrak{g}, \mathfrak{h}$  and a Lie group morphism  $\varphi: G \rightarrow H$ . A vector lies in the kernel of  $\varphi'(1)$  just when its associated one parameter subgroup lies in the kernel of  $\varphi$ .*

*Proof.* Take  $A \in \mathfrak{g}$  and let  $B = \varphi'(1)A$ . The flow line of  $\vec{A}$  through 1 is just the one parameter subgroup generated by  $A$ . The image of that flow line is just the one parameter subgroup generated by  $B$ , i.e. the flow line of  $\vec{B}$ . So  $B = 0$  iff  $\vec{B} = 0$  iff  $e^{tB} = 1$  for all  $t$  iff  $\varphi(e^{tA}) = 1$  for all  $t$ .  $\square$

**Lemma 10.4.** *Take Lie groups  $G, H$  with Lie algebras  $\mathfrak{g}, \mathfrak{h}$  and a Lie group morphism  $\varphi: G \rightarrow H$ . The map  $\varphi$  is an open map if and only if  $\varphi'(1)$  is onto.*

*Proof.* The exponential map is a local diffeomorphism near the identity; apply this to the equation  $\varphi(e^A) = e^{\varphi'(1)A}$  to see that  $\varphi$  is open near the identity element just when  $\varphi'(1)$  is surjective. Away from the identity element, apply a left translation:  $\varphi(g_0 g) = \varphi(g_0) \varphi(g)$ , and use the fact that left translation is a diffeomorphism.  $\square$

**Theorem 10.5.** *If  $\varphi: G \rightarrow H$  is a Lie group morphism then  $\varphi'(1)$  an isomorphism just when  $\varphi$  is a covering map to its image, which is a union of path components of  $H$ . In particular,  $\varphi$  is then an isomorphism if and only if  $\varphi$  has trivial kernel and has image in every component of  $H$ .*

*Proof.* The image is an open subgroup; by lemma 9.1 on page 93, it is a union of path components of  $H$ , so we can replace  $H$  by this union and assume that  $\varphi$  is onto. We can then replace  $G$  by the quotient by the kernel, by theorem 15.7 on page 139, since this is a covering map through which  $\varphi$  quotients. So now  $\varphi$  is bijective. Near the identity element of  $G$ ,

$$\varphi = \exp \circ \varphi'(1) \circ \log,$$

so near the identity element of  $H$ ,

$$\varphi^{-1} = \exp \circ \varphi'(1)^{-1} \circ \log$$

is smooth. Again by translation,  $\varphi^{-1}$  is smooth everywhere.  $\square$

**10.6** Classify one dimensional connected Lie groups up to isomorphism.

**10.7\*** The intersection of any two Lie subgroups of a Lie group is a Lie subgroup, for a unique topology and smooth structure, with Lie algebra the intersection of the Lie algebras. (Careful: connected Lie subgroups may have disconnected intersection.)

**10.8\*** The intersection of any collection of Lie subgroups of a Lie group is a Lie subgroup, for a unique topology and smooth structure, with Lie algebra the intersection of the Lie algebras.

## Abelian Lie groups

The Lie algebra of a Lie group  $G$  is *abelian* if the bracket of any two left invariant vector fields is zero.

**Lemma 10.6.** *The Lie algebra of a connected Lie group  $G$  is abelian just when  $G$  is an abelian group.*

*Proof.* Suppose that  $G$  is abelian. The flows of left invariant vector fields are right translations, so commute, so their brackets are zero. Suppose that the Lie algebra is abelian. Then the flows of left invariant vector fields commute, i.e. right translations commute, so all elements of  $G$  near the identity element commute. Apply lemma 9.1 on page 93.  $\square$

**Lemma 10.7.** *Suppose that  $\Lambda \subset V$  is a discrete subgroup of a finite dimensional real vector space. Then up to linear isomorphism,  $V = \mathbb{R}^n$  for some  $n$  and  $\Lambda = \mathbb{Z}^k \oplus 0$  for some  $0 \leq k \leq n$ .*

*Proof.* Take any positive definite inner product on  $V$ , and let  $v_1$  be a nonzero element of  $\Lambda$  of smallest norm in the inner product. Perhaps rescaling the inner product, we can arrange that  $V = \mathbb{R}^n$  with usual inner product and that  $v_1 = e_1$ . If  $n = 1$ , we are done, so suppose  $n > 1$ . Let  $p: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$  be projection normal to  $e_1$ . Each element  $x \in \mathbb{R}^n$  projecting to a given element  $y \in \mathbb{R}^{n-1}$  can be replaced by  $me_1 + x$  to arrange that  $|x_1| \leq \frac{1}{2}$ , so that

$$|x|^2 \leq \frac{1}{4} + |y|^2.$$

If we take some nonzero  $x \in \Lambda$ , then  $|x| \geq 1$ , so

$$|y|^2 \geq \frac{3}{4}.$$

Hence  $p\Lambda \subset \mathbb{R}^{n-1}$  is a discrete subgroup. Apply induction.  $\square$

**Theorem 10.8.** *Suppose that  $G$  is a connected abelian Lie group. Then  $G = \mathbb{R}^k \times T^\ell$  where  $\mathbb{R}^k$  is a finite dimensional vector space, and  $T^\ell$  is a torus, i.e. the direct product of  $\ell$  circles.*

*Proof.* The Lie algebra of  $G$  is trivial, so the exponential map  $\exp: \mathfrak{g} \rightarrow G$  is a Lie group homomorphism, surjective by lemma 9.1 on page 93. Hence  $G = \mathfrak{g}/\Gamma$  where  $\Gamma \subset \mathfrak{g}$  is a discrete abelian subgroup. Apply lemma 10.7 on the previous page.  $\square$

**Corollary 10.9.** *Every finite dimensional field extension of the real numbers is isomorphic as a field to the real numbers or to the complex numbers.*

*Proof.* Suppose that  $\mathbb{C} \subset k$  is a finite field extension, i.e.  $k$  is a field containing  $\mathbb{C}$  and a finite dimensional vector space over  $\mathbb{C}$ . Then  $k$  is generated by finitely many elements  $\theta_i$ , and the multiplication operation in  $k$  is

$$\theta_i \theta_j := \sum c_{ijk\ell} \theta_k^\ell,$$

for some complex number constants  $c_{ijk\ell}$ . The group of units  $k^\times$  is an abelian group, and a Lie group, because its group operation is just the resulting polynomial expression in the various generators. If  $k \neq \mathbb{C}$ , i.e.  $k$  is a complex vector space of dimension two or more, then  $k^\times$  is simply connected as it is a punctured complex vector space  $k^\times = k - \{0\}$ . But  $k^\times$  is diffeomorphic to a finite dimensional real vector space, by theorem 10.8 on the preceding page, being a simply connected abelian Lie group. No finite dimensional real vector space is diffeomorphic to itself punctured by problem 27.12 on page 226. (Various topological results can prove this: the sphere has nonvanishing homotopy or homology or cohomology groups, and hence  $\mathbb{R}^n$  is not even homeomorphic to  $\mathbb{R}^n - \{0\}$ .) Hence  $\mathbb{C}$  admits no finite dimensional field extensions.

Apply the same reasoning to  $\mathbb{R}$  instead of  $\mathbb{C}$ : the only finite field extensions are of dimensions 1 or 2, because  $\mathbb{R}^n - \{0\}$  is simply connected for  $n \geq 3$ , but not for  $n = 2$ .

**10.9** Prove that the only 2-dimensional field extension of  $\mathbb{R}$  is  $\mathbb{C}$ .

Hence the finite dimensional field extensions of  $\mathbb{R}$  are  $\mathbb{R}$  and  $\mathbb{C}$ .  $\square$

**Theorem 10.10** (Fundamental Theorem of Algebra). *Every nonconstant complex polynomial  $p(z)$  can be factored into linear factors. More specifically,*

$$p(z) = c(z - z_1)^{d_1}(z - z_2)^{d_2} \dots (z - z_k)^{d_k}$$

where  $c$  is a constant,  $z_1, z_2, \dots, z_k$  are the roots of  $p(z)$ , and  $d_1, d_2, \dots, d_k$  are positive integers.

*Proof.* If a nonconstant polynomial  $p(z)$  over  $\mathbb{C}$  has no root, we construct a finite field extension  $k := \mathbb{C}[z]/(p(z))$  in which it has a root.  $\square$

**10.10** Prove that  $\mathbb{R}$  and  $\mathbb{C}$  are the only fields that are also positive dimensional manifolds which have smooth addition and multiplication.

## Chapter 11

### Closed subgroups of Lie groups

To understand non-Abelian Lie groups better, we investigate the relation between Lie bracket in the Lie algebra and commutation in the group. We prove that every closed subgroup is an embedded Lie subgroup.

#### The commutator formula

We have to fight with noncommutativity banjaxing the usual laws of exponents. Define a “weird sum”:  $A \dot{+} B := \log(e^A e^B)$ , defined near the origin in  $\mathfrak{g}$ . Note that  $0 \dot{+} 0 = 0$ , and  $A \dot{+} 0 = 0 \dot{+} A = A$ . The flow line of  $\vec{A}$  through 1 is the one parameter subgroup  $e^{tA}$ , so  $e^{sA} e^{tA} = e^{(s+t)A}$  for all  $s, t \in \mathbb{R}$ , so  $(sA) \dot{+} (tA) = (s+t)A$ . Expand the weird sum in a Taylor series:

$$A \dot{+} B = 0 + (\text{linear in } A) + (\text{linear in } B) + (\text{quadratic in } A) + \dots$$

Compare term by term with  $(sA) \dot{+} (tA) = sA + tA$ :

$$(sA) \dot{+} (tA) = 0 + s(\text{linear in } A) + t(\text{other linear in } A) + s^2(\text{quadratic in } A) + \dots$$

to see that the linear term in  $A$  is  $A$ , higher terms in  $A$  vanish, and the same for  $B$ :

$$A \dot{+} B = A + B + (\text{bilinear in } A, B) + \dots,$$

and the bilinear term vanishes if  $A = B$  (as do all higher order terms).

**11.1** Prove that the bilinear term is antisymmetric.

**Lemma 11.1.** *For any Lie group,*

$$e^A e^B = e^{A+B+\frac{1}{2}[AB]+\dots}.$$

*Proof.* The exponential map identifies some open neighborhood of  $0 \in \mathfrak{g}$  with an open neighborhood of  $1 \in G$ . It identifies the left invariant vector field  $\vec{A}$  on  $G$  with some vector field, which we denote by the same symbol, near  $0 \in \mathfrak{g}$ . It identifies vector fields, so identifies their Lie brackets as vector fields:

$$\exp_* [\vec{A}\vec{B}] = [\vec{A}\vec{B}] = [\overline{A}, \overline{B}].$$

The flow line of  $\vec{A}$  through  $0 \in \mathfrak{g}$  is  $tA$ , as it matches up by  $\exp$  with the flow line of  $\vec{A}$  through  $1 \in G$ . Differentiate:  $\exp'(0)A = A$ , so  $\vec{A}(0) = A$ . So  $[\vec{A}\vec{B}](0) = [AB]$  on  $\mathfrak{g}$ , for any  $A, B \in \mathfrak{g}$ .

On  $G$ , left invariant vector fields generate the right action: the flow of  $\vec{A}$  through  $g_0$  is

$$e^{t\vec{A}}g_0 = g_0 \exp(tA),$$

where we write  $\exp$  for the exponential map and  $e^{t\vec{A}}$  for the flow of the vector field  $\vec{A}$  on  $G$ . So if  $g_0 = \exp(B)$  then

$$e^{t\vec{A}}\exp(B) = \exp(B)\exp(tA).$$

Taking log, the flow line of  $\vec{A}$  on  $\mathfrak{g}$  is

$$e^{t\vec{A}}B = B \dot{+} (tA).$$

Let  $A \dot{+} B \dot{+} C$  mean  $(A \dot{+} B) \dot{+} C$ . Composing flows in  $\mathfrak{g}$ :

$$e^{-t\vec{B}}e^{-s\vec{A}}e^{t\vec{B}}e^{s\vec{A}}0 = (sA) \dot{+} (tB) \dot{+} (-sA) \dot{+} (-tB).$$

Computing out brackets of any vector fields  $X$  and  $Y$  on any manifold, by lemma 7.1 on page 78,

$$I - e^{-tY}e^{-sX}e^{tY}e^{sX} = -st[XY] + O(s, t)^3.$$

Therefore if we let  $X = \vec{A}$  and  $Y = \vec{B}$ ,

$$(sA) \dot{+} (tB) \dot{+} (-sA) \dot{+} (-tB) = e^{-tY}e^{-sX}e^{tY}e^{sX}0 = st[AB] + O(s, t)^3.$$

Plugging in our Taylor expansion above, the bilinear term is  $\frac{1}{2}[AB]$ .  $\square$

**Lemma 11.2** (Trotter product formula). *In any Lie group  $G$  with Lie algebra  $\mathfrak{g}$ , for any  $A, B \in \mathfrak{g}$ ,*

$$e^{A+B} = \lim_{k \rightarrow \infty} \left( e^{A/k} e^{B/k} \right)^k.$$

*Proof.* Up to quadratic terms:

$$\begin{aligned} k \left( \frac{A}{k} \dot{+} \frac{B}{k} \right) &= k \left( \frac{A}{k} + \frac{B}{k} + \frac{1}{2}[A/k, B/k] + \dots \right), \\ &= A + B + \frac{1}{2k}([AB] + \dots). \end{aligned}$$

Exponentiate.  $\square$

For elements  $g, h$  of any group, let  $[gh] = ghg^{-1}h^{-1}$ .

**Lemma 11.3** (The commutator formula). *For  $A, B$  in the Lie algebra of a Lie group,*

$$e^{[AB]} = \lim_{k \rightarrow \infty} \left[ e^{A/k} e^{B/k} \right]^{k^2}.$$

*Proof.*

$$(A/k) \dot{+} (B/k) \dot{+} (-A/k) \dot{+} (-B/k) = [AB]/k^2 + O(1/k)^3.$$

Repeat  $k^2$  times:

$$k^2((A/k) \dot{+} (B/k) \dot{+} (-A/k) \dot{+} (-B/k)) = [AB] + O(1/k).$$

Exponentiate.  $\square$

## Closed subgroups are Lie subgroups

**11.2\*** Prove that every embedded Lie subgroup of a Lie group is closed.

**Theorem 11.4.** *Any subgroup of any Lie group is closed as a subset if and only if it is an embedded submanifold.*

*Proof.* Take a closed subgroup  $H \subseteq G$  of a Lie group  $G$ . Let  $\mathfrak{h}$  be the set of all left invariant vector fields  $\vec{A}$  on  $G$  whose flow preserves  $H$ . By left invariance,  $\mathfrak{h}$  consists precisely of those  $A$  for which the flow of  $\vec{A}$  through 1 stays in  $H$ , i.e.  $e^{tA} \in H$  for all  $t$ . Clearly  $\mathfrak{h}$  is closed under rescaling by real numbers. By the Trotter product formula (lemma 11.2 on the preceding page), since  $H$  is closed, taking limits,  $\mathfrak{h}$  is closed under addition. By the commutator formula (lemma 11.3 on the facing page),  $\mathfrak{h}$  is closed under brackets. Hence  $\mathfrak{h} \subseteq \mathfrak{g}$  is a Lie subalgebra.

Take a sequence  $h_1, h_2, \dots \rightarrow 1$  of elements of  $H$ . Eventually, they get close enough that we can write them as exponentials:  $h_j = \exp a_j$  for some  $a_j \in \mathfrak{g}$ . Take any inner product on  $\mathfrak{g}$ . The unit sphere of  $\mathfrak{g}$ , in our inner product, is compact. So the unit vectors  $a_j/|a_j|$  have a convergent subsequence, say converging to a unit vector  $u \in \mathfrak{g}$ . As the vector  $a_j$  gets small, we can approximate  $a_j/|a_j|$  by some integer multiple of  $a_j$ , so approximate  $u$  by some elements  $n_j a_j$ , so approximate  $\exp u$  by  $\exp(a_j)^{n_j} \in H$ , i.e.  $\exp(u) \in H$ . This argument holds just as well for any rescaling of  $u$ , so  $u \in \mathfrak{h}$ . So the directions of some subsequence of the  $a_j$  approach a direction in  $\mathfrak{h}$ . Conversely, if we have a sequence of vectors in  $\mathfrak{g}$ , approaching the origin, whose directions stay away from  $\mathfrak{h}$ , then only finitely many of their exponentials enter  $H$ .

Take some linear subspace  $V \subseteq \mathfrak{g}$  complementary to  $\mathfrak{h}$ . Any sequence of vectors in  $V$ , approaching the origin, has direction staying away from  $\mathfrak{h}$ , so its exponentials stay outside  $H$ . Therefore there is a ball near 0 in  $\mathfrak{g}$  on which the exponential map keeps the points of  $V$  outside of  $H$ . For small enough balls  $B_{\mathfrak{h}} \subseteq \mathfrak{h}$  and  $B_V \subseteq V$ ,

$$(A, B) \in B_{\mathfrak{h}} \times B_V \mapsto \exp(A)\exp(B) \in G,$$

is a diffeomorphism identifying  $H$  near  $1 \in G$  with an embedded submanifold  $B_{\mathfrak{h}} \times \{0\} \subseteq B_{\mathfrak{h}} \times B_V$ . Hence  $H \subseteq G$  is an embedded submanifold near  $1 \in G$ . By left translation  $H \subseteq G$  is an embedded submanifold.  $\square$

The symmetries of any finite dimensional algebra, or stabilizer of any tensors or polynomials on any finite dimensional vector space, form a Lie group, by Cartan's theorem.

This proves that all of our examples of Lie groups are in fact Lie groups.

The *center*  $Z_G$  of a group  $G$  is the set of elements  $g \in G$  which commute with all other elements:  $gh = hg$  for every  $h \in G$ . The *centralizer* of a subset  $S \subseteq G$  of a Lie group  $G$  is the set  $Z_S$  of all elements  $g \in G$  so that  $gs = sg$  for every  $s \in S$ . In particular, the center is  $Z_G$ . The *normalizer* of a subset  $S \subseteq G$  of a Lie group  $G$  is the set  $N_S$  of all elements  $g \in G$  so that  $gSg^{-1} = S$  (i.e.  $gs g^{-1} \in S$  for any  $s \in S$ ).

**11.3** Prove that  $Z_S \subseteq G$  and  $N_S \subseteq G$  are embedded Lie subgroups. Prove that  $Z_G \subseteq G$  is a closed normal subgroup. If  $S \subseteq G$  is a subgroup, prove that  $S \subseteq N_S$  is a normal subgroup.

**11.4** Compute centres of  $\mathrm{SO}(n)$ ,  $\mathrm{SU}(n)$ ,  $\mathrm{Sp}(n)$ . Compare the dimensions and the centres; which of these Lie groups might be isomorphic?

**11.5** Suppose that  $A \subset G$  is an abelian subgroup of a Lie group  $G$ . Prove that the closure  $\bar{A}$  of  $A$  is a closed abelian Lie subgroup. Prove that every Lie group contains a positive dimensional abelian Lie subgroup.

Any Lie group morphism  $\varphi: G \rightarrow H$  has *kernel*  $\varphi^{-1}\{1\} = \{g \in G \mid \varphi(g) = 1\}$  a closed normal subgroup, hence an embedded Lie subgroup.

$\mathrm{SL}(n, k)$  is the kernel of  $\det: \mathrm{GL}(n, k) \rightarrow k^\times$ ,  $k = \mathbb{R}, \mathbb{C}$ , hence an embedded Lie subgroup.

$\mathrm{SO}(n, k)$  is the kernel of  $\det: \mathrm{GL}(n, k) \rightarrow k^\times$ ,  $k = \mathbb{R}, \mathbb{C}$ .

**Lemma 11.5.** *The Lie algebra of the kernel of a Lie algebra morphism  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$  is the kernel of the derivative  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ .*

*Proof.* By theorem 10.2 on page 100,  $\varphi(e^{tA}) = e^{t\varphi A}$ , so  $\varphi A = 0$  if and only if  $e^{tA}$  lies in the kernel. Since the kernel is an embedded subgroup (and the exponential is a diffeomorphism near the identity) its Lie algebra consists of the vectors in  $\mathfrak{g}$  whose flows lie in it.  $\square$

## The adjoint representation

Define  $\mathrm{Ad}_x y = xyx^{-1}$  for  $x, y \in G$ , and  $\mathrm{Ad}_x A = L_{x*} R_{x*}^{-1} A$  for  $x \in G$ ,  $A \in \mathfrak{g}$ .

**11.6** Prove that  $R_{x*} \vec{A} = \overrightarrow{\mathrm{Ad}_x^{-1} A}$ .

Define  $\mathrm{ad}_A B = [AB]$ .

**11.7** Prove that  $\mathrm{Ad}: G \rightarrow \mathrm{GL}(\mathfrak{g})$  is a representation, the *adjoint representation* and  $\mathrm{Ad}'_1 A = \mathrm{ad} A$ .

**11.8** What is the adjoint representation of  $\mathrm{Sp}(1)$ ?

**11.9** What is the adjoint representation of  $\mathrm{O}(3)$ ?

An *ideal* of a Lie algebra  $\mathfrak{g}$  is a Lie subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  which is “sticky”:  $[\mathfrak{h}\mathfrak{g}] \subseteq \mathfrak{h}$ .

**11.10** Suppose that  $H \subset G$  is a connected Lie subgroup of a connected Lie group  $G$ . Prove that  $H \subset G$  is a normal subgroup if and only if its Lie algebra  $\mathfrak{h} \subset \mathfrak{g}$  is an ideal.



## Continuity and smoothness

This section can be omitted without loss of continuity.

**Lemma 11.6.** *Every continuous group morphism  $\mathbb{R} \rightarrow G$  to a Lie group is a one parameter subgroup.*

*Proof.* Take a continuous group morphism  $\varphi: \mathbb{R} \rightarrow G$ . Let  $A(t) = \log \varphi(t)$  where this is defined. For any fixed  $t_0$ , the flow of  $\vec{A}(t_0)$  for time  $t_0$  takes us from 1 to  $\varphi(t_0)$ . The flow of  $2\vec{A}(t_0)$  takes us from 1 to  $\varphi(t_0)^2$ , as we follow the same vector field for twice the time. Similarly, for any integer  $k$ ,

$$e^{kA(t_0)} = \varphi(t_0)^k = \varphi(kt_0) = e^{A(kt_0)}.$$

Hence  $e^{kA(t/k)} = \varphi(t)$  is independent of  $k$ , and so  $kA(t/k) = A(t)$  for any  $t$  near zero and integer  $k$ . So

$$A(pt_0/q) = pA(t_0/q) = \frac{p}{q}A(t_0),$$

for all sufficiently small  $t_0$ . By continuity  $A(st) = sA(t)$  for all sufficiently small  $s, t$ , i.e.  $A$  is linear near the origin, so  $\varphi$  is a one parameter subgroup near the origin.  $\square$

**Theorem 11.7.** *Every continuous group morphism of Lie groups is smooth.*

*Proof.* Take a continuous group morphism  $\varphi: G \rightarrow H$ . Take a one parameter subgroup  $t \mapsto e^{tA} \in G$ . By lemma 11.6,  $\varphi(e^{tA}) = e^{tB}$  for some  $B \in \mathfrak{h}$ ; denote  $B$  as  $B = \psi(A)$ . From  $\varphi(e^{tA}) = e^{t\psi(A)}$ ,  $\psi(tA) = t\psi(A)$ . Clearly  $\psi = \log \varphi \exp$  is continuous. From lemma 11.2 on page 104, we find  $\psi$  is additive, so linear, so smooth. Since  $\psi = \log \varphi \exp$ , and  $\log$  and  $\exp$  are local diffeomorphisms,  $\varphi$  is smooth near the origin, and so (by translation) smooth everywhere.  $\square$

A *topological group* is a topological space which is also a group, so that the group multiplication operation and the group inverse operation are smooth maps.

**Corollary 11.8.** *Any topological group admits at most one smooth structure making it a Lie group. If there is such a smooth structure, its closed subgroups are precisely the embedded Lie subgroups in that smooth structure, and its continuous morphisms to and from Lie groups are smooth maps in that smooth structure.*

## Notes

The weird sum is not a standard notation or terminology, and we won't use it again.



## Chapter 12

### Lie algebras

#### Vector fields arising from actions

Given an action of a Lie group  $G$  on a manifold  $M$ , associate to any  $A \in \mathfrak{g}$  the vector field  $\tilde{A}$  on  $M$  defined by

$$\tilde{A}(m) := \left. \frac{d}{dt} \right|_{t=0} e^{tA} m.$$

If we have  $G$  acting on  $M := G$ , by left translation, then  $\tilde{A}$  is the right invariant vector field of  $A$ , as the flow of the right invariant vector field  $\tilde{A}$  is

$$e^{t\tilde{A}} g_0 = e^{tA} g_0,$$

the left translation.

If  $G = \mathrm{GL}(n, k)$  acts on  $M = k^n$  by usual matrix multiplication, then  $A \in \mathfrak{gl}(n, k)$  gives a vector field  $\tilde{A}(x)$  for  $x \in k^n$  by

$$\tilde{A}(x) = \left. \frac{d}{dt} \right|_{t=0} e^{tA} x = Ax.$$

**12.1** Prove that for any Lie group action

$$[\tilde{A}\tilde{B}] = -[\overleftarrow{AB}].$$

The affine group  $G$  acts on  $M := \mathbb{R}^n$  by affine transformations  $gx = ax + b$ , for  $a$  any invertible  $n \times n$  matrix, and  $b \in \mathbb{R}^n$  any constant vector. We write elements of the affine group as matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

The Lie algebra of the affine group of  $\mathbb{R}^n$  is the Lie algebra  $\mathfrak{g}$  of matrices

$$\begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix}.$$

The elements of  $\mathfrak{g}$  of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

have

$$\exp \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^A & 0 \\ 0 & 1 \end{pmatrix},$$

so vector field

$$\left. \frac{d}{dt} \right|_{t=0} e^{tA} x + 0 = Ax.$$

The elements of  $\mathfrak{g}$  of the form

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

have

$$\exp \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & b \\ 0 & 1 \end{pmatrix},$$

so vector field

$$\left. \frac{d}{dt} \right|_{t=0} x + tb = b.$$

In other words, the vector fields of the Lie algebra are the linear combinations of (1) translations  $\partial_{x_i}$  and (2) the linear vector fields  $x_j \partial_{x_i}$ .

The Lie algebra  $\mathfrak{g} := \mathfrak{so}(3)$  of  $G := \text{SO}(3)$  consist of the antisymmetric matrices

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

Identifying such a matrix with the vector  $v = (a, b, c) \in \mathbb{R}^3$  turns Lie bracket into cross product. The vector fields are

$$\tilde{A}(x) = \left. \frac{d}{dt} \right|_{t=0} e^{tA} x = Ax = v \times x.$$

In particular, for the standard basis vector  $e_1, e_2, e_3$ , we find the vector fields

$$e_1 \times x = \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} = -x_3 \partial_{x_2} + x_2 \partial_{x_3},$$

$$e_2 \times x = \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix} = x_3 \partial_{x_1} - x_1 \partial_{x_3},$$

$$e_3 \times x = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} = -x_2 \partial_{x_1} + x_1 \partial_{x_2}.$$

Note that each of the vector fields is tangent to every sphere around the origin.

**Lemma 12.1.** *If a Lie group  $G$  acts on a manifold  $M$ , the flow of the vector field  $\tilde{A}$  on  $M$  is*

$$e^{t\tilde{A}} m_0 = e^{tA} m_0.$$

*Proof.* These two expressions move any point of  $M$  with the same velocity, and both satisfy the definition of a flow of a vector field. By uniqueness of the flow of a vector field, they are equal.  $\square$

**12.2** For any action of a Lie group  $G$  on a manifold  $M$ , and point  $m_0 \in M$ , prove that the Lie algebra  $\mathfrak{g}^{m_0}$  of the stabilizer subgroup  $G^{m_0}$  is the set of all  $A \in \mathfrak{g}$  so that  $\tilde{A}(m_0) = 0$ .

If  $X$  is a  $G$ -homogeneous space and  $x_0 \in X$ , then the stabilizer subgroup  $H := G^{x_0} \subseteq G$  has Lie algebra  $\mathfrak{h} = \mathfrak{g}^{x_0}$  set of all  $A \in \mathfrak{g}$  so that  $\tilde{A}(x_0) = 0$ . Hence the identification  $T_{x_0}X \cong \mathfrak{g}/\mathfrak{h}$ . So the vector fields  $\tilde{A}$  of the  $G$ -action span the tangent space at every point  $x_0 \in X$ .

### Example: linear fractional transformations

Recall that every Lie group action induces a Lie algebra action. The group  $\mathrm{SL}(2, \mathbb{R})$  acts on the manifold  $M = \mathbb{RP}^1$ , by linear transformations acting on lines through the origin. As the subgroup  $\pm I$  takes every line to itself, the quotient group  $\mathbb{P}\mathrm{SL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R})/\pm I$  also acts. As in chapter 1, we can write any line through the origin as the span of a vector in the plane; denote the span of a vector by square braces: the symbol

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

represents the line spanned by the vector

$$\begin{pmatrix} x \\ y \end{pmatrix}.$$

Each matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

acts on each line by linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

In chapter 1 we found a chart, given by

$$\begin{bmatrix} x \\ 1 \end{bmatrix} \mapsto x$$

defined on the open set of all nonhorizontal lines through the origin. In this chart, the action is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{ax + b}{cx + d},$$

the *linear fractional transformations*.

The Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$  is the collection of traceless  $2 \times 2$  matrices, and has as a basis

$$X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with brackets  $[XY] = H$ ,  $[HX] = 2X$ ,  $[HY] = -2Y$ . Exponentiate:

$$e^{tX} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, e^{tH} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, e^{tY} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Hence, as linear fractional transformations,

$$e^{tX}x = x + t, e^{tH}x = e^{2t}x, e^{tY}x = \frac{x}{tx + 1}.$$

Differentiate at  $t = 0$  to find the vector fields:

$$\tilde{X} = \partial_x, \tilde{H} = 2x\partial_x, \tilde{Y} = -x^2\partial_x.$$

The Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$  arises as the quadratic vector fields

$$aX + bH + cY = \begin{pmatrix} b & a \\ c & -b \end{pmatrix} \mapsto a\tilde{X} + b\tilde{H} + c\tilde{Y} = (a + 2bx - cx^2)\partial_x.$$

### Characters and cocharacters

A *character* of a Lie group  $G$  is a Lie group morphism  $G \rightarrow \mathbb{C}^\times$ . A *cocharacter* of a Lie group  $G$  is a Lie group morphism  $S^1 \rightarrow G$ .

**12.3** Prove that every character and every cocharacter of  $S^1$  is precisely one of the maps  $e^{i\theta} \mapsto e^{ik\theta}$ , for an integer  $k$ .

**12.4** Prove that the cocharacters of  $\mathrm{GL}(n, \mathbb{R})$  are the maps  $e^{i\theta} \mapsto ge^{\theta A}g^{-1}$  where  $g \in \mathrm{GL}(n, \mathbb{R})$  and  $A$  is a block matrix, made of  $2 \times 2$  blocks of the form

$$A = \begin{pmatrix} 0 & -k_1 & & & & \\ k_1 & 0 & & & & \\ & & 0 & -k_2 & & \\ & & k_2 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & -k_r \\ & & & & & k_r & 0 \end{pmatrix}$$

if  $n$  is even or

$$A = \begin{pmatrix} 0 & -k_1 & & & & \\ k_1 & 0 & & & & \\ & & 0 & -k_2 & & \\ & & k_2 & 0 & & \\ & & & & \ddots & \\ & & & & & 0 & -k_r \\ & & & & & k_r & 0 \\ & & & & & & & 0 \end{pmatrix}$$

if  $n$  is odd, with  $k_1, k_2, \dots, k_r$  integers.

The cocharacters of  $\mathrm{GL}(n, \mathbb{C})$ , by the same reasoning, are  $e^{i\theta} \mapsto ge^{i\theta A}g^{-1}$  where  $g \in \mathrm{GL}(n, \mathbb{R})$  and  $A$  is diagonal, with integers down the diagonal.

The cocharacters of any subgroup of  $\mathrm{GL}(n, \mathbb{R})$ , by the same reasoning, are those cocharacters of  $\mathrm{GL}(n, \mathbb{R})$  which happen to land in that subgroup. For example, the group  $B \subset \mathrm{GL}(n, \mathbb{R})$  of upper triangular matrices, with positive entries on the diagonal, contains no elements with imaginary eigenvalues, so the only cocharacter of  $B$  is the trivial map  $e^{i\theta} \mapsto 1$ .

**12.5** If  $G$  is a compact Lie group, prove that every character of  $G$  is valued in the circle  $S^1 \subset \mathbb{C}$ . In particular, every real-valued character of a compact connected Lie group  $G$  is trivial.

**12.6** If  $G$  is a compact connected Lie group with finite fundamental group, prove that every character of  $G$  is trivial.

**12.7** For any representation  $G \rightarrow \mathrm{GL}(V)$  of any compact connected Lie group  $G$ ,  $G$  preserves every constant coefficient volume form on  $V$ .

## Abstract Lie algebras

A *Lie algebra* is a vector space  $\mathfrak{g}$  equipped with a bilinear antisymmetric operation  $x \in \mathfrak{g}, y \in \mathfrak{g} \mapsto [xy] \in \mathfrak{g}$  so that, if we define  $\mathrm{ad}_x y := [xy]$ , then  $[\mathrm{ad}_x \mathrm{ad}_y] = \mathrm{ad}_{[xy]}$  (the *Jacobi identity*).

**12.8** Prove that we can also write the Jacobi identity as

$$0 = [x[yz]] + [z[xy]] + [y[zx]],$$

for any  $x, y, z \in \mathfrak{g}$ .

The Lie brackets of any 3 vector fields on a manifold satisfy the Jacobi identity.

So does the bracket  $[xy] = xy - yx$  in any associative algebra.

The Lie bracket on the left invariant vector fields on any Lie group  $G$  is the bracket of vector fields, so satisfies the Jacobi identity; in this way  $\mathfrak{g} = T_1 G$  is a Lie algebra.

If  $\mathfrak{g}, \mathfrak{h}$  are Lie algebras, then  $\mathfrak{g} \oplus \mathfrak{h}$  is a Lie algebra, with brackets given by the usual brackets on  $\mathfrak{g}$  and  $\mathfrak{h}$ , and by  $0 = [xy]$  if  $x \in \mathfrak{g}$  and  $y \in \mathfrak{h}$ . If  $\mathfrak{g}, \mathfrak{h}$  are the Lie algebras of  $G, H$ , then  $\mathfrak{g} \oplus \mathfrak{h}$  is the Lie algebra of  $G \times H$ .

An *action* of a Lie algebra  $\mathfrak{g}$  on a manifold  $M$  is a choice of vector field  $\tilde{A}$  on  $M$  for each  $A \in \mathfrak{g}$ , linear in  $A$ , so that

$$[\tilde{A}\tilde{B}] = -[\overline{AB}],$$

for any  $A, B \in \mathfrak{g}$ . Problem 12.1 on page 109 proves that every Lie group action induces a Lie algebra action.

A *Lie algebra morphism*  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$  is a linear map so that  $\varphi[AB] = [\varphi A \varphi B]$ . Suppose that  $\varphi: G \rightarrow H$  is a Lie group morphism. The group morphism takes left translations to left translations, so takes left invariant vector fields to left invariant vector fields, brackets to brackets, defining a map (also denoted  $\varphi$ )  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ , which agrees with  $\varphi'(1)$  at  $T_1G$ , the associated Lie algebra morphism.

## Low dimensional Lie groups

This section can be omitted without loss of continuity.

By a theorem of Ado, every finite dimensional Lie algebra is the Lie algebra of a Lie subgroup of  $\mathrm{GL}(n, \mathbb{R})$  for some integer  $n \geq 0$ . However, there are Lie groups  $G$  so that  $G$  is not a Lie subgroup of  $\mathrm{GL}(n, \mathbb{R})$  for any integer  $n$ . Ado's theorem ensures that every Lie algebra is the Lie algebra of a unique connected and simply connected Lie group.

We take Ado's theorem as a motivation to survey the low dimensional Lie algebras, giving the classification of low dimensional connected and simply connected Lie groups.

**12.9** Suppose that  $V$  is a finite dimensional real vector space, and  $T: V \rightarrow V$  is a linear map. Let  $\mathfrak{g}_T := V \oplus \mathbb{R}$ , with bracket zero on  $V$ , and on  $\mathbb{R}$ , and bracket  $[xv] = xTv = -[vx]$  for  $x \in \mathbb{R}$  and  $v \in V$ . For which linear maps  $T$  is this a Lie algebra?

We can write out any Lie algebra  $\mathfrak{g}$ , using any choice of basis  $e_1, e_2, \dots, e_n$  for  $\mathfrak{g}$ , by describing the brackets  $[e_i e_j]$  in our basis. Note that the Lie bracket is antisymmetric, so we only need bracket  $[e_i e_j]$  for  $i < j$ .

**12.10** Using problem 12.9, we can construct some 3-dimensional Lie algebras: just take  $V = \mathbb{R}^2$ , and  $T$  any  $2 \times 2$  matrix. Find as many nonisomorphic 3-dimensional Lie algebras as you can by this construction.

A *solvable* Lie group is a Lie group which, as a group, is a solvable group, and similarly for the term *nilpotent* Lie group, and so on for most of the terminology of group theory.

**12.11\*** Prove that the group of all invertible upper triangular matrices with entries in  $k$  is a solvable subgroup of  $\mathrm{GL}(n, k)$ , for  $k = \mathbb{R}$  or  $k = \mathbb{C}$ , but *not* for  $k = \mathbb{H}$ .

**12.12\*** Prove that the group of all invertible strictly upper triangular matrices (i.e. with diagonal entries all equal to 1) with entries in  $k$  is a nilpotent subgroup of  $\mathrm{GL}(n, k)$ .



Denote by  $\mathfrak{aff}_n$  the Lie algebra of the group of affine transformations of  $\mathbb{R}^n$ . Here is the classification of Lie algebras in low dimension, up to isomorphism, first worked out by Bianchi:  $\sigma > 0$  can be any positive constant.

| dim | type             | Lie algebra                       | brackets    |                    |                     | properties |
|-----|------------------|-----------------------------------|-------------|--------------------|---------------------|------------|
| 1   |                  | abelian                           | 0           |                    |                     | abelian    |
| 2   |                  |                                   | $[e_1 e_2]$ |                    |                     |            |
|     | I                | abelian                           | 0           |                    |                     | abelian    |
|     | III              | $\mathfrak{aff}_1$                | $e_1$       |                    |                     | solvable   |
| 3   |                  |                                   | $[e_1 e_2]$ | $[e_2 e_3]$        | $[e_3 e_1]$         |            |
|     | I                | abelian                           | 0           | 0                  | 0                   | abelian    |
|     | II               | Heisenberg                        | 0           | $e_1$              | 0                   | nilpotent  |
|     | III              | abelian $\oplus \mathfrak{aff}_1$ | 0           | $e_3$              | 0                   | solvable   |
|     | IV               |                                   | 0           | $e_1 + e_2$        | $-e_1$              | solvable   |
|     | V                |                                   | 0           | $e_2$              | $-e_1$              | solvable   |
|     | VI <sub>0</sub>  |                                   | 0           | $e_2$              | $e_1$               | solvable   |
|     | VI               |                                   | 0           | $\sigma e_1 + e_2$ | $-e_1 + \sigma e_2$ | solvable   |
|     | VII              |                                   | 0           | $\sigma e_1 + e_2$ | $-e_1 - \sigma e_2$ | solvable   |
|     | VII <sub>0</sub> |                                   | 0           | $e_1$              | $e_2$               | solvable   |
|     | VIII             | $\mathfrak{sl}(2, \mathbb{R})$    | $e_3$       | $e_2$              | $e_1$               | simple     |
|     | IX               | $\mathfrak{so}(3)$                | $e_3$       | $e_1$              | $e_2$               | simple     |

## Notes

By a theorem of Palais, if a collection of smooth complete vector fields on a connected manifold generates a finite dimensional Lie algebra of vector fields, then there is a Lie group action inducing that Lie algebra action.



## Chapter 13

### Foliations

We imagine slicing manifolds up, like salami.

#### Foliations

Picture a *mille-feuille*, a French dessert made of a large number of very thin layers of pastry.



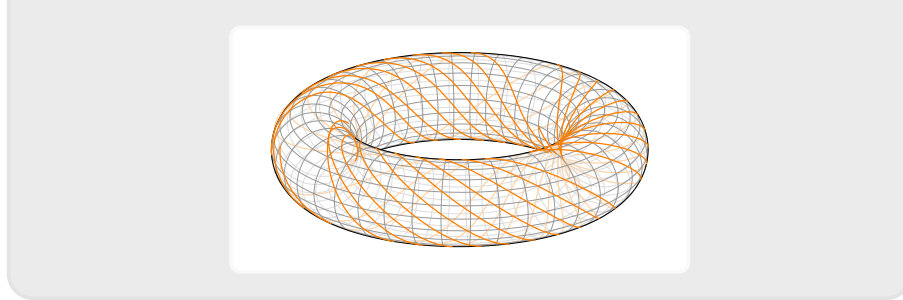
Mille-feuille, public domain, Wikimedia Commons, Attribution: Georges Seguin (Okki), 2010

These layers carve up a rectangular block of dessert into a large number of *feuilles* (leaves).

Draw the lines parallel to a given line to carve up the plane. Pick a rectangle in the plane.



Identify opposite sides to make a torus. Our drawing of parallel lines carves up the torus into a family of curves.

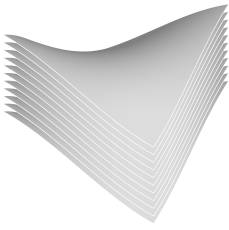


**13.1** Prove that a diffeomorphism  $\varphi$  between connected open sets of the plane, say written as  $(u, v) = \varphi(x, y)$ , takes horizontal line segments to horizontal line segments just when  $v = v(y)$  is independent of  $x$ .

A *p*-foliated diffeomorphism  $(u, v) = \varphi(x, y)$  between connected open sets of  $\mathbb{R}^{p+q}$ , with  $u, x \in \mathbb{R}^p$  and  $v, y \in \mathbb{R}^q$  is one which takes constant  $y$  to constant  $v$ . A *p*-foliation atlas of a manifold  $M$  is an atlas whose transition maps are *p*-foliated. A *rank p foliation* is a maximal *p*-foliation atlas; its charts are *adapted* to the foliation. A *plaque* of a foliation is a path connected submanifold  $P \subset M$  which is identified by some adapted chart  $(U, \varphi)$ , say  $(x, y) = \varphi(m)$ , with a subset of constant  $y$ , i.e.

$$P = \varphi^{-1}(\varphi(U) \cap (\mathbb{R}^p \times \{y_0\})).$$

These plaques are just little pieces of the true submanifolds of interest: a *leaf* is a maximal path connected union of plaques. Any two plaques which intersect at a point are both identified, in any chart, with subsets of constant  $y$ , and hence the same constant near that point. Hence they intersect in a set which is open in both of them. Consequently, every leaf is a *p*-dimensional submanifold (with topology and smooth functions induced from the plaques) but perhaps not an embedded submanifold.



**Theorem 13.1** (The implicit function theorem). *Take each fiber  $\varphi^{-1}\{q\}$  of a submersion  $\varphi: P \rightarrow Q$  and split it into its components. The components of the fibers are the leaves of a foliation of  $P$ , the associated foliation, of rank  $\dim P - \dim Q$ .*

**13.2** Generalise the implicit function theorem: suppose that  $\varphi: P \rightarrow Q$  is a submersion and  $Q$  has a foliation. Prove that there is a unique foliation of  $P$  so that a subset of  $P$  is a leaf just when it is a preimage  $\varphi^{-1}L$  of a leaf  $L \subset Q$ .

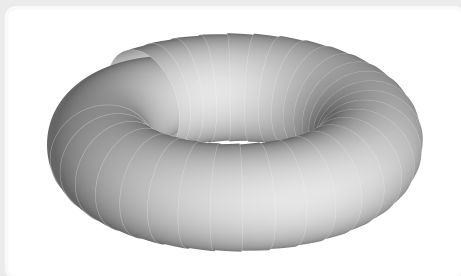
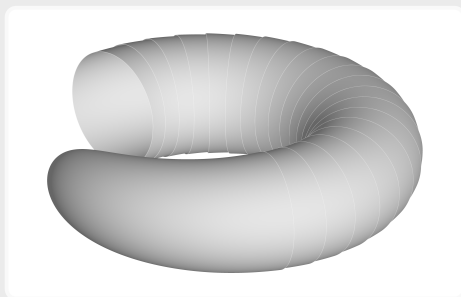
Take the functions

$$z = \text{constant} + \frac{1}{1 - x^2 - y^2},$$

whose graphs foliate the solid open cylinder  $U \subset \mathbb{R}^3$  of points where  $x^2 + y^2 < 1$ :



Map  $(x, y, z) \mapsto ((4+x)\cos z, (4+x)\sin z, y)$  to map the solid open cylinder to a solid torus. The foliation is mapped to the *Reeb foliation*, filling the solid torus with surfaces which look like paraboloids clinging asymptotically to the torus boundary of the solid torus. Each of these “curved paraboloids” is infinitely long, wrapping infinitely many times around the torus; we only draw a finite piece of a few of these.



**13.3** Prove that every leaf of a foliation is weakly embedded.

## The leaf space

The *leaf space*  $M/F$  of a foliation  $F$  on a manifold  $M$  is the set of leaves, equipped with the quotient map  $M \rightarrow M/F$ , associating to each point the leaf through that point, and with the quotient topology.

A *local complement* of a foliation is an embedded submanifold  $S \subset M$  transverse to every leaf so that  $\dim M - \dim S$  is the rank of  $F$ . A local complement is a *local section* of the foliation if it strikes every leaf at most once. Any foliation  $F$  admits many local complements through every point  $m \in M$ . Take an adapted chart near a point  $m_0$ , say  $x, y$ . The plaques of any local complement, in the domain of the chart, are locally identified by the chart with the graphs  $x = x(y)$  of smooth functions. Conversely, any such graph is locally identified by the chart with a local complement.

Suppose that  $m_0, m_1 \in M$  lie in the same leaf, and both in the domain  $U$  of the same foliated chart, on which they lie in the same plaque, some  $y$  level set. Suppose the same for  $m_1, m_2$ , perhaps in a different foliated chart, and so on. Pick local complements  $S_{m_i}$ . Each  $S_{m_i}$  is a graph  $x = x_i(y)$  near  $m_i$ . Define the *holonomy*: the diffeomorphism

$$(x_0(y), y) \in U \cap S_{m_0} \mapsto (x_1(y), y) \in U \cap S_{m_1}.$$

Geometrically, we are taking each point of  $S_{m_0}$  near  $m_0$  to the unique point of  $S_{m_1}$  near  $m_1$  on the same plaque in that chart. The map depends only on the choice of the domain  $U$  of the chart, not on chart itself; say that  $U$  is an *adapted* open set.

Suppose that the points  $m_i$  all lie on some path. Replace  $U$  by any other adapted open set containing that path, and get the same holonomy near  $m_0$  in  $S_{m_0}$ , as the two adapted open sets will both contain some adapted open set around that path. So holonomy is independent of adapted open set, at least near enough to  $m_0$  on  $S_{m_0}$ . If we put in place more local complements  $S_{m_j}$  at more points  $m_j$  of that path, the holonomies between the local complements compose to give the holonomy between the end points of the path, as each holonomy map is given by just intersecting with each plaque of the foliation.

The holonomy is a diffeomorphism of some open sets around  $p_0 \in S_{p_0}$  and  $q_0 \in S_{q_0}$ , if  $p_0$  and  $q_0$  are the ends of the path.

Take two continuous paths  $x, y: [a, b] \rightarrow M$  in a manifold, both starting at the same point  $p_0 = x(a) = y(a)$  and ending at the same point  $x(b) = y(b) = q_0$ . A *homotopy with fixed end points* between these paths is a continuous map  $z: [a, b] \times [0, 1] \rightarrow M$ , denoted  $z_t(s)$  instead of  $z(s, t)$ , so that  $x = z_0$  and  $y = z_1$  and  $z_t(a) = p_0$  and  $z_t(b) = q_0$  for all  $0 \leq t \leq 1$ .

**13.4** Under a homotopy of paths with fixed end points, with all paths lying in the same leaf, prove that the holonomy is unchanged, at least near enough to  $p_0$  on  $S_{p_0}$ .

Taking any local complement  $S \subset M$ , consider the set  $U \subset M$  which is the union of all leaves of  $M$  striking  $S$ . Near any point of  $S$ , we see in adapted charts that  $U$  contains an open subset of  $M$ . Take any point  $q_0 \in U$ . So  $q_0$  lies in the same leaf as some  $p_0 \in S$ ; write  $S$  as  $S_{p_0}$  and pick some local complement  $S_{q_0}$  near  $q_0$ . The holonomy  $\text{open} \subset S_{p_0} \rightarrow \text{open} \subset S_{q_0}$  identifies points on the same leaf, so  $U$  contains an open set around  $q_0$ .

**Theorem 13.2.** *A foliation is associated to a submersion if and only if its leaf space (1) is Hausdorff and (2) admits a local section near any point. If this occurs then, up to diffeomorphism, it is associated to a unique submersion with connected fibers.*

*Proof.* If  $F$  is associated to a submersion  $p: M \rightarrow N$ , then the implicit function theorem gives local coordinates in which  $F$  is linear, and hence admits local sections  $s: \text{open} \subset N \rightarrow M$ , the image of which is a local section of the foliation. The map  $M \rightarrow N$  descends to a map  $M/F \rightarrow N$ , taking the leaves. The local section  $s$  gives a local homeomorphism inverting  $M/F \rightarrow N$  locally, and therefore the charts on  $N$  pull back to charts on  $M/F$  making  $M/F$  a manifold and making the map  $M \rightarrow M/F$  agree with  $M \rightarrow N$  in local charts, hence a submersion with connected fibers. If  $M \rightarrow N$  has connected fibers, then  $M/F \rightarrow N$  is a bijective local diffeomorphism, so a diffeomorphism.

Each local complement  $S = S_{m_0} \subset M$  has, by composing the inclusion map  $S \rightarrow M$  with the quotient map  $M \rightarrow M/F$ , a continuous map  $S \rightarrow M/F$ . For a local section, this map is injective. Suppose that  $M/F$  is Hausdorff. From topology, a proper injection to a locally compact Hausdorff spaces is a homeomorphism to its image [13] p.20, Corollary 2.6 . So perhaps shrinking  $S$  to become a smaller local section,  $S \rightarrow M/F$  is a homeomorphism to its image. Hence if , we compose the inverse of  $S \rightarrow M/F$  with any chart on  $S$  to make a chart on  $M/F$  near the leaf of  $m_0$ . The transition map between these charts is the holonomy  $S_{m_0} \rightarrow S_{m_1}$  of the foliation, so smooth.  $\square$

## Plane fields

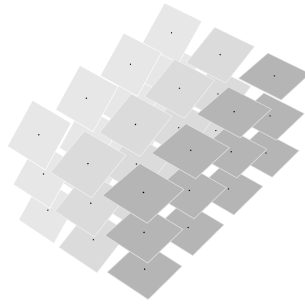
A *plane field* is a smoothly varying family of planes in an open subset of  $\mathbb{R}^3$ , one plane  $P_p \subset T_p \mathbb{R}^3$  in the tangent space of each point  $p = (x, y, z)$ . We can write out each plane as the solutions of some linear equation, say as the set of vectors

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for which

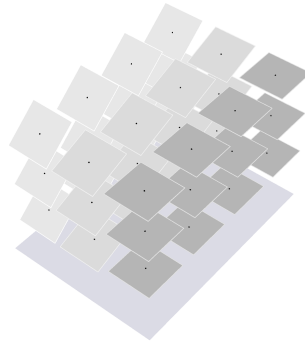
$$c = f(x, y, z) a + g(x, y, z) b,$$

for example, or perhaps the same with the roles of the variables  $x, y, z$  permuted in some way. To say that a plane is *smoothly varying* we mean that the functions  $f$  and  $g$  are smooth.

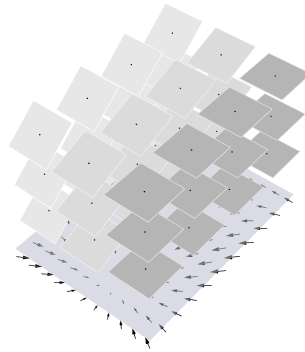


Any foliation  $F$  has associated plane field given the tangent spaces of its leaves. On the other hand, if the planes of our plane field rotate “like propellers” as we move along in some direction, we would not expect the plane field to be tangent to any foliation  $F$  by surfaces. We want to make this idea rigorous.

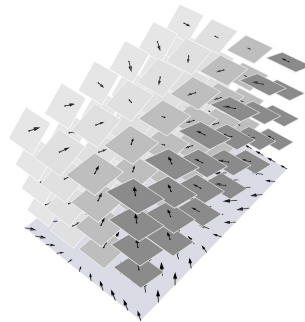
Draw a plane field as above in a box in  $\mathbb{R}^3$ , and pick a point, and suppose that the planes are never tangent to the vertical anywhere near this point. (If they are, then permute variables so that they are not, at least near our chosen point.) Picture the projection map  $\pi(x, y, z) = (x, y)$ .



Draw a vector field  $X$  on the plane:



Any vector field  $X$  on the plane determines a unique vector field  $\hat{X}$  on our box, lying in the planes of our plane field, so that  $\hat{X}$  projects down to  $X$ , i.e.  $\pi'(x, y, z)\hat{X}(x, y, z) = X(x, y)$ :



In terms of our equation for the plane field, if

$$X(x, y) = a(x, y)\partial_x + b(x, y)\partial_y,$$



then clearly, to get

$$c = f(x, y, z)a + g(x, y, z)b$$

satisfied on  $\hat{X}$  (i.e. to get  $\hat{X}$  to lie in the planes) we need

$$\hat{X}(x, y, z) = a(x, y)\partial_x + b(x, y)\partial_y + (f(x, y, z)a(x, y) + g(x, y, z)b(x, y))\partial_z.$$

Flowing any point along the flow of  $X$  downstairs gives us an associated flow line of  $\hat{X}$  in our box upstairs, so  $\pi$  relates the flow lines. In particular, if we just look at  $X = (1, 0)$  and  $Y = (0, 1)$ , we get two vector fields  $\hat{X}, \hat{Y}$  on our box.

If our plane field is the tangent plane field to a foliation  $F$  by surfaces, then the surfaces are projected by  $\pi$  diffeomorphically to the plane, at least near our chosen point. The vector fields  $\hat{X}, \hat{Y}$  are then just tangent vector fields to each those surfaces, taken by that diffeomorphism to  $X, Y$ . A diffeomorphism preserves Lie brackets of vector fields. But  $[XY] = 0$  since  $X, Y$  are constant vector fields. Therefore  $[\hat{X}\hat{Y}] = 0$  too, along each surface of the foliation.

Generalise to all dimensions: a subset  $V \subset TM$  is a *plane field* of rank  $k$  if, near each point  $m_0 \in M$  there are linearly independent smooth vector fields  $X_1, X_2, \dots, X_k$  (a *local basis*) so that for every point  $m \in M$  close enough to  $m_0$ ,  $V \cap T_m M$  is the span of the vectors  $X_1(m), X_2(m), \dots, X_k(m)$ . Let  $V_m = V \cap T_m M$ .

For a foliation  $F$  on a manifold  $M$ , write  $T_m F$ , the *tangent space* to the foliation at  $m$ , to mean the tangent space to the leaf of  $F$  through  $m$ . Let  $TF$ , the *tangent bundle* of the foliation, be the union of these  $T_m F$  for all points  $m \in M$ . An *integral manifold* of a plane field is a submanifold whose every tangent space is one of the planes of the plane field. A *section* of a plane field  $V$  is a smooth vector field  $X$  defined on an open subset of  $M$  so that  $X(m) \in V_m$  for all  $m \in M$ . A plane field is *bracket closed* if the Lie bracket of any two sections is a section (where it is defined).

**Theorem 13.3** (The Frobenius theorem). *A plane field is the tangent bundle of a foliation just when it is bracket closed and also just when it has an integral manifold through each point. There is then a unique such foliation, and the plane field has a unique maximal connected integral manifold through each point: the leaf of the foliation.*

*Proof.* The problem is local: we work near some point, take some local coordinates  $x, y$  near that point, taking the point to the origin, and take some projection map  $\pi: (x, y) \in \mathbb{R}^{p+q} \mapsto x \in \mathbb{R}^p$  so that this map is a linear isomorphism on  $\pi'(0): V_0 \rightarrow \mathbb{R}^p$  for our plane field  $V$ . Each vector field  $X$  near  $0 \in \mathbb{R}^p$  lifts uniquely to a section  $\hat{X}$  of  $V$  defined near  $0 \in \mathbb{R}^{p+q}$  so that  $\pi'(x, y)\hat{X}(x, y) = X(x)$ . So  $\hat{X}$  and  $X$  correspond under  $\pi$ . If we pick two commuting vector fields  $X, Y$  on  $\mathbb{R}^p$ , their lifts  $\hat{X}, \hat{Y}$  have bracket  $Z = [\hat{X}, \hat{Y}]$  with corresponding vector field  $0 = [X, Y]$ . So  $Z$  is tangent to the fibers of  $\pi$ , i.e.  $\pi'(x, y)Z(x, y) = 0$ .

Suppose  $V$  is bracket closed. Since  $\hat{X}$  and  $\hat{Y}$  are sections of  $V$ ,  $Z$  is also a section of  $V$ . But  $V$  has zero intersection with the fibers of  $\pi$ . Therefore  $Z = 0$ , i.e.  $0 = [\hat{X}, \hat{Y}]$ , so commuting vector fields on  $\mathbb{R}^p$  lift to commuting vector fields upstairs. So if we take  $X_j(x) = e_j$  the standard basis vectors in  $\mathbb{R}^p$ , their lifts  $\hat{X}_j$  all commute upstairs and span  $V$ . Apply the simultaneous flow box theorem (theorem 7.5 on page 82) to get new variables in which all  $\hat{X}_j$  are coordinate vector fields. In particular, this new variables identify the plane field of the model foliation of  $\mathbb{R}^{p+q}$  with  $V$ , and therefore  $V$  is locally the tangent bundle of a foliation. By uniqueness of that foliation, clearly  $V$  is everywhere the tangent bundle of a unique foliation.

Suppose that  $V$  is the tangent bundle of a foliation. Then every leaf is an integral manifold.

Suppose that there is an integral manifold through each point. The brackets of vector fields tangent to the integral manifold are tangent by problem 7.15 on page 81. So the brackets of any sections  $X, Y$  of  $V$  are also tangent to all of the integral manifolds, so tangent to the plane field.  $\square$

**13.5** On  $M = \mathbb{R}^3_{x,y,z}$ , define vector fields

$$X(x, y, z) = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y},$$

$$Y(x, y, z) = z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}.$$

Prove that on the set where  $y \neq 0$ ,  $X, Y$  span a plane field. What does the Frobenius theorem tell you about that plane field? Can you find a foliation that the plane field is tangent to?

### Orbits of families of vector fields

Suppose that  $\Gamma$  is a family of smooth vector fields on a manifold  $M$ . The *orbit* of  $\Gamma$  through a point  $m \in M$  is the set of all points  $e^{t_1 X_1} e^{t_2 X_2} \dots e^{t_k X_k}(m)$  for any vector fields  $X_j \in \Gamma$  and numbers  $t_j$  (positive or negative) for which this is defined.

The vector field  $\frac{\partial}{\partial \theta}$  on the Euclidean plane (in polar coordinates) has orbits the circles around the origin, and the origin itself.

The set of smooth vector fields supported in a disk has as orbits the open disk (a 2-dimensional orbit) and the individual points outside or on the boundary of the disk (zero dimensional orbits).

On Euclidean space, the set of vector fields supported inside a ball, together with the radial vector field coming from the center of the ball, forms a set of vector fields with a single orbit.

Translation in a generic direction on a torus has densely winding orbits.

**Theorem 13.4.** *The orbit of any point under any family of smooth vector fields is an immersed submanifold. If two orbits intersect, then they are equal. Let  $\langle \Gamma \rangle$  be the largest family of smooth vector fields which have the same orbits as the given family  $\Gamma$ . For any  $X, Y \in \langle \Gamma \rangle$ ,  $[X, Y] \in \langle \Gamma \rangle$ , and for any smooth functions  $f, g$ ,  $fX + gY \in \langle \Gamma \rangle$ .*

*Proof.* We can replace  $\Gamma$  by  $\langle \Gamma \rangle$  without loss of generality. If  $X, Y \in \Gamma$ , we can suppose that  $e_*^X Y \in \Gamma$  since the flow of  $e_*^X Y$  is

$$e^{t(e_*^X Y)} = e^X e^{tY},$$

which preserves orbits. We refer to this process as *pushing around* vector fields.

Fix attention on a specific orbit. For each point  $m_0 \in M$ , take as many vector fields as possible  $X_1, \dots, X_k$ , out of  $\Gamma$ , which are linearly independent at  $m_0$ . Refer to the number  $k$  of vector fields as the *orbit dimension*. Pushing around convinces us that the orbit dimension is a constant throughout the orbit. The map

$$(t_1, \dots, t_k) \in \text{open} \subset \mathbb{R}^k \mapsto e^{t_1 X_1} \dots e^{t_k X_k} m_0 \in M$$

(which we will take to be defined in some open set on which it is an embedding) is a *distinguished parameterization*, its image is a *distinguished set*, and its inverse is a *distinguished chart*. The tangent space to each point  $e^{t_1 X_1} \dots e^{t_k X_k} m_0$  of a distinguished set is spanned by the linearly independent vector fields

$$X_1, e_*^{t_1 X_1} X_2, \dots, e_*^{t_1 X_1} \dots e_*^{t_{k-1} X_{k-1}} X_k,$$

which belong to  $\Gamma$ , since they are just pushed around copies of the  $X_j$ . Let  $\Omega$  be a distinguished set. Suppose that  $Y \in \Gamma$  is a vector field, which is not tangent to  $\Omega$ . Then at some point of  $\Omega$ ,  $Y$  is not a multiple of those pushed around vector fields, so the orbit dimension exceeds  $k$ .

Therefore all vector fields in  $\Gamma$  are tangent to all distinguished sets. So any point inside any distinguished set stays inside that set under the flow of any vector field in  $\Gamma$ , at least for a short time. So such a point must also stay inside the distinguished set under compositions of flows of the vector fields, at least for short time. Therefore a point belonging to two distinguished sets must remain in both of them under the flows that draw out either of them, at least for short times. Therefore that point belongs to a smaller distinguished set lying inside both of them: the intersection of distinguished sets is a distinguished set.

With distinguished charts as atlas, an open set of an orbit is any union of distinguished sets. Pick a countable collection of distinguished charts covering the orbit. Every open subset of  $M$  intersects every distinguished set in a distinguished set, so intersects every open set of the orbit in an open set of the orbit. Thus the inclusion mapping of the orbit into  $M$  is continuous. Since  $M$  is Hausdorff, the orbit is also Hausdorff, so a submanifold of  $M$ . The distinguished charts are smoothly mapped into  $M$ , ensuring that the inclusion is a smooth map.

Each distinguished set is an embedded submanifold, so locally closed. If  $X, Y \in \Gamma$ , then the flow of  $e_*^{tX} Y$  preserves the submanifold for all  $t$ , and so the limit as  $t \rightarrow 0$  does as well, i.e.  $[Y, X] \in \Gamma$ , and similarly  $fX, X + Y \in \Gamma$ .  $\square$

Consider the plane field consisting of vectors

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

with  $b = za$  in  $\mathbb{R}^3$ . The vector fields tangent to that plane field have one orbit: all of  $\mathbb{R}^3$ , since they include  $\partial_z, \partial_x + z\partial_y$ , and therefore include the bracket:

$$[\partial_z, \partial_x + z\partial_y] = \partial_y.$$

## Notes

Plane fields are often called *distributions* or *subbundles of the tangent bundle*. A plane field is sometime called *Frobenius*, or also often called *integrable*, if it is the tangent bundle of a foliation.

## Chapter 14

### Quotients by Lie group actions

We use the theory of foliations to finish our work on building up global Lie group structure from microscopic Lie algebra structure. We use this to construct quotients of Lie group actions.

#### Lie algebra to Lie group

**Theorem 14.1.** *If  $G$  is a Lie group with Lie algebra  $\mathfrak{g}$ , then every Lie subgroup  $H \subseteq G$  has a Lie subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$ , and every Lie subalgebra  $\mathfrak{h} \subseteq \mathfrak{g}$  arises from a unique connected Lie subgroup  $H \subseteq G$ .*

Warning:  $H$  might not be an embedded subgroup; think of irrational slope lines on a torus  $\mathbb{R}^2/\mathbb{Z}^2$ .

*Proof.* The left translates of  $\mathfrak{h}$  form a plane field, spanned by the left invariant vector fields  $\bar{A}$  for  $A \in \mathfrak{h}$ . The plane field is bracket closed precisely because  $\mathfrak{h} \subset \mathfrak{g}$  is bracket closed, i.e. a Lie subalgebra. Applying the Frobenius theorem (theorem 13.3 on page 123) to the plane field, there is a unique maximal connected integral manifold through 1; call it  $H$ . By left invariance of the plane field, left translation permutes the various leaves of the associated foliation, i.e. the maximal connected integral manifolds. So left translation by any element  $g \in G$  takes leaves to leaves, taking the leaf  $H$  through 1 to the leaf  $gH$  through  $g$ . In particular, if  $g \in H$ , then  $gH = H$ , i.e.  $H$  is closed under multiplication. So if  $g \in H$ , then  $gH = H$  and  $1 \in H$  so  $1 \in gH$ , i.e.  $g^{-1} \in H$ . So  $H$  is closed under taking inverses.  $\square$

**14.1** A Lie algebra  $\mathfrak{g}$  is *semisimple* if, for any  $A \in \mathfrak{g}$  not zero, there is some  $B \in \mathfrak{g}$  so that  $[AB] \neq 0$ . Prove that every semisimple Lie algebra is the Lie algebra of a Lie group.

**Theorem 14.2.** *Suppose that  $G$  and  $H$  are Lie groups with Lie algebras  $\mathfrak{g}$  and  $\mathfrak{h}$ , and that  $G$  is connected. Any Lie algebra morphism  $\mathfrak{g} \rightarrow \mathfrak{h}$  arises from a Lie group morphism  $G' \rightarrow H$  of some covering group of  $G$ .*

*Proof.* Make  $\mathfrak{g} \oplus \mathfrak{h}$  into a Lie algebra, using the brackets on  $\mathfrak{g}, \mathfrak{h}$ , and making  $0 = [\mathfrak{g}\mathfrak{h}]$ . This is the Lie algebra of  $G \times H$ . Take a linear map  $\varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ . Let  $\mathfrak{g}' \subseteq \mathfrak{g} \oplus \mathfrak{h}$  be the set of all pairs  $(A, \varphi(A))$  so that  $A \in \mathfrak{g}$ . Check that  $\mathfrak{g}'$  is a Lie subalgebra just when  $\varphi$  is a Lie algebra morphism; assume this. Let  $G' \subset G \times H$  be the associated connected Lie subgroup. The maps  $(g, h) \in G \times H \mapsto g \in G$  and  $(g, h) \in G \times H \mapsto h \in H$  are Lie group morphisms, so restrict to  $G'$  to Lie group morphisms  $p: G' \rightarrow G$  and

$q: G' \rightarrow H$ . The associated Lie algebra morphisms  $p', q'$  are just  $p'(A, \varphi(A)) = A$  and  $q'(A, \varphi(A)) = \varphi(A)$ . Since  $p'$  is a linear isomorphism,  $p: G' \rightarrow G$  is a covering map (theorem 10.5 on page 100).  $\square$

**Corollary 14.3.** *Any two connected Lie groups with isomorphic Lie algebras have isomorphic covering groups.*

To classify connected Lie groups up to covering, classify Lie algebras, i.e. tensors  $\square \in \Lambda^2 V^* \otimes V$  satisfying Jacobi identity, up to  $\text{GL}(V)$ . This is not possible, after about dimension 5 or 6, as the classification get too complicated to work with.

**14.2** Suppose that  $H \subset G$  is an immersed submanifold, and that  $H$  is also a subgroup. Prove that  $H$  is a Lie group and that the inclusion map  $h \in H \mapsto h \in G$  is a Lie group morphism.

If a Lie group  $G$  acts on a manifold  $M$ , and a Lie group  $G'$  acts on a manifold  $M'$ , a smooth map  $\Phi: M \rightarrow M'$  is *equivariant* for a Lie group morphism  $\varphi: G \rightarrow G'$  if  $\varphi(g)\Phi(m) = \Phi(gm)$  for all  $g \in G$  and  $m \in M$ . We can also say that  $(\Phi, \varphi)$  is a *morphism* of Lie group actions. A *morphism* is a morphism of their group actions. A *covering homogeneous space* of a homogeneous space  $(X, G)$  is a morphism from a homogeneous space  $(\hat{X}, \hat{G})$  to  $(X, G)$  consisting of covering maps  $\hat{X} \rightarrow X$  and  $\hat{G} \rightarrow G$ .

**14.3\*** Suppose that  $(X, G)$  and  $(X', G')$  are homogeneous spaces. Suppose that  $G$  is connected. Pick points  $x_0 \in X$ ,  $x'_0 \in X'$ , with stabilizers  $H \subset G$  and  $H' \subset G'$ . Suppose we have a Lie algebra morphism  $\mathfrak{g} \rightarrow \mathfrak{g}'$  under which  $\mathfrak{h}$  has image inside  $\mathfrak{h}'$ . Prove that there is a covering homogeneous space  $(\hat{X}, \hat{G})$  of  $(X, G)$  and morphism of  $(\hat{X}, \hat{G})$  to  $(X', G')$  inducing  $\varphi$  on the Lie algebras  $\hat{\mathfrak{g}} = \mathfrak{g} \rightarrow \mathfrak{g}'$ .

**Theorem 14.4.** *Suppose that  $\mathfrak{g}$  is the Lie algebra of a connected and simply connected Lie group  $G$ . A Lie algebra action of  $\mathfrak{g}$  arises from a Lie group action of  $G$  if and only if the Lie algebra action takes all elements of  $\mathfrak{g}$  to complete vector fields.*

*Proof.* Take a Lie algebra action associating to each  $A \in \mathfrak{g}$  a vector field  $\tilde{A}$  on a manifold  $M$ . Define a Lie algebra action on  $G \times M$  by associating to each  $A \in \mathfrak{g}$  the complete vector field  $\tilde{A} := \tilde{A}_G + \tilde{A}_M$ . All  $\tilde{A}_G$  are linearly independent (but perhaps not all  $\tilde{A}_M$ !),  $\tilde{A}$  spanning a bracket closed plane field on  $G \times M$ . Pick a point  $m_0 \in M$ . Applying the Frobenius theorem (theorem 13.3 on page 123) to the plane field, there is a unique connected integral manifold through  $(1, m_0)$ ; call it  $G'$ . The map  $G' \rightarrow G$  takes  $\tilde{A}$  to  $\tilde{A}_G$ , so is a covering map by corollary 51.2 on page 415. But  $G'$  is connected and  $G$  is connected and simply connected, so  $G' \rightarrow G$  is a diffeomorphism, say with inverse  $g \mapsto (g, gm_0) \in G' \subset G \times M$ , defining  $gm_0 \in M$ . By invariance of the plane field under the flows of the  $\tilde{A}$ ,  $(e^{tA}g, e^{tA}(gm_0)) \in G'$ , i.e.  $e^{tA}(gm_0) = (e^{tA}g)m_0$ . Repeat this to see the same for a finite product of exponentials. By problem 10.3 on page 99, for any  $g_1, g_2 \in G$ :  $g_0(g_2m_0) = (g_0g_2)m_0$ . So  $G$  acts on the  $\mathfrak{g}$ -orbit through  $m_0$  as a Lie group action. The value of  $gm_0$  depends smoothly on  $g$  and  $m_0$ , as it is given by integration of flows of vector fields.  $\square$

Try to work as much as possible with Lie algebras instead of Lie groups; we are better at linear algebra than we are at geometry.

## Quotient spaces

Consider an action of a Lie group  $G$  on a manifold  $M$ . Any point  $m_0 \in M$  has  $G$ -stabiliser

$$G^{m_0} := \{g \in G \mid gm_0 = m_0\} \subset G$$

clearly a closed subgroup.

The *quotient space*  $G \backslash M$  of a Lie group action is the set of all orbits, with the quotient topology; the *quotient map* is  $m \in M \mapsto Gm \in G \backslash M$ . A *right action* of a Lie group  $G$  on a smooth manifold  $M$  is a smooth right group action, i.e. a smooth map  $M \times G \rightarrow M$ , denoted  $m, g \mapsto mg$ , so that  $m1 = m$  for all  $m \in M$  and  $m(gh) = (mg)h$  for all  $g, h \in G$  and  $m \in M$ . The *quotient space*  $M/G$  of a right Lie group action is the set of all orbits, with the quotient topology, and the quotient map is  $m \in M \mapsto mG \in M/G$ . Every right action determines a left action  $g * m := mg^{-1}$ , so the theory is identical.

Rotating the 2-sphere, fixing the north and south poles, is an action of the circle on the 2-sphere. The quotient space is a closed interval, not a smooth 1-dimensional manifold.

Note that  $M \rightarrow G \backslash M$  is continuous by definition. If  $G \backslash M$  is Hausdorff (a *closed action*) then each point of  $G \backslash M$  is closed, so the preimage in  $M \rightarrow G \backslash M$  is closed, i.e. each orbit is closed.

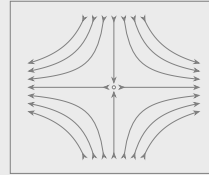
On the plane  $M = \mathbb{R}^2$ , take the vector field  $X(x, y) = (1 - x^2 - y^2)(x\partial_y - y\partial_x)$ . Let  $G = \mathbb{R}$  act on  $M = \mathbb{R}^2$  by letting  $t \in G$  take any point  $(x, y) \in M$  by following the flow of  $X$  for time  $t$ .

**14.4** What are the orbits? Are they closed? Is the quotient space  $G \backslash M$  Hausdorff?

Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . The action of  $G := k^\times := k - \{0\}$  on  $M := k^2$ :

$$t(x, y) = (tx, y/t)$$

is *not* closed: the orbits are not closed. So the quotient space is not Hausdorff.



The *incidence relation*  $R \subset M \times M$  is the set of pairs of the form  $(m, gm)$  for any  $m \in M$  and  $g \in G$ .

**Theorem 14.5.** *The incidence relation of an action is a closed set if and only if the quotient space is Hausdorff.*

*Proof.* Recall that the topology of  $M \times M$  is generated by product open sets. If we let  $U = M \times M - R$ , then  $R$  is closed just when, for any  $m_0, m_1 \in M$ , if  $(m_0, m_0) \in U$  then there is an open subset of the form  $U_0 \times U_1 \subset U$  containing  $(m_0, m_1)$ . In other words, two points  $m_0, m_1 \in M$  are not in the same  $G$ -orbit just when there are open sets  $U_0, U_1 \subset M$  so that  $m_0 \in U_0$  and  $m_1 \in U_1$  and for any point  $m \in U_0$ ,  $gm$  is not

in  $U_1$ , i.e.  $GU_0$  is disjoint from  $U_1$ , i.e.  $GU_0, GU_1$  are disjoint  $G$ -invariant open sets around  $m_0, m_1$ .  $\square$

**14.5** Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Prove that quotient space of the action of  $G := k^\times := k - \{0\}$  on  $M := k^2 - \{0\}$ :

$$t(x, y) = (tx, y/t)$$

is not Hausdorff.

**14.6** Prove that every Lie group action of any compact Lie group has Hausdorff quotient space.

**14.7\*** Let  $G$  be the group of diffeomorphisms of the unit sphere  $S^n \subset \mathbb{R}^{n+1}$  generated by the reflection map

$$r(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1}).$$

Let  $s: S^n \rightarrow \bar{B}^n$  be the “squishing flat map”

$$s(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n).$$

Prove that the quotient  $G \backslash S^n$  is homeomorphic to the closed ball  $\bar{B}^n \subset \mathbb{R}^n$ , by a homeomorphism identifying the quotient map with  $s$ .

**14.8** Suppose that the quotient map of some Lie group action is a smooth submersion, for some smooth structure on the quotient space. Prove that the smooth structure is unique, the quotient map is surjective, and the incidence relation is a closed smooth submanifold.

**Theorem 14.6.** *Suppose that a Lie group  $G$  acts smoothly on a manifold  $M$  with quotient  $\bar{M} := G \backslash M$ . The incidence relation  $R \subset M \times M$  is a closed embedded submanifold just when the quotient map  $M \rightarrow \bar{M}$  is a smooth submersion of manifolds, for some smooth structure on  $\bar{M}$ . This smooth structure is then unique.*

*Proof.* Suppose that  $q: M \rightarrow \bar{M}$  is a smooth submersion, for some smooth structure on  $\bar{M}$ . Let

$$\Delta := \{(x, x) \mid x \in \bar{M}\} \subset \bar{M} \times \bar{M}.$$

Map  $\Phi: (m, n) \in M \times M \mapsto (q(m), q(n)) \in \bar{M} \times \bar{M}$ , a submersion. Clearly  $R = \Phi^{-1} \Delta$ , so a closed embedded submanifold.

Suppose on the other hand that  $R \subset M \times M$  is a closed embedded submanifold. By theorem 14.5 on the previous page,  $\bar{M}$  is Hausdorff, and so the  $G$ -orbits in  $M$  are closed subsets. The permutation  $(p, q) \in M \times M \mapsto (q, p) \in M \times M$  preserves  $R$ , as  $(gm, m) = ((gm), g^{-1}(gm))$ . Define maps  $\pi_1: (p, q) \in M \times M \mapsto p \in M$  and  $\pi_2: (p, q) \in M \times M \mapsto q \in M$ . For any fixed  $g_0 \in G$ , composing the map  $p \in M \mapsto (p, g_0 p) \in R$  with the map  $\pi_1(p, g_0 p) = p \in M$  gives the identity map. So  $\pi_1$  restricts to  $R$  to a submersion  $R \rightarrow M$ . Permute: both  $\pi_1$  and  $\pi_2$  restrict to  $R$  to be submersions. The fibers of any submersion are closed embedded submanifolds, all of the same dimension. For each point  $p \in M$ , let  $R_p := \pi_1^{-1}\{p\}$ , a closed embedded submanifold of  $M \times M$ . Since  $(\pi_1, \pi_2)$  is the identity map,  $\ker \pi_1' \cap \ker \pi_2' = 0$ , i.e.  $\pi_2$  maps  $R_p \rightarrow M$  by an injective immersion. Note that

$$R_p = \pi_1^{-1}\{p\} = \{(p, gp) \mid g \in G\} = \{p\} \times Gp,$$



is an embedded submanifold in  $R$  and so in  $M \times M$ , so  $Gp \subset M$  is an embedded submanifold. So each  $G$ -orbit  $Gp \subset M$  is a closed embedded submanifold. Note that  $Gp \cong R_p$  has dimension equal to  $\dim R - \dim M$ , a constant independent of  $p$ .

As the  $R_p$  are the fibers of a submersion, their tangent spaces vary smoothly in  $R$ . At each point  $r = (p, gp) \in R$ , the tangent space of the fiber is  $T_r R_p = 0 \oplus T_{gp} Gp$ . If we write this as  $r = (g^{-1}q, q)$ , then  $T_r R = 0 \oplus T_q Gq$ , varies smoothly with  $q$  for any fixed  $g$ . So these linear subspaces  $T_q Gq$  form a smooth plane field  $F_q = T_q Gq$  on  $M$ . The  $G$ -orbits on  $M$  are the leaves of a smooth foliation  $F$  on  $M$ .

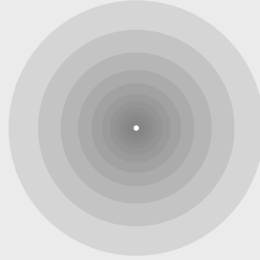
Take a point  $r_0 = (p_0, g_0 p_0) \in R$ . The two linear surjections  $\pi'_1(r_0), \pi'_2(r_0)$  on  $T_{r_0} R$  have kernels of the same dimension:  $\dim R - \dim M$ . The generic linear subspace  $V \subset T_{r_0} R$  complementary to one kernel is complementary to both kernels, so injectively mapped by both  $\pi'_1(r_0)$  and  $\pi'_2(r_0)$ . Take an embedded submanifold  $S \subset R$  with  $T_{r_0} S = V$ . Then  $\pi_1$  and  $\pi_2$  are immersions of  $S$  near  $r_0$ . Every immersion is locally an embedding, by the rank theorem. Making  $S$  smaller, we can make both  $\pi_1$  and  $\pi_2$  embeddings  $S \rightarrow M$ . Since  $\pi_1$  embeds  $S \rightarrow M$ , every fiber  $R_p = \pi_1^{-1}\{p\}$  intersects  $S$  at most once. Each fiber  $R_p$  of  $\pi_1$  projects under  $\pi_2$  to a  $G$ -orbit  $Gp = \pi_2 R_p$ , i.e. a leaf of  $F$ . So  $\pi_2 S \subset M$  is a local section of the foliation  $F$ . By theorem 13.2 on page 120, this foliation is a submersion, say  $M \rightarrow N$ , with fibers the connected components of the  $G$ -orbits.

The continuous map  $N \rightarrow \bar{M}$  is a surjective local homeomorphism, since the local sections  $S$  (after perhaps shrinking them a bit) each map to  $\bar{M}$  by local homeomorphism (as in the proof of theorem 13.2 on page 120). Any two points of  $N$  on the same fiber of  $N \rightarrow \bar{M}$  are images of points of  $M$  in the same  $G$ -orbit. The group action smoothly moves local sections around to all points of each  $G$ -orbit on  $M$ , so the resulting smooth charts on  $N$  are taken to one another, and hence the smooth structure of  $N$  descends to one on  $\bar{M}$ , for which  $N \rightarrow \bar{M}$  is a local diffeomorphism.  $\square$

Over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , take a diffeomorphism  $\varphi: k^n \rightarrow k^n$  fixing the origin. Let  $M := k^n - \{0\}$ . Let  $G$  be the group of diffeomorphisms generated by  $\varphi$ :

$$G = \{ \varphi^j \mid j \in \mathbb{Z} \},$$

where  $\varphi^j$  means  $j$ -fold composition of  $\varphi$ . Suppose that  $G$  has no finite orbits on  $M$ , i.e. every  $\varphi^j$  has no fixed points except at the origin. The *Hopf manifold*  $M_\varphi$  is the quotient of  $M_\varphi = M/G$ .



For a particular case, try  $n := 2$  and

$$\varphi(x, y) = \left( \frac{x}{4} + y^2, \frac{y}{2} \right).$$

Let's prove that the action quotients. For each  $j$ , let  $R_j$  be the image of the embedding

$$x \in M = k^n - \{0\} \mapsto (x, \varphi^j(x)) \in M \times M.$$

The incidence relation  $R$  is the union of the embedded submanifolds  $R_j \subset M \times M$ , each closed by continuity of  $\varphi$ . These  $R_j$  are disjoint: if  $(x, \varphi^i(x)) = (x, \varphi^j(x))$ , some  $i \neq j$ , then  $\varphi^i(x)$  is fixed by  $\varphi^{(j-i)}$ , impossible. If a sequence of points of  $R$  converges, say  $(x_i, \varphi^{j_i}(x_i)) \rightarrow (x, y)$ , then  $x_i \rightarrow x$  and so  $\varphi^{j_i}(x) \rightarrow y$ . To have  $y \neq 0$ , the orders  $j_i$  are bounded; after taking a subsequence, they are constant, say  $j$ . Then  $(x, y)$  lies on one  $R_j$  along with all  $(x_i, \varphi^{j_i}(x_i))$ . So  $R$  is a closed submanifold. So the quotient, the Hopf manifold, is a manifold.

**14.9\*** Compute the incidence correspondence for the action of rotations around the origin on the plane. Prove that this incidence correspondence is not an embedded submanifold. Hint: it is a cone; see on page 52.

**14.10** Let  $M = S^2 \times S^2$ , a product of spheres. Let  $\alpha: x \in S^2 \mapsto -x \in S^2$ , the antipodal map. Let

$$r: (x, y, z) \in S^2 \mapsto (x, y, -z) \in S^2,$$

the reflection map through the horizontal plane. Consider the map  $\varphi: (x, y) \in M \rightarrow (r(y), \alpha(x)) \in M$ . Let  $G$  be the group of diffeomorphisms of  $M$  generated by  $\varphi$ . Prove that the quotient  $\bar{M} = G \backslash M$  is a smooth 4-dimensional manifold.

**14.11** Suppose that a Lie group  $G$  acts smoothly on a manifold  $M$  with incidence relation a closed embedded submanifold. Prove that any smooth map  $M \rightarrow N$  to any manifold  $N$  is constant on  $G$ -orbits just when it is the composition of  $M \rightarrow G \backslash M$  with a unique smooth map  $G \backslash M \rightarrow N$ . We say that  $M \rightarrow N$  *descends* to  $G \backslash M \rightarrow N$ .

**14.12** Suppose that a Lie group  $G$  acts smoothly on a manifold  $P$  and that  $\varphi: P \rightarrow Q$  is a smooth surjective submersion, taking each  $G$ -orbit to a single point, and taking different  $G$ -orbits to different points. Prove that  $Q$  is diffeomorphic to  $G \backslash P$  by a unique diffeomorphism taking  $\varphi(p) \mapsto Gp$  for any  $p \in P$ .

Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ; let  $d$  be the real dimension of  $k$ . Recall the map

$$\varphi: \begin{pmatrix} a \\ b \end{pmatrix} \in k^2 \rightarrow \begin{pmatrix} 2a\bar{b} \\ |a|^2 - |b|^2 \end{pmatrix} \in k \oplus \mathbb{R}.$$

**14.13** Prove that  $\varphi$  is a submersion away from the origin.

Write the unit elements of  $k$  as  $S^{d-1} \subset k$ . In problem 8.35 on page 90 and subsequent problems, we checked that this map maps to the unit sphere and is invariant under

$$g \in S^{d-1}, \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2d-1} \mapsto \begin{pmatrix} a\bar{g} \\ b\bar{g} \end{pmatrix}$$

and that the orbits of that group action are precisely the fibers of  $\varphi$ . By problem 14.12,  $S^d$  is the quotient space  $S^d = S^{d-1} \backslash S^{2d-1}$ .

Take  $M := S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$  the unit sphere, by elements of  $G := S^1 \subset \mathbb{C}^\times$ . We expect the quotient to be  $\mathbb{CP}^n$ , with quotient map  $\varphi(z) = [z]$ . Note that  $\varphi$  is a submersion (using our usual charts on  $\mathbb{CP}^n$ ), constant on  $G$ -orbits. So the quotient  $G \backslash M$  as defined above is  $\mathbb{CP}^n$ , up to a unique  $G$ -equivariant diffeomorphism.

An important example where our theory of quotients doesn't quite apply. Let  $M = \mathbb{C}$ ,  $G = \{\pm 1\}$  acting as usual by complex number multiplication. We can guess that the quotient is  $G \backslash M = \mathbb{C}$ , with quotient map  $z \mapsto w = p(z) = z^2$ . Let's prove it. This is a smooth map, and the fiber over each point  $w = z^2$  consists of two values of the form  $z, -z$ , except if  $w = 0$ , where find only  $z = 0$ . So the fibers are the group orbits. On the other hand, the map has the origin as its only critical point, where  $p'(0) = 0$ . So this map  $p$  is not a submersion, but is a local diffeomorphism away from the origin.

Consider the invariant open sets. Where  $p$  is a local diffeomorphism, any open set has open image. Any open set which contains the origin also contains a small disk around the origin, say  $|z| < \varepsilon$ . This disk maps by  $p(z) = w = z^2$  precisely to the points  $w$  of the complex plane with  $|w| < \varepsilon^2$ , as the reader can verify, an open set. So open sets have open images, and so the invariant open sets are precisely the  $p$ -preimages of open sets. Hence  $p$  is the quotient map, and the quotient topology is the usual topology on the complex plane [13] p.20, Corollary 2.6. Note that the function  $|z|^2$  is smooth, and invariant under the group action, but is not a smooth function of  $w = z^2$ .

**14.14** Carry out a similar analysis of the quotient of the complex plane by the action of  $1 \cdot z = z$ ,  $(-1) \cdot (x + iy) := (y + ix)$ .



## Chapter 15

### Homogeneous spaces

#### Homogeneous spaces

Recall that a homogeneous space of a Lie group is a manifold on which the Lie group acts smoothly and transitively. Homogeneous spaces of a Lie group are classified by classifying closed subgroups:

**Theorem 15.1.** *Suppose that  $(X, G)$  is a homogeneous space. Then the stabilizer  $H = G^{x_0}$  of any point  $x_0 \in X$  is a closed subgroup of  $G$  and  $G/H$  is a manifold and  $gH \in G/H \mapsto gx_0 \in X$  is a well defined  $G$ -equivariant diffeomorphism, with derivative at  $1H$  a linear isomorphism  $\mathfrak{g}/\mathfrak{h} \rightarrow T_{x_0}X$ . Conversely, if  $H \subset G$  is a closed subgroup, and  $X := G/H$ , then  $(X, G)$  is a homogeneous space for a unique smooth structure on  $X$ .*

*Proof.* We know that  $H$  is an embedded Lie subgroup of  $G$  by theorem 11.4 on page 105. The  $H$ -action has incidence relation  $R = \{(g, gh) \mid g \in G, h \in H\} \subset G \times G$ , which is diffeomorphic to  $G \times H$  by

$$(a, b) \in G \times G \mapsto (a, a^{-1}b) \in G \times G,$$

so is a closed submanifold. Hence  $X = G/H$  is a smooth manifold and  $G \rightarrow G/H$  is a smooth map. This map commutes with left action of  $G$ . The action is smooth on  $G/H$ , because it is smooth on  $G$ .

Taking any homogeneous space  $(X, G)$ , we construct  $H$  as the stabiliser of some point  $x_0$ , and smoothly map

$$g \in G \mapsto gx_0 \in X.$$

This map descends to a smooth bijection  $gH \in G/H \mapsto gx_0 \in X$ . From topology, a proper injection to a locally compact Hausdorff spaces is a homeomorphism to its image [13] p.20, Corollary 2.6. So this map is a local homeomorphism, so a homeomorphism. As smooth functions on  $X$ , and on  $G/H$ , are precisely those continuous functions which pull back to smooth  $H$ -invariant functions on  $G$ , this map is a diffeomorphism.  $\square$

Over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , the subgroup  $H \subset G$  of  $G = \mathrm{GL}(n, k)$  preserving a  $k$ -linear subspace is a closed subgroup. The associated homogeneous space  $X = G/H$  is the Grassmannian, giving the Grassmannian a smooth structure. If  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$  respectively, then let  $K = \mathrm{O}(n), \mathrm{U}(n)$  or  $\mathrm{Sp}(n)$  respectively. So  $K \subset G$  is a compact Lie subgroup, and hence the Grassmannian  $X$  is also homogeneous under  $K$ , with the same smooth structure as the action is also smooth, and hence  $X$  is a compact manifold. If we let  $V = k^n$ : the Grassmannian  $\mathrm{Gr}_p V = G/H$  is a homogeneous space, and therefore a manifold. Note how much easier this is than in theorem 2.2 on page 21. In particular, if  $p = 1$ ,  $\mathrm{Gr}_1 V = \mathbb{P}V$  is a manifold.

**15.1** Find the dimensions of all homogeneous spaces  $(X, G)$  with  $G = \mathrm{SO}(3)$ . Give some examples of such homogeneous spaces.

**15.2** If  $(X, G)$  and  $(X', G')$  are homogeneous spaces, with points  $x_0 \in X$  and  $x'_0 \in X'$  and stabilizer subgroups  $H := G^{x_0} \subset G$  and  $H' := G'^{x'_0} \subset G'$ . Suppose that there is a Lie group morphism  $\varphi: G \rightarrow G'$  so that  $\varphi(H) \subseteq H'$ . Prove that there is a unique  $\varphi$ -equivariant map  $\Phi: X \rightarrow X'$  so that  $\Phi(x_0) = x'_0$ , and that  $\Phi$  is smooth. Conversely, given any continuous  $\varphi$ -equivariant map  $\Phi: X \rightarrow X'$  of homogeneous spaces  $(X, G)$  and  $(X', G')$ , if  $x'_0 := \Phi(x_0)$ , prove that  $\varphi(G^{x_0}) \subset G'^{x'_0}$ , and so  $\Phi$  is smooth. If  $\varphi$  is a submersion, conclude that all  $\varphi$ -equivariant maps of homogeneous spaces are submersions.

The Hopf fibration from problem 3.4 on page 24 is equivariant and therefore a smooth submersion.

**15.3** Prove that for any Lie group action, the stabilizer subgroup  $G^m$  of any point  $m \in M$  has Lie algebra  $\mathfrak{g}^m$  consisting of the  $A \in \mathfrak{g}$  so that  $\tilde{A}(m) = 0$ .

**Theorem 15.2.** Suppose that a Lie group  $G$  acts on a manifold  $M$ . For each  $m \in M$ , the map  $gG^m \in G/G^m \mapsto gm \in Gm \subset M$  is a diffeomorphism of  $G/G^m$  to the orbit  $Gm$ , and an immersion to  $M$ , so every orbit is a immersed submanifold and a homogeneous space.

*Proof.* Clearly  $G^m \subset G$  is a closed subgroup. The smooth map  $g \in G \mapsto gm \in M$  descends to a smooth map  $\varphi: G/G^m \mapsto Gm \subset M$ . Check that this map is a bijection to the orbit. Differentiating:  $\varphi'(g)\tilde{A} = 0$  just when

$$\begin{aligned} 0 &= \varphi'(g)\tilde{A}, \\ &= \left. \frac{d}{dt} \right|_{t=0} \varphi(e^{tA}g), \\ &= \left. \frac{d}{dt} \right|_{t=0} e^{tA}gm. \end{aligned}$$

This vanishes if and only if

$$0 = \left. \frac{d}{dt} \right|_{t=0} g^{-1}e^{tA}gm,$$

i.e. if and only if  $\mathrm{Ad}_{g^{-1}}A \in \mathfrak{g}^m$ , i.e.  $A \in \mathrm{Ad}_g \mathfrak{g}^m$ . Hence the kernel of  $\varphi$  on  $G$  has constant rank, with kernel  $\varphi'(g) = \mathrm{Ad}_g \mathfrak{g}^m$ . Thus on  $G/G^m$ ,  $\varphi$  is an immersion at  $1G^m$ , and so an immersion everywhere.  $\square$

Rotations act transitively on the unit sphere, i.e.  $\mathrm{SO}(n+1)$  acts transitively on  $S^n$ , with stabilizer  $\mathrm{SO}(n)$ , the rotations of the perpendicular plane to a given unit vector. So  $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ . By induction,  $\mathrm{SO}(n)$  is connected for all  $n$ .

**15.4** Use the same trick to find the numbers of components of various Lie groups.

For the reader familiar with tensors in linear algebra. Rotations act transitively on the unit sphere, i.e.  $\mathrm{SO}(n+1)$  acts transitively on  $S^n$ , with stabilizer  $\mathrm{SO}(n)$ , the rotations of the perpendicular plane to a given unit vector. Let  $V := \mathbb{R}^{n+1}$ ; in an orthonormal basis  $e_0, \dots, e_n$ ,  $\mathrm{SO}(n)$  stabilizes  $e_0$ . So  $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ . Let  $W := S^2V$  the space of symmetric tensors on  $V$ , i.e. sums of tensors  $v \otimes v$  for  $v \in V$ . Let  $N := \dim W = n(n+1)/2$ . Clearly  $\mathrm{SO}(n+1)$  also acts as linear transformations of the tensors. The stabilizer in  $\mathrm{SO}(n+1)$  of  $e_0 \otimes e_0$  is a group  $H \subset \mathrm{SO}(n+1)$  which includes in it the linear maps which change signs of any even number of  $e_j$ . The map  $\varphi(x) = x \otimes x$  is  $\mathrm{SO}(n+1)$ -equivariant, therefore a morphism of homogeneous spaces

$$\varphi: S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n) \rightarrow \mathrm{SO}(n+1)/H.$$

Since  $\varphi(-x) = \varphi(x)$ ,  $\varphi$  descends from  $S^n$  to  $\mathbb{RP}^n = \mathrm{SO}(n+1)/H$ , giving an embedding as a submanifold of the unit sphere  $S^{N-1} \subset W$ :

$$\varphi: \mathbb{RP}^n \rightarrow S^{N-1}.$$

So the real projective spaces, while defined abstractly, embed quite naturally in spheres of some high dimension.

## Normal subgroups and quotient groups

**Theorem 15.3.** *If  $N \subseteq G$  is a closed normal subgroup of a Lie group, then the quotient group  $Q := G/N$  has a unique Lie group structure so that the quotient morphism  $g \in G \mapsto gN \in Q$  is smooth. Conversely, every Lie group morphism has closed normal kernel, and in particular if  $\varphi: G \rightarrow Q$  is a surjective Lie group morphism, with kernel  $N \subseteq G$ , then the map  $gN \in G/N \mapsto \varphi(gN) \in Q$  is a Lie group isomorphism.*

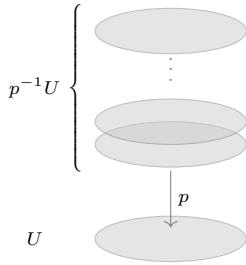
*Proof.* Since  $N \subseteq G$  is closed, the homogeneous space  $Q := G/N$  is a smooth manifold, and the map  $g \in G \mapsto gN \in Q$  is a smooth submersion. Because it is a submersion, for any point  $q_0 \in Q$ , we can make a smooth map  $s: U_Q \rightarrow G$  of an open set  $U_Q \subseteq Q$  around  $q_0$  so that  $s(q)N = q$  for  $q \in U_Q$ . Make such maps  $s$  around any two points  $q_0, q_1 \in Q$ . Then for  $x$  near  $q_0$  and  $y$  near  $q_1$  in  $Q$ ,  $xy = (s(x)N)(s(y)N) = s(x)s(y)N$  is smooth in  $x, y$ . So the quotient group structure on  $Q$  is smooth.

Suppose we have a Lie group submersion  $\varphi: G \rightarrow Q$ . The map is constant on translates  $gN$ , i.e. on fibers of the smooth submersion  $G \rightarrow G/N$ , so drops to a smooth map  $G/N \rightarrow Q$ , a smooth bijection. We can replace  $G$  by  $G/N$  so assume that  $N = \{1\}$ , so  $Q$  is a homogeneous space of  $G$  with stabilizer 1. By theorem 15.1 on page 135, the map  $G \rightarrow Q$  is a diffeomorphism.  $\square$

The *centre* of  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$  is the set of elements which commute with all other elements:

$$Z_{\mathbb{R}} = \mathbb{R}, Z_{\mathbb{C}} = \mathbb{C}, Z_{\mathbb{H}} = \mathbb{R}.$$

Write  $Z_k^\times$  for the set of nonzero elements of  $Z_k$ , and identify elements of  $k$  with multiples of the identity matrix.



The *projective linear group*

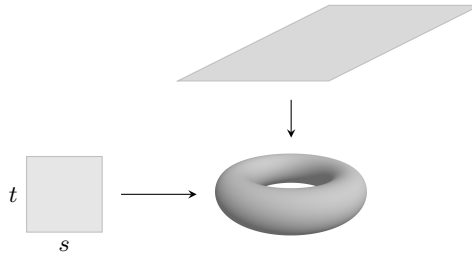
$$\mathbb{P}\mathrm{SL}(n, k) = \mathrm{GL}(n, k) / Z_k^\times I = \mathrm{SL}(n, k) / \{ z \in Z_k \mid z^n = 1 \} I$$

acts on projective space  $\mathbb{P}^{n-1}$  and on the various Grassmannians  $\mathrm{Gr}_p k^n$ .

### Covering maps and fundamental groups

**Theorem 15.4.** *Every connected Lie group has abelian fundamental group.*

*Proof.* Draw paths  $x(s), y(t) \in G$ . Define the map  $(s, t) \mapsto x(s)y(t)$ .



The bottom and right of the square map to  $x(s)$  followed by  $y(t)$ , while the left and top map to  $y(t)$  followed by  $x(s)$ . So the square is a homotopy. In particular, if  $x(s)$  and  $y(t)$  are loops, we see that  $\pi_1(G)$  is abelian.  $\square$

**Theorem 15.5.** *If a Lie group  $G$  is connected, then its universal covering space  $\tilde{G}$  has a unique Lie group structure for which the universal covering map  $\tilde{G} \rightarrow G$  is a smooth group morphism and a local diffeomorphism. The fundamental group  $\pi_1(G)$  is embedding into  $\tilde{G}$  as the kernel of  $\tilde{G} \rightarrow G$  and is a discrete central subgroup.*

*Proof.* Denote the universal covering space by  $\tilde{G} \rightarrow G$ , with some point  $1 \in \tilde{G}$  mapping to  $1 \in G$ . We don't assume that  $\tilde{G}$  is a Lie group, just a manifold, as every connected covering space of a manifold is a manifold. Paths in  $G$  from 1 lift to paths in  $\tilde{G}$  from 1. Take any paths  $\tilde{x}(s), \tilde{y}(t)$  in  $\tilde{G}$  starting at 1, defined for some times  $0 \leq s \leq 1$  and  $0 \leq t \leq 1$ . Take their images in  $G$ , paths  $x(s), y(t)$ . By the covering property, the map  $(s, t) \mapsto x(s)y(t)$  lifts to a unique continuous map  $\varphi: [0, 1]^2 \rightarrow \tilde{G}$  so that  $\varphi(s, 0) = \tilde{x}(s)$  and  $\varphi(0, t) = \tilde{y}(t)$  and  $\varphi(s, t)$  maps to  $x(s)y(t)$ . Deform  $\tilde{x}(s)$  and  $\tilde{y}(t)$  through homotopies without changing their endpoints:  $x(s)$  and  $y(t)$  deform by homotopies without changing their endpoints. The map  $(s, t) \mapsto x(s)y(t)$  deforms through a homotopy without changing endpoints at every corner of the square, and so its lift  $\varphi$  deforms through a homotopy with all corners unchanged. So the location of the corner  $\varphi(1, 1)$  depends only on the homotopy classes of the paths  $\tilde{x}(s)$  and  $\tilde{y}(t)$  with fixed endpoints. Since  $\tilde{G}$  is simply connected, those homotopy classes depend only on end points. So we have defined an operation on  $\tilde{G}$ , which we will call multiplication, taking  $\tilde{x}(1), \tilde{y}(1) \mapsto \varphi(1, 1)$ . The covering map  $\tilde{G} \rightarrow G$  locally identifies the multiplication operations, so that the multiplication on  $\tilde{G}$  is smooth. A similar story gives an inverse operation, making  $\tilde{G}$  a group.



The kernel  $\Gamma$  of  $\tilde{G} \rightarrow G$  is the preimage of  $1 \in G$ . Since  $\tilde{G} \rightarrow G$  is a covering map,  $\Gamma$  is a discrete subgroup of  $\tilde{G}$ , acting transitively on the sheets of  $\tilde{G} \rightarrow G$ . The action of  $\Gamma$  on  $\tilde{G}$  is a covering action, so  $\Gamma$  is identified with  $\pi_1(G)$ .

Take a path  $\tilde{x}(t)$  from  $\tilde{x}(0) = 1$  to an arbitrary element  $g \in \tilde{G}$ , projecting to a path  $x(t) \in G$ . Take an element  $h \in \Gamma$ . Consider the path  $t \mapsto h\tilde{x}(t)h^{-1}$ . Since  $h$  lies in the kernel of  $\tilde{G} \rightarrow G$ , this path maps also maps to  $x(t)$ . At  $t = 0$ ,  $h\tilde{x}(0)h^{-1} = h1h^{-1} = 1$ . So both paths  $\tilde{x}(t)$ ,  $h\tilde{x}(t)h^{-1}$  start at 1 and both lift  $x(t)$ . So they are equal. Therefore at time  $t = 1$ ,  $g = hgh^{-1}$ . Since  $g \in \tilde{G}$  is arbitrary,  $\Gamma \subset \tilde{G}$  lies in the center of  $\tilde{G}$ .  $\square$

**Corollary 15.6.** *Suppose that  $\mathfrak{g}$  is the Lie algebra of some Lie group. An action of  $\mathfrak{g}$  arises from an action of some connected Lie group with Lie algebra  $\mathfrak{g}$  if and only if the  $\mathfrak{g}$ -action takes all elements of  $\mathfrak{g}$  to complete vector fields.*

**Theorem 15.7.** *If  $G$  is a Lie group and  $\Gamma \subset G$  is any discrete central subgroup, then  $G/\Gamma$  has a unique structure of a Lie group for which  $G \rightarrow G/\Gamma$  is a Lie group morphism and a covering map.*

*Proof.* Clearly  $\Gamma$  acts freely on  $G$  by  $\gamma \in \Gamma, g \in G \mapsto g\gamma^{-1}$ . Note that this peculiar action is the action for which the quotient is the quotient group  $G/\Gamma$ . Take any two points  $g_0, g_1 \in G$  from different orbits. Applying the diffeomorphism  $g \in G \mapsto gg_0 \in G$  to  $\Gamma$ , we see that the orbits of  $\Gamma$  are discrete. Hence there is an open set  $g_0U$  around  $g_0$  containing no point of  $\Gamma g_1$ . Take an open set  $V \subset G$  around 1 so that  $VV^{-1} \subset U$ . If  $\Gamma g_0V$  intersects  $\Gamma g_1V$ , say at some point  $\gamma_0 g_0 v_0 = \gamma_1 g_1 v_1$ , then

$$g_0 v_0 v_1^{-1} = \gamma_0^{-1} \gamma_1 g_1,$$

i.e.  $g_0U$  intersects  $\Gamma g_1$ , a contradiction. Hence the action of  $\Gamma$  is free and proper. By lemma 3.4 on page 29,  $G/\Gamma$  is a manifold and  $G \rightarrow G/\Gamma$  is a local diffeomorphism.  $\square$

So we have an isomorphism: every connected Lie group  $G$  has universal covering group  $\tilde{G}$  a simply connected Lie group, and the set of Lie groups with given universal covering group  $\tilde{G}$  is isomorphic to the set of discrete central subgroups of  $\tilde{G}$ .

In chapter 8, we saw that  $\mathrm{SU}(2) = \mathrm{Sp}(1) = S^3$  is the unit sphere in  $\mathbb{R}^4 = \mathbb{H}$ , so is simply connected.

We also saw a Lie group morphism  $\mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$  with kernel  $\pm 1$ , so  $\mathrm{SO}(3) = \mathbb{RP}^3$  and so  $\pi_1(\mathrm{SO}(3)) = \{\pm 1\}$ .

We also saw a Lie group morphism  $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \rightarrow \mathrm{SO}(4)$  with kernel  $\pm 1$ , so  $\mathrm{SO}(4) = (S^3 \times S^3)/\pm 1$  and so  $\pi_1(\mathrm{SO}(4)) = \{\pm 1\}$ .

**15.5** Find all discrete central subgroups of  $\tilde{G} = \mathrm{SU}(2)$ , and find all of the quotient groups  $G$ .

**15.6** Prove that any discrete normal subgroup of a connected Lie group is central.

To classify connected Lie groups:

- a. classify simply connected Lie groups  $\tilde{G}$ ,
- b. find all discrete central subgroups of  $\tilde{G}$ .

The first step is impossible, but the second is usually easy: the center is a connected abelian Lie group, and every connected abelian Lie group is a product of a vector space and finitely many circles.

Two connected Lie groups are almost the same if they have the same universal covering group.

**15.7** Prove that in any connected Lie group, any neighborhood of the identity generates the group.

## Abelian Lie groups

This section can be omitted without loss of continuity.

**Theorem 15.8.** *Suppose that  $G$  is a Lie group with abelian Lie algebra and abelian group of components  $G/G^0$ . Then  $G$  is an abelian Lie group and a product  $F \times \mathbb{R}^k \times T^\ell$  where  $F$  is an abelian group with the discrete topology,  $\mathbb{R}^k$  is a finite dimensional vector space, and  $T^\ell$  is a torus, i.e. the direct product of  $\ell$  circles.*

*Proof.* Use additive notation in  $G$ . Let  $F := G/G^0$  be the group of components of  $G$ , i.e. the quotient by the path component  $G^0$  of the identity element of  $G$ . Recall that in our definition of manifold, every manifold has a countable basis of open sets, and so countably many components, so we can write down every element of  $F$  in a sequence. Going through that sequence, let  $f_1$  be the first nonidentity element, and then let  $f_2$  be the first element not a multiple of  $f_1$ , and so on, so that  $f_{k+1}$  is the first element not in the subgroup generated by  $f_1, \dots, f_k$ . For each element  $f_i$ , pick some element  $g_i \in G$  mapping to  $f_i \in F$ . Take any word  $w$  in the  $g_i$  which is satisfied by the  $f_i$ , i.e. any sum

$$m_1 g_1 + m_2 g_2 + \dots + m_n g_n$$

with integer coefficients  $m_1, m_2, \dots, m_n$ , not all zero, so that

$$m_1 f_1 + m_2 f_2 + \dots + m_n f_n = 0.$$

Since  $F = G/G^0$ , the elements of  $G$  mapping to zero in  $F$  are precisely the elements of  $G^0$ , i.e.

$$m_1 g_1 + m_2 g_2 + \dots + m_n g_n \in G^0.$$

We can assume, by perhaps reordering the  $g_i$ , that  $m_n \neq 0$ . Suppose by induction that there are no such words in  $g_1, g_2, \dots, g_{n-1}$ . Taking all such words, we can replace them by linear combinations of them until we arrange that  $m_n$  is the greatest common divisor of all of the  $g_n$  coefficients from all of the words. In particular, there is a unique such word with  $m_n > 0$  from which all other such words are nonzero integer multiples. Recall that by definition  $G^0$  is path connected. Vary the choice of  $g_n$  in its path component, say replacing  $g_n$  by  $g_n + h$  for any  $h \in G^0$ , and thereby vary the word by an amount  $m_n h$ . The map  $h \in G^0 \mapsto m_n h \in G^0$  is surjective, since  $G^0$  is a product of a vector space and a torus. So we can vary  $g_n$  by replacing it by  $g_n + h$ , continuously varying  $h$ , until the word becomes zero in  $G^0$ . So we can choose  $g_1, g_2, \dots, g_n$  to have any word satisfied by  $f_1, f_2, \dots, f_n$  also satisfied by the  $g_i$ . Do this by induction for  $n = 1, 2, \dots$ , using the same  $g_1, g_2, \dots, g_{n-1}$  at each step. Lift  $f_i \mapsto g_i$ , well defined

because by definition all of these  $f_i$  are distinct. Any elements of  $F$  are expressible as words in these  $f_i$ , and it is a word to state that two elements have some third element as their product, so this map extends uniquely to a morphism of abelian groups  $F \subset G$ , splitting  $G = F \times G^0$ .  $\square$

## Topology determines smooth structure

This section can be omitted without loss of continuity.

**Theorem 15.9.** *Every continuous bijective group morphism of Lie groups is an isomorphism of Lie groups.*

*Proof.* Take a continuous bijective group morphism  $\varphi: G \rightarrow H$ . Take some open set  $U \subset G$ . We need to prove that  $\varphi(U) \subset H$  is open. Pick any  $h_0 \in \varphi(U)$ . We need to prove that  $h_0$  lies in the interior of  $\varphi(U)$ . Pick some  $g_0 \in U$  so that  $h_0 = \varphi(g_0)$ . Replace  $U$  by  $g_0^{-1}U$ ; so we can assume that  $h_0 = 1 \in U$ , and we only need to prove that 1 lies in the interior of  $\varphi(U)$ . Take some compact set  $K \subset U$  with 1 in the interior of  $K$  and with  $KK^{-1} \subset U$ . Take a dense sequence  $g_1, g_2, \dots \in G$ . For any point  $g \in G$ , infinitely many points  $g_j$  of the sequence lie in the nonempty interior of  $gK^{-1}$ , i.e.  $g \in g_jK$ . So  $G = g_1K \cup g_2K \cup \dots$ . Let  $h_j := \varphi(g_j)$ . Since  $\varphi$  is onto,  $H = h_1\varphi(K) \cup h_2\varphi(K) \cup \dots$ . By the Baire category theorem, some  $h_j\varphi(K)$  has nonempty interior, and so  $\varphi(K)$  does too. Pick  $g \in G$  with  $\varphi(g)$  in the interior of  $K$ . Then  $1 = \varphi(g)\varphi(g)^{-1}$  lies in the interior of  $\varphi(KK^{-1}) \subset \varphi(U)$ . So  $\varphi^{-1}$  is continuous, and the result follows from theorem 11.7 on page 107.  $\square$

A *topological homogeneous space*  $(X, G)$  is a topological space  $X$  and a topological group  $G$  acting on  $X$  by a continuous and transitive action  $G \times X \rightarrow X$ .

**15.8** Prove: a topological homogeneous space  $(X, G)$  arises from a smooth homogeneous space, for some smooth structures on  $X$  and  $G$ , if and only if  $G$  has a smooth structure as a Lie group, in which case the smooth structures on both  $X$  and  $G$  are uniquely determined.



## Chapter 16

### Tangent and cotangent spaces and bundles

#### Cotangent spaces

A *covector* or *cotangent vector*  $\xi$  at a point  $m_0 \in M$  of a manifold is a linear map  $\xi: T_{m_0}M \rightarrow \mathbb{R}$ . The *cotangent space*  $T_{m_0}^*M$  of a manifold  $M$  at a point  $m_0 \in M$  is the dual space of the tangent space  $T_{m_0}M$ , i.e. the set of covectors. Take a chart. Denote by  $dx^i$  the linear function  $dx^i(v) = v^i$  on vectors

$$v = v^i \frac{\partial}{\partial x^i}.$$

These linear functions form a basis for  $T_{m_0}^*M$ : the dual basis to the  $\frac{\partial}{\partial x^i}$ . In any chart, each element of  $T_{m_0}^*M$  is uniquely  $a_i dx^i$ . The coefficients  $a_i$  are constant numbers, because the cotangent vector is defined only at a single point  $m_0 \in M$ .

The cotangent vectors  $dx, dy$  span the cotangent spaces of  $M := \mathbb{R}^2$  at every point, so each cotangent space is a 2-dimensional vector space consisting of cotangent vectors like  $dx$ ,  $2dx - dy$ ,  $7dx + 4dy$ .

Suppose that  $f$  is a smooth real valued function defined near a point  $m_0$  of a manifold  $M$ . The map  $v \in T_{m_0}M \mapsto vf \in \mathbb{R}$  is a linear map on tangent vectors  $v \in T_{m_0}M$ , so is an element of  $T_{m_0}^*M$ , denoted  $df$ , the *differential* of  $f$ . In any coordinates  $x^1, \dots, x^n$ , if

$$v = v^i \frac{\partial}{\partial x^i},$$

then

$$df(v) = vf = v^i \frac{\partial f}{\partial x^i} = \frac{\partial f}{\partial x^i} dx^i(v),$$

so

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

The function  $f(x, y) = x^2y + y^3$  has differential  $df = 2xy dx + (x^2 + 3y^2) dy$  at each point  $(x, y)$ . The differential at  $(x, y) = (4, -5)$  is

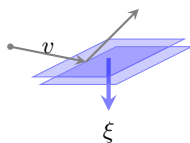
$$df|_{(x,y)=(4,-5)} = 2(4)(-5) dx + ((4^2 + 3(-5)^2) dy = -40 dx + 91 dy,$$

having constant coefficients since we made a particular choice of point  $(x, y) = (4, -5)$ . Note that we cannot simplify any further:  $dx$  and  $dy$  are linearly independent linear functions on this tangent space  $T_{(4,-5)}\mathbb{R}^2$ .

We usually write down submanifolds by writing down equations they satisfy, expressed in functions. It is easy to differentiate those functions, but quite a bit of

linear algebra to write out, for example, a basis for a tangent space. So we prefer to work directly with the equations cutting out a manifold and the differentials of the functions in those equations: we prefer cotangent vectors to tangent vectors.

Geometrically, a tangent vector is the velocity of a curve. Picture a metal plate, allowed to slide only in the direction perpendicular to the plate. As a particle hits the plate with a velocity  $v$ , it moves the plate. The velocity of the plate is a linear function of  $v$ , say  $\xi(v)$ .



Intuitively we can think of a cotangent vector as a momentum vector, given by some such measuring apparatus, measuring the tendency of a velocity vector to point in a particular direction. (This is a little misleading, as the motion of the plate clearly depends on the mass of the particle, but it can be a helpful intuition.)

If  $\varphi: P \rightarrow Q$  is a continuously differentiable map, and if  $q_0 = \varphi(p_0)$ , we can define a linear map  $\varphi^*: T_{q_0}^*Q \rightarrow T_{p_0}^*P$  by  $(\varphi^*\eta)(v) = \eta(\varphi'(p_0)v)$  for any vector  $v \in T_{p_0}P$ . In charts, if we write  $\eta = a_i dy^i$  and  $\varphi$  is  $y = \varphi(x)$  then

$$\varphi^*\eta = a_i \frac{\partial y^i}{\partial x^j} dx_j.$$

Let  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the map  $\varphi(x, y) = (u, v) = (x^2 - y^2, 2xy)$ . So  $\varphi^*du = d(x^2 - y^2) = 2x dx - 2y dy$ , and  $\varphi^*dv = 2y dx + 2x dy$ . For example, at any point  $(x, y) = (x_0, y_0)$ , if we let  $(u_0, v_0) = \varphi(x_0, y_0)$ , we have

$$\varphi^*|_{(x_0, y_0)}: T_{(u_0, v_0)}^*\mathbb{R}^2 \rightarrow T_{(x_0, y_0)}^*\mathbb{R}^2,$$

given by

$$\begin{aligned}\varphi^*|_{(x_0, y_0)} du &= 2x_0 dx - 2y_0 dy, \\ \varphi^*|_{(x_0, y_0)} dv &= 2y_0 dx + 2x_0 dy.\end{aligned}$$

Note that we don't write  $dx_0$  here: first, because  $x_0$  is a constant chosen numerical value for the variable  $x$ , so  $dx_0 = 0$ . Second, because  $\varphi^*$  is mapping to the 2-dimensional vector space  $T_{(x_0, y_0)}^*\mathbb{R}^2$  spanned by the basis  $dx, dy$ . Each element of  $T_{(x_0, y_0)}^*\mathbb{R}^2$  is a unique constant coefficient linear combination of  $dx, dy$ . In particular, at  $(x_0, y_0) = (0, 0)$ , we find  $(u_0, v_0) = (0, 0)$ , and  $\varphi^*|_{(x_0, y_0)} = 0$ . On the other hand, at  $(x_0, y_0) = (3, 4)$ , we find

$$(u_0, v_0) = (3^2 - 4^2, 2 \cdot 3 \cdot 4) = (-7, 24),$$

and

$$\begin{aligned}\varphi^*|_{(x_0, y_0)} du &= 6 dx - 8 dy, \\ \varphi^*|_{(x_0, y_0)} dv &= 8 dx + 6 dy.\end{aligned}$$

An almost trivial example:  $\varphi: P = \mathbb{R} \rightarrow Q = \mathbb{R}$  is just  $y = \varphi(x) = 2x$ . Then  $\varphi$  is a diffeomorphism with  $\varphi'(x) = 2$  constant, so as a linear map

$$\varphi'(x_0): T_{x_0}P \rightarrow T_{y_0}Q$$

is just

$$\frac{\partial}{\partial x} \mapsto \varphi'(x_0) \frac{\partial}{\partial x} = 2 \frac{\partial}{\partial y},$$

multiplication by 2. So then

$$\varphi^* dy = \frac{dy}{dx} dx = 2 dx$$

is also multiplication by 2. If we think of  $\varphi$  not as a map, but as a change of variables, i.e. a transition map between charts, then we can write this as  $dy = 2 dx$ , or as

$$dx = \frac{1}{2} dy.$$

(We could also think of  $\varphi^{-1*}$  as identifying  $dx$  with  $(1/2)dy$ .) So when we change variables from  $x$  to  $y$  by  $y = 2x$ , we change tangent vectors by doubling, but cotangent vectors by dividing in half. So they really are very different objects.

Why have we managed to get so far into several variable calculus without hitting cotangent vectors yet? Because under rotations and translations, they transform in the same way, and these are the natural symmetries of Euclidean geometry. So in Euclidean space, tangent and cotangent vectors look the same: just vectors. But when we allow nonlinear changes of variable, or even just rescalings of variables, they look very different.

## Hyperplanes

This section can be omitted without loss of continuity.

A *hyperplane* in a vector space  $V$  is a linear subspace  $W \subset V$  so that  $V/W$  is 1-dimensional. Every line  $\ell$  through the origin in  $V^*$  is perpendicular to a hyperplane  $W = \ell^\perp$  and conversely. To write down a hyperplane  $W$ , it is more convenient to write down a nonzero element of the perpendicular line  $\ell \subset V^*$ . Recall that the set of lines in a vector space is the *projective space* on that vector space. So the set of hyperplanes in  $V$  is identified with the projective space on  $V^*$ . A *cotangent line*  $\ell \subset T_x^*M$  is a line in  $T_x^*M$ .

At any point of the plane, the multiples of  $3dx + 4dy$  form a cotangent line, and the associated hyperplane is the line of tangent vectors

$$a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$$

so that  $3a + 4b = 0$ .

At a point  $(x, y, z) \in \mathbb{R}^3$ , the multiples of  $3dx + 4dy - 2dz$  form a cotangent line. The associated hyperplane is the collection of tangent vectors

$$v = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}$$

so that  $3a + 4b - 2c = 0$ . In other words, this hyperplane is the plane spanned by  $(2, 0, 3)$ ,  $(0, 1, 2)$ , which we can write also as

$$2 \frac{\partial}{\partial x} + 3 \frac{\partial}{\partial z}, \frac{\partial}{\partial y} + 2 \frac{\partial}{\partial z}.$$

Since the cotangent line has dimension 1, while the hyperplane has dimension 2, it is more convenient to write down the cotangent line, or just the equation  $0 = 3dx + 4dy - 2dz$ .

The *projectivised cotangent space* at a point  $x \in M$  in a manifold  $M$ , denoted  $\mathbb{P}T_x^*M$ , is the set of all choices of cotangent line  $\ell$  at a point  $x \in M$ .

## The tangent bundle

This section can be omitted without loss of continuity.

Suppose that  $M$  is manifold. Let  $TM$  be the set of all pairs  $(m, v)$  where  $m \in M$  and  $v \in T_x M$  and call  $TM$  the *tangent bundle*. We usually picture elements of  $TM$  as tangent vectors  $v$ , and informally say  $v \in TM$ , but it is useful to keep track of the point  $m$  where the tangent vector lives as well, so our formal and precise definition makes use of pairs  $(m, v)$ . Let  $\pi: (m, v) \in TM \rightarrow m \in M$ . So for any open set  $U \subset M$ , the set  $\pi^{-1}U$  is the set of pairs  $(m, v)$  with  $m \in U$  and  $v \in T_m M$ . This  $v$  is itself an equivalence class of tangent vector representative, say the equivalence class of  $(\varphi, w)$  where  $(U, \varphi)$  can be any chart and  $w \in \mathbb{R}^n$ .

Take a chart  $(U, \varphi)$  on  $M$ . Define a chart  $(\pi^{-1}U, \Phi)$  on  $TM$  by

$$\Phi(m, v) = (\varphi(m), w),$$

if  $v$  is the equivalence class of  $(\varphi, w)$ .

**16.1** If we have two charts  $\varphi_0, \varphi_1$  for  $M$ , and we let  $t = \varphi_1^{-1} \circ \varphi_0$ , find the transition map between the associated charts  $\Phi_0, \Phi_1$  for  $TM$ . Use this to prove that, for any  $C^k$  chart on  $M$ , the associated charts constitute a  $C^{k-1}$ -structure on  $TM$ .

So  $TM$  is a manifold of dimension  $\dim TM = 2 \dim M$ , and the *projection map*  $\pi: TM \rightarrow M$  is  $C^{k-1}$ .

**16.2** Suppose that  $M$  is the unit sphere in  $\mathbb{R}^n$ . Find a bijection between the points of  $TM$  and the lines (not necessarily through the origin) of  $\mathbb{R}^n$ .

**16.3** Prove that each continuously differentiable vector field  $X$  on a manifold  $M$  gives a continuously differentiable map  $x \in M \mapsto (x, X(x)) \in TM$ . Prove that any continuously differentiable map  $s: M \rightarrow TM$  arises in this way from a vector field if and only if  $\pi \circ s = \text{id}$  is the identity map.



## The cotangent bundle

This section can be omitted without loss of continuity.

The *cotangent bundle*  $T^*M$  is the set of all pairs  $(x, \xi)$  where  $x \in M$  and  $\xi \in T_x^*M$  is a cotangent vector.

**16.4** Suppose that  $\varphi$  is a chart on a manifold  $M$ , associating to each point  $m \in U_M$  in some open set  $U_M \subset M$  some point  $x = f(m) \in U_{\mathbb{R}^n}$  where  $U_{\mathbb{R}^n} \subset \mathbb{R}^n$  is open. Each cotangent vector  $\alpha$  on  $U_{\mathbb{R}^n}$  pulls back to a cotangent vector on  $M$ :  $\varphi^*\alpha$  is the cotangent vector which eats a tangent vector  $v$  to give the number  $\alpha(\varphi_*v)$ . Prove that  $\varphi^*dx^1, \dots, \varphi^*dx^n$  is a basis of the cotangent space  $T_m^*M$ . Hence every cotangent vector at  $m$  has the form  $a_i\varphi^*dx^i$ . Prove that the map  $a_i\varphi^*dx^i \mapsto (\varphi(m), a_1, \dots, a_n)$  is a chart on  $T^*M$ . Use such charts to prove that  $T^*M$  is a manifold of dimension  $\dim T^*M = 2 \dim M$ . What are the transition maps? Prove that the *projection map*  $\pi: (x, \xi) \in T^*M \mapsto x \in M$  is  $C^{k-1}$ . Do the same for the tangent bundle.

## The hyperplane bundle

This section can be omitted without loss of continuity.

The *projectivised cotangent bundle* of a manifold  $M$ , denoted  $\mathbb{P}T^*M$ , is the set of all choices of point  $x \in M$  and of cotangent line  $\ell$  at  $x$ . The projectivised cotangent bundle  $\mathbb{P}T^*M$  is also called the *bundle of tangent hyperplanes*; we can draw  $(x, \ell)$  as a point  $x$  and a hyperplane  $\ell^\perp \subset T_xM$ . A choice of hyperplane  $H_x \subset T_xM$  at every  $x \in M$  is a *hyperplane field*; it is  $C^k$  if, near each point of  $M$ , there is some  $C^k$  1-form  $\xi$  so that  $\xi_x^\perp = H_x$ .

**16.5** Explain why  $\mathbb{P}T^*M$  is a manifold so that the map  $\mathbb{P}T^*M \rightarrow M$  is  $C^{k-1}$ , if  $M$  is  $C^k$ , and the fibers of this map are submanifolds.

**16.6** The spheres centered at the origin of Euclidean space  $\mathbb{R}^n$  have tangent spaces which are hyperplanes, giving a hyperplane field on  $\mathbb{R}^n - \{0\}$ . Write down a choice of 1-form  $\xi$  on  $\mathbb{R}^n - \{0\}$  so that this hyperplane field is  $\xi^\perp$ .



## Chapter 17

### Transverse submanifolds

This chapter can be omitted without loss of continuity. We prove that submanifolds whose tangent spaces are suitably skew at one point intersect smoothly near that point.

#### The implicit function theorem

**Theorem 17.1.** *If some  $C^k$  functions  $f_1, \dots, f_p$  on a manifold  $M$  have linearly independent differentials at every point where they all vanish, then the set of points where they vanish is a codimension  $p$  embedded  $C^k$  submanifold.*

*Proof.* In any coordinates,

$$df_i = \frac{\partial f_i}{\partial x^j} dx^j,$$

has coefficients a row of the matrix

$$\left( \frac{\partial f_i}{\partial x^j} \right).$$

Linear independence is precisely linear independence of those rows; apply the usual implicit function theorem.  $\square$

#### Transverse submanifolds

These two curves  intersect in a point. If we slightly perturb them,  they still intersect in a nearby point, and the picture remains diffeomorphic.

Curves in the plane that have a tangent point  might be pulled apart by small perturbation:  or pushed together to have two points of intersection:  intersection:

We want to improve the implicit function theorem slightly, to understand submanifold intersections. The *codimension* of a submanifold  $S$  of a manifold  $M$  is  $\dim M - \dim S$ . The implicit function theorem tells us that  $C^k$  functions  $f_1, f_2, \dots, f_p$

on a manifold  $M$ , with linearly independent differentials  $df_1, df_2, \dots, df_p$ , have common level set

$$S := \{m \in M \mid 0 = f_1(m) = f_2(m) = \dots = f_p(m)\}$$

a submanifold of  $M$  of codimension  $p$ . A *conormal* of a submanifold  $S \subset M$  at a point  $m \in S$  is a cotangent vector in  $T_m^*M$  vanishing on the tangent space  $T_m S$ . The conormal space  $S_{s_0}^\perp \subset T_m^*M$  is the set of conormals, i.e. the differentials of functions  $f$  constant on  $S$  and defined on a neighborhood of  $m$  in  $M$ . Two submanifolds  $P, Q \subset M$  are *transverse* if they have no common conormals: at any point  $m \in P \cap Q$ ,  $P_m^\perp \cap Q_m^\perp = \{0\}$ .

**Theorem 17.2.** *Any two transverse embedded  $C^k$  submanifolds intersect in an embedded  $C^k$  submanifold. Near any point of the intersection, there is a chart in which the submanifolds are transverse linear subspaces. The codimension of the intersection is the sum of the codimensions of the submanifolds.*

*Proof.* We can write  $P$  as locally the zero set of some functions  $f_i$ , with linearly independent differentials, and  $Q$  as locally the zero set of some functions  $g_j$ , also with linearly independent differentials. Then  $P_{m_0}^\perp$  is the span of

$$df_1(m_0), \dots, df_p(m_0)$$

while  $Q_{m_0}^\perp$  is the span of

$$dg_1(m_0), \dots, dg_q(m_0).$$

Being transverse means precisely that all  $df_i, dg_j$  put together are linearly independent i.e.  $P$  and  $Q$  are cut out by constraints with linearly independent differentials. Hence, putting the constraints together gives a submanifold.  $\square$

More generally, any collection of submanifolds are *transverse* if, at every point where they all intersect, the conormal spaces of any two of them intersect only at the zero cotangent vector. Clearly their intersection is then also a submanifold.

**17.1** Given two vector subspaces  $V, W \subset U$  of a vector space  $U$ , denote by  $V + W$  the set of vectors of the form  $v + w$  for  $v \in V$  and  $w \in W$ . Note that  $V + W$  is not necessarily a direct sum, so if  $V + W = U$  then  $\dim V + \dim W \geq \dim U$ ; for example  $U + U = U$ . Prove that two embedded submanifolds  $P, Q \subset M$  are transverse at a point  $m \in P \cap Q$  just when  $T_m P + T_m Q = T_m M$  (not necessarily a direct sum).

If  $P$  doesn't intersect  $Q$ , then  $P$  and  $Q$  are transverse.

Two circles in the plane are transverse as long as they are not tangent anywhere: they can be either disjoint, or overlap at two points.



In  $\mathbb{R}^3$ , the graph of any smooth function  $z = f(x, y)$  is transverse to any vertical line.

Two curves in  $\mathbb{R}^3$  are transverse just if they don't intersect at all.

An *exact sequence* is a sequence of linear maps of vector spaces in which image of each map is the kernel of the next. If embedded submanifolds  $P, Q$  meet at a point  $m$ , we find a sequence of linear maps

$$0 \rightarrow (T_m P) \cap (T_m Q) \rightarrow T_m P \oplus T_m Q \rightarrow T_m M \rightarrow 0$$

where the maps are

$$v \in (T_m P) \cap (T_m Q) \mapsto (v, -v) \in T_m P \oplus T_m Q$$

and

$$(v, w) \in T_m P \oplus T_m Q \mapsto v + w \in T_m M.$$

**17.2** Prove that this sequence is exact just when the submanifolds  $P$  and  $Q$  are transverse.

## Transverse maps

A *conormal* to a map  $\varphi: P \rightarrow M$  of manifolds at a point  $p \in P$  is a cotangent vector  $\xi \in T_{\varphi(p)}^* M$  for which  $\varphi^* \xi = 0$  on  $P$ . The *conormal space*  $\varphi_p^\perp$  is the set of conormals at  $p$ .

The map

$$(u, v, w) = \varphi(x, y) = (x, xy, x^2)$$

pulls back  $du = dx$ ,  $dv = x dy + y dx$ ,  $dw = 2x dx$ . So conormals at a point  $(x, y, z)$  are the cotangent vectors  $\xi = a du + b dv + c dw$  so that  $0 = a dx + b(x dy + y dx) + 2cx dx$ . At the origin  $(x, y, z) = (0, 0, 0)$ , the conormals are therefore

$$\varphi_0^\perp = \langle dv, dw \rangle.$$

A continuously differentiable map  $\varphi: P \rightarrow M$  of manifolds is *transverse* to an embedded submanifold  $Q \subset M$  if  $\varphi$  is transverse to the inclusion map  $\iota(q) = q$ ,  $\iota: Q \rightarrow M$ . The *codimension* of a map  $P \rightarrow M$  is the difference  $\dim M - \dim P$ . Note that codimension can be negative, but if  $P \rightarrow M$  is an immersion, then the codimension of  $P \rightarrow M$  is not negative.

**Theorem 17.3.** If a  $C^k$  map  $\varphi: P \rightarrow M$  of manifolds,  $k \geq 1$ , is transverse to an embedded  $C^k$  submanifold  $Q \subset M$  then  $\varphi^{-1}Q \subset P$  is an embedded  $C^k$  submanifold of codimension equal to the sum of the codimensions of  $P$  and  $Q$ .

*Proof.* The problem is clearly local on  $P$ , so we can assume that  $M$  is a vector space, and that  $Q \subset M$  is a linear subspace. Let  $\pi: M \rightarrow M/Q$  be the linear projection map. Transversality is just the requirement that  $\pi \circ \varphi'(p)$  is a linear surjection, i.e. that  $\pi \circ \varphi$  is a submersion. But then  $\varphi^{-1}Q = (\pi \circ \varphi)^{-1} \{0\}$ , an embedded submanifold by the implicit function theorem, and

$$T_q \varphi^{-1}Q = \ker \pi \circ \varphi'(p) = \varphi'(p)^{-1} T_q Q.$$

With a little linear algebra, we can work out the dimensions: if  $n = \dim M$  then

$$n - \dim \varphi^{-1}Q = (n - \dim P) + (n - \dim Q).$$

□

Two smooth maps  $\varphi: P \rightarrow M$  and  $\psi: Q \rightarrow M$  of manifolds are *transverse* if they have no common conormals, i.e. for any point  $p \in P$  and  $q \in Q$ , if  $\varphi(p) = \psi(q)$  then  $0 = \varphi_p^\perp \cap \psi_q^\perp$ . Equivalently, if  $m = \varphi(p) = \psi(q)$  then  $T_m M = \varphi'(p)T_p P + \psi'(q)T_q Q$ , i.e. their tangent spaces span the whole ambient tangent space. For example, if  $\varphi(P)$  doesn't intersect  $\psi(Q)$ , then  $\varphi$  and  $\psi$  are transverse. The *incidence* is

$$T_{p,q} := \{ (u, v) \in T_p P \oplus T_q Q \mid \varphi'(p)u = \psi'(q)v \}.$$

Take the sequence of linear maps

$$0 \rightarrow T_{p,q} \rightarrow T_p P \oplus T_q Q \rightarrow T_m M \rightarrow 0$$

given by inclusion and by  $(u, v) \mapsto \varphi'(p)u - \psi'(q)v$ . This sequence is exact just when the maps are transverse.

**Theorem 17.4.** *If two  $C^k$  maps  $\varphi: P \rightarrow M$  and  $\psi: Q \rightarrow M$  of manifolds are transverse,  $k \geq 1$ , then the fiber product*

$$P \times_M Q := \{ (p, q) \mid \varphi(p) = \psi(q) \} \subset P \times Q$$

*is an embedded  $C^k$  submanifold of codimension equal to the sum of the codimensions of  $\varphi$  and  $\psi$ , with the obvious  $C^k$  maps:*

$$\begin{array}{ccc} P \times_M Q & \longrightarrow & P \\ \downarrow & \searrow & \downarrow \\ Q & \longrightarrow & M. \end{array}$$

*The tangent spaces of the fiber product are then the incidence spaces, hence the exact sequence*

$$0 \rightarrow T_{(p,q)} P \times_M Q \rightarrow T_p P \oplus T_q Q \rightarrow T_m M \rightarrow 0$$

*(i.e. the image of each linear map is the kernel of the next).*

*Proof.* Take local coordinates in  $P, Q, M$  about  $p, q, m$ , say  $x, y, z$ . Our map  $\varphi$  is  $z = z(x)$ , while  $\psi$  is  $z = z(y)$ . Let  $z(x, y) := z(x) - z(y)$ : its derivative is the matrix

$$z'(x, y) = (z'(x) \quad -z'(y)).$$

To say that the maps  $\varphi, \psi$  are transverse is to say that this matrix is surjective. Clearly the fiber product is the set of points where  $z(x, y) = 0$ . Apply the implicit function theorem to see that the fiber product is a manifold of expected dimension with tangent space given by differentiating the equation  $z(x) - z(y) = 0$ , i.e. the kernel of that matrix. □

## Orientations

Orientations are tricky. It is often more important to know that there is a natural choice of orientation than to explicitly compute it. To organize orientations, it helps to work directly with linear algebra. This is why we emphasized exact sequences above.

**17.3** Use elementary linear algebra to prove that an invertible matrix is connected by a continuous path to another just when their product has positive determinant.

An *orientation* of a finite and positive dimensional real vector space  $V$  is a choice of linear isomorphism  $V \rightarrow \mathbb{R}^n$  up to equivalence, two being equivalent if they agree up to a positive determinant linear isomorphism of  $\mathbb{R}^n$ . In other words, an orientation is a basis, but only defined up to moving through a continuous path of bases, or in other words, an orientation is a path component in the manifold of bases of  $V$ . An *orientation* of a zero dimensional vector space is just a choice of a sign  $+$  or  $-$ .

If  $V$  and  $W$  are vector spaces, every linear isomorphism  $V \rightarrow W$  identifies bases, and so orientations. But if we pick two vector spaces, of perhaps different dimensions, and a linear map  $V \rightarrow W$ , the relation between bases is more complicated.

An *exact sequence* of vector spaces is a sequence of linear maps from one vector space to the next

$$\dots \longrightarrow V_i \longrightarrow V_{i+1} \longrightarrow \dots$$

so that the image of each map is the kernel of the next. We will only want to consider *short* exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

So  $0$  is the kernel of  $A \rightarrow B$ ,  $A$  is the kernel of  $B \rightarrow C$ , and  $B$  is the kernel of  $C \rightarrow 0$ . In other words,  $A \rightarrow B$  is injective,  $B \rightarrow C$  is surjective, and  $A$  is isomorphically mapped by  $A \rightarrow B$  to the kernel of  $B \rightarrow C$ .

Pick any ordered basis of  $A$ , and then send it to  $B$ , to a linearly independent set of vectors. Then extend to an ordered basis of  $B$ . The vectors from  $A$  map to zero in  $C$ , so the vectors we extended with map to  $C$  to give a basis of  $C$ . These bases of  $A, B, C$  are now uniquely determined by the basis of  $B$ , since its first so many vectors arise uniquely from vectors in  $A$ , and its other vectors map to the basis of  $C$ . An ordered basis of  $B$  arising in this way is a *basis* of the short exact sequence.

An *orientation* of the exact sequence is an orientation of all three vector spaces for which there is a basis of the short exact sequence which gives negatively oriented bases of an even number of the vector spaces, i.e. zero or two of them. So an even number of “sign mistakes”.

**17.4** For any short exact sequence, if we orient two of the vector spaces, there is a unique compatible orientation of third. If the dimensions of all three vector spaces are positive we can find a basis of the exact sequence which yields positively oriented bases for all three.

On a manifold  $M$ , an orientation of  $M$  determines an orientation of each tangent space  $T_m M$ . Note that there are only two orientations of an orientable manifold. Conversely, if  $M$  is a connected manifold, and we pick a point  $m \in M$ , each orientation of  $T_m M$  picks out at most one orientation of  $M$ , and indeed picks out an orientation of every connected open orientation subset of  $M$  containing  $m$ .

If  $\varphi: P \rightarrow Q$  is a  $C^1$  map with a regular value  $q_0 \in Q$ , and we let  $S := \varphi^{-1}\{q_0\}$ , we have a short exact sequence

$$0 \rightarrow T_p S \rightarrow T_p P \rightarrow T_{q_0} Q,$$

for every  $p \in S$ . Therefore an orientation of  $P$  and of  $Q$  determines an orientation of  $T_p S$  for every  $p \in S$ . Recall that the implicit function theorem shows that there are local coordinates on  $P$  and  $Q$  near  $p$  and  $q_0$  in which  $\varphi$  is a linear map, with  $S$  a linear subspace. Hence the orientation induced at each point of  $S$  agrees with that of all of the nearby points, so consistently orients  $S$ .

From theorem 17.4 on page 152, transverse  $C^1$  maps  $\varphi: P \rightarrow M$ ,  $\psi: Q \rightarrow M$  of oriented manifolds have fiber product  $P \times_M Q$

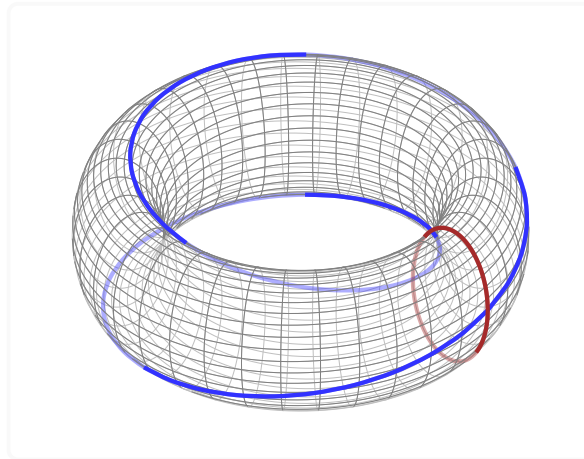
$$P \times_M Q := \{(p, q) \mid \varphi(p) = \psi(q)\} \subset P \times Q$$

oriented by the exact sequence

$$0 \rightarrow T_{(p,q)} P \times_M Q \rightarrow T_p P \oplus T_q Q \rightarrow T_m M \rightarrow 0.$$

Take transverse maps  $\varphi: P \rightarrow M$ ,  $\psi: Q \rightarrow M$  where one is proper and the other maps from a compact manifold. Suppose that  $P, Q, M$  are all oriented. Suppose that  $\dim P + \dim Q = \dim M$ ; we say the dimensions are *complementary*. The fiber product  $P \times_M Q$  is then a finite sum of points, each equipped with a sign by the orientation. The *intersection number*  $\varphi \cdot \psi$  is the sum of the  $+$  signs minus the sum of the  $-$  signs of the fiber product.

**17.5** Pick directions for the curves, an orientation for the torus, and count the intersection number:





## Chapter 18

### Sard's theorem

This chapter can be omitted without loss of continuity. You know that for max and min problems, we only need to look at critical points. We will see that for many problems in analysis, we only need to look away from critical points.

#### Measure zero

A set of points  $A \subset \mathbb{R}^n$  has *measure zero* if for any  $\varepsilon > 0$  there is a sequence of balls covering  $A$  whose volumes sum to less than  $\varepsilon$ . (The word “measure” just means “volume”, but it is tradition to say “measure zero” instead of saying “no volume”.)

**18.1** Prove that a line is a measure zero subset of the plane.

**18.2** Prove that we could use boxes instead of balls in the definition of measure zero.

**18.3** Use the previous problem to prove that we could use arbitrarily rotated boxes instead of balls in the definition of measure zero.

For more about measure zero sets see [10] or any textbook covering measure theory.

**18.4** Prove that the union of any sequence of measure zero sets has measure zero.

**18.5** Suppose that  $U \subset \mathbb{R}^n$  is an open set. Prove that a subset  $A \subset U$  has measure zero just when, for any open cover of  $U$ , say by sets  $W_\alpha$ ,  $A \cap W_\alpha$  has measure zero for all  $\alpha$ .

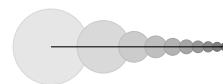
We define measure zero using coverings by *infinite* sequences of boxes. For compact sets, we can use finite sequences.

**18.6** Prove that a compact set  $K \subset \mathbb{R}^n$  has measure zero just when, for any number  $\varepsilon > 0$ , we can cover  $K$  by finitely many balls (or finitely many boxes) whose volumes sum to less than  $\varepsilon$ .

If a box  $X$  can be covered by boxes  $X_1, X_2, \dots$ , then

$$\text{Vol } X \leq \sum_i \text{Vol } X_i.$$

For proof see [10]. In particular, a box has measure zero just when one of its sides has length zero.



The real number line is a measure zero subset of the plane.



A measure zero set might look like a very skinny tree or like smoke



**Theorem 18.1.** *Take a closed set  $A \subset \mathbb{R}^{p+q}$ . For any  $x \in \mathbb{R}^p$ , let  $A_x$  be the set of points  $y \in \mathbb{R}^q$  for which  $(x, y) \in A$ . Then  $A$  has measure zero just when there is a set  $A' \subset \mathbb{R}^p$  of measure zero so that, for any  $x \notin A'$ ,  $A_x \subset \mathbb{R}^q$  has measure zero.*



For proof see [10]; this result is a trivial consequence of the famous Fubini–Tonelli theorem for measurable functions.

### Measure zero sets on manifolds

A set of points  $A \subset M$  on a manifold  $M$  has *measure zero* for that manifold if for any chart  $(U, \varphi)$  for  $M$ , the set  $\varphi(U \cap A)$  has measure zero. By the above lemmas, this definition is independent of the choice of charts.

**Lemma 18.2.** *The empty set is the only measure zero open subset of any manifold.*

*Proof.* Every ball in  $\mathbb{R}^n$  contains a box of positive volume, so no ball has measure zero. Every nonempty open subset of  $\mathbb{R}^n$  is a union of a sequence of open balls, so does not have measure zero. Charts identify open sets locally.  $\square$

**18.7** Prove that the set of rational numbers has measure zero in  $\mathbb{R}$ .

**18.8** Suppose that  $\varphi: P \rightarrow Q$  is a  $C^1$  of manifolds of equal dimension. Prove that every measure zero set in  $P$  has measure zero image in  $Q$ .

**18.9** Suppose that  $\varphi: P \rightarrow Q$  is  $C^1$ . If  $P$  has lower dimension than  $Q$ , prove that  $\varphi$  has image of measure zero.

### Embedding compact manifolds

**Theorem 18.3.** *Take a compact  $n$ -dimensional manifold  $M$  and a  $C^k$  map  $\varphi: M \rightarrow \mathbb{R}^N$ ,  $N \geq 2n$ ,  $2 \leq k \leq \infty$ . There is a  $C^k$  immersion as close as we like to  $\varphi$ . To be precise, for any  $\varepsilon > 0$  we can pick a smooth immersion  $\psi: M \rightarrow \mathbb{R}^N$  so that, for any point  $p \in M$ ,  $|\psi(p) - \varphi(p)| < \varepsilon$ .*

*Proof.* Take an open set  $U$  whose closure lies inside another open set  $W$  which is the domain of a chart. We first want to push  $\varphi$  very slightly into a map  $\psi$  which is an immersion on  $\bar{U}$ . Pick some injective linear map  $A: \mathbb{R}^n \rightarrow \mathbb{R}^N$  very close to the zero linear map, and a smooth bump function  $f$  supported on  $W$  and equal to 1 on  $U$ . Let  $\psi := \varphi$  except on  $W$  where, in the coordinates of its chart,

$$\psi(x) := \varphi(x) + f(x)Ax.$$

Let's see why we can choose  $A$  to make  $\psi$  an immersion on  $\bar{U}$ . On  $\bar{U}$ ,  $f = 1$  so

$$\psi'(x) = \varphi'(x) + Ax.$$

We have to pick some  $A$  so that  $\varphi'(x) + A$  has full rank for every  $x \in \bar{U}$ .

We want to avoid low rank linear maps. Let  $V$  be the determinantal variety consisting of all linear maps  $\mathbb{R}^n \rightarrow \mathbb{R}^N$  of rank at most  $n - 1$ . So  $V$  is the union of the determinantal manifolds  $V_r$ , each consisting of the linear maps of a given rank  $r = 0, 1, 2, \dots, n - 1$ . As we saw in problem 1.36 on page 13, each  $V_r$  is a manifold of dimension  $r(n + N - r)$ .

**18.10** Prove that these  $V_r$  increase in dimension for  $r = 0, 1, 2, \dots, n - 1$ , with  $V_{n-1}$  having dimension  $(n - 1)(N + 1)$ .

Consider the map

$$(x, B) \mapsto B - \varphi'(x),$$

defined on points  $x \in W$  and linear maps  $B: \mathbb{R}^n \rightarrow \mathbb{R}^N$ , mapping to  $\mathbb{R}^{N \times n}$ . Restrict this map to  $B \in V$ .

**18.11** Why is the image of this map of measure zero?

Any open set have positive measure, so contains an injective  $A$  so that

$$B - \varphi'(x) \neq A,$$

for any  $x \in W$  and any  $B$  not of full rank. Turning this around,

$$\varphi'(x) + A \neq B$$

for any  $x \in U$  and any  $B$  not of full rank. So  $\psi'(x)$  has full rank on  $\bar{U}$ .

Suppose that  $\varphi$  was already an immersion on some compact set  $K$ . Everywhere, in any charts,  $\psi'$  is as close to  $\varphi'$  as we like by choice of  $A$ , so we can make  $A$  small enough that  $\psi$  remains an immersion on  $K$ , so  $\psi$  immerses  $K \cup \bar{U}$ .

Repeat this process, starting with  $K$  empty, adding some new compact set  $\bar{U}$  with nonempty interior at each step. Since  $M$  is compact, we can cover  $M$  in finitely many open sets  $U$  each of which has compact closure in the domain of a chart; we can even ensure that each has some bump function equal to 1 on  $\bar{U}$  and zero outside the chart domain. At each step, we control how far the next map is from the previous one by making our choice of  $A$  small enough.  $\square$

**Theorem 18.4** (Whitney). *Take a compact  $n$ -dimensional manifold  $M$  and a  $C^k$  map  $\varphi: M \rightarrow \mathbb{R}^N$ , for some  $N \geq 2n + 1$ ,  $2 \leq k \leq \infty$ . There is a  $C^k$  embedding as close as we like to  $\varphi$ . To be precise, for any  $\varepsilon > 0$  we can pick a smooth embedding  $\psi: M \rightarrow \mathbb{R}^N$  so that, for any point  $p \in M$ ,  $|\psi(p) - \varphi(p)| < \varepsilon$ .*

*Proof.* We can assume that  $\varphi$  is an immersion by theorem 18.3 on the preceding page. Take an open set  $U$  whose compact closure lies in an open set  $W$  so that  $\varphi$  is injective on  $W$ . We first want to push  $\varphi$  very slightly into a map  $\psi$ . Pick some vector  $c \in \mathbb{R}^N$  very close to the zero vector, and a smooth compactly supported function  $f$ , whose support lies inside  $W$ , equal to 1 on  $\bar{U}$ . Let  $\psi := \varphi + cf$ , so  $\psi' = \varphi' + cf'$  is as close to  $\varphi'$  as we like if we pick  $c$  small enough, so an immersion for small enough  $c$ .

Consider the open set in  $M \times M$  of pairs  $(x, y)$  of points  $x, y \in M$  on which  $f(x) \neq f(y)$ . On that open set, the map

$$(x, y) \mapsto -\frac{\varphi(x) - \varphi(y)}{f(x) - f(y)} \in \mathbb{R}^N$$

has image of measure zero, by problem 18.9 on page 156, so we can pick  $c$  not of this form, i.e.

$$\varphi(x) + f(x)c \neq \varphi(y) + f(y)c.$$

In other words, if  $f(x) \neq f(y)$  then  $\psi(x) \neq \psi(y)$ . Conversely, if  $\psi(x) = \psi(y)$  then  $\varphi(x) = \varphi(y)$ :  $\psi$  injects any set  $\varphi$  does.

Suppose that  $S$  is a set  $\varphi$  injects, and let  $S' = S \cup \{f > 0\}$ . Claim:  $\psi$  injects  $S'$ . Proof: on  $S$ ,  $\psi$  injects. So if  $\psi(x) = \psi(y)$  for some  $x, y \in S'$ , then without loss of generality  $f(x) > 0$ . But  $f(x) = f(y)$  forces  $f(y) > 0$  so  $y \in W$ ;  $\varphi$  injects  $W$ , so  $\psi$  does too, so  $x = y$ . In particular, if  $\varphi$  injects  $S$  then  $\psi$  injects  $S \cup \bar{U}$ .

Repeat this process, starting with  $S$  empty, adding some new compact set  $\bar{U}$  with nonempty interior at each step. Since  $M$  is compact, we can cover  $M$  in finitely many open sets  $U$  each of which has closure in an open set on which  $\varphi$  is injective. At each step, we control how far the next map is from the previous one by making our choice of  $A$  small enough. Eventually we produce an injective immersion, an embedding since  $M$  is compact.  $\square$

**18.12** Prove that every compact manifold admits a smooth embedding into Euclidean space.

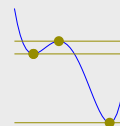
**18.13** An *exhaustion* of a manifold is a sequence of open sets, each with compact closure contained inside the next open set, whose union is the manifold. If a manifold of dimension  $n$ , perhaps not compact, admits an exhaustion, repeat the proofs above to prove that it immerses into  $\mathbb{R}^{2n}$ . If in addition it bears a proper function, prove it admits an embedding in  $\mathbb{R}^{2n+1}$ .

## Sard's theorem

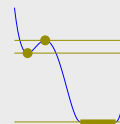
Consider one function of one variable  $y = f(x)$ . At a critical point, the rate of change is 0, so small changes in  $x$  make almost no change in  $y$ : a small interval of  $x$  values is mapped to a very small interval of  $y$  values.

**Theorem 18.5** (Sard's theorem). *Suppose that  $P$  and  $Q$  are manifolds of the same dimension and  $\varphi: P \rightarrow Q$  is a continuously differentiable map. The set of critical values of  $\varphi$  has measure zero; in particular,  $\varphi$  has regular values.*

Here there are only 3 critical values (3 heights where critical points occur).



You could have a long interval of critical points, but still only have 3 critical values—the set of critical points might *not* have measure zero, but the set of critical values does.



*Proof.* By the above results, it is enough to work in a single chart on  $P$  and a single chart on  $Q$ , so we can assume that  $P$  and  $Q$  are bounded open subsets of  $\mathbb{R}^n$ . For simplicity, we can assume that  $P$  lies inside an open box in  $\mathbb{R}^n$ , say of volume  $V$ , or even that  $P$  is such a box. Let

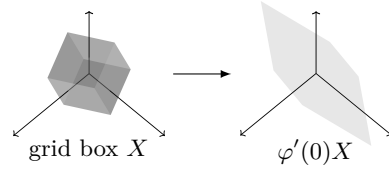
$$M = \max_{x \in P} |\varphi'(x)|;$$

so  $\varphi$  stretches distances by at most  $M$ .

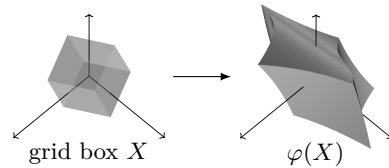
Cover  $P$  in a grid. For any  $\varepsilon$ , if we make the grid fine enough, say of mesh size at most  $\delta$ , then we can ensure that for any two points  $x$  and  $y$  in the same grid box,

$$|\varphi(y) - \varphi(x) - \varphi'(x)(y - x)| < \varepsilon|x - y| < \varepsilon\delta,$$

so the nonlinear map  $\varphi$  changes distances by about the same as its linear approximation. Take a critical point  $x$  of  $\varphi$  inside a grid box  $X$ ; we want to estimate  $\text{Vol } \varphi(X)$ . As usual we can arrange  $x = 0$  and  $\varphi(0) = 0$  by translation without changing the volume of  $\varphi(X)$ . The linear map  $\varphi'(0)$  squishes the box flat: the image of  $\varphi'(0)$  lies in a linear subspace  $W \subset \mathbb{R}^n$  of dimension at most  $n - 1$ , since the image of  $\varphi'(0)$  is not all of  $\mathbb{R}^n$ .



Comparing to the linear approximation, the set  $\varphi(X)$  lies within distance at most  $\varepsilon\delta$  of  $\varphi'(0)X$ , so  $\varphi(X)$  might look like a very slight thickening of  $\varphi'(0)X$ :



Rotation doesn't affect volume; rotate the picture to arrange that  $W = \{0\} \times \mathbb{R}^{n-1}$ . For the duration of this proof, let's adopt a notation in which  $X_r$  means a box all of whose sides have length  $r$ , centered at an unspecified point, and  $\bar{B}_r$  similarly means a ball of radius  $r$  around an unspecified point. So if we write  $B_r \subset X_{2r}$ , we mean that every ball of radius  $r$  lies inside a box all of whose sides have length at most  $2r$ . Our box  $X$  is  $X = X_\delta \subset B_{\delta\sqrt{n}/2}$ , so

$$\varphi'(0)X_\delta \subset \varphi'(0)B_{\delta\sqrt{n}/2},$$

and  $\varphi'(x)$  stretches by at most a factor of  $M$ :

$$\varphi'(0)X_\delta \subset \{0\} \times B_{M\delta\sqrt{n}/2} \subset \{0\} \times X_{M\delta\sqrt{n}}.$$

But  $\varphi(X_\delta)$  lies within  $\varepsilon\delta$  of  $\varphi'(0)X_\delta$ , so

$$\varphi(X_\delta) \subset [-\varepsilon\delta, \varepsilon\delta] \times X_{M\delta\sqrt{n}+2\varepsilon\delta},$$

and therefore  $\varphi(X_\delta)$  lies in box of volume at most

$$2\varepsilon\delta \operatorname{Vol} X_{M\delta\sqrt{n}+2\varepsilon\delta} = 2\varepsilon\delta^n (M\sqrt{n} + 2\varepsilon)^{n-1}.$$

The factor of  $\varepsilon$  is crucial—it describes the thinness of the image  $\varphi(X)$  of the grid box  $X$  and it doesn't show up if we work near a regular point.

Adding up over all of the grid boxes, each of which we can assume has volume  $\delta^n$ , the critical values lie in various rotated boxes whose total volume is at most

$$2\varepsilon V (M\sqrt{n} + 2\varepsilon)^{n-1},$$

which we can make arbitrarily small by choosing  $\delta$  small enough to make  $\varepsilon$  as small as we need.  $\square$

**18.14** Suppose that  $f_1, f_2, f_3, \dots$  is a sequence of continuously differentiable functions  $\mathbb{R} \rightarrow \mathbb{R}$ . Prove that there is a value  $y_0 \in \mathbb{R}$  so that none of these functions has  $y_0$  as a critical value.

**18.15** Prove that there are real numbers  $x_0, y_0$  so that, for any pair of polynomials  $p(x, y), q(x, y)$  in two variables, of any degree, with rational number coefficients, the set of points

$$\{ (x, y) \mid p(x, y) = x_0 \text{ and } q(x, y) = y_0 \}$$

is discrete (perhaps empty).

**18.16** Suppose that  $\varphi: P \rightarrow Q$  is a continuously differentiable map of manifolds and that  $\dim P < \dim Q$ . Use Sard's theorem to prove that the image of  $\varphi$  has measure zero in  $Q$ .

**18.17\*** Suppose that  $M \subset \mathbb{R}^{p+q}$  is a smooth embedded submanifold of dimension  $p$  and that  $p < q$ .

- Prove that there is a unit vector  $u \in \mathbb{R}^{p+q}$  so that the linear projection map taking  $\mathbb{R}^{p+q}$  to the perpendicular space to  $u$  is a local diffeomorphism on  $M$ .
- Prove that there is a unit vector  $u \in \mathbb{R}^{p+q}$  so that the linear projection map taking  $\mathbb{R}^{p+q}$  to the perpendicular space to  $u$  is injective on  $M$ .
- Prove that there is a unit vector  $u \in \mathbb{R}^{p+q}$  so that the linear projection map taking  $\mathbb{R}^{p+q}$  to the perpendicular space to  $u$  is a diffeomorphism on  $M$ .
- Prove that every embedded submanifold of dimension  $p$  inside any Euclidean space  $\mathbb{R}^n$  is diffeomorphic to a manifold inside  $\mathbb{R}^{2p+1}$ . For example, every surface  $M \subset \mathbb{R}^{100}$  is diffeomorphic to a surface  $M \subset \mathbb{R}^5$ .

**18.18\*** Give a counterexample to Sard's theorem among smooth maps of manifolds in the broad sense.

## Different dimensions

**Theorem 18.6** (Sard's theorem). *Suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  map of manifolds, perhaps of different dimensions, with  $k \geq 1$  and also with  $k \geq \dim P - \dim Q + 1$ . Then the set of critical values of  $\varphi$  is a measure zero subset of  $Q$ .*

*Proof.* If  $P$  has dimension at most that of  $Q$ , we have the proof above. As before we can assume that  $P$  and  $Q$  are open subsets of  $\mathbb{R}^{p+q}$  and  $\mathbb{R}^q$  respectively. Let  $C \subseteq P$  be the set of critical points of  $\varphi$ . Let  $C_i \subseteq C$  be the set of points at which all partial derivatives of order up to  $i$  of all components of  $\varphi$  vanish.

First, we prove that  $\varphi(C - C_1)$  has measure zero. For each  $x \in C - C_1$ ,  $x$  is a critical point of  $\varphi$ , but some first derivative of  $y = \varphi(x)$  is not zero, say  $\frac{\partial y^1}{\partial x^1}$ . We can then replace  $x^1$  by  $y^1$  as a coordinate function on  $P$ . So then  $y = \varphi(x)$  has  $y^1 = x^1$ , and so  $\varphi'(x)$  has full rank unless  $y^2, \dots, y^q$  drop rank in the variables  $x^2, \dots, x^{p+q}$ . By induction on dimension, for any fixed value of  $x^1$ , this set is of measure zero. By theorem 18.1 on page 156,  $C - C_1$  is of measure zero,

We want to prove that each  $C_k - C_{k+1}$  has measure zero. Each nonzero partial derivative of order  $k+1$  is a derivative, say  $\frac{\partial f}{\partial x^1}$ , of a partial derivative  $f$  of order  $k$  of some  $y^i = \varphi^i(x)$ . Consider the map  $\psi(x) = (f(x), x^2, \dots, x^n)$ , a local diffeomorphism. Restricting to a smaller domain  $P$ , we can assume that  $\psi$  is a diffeomorphism. On  $C_k - C_{k+1}$ ,  $f = 0$ . So  $\psi$  moves  $C_k - C_{k+1}$  into the hyperplane  $H = \{0\} \times \mathbb{R}^{n-1}$ . Replace  $\varphi$  by  $\varphi \circ \psi^{-1}$ . The critical values of  $\varphi$  don't move. Points of the old  $C_k$  map to points of the new  $C_k$ , by the chain rule.

Let  $\rho := \varphi|_H$ . By induction on dimension,  $\rho$  has measure zero critical values. The image of  $\rho'$  lies inside the image of  $\varphi'$  at each point of  $H$ , so if  $\varphi$  is critical (i.e.  $\varphi'$  is not surjective) then  $\rho$  is critical. At each point of  $C_k \subset H$ ,  $\varphi$  is critical, so  $\psi$  is too. Hence these points have measure zero image in  $Q$ .  $\square$

**18.19** Prove that any smooth map  $\varphi: P \rightarrow Q$  of constant rank is a submersion if and only if its image is not of measure zero.

**18.20\*** Prove that any  $C^k$  bijective map of manifolds,  $k \geq 1$ , is a  $C^k$  diffeomorphism just when it is either constant rank or a submersion or an immersion.

## Avoiding sets

**18.21** Suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  map of manifolds, with  $k \geq 1$  and also with  $k \geq \dim P - \dim Q + 1$ . Take a closed set  $A \subset P$ . Prove that  $A$  has measure zero just when there is a measure zero set  $A' \subset Q$  so that, for any  $q \in Q - A'$ , the fiber  $P_q := \varphi^{-1}\{q\}$  has intersection  $A \cap P_q$  of measure zero.

Take a vector space  $V$  of finite positive dimension over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take any measure zero subset  $A \subset V$  of the vector space. Let us prove that “most” points of “most” lines through the origin miss  $A$ . Let  $P := V - \{0\}$ . Let  $Q$  be the projective space  $Q := \mathbb{P}V$ . Map  $\varphi: P \rightarrow Q$  by taking each nonzero vector  $v \in P$  to the line  $\varphi(v) = [v]$  through the origin and  $V$ . Each fiber  $P_q$  is punctured line  $tv$ ,  $t \in k - \{0\}$ , for a vector  $v \neq 0$ . Away from some measure zero in  $Q = \mathbb{P}V$ , i.e. a measure zero set of lines through the origin, the line  $P_q$  strikes the set  $A$  only on a set of measure zero inside that line.

A subset  $A \subseteq M$  of a manifold  $M$  is of *dimension at most  $k$*  if  $A$  lies in the union of a sequence of  $C^1$  embedded submanifolds of dimension at most  $k$ . A set is of *dimension  $k$*  if it is of dimension at most  $k$  but not at most  $k-1$ . A set  $A \subseteq M$  is of *codimension  $k$*  if it is of dimension  $\dim M - k$ .

The set of rational numbers has dimension zero in the real numbers.

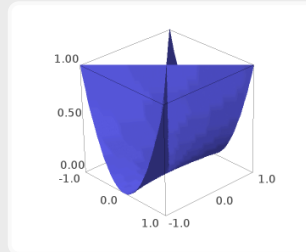
The Cantor set has dimension one in the real numbers.

Every subset of any manifold  $M$  has at most the same dimension as  $M$ .

A *rational line* is a line in the plane which passes through at least two distinct points with rational coordinates. Clearly the set of rational lines is countable, so the union of all of the rational lines has dimension one.

The union of all rational radius circles in the plane centered at rational points has dimension one.

The *Whitney umbrella*  $W$  consists of the points  $(x, y, z) \in \mathbb{R}^3$  where  $x^2 = y^2 z$ .



The solutions do not form a submanifold. The points of  $W$  at which  $y \neq 0$  form the graph of  $z = x^2/y^2$ , a smooth surface. The points of  $W$  at which  $y = 0$  form the line  $x = y = 0$ . So the Whitney umbrella is not a smooth surface, but is of dimension 2.

Recall that a function is *analytic* if it is locally the sum of a convergent Taylor series expansion. The solutions of any collection of analytic equations on several real or complex variables form a locally finite union of embedded submanifolds. See [4] chapter II for a complete proof of a stronger result [and [4] theorem 4.19 p. 95 for a statement of the result], by first complexifying the variables and then applying advanced methods of complex analytic geometry. See [16] for a more elementary proof for algebraic rather than analytic equations.

**18.22** The *determinantal variety*  $V = V_{p,q,r}$  is the set of all  $p \times q$  real matrices of rank at most  $r$ . Prove that  $V$  has dimension  $r(p + q - r)$ .

A map  $\varphi: P \rightarrow Q$  is *transverse* to a set  $S \subset Q$  if it is transverse to every one of the embedded submanifolds in a sequence covering  $S$ .



**Lemma 18.7.** *Suppose that  $S \subset P$  is a subset of a manifold, and that  $\varphi: P \rightarrow Q$  is a smooth map. For any point  $q \in Q$  outside some measure zero set, the fiber  $P_q := \varphi^{-1}\{q\}$  is transverse to  $S$ .*

*Proof.* The union of a sequence of measure zero sets is measure zero, so it suffices to prove the result for  $S$  an embedded submanifold. Cut out the measure zero set of points of  $Q$  above which  $\varphi$  does not have full rank, and the measure zero set of points of  $Q$  above which  $\varphi|_S: S \rightarrow Q$  is not of full rank. So we can assume that  $\varphi$  is a submersion on  $P$  and on  $S$ . Coordinates on  $Q$  vanishing at some point  $q \in Q$  vanish just along the fiber  $P_q$ . Their differentials are linearly independent on  $S$ , so linearly independent of the conormals of  $S$ , so  $P_q$  and  $S$  are transverse.  $\square$

**18.23\*** Suppose that  $\varphi: P \rightarrow Q$  is an analytic map of analytic manifolds. (We leave the reader to define *analytic* manifolds.) Suppose that  $\varphi$  is not constant on any component of  $P$ . Prove that, for any  $q_0 \in Q$ ,  $\varphi^{-1}\{q_0\} \subset P$  is of measure zero.



## Chapter 19

### Corners

We define the notion of a manifold with corners and reconstruct our theorems in this larger context.

#### Corners

We can relate global problems to local problems by cutting up our manifolds, like we might cut up a cake. Cutting introduces corners. The *n-dimensional standard corner* is  $[0, \infty)^n$ .

The standard corner in the plane is where  $x \geq 0$  and  $y \geq 0$



#### Smoothness

A map between subsets (perhaps not open) of Euclidean space is called  $C^k$  (or *smooth* or  $C^\infty$ ) if it locally admits extension to a  $C^k$  (or smooth) map on some open set. The map is a  $C^k$  *diffeomorphism* if it also admits a  $C^k$  inverse.

#### Domains and regions

For a bit of concreteness, before we generalize the concept of manifold to allow corners, let us consider a simpler concept of a region inside a given manifold. Take a  $C^k$  manifold  $M$  of dimension  $n$ . A set  $X \subset M$  is a  $C^k$  *region with corners* if, for each point  $x_0 \in X$ , there are some  $C^k$  functions  $f^1, \dots, f^q$  defined in an open subset  $U \subset M$  containing  $x_0$ , so that  $df^1, \dots, df^q$  are linearly independent, and  $X \cap U$  is the set of points  $x \in U$  on which all  $f^i(x)$  are not negative:

$$X \cap U = \{x \in U \mid f^1(x) \geq 0, \dots, f^q(x) \geq 0\}.$$

These  $f^i$  are *boundary functions* of  $X$ . (The topological interior of  $X$ , cut out by the same inequalities with  $\geq$  changed to  $>$ , is a  $C^k$  *domain with corners*. Unfortunately, while the terminology *domain* is standard, there is no completely standard terminology for what we here call *region*.)

Without changing  $X$ , clearly we can change these  $f^i$  to any other functions  $g^i$  with linearly independent differentials as long as, after perhaps reordering,  $g^i \geq 0$  just when  $f^i \geq 0$ .

The *standard  $n$ -dimensional corner*  $X \subset \mathbb{R}^n$  is the set  $X = [0, \infty)^n$ , i.e. the set of points  $x \in \mathbb{R}^n$  at which  $x^1, \dots, x^n \geq 0$ . Clearly the standard corner is a smooth region with corners.

**19.1** Every  $C^k$  region with corners  $X$  is locally diffeomorphic to the standard corner.

**19.2** Is the set of points  $\{(x, y) \mid xy \geq 0\}$  locally diffeomorphic to the standard corner?

**19.3** Is the set of points  $\{(x, y) \mid x \leq 0 \text{ or } y \leq 0\}$  locally diffeomorphic to the standard corner?

**19.4** For any  $C^k$  region with corners  $X$ , some  $k \geq 1$ , prove that a point  $x_0 \in X$  is a boundary point of  $X$  (i.e. one of the boundary functions vanishes there) just when there is a  $C^1$  function  $f$  defined near that point, with  $df \neq 0$ , so that  $f(x_0) \leq f(x)$  for all  $x \in X$ .

**19.5** Suppose that  $X$  and  $Y$  are  $C^k$  regions with boundary. Prove that for every  $C^k$  diffeomorphism  $\varphi: X \rightarrow Y$  takes boundary points to boundary points.

**19.6** Suppose that  $f$  is a  $C^1$  function defined on an open subset of the standard corner  $[0, \infty)^n$ , and that  $f \geq 0$ . Then at every point where  $f$  is defined, if some variable  $x^i$  vanishes at that point then either  $f > 0$  or

$$\frac{\partial f}{\partial x^i} \geq 0.$$

**19.7** Suppose that  $X$  and  $Y$  are  $C^k$  regions with corners, where  $X$  is an open subset of the standard corner in  $\mathbb{R}^n$ . Prove that a  $C^k$  function  $f$  defined on an open subset of  $Y$  is a boundary function just when  $f \circ \varphi$  is a boundary function.

**19.8** Take a  $C^k$  region with corners  $X$  given by boundary function inequalities

$$g_1 \geq 0, \dots, g_n \geq 0.$$

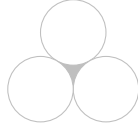
Prove that a  $C^k$  function  $f \geq 0$  on a  $C^k$  region with corners is a boundary function for  $X$  near some chosen point if and only if  $df \neq 0$  and  $f = e^u g^i$  for one of these functions  $g^i$ , where  $u$  is  $C^{k-1}$  near  $g^i = 0$ , and  $C^k$  away from  $g^i = 0$ .

**Corollary 19.1.** Any  $C^k$  diffeomorphism  $\varphi: X \rightarrow Y$  of  $C^k$  regions with corners, say written as  $y = \varphi(x)$  in terms of boundary functions, is expressed as  $y^i = e^{u^i(x)} x^j$  for a unique permutation  $j = \pi(i)$  of  $1, \dots, n$  and some functions  $u^i(x)$ ,  $C^{k-1}$  near  $x^j = 0$  and  $C^k$  elsewhere.

## Charts

We now step into abstraction. An  *$n$ -dimensional corner chart* on a set  $M$  (often just called a *chart*) is a pair  $(U, \varphi)$  where  $U \subset M$  is a subset and  $\varphi$  is a bijection to an open subset of  $[0, \infty)^n$ . Denote each corner chart as  $x = (x^1, x^2, \dots, x^n)$ : we say we have *coordinates*  $x^1, \dots, x^n$  on  $M$ , *centred* at a point  $m_0 \in M$  if they vanish at  $m_0$ .

We repeat without change the definitions of *transition map*, *compatibility*, *atlas*, *maximal atlas*, *smooth structure*, *open set*, *manifold* (but call it a *manifold with corners*), *smooth map*, *coordinates*, *tangent vector*, *tangent space*, *vector field*, *cotangent space*, and *bundle of tangent hyperplanes*.



**19.9** Explain why is *not* a manifold with corners.

**19.10** Suppose that  $M$  is a surface with corners embedded in  $\mathbb{R}^3$ . Do the angles at the corners have to be right angles?

**19.11** The *standard  $n$ -simplex* is

$$X = \{x \in [0, 1]^{n+1} \mid x_1 + \cdots + x_{n+1} = 1\}.$$

Explain why  $X$  is an  $n$ -dimensional manifold with corners.

**19.12** Prove that a hollow cylinder in  $\mathbb{R}^3$  with closed top and bottom is not a manifold with corners.

**19.13** Prove that a solid cylinder in  $\mathbb{R}^3$  is a manifold with corners.

**19.14\*** Is an octahedron a manifold with corners?

**19.15\*** Is a cone a manifold with corners?

**19.16** Suppose that  $U \subset \mathbb{R}^n$  is open and  $f: U \rightarrow \mathbb{R}^n$  is a  $C^1$  map, and that  $\det f'(x) \neq 0$  for every  $x \in U$ . Prove that the set

$$M := \{x \in U \mid f_1(x) \geq 0 \text{ and } f_2(x) \geq 0 \text{ and } \dots \text{ and } f_n(x) \geq 0\} \subset \mathbb{R}^n$$

is a manifold with corners.

**19.17\*** Prove that every  $C^k$  compact curve with corners is  $C^k$  diffeomorphic to a finite union of (1) circles and (2) closed intervals of the real number line.

**19.18** Suppose that  $G$  is a manifold with corners, and also a group, and the group operation is a smooth map. Prove that  $G$  has empty boundary.

A *manifold with boundary* is a manifold with corners so that any boundary point lies interior to a wall, i.e. any two boundary functions have linearly dependent differentials at every boundary point. (The terminology is terrible: the standard corner of any dimension 2 or more is a manifold with corners, has a boundary, but is *not* a manifold with boundary.)

**19.19** Construct a 3-dimensional manifold with boundary whose boundary has two components: one diffeomorphic to the compact oriented surface of genus  $g$ , and the other diffeomorphic to the sphere. Construct a 3-dimensional manifold with boundary whose boundary has two components: one diffeomorphic to the compact unoriented surface of genus  $g$ , and the other diffeomorphic to the real projective plane.

A *cowall* of a manifold with corners  $M$  at a point  $m_0$  is a ray of covectors  $\langle \xi \rangle = \mathbb{R}_{\geq 0} \xi \subset T_{m_0}^* M$  so that, in some coordinates,  $\xi = dx^i$  for some  $x^i$  vanishing at  $m_0$ , i.e. the positive scalings of the differential of a boundary function.



The set  $X = [0, \infty) \times [0, \infty) \times [0, \infty)$  is a corner in 3-dimensional space, consisting of the points  $(x, y, z)$  with  $x, y, z \geq 0$ . We can replace the inequalities  $x, y, z \geq 0$  by any equivalent inequalities, say  $2x, (1+x^2)y, 4z \geq 0$  and this will change  $dx, dy, dz$  at the corner point to  $2dx, dy, 4dz$ , i.e. by positive rescaling.

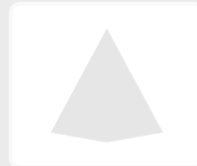
Similarly, suppose that a manifold with corner is cut out by some inequality  $f(x, y, z) \geq 0$  with  $df \neq 0$ . We can replace  $f(x, y, z) \geq 0$  by an equivalent inequality, say  $3(1+x^2)f(x, y, z) \geq 0$ , and this will change  $df$  to  $3(1+x^2)df$ , at points where  $f = 0$ , i.e. by positive rescaling.

Intuitively, we want to picture that the boundary of a manifold with corners consists of walls, and the cowalls are the covectors vanishing on the tangent spaces of the walls, hence the term *cowall*.

The corner of the teardrop has two cowalls: .

We draw the cowalls *not* as the ray of tangent vectors to the corner, but as the ray spanned by the *normal* vectors to the corner, since any cowall differential  $dh$  is identified in any coordinates with the gradient vector  $\nabla h$ .

The cowalls of

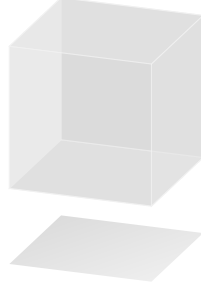


at its lowest point are



## The implicit function theorem for manifolds with corners

Intuitively, a *submersion* means a map that looks (locally) like this:



**19.20** Write this map down in coordinates.

A  $C^k$  map  $\varphi: P \rightarrow Q$  of manifolds with corners is *tidy* if the pull back of any boundary function is, near any boundary point, either a boundary function or everywhere zero. A point  $p \in P$  is a *regular point* of a map if at  $q := f(p)$ :

- a. boundary functions pull back to boundary functions (hence tidy) and
- b.  $\varphi^*$  is injective on cowalls and
- c.  $\varphi^*$  is injective on the quotient of the cotangent space by the span of the cowalls.

A point  $q \in Q$  is a *regular value* of  $f$  if every point of  $\varphi^{-1}\{q\}$  is a regular point. Points which are not regular are *critical points*. A map is a *submersion* if all points of its domain are regular.

Consider the map

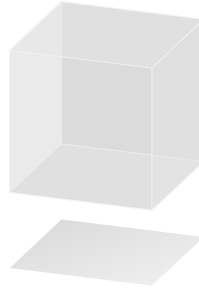
$$f: P := [0, \infty)^3 \rightarrow Q := [0, \infty) \times \mathbb{R}$$

defined by

$$f(x, y, z) = (s, t) := (x, y).$$

This map is nasty: it maps a variable  $y \geq 0$  to a variable  $t$  which is not required to be  $\geq 0$ . Let  $p_0 := 0 \in P$  and  $q_0 := 0 \in Q$ . The unique cowall on  $Q$  at  $q_0$  is  $ds$ . (We write  $ds$  when really we mean the line spanned by  $ds$ .) There are three cowalls at  $p_0$ :  $dx, dy, dz$ . The map  $f$  pulls back as  $f^*ds = dx$  so injectively maps cowalls, as required of a submersion. The cotangent spaces, i.e. spaces spanned by all cotangent vectors, are  $T_{q_0}^*Q = \langle ds, dt \rangle$  and  $T_{p_0}^*P = \langle dx, dy, dz \rangle$ . So the quotient spaces are  $T_{q_0}^*Q / \langle \text{cowalls} \rangle = \langle ds, dt \rangle / \langle ds \rangle \cong \langle dt \rangle$  and  $T_{p_0}^*P / \langle \text{cowalls} \rangle = 0$ . So  $f^*(p_0)$  is *not* an injection on these quotient spaces, so  $p_0$  is *not* a regular point,  $q_0$  is *not* a regular value and  $f$  is *not* a submersion.

**19.21** Draw the cowalls, and check if this is a submersion:



**19.22** Suppose that  $P$  and  $Q$  are manifolds without boundary or corners. Prove that a  $C^k$  map  $\varphi: P \rightarrow Q$  is a  $C^k$  submersion just when  $\varphi'(p_0): T_{p_0}P \rightarrow T_{q_0}Q$  is a linear surjection, for every point  $p_0 \in P$ .

**19.23** Is this map a submersion of compact manifolds with corners?

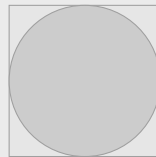


**19.24** Is this map a submersion of compact manifolds with corners?

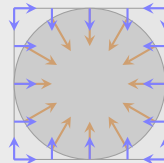


**19.25** Suppose that  $P$  and  $Q$  are manifolds with corners. In terms of the cowalls at  $p \in P$  and  $q \in Q$ , what are the cowalls at  $(p, q) \in P \times Q$ ? Prove that the map  $f: (p, q) \in P \times Q \mapsto q \in Q$  is a submersion.

The inclusion of the closed disk inside the closed square



is *not* a submersion. It pulls back cowalls to cowalls:



But some boundary functions do not pull back to boundary functions.



**Theorem 19.2** (The implicit function theorem). *Every  $C^k$  submersion,  $k \geq 1$ , is locally identified in  $C^k$  coordinates with a linear submersion of corners, restricted to open subsets of those corners. To be precise: if  $p_0 \in P$  is a regular point of a  $C^k$  map  $\varphi: P \rightarrow Q$  of manifolds with corners, then  $p_0$  lies in the domain of a coordinate chart  $x, y$  valued in an open subset of  $[0, \infty)^{a+b}$  and  $q_0 := \varphi(p_0)$  lies in the domain of a coordinate chart  $z$  valued in an open subset of  $[0, \infty)^a$ , by which the map  $\varphi$  is identified with the linear submersion  $z = x$ .*

*Proof.* Write the map in coordinates as  $(x, y) \mapsto (s, t)$ , supposing that the cowalls are the  $\langle dx^i \rangle$  and the  $\langle ds^j \rangle$ . Each  $s^j$  is a boundary function, so pulls back to some  $e^{u^i(x)} x^i$ . The differentials are cowalls, injectively mapped, so we can permute the  $x^i$  to arrange that  $i = j$ . Replace  $x^i$  by the function  $s^i$  as a coordinate function. So  $s^i = x^i$  for all  $s^i$ . Pull back the functions  $t^j$ , i.e. functions with differentials linearly independent of the cowalls. Their differentials remain independent of the cowalls, so their differentials  $dt^j$  are linearly independent of the various  $dx^i$ . Replace various  $y^j$  coordinate functions by these  $t^j$  functions.  $\square$

**Corollary 19.3** (The inverse function theorem). *Suppose that  $f: P \rightarrow Q$  is a  $C^k$  map of manifolds with corners, of equal dimension, and  $p_0 \in P$  is a regular point of  $f$ . Then, after perhaps replacing  $P$  and  $Q$  by neighborhoods of  $p_0$  and  $f(p_0)$ ,  $f$  is a  $C^k$  diffeomorphism.*

**Theorem 19.4** (Sard's theorem). *Suppose that  $\varphi: P \rightarrow Q$  is a continuously differentiable map of manifolds with corners of equal dimension. The set of critical values of  $\varphi$  has measure zero in  $Q$ . In particular,  $\varphi$  has regular values.*

**19.26** Prove Sard's theorem.

**Theorem 19.5.** [The Brouwer fixed point theorem] *Every smooth map  $\varphi: \bar{B} \rightarrow \bar{B}$  of a closed ball  $\bar{B}$  in Euclidean space has a fixed point.*

*Proof.* Let  $S := \partial B$  be the boundary sphere. Suppose  $\varphi: \bar{B} \rightarrow \bar{B}$  has no fixed points. Let  $g: \bar{B} \rightarrow S$  take each point  $x \in \bar{B}$  to the point  $y = g(x) \in S$  at which the line from  $\varphi(x)$  through  $x$  hits  $S$ . The reader can prove that  $g$  is a smooth map with  $g(x) = x$  for  $x \in S$ . By Sard's theorem,  $g$  has a regular value, say  $y_0 \in S$ , and the level set  $C := \{x \in \bar{B} \mid g(x) = y_0\}$  is a smooth curve. Because  $\bar{B}$  is compact and  $C \subset \bar{B}$  is closed,  $C$  is compact. By our classification of compact curves in problem 19.17 on page 167,  $C$  is a union of closed intervals and circles, so  $C$  has an even number of ends. By the rank theorem, near any point of  $C$ ,  $g$  is locally expressible in some coordinates as a linear surjective map, so the level sets of  $g$  have ends only on the boundary  $S$ , so  $C$  has end points only on  $S$ . Note that  $C \cap S$  contains  $y_0$  as an end point of  $C$ . If  $x \in C \cap S$ , then  $x \in S$  so  $g(x) = x$ , but  $x \in C$  so  $g(x) = y_0$ , so  $x = y_0$  is the unique point in  $C \cap S$ . Hence  $C$  has only one end, a contradiction.  $\square$

## Immersion and embeddings

A  $C^k$  immersion  $\varphi: S \rightarrow M$  of manifolds with corners is a  $C^k$  tidy map with injective differential. An *embedding* is a proper injective immersion, hence its image is an embedded submanifold with corners.

**19.27** Tidy? Immersed? Embedded?

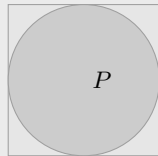


**19.28** Tidy? Immersed? Embedded?



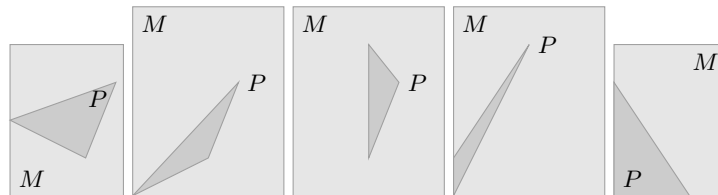
A  $C^k$  *embedded submanifold* is a subset which can be equipped with the structure of a  $C^k$  manifold with corners, so that the inclusion map is an embedding; it follows that the  $C^k$  functions on an embedded submanifold are the pulled back  $C^k$  functions.

A topological embedding given by a smooth proper injective map with injective differential might not be an embedding, or even an immersion, according to



these definitions.  $M$

**19.29** Which are embedded?



**19.30** Suppose that  $M$  is a closed unit square in the plane, and that  $S \subset M$  is the intersection of  $M$  with a line  $\ell$ . For which lines  $\ell$  is  $S \subset M$  embedded?

**19.31** Prove that the line segment between any two corners of a box is embedded.

**19.32** If we slice a box by a plane, the intersection is embedded in the box just when the plane misses the vertices.

**Proposition 19.6.** A  $C^k$  submanifold with corners  $S \subset M$ ,  $k \geq 1$ , is embedded just when, near any point  $p_0 \in S$ , there are  $C^k$  coordinates  $x^1, x^2, \dots, x^n$  on  $M$  valued in the standard corner, i.e. with  $x^i \geq 0$ , in which  $S$  is a locally cut out by equations and inequalities each of the form  $x^i = x^j$ ,  $x^i = 0$ ,  $x^i = c^i$  or  $x^i \geq c^i$  for some  $c^i > 0$ .

*Proof.* We need only prove the result locally, so can assume that  $S \subseteq [0, \infty)^p$  and  $M \subseteq [0, \infty)^n$  are open subsets and we have an embedding  $\varphi: S \rightarrow M$ . If  $S$  satisfies  $x^i = 0$ , replace  $M$  by the submanifold of  $M$  in which that equation is satisfied, so we can assume that  $S$  does not satisfy any such equations. In particular, every  $x^i$

pulls back to a boundary function on  $S$ . There might be several  $x^i$  pulling back to boundary functions vanishing along the same wall of  $S$ . Suppose for example that  $x^1$  and  $x^2$  pull back to  $S$  to become boundary functions for the same wall. Then on  $S$ ,  $x_2 = e^u x_1$ , some  $C^{k-1}$  function  $u$  on  $S$ . We can extend  $u$  to be defined nearby in  $M$ . By problem 19.8 on page 166, we can make new coordinates, replacing  $x^2$  by  $\tilde{x}^1 := e^u x^1$ . We have arranged that on  $S$ ,  $x^2 = x^1$ . Replace  $M$  by the submanifold of  $M$  on which  $x^2 = x^1$ . So we can assume that each  $x^i$  pulls back to  $S$  to become a boundary function on a wall of  $S$  or to be positive. The submanifold with corners  $S$  is cut out by  $C^k$  inequalities  $f^j \geq 0$  and equations  $g^k \geq 0$  near our given point, for  $C^k$  functions  $f^j, g^k$  defined nearby on  $M$ . Add a constant to each  $f^j$  and to each  $g^k$  so that we can rewrite these as  $f^j \geq c^j$  or  $g^k = c^k$  for positive constants. After perhaps adding a constant, each  $f^j$  and each  $g^k$  is a boundary function. So change variables on  $M$  to make each  $f^j$  and each  $g^k$  one of the  $x^i$  not vanishing at that point of  $S$ .  $\square$

For any  $C^k$  submersion  $\varphi: P \rightarrow Q$ , the preimage  $S_P := \varphi^{-1}S_Q$  of any  $C^k$  embedded submanifold  $S_Q \subseteq Q$  with corners is a  $C^k$  embedded submanifold with corners and

$$\dim Q - \dim S_Q = \dim P - \dim S_P.$$

The boundary  $\partial S_P$  consists of the preimage  $f^{-1}\partial S_Q$  together with the intersection of  $S_P$  with the various boundary points of  $P$  which  $S_P$  shares.

**Theorem 19.7.** *Take a compact manifold  $M$  with corners; let  $n$  be the dimension of  $M$ . Take an integer  $N \geq 2n$ , a smooth map  $\varphi: M \rightarrow \mathbb{R}^N$ , and a real number  $\varepsilon > 0$ . There is a smooth immersion as close as we like to  $\varphi$ , i.e. for any  $\varepsilon > 0$  we can pick an immersion  $\psi: M \rightarrow \mathbb{R}^N$  so that, for any point  $p \in M$ ,  $|\psi(p) - \varphi(p)| < \varepsilon$ . If  $N \geq 2n + 1$ , we can ensure that  $\psi$  is an embedding.*

The proof is identical to that of theorems 18.3 on page 156 and 18.4 on page 157.

## Seeley's extension theorem

This section can be omitted without loss of continuity.

It is difficult to make precise the conditions under which a smooth function on an open set in Euclidean space extends to be smooth to the boundary (i.e., by our definition of smooth, extends to a larger open set containing the boundary of the original open set). We discuss the problem only for extending across a boundary which is cut out locally by a single boundary function.

**19.33** Take a homogeneous polynomial  $p$ , say of degree  $d$  on  $\mathbb{R}^n$ , and a positive number. Prove that there is a smooth function of compact support  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , supported inside any open set we choose containing the origin, so that  $f = p$  near the origin and all derivatives of  $f$  of order up to  $d - 1$  are bounded in absolute value throughout  $\mathbb{R}^n$  by that positive number.

**Theorem 19.8** (Borel). *Every power series, in finitely many variables, arises as the Taylor series of a smooth function.*

*Proof.* Any formal power series is a sum of homogeneous polynomials  $p_\ell$  of degree  $\ell = 0, 1, 2, \dots$ . Choose some  $f_\ell$  so that  $f_\ell = p_\ell$  near the origin and all derivatives of  $f_\ell$  of order at most  $\ell - 1$  are at most some  $\varepsilon_\ell$ , by problem 19.33 on the preceding page. Pick these  $\varepsilon_\ell$  as you like so that their sum is finite. Let

$$f = f_0 + f_1 + f_2 + \dots$$

□

Problem 19.33 on the previous page also works for polynomials with smooth function coefficients:

**19.34** Take

- a number  $\varepsilon > 0$  and
- an open set  $U \subset \mathbb{R}^q$  containing the origin and
- a smooth function  $g: \mathbb{R}^p \times U \rightarrow \mathbb{R}$ , a homogeneous polynomial of degree  $d$  in the first  $p$  variables, with coefficients smooth functions on  $U$ .

Prove that there is a smooth function of compact support  $f: \mathbb{R}^{p+q} \rightarrow \mathbb{R}$  so that  $f = g$  near the origin and all derivatives of  $f$  of order up to  $d - 1$  in the first  $p$  variables are bounded throughout  $\mathbb{R}^{p+q}$  in absolute value by  $\varepsilon$ .

The proof of Borel's theorem (theorem 19.8 on the preceding page) then immediately gives

**Theorem 19.9** (Borel II). *Every power series on  $\mathbb{R}^p$  with coefficients smooth functions in some open set  $U \subseteq \mathbb{R}^q$  arises as the Taylor series of a smooth function on  $\mathbb{R}^{p+q}$ .*

**Theorem 19.10** (Seeley extension theorem). *Take an open set  $U \subset \mathbb{R}^n$  with smooth boundary. A smooth function  $f: U \rightarrow \mathbb{R}$  extends to a smooth function on  $\bar{U}$  just when its value and partial derivatives of all orders have smooth limits on  $\partial U$ .*

*Proof.* The problem is local, since we can patch together smooth functions which agree on overlaps of their domains. We can apply a diffeomorphism to some open set to arrange that  $U$  is an open subset of a half-space  $t > 0$  in  $\mathbb{R}^n$  with points written as  $(t, x) \in \mathbb{R}^n$ , and we need only extend to  $t = 0$ . We take the limits of these derivatives to form a formal power series in  $t$ , with coefficients smooth functions of  $x$ ; apply Borel's theorem (theorem 19.9). □

Seeley proved a stronger result, as did Whitney; see [6] for an overview of extension theorems.

**19.35** Prove that every smooth function in Euclidean space invariant under rotations around the origin is a smooth function of squared distance to the origin.

**19.36** Prove that the smooth vector fields on Euclidean space invariant under rotations are those of the form  $fX$  where  $X(x) = x$  is the radial vector field and  $f$  is a smooth function of squared radius.

## Notes

We recommend the terminology *manifold with corners*. The term *manifold with boundary and corners* is often used for what we have called *manifold with corners*; a *manifold with boundary* is then a manifold with corners which has at most one wall through each point. Corners allow cutting and pasting, while the theory of *manifolds with boundary* in that sense is not nearly so useful.



## Chapter 20

### Transverse submanifolds with corners

This chapter can be omitted without loss of continuity. We define a notion of transversality for corners, so that transverse submanifolds with corners intersect in a manifold with corners.

#### Conormals and cofaces: intuition

Take a manifold with corners  $M$  and an embedded submanifold with corners  $S \subseteq M$ . Locally  $S$  consists of a submanifold, cut out by equations  $f^i = 0$ , which ends in some walls of  $S$ , cut out by inequalities  $g^j \geq 0$ . We need to keep track of the differentials  $df^i$  of these equations, the conormals. Each inequality  $g^j \geq 0$  ends in a wall of  $S$ :  $g^j = 0$ . These walls intersect in a face of  $S$ . The smallest face of  $S$  through a given point is the intersection of the walls through that point. In other words, that smallest face arises when we turn all of the inequalities  $g^j \geq 0$  into equations  $g^j = 0$ . The differentials  $df^i, dg^j$  of all equations and inequalities form the conormal *not* of  $S$  but *of that face*; we call the set of these differentials the *coface*.

#### Conormals and cofaces: definitions

Now we make precise definitions, and forget about walls and faces. The definition of conormal is unchanged: a *conormal* of a submanifold with corners  $S \subset M$  at a point  $m \in S$  is a cotangent vector in  $T_m^*M$  vanishing on the tangent space  $T_mS$ . The conormal space  $S_{M,s_0}^\perp \subset T_m^*M$  is the set of conormals, i.e. the differentials of functions  $f$  constant on  $S$  and defined on a neighborhood of  $m$  in  $M$ . If the ambient manifold  $M$  is obvious, let  $S_{s_0}^\perp := S_{M,s_0}^\perp$ .

A *coface differential* is a covector  $\xi \in T_{s_0}^*M$  so that  $\xi$  restricted to  $T_{s_0}S$  is a linear combination of cowalls of  $S$ . The *coface*  $S_{M,s_0}^+$  is the set of coface differentials. So  $S_{s_0}^\perp \subseteq S_{M,s_0}^+$ . The coface of  $S$  as a submanifold of  $S$  itself might not be trivial, so we need to specify the ambient manifold  $M$  in the notation unless  $S = M$ .

**20.1** Take a manifold  $M$  without boundary or corners. Take an embedded submanifold  $S = \{f_i \geq 0, g_j = 0\} \subset M$ , where the various  $f_i, g_j$  have linearly independent differentials. Prove that the coface  $S_{M,s_0}^+$  is precisely the span of the  $df_i(s_0)$  and  $dg_j(s_0)$ .

For an embedded submanifold  $S \subset M$ ,  $M_{M,s_0}^+ \subset S_{M,s_0}^+$ .

### Transverse embedded submanifolds

In  $\mathbb{R}^3$ , the sets  $P := \{x, y \geq 0\}$  and  $Q := \{z \geq 0\}$  are submanifolds with corners. Their intersection is also a submanifold with corners:  $P \cap Q = \{x, y, z \geq 0\}$ .

**20.2** If  $M$  is any manifold and  $f, g, h$  are three smooth functions with linearly independent differentials, prove that  $P := \{f, g \geq 0\}$  and  $Q := \{h \geq 0\}$  are manifolds with corners as is their intersection.

If  $M = \{x \geq 0\} \subset \mathbb{R}^3$  and  $P = \{x, y \geq 0\}$  and  $Q = \{x, z \geq 0\}$ , it is clear that  $P \cap Q = \{x, y, z \geq 0\}$  is a submanifold with corners of  $M$ .

Suppose that  $M$  is any manifold with boundary and  $f, g, h$  are three smooth functions with linearly independent differentials, so that  $f \geq 0$  on  $M$  and the boundary of  $M$  is  $(f = 0)$ . Then  $P := \{f, g \geq 0\}$  and  $Q := \{f, h \geq 0\}$  are manifolds with corners and their intersection  $P \cap Q = \{f, g, h \geq 0\}$  is a manifold with corners. We only need to have independent differentials for those equations and inequalities that cut out  $P$  and  $Q$ , modulo those that cut out  $M$ .

Embedded submanifolds  $P, Q \subset M$  with corners are *transverse* at some point  $m_0$  if

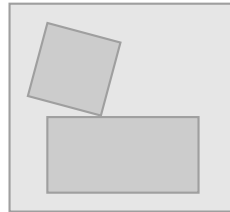
- a.  $P_{M, m_0}^+ \cap Q_{M, m_0}^+ = M_{M, m_0}^+$  and
- b.  $P_{m_0}^\perp \cap Q_{m_0}^\perp = 0$ .

Submanifolds are *transverse* if transverse at every intersection point.

If  $P$  and  $Q$  don't intersect, then they are transverse.

**20.3** Which are transverse?

a.

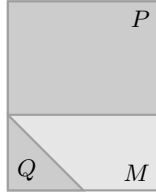


- b. Inside  $M = (0 \leq x, y, z \leq 1) \subset \mathbb{R}^3$  the unit cube, let  $P$  be the unit square in the horizontal plane with corners at the origin and at  $(1, 1, 0)$  and let  $Q$  be the unit square in the vertical plane  $y = 0$  with corners at the origin and at  $(1, 0, 1)$ .





c. Here  $P$  and  $Q$  sit inside a rectangle  $M$ .



d. The planes  $P := \{x = 0, y \geq 0\}$  and  $Q := \{z = 0, y \leq 0\}$  inside  $M := \mathbb{R}^3$ .

e. The planes  $P := \{x = 0, y \geq 0\}$  and  $Q := \{z = 0, y \leq 1\}$  inside  $M := \mathbb{R}^3$ .

**Theorem 20.1.** *Transverse embedded  $C^k$  submanifolds with corners intersect in a  $C^k$  embedded submanifolds with corners,  $k \geq 1$ . The codimension of the intersection is the sum of the codimensions of the submanifolds. There is an exact sequence of vector spaces*

$$0 \rightarrow T_m(P \cap Q) \rightarrow T_m P \oplus T_m Q \rightarrow T_m M \rightarrow 0.$$

*Proof.* By proposition 19.6 on page 172, in some coordinates  $x^1, \dots, x^n$  on  $M$ , we can write  $P$  as cut out by equations and inequalities of the form  $x^i = 0$ ,  $x^i = x^j$ ,  $x^i = c > 0$  and  $x^i \geq c > 0$  near some given point, for various positive constants all given the same name  $c$ . Similarly, in some coordinates  $y^1, \dots, y^n$  on  $M$ ,  $Q$  is cut out by similar equations and inequalities in those variables near that same point. The  $y^i$  are boundary functions so, up to reordering, positive multiples of  $x^i$ . So  $Q$  cut out by equations and inequalities each of the form  $x^i = 0$ ,  $x^i = e^u x^j$ ,  $e^u x^i = c > 0$  and  $e^u x^i \geq c > 0$  for various functions all called  $u$  and various constants  $c > 0$ .

Suppose we have some equation  $x^i = x^j$  on  $P$  and also an equation  $x^i = e^u x^j$  on  $Q$  on the same variables. Since the conormals intersect only at zero,  $u \neq 0$  at our point. So the pair of equations is equivalent to  $x^i = 0$  and  $x^j = 0$ .

Similarly an inequality of the form  $e^u x^i \geq c$  with differential independent of all of the others allows us to make  $e^u x^i$  one of our coordinate functions on  $M$  near our point, so becomes  $x^i \geq c$ .

So all of the equations put together cut out a smooth embedded submanifold of expected dimension.

The equations and inequalities on  $P$  and on  $Q$  have no differentials in common, except for those of  $x^i \geq 0$ , satisfied on  $M$  so on both  $P$  and  $Q$ . By linear independence of their differentials, put together they produce a corner on our embedded submanifold. The explicit form of the equations and inequalities ensure that the resulting corner is tidy so embedded.  $\square$

**20.4** Give an example where  $P, Q \subset M$  are transverse embedded submanifolds with intersection having components of different dimension.

## Transverse maps

A *conormal* of a map is a cotangent vector pulling back to zero; the conormals span the *conormal space*. A *coface differential* is a cotangent vector pulling back to a linear combination of cowalls; coface differentials at a point span the *coface*, denoted  $\varphi_{p_0}^+$  for a map  $\varphi: P \rightarrow Q$  and a point  $p_0 \in P$ . Two  $C^1$  tidy maps  $\varphi: P \rightarrow M$ ,  $\psi: Q \rightarrow M$  are *transverse* at points of  $P$  and  $Q$  if either  $\varphi \neq \psi$  or

- $0 = \ker \varphi^* \cap \ker \psi^*$  and
- $M^+ = \varphi^+ \cap \psi^+$ .

They are *transverse* if transverse everywhere.

**Theorem 20.2.** *If two  $C^k$  maps  $\varphi: P \rightarrow M$  and  $\psi: Q \rightarrow M$  of manifolds with corners are transverse,  $k \geq 1$ , then the fiber product*

$$P \times_M Q := \{(p, q) \mid \varphi(p) = \psi(q)\} \subset P \times Q$$

*is an embedded closed  $C^k$  submanifold with corners of codimension equal to the sum of the codimensions of  $\varphi$  and  $\psi$ .*

*Proof.* Pick local coordinates  $x$  on  $P$ ,  $y$  on  $Q$ ,  $z$  on  $M$ , valued in open subsets of the standard corners  $[0, \infty)^p$ ,  $[0, \infty)^q$ ,  $[0, \infty)^n$ . Extend the maps  $z = \varphi(x)$ ,  $z = \psi(y)$  to be defined on open sets in  $\mathbb{R}^p, \mathbb{R}^q$  mapping to an open subset of  $\mathbb{R}^n$ . These maps are transverse by hypothesis, so the set of points  $(x, y)$  so that  $\varphi(x) = \psi(y)$  is a submanifold of  $\mathbb{R}^{p+q}$  of dimension  $p + q - n$ . This is the fiber product of the extended maps.

Consider the various inequalities  $x^i \geq 0, y^j \geq 0$  that cut out our standard corners in  $P$  and  $Q$ . We need precisely to prove that each of these  $x^i, y^j$  pulls back to our fiber product to vanish or to be a boundary function.

The imposition of an equation  $z = \varphi(x)$ , because  $\varphi$  is tidy, forces each  $z^k$  to pull back to zero or a boundary function. If  $z^k$  pulls back to zero via either  $\varphi$  or  $\psi$ , then it does under both, and our extended fiber product lies inside  $z^k = 0$ , and we can reduce the problem to eliminate that  $z^k$  variable. So we can assume that  $\varphi$  and  $\psi$  both pull back every  $z^k$  function to become a boundary function of  $x$  and also of  $y$ .

Take some point of  $M$  in the image of  $\varphi$  and  $\psi$ . For each inequality  $z^k \geq 0$  we find there,  $z^k \geq 0$  pulls back by  $\varphi$  to be some wall  $x^i \geq 0$ , and similarly to be some wall  $y^j \geq 0$ . These differentials  $dz^k$  then span precisely the common cowalls of  $\varphi, \psi$ , i.e. we can write out various relations  $dy^j = dx^i$  (up to positive multiple) for various  $y^j$ , one for each  $z^k$ . So the inequalities of our fiber product can all be rewritten as equations that force each  $y^j$  to vanish just where that  $z^k$  does, which is just where that  $x^i$  does, i.e. an equation  $y^j = y^j(x)$  making  $y^j$  a tidy function.  $\square$

## Avoiding sets

A subset  $S \subseteq M$  of a manifold with corners  $M$  is of *dimension at most  $k$*  if  $S$  lies in the union of a sequence of embedded submanifolds with corners of dimension at most  $k$ . A set is of *dimension  $k$*  if it is of dimension at most  $k$  but not at most  $k - 1$ . A set  $S \subseteq M$  is of *codimension  $k$*  if it is of dimension  $\dim M - k$ .

Any manifold with corners  $M$  has such a covering by embedded submanifolds: its interior, the interior of its boundary, and so on. Hence we can assume that  $S$  is covered by embedded submanifolds without corners.

A map  $\varphi: P \rightarrow Q$  is *transverse* to a set  $S \subset Q$  if it is transverse to every one of the embedded submanifolds in a sequence covering  $S$ .

**Lemma 20.3.** *Suppose that  $S \subset P$  is a subset of a manifold with corners, and that  $\varphi: P \rightarrow Q$  is a smooth map. For any point  $q \in Q$  outside some measure zero set, the fiber  $P_q := \pi^{-1}\{q\}$  is transverse to  $S$ .*

The proof is the same as for lemma 18.7 on page 163, after cutting out the (measure zero) corners of  $Q$ .



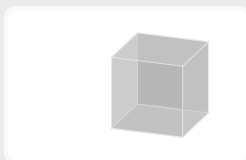
## Chapter 21

### Boundaries and corners

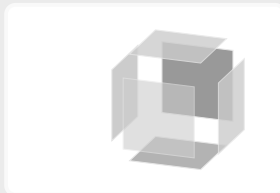
#### What is wrong with boundaries of corners

The *naive boundary* of a manifold with corners is the set of points where one or more of the coordinates vanish, in a chart valued in the standard corner  $[0, \infty)^n$ . The problem: the naive boundary is usually *not* itself a manifold with corners.

The boundary of a solid cube is *not* a surface with corners.



Picture it as six squares glued together along edges:



These squares *are* manifolds with corners, so the gluing is causing the trouble.

#### What is a boundary point

The naive boundary of the teardrop  is , which is *not* a 1-dimensional manifold with corners, as it contains a sharp “bend”. We want to treat the corner point as a “multiple point”, as if it were two different points “glued together”. If you stand near the corner point , you seem to see two walls meeting the corner point, but the boundary consists of a single connected wall.

A *boundary point* of a manifold  $M$  with corners is a pair  $(m, \ell)$  so that  $m \in M$  and  $\ell$  is a cowall at  $m$ , i.e. the span  $\ell = \langle dx^i \rangle$  of some  $dx^i$  for some coordinate  $x^i$  vanishing at  $m$ . Note that  $\ell$  is *not* a covector, but a ray spanned by a covector. The *boundary*  $\partial M$  is the set of boundary points. The *boundary attaching map* is

$(m, \ell) \in \partial M \mapsto m \in M$ ; its image is the *naive boundary*, also called the *attached boundary*.

The square  $X = [0, 1] \times [0, 1]$  has boundary  $\partial X$  consisting of the four *disjoint* curves

$$\begin{aligned} & \{ ((0, y), \langle dx \rangle) \mid 0 \leq y \leq 1 \}, \\ & \{ ((1, y), \langle -dx \rangle) \mid 0 \leq y \leq 1 \}, \\ & \{ ((x, 0), \langle dy \rangle) \mid 0 \leq x \leq 1 \}, \\ & \{ ((x, 1), \langle -dy \rangle) \mid 0 \leq x \leq 1 \}, \end{aligned}$$

Each of these curves has a boundary consisting of two points, and those points are in total 8 distinct points.

For each point  $(m, \ell) \in \partial M$ , take coordinates  $x^1, \dots, x^n$  centered at  $m$ , so that  $\ell = \langle dx^1 \rangle$ , and then  $x^2, \dots, x^n$  are coordinates on  $\partial M$  near  $(m, \ell)$ , the associated *boundary coordinates*.

**Theorem 21.1.** *Given any atlas for a  $C^k$  manifold with corners  $M$ , the set of its boundary charts is an atlas for  $\partial M$ , making  $\partial M$  a  $C^k$  manifold with corners. The boundary attaching map is  $C^k$ , of full rank, proper, and locally injective.*

**21.1** Prove that the boundary attaching map is a smooth immersion. Give an example of a manifold with corners  $M$  for which the attaching map is not an injection.

**21.2** Prove that the boundary attaching map is proper.

**21.3** Prove that  $\partial M \subset \mathbb{P}T^*M$  is an embedded submanifold with corners.

The connected components of  $\partial M$  are the *walls* of  $M$ .

**Theorem 21.2.** *Suppose that  $M$  is a manifold with corners and that  $p: \hat{M} \rightarrow M$  is a covering map. Suppose that the preimage in  $\hat{M}$  of each path component of  $M$  has countably many path components. Then  $\hat{M}$  admits a unique smooth structure as a manifold with corners, compatible with its topology, for which the map  $p$  is a local diffeomorphism. The map  $\partial \hat{M} \rightarrow \hat{M}$ , defined in the obvious way, is also a smooth covering map.*

**Lemma 21.3.** *If a group  $\Gamma$  acts on a manifold  $M$  with corners, by diffeomorphisms, freely and properly, then the quotient space  $\bar{M} := \Gamma \backslash M$  has a unique manifold structure with corners, so that the quotient map  $M \rightarrow \bar{M}$  is a local diffeomorphism and a covering map. The map  $\partial M \rightarrow \partial \bar{M}$  is also a smooth covering map.*

The proof is as for lemma 3.4 on page 29.

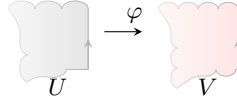
**21.4** Take a square  $X$ ; what is  $\partial X$ , and what is  $\partial^2 X$ ? Now the same for a cube.



## Orientation

If a manifold  $M$  with corners has an orientation, we already know how to orient the boundary: just follow the recipe for a box, and then apply that recipe to a chart near some point. In other words, pick coordinates  $x^1, \dots, x^n$  on  $M$  associated to a positively oriented chart, and take  $x^2, \dots, x^n$  as coordinates on the boundary wall  $x^1 = 0$ , i.e. on the set of points  $(m, \ell)$  for which  $x^1(m) = 0$  and  $\ell = \langle dx^1 \rangle$ .

**Lemma 21.4.** *If  $\varphi: U \rightarrow V$  is a positively oriented diffeomorphism between two open subsets of the standard corner, then the induced map  $\partial U \rightarrow \partial V$  is orientation preserving. Consequently, along each wall of an oriented manifold with corners an induced orientation is consistently defined.*



*Proof.* For simplicity, we assume that  $U$  and  $V$  are open subsets of the left half space in the plane  $x \leq 0$ . The proof in any dimension is similar. Write our map as

$$(s, t) = \varphi(x, y).$$

Then  $U$  lies in the set  $x \leq 0$ , and  $V$  lies in  $s \leq 0$ . The map along the wall is just

$$(0, y) \mapsto (0, t(0, y)),$$

i.e.  $s(0, y) = 0$ . Therefore  $\frac{\partial s}{\partial y} = 0$  along the wall. Along the wall,  $\frac{\partial s}{\partial x} \geq 0$ ; otherwise moving  $x$  to the left a little would make  $s > 0$ . So

$$\varphi'(0, y) = \begin{pmatrix} \frac{\partial s}{\partial x} & 0 \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix}$$

and  $0 < \det \varphi'(0, y) = \frac{\partial s}{\partial x} \frac{\partial t}{\partial y}$ , so  $\frac{\partial s}{\partial x} > 0$  and  $\frac{\partial t}{\partial y} > 0$ .  $\square$

For any manifold with corners  $M$ , we can obviously identify  $\partial^2 M$  with the set of tuples  $(m, \ell_1, \ell_2)$  for which  $m \in M$  and  $\ell_1$  and  $\ell_2$  are distinct conormal lines at  $m$ . Similarly,  $\partial^k M$  is identified with the set of all tuples  $(m, \ell_1, \ell_2, \dots, \ell_k)$  for which  $m \in M$  and  $\ell_1, \ell_2, \dots, \ell_k$  are distinct conormal lines at  $m$ . There is an obvious map  $\partial^2 M \rightarrow \partial^2 M$  given by  $(m, \ell_1, \ell_2) \mapsto (m, \ell_2, \ell_1)$ , reversing orientation.

**21.5** Given an oriented manifold with corners  $M$ , write  $-M$  for the same manifold with corners but with the opposite orientation. Suppose that  $P$  and  $Q$  are manifolds with corners, of dimensions  $p$  and  $q$ . Prove that  $-(P \times Q) = (-P) \times Q = P \times (-Q)$ . Denote disjoint unions of sets as  $\sqcup$ . Prove the *Leibnitz rule*

$$\partial(P \times Q) = ((\partial P) \times Q) \sqcup (-1)^p (P \times (\partial Q)).$$





## Chapter 22

### Homotopy and degree

We count how many times a point  $y$  is struck by a map  $y = \varphi(x)$  and watch how this changes when we deform the map.

#### Degree

Suppose that  $\varphi: P \rightarrow Q$  is a continuously differentiable map between two oriented manifolds with corners, of equal dimension. The *sign* of  $\varphi$  at a regular point  $p \in P$  is 1 if  $\varphi$  is orientation preserving near  $p$ , and  $-1$  if  $\varphi$  is orientation reversing near  $p$ . Suppose that  $q_0 \in Q$  is a regular value and that there are only finitely points  $p \in P$  mapped to  $q_0$  by  $\varphi$ . The *degree* of  $\varphi$  above  $q_0$  is the sum of signs at these points  $p$ :

$$\deg_{q_0} \varphi = \sum_{\varphi(p)=q_0} (\text{sign of } \varphi \text{ at } p).$$

Take a  $C^1$  map  $\varphi: S^1 \rightarrow S^1$  of the circle to itself. Recall that  $\varphi$  has an *angle function*  $\Phi$  so that  $\varphi(e^{i\theta}) = e^{i\Phi(\theta)}$  as in problem 3.3 on page 24. We will see that the degree of  $\varphi$  at any regular point is  $\Phi(2\pi) - \Phi(0)$ .

The identity map  $M \rightarrow M$  of any manifold with corners has degree one at every point.

A constant map  $P \rightarrow Q$  of any manifold with corners, mapping to a single point  $q_0 \in Q$ , has degree zero at all points but  $q_0$ .

#### Proper maps

A map  $\varphi: P \rightarrow Q$  between manifolds with corners is *proper* if every compact set  $K \subset Q$  has compact preimage  $\varphi^{-1}K \subset P$ .



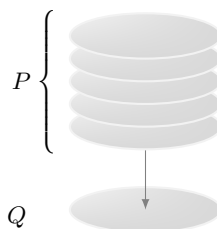
**22.1** Prove that every continuous map from a compact manifold with corners is proper.

**22.2** Which polynomial functions  $y = p(x)$  of one variable are proper?

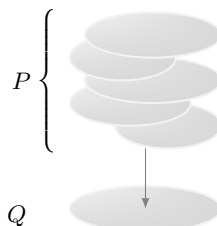
**Lemma 22.1.** *For any regular value  $q_0$  of any continuously differentiable proper map  $\varphi: P \rightarrow Q$  between equal dimensional manifolds with corners, there are only finitely many points  $p \in P$  mapping to  $q_0$ .*

*Proof.* If  $\varphi(p_0) = q_0$ ,  $\varphi$  is a diffeomorphism near  $p_0$ , and therefore no point close to  $p_0$  maps to  $q_0$  other than  $p_0$ . So every point of  $\varphi^{-1}\{q_0\}$  lies in an open set containing only that point; these open sets cover. By properness,  $\varphi^{-1}\{q_0\}$  is compact, so has a finite subcover.  $\square$

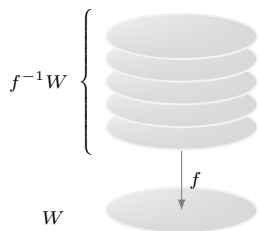
Imagine a map  $\varphi: P \rightarrow Q$  as a projection of a surface  $P$  to its shadow in a plane  $Q$ . “Above” each of those regular points in  $Q$ , there are various layers of  $P$ , known as *sheets*.



We have a local inverse map for each sheet. The degree of a proper map cannot change as we move along a curve in  $Q$ , as long as that curve only passes through regular points. If  $\varphi$  is an improper map, the sheets up in  $P$  might “run out”:

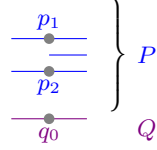


Then we don’t have an inverse map defined for that sheet. A *covering map*  $\varphi: P \rightarrow Q$  is a smooth map between manifolds of the same dimension (perhaps with corners) so that every point of  $Q$  lies in an open set  $W$  so that  $\varphi^{-1}W$  is a union of disjoint open sets called *sheets*, each mapped diffeomorphically to  $W$ . In a picture,  $P$  “above” the open set  $W \subset Q$  looks like a collection of sheets, all of which are diffeomorphic copies of  $W$ : the sheets don’t run out. If there are  $m$  sheets, the covering map is  $m$  to 1 (a finite covering); otherwise it is  $\infty$  to 1.



**Lemma 22.2.** *Suppose that  $\varphi: P \rightarrow Q$  is a proper continuously differentiable map between manifolds with corners of the same dimension. The set  $Q_{\text{reg}} \subset Q$  of regular values of  $\varphi$  is open. On the set  $P_{\text{reg}} = \varphi^{-1}Q_{\text{reg}}$  of regular points,  $\varphi|_{P_{\text{reg}}}: P_{\text{reg}} \rightarrow Q_{\text{reg}}$  is a finite covering map.*

*Proof.* Suppose that  $q_0 \in Q$  is a regular value. By lemma 22.1, there are finitely many points  $p \in P$  at which  $\varphi(p) = q_0$ , say  $p = p_1, p_2, \dots, p_m$ .



By the inverse function theorem, around each of these points  $p_i$ , there is some open set  $U_i \subset P$  and there is some open set  $W_i \subset Q$  so that

$$\varphi|_{U_i}: U_i \rightarrow W_i$$

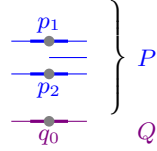
is a diffeomorphism. Let

$$W = W_1 \cap W_2 \cap \cdots \cap W_m.$$

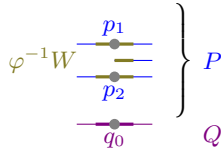
Replace  $U_i$  by  $U_i \cap \varphi^{-1}W$  so that

$$\varphi|_{U_i}: U_i \rightarrow W$$

is a diffeomorphism for each  $i$ .



In our picture,  $W$  is the thick purple line,  $U_1$  and  $U_2$  are the thick blue lines. The preimage  $\varphi^{-1}W$  consists of  $U_1$ ,  $U_2$ , and a little bit of the middle blue line:



We know that

$$U_1 \cup U_2 \cup \cdots \cup U_m \subseteq \varphi^{-1}W,$$

and we need only justify replacing  $\subseteq$  by  $=$ . Suppose that this fails. We can replace  $W$  by any smaller open set  $W' \subseteq W$  containing  $q_0$ , and take  $U'_i := U_i \cap \varphi^{-1}W'$ , and try again. So we can assume that  $W$  lies in a chart domain. Replacing  $Q$  by  $W$  and  $P$  by  $\varphi^{-1}W$ , we can even assume that  $Q$  is just an open set in Euclidean space. Replacing  $W$  by a smaller open set, we can assume that  $W$  is a ball whose closure lies in  $Q$ .

Take balls  $W_i$  around the origin, of radius going to zero. If we replace  $W$  by such a  $W_i$  and fail again, then there is some point  $p_i \in P$  with  $\varphi(p_i) \in W_i$  and  $p_i$  not in any  $U_j$ . These  $p_i$  all lie in the compact set  $\varphi^{-1}\bar{W}$ , so we can replace them by a convergent subsequence, say converging to a point  $p_0$ . By continuity of  $\varphi$ ,  $\varphi(p_0) = q_0$ . But then  $p_0$  is one of the points  $p_1, \dots, p_m$ , a contradiction.  $\square$

**22.3** Suppose that

- $\varphi: P \rightarrow Q$  is a proper continuously differentiable map,

- $P$  and  $Q$  are manifolds of equal dimension,
- this dimension is 2 or more,
- $Q$  is connected, and
- $\varphi$  has only finitely many critical points in any compact subset of  $P$ .

Prove that the set of regular values is path connected, and that any two regular values have the same number of preimages. Prove that  $\varphi$  is surjective. Use this to prove the *fundamental theorem of algebra*: that any nonconstant complex polynomial function of a complex variable has a root.

**22.4** Suppose that  $\varphi: P \rightarrow Q$  is smooth and that  $P$  and  $Q$  have equal dimension. Take embedded submanifolds  $P_0 \subset P$  and  $Q_0 \subset Q$  with corners. Suppose that  $\varphi$  restricts to  $P_0$  to give a finite covering map  $\varphi_0: P_0 \rightarrow Q_0$ , and that every point of  $P_0$  is a regular value of  $\varphi$ . Prove that  $\varphi$  restricts to a covering map between some open set  $U_P \subset P$  containing  $P_0$  and some open set  $U_Q \subset Q$  containing  $Q_0$ , with same number of sheets as  $P_0$  over  $Q_0$ .

A  $C^k$  homotopy between two  $C^k$  maps  $\varphi, \psi: P \rightarrow Q$  of manifolds,  $k \geq 0$ , is a  $C^k$  map

$$\Phi: [0, 1] \times P \rightarrow Q,$$

denoted  $\Phi_t(p)$  to mean  $\Phi(t, p)$ , so that  $\Phi_0 = \varphi$  and  $\Phi_1 = \psi$ .

**22.5** Suppose that  $\varphi: P \rightarrow S^n$  is a  $C^1$  map of a manifold  $P$  to the  $n$ -dimensional sphere. Suppose that  $\varphi$  is not onto. Prove that there is a  $C^1$  homotopy between  $\varphi$  and a constant map.

**22.6** Use the result of problem 3.3 on page 24 to prove that, for any  $C^k$  map  $\varphi: S^1 \rightarrow S^1$ , there is a  $C^k$  homotopy between  $\varphi$  and a map  $\psi(e^{i\theta}) = e^{in\theta}$ .

**22.7** Prove that, for any integers  $p \neq q$ ,  $e^{i\theta} \mapsto e^{ip\theta}$  is not  $C^0$  homotopic to  $e^{i\theta} \mapsto e^{iq\theta}$ .

**22.8** Suppose that  $P$  is a product of  $p$  circles,  $Q$  a product of  $q$  circle, and  $R$  a product of  $r$  circles. Associate to each continuous map  $P \rightarrow Q$  an integer matrix, determining the map up to homotopy, so that when continuous maps  $P \rightarrow Q$  and  $Q \rightarrow R$  compose, the associated matrices multiply.

**Lemma 22.3.** Suppose that  $P$  and  $Q$  are oriented manifolds with corners, of equal dimension. Suppose that  $\varphi: P \rightarrow Q$  and  $\psi: P \rightarrow Q$  are proper continuously differentiable maps taking attached boundary points to attached boundary points. Suppose that there is a proper continuously differentiable homotopy  $\Phi$  between  $\varphi$  and  $\psi$ , so that every  $\Phi_t$  takes attached boundary points to attached boundary points. Suppose that  $q_0 \in Q$  is a regular value for  $\varphi$  and for  $\psi$ . Then

$$\deg_{q_0} \varphi = \deg_{q_0} \psi.$$

*Proof.* Take a proper continuously differentiable homotopy  $\varphi \xrightarrow{\Phi} \psi$ . Suppose first that  $q_0$  is a regular value for  $\Phi$ , not lying on the boundary of  $Q$ . By the implicit function theorem,  $\Phi^{-1}(q_0) \subset [0, 1] \times P$  is a curve with corners:

$$\partial \Phi^{-1}\{q_0\} = \varphi^{-1}\{q_0\} \times \{0\} \cup \psi^{-1}\{q_0\} \times \{1\}.$$

This manifold is oriented, using the orientations for  $[0, 1]$ ,  $P$  and  $Q$ . By the solution of problem 19.17 on page 167, the curve is a union of circles and closed intervals, up to diffeomorphism. Each closed interval starts on  $\varphi^{-1}\{q_0\}$  and ends on  $\psi^{-1}\{q_0\}$ , drawing a path from one sheet to the other, and preserving orientation as we go along it.

Suppose that  $q_0$  is not a regular value of  $\Phi$ , or lies on the boundary of  $Q$ . Since  $q_0$  is regular for  $\varphi$ , by lemma 22.2 on page 188, every point close enough to  $q_0$  is regular for  $\varphi$ . Similarly all points close enough to  $q_0$  are regular for  $\psi$ . The set of critical points of  $\Phi$  cannot contain an open set by Sard's theorem and lemma 18.2 on page 156. Therefore  $\Phi$  and  $\varphi$  and  $\psi$  are all regular at points arbitrarily close to  $q_0$ , where our argument above tells us that  $\varphi$  and  $\psi$  have the same degree. The degrees of  $\varphi$  and  $\psi$  are therefore equal at all of those points. But the degree of  $\varphi$  doesn't change near  $q_0$ , since  $\varphi$  is regular at  $q_0$ , and the same is true for  $\psi$ .  $\square$

## The diffeomorphism group

The *reverse* of a homotopy  $\Phi_t(x)$  is  $\Phi_{1-t}(x)$ . A  $C^k$  homotopy  $\Phi_t: P \rightarrow Q$  is an *diffeotopy* from  $\Phi_0$  to  $\Phi_1$  if it is an embedding for all values of  $t$ . (An *isotopy* is a  $C^0$  diffeotopy.) Embeddings are *diffeotopic* if they occur as  $\Phi_0$  and  $\Phi_1$  for some diffeotopy.

**22.9** Prove that diffeotopic maps of oriented manifolds of equal dimension are either both orientation preserving or both orientation reversing.

Suppose that  $\Phi_t(x)$  is some homotopy. Pick a closed interval  $I = [a, b] \subset (0, 1)$ . Pick a smooth increasing function  $\tau(t)$ , constant equal to 0 for  $t \leq a$ , constant equal to 1 for  $t \geq b$ . Let  $\check{\Phi}(t, x) := \Phi(\tau(t), x)$ . So  $\check{\Phi}$  moves each point just where  $\Phi$  does, but is constant in  $t$  near  $t = 0$  and near  $t = 1$ ;  $\check{\Phi}$  is  $\Phi$  *made steady* outside  $I$ . If  $\Phi$  is an diffeotopy, so is  $\check{\Phi}$ .

Take two homotopies  $\Phi, \Psi$ . Suppose that  $\Phi_1 = \Psi_0$ . Imagine we naively compose them, into a homotopy which follows  $\Phi$  and then  $\Psi$ , following each at twice the usual speed to make sure that we travel along  $0 \leq t \leq 1$ . This might not be smooth when we switch from  $\Phi$  to  $\Psi$ . Let  $\check{\Phi}$  be  $\Phi$  made steady outside some interval and  $\check{\Psi}$  be  $\Psi$  made steady outside some interval. Let

$$(\Psi * \Phi)(t, x) := \begin{cases} \check{\Phi}_{2t}(x), & \text{if } 0 \leq t \leq 1/2, \\ \check{\Psi}_{2t-1}(x), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

This homotopy is as smooth as either  $\Phi$  or  $\Psi$ . Hence we can “smoothly compose smooth homotopies”.

**22.10** Prove that the diffeomorphisms of a manifold with corners which are diffeotopic to the identity form a group.

**22.11** Prove that every diffeomorphism  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  is diffeotopic to precisely one of:  $x \mapsto x$  or  $x \mapsto -x$ .

**22.12** Prove that every diffeomorphism  $\varphi: [0, 1] \rightarrow [0, 1]$  is diffeotopic to precisely one of:  $x \mapsto x$  or  $x \mapsto 1 - x$ .

**22.13** Prove that every diffeomorphism  $\varphi: [0, \infty) \rightarrow [0, \infty)$  is diffeotopic to  $x \mapsto x$ .

**22.14\*** Prove that every diffeomorphism of the circle is diffeotopic to precisely one of: the identity map or complex conjugation.

**22.15\*** Prove that every diffeomorphism of the unit circle arises from a diffeomorphism of the closed unit disk.

**22.16\*** The Alexander trick: Prove that every homeomorphism of a closed ball equal to the identity map on the boundary is diffeotopic to the identity map.

**22.17\*** Prove that any two homeomorphisms of a closed ball equal near the boundary are diffeotopic.

**22.18\*** Prove that every diffeomorphism of Euclidean space is diffeotopic to the identity map or reflection in a hyperplane.

**22.19** Suppose that  $P$  is a ball and  $Q$  is Euclidean space of the same dimension. Take two smooth embeddings  $\varphi, \psi: P \rightarrow Q$ . Take a reflection  $r: Q \rightarrow Q$ . Prove that there is a smooth homotopy through embeddings from  $\varphi$  to  $\psi$ , or else from  $\varphi$  to  $r \circ \psi$ .

The *support* of a diffeomorphism of a manifold with corners  $M$  is the closure of the set of points not fixed by the diffeomorphism. The *support* of an diffeotopy  $\Phi$  of a manifold with corners  $M$  is the closure of the set of points  $x$  on which  $\Phi_t(x) \neq x$  for some  $t$ .

**Lemma 22.4.** *For any two points  $m_0, m_1 \in M$  of a connected smooth manifold with corners, neither point contained in the boundary, there is a smooth compactly supported diffeotopy  $\Phi: [0, 1] \times M \rightarrow M$  between the identity map and a map  $\Phi_1$ , whose support does not touch the boundary of  $M$ , so that  $\Phi_1(m_0) = m_1$ .*

*Proof.* If  $m_0$  and  $m_1$  lie inside the domain  $U$  of some chart  $(U, \varphi)$ , and there is a closed ball inside  $\varphi(U)$  containing both  $m_0$  and  $m_1$ , then the result follows from problem 6.9 on page 70.

Let  $P \subseteq M \times M$  be the set of all pairs  $(m_0, m_1)$  so that there is such an diffeotopy  $\Phi$ . We have seen that  $P$  contains all points  $(m, m)$  and indeed contains an open subset around those points. Compose isotopies to see that if  $P$  contains some point  $(m_0, m_1)$ , it also contains all nearby points:  $P \subset M \times M$  is open. The same argument shows that the complement of  $P$  is open. Since  $M$  is connected, so is  $M \times M$ . Therefore the only nonempty subset of  $M \times M$  which is both open and closed is  $P = M \times M$ .  $\square$

**22.20** You pick finitely many points on a connected manifold, and I pick the same number of points of the same manifold. Prove that there is a diffeomorphism taking your points to mine.

**22.21** Suppose that  $\varphi_0, \varphi_1: \bar{B} \rightarrow M$  are two orientation preserving embeddings of a closed ball  $\bar{B} \subset \mathbb{R}^n$  to the interior of an oriented  $n$ -dimensional manifold with corners  $M$ . Prove that there is a compactly supported diffeotopy  $\mu$  of  $M$ , fixing every boundary point of  $M$ , with  $\mu_0$  the identity, and an diffeotopy  $\beta$  of  $\bar{B}$  with  $\beta_0$  the identity, so that  $\varphi_1 \circ \beta_1 = \mu_1 \circ \varphi_0$ .

An immediate consequence:

**Theorem 22.5.** *The connected sum  $M = M_0 \# M_1$  of two connected manifolds, obtained from the “gluing plan” of any ball embeddings  $M_0 \leftarrow B \rightarrow M_1$  that extend to closed ball embeddings  $M_0 \leftarrow \bar{B} \rightarrow M_1$ , is defined up to diffeomorphism independently of the embeddings, depending only on the diffeomorphism types of the summands  $M_0, M_1$ .*

It is this connected sum that is conventionally called *the* connected sum, although in principle we might use connected sums arising from ball embeddings that do not extend to the closed ball.

## Degree invariance

A map  $\varphi$  has *degree*  $k$  if its degree is  $k$  at every regular value. We can then replace the notation  $\deg_{q_0} \varphi$  by the notation  $\deg \varphi$ .

**Theorem 22.6.** *Consider a proper continuously differentiable map  $\varphi: P \rightarrow Q$  between oriented manifolds with corners, taking attached boundary points to attached boundary points. Suppose that  $Q$  is connected. The degree of  $\varphi$  is the same at all regular values, so the map has a degree.*

*Proof.* Pick two regular points  $q_0 \in Q$  and  $q_1 \in Q$  not in the boundary. By lemma 22.4 on the facing page, we can make an diffeotopy  $\Phi_t: Q \rightarrow Q$  so that  $\Phi_0(q) = q$  for all  $q$ , and  $\Phi_1(q_0) = q_1$ , and so that outside some compact set  $K$ ,  $\Phi_t(q) = q$  for all  $t$  and  $q$ . The maps  $\Phi_t \circ \varphi$  are proper and

$$\deg_{q_0} \varphi = \deg_{\Phi_t(q_0)} \Phi_t \circ \varphi$$

for any  $t$ , since these are counting the same points of  $P$ , and the same signs. By lemma 22.3 on page 190,

$$\deg_{q_1} \Phi_1 \circ \varphi = \deg_{q_1} \varphi,$$

since these are homotopic maps.

Regular values in the boundary lie in open sets of regular values, hence regular values in the interior, with the same degree.  $\square$

**22.22** Prove that the degree as a map of a nonconstant complex polynomial  $w = p(z)$  function of a complex variable is its degree as a polynomial.

**Theorem 22.7** (The hairy ball theorem). *A sphere  $S^n \subset \mathbb{R}^{n+1}$  admits a nowhere vanishing smooth vector field just when  $n$  is odd.*

*Proof.* Note that the tangent space to a sphere  $S^n$  at any point  $x \in S^n$  consists precisely of the vectors in  $\mathbb{R}^{n+1}$  perpendicular to  $x$ . If  $n$  is odd, say  $n = 2k + 1$ , then  $\mathbb{R}^{n+1} = \mathbb{R}^{2k+2} = \mathbb{C}^{k+1}$  admits the vector field, expressed in complex variables  $z = (z^1, \dots, z^{k+1})$ , as

$$X(z) = iz.$$

The reader can check that this is tangent to the unit sphere  $S^{2k+1}$ .

If  $n$  is even, suppose that there is a smooth vector field  $X$  tangent to  $S^n$ . Replace  $X$  by  $X/|X|$  to arrange that  $X$  is unit length. Define a map

$$\varphi_t(x) = \cos(2\pi t)x + \sin(2\pi t)X(x).$$

Then  $\varphi: S^n \times [0, 1] \rightarrow S^n$  is a homotopy between the identity map, at  $t = 0$ , and the antipodal map, at  $t = \pi$ . The degree of the antipodal map is  $-1$  for  $n$  even,  $1$  for  $n$  odd, as the reader can check.  $\square$

Another proof of the fundamental theorem of algebra. This is a very sophisticated example, for which we will not give many details. Write each element of the complex projective line as

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{CP}^1.$$

To each complex coefficient polynomial

$$p(z) = c_0 + c_1 z + \cdots + c_n z^n,$$

in a complex variable  $z$ , not everywhere zero, associate the point

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{CP}^n.$$

To any point

$$t = \left( \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}, \dots, \begin{bmatrix} a_n \\ b_n \end{bmatrix} \right) \in \mathbb{CP}^1 \times \cdots \times \mathbb{CP}^1 = (\mathbb{CP}^1)^n$$

associated first the polynomial

$$p(z) = (a_1 z - b_1)(a_2 z - b_2) \cdots (a_n z - b_n)$$

with roots

$$z = \frac{b_1}{a_1}, \dots, \frac{b_n}{a_n}$$

for all nonzero  $a_1, \dots, a_n$ . Note that  $p(z)$  simply drops degree for any  $a_k = 0$ . Expand  $p(z)$  out in coefficients

$$p(z) = c_0 + c_1 z + \cdots + c_n z^n,$$

and let  $\varphi(t) \in \mathbb{CP}^n$  be the point associated to those coefficients as above. The map

$$\varphi: (\mathbb{CP}^1)^n \rightarrow \mathbb{CP}^n$$

is very complicated. Here is the difficult bit: one can differentiate  $\varphi(t)$  near any point  $t$  at which all of the roots are distinct, with all  $a_k \neq 0$ , to find it has invertible and orientation preserving differential, a local diffeomorphism preserving orientation. The roots are unique up to reordering, so the degree of  $\varphi$  is  $n!$  above such a point. Hence  $\varphi$  is onto: every complex coefficient nonconstant polynomial  $p(z)$  splits into linear factors. This proof fails for the real numbers because the map preserves orientation on some open set and reverses it on some other open set.

**22.23** Following through the steps in the example above, using real numbers instead of complex, draw the image of the map  $\varphi: (\mathbb{RP}^1)^2 \rightarrow \mathbb{RP}^2$ . What is the preimage of each point of  $\mathbb{RP}^2$ ?



**22.24** Let  $M$  be the unit sphere around the origin in Euclidean space  $\mathbb{R}^{n+1}$ . Suppose that  $\varphi: M \rightarrow M$  is a continuous map. Suppose that, for any two points  $x, y \in M$ , the distance between  $\varphi(x), \varphi(y)$  is never more than that between  $x, y$ . Prove that  $\varphi$  is either (i) a rotation, of degree 1 or (ii) a composition of a rotation and a reflection, of degree  $(-1)^n$  or (iii) has image contained in a single hemisphere, of degree 0.

### Notes

We have seen that the group of diffeomorphisms of any connected manifold  $M$  acts transitively; indeed we see that this is already the case for the group of all diffeomorphisms  $\varphi: M \rightarrow M$  which are equal to the identity outside some compact set. The diffeomorphism group is a mystery even for some well known manifolds.



## Chapter 23

### Differential forms

We define differential forms and their algebraic operations. Differential forms are objects that can be integrated over manifolds, much like integrating functions in calculus, but the results are unchanged under diffeomorphism.

#### Integrals

Return to calculus of one variable: consider the integral of a function over a closed interval. The integral is the product of the average value of the function and the length of the interval. Even for a constant function, the integral depends on how long the interval is: integrals of functions are not diffeomorphism invariant.

On an interval  $I = [a, b]$ , write

$$\int_I f(x) dx$$

to mean

$$\int_a^b f(x) dx.$$

Think of the integral geometrically, so that we think of  $I$  as an abstract compact curve with boundary, and  $f: I \rightarrow \mathbb{R}$  a smooth function on that curve. Think of  $x$  as a chart diffeomorphically mapping  $I$  to an interval  $[a, b]$  of the real number line. If we change the chart  $x$  on  $I$  to another chart, we typically change the value of the integral. Changing chart, we could change both the length of the interval and the average value of the function, since we can squish and stretch portions of the interval; the stretched bits count more for the average of the function, while the squished bits count less, as we stretch and squish the graph of the function in horizontal directions. So integrals of functions depend on the chart in which we compute them.

But we *know* how to change variables in integrals. We change of variables by

$$\int_I f(x) dx = \int_J f(x(y)) \left| \frac{dx}{dy} \right| dy.$$

where  $f: I \rightarrow \mathbb{R}$  is our function,  $I$  our interval, and  $\varphi: I \rightarrow J$  is a diffeomorphism of intervals, which we write as  $y = y(x) = \varphi(x)$  and so  $x = x(y) = \varphi^{-1}(y)$ .

The reason for the absolute value is that our diffeomorphism might alter the direction of the intervals, i.e. the orientation. There is a serious problem here: integrals with signs that depend on orientation arise naturally in fluid mechanics, electrodynamics and topology, so might prefer that the signs reverse when we reverse orientation. So we might want to drop the absolute value. For the moment, ignore

this problem by supposing that  $\varphi$  is orientation preserving, and then the absolute value drops out because  $dx/dy > 0$ .

Think of  $I$  and  $J$  as abstract compact oriented curves with boundary, and of  $x$  and  $y$  as positively oriented charts diffeomorphically mapping  $I$  and  $J$  to intervals of the real number line and  $\varphi: I \rightarrow J$  an orientation preserving diffeomorphism, so

$$\int_I f(x) dx = \int_J f(x(y)) \frac{dx}{dy} dy.$$

- A function  $f(x)$  changes to  $f(x(y))$  in new variables: it doesn't have the correct factor of  $dx/dy$  to fit the change of variables rule for integrals. We *can't* integrate functions invariantly under even very simple diffeomorphisms.
- Perhaps the correct object to integrate is not the function  $f(x)$  but the “differential”  $f(x)dx$ , which we want to insist changes to

$$f(x(y)) \frac{dx}{dy} dy$$

when we change variables preserving orientation.

- Perhaps the correct object to integrate over is not an interval but an oriented curve.

This is just how mathematicians integrated centuries ago, recognizing that we don't integrate functions: we integrate “differentials”. If only we had something that changes according this rule when we change variables, we could call it a differential, and we would recognize how to integrate it. Similarly, if  $C$  is an oriented curve in the plane, closed or not, mathematicians centuries ago would integrate something like

$$\int_C f(x, y)dx + g(x, y)dy.$$

But they never explained what  $dx$  is supposed to mean: what is a “differential”? They also didn't know what an orientation is, so sometimes wrote

$$\pm \int_C f(x, y)dx + g(x, y)dy.$$

I learned calculus from a book that used differentials without ever explaining even roughly what they were; I was embarrassed because I assumed I was supposed to have seen them before, and I couldn't figure out what they were. But in fact they are difficult to define with precision.

## The differential of a function

We defined the differential  $df$ , of a function  $f$  on a manifold with corners  $M$ , as the operation which associates to any tangent vector  $v$  the number  $df(v) := vf$ . At a point  $m \in M$ , we might write  $df_m$  to mean this operation on vectors  $v \in T_m M$ , so  $df_m \in T_m^* M$ . We found that in any coordinates,

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

Clearly  $df$  is a field of cotangent vectors, also called a 1-form, because it eats one vector. If  $f$  is  $C^k$  and  $X$  is  $C^\ell$  vector field then  $df(X)$  is  $C^m$  where  $m$  is the smaller

of  $k-1, \ell$ , because we differentiate  $f$  but not  $X$ . This  $df$  gives the same as the push forward  $f'(m)$ : since  $f: M \rightarrow \mathbb{R}$ ,  $f'(m): T_m M \rightarrow \mathbb{R}$  is a linear map and  $f'(m)v$  is a number, which is  $df_m(v)$ . So we now have several notations for the same thing. This  $df$  has an advantage over the usual notion of gradient of a function: it is well defined independent of any choice of chart.

### Defining 1-forms

A  $C^k$  1-form on a manifold with corners  $M$  is a map  $\omega$  that eats any vector  $v$  at any point  $m \in M$ , and spits out a number  $\omega_m(v)$ , linear in the choice of vector  $v$  at each point  $m$ , and is a  $C^k$  function as we vary  $m$  and  $v$ . Take any  $C^k$  vector field  $X$  on  $M$ ; then  $\omega$  eats each vector  $v = X(m)$  at each point  $m \in M$ , and splits out a  $C^k$  function, denoted  $\omega(X)$ , on  $U$ , with  $\omega(X)_m := \omega_m(v)$ .

In coordinates  $x^1, \dots, x^n$ , each differential  $dx^i$  is a 1-form, so that

$$dx^1(v) = v_1,$$

for any vector field

$$v = v^i \frac{\partial}{\partial x^i}.$$

**Lemma 23.1.** Suppose that we write points of  $\mathbb{R}^2$  as  $(x, y)$ . Every  $C^k$  1-form  $\omega$  on any open set in  $\mathbb{R}^2$  can be written uniquely as

$$\omega = f(x, y) dx + g(x, y) dy,$$

for some  $C^k$  functions  $f(x, y)$  and  $g(x, y)$ .

*Proof.* Define

$$\begin{aligned} f(x, y) &:= \omega_{(x, y)} \left( \frac{\partial}{\partial x} \right), \\ g(x, y) &:= \omega_{(x, y)} \left( \frac{\partial}{\partial y} \right). \end{aligned}$$

Let  $\alpha := f(x, y) dx + g(x, y) dy$ . We claim  $\omega = \alpha$ . To check, pick a point  $(x, y)$  and a vector

$$\begin{aligned} v &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \\ &= v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}. \end{aligned}$$

Plug in:

$$\begin{aligned} \omega_{(x, y)}(v) &= v_1 \omega_{(x, y)} \left( \frac{\partial}{\partial x} \right) + v_2 \omega_{(x, y)} \left( \frac{\partial}{\partial y} \right), \\ &= v_1 f(x, y) + v_2 g(x, y) \\ &= \alpha_{(x, y)}(v). \end{aligned}$$

□

The same works in any manifold: in coordinates, every  $C^k$  1-form is

$$\omega = f_i(x)dx^i$$

in coordinates  $x^1, \dots, x^n$ , where these  $f_i(x)$  can be any  $C^k$  functions.

**23.1** Prove that each  $C^k$  1-form  $\xi$  on  $M$  gives a  $C^k$  map  $x \in M \mapsto \xi(x) \in T^*M$ . Prove that a  $C^k$  map  $\xi: M \rightarrow T^*M$  arises in this way from a 1-form if and only if  $\pi \circ \xi = \text{id}$  is the identity map.

### Higher degree forms

A  $C^k$  0-form on a manifold with corners  $M$  is just a  $C^k$  function  $f: M \rightarrow \mathbb{R}$ . For any positive integer  $p$ , a  $p$ -form on  $M$  is an operation  $\omega$  that eats  $p$  vectors, say  $v_1, v_2, \dots, v_p$ , at each point  $m \in M$ , and spits out a number, which we denote

$$\omega_m(v_1, v_2, \dots, v_p),$$

so that:

- a. The result changes sign if we swap any two of the vectors and
- b. the result is linear, so

$$\omega_m(u + w, v_2, v_3, \dots, v_p) = \omega_m(u, v_2, v_3, \dots, v_p) + \omega_m(w, v_2, v_3, \dots, v_p)$$

for any vectors  $u$  and  $w$ , and

- c. the result is linear, so

$$\omega_m(cv_1, v_2, \dots, v_p) = v\omega_m(v_1, v_2, \dots, v_p).$$

d.

Given vector fields  $X_1, X_2, \dots, X_p$  on  $M$ , plug in to  $\omega$  to spit out a function, which we denote

$$\omega(X_1, X_2, \dots, X_p).$$

We say that  $\omega$  is  $C^k$  to mean that this function is  $C^k$ , for any  $C^k$  vector fields. A  $p$ -form is also called a *differential form* of degree  $p$ .

In the plane, define  $dx \wedge dy$  to be the 2-form

$$dx \wedge dy(X, Y) = dx(X)dy(Y) - dx(Y)dy(X).$$

(Naturally, we want to just use the obvious product formula: the first term. But differential forms must change sign when you swap vector fields, so we need the second term to fix that.) If  $\alpha^1, \alpha^2, \dots, \alpha^p$  are any 1-forms, we define a  $p$ -form, denoted

$$\alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^p$$

by asking that its value on any vector fields  $X_1, X_2, \dots, X_p$  be

$$\alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^p(X_1, X_2, \dots, X_p) = \det \begin{pmatrix} \alpha^1(X_1) & \alpha^1(X_2) & \dots & \alpha^1(X_p) \\ \alpha^2(X_1) & \alpha^2(X_2) & \dots & \alpha^2(X_p) \\ \vdots & \vdots & \dots & \vdots \\ \alpha^p(X_1) & \alpha^p(X_2) & \dots & \alpha^p(X_p) \end{pmatrix}.$$

For example, this recovers  $dx \wedge dy$  as above.

If we have a list of increasing indices, say  $I = (1, 3, 5, 7)$ , write  $dx^I$  to mean the form

$$dx^I = dx^1 \wedge dx^3 \wedge dx^5 \wedge dx^7,$$

etc.

**23.2\*** Prove that these  $dx^I$  are linearly independent, in the sense that if there is a linear relation, say

$$0 = a_I(x)dx^I,$$

where the Einstein notation sum is over all lists of increasing indices, then all of the functions  $a_I(x)$  vanish.

**Lemma 23.2.** Every differential form  $\omega$ , say of degree  $p$  on an open set in  $[0, \infty)^n$ , can be written uniquely as  $\omega = a_I(x)dx^I$ , where the Einstein notation sum is over all possible lists  $I$  of  $p$  indices, say

$$I = (i_1, i_2, \dots, i_p),$$

a list of increasing integers from 1 to  $n$ .

The proof is the same as that of lemma 23.1 on page 199, with more notation.

You won't really need to use this complicated mess, but more generally, if we have any differential forms, say  $\alpha$  and  $\beta$ , of degrees  $a$  and  $b$ , we define a new differential form of degree  $a + b$ , called the *wedge product* by asking that if  $\alpha = a_I(x)dx^I$  and  $\beta = b_J(x)dx^J$  then

$$\alpha \wedge \beta = a_I(x)b_J(x)dx^{IJ},$$

where  $IJ$  means to write out the indices of  $I$  and then of  $J$  (which might not be strictly increasing indices anymore).

What we need to know is that this wedge product operation satisfies

$$\begin{aligned} (\alpha + \beta) \wedge \gamma &= \alpha \wedge \gamma + \beta \wedge \gamma, \\ \alpha \wedge (\beta + \gamma) &= \alpha \wedge \beta + \alpha \wedge \gamma, \\ (f\alpha) \wedge \beta &= f(\alpha \wedge \beta) \\ &= \alpha \wedge (f\beta), \\ \alpha \wedge \beta &= (-1)^{ab}\beta \wedge \alpha \end{aligned}$$

with  $\alpha, \beta, \gamma$  forms and  $f$  a real-valued function. Moreover, the wedge product is associative:  $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ . We won't prove any of these identities; the proofs are just long manipulations of permutations. We also write multiplication by a function, say  $f\omega$ , as  $f \wedge \omega$ , treating  $f$  as a 0-form.

**23.3** Use the identities to prove that  $dx \wedge dx = 0$ .

**23.4** Writing points of  $\mathbb{R}^4$  as  $(x, y, u, v)$ , let  $\alpha = dx \wedge dy + du \wedge dv$ . Simplify the expression  $\alpha \wedge \alpha$ .

**23.5\*** Suppose that  $A$  is an  $n \times n$  matrix, with indices arranged as  $A = (A_j^i)$ . Let  $\alpha^i = A_j^i dx^j$ , for  $i = 1, 2, \dots, n$ ; these  $\alpha^i$  are 1-forms on  $\mathbb{R}^n$ . Use the identities and induction to prove that

$$\alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^n = \det(A)dx^1 \wedge dx^2 \wedge \dots \wedge dx^n.$$

**23.6** If  $p > n$ , prove that the only  $p$ -form on any open subset of  $\mathbb{R}^n$  is the one that vanishes everywhere:  $\omega = 0$ . Hint: use lemma 23.2 on the previous page and problem 23.3 on the preceding page.

**23.7** Suppose that  $\alpha^1, \alpha^2, \dots, \alpha^p$  are 1-forms on  $\mathbb{R}^n$ . Prove that they are linearly independent at every point of  $\mathbb{R}^n$  just when  $0 \neq \alpha^1 \wedge \alpha^2 \wedge \dots \wedge \alpha^p$ .

## Notes

The algebraic manipulation of differential forms seems daunting, when faced with the definition of the wedge product. Actually the algebra is very easy:

- a.  $dx \wedge dx = 0$ , and similarly for any variable in place of  $x$ ,
- b.  $dx \wedge dy = -dy \wedge dx$ , and similarly for any variables in place of  $x$  and  $y$ ; for example, putting  $y$  equal to  $x$  gives  $dx \wedge dx = 0$ ,
- c. wedge product is associative and distributive over  $+$ , just like usual multiplication,
- d. nothing else: all identities on differential forms are consequences of the above.



## Chapter 24

### Differentiating differential forms

We explain how differential forms transform under maps and how to take the derivative of a differential form.

#### Pullback

If  $F: P \rightarrow Q$  is a continuously differentiable map of manifolds with corners and  $\omega$  is a  $p$ -form on  $Q$ , then the *pullback* of  $\omega$  by  $F$ , denoted  $F^*\omega$ , is the  $p$ -form

$$F^*\omega(v_1, v_2, \dots, v_p) = \omega(F_*v_1, F_*v_2, \dots, F_*v_p),$$

by pushing forward the vectors forward at each point. The pullback of any  $C^k$  differential form by a  $C^{k+1}$  map is a  $C^k$  form. The *pullback* of a  $C^k$  function  $f$  (in other words, a 0-form) by a  $C^k$  map  $F$  is the  $C^k$  function  $F^*f$  defined by  $F^*f = f \circ F$ .

**24.1** For the map  $(u, v) = F(x, y) = (x + y, y - x^2)$ , what is the pullback  $F^*du$ ? What is  $F^*dv$ ?

**24.2** For the map  $F(x, y) = x$ , what is the pullback  $F^*x^3 dx$ ?

**24.3** Prove that for any diffeomorphism  $F: P \rightarrow Q$  of manifolds with corners and forms  $\alpha$  and  $\beta$  on  $Q$ ,

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta).$$

**24.4** Suppose that  $F$  and  $G$  are two diffeomorphisms. Prove that  $(F \circ G)^*\omega = G^*F^*\omega$  as long as both sides are defined.

**24.5** An explicit recipe to compute pullback: write a map in coordinates as  $y = f(x)$ . Prove that  $f^*dy^i = \frac{\partial y^i}{\partial x^j} dx^j$ . Prove that any differential form

$$\omega = a_{i_1 i_2 \dots i_p}(y) dy^{i_1} \wedge dy^{i_2} \wedge \dots \wedge dy^{i_p},$$

pulls back as

$$f^*\omega = a_{i_1 i_2 \dots i_p}(f(x)) \frac{\partial y^{i_1}}{\partial x^{j_1}} dx^{j_1} \wedge \frac{\partial y^{i_2}}{\partial x^{j_2}} dx^{j_2} \wedge \dots \wedge \frac{\partial y^{i_p}}{\partial x^{j_p}} dx^{j_p}.$$

**Lemma 24.1.** If  $f: Q \rightarrow \mathbb{R}$  is a  $C^1$  function and  $\varphi: P \rightarrow Q$  is a  $C^1$  map then

$$F^*df = d(F^*f).$$

*Proof.* Pick a vector  $v$  at a point  $p_0 \in P$ , say the velocity at  $t = 0$  of some path  $p(t)$  with  $p(0) = p_0$ . Let  $q_0 := F(p_0)$  and let  $q(t) = F(p(t))$ :

$$\begin{aligned} (F^* df)(v) &= df(F_* v) \\ &= \left. \frac{d}{dt} \right|_{t=0} f(q(t)) \\ &= \left. \frac{d}{dt} \right|_{t=0} f(F(p(t))) \\ &= d(f \circ F)(v) \\ &= d(F^* f)(v). \end{aligned}$$

□

**24.6** Which of the following are  $df$  for some continuously differentiable function  $f$ ?

- a.  $2x \, dx$
- b.  $x \, dx$
- c.  $x \, dx + y \, dy$
- d.  $y \, dx + x \, dy$
- e.  $y \, dx - x \, dy$

### Exterior derivative

If  $\omega = a_I(x) \, dx^I$  is a  $C^k$   $p$ -form, then we denote by  $d\omega$  the  $C^{k-1}$   $(p+1)$ -form

$$d\omega = da_I \wedge dx^I = \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I.$$

If  $\omega = xy \, dx + x \, dy$ , then

$$\begin{aligned} d\omega &= d(xy) \wedge dx + d(x) \wedge dy, \\ &= y \, dx \wedge dx + x \, dy \wedge dy + dx \wedge dy, \\ &= (1 - x) \, dx \wedge dy. \end{aligned}$$

**Theorem 24.2.** The operation  $d$  is the unique operation taking  $C^{k+1}$  differential forms to  $C^k$  differential forms, so that:

- a. If  $f$  is any  $C^{k+1}$  function, then  $df$  is as defined above.
- b. For any  $C^{k+1}$  differential forms  $\alpha$  and  $\beta$ ,

$$d(\alpha + \beta) = d\alpha + d\beta.$$

- c. The product rule (also called the Leibnitz rule): if  $\alpha$  is any  $a$ -form and  $\beta$  is any differential form, both  $C^{k+1}$ , then

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^a \alpha \wedge d\beta.$$

- d. If  $\omega$  is  $C^2$  then  $d(d\omega) = 0$ . (This is often expressed by writing  $d^2 = 0$ .)

*Proof.* Lets try some examples on the  $x, y$  plane to see why there is only one operation  $d$  with those four properties. First, we know  $d$  on any function  $f$  is  $df$ . For example,  $d$  on the function  $f(x, y) = x$  is

$$\begin{aligned} d(x) &= \frac{\partial x}{\partial x} dx + \frac{\partial x}{\partial y} dy, \\ &= dx. \end{aligned}$$

On something like  $f(x, y) dx$  the product rule applies:

$$d(f(x, y) dx) = (df) \wedge dx + f(x, y) d(dx),$$

since  $f(x, y)$  is a 0-form; but  $d^2 = 0$  so we get

$$\begin{aligned} &= \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) \wedge dx + 0 \\ &= \frac{\partial f}{\partial y} dy \wedge dx. \end{aligned}$$

The same idea works more generally. Take any differential form  $\omega$ . By lemma 23.2 on page 201, we can write  $\omega = a_I(x) dx^I$ . But as above,

$$da_I = \frac{\partial a_I}{\partial x^j} dx^j,$$

and  $d(dx^I) = 0$ . We get the explicit formula

$$d\omega = \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I.$$

So if there is an operation  $d$  with the four properties listed above, then it is the operation  $d$  that we defined above.

We need to check that this operation  $d$  has the four required properties. The first two are obvious. For the third property, we can explicitly write out our forms  $\alpha$  and  $\beta$  in terms of  $dx^I$  and expand, so that it is enough to check  $\alpha = dx^I$ , and  $\beta = dx^J$ , for some index lists  $I$  and  $J$ , for which the result is an easy explicit calculation. Similarly, we can check  $d^2 = 0$  explicitly.  $\square$

**24.7** Use commutativity of partial derivatives to show that

$$d \circ d(f(x) dx^I) = 0,$$

for any twice continuously differentiable function  $f$ . Explain why this proves that  $d^2 = 0$ .

**24.8** If  $\alpha = f(x) dx^I$ , and  $\beta = g(x) dx^J$ , for some index lists  $I$  and  $J$ , and  $I$  has  $a$  elements in it, then check the Leibnitz rule, i.e. that

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^a \alpha \wedge d\beta.$$

Hint: you could try induction on  $a$ .

**Theorem 24.3.** If  $F: P \rightarrow Q$  is a  $C^2$  map of manifolds with corners then  $F^*d\omega = dF^*\omega$ , for any  $C^1$  form  $\omega$  on  $Q$ .

*Proof.* By lemma 24.1 on page 203, this is true if  $\omega$  is a 0-form. It is also trivially true if  $\omega$  is  $\omega = df$  for some  $C^2$  function  $f$ , since  $d^2 = 0$ . If it is true for two differential forms  $\alpha$  and  $\beta$ , then it is true for  $\alpha \wedge \beta$  by the result of problem 24.3 on page 203. Any differential form is a sum of wedge products of 0-forms and 1-forms of the form  $df$ .  $\square$

**24.9** Is there an analogue of the flow box theorem (theorem 6.3 on page 70) for 1-forms?

### Polar coordinates

As an example, consider the map  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

$$\begin{aligned}(x, y) &= F(r, \theta), \\ &= (r \cos \theta, r \sin \theta).\end{aligned}$$

Then  $F^*x = r \cos \theta$ . We can write this as  $x = r \cos \theta$ . Similarly,  $F^*y = r \sin \theta$ . We can write this as  $y = r \sin \theta$ . The complicated Leibnitz rule becomes just the simple product rule on functions:

$$\begin{aligned}dx &= d(r \cos \theta), \\ &= dr \cos \theta + r d(\cos \theta), \\ &= \cos \theta dr - r \sin \theta d\theta.\end{aligned}$$

Similarly,

$$dy = \sin \theta dr + r \cos \theta d\theta.$$

Taking  $\wedge$ :

$$dx \wedge dy = r dr \wedge d\theta.$$

**24.10** In polar coordinates, find  $dr$  and  $d\theta$  in terms of  $dx$  and  $dy$ .

### Closed and exact forms

A differential form  $\omega$  is *exact* if  $\omega = d\xi$  for some continuously differentiable differential form  $\xi$ .

**24.11** Find an example of a differential form which is not exact.

A continuously differentiable differential form  $\omega$  is *closed* if  $d\omega = 0$ . For example, on  $\mathbb{R}^2 - \{0\}$ , the 1-form

$$\omega = \frac{-y dx + x dy}{x^2 + y^2}$$

is closed, but turns out not to be exact, as we will see.

**24.12** Prove that continuously differentiable exact forms are closed.

**24.13\*** Suppose that  $\omega = f(x, y, z)dx \wedge dy \wedge dz$  is a continuous 3-form defined in all of  $\mathbb{R}^3$ . Prove that  $\omega$  is exact. Hint: try something like  $\xi = g(x, y, z)dy \wedge dz$ .

**24.14** Suppose that  $\omega$  is a continuous  $n$ -form defined in all of  $\mathbb{R}^n$ . Prove that  $\omega$  is exact.

**24.15\*** If  $\omega$  is a continuous  $n$ -form in  $\mathbb{R}^n$ , vanishing for  $x^1 \leq 0$ , then prove that  $\omega$  is exact, and that  $\omega = d\xi$  for some continuously differentiable  $(n-1)$ -form vanishing for  $x^1 \leq 0$ .

**24.16** Prove that a continuously differentiable 2-form of the form  $\omega = f(x, y, z)dy \wedge dz$  defined in  $\mathbb{R}^3$  is closed just when it is exact.

**24.17\*** Prove that every continuously differentiable 2-form defined in  $\mathbb{R}^3$  is closed just when it is exact.

## Notes

- Differential forms satisfy the algebraic properties that “infinitesimals” should have. We could have used any one of many other definitions.
- When working in  $\mathbb{R}^3$ , scientists and engineers identify any 1-form

$$\xi = f(x, y, z) dx + g(x, y, z) dy + h(x, y, z) dz$$

with a vector field

$$A = \begin{pmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{pmatrix}.$$

This identification is invariant under any translation, rotation or reflection. It is not invariant under rescaling, for example. They identify any 2-form

$$\eta = f(x, y, z) dy \wedge dz + g(x, y, z) dz \wedge dx + h(x, y, z) dx \wedge dy,$$

with a vector field

$$F = \begin{pmatrix} f(x, y, z) \\ g(x, y, z) \\ h(x, y, z) \end{pmatrix}.$$

This identification is invariant under any translation or rotation, and not under either rescaling or reflection. Finally, they identify any 3-form

$$\omega = f(x, y, z) dx \wedge dy \wedge dz$$

with a function  $f(x, y, z)$ . This identification is again invariant under any translation or rotation, and not under either rescaling or reflection.

These identifications then turn the exterior derivative into an operator on vector fields. If  $f$  is a function and we let  $\xi = df$ , then the corresponding vector field  $A$  is denoted

$$A = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

called the *gradient* of  $f$ . Similarly if  $A$  is any vector field, which we write as

$$A = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

with corresponding 1-form  $\xi$ , and we let  $\eta = d\xi$ , then the vector field  $F$  corresponding to  $\eta$  is denoted

$$F = \nabla \times A = \begin{pmatrix} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix},$$

called the *curl* of  $A$ . Finally, if  $F$  is any vector field, written as

$$F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix},$$

with corresponding 2-form  $\eta$ , and  $\omega = d\eta$ , then  $\omega = f dx \wedge dy \wedge dz$  where  $f$  is a function denoted

$$f = \nabla \cdot F,$$

called the *divergence* of  $F$ , and given explicitly by

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

We will avoid all of this notation, which is special to 3 dimensions and is not invariant under diffeomorphism.

## Chapter 25

### Integrating differential forms

We define the integral of a differential form over a box.

#### Definition

We suppose that you have already encountered several variable integration in a previous multivariable calculus class, including the formula for changing variables; see any rigorous multivariable calculus textbook for such a theory [9, 12].

Take a  $C^0$   $n$ -form  $\omega$  with compact support inside an open set  $U \subseteq \mathbb{R}^n$ :

$$\omega = f(x)dx^{1\dots n}.$$

Note that the dimension of  $U$  is  $n$ , equal to the degree  $n$  of the form  $\omega$ ;  $\omega$  is *top degree*. The *integral* of  $\omega$  on  $U$  is

$$\int_U \omega := \int_U f.$$

**25.1** Let  $\omega = (1 + x^2y^4) dx \wedge dy$  and  $X = [0, 1]^2$ ; compute  $\int_X \omega$ .

#### Change of variables

If we write each point of  $U$  as  $y = (y^1, \dots, y^n)$ , then

$$\omega = f(y)dy^{1\dots n},$$

for some continuous function  $f: U \rightarrow \mathbb{R}$ . Under a  $C^1$  diffeomorphism  $y = \varphi(x)$ ,  $\varphi: W \rightarrow U$ , for some open set  $W \subseteq [0, \infty)^n$ , then

$$\varphi^*\omega = f(y(x)) \det y'(x) dx^{1\dots n}.$$

As long as  $x \mapsto y(x)$  is orientation preserving, this looks just like the change of variables formula for integrals:

$$\int_U f(y)dy^1 \dots dy^n = \int_W f(y(x)) \det y'(x) dx^1 \dots dx^n.$$

If  $y = \varphi(x)$  swaps orientation, then this equation on integrals won't hold: it needs an absolute value wrapped around  $\det y'(x)$ . So we see that integrals of compactly supported top degree forms are defined on open subsets of the standard corner  $[0, \infty)^n$ , and are invariant under precise the orientation preserving diffeomorphisms.

**Lemma 25.1** (The Leibnitz rule). *If  $X$  and  $Y$  are boxes then*

$$\int_{\partial(X \times Y)} \omega = \int_{(\partial X) \times Y} \omega + (-1)^{\dim X} \int_{X \times \partial Y} \omega,$$

*for any continuous differential form  $\omega$  defined near  $\partial(X \times Y)$ .*

This is just restating the Leibnitz rule  $\partial(X \times Y) = (\partial X) \times Y \sqcup (-1)^{\dim X} X \times \partial Y$ .

### Stokes's theorem in a box

**Theorem 25.2** (Stokes's theorem in a box). *Suppose that  $X \subset \mathbb{R}^n$  is a box and  $\omega$  is a continuously differentiable  $(n-1)$ -form defined on  $X$ . Then*

$$\int_X d\omega = \int_{\partial X} \omega.$$

*Proof.* Let's do the proof for the box

$$X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$$

in the plane. (The general proof is just more notation.) We can write any 1-form in the plane as

$$\omega = f(x, y) dx + g(x, y) dy.$$

It is enough to separately prove the result for each of  $f(x, y) dx$  and  $g(x, y) dy$ . We will do the first; the second follows by the same method, just switching the roles of  $x$  and  $y$ . So assume that  $\omega = f(x, y) dx$  so that

$$d\omega = -\frac{\partial f}{\partial y} dx \wedge dy$$

Compute

$$\begin{aligned} \int_X d\omega &= \int_X \left( -\frac{\partial f}{\partial y} \right) dx \wedge dy, \\ &= - \int_0^1 \int_0^1 \frac{\partial f}{\partial y} dy dx, \end{aligned}$$

and by the fundamental theorem of calculus this is

$$\begin{aligned} &= - \int_0^1 (f(x, 1) - f(x, 0)) dx, \\ &= \int_0^1 f(x, 0) dx - \int_0^1 f(x, 1) dx, \end{aligned}$$

and we leave the reader to check that this is precisely

$$= \int_{\partial X} \omega.$$

□



**25.2** Suppose that  $U \subseteq \mathbb{R}^n$  and  $W \subseteq \mathbb{R}^n$  are open sets. Suppose that  $F: U \rightarrow W$  and  $G: U \rightarrow W$  are continuously differentiable maps. Take a box  $X \subset U$ . Suppose that near  $\partial X$ ,  $F = G$ . Use the result of problem 24.15 on page 207 to prove that

$$\int_X F^* \omega = \int_X G^* \omega$$

for any  $n$ -form defined on  $W$ .

**25.3** In one dimension, i.e.  $n = 1$ , solve the last problem more directly.

**25.4\*** Suppose that  $X$  is a box and  $F: X \rightarrow X$  is a continuously differentiable map and  $F(x) = x$  for all  $x \in \partial X$ . Use the result of problem 25.2 to prove that  $F$  is onto. Hint: take  $G(x) = x$ , and take lots of “bump functions”  $f$ , letting

$$\omega = f(x) dx^{1 \cdots n}.$$

**25.5** Do you see any (vague) similarity between the boundary operation  $\partial$  and the exterior derivative operation  $d$ ?

## Partitions of unity

Now we can integrate differential forms on open sets of Euclidean space. We want to integrate over manifolds. We will prove that we can write any compactly supported differential form as a finite sum of differential forms each supported in the domain of a chart, and use our chart. This is difficult to prove. Consider first how we could prove this for functions instead of differential forms.

**25.6** Give an example of a continuous function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  so that  $f(x) = 1$  if  $|x| \leq 1$  and  $f(x) = 0$  if  $|x| \geq 2$ .

An *open cover*  $\{M_\alpha\}$  of a manifold with corners  $M$  is a collection of open subsets  $M_\alpha \subseteq M$  whose union is  $M$ . A  $C^k$  *partition of unity* on a compact set  $K \subset M$  of a  $C^k$  manifold with corners  $M$  is a collection of  $C^k$  functions  $f_1, f_2, \dots, f_N$  so that

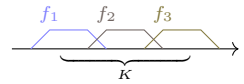
- a.  $0 \leq f_1, f_2, \dots, f_N \leq 1$  and
- b.  $f_1(x) + f_2(x) + \dots + f_N(x) = 1$  for each  $x \in K$ .

In picture, where one  $f_i$  goes down, various other  $f_j$  go up to take its place, so the sum stays 1.

A partition of unity  $f_1, f_2, \dots, f_N$  is *subordinate* to an open cover  $\{M_\alpha\}$  if for every function  $f_i$ , there is some open set  $M_\alpha$  from the open cover, so that  $f_i$  is supported on  $M_\alpha$ . We will discuss partitions of unity in general later; for now (and for most problems in differential geometry) we only need partitions of unity over compact sets.

**Theorem 25.3.** *For any open cover of any compact set on any smooth (or  $C^k$ ,  $k \geq 1$ ) manifold with corners, there is a smooth (or  $C^k$ ) partition of unity subordinate to it.*

*Proof.* Each point  $m_0 \in K$  lies in one of these  $M_\alpha$ . Since  $M_\alpha$  is open, we can pick a chart  $(U, \varphi)$  with domain  $U$  containing  $m_0$ , mapping to an open set  $\varphi(U) \subset [0, \infty)^n$ . Let  $x_0 := \varphi(m_0)$ . Pick a smooth function  $g$  so that  $g(x) = 1$  near  $x = x_0$  and  $g = 0$  outside of a compact subset  $C \subseteq \varphi(U)$ . Replacing  $g$  by  $g \circ \varphi$ , we have a continuous function  $g$  so that  $g = 1$  on an open subset  $W \subseteq U$  containing  $m_0$ , and  $g = 0$  outside of a compact set  $C \subset U$ .



A partition of unity on a compact set  $K$

We can pick one such  $g$  for each point  $m_0$ , with various open sets  $W$  and compact sets  $C$ . These various open sets form an open cover of  $K$ ; pick a finite subcover  $W_1, \dots, W_N$ . So  $K$  lies in the union of open sets  $W_1, \dots, W_N \subseteq M$ , with each  $W_i$  lying inside a compact set  $C_i$ . We have continuous functions  $g_1, \dots, g_N: M \rightarrow \mathbb{R}$  so that  $g = 1$  inside  $W_i$  and  $g_i = 0$  outside  $C_i$ .

These  $g_i$  might not add up to 1. Let  $f_1 := g_1$  and let

$$f_{i+1} := (1 - g_1)(1 - g_2) \dots (1 - g_i)g_{i+1}.$$

By induction, check that

$$\begin{aligned} f_1 &= 1 - (1 - g_1), \\ f_1 + f_2 &= 1 - (1 - g_1)(1 - g_2), \\ &\vdots \\ f_1 + f_2 + \dots + f_N &= 1 - \prod_i (1 - g_i). \end{aligned}$$

On  $C_i$ ,  $g_i = 1$  so  $f_1 + f_2 + \dots + f_N = 1$ . The various  $C_i$  cover  $K$ , so  $f_1 + f_2 + \dots + f_N = 1$  everywhere on  $K$ .  $\square$

**25.7** Take a manifold with corners  $M$ , a compact set  $K \subset M$ , and a closed set  $A \subset M$  disjoint from  $K$ . Prove that there is a continuous function  $f: M \rightarrow \mathbb{R}$  so that  $f = 1$  on  $K$  and  $f = 0$  on  $A$ .

**25.8\*** Prove that every smooth compact manifold has a smooth embedding into Euclidean space.

## Integration on manifolds

Suppose that  $M$  is a  $C^1$   $p$ -dimensional oriented manifold with corners, and that  $\omega$  is a  $C^0$  compactly supported  $p$ -form on  $M$ . We want to define  $\int_M \omega$ . We can easily define integrals over any open set lying in the domain of a *single* chart: use the chart to identify with an open set in Euclidean space.

Consider the open cover of the support of  $\omega$  consisting of all open sets  $M_\alpha \subseteq M$  so that  $M_\alpha$  lies inside the domain of a chart. Pick a partition of unity  $f_1, f_2, \dots, f_N$  (see chapter 33) subordinate to that open cover. So each  $f_i$  vanishes at every point of  $M$  outside the domain of a single chart, say  $(M_i, \varphi_i)$ , and these  $f_i$  add up to 1 at every point of  $M$ . We can assume that  $\varphi_i$  preserves orientation. Let

$$\int_M \omega := \sum_i \int_{\varphi_i(M_i)} \varphi_i^{-1*}(f_i \omega).$$

**Lemma 25.4.** *The integral  $\int_M \omega$  defined above does not depend on the choice of partition of unity or charts.*

*Proof.* Suppose that you pick a partition of unity  $f_1, f_2, \dots, f_N$  and charts  $(U_i, \varphi_i)$ . I pick another partition of unity, say  $g_1, g_2, \dots, g_N$ , and charts  $(W_j, \psi_j)$ . For any  $i$  and  $j$ , consider the 1-form  $f_i g_j \omega$ . Consider the integral

$$\int_{\varphi(U_i \cap W_j)} \varphi_i^{-1*}(f_i g_j \omega).$$

By the change of variables formula in Euclidean space, the value of the integral doesn't change when we apply the transition map to move to another subset of Euclidean space:

$$\begin{aligned} \int_{\varphi(U_i \cap W_j)} \varphi_i^{-1*}(f_i g_j \omega) &= \int_{\psi \circ \varphi^{-1} \varphi(U_i \cap W_j)} (\psi \circ \varphi^{-1})^{-1*} \circ \varphi_i^{-1*}(f_i g_j \omega), \\ &= \int_{\psi(U_i \cap W_j)} (\psi)^{-1*}(f_i g_j \omega). \end{aligned}$$

But then add up

$$\sum_{ij} \int \varphi_i^{-1*}(f_i g_j \omega) = \sum_{ij} \int \psi_j^{-1*}(f_i g_j \omega).$$

Clearly the left hand side is

$$\sum_i \int \varphi_i^{-1*} \left( \sum_j f_i g_j \omega \right) = \sum_i \int \varphi_i^{-1*}(f_i \omega),$$

while the right hand side is

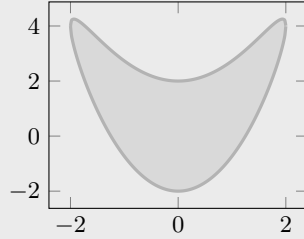
$$\sum_j \int \psi_j^{-1*} \left( \sum_i f_i g_j \omega \right) = \sum_j \int \psi_j^{-1*}(g_j \omega).$$

□

**Lemma 25.5.** *Orientation preserving diffeomorphisms preserve the integral of any top degree form over any closed or open subset on which the form has compact support.*

The *smile*  $M$  is the set of points  $(x, y)$  of the plane for which

$$x^4 + x^2 - 2x^2y + y^2 \leq 4.$$



The map

$$f(u, v) = (x, y) = (u, v + u^2)$$

is clearly a diffeomorphism, with inverse

$$f^{-1}(x, y) = (u, v) = (x, y - x^2).$$

The smile is  $M = f(D)$  where  $D$  is the closed disk of radius 2 about the origin:

$$4 = u^2 + v^2 = x^2 + (y - x^2)^2 = x^4 + x^2 - 2x^2y + y^2.$$

Note that  $du = dx$  and  $dv = dy + 2x dx$ , so that  $du \wedge dv = dx \wedge dy$ , i.e.  $f^*dx \wedge dy = du \wedge dv$ , so the area inside the smile is

$$\int_M dx \wedge dy = \int_D du \wedge dv = 4\pi.$$

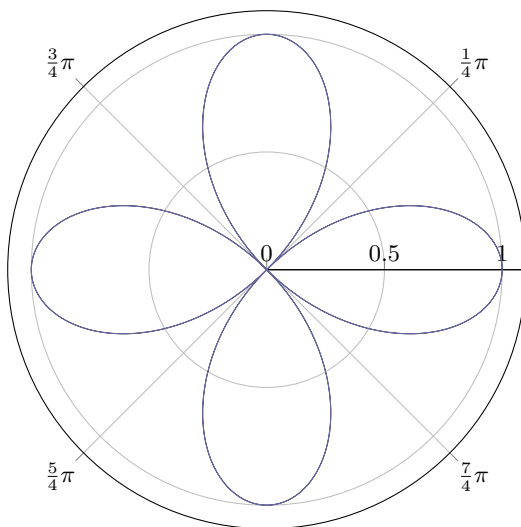
**25.9** Use the change of variable  $(u, v) = F(x, y) = (x + y^2, y)$  to calculate the area of the domain  $u \geq v^2, v \geq 0, u + v \leq 1 + v^2$ . Note: the area of a domain in the  $(u, v)$ -plane is the integral of  $du \wedge dv$  over that domain.

**25.10** Use a change of variables to compute the area inside the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1$$

using the well-known area inside a circle.

**25.11** Find the area of one leaf of the 4-petal rose.



The rightmost leaf is given in polar coordinates by  $-\pi/4 \leq \theta \leq \pi/4$  and  $0 \leq r \leq \cos(2\theta)$ .

**25.12** Map the solid cylinder

$$C = \{ (x, y, z) \mid -1 \leq z \leq 1, x^2 + y^2 \leq 1 \}$$

to the closed ball

$$B = \{ (X, Y, Z) \mid X^2 + Y^2 + Z^2 \leq 1 \}$$

by the map

$$(X, Y, Z) = F(x, y, z) = (f(z)x, f(z)y, z),$$

where

$$f(z) = \sqrt{1 - z^2}.$$



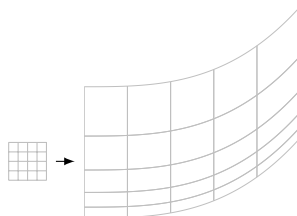
Compute the volume of the ball

$$\int_B dX \wedge dY \wedge dZ$$

in terms of the area of the flat unit disk by computing

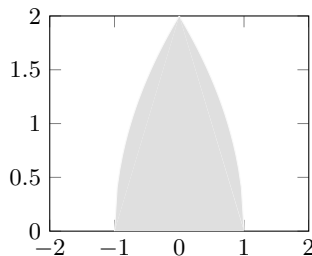
$$\int_C F^*(dX \wedge dY \wedge dZ).$$

**25.13** The map  $(u, v) = F(x, y) = (x, y + x^3 + y^3)$  deforms boxes in the  $x, y$ -plane into some curved shapes in the  $u, v$ -plane:



Prove that  $F$  is a diffeomorphism. What is the area of  $F(X)$  for a box  $X = [x_0, x_1] \times [y_0, y_1]$ ?

**25.14** A window in a gothic church: let  $M \subset \mathbb{R}^2$  be the set of points  $(u, v) \in \mathbb{R}^2$  which lie above the  $u$ -axis, to the left of the curve  $u = 1 - (v/2)^2$  for  $0 \leq v \leq 2$ , to the right of the curve  $u = (v/2)^2 - 1$  for  $0 \leq v \leq 2$ .



Use the parameterization  $F(x, y) = (u, v) = (x^2 - y^2, 2xy)$  to compute the area of  $M$ .

**25.15** The *Gaussian* is the function  $f(x, y) = e^{-x^2 - y^2}$ . By changing to polar coordinates, find the integral of the Gaussian over the disk of radius  $R$  about the origin in  $\mathbb{R}^2$ .

**25.16** For those who like long calculations: compute

$$\int_X \left(1 - \sqrt{x^2 + y^2 + z^2}\right) dx dy dz$$

over the hemisphere

$$X = \{ (x, y, z) \in \mathbb{R}^3 \mid z \geq 0, x^2 + y^2 + z^2 \leq 1 \}.$$

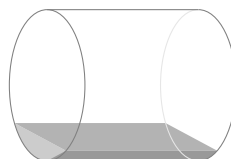
Hint: try *spherical coordinates*

$$x = r \sin \alpha \cos \beta,$$

$$y = r \sin \alpha \sin \beta,$$

$$z = r \cos \alpha.$$

**25.17** Water sits to a height  $h$  in a cylindrical drum of radius  $r$  and length  $\ell$  like this:



Find the volume of water sitting in the tank.

**25.18** Let  $X \subset \mathbb{R}^3$  be the set of points  $(x, y, z)$  bounded by the cylinders  $x^2 + y^2 \geq 4$ ,  $x^2 + y^2 \leq 9$ , and with  $0 \leq z \leq 1$  and with  $0 \leq y \leq x$ . Draw  $X$ . Using cylindrical coordinates

$$x = r \cos \theta,$$

$$y = r \sin \theta,$$

$$z = z,$$

i.e. the map  $F(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$ , compute

$$\int_X z(x^2 + y^2) dx dy dz.$$

**25.19** Generalise the method of problem 25.12 on page 214 to prove that the volume of the unit ball  $B \subset \mathbb{R}^n$  is

$$\text{Vol } B = \prod_{i=1}^{n-1} \int_{z=-1}^{z=1} (1 - z^2)^{i/2}.$$

**25.20** Make big bumps: suppose that  $K \subset U \subseteq M$  are subsets of a  $C^1$  manifold,  $K$  compact and  $U$  open. Prove that there is a  $C^\infty$  function  $\beta \geq 0$  defined on  $M$ , vanishing outside a compact subset of  $U$  and equal to 1 on  $K$ .

## Notes

Imagine a simple but geometric definition of integration on manifolds with corners: cut the manifold into small “pieces”, each piece being diffeomorphic to a box, so that any two pieces overlap only on a smaller dimensional manifold. Pull back the form by inverse of each chart, and integrate in Euclidean space. This naive definition is not so easy. You need to show that there is a way to cut up the manifold into such

pieces, and that the result doesn't depend on how you cut up the manifold. Partitions of unity are a trick to avoid cutting manifolds. Instead we work with functions (or forms) defined on a single chart domain first. Then we add up such functions or forms. Existence of a partition of unity ensures that we can build up any function or form out of these local contributions.





## Chapter 26

### Stokes's theorem

We prove Stokes's theorem for manifolds with corners.



**Theorem 26.1** (Stokes's theorem). *On an  $n$ -dimensional  $C^2$  manifold  $M$  with corners, take a  $C^1$   $(n-1)$ -form  $\omega$  with compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

*Proof.* It works in a box, and any integral is a finite sum of integrals over boxes.  $\square$

**26.1** We can use integrals around curves to compute areas. Prove that the area of any domain  $U \subseteq \mathbb{R}^2$  is

$$\int_{\partial U} x \, dy.$$

Use this to compute the area of a disk of radius  $R$ .

**26.2** Use Stokes's theorem to find the area inside the curve

$$x^{2/5} + y^{2/5} = 1.$$

**26.3** Take a positive continuously differentiable function  $y = f(x)$  of one variable, and two real numbers  $a < b$ . Let  $M$  be the manifold with boundary and corners consisting of all points  $(x, y)$  with  $a \leq x \leq b$  and  $0 \leq y \leq f(x)$ . Draw a picture. Use Stokes's theorem applied to  $\omega = y \, dx$  to recover the usual procedure for calculating area under the graph of a function from one variable calculus. Be careful with minus signs.

**26.4** Suppose that  $\omega$  is a differential form on  $\mathbb{R}^3$  and  $M$  is a sphere in  $\mathbb{R}^3$ . What can you say about  $\int_M d\omega$ ?

**26.5** Suppose that  $M$  is a compact oriented surface in  $\mathbb{R}^3$  whose boundary is the unit circle in the plane  $z = 0$ . Let  $\omega = xz \, dx + (e^z - 1)xy^2 \, dy + \sin(ze^{xy}) \, dz$ . Calculate  $\int_M d\omega$ .

**26.6** Take a bounded open set  $U \subset \mathbb{R}^2$  containing the origin, and with continuously differentiable boundary. Suppose that the boundary is parameterised as a continuously differentiable curve  $(x(s), y(s))$  at unit speed,  $0 \leq s \leq \ell$ . Explain why  $\ell$  is the length of the boundary. In polar coordinates, prove that

$$\int_{s=0}^{s=\ell} r \, ds \geq 2 \, \text{Area } U$$

with equality only when  $U$  is a disk centred at the origin.

**26.7\*** For any positive constant  $a > 0$ , consider the cone  $z = a\sqrt{x^2 + y^2}$ . The map  $(x, y, z) \mapsto (zx/a, zy/a, z)$  maps the cylinder  $x^2 + y^2 \leq 1, z \geq 0$  to the cone. Use this map to prove Stokes's theorem on the cone.

**26.8** Suppose that  $S$  is a  $C^2$   $n$ -dimensional manifold with corners,  $\omega$  is a continuously differentiable  $(n-1)$ -form on  $S$  and  $f$  is a continuously differentiable function on  $S$ . Suppose that  $f\omega$  vanishes everywhere except in a compact set. Prove the equation of integration by parts:

$$\int_S f d\omega = \int_{\partial S} f\omega - \int_S df \wedge \omega.$$

Explain how this generalises integration by parts from one variable calculus.

**26.9** Let  $M$  be the plane with a disk removed, say of radius  $r_0$  about the origin. Consider the 1-form  $\omega = f(r)d\theta$ . If  $f(r)$  does not have compact support, then neither does  $\omega$ , so we can't justify application of Stokes's theorem. If  $f(r) \rightarrow 0$  as  $r \rightarrow \infty$ , prove that  $\int_M d\omega = \int_{\partial M} \omega$ . What about if  $f(r) \rightarrow 1$  as  $r \rightarrow \infty$ ?

## Integrating over curves

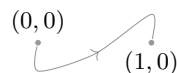
This section can be omitted without loss of continuity.

If  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuously differentiable function and  $C \subset \mathbb{R}^n$  is a compact curve with boundary, say leaving a point  $p$  and going to a point  $q$ , Stokes's theorem says

$$\int_C df = f(q) - f(p).$$

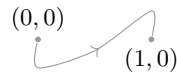
In particular, the integral depends only on the end points  $p$  and  $q$  of the curve  $C$ : it comes out the same for any two paths which start at  $p$  and end at  $q$ .

**26.10** Calculate the integral of  $\eta = 2xy dx + x^2 dy$  over the curve:



Hint: can we write  $\eta = df$  for some function  $f$ ?

**26.11** Calculate the integral of  $\eta = -2\pi x \sin(\pi x^2 + y) dx - \sin(\pi x^2 + y) dy$  over the curve:



Hint: can we write  $\eta = df$  for some function  $f$ ?

**26.12** Let

$$\omega = \frac{-y dx + x dy}{x^2 + y^2},$$

the *angle form*. Prove that the integral along a curve in the plane, *not* passing through the origin, is the total angle the curve travels around the origin.

The integral of the angle form *depends* on the curve; you don't know how many times a curve winds around the origin if you only know where it starts and where it stops.

## The Cauchy integral theorem

This section can be omitted without loss of continuity.

Let  $\mathbb{C}$  be the set of complex numbers, with  $i = \sqrt{-1}$ . Each complex number  $z$  splits uniquely into  $z = x + iy$ , with  $x$  and  $y$  real, and this identifies  $\mathbb{C} = \mathbb{R}^2$ . Write  $\bar{z}$  to mean  $x - iy$ , and write  $dz$  to mean  $dx + i dy$ , etc.

Every complex-valued 1-form can be written in terms of  $dx$  and  $dy$ , with complex number coefficients. Solve the equations

$$\begin{aligned} dz &= dx + i dy, \\ d\bar{z} &= dx - i dy, \end{aligned}$$

to find that

$$\begin{aligned} dx &= \frac{1}{2} dz + \frac{1}{2} d\bar{z}, \\ dy &= -\frac{i}{2} dz + \frac{i}{2} d\bar{z}. \end{aligned}$$

Therefore every complex valued differential form on any open subset of  $\mathbb{C}$  can be expressed in terms of  $dz$  and  $d\bar{z}$ . Each complex valued 1-form looks like  $\omega = f(x, y) dz + g(x, y) d\bar{z}$ , for some continuous functions  $f(x, y)$  and  $g(x, y)$ . We take exterior derivative of a complex valued form by  $d(\xi + i \eta) = d\xi + i d\eta$ , for the real and imaginary parts  $\xi$  and  $\eta$ . A continuously differentiable complex valued function  $f$  is *holomorphic* if  $df \wedge dz = 0$ .

**Theorem 26.2.** *Suppose that  $U \subset \mathbb{C}$  is a bounded domain with continuously differentiable boundary, perhaps with corners. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(w)}{w - z} dw,$$

for every point  $z \in U$  and every holomorphic function  $f$  on  $\bar{U}$ . In particular, the holomorphic function  $f$  is completely determined by its values on  $\partial U$ .

*Proof.* Let

$$\omega = \frac{1}{2\pi i} \frac{f(w)}{w - z} dw.$$

Calculate  $d\omega = 0$ . Pick  $\varepsilon > 0$  small enough that  $\bar{B}_\varepsilon z \subseteq U$ . Let  $U_\varepsilon = U - B_\varepsilon z$ .

By Stokes's theorem,

$$\int_{\partial U_\varepsilon} \omega = 0.$$

If  $\varepsilon$  is small enough that  $B_\varepsilon 0$  does not intersect  $\partial U$ , then

$$\partial U_\varepsilon = \partial U \cup \partial B_\varepsilon 0,$$

with opposite orientations on the two boundary components, so

$$\begin{aligned} \int_{\partial U} \omega &= \int_{\partial B_\varepsilon 0} \omega, \\ &= \frac{1}{2\pi i} \int_{\partial B_\varepsilon 0} \frac{f(w)}{w - z} dz, \end{aligned}$$



to which we apply the change of variable  $w = z + \varepsilon e^{i\theta}$  to get

$$\begin{aligned} &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta, \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta}) d\theta, \end{aligned}$$

just giving the average value of  $f$  over the circle of radius  $\varepsilon$  around  $z$ , which by continuity approaches  $f(z)$  as  $\varepsilon \rightarrow 0$ .  $\square$

**26.13** Recall that a polynomial  $p(z)$  has a zero of *multiplicity*  $k$  at  $z = z_0$  if  $p(z)/(z - z_0)^k$  is a polynomial not vanishing at  $z = z_0$ . Suppose that  $p(z)$  is a nonzero complex polynomial of a complex variable, and that  $U$  is a bounded domain with continuously differentiable boundary and corners. Suppose that  $p(z) \neq 0$  for any  $z \in \partial U$ . Prove the *argument principle* for polynomials: that the number of zeroes of  $p(z)$  inside  $\bar{U}$ , counted with multiplicity, is

$$\frac{1}{2\pi i} \int_{\partial U} \frac{dp}{p}.$$

Use this to prove that every polynomial  $q(z)$  with coefficients close enough to those of  $p(z)$  has the same number of zeroes inside  $\bar{U}$ , counted with multiplicity.

**26.14** Suppose that  $p(z) = a_n z^n + \cdots + a_1 z + a_0$  is a polynomial with no roots. Use the Cauchy integral theorem to prove that, for any  $\varepsilon$  with  $0 < \varepsilon < 1$ , for all sufficiently large  $r > 0$ ,

$$\frac{2\pi}{|p(0)|} \geq \frac{2\pi}{|a_n|(1 - \varepsilon)r^{n-1}}.$$

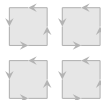
Prove the fundamental theorem of algebra.

## Notes

If we cut up an oriented box



into a grid, the grid boxes orient like:



(We insert a little space between the grid boxes to make room to draw the arrows, but there really isn't any such space.) When two grid boxes meet, they meet along a pair of sides that run in opposite directions. The integrals over these sides cancel each other out. Therefore the integral over the boundary of the original box is the sum of the integrals over the grid boxes.

## Chapter 27

### The Brouwer fixed point theorem

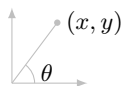
This chapter can be omitted without loss of continuity. We prove a fundamental result in topology: any map from a closed ball to itself has a fixed point.

#### The angle that a curve winds around the origin

Recall polar coordinates:  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$(x, y) = F(r, \theta) = (r \cos \theta, r \sin \theta).$$

The quantity  $\theta$  is the angle of a point  $(x, y)$  measured from the origin, starting at the positive  $x$ -axis.



But  $\theta$  is not really a *function* of  $(x, y)$ , because  $\theta$  represents the same angle if you increase  $\theta$  by  $2\pi$ . This map  $F$  is not a diffeomorphism for 2 reasons:

- a.  $F(r, \theta) = F(r, \theta + 2\pi)$ , so different inputs with same output, and
- b. check that  $F'(0, \theta) = 0$ .

Recall

$$\begin{aligned} F^* dx &= \cos \theta dr - r \sin \theta d\theta, \\ F^* dy &= \sin \theta dr + r \cos \theta d\theta. \end{aligned}$$

Of course  $r = \sqrt{x^2 + y^2}$ . The value of  $\theta$  cannot be expressed in terms of  $x$  and  $y$ , as we said:  $F(r, \theta + 2\pi) = F(r, \theta)$ . But strangely  $d\theta$  is expressible in terms of  $x$  and  $y$ :

$$d\theta = F^* \frac{-y dx + x dy}{x^2 + y^2}.$$

Therefore, even in the  $x, y$  variables, we always write  $d\theta$  to mean  $\frac{-y dx + x dy}{x^2 + y^2}$ . If  $(x(t), y(t))$  is a curve winding around the origin, then the total angle it winds is

$$\int d\theta.$$

**27.1** Calculate the integral of  $d\theta$  around a circle centered at the origin. Why is  $d\theta$  not exact (i.e. not really  $d$  of something) in the  $x, y$ -plane minus the origin?

To compute integrals of areas of regions  $X$  in the plane, we need to integrate

$$\int_X dx \wedge dy.$$

If we can write  $X$  in terms of polar coordinates, it might be easier.

**27.2** How to compute areas: calculate that

$$F^* dx \wedge dy = r dr \wedge d\theta.$$

**27.3** Find the area of the disk of radius  $R$  in the plane around the origin, using polar coordinates.

**27.4** Calculate the area of the disk by using Stokes's theorem applied to the 1-form  $\xi = y dx$  and computing  $\int \xi$  around the boundary of the disk.

**27.5** Prove that  $\mathbb{R}^2$  is not  $C^2$  diffeomorphic to  $\mathbb{R}^2 - \{0\}$ .

### Solid angles in three dimensional space

In spherical coordinates:

$$x = r \sin \alpha \cos \beta,$$

$$y = r \sin \alpha \sin \beta,$$

$$z = r \cos \alpha$$

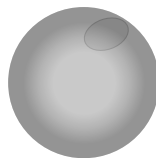
the *solid angle form* is  $\Omega := \sin \alpha d\alpha \wedge d\beta$ . Clearly  $d\Omega = 0$ . A boring calculation gives

$$dx \wedge dy \wedge dz = r^2 dr \wedge \Omega.$$

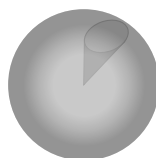
Another one gives us the expression for  $\Omega$  in  $(x, y, z)$ -coordinates:

$$\Omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

**Lemma 27.1.** Take an open subset  $U \subset S$  of the unit sphere  $S \subset \mathbb{R}^3$ .



Let  $C$  be the solid cone that extends from the origin out to  $U$ .



Then

$$\int_U \Omega = 3 \operatorname{Vol} C.$$

*Proof.* For any two radii  $R_1 < R_2$ , let  $U_{R_1, R_2}$  be the set of points  $(x, y, z)$  so that  $R_1 < |(x, y, z)| < R_2$  and

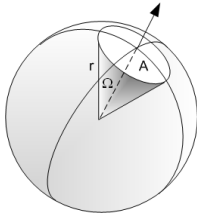
$$\frac{(x, y, z)}{|(x, y, z)|} \in U.$$

Clearly  $U_{R_1, R_2}$  is an open subset of  $\mathbb{R}^3$ .

$$\begin{aligned} \operatorname{Vol} U_{R_1, R_2} &= \int_{U_{R_1, R_2}} r^2 dr \wedge \Omega, \\ &= \int_{R_1}^{R_2} r^2 dr \int_U \Omega, \\ &= \frac{r^3}{3} \Big|_{R_1}^{R_2} \int_U \Omega, \\ &= \frac{R_2^3 - R_1^3}{3} \int_U \Omega. \end{aligned}$$

Take the limit as  $R_1 \rightarrow 0$  and set  $R_2 = 1$ . □

We call  $\int_U \Omega$  the *solid angle* of a subset  $U$  of the sphere:



**27.6** What is the solid angle of the entire unit sphere about the origin in  $\mathbb{R}^3$ ? What about the sphere of radius  $r$  about the origin?

**27.7** Prove that  $\mathbb{R}^3$  is not  $C^2$  diffeomorphic to  $\mathbb{R}^3 - \{0\}$ .

Hook

If you plug just one vector field into a 2-form, you are left with a 1-form:

$$dx \wedge dy(e_1, \cdot) = dy.$$

If  $\omega$  is a  $p$ -form and  $X$  is a vector field, we denote by  $X \lrcorner \omega$  the  $(p-1)$ -form

$$\omega(X, \cdot, \dots, \cdot).$$

In other words,  $X \lrcorner \omega$  is the  $(p-1)$ -form that eats any vector fields  $Y_1, Y_2, \dots, Y_{p-1}$  and splits out

$$X \lrcorner \omega(Y_1, Y_2, \dots, Y_{p-1}) = \omega(X, Y_1, Y_2, \dots, Y_{p-1});$$

just stick an  $X$  in the first slot, and you have a  $(p-1)$ -form.

The *solid angle form*  $\Omega$  on  $\mathbb{R}^n$  is the  $(n-1)$ -form which, at any point  $x \in \mathbb{R}^n - \{0\}$ , is

$$\Omega = \frac{1}{|x|^n} x \lrcorner dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

**27.8** Prove that  $d\Omega = 0$ .

**27.9** For  $x \in \mathbb{R}^n$ , let  $r = |x|$ . Prove that

$$d(r^2) \wedge \Omega = 2r^{3-n} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Conclude that in “polar coordinates” in  $\mathbb{R}^n$

$$dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = r^{n-1} dr \wedge \Omega.$$

Repeating the integration above,

$$\int_U \Omega = n \text{Vol } U_{0,1} > 0,$$

in  $\mathbb{R}^n$  just as in  $\mathbb{R}^3$ , for any open subset  $U \subset S$  of the unit sphere  $S \subset \mathbb{R}^n$ , or of any sphere around the origin.

**27.10** Prove that the number

$$\int_S \Omega$$

is the same number for any sphere  $S$  which surrounds the origin, and that it is 0 for any sphere which is the boundary of a ball not containing the origin.

**27.11** Check that the solid angle of a piece of a circle around the origin is the usual notion of angle.

**27.12** Prove that  $\mathbb{R}^n$  is not  $C^2$  diffeomorphic to  $\mathbb{R}^n - \{0\}$ .

## The Brouwer fixed point theorem

**Theorem 27.2.** [The Brouwer fixed point theorem] Let  $\bar{B}$  be a closed ball in  $\mathbb{R}^n$ . Suppose that  $F: \bar{B} \rightarrow \bar{B}$  is a  $C^2$  map. Then  $F$  has a fixed point.

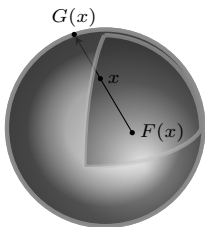
*Proof.* Suppose not. Assume that  $\bar{B} = \bar{B}_1 0$ . Assume that  $F$  has no fixed point. Let  $S = \partial B$ . Take any point  $x \in \bar{B}$ . Since  $F(x) \neq x$ , the two points  $x$  and  $F(x)$  lie on a unique line. Follow that line starting from  $F(x)$  in the direction of  $x$  until you get to a point of  $S$ . Call that point  $G(x)$ .

This defines a map  $G: \bar{B} \rightarrow S$ , so that  $G(x) = x$  for  $x \in S$ . We let the reader prove that  $G$  is continuously differentiable on  $\bar{B}$ . Compute solid angles:

$$0 < \int_S \Omega,$$

but  $G(x) = x$  on  $S$  so

$$= \int_S G^* \Omega,$$





to which we apply Stokes's theorem

$$\begin{aligned} &= \int_{\bar{B}} dG^* \Omega, \\ &= \int_{\bar{B}} G^* d\Omega, \\ &= 0 \end{aligned}$$

because  $d\Omega = 0$ . □

**27.13\*** Prove that the map  $G$  described in the proof of the Brouwer fixed point theorem is well defined and  $C^2$ .

**27.14** In the proof of the Brouwer fixed point theorem: the form  $\Omega$  is not defined at 0, so is only defined on  $\mathbb{R}^n - \{0\}$ . So why is  $G^* \Omega$  defined and continuously differentiable on all of  $\bar{B}$ ?

**27.15** The Brouwer fixed point theorem holds true for merely continuous maps, not necessarily  $C^2$ . Use that fact (which we prove in theorem 36.1 on page 277) to solve this problem.

- a. Take two identical pieces of graph paper of the same size and shape. Pick up one piece of paper and crumple it. Put it on top of the other piece of paper. Prove that some point of the crumpled piece of paper is now lying exactly on top of the corresponding point of the lower sheet. (Assume that crumpling is continuous.)
- b. Take a still cup of coffee and stir it. Wait until the coffee is still again. Prove that some point inside the fluid of the coffee is now located exactly where it was before you stirred the coffee. (Assume that stirring is continuous.)

**27.16** Suppose that  $X$  is a  $C^2$  vector field on the unit closed ball  $\bar{B} \subset \mathbb{R}^n$ . Suppose that  $X(x) = x$  for all  $x \in \partial\bar{B}$ ; in other words  $X$  points “straight out of the sphere” at all points of the unit sphere. Prove that  $X$  has a zero somewhere inside the ball  $\bar{B}$ . Hint: use the map

$$F(x) = \frac{X(x)}{\|X(x)\|},$$

integration of the solid angle form  $\Omega$ , Stokes's theorem, and the fact that  $d\Omega = 0$ .

**27.17** Write down a smooth diffeomorphism  $F: \bar{B} \rightarrow \bar{B}$  of a closed ball  $\bar{B} \subset \mathbb{R}^n$  so that every fixed point of  $F$  lies on the boundary.

## Notes

The Brouwer fixed point theorem is also true for merely continuous maps; one can “smooth out” continuous maps of manifolds to prove the result; see theorem 36.1 on page 277. The Brouwer fixed point theorem differs from the contraction mapping theorem because the map in Brouwer's theorem might stretch some places and squish others, just as long as it fits the closed ball inside itself. A contraction map has to squish everywhere, stretch nowhere.



## Chapter 28

### Maps and integrals

#### Maps

**Theorem 28.1.** *Suppose that  $\varphi: P \rightarrow Q$  is a proper continuously differentiable map between oriented manifolds with corners of equal dimension, say  $n$ . For any compactly supported  $n$ -form  $\omega$  on  $Q$ ,*

$$\int_P \varphi^* \omega = \deg \varphi \int_Q \omega.$$

*Proof.* If  $\omega$  is supported in a small enough open set around a regular value, then the result is clear by lemma 22.2 on page 188. Using a partition of unity, the result becomes clear if  $\omega$  vanishes near the critical values. Again using a partition of unity, we only need to prove the result for every  $n$ -form  $\omega$  supported in arbitrarily small neighborhood of each critical value.

Pick a critical value  $q_0 \in Q$  away from  $\partial Q$ . We leave the reader to prove that there is a continuously differentiable proper homotopy  $\Phi: [0, 1] \times P \rightarrow Q$  so that  $\Phi_0 = \varphi$  and so that  $\Phi_1$  has  $q_0$  as a regular value, and  $\Phi_t = \varphi$  on  $\partial P$  for all  $t$ . Pick an  $n$ -form  $\omega$  on  $Q$  with compact support contained in the regular values of  $\Phi_1$ . Because  $d\omega = 0$ ,

$$\begin{aligned} 0 &= \int_{[0,1] \times P} \Phi^* d\omega, \\ &= \int_{[0,1] \times P} d\Phi^* \omega, \end{aligned}$$

to which we apply Stokes's theorem:

$$= \int_{\{1\} \times P} \Phi^* \omega - \int_{\{0\} \times P} \Phi^* \omega - \int_{[0,1] \times \partial P} \Phi^* \omega,$$

but on  $[0, 1] \times \partial P$ ,  $\Phi^* \omega = \varphi^* \omega$  is independent of  $t$ , so has no  $dt$  component, so the last term above vanishes

$$\begin{aligned} &= \int_P \Phi_1^* \omega - \int_P \varphi^* \omega, \\ &= \deg \Phi_1 \int_Q \omega - \int_P \varphi^* \omega, \\ &= \deg \varphi \int_Q \omega - \int_P \varphi^* \omega. \end{aligned}$$

By the same partition of unity argument, our result now holds for any  $n$ -form  $\omega$  whose support stays away from  $\partial Q$ . Pick a continuous function  $h$  on  $Q$ , vanishing in a thin layer near  $\partial Q$ , and increasing to 1 on slightly thicker layer near those points of  $\partial Q$  where  $\omega \neq 0$ . If we make these layers thin enough, their preimages will be as thin

as we want, lying close to  $\partial P$ , by continuity. Therefore we can arrange that  $\int_Q h\omega$  is as close to  $\int_Q \omega$  as we want, and simultaneously that  $\int_P \varphi^* h\omega$  is as close to  $\int_P \varphi^* \omega$  as we want. By the above,

$$\int_P \varphi^* h\omega = \deg \varphi \int_Q h\omega.$$

Taking the limit as our layers get thin,

$$\int_P \varphi^* \omega = \deg \varphi \int_Q \omega.$$

□

**Corollary 28.2.** *If  $\varphi: P \rightarrow Q$  is a proper continuously differentiable map between compact oriented manifolds with corners, of equal dimension, then the degree is an integral:*

$$\deg \varphi = \frac{\int_P \varphi^* \omega}{\int_Q \omega},$$

for any differential form  $\omega$  with  $0 \neq \int_Q \omega$ .

Suppose that  $\varphi: S^1 \rightarrow S^1$  is a  $C^1$  map of the circle  $S^1$ . Problem 22.6 on page 190 gives a  $C^1$  homotopy taking  $\varphi$  to the map  $e^{i\theta} \mapsto e^{in\theta}$ , for a unique integer  $n$ . Calculate that this map has degree  $n$  at every point. Apply the homotopy invariance of lemma 22.3 on page 190 to conclude that  $\varphi$  has degree  $n$  above every regular value. Take an angle function  $\Phi$  for  $\varphi$ :

$$\deg \varphi = \frac{1}{2\pi} (\Phi(2\pi) - \Phi(0)) = \frac{1}{2\pi} \int_0^{2\pi} d\Phi = \frac{1}{2\pi} \int_0^{2\pi} \varphi^* d\theta.$$

**28.1** Suppose that  $\varphi: P \rightarrow Q$  is a continuously differentiable map and that  $P$  and  $Q$  are manifold with corners equipped with volume forms  $dV_P$  and  $dV_Q$ . The *Jacobian* of the map is the function  $J_\varphi$  so that  $\varphi^* dV_Q = J_\varphi dV_P$ . If  $P$  and  $Q$  are compact and have equal volume and  $|J_\varphi| \leq 1$  (the map locally contracts volumes) and there is a point where  $|J_\varphi| < 1$ , then prove that  $\varphi$  is not surjective.

**28.2** Consider the map  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,

$$(x, y) = \varphi(s, t) = \left( \frac{s}{\sqrt{1+t^2}}, \frac{st}{\sqrt{1+t^2}} \right).$$

Prove that every disk around the origin in the  $x, y$ -plane has infinite area preimage in the  $s, t$ -plane. So there is no easy change of variables formula for integrals for improper maps.

### The Gauss map

Suppose that  $P \subset \mathbb{R}^3$  is an oriented surface. At each point  $p \in P$ , let  $\nu(p)$  be the positively oriented unit normal vector. This  $\nu(p)$  is a unit length vector, so lies in the sphere  $S^2 \subset \mathbb{R}^3$ . The *Gauss map* is the map

$$\nu: P \rightarrow S^2, p \mapsto \nu(p).$$

Careful: of course  $\nu$  is just the unit normal vector  $\nu$ , but now we think of  $\nu$  now as a map to a different manifold, while we previously thought of the unit vector  $\nu$  as a normal vector field.

**28.3** If  $P$  is a  $C^k$  surface, some  $k \geq 1$ , prove that  $\nu$  is  $C^{k-1}$ .

If  $dA_P$  is the area form on  $P$ , every 2-form on  $P$  can be written uniquely as  $\omega = f dA_P$ , for some function  $f$ . If  $P$  is a twice continuously differentiable surface, then  $\nu$  is continuously differentiable, so we can write

$$\nu^* dA_{S^2} = K dA_P,$$

for a unique continuous function  $K$ , called the *Gauss curvature* of  $P$ . The Gauss curvature is large if the normal vector field changes rapidly in different directions, so that the Gauss map covers a large area of the sphere from a small area of  $P$ .

By corollary 28.2 on the preceding page, we have the *Gauss–Bonnet theorem*:

$$\deg \nu = \frac{\int_P K dA}{4\pi} :$$

the total curvature added up over the surface (and suitably scaled) is an integer, representing the degree of the Gauss map. For example, if  $P$  is a sphere centered at the origin, then

$$\nu(x) = \frac{x}{|x|}$$

has degree 1.

**28.4** The same ideas work for curves in the plane. Define the Gauss map of a plane curve. Draw pictures to explain what the Gauss map of a figure 8 looks like. Find the degree of the Gauss map of a figure 8.

**28.5\*** Find the Gauss map and the degree of the Gauss map of the compact oriented surface of genus  $g$  from problem 4.26 on page 47.

## Closed flow lines on surfaces

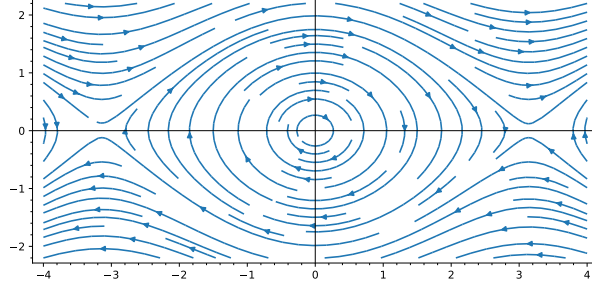
This section can be omitted without loss of continuity.

Take a vector field  $X$  on a oriented surface  $S$ . If some function  $f$  increases along the flow of  $X$ , then clearly there are no closed flow lines of  $X$ . Moreover, there is no continuous loop formed from flow lines.

We want to generalize this elementary observation. Picture a compact portion  $D$  of the surface  $S$  bounded by flow lines of  $X$ . We want to find a test to see that there are no such  $D$ . We further generalize this picture. A *trap* for  $X$  is a compact oriented surface  $D$  with corners and a  $C^1$  map  $\varphi: D \rightarrow S$ , so that

- $\varphi$  has nonnegative degree over its regular values, somewhere positive, and
- the restriction  $\partial\varphi$  of  $\varphi$  to the boundary has nonnegative degree over its regular values, somewhere positive, and
- each regular value of  $\partial\varphi$  has a neighborhood whose  $\partial\varphi$ -preimage maps under  $\partial\varphi$  to a flow line of  $X$ .

An immersive orientation preserving trap (for example, any embedded trap) clearly has image invariant under the flow of  $X$ , and this is the sort of trap we have in mind in formulating the concept. A vector field is *untrapped* if it has no traps. In a simple example, we can spot a trap, an embedded surface, shaped like an eye, with two corners:



A *diverging form*  $\omega$  for a vector field  $X$  on an oriented surface  $S$  is 1-form  $\omega$  so that

- if we let  $f := \omega(X)$  then  $f \leq 0$  and
- $d\omega \geq 0$ , i.e.  $d\omega$  is a nonnegative multiple of a positively oriented area form near each point of  $S$  and either
  - $d\omega > 0$  on a dense set or
  - the set  $f = 0$  contains no nonconstant flow line of  $X$ .

**Theorem 28.3.** *On an oriented surface, any vector field with a diverging form is not trapped. In particular, no closed flow line bounds a compact surface. More generally, no finite collection of flow lines bounds a compact surface with corners.*

*Proof.*

$$0 \geq \int_{\partial D} \varphi^* \omega = \int_D d\omega \geq 0,$$

so both equal zero. We cannot have  $d\omega > 0$  on a dense open set. Therefore the set  $f = 0$  contains no nonconstant flow line of  $X$ . But  $\partial\varphi$  maps some open set  $C \subseteq \partial D$  by an embedding to a flow line of  $X$ , with positive degree. From our integral,  $f = 0$  along that flow line, a contradiction.  $\square$

In the plane, if

$$X = X^1 \frac{\partial}{\partial x^1} + X^2 \frac{\partial}{\partial x^2},$$

has a discrete set of zeroes, and we take

$$\omega = -X^2 dx^1 + X^1 dx^2,$$

then

$$d\omega = \left( \frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2} \right) dx^1 \wedge dx^2.$$

So if the “divergence”

$$\frac{\partial X^1}{\partial x^1} + \frac{\partial X^2}{\partial x^2}$$

is positive, then  $X$  has no closed flow lines. In fact, even if there were some fixed points of  $X$ , i.e. where  $X = 0$ , we could not connect them with finitely many flow lines, forming the boundary of a compact surface with corners.

In the torus  $S = \mathbb{R}^2/\mathbb{Z}^2$ , any smooth vector field of the form

$$X = P(x, y)\partial_x - e^{Q(x, y)} \cos^2 \pi(x - 3y)\partial_y$$

is not trapped, for any functions  $P, Q$  on the torus, unless

$$P = 3 \cos^2 \pi(x - 3y)e^Q.$$

Take  $\omega = dy$  so  $d\omega = 0$  and  $f = -e^Q \cos^2 \pi(x - 3y)$  vanishes only along  $x = 3y$  (up to integer multiples). But  $x = 3y$  contains no nonconstant flow line of  $X$ .





## Chapter 29

### Integration over chains

#### Chains

Let us reconsider the objects we want to integrate over. Rather than integrating over submanifolds, we prefer to allow maps of manifolds, but picture them in our minds as if they were submanifolds; a map of manifolds is a slight generalization of a submanifold. Picture each map to  $M$  as a collection of points, curves, surfaces, etc. inside the manifold  $M$ . We want to say that two maps over which all integrals are the same are equivalent for purposes of integration, and call the equivalence classes *chains*. Careful: we pay attention not only to integrals over the interior, but also to boundary integrals. A *p-body* in a manifold  $M$  is a  $C^1$  map  $\varphi: P \rightarrow M$  from a compact oriented  $p$ -dimensional manifold  $P$  with corners.

If  $M = M_0 \times M_1$ , a  $p$ -body in  $M$  is precisely a  $p$ -body in each of  $M_0, M_1$ .

If we draw a curve in a plane,  we can imagine integrating a 1-form over it:  $\int_{\gamma} \omega$ . We can picture  $2 \int_{\gamma} \omega$  as integrating twice along the same curve  $\gamma$ , and formally write it as  $\int_{2[\gamma]} \omega$ . We imagine that  $2[\gamma]$  means the same curve but travelled along twice.

Take two  $p$ -bodies  $\varphi: P \rightarrow M$  and  $\psi: Q \rightarrow M$ . The *sum*  $\varphi + \psi$  is the map  $\varphi + \psi: P \sqcup Q \rightarrow M$  on the disjoint union, agreeing with  $\varphi$  on  $P$  and  $\psi$  on  $Q$ . Let  $2\varphi := \varphi + \varphi$ , and so on. Write  $-\varphi$  to mean  $\varphi$  with opposite orientation. Let  $\varphi - \psi := \varphi + (-\psi)$ . Let  $0\varphi := \varphi - \varphi$ .

Integrating over a sphere is the same as integrating over its top half and also its bottom half and adding the results. The two halves have disjoint union diffeomorphic to a pair of closed disks, not to a sphere. The two halves are attached by a map which turns the boundaries around the equator in opposite directions, so boundary integrals cancel out, as they should since the sphere has no boundary. So we can rip a sphere into halves, without changing the results of integrals.

A *k-sheet* of a  $p$ -body  $Q \rightarrow M$  is a  $p$ -body  $P \rightarrow Q$  with degree  $k$  at every point of  $Q$  where  $Q \rightarrow M$  has injective differential, except perhaps for a subset of measure zero.

Take a box  $Q$  in the plane, oriented as usual. Slice  $Q$  into two boxes meeting only on their boundary. Pull the two apart to become disjoint; let  $P$  be their union. Map  $p \in P \mapsto q = \varphi(p)$  by sliding the two boxes back together again. So  $\varphi: P \rightarrow Q$  is a 1-sheet.

We want  $k$ -sheets to multiply integrals by  $k$ . Consider the equivalence relation on bodies generated by requiring that, for any bodies,

- $0\varphi \sim 0\psi$ , both denoted 0,
- $\varphi + 0\psi \sim \varphi$ ,
- $\varphi \circ \sigma \sim k\varphi$  for any  $k$ -sheet  $\sigma$ .

Changing orientations,  $-\varphi \sim (-1)\varphi$ .

**29.1** We want integrals over any sum to be the sum of the integrals. If  $\varphi + \psi \sim 0$ , and  $\varphi \sim 0$ , prove that  $\psi \sim 0$ .

If  $M$  is a manifold with corners of dimension  $n$  and  $Q \rightarrow M$  is a  $p$ -body and  $n < p$  then this  $p$ -body is equivalent to zero, because a constant map  $P \rightarrow Q$  is a 1-sheet.

For a body  $\varphi: X \rightarrow M$ , let  $\partial\varphi: \partial X \rightarrow M$  be the composition  $\varphi \circ \iota$  with the attaching map  $\iota: \partial X \rightarrow X$ .

**29.2** Prove that, for any body  $\varphi: X \rightarrow M$ ,  $\partial^2\varphi \sim 0$ .

A  $p$ -chain is the equivalence class of a  $p$ -body. Add chains by disjoint union of maps. Take boundary by boundary of maps. Take negative by negative of maps. The set of  $p$ -chains is the  $p$ -th chain group of  $M$ , denoted  $C_p(M)$ . Denote by 0 the equivalence class of null maps.

**29.3** Work through the definitions to give an elementary description of 0-chains.

**Theorem 29.1.** *The chain groups are abelian groups, and the boundary is a morphism of abelian groups*

$$\dots \xrightarrow{\partial} C_p(M) \xrightarrow{\partial} C_{p-1}(M) \xrightarrow{\partial} \dots$$

with  $\partial^2 = 0$ .

*Proof.* Disjoint union of maps is commutative and associative up to obvious isomorphism.  $\square$

Stokes's theorem:  $\int_{\partial C} \omega = \int_C d\omega$  for any chain  $C$  on a manifold with corners  $M$  and any  $C^1$  differential form  $\omega$  on  $M$ . A *cycle* is a chain with zero boundary.

If  $M$  is a compact oriented manifold, then the identity map  $\text{id}: M \rightarrow M$  has associated chain  $[\text{id}]$ , denoted  $[M]$ , called the *fundamental cycle*

If  $\psi: M \rightarrow N$  is a  $C^1$  map of manifolds with corners and  $\varphi: X \rightarrow M$ , let  $\psi_*\varphi := \psi \circ \varphi$ . This descends to an abelian group morphism on chains:

$$C_*(M) \xrightarrow{\psi_*} C_*(N) ,$$

commuting with  $\partial$ .

## Long chains

Let us remove the compactness hypotheses on chains. For any continuously differentiable proper map  $\varphi: X \rightarrow M$  of an oriented manifold with corners  $X$  and any compactly supported differential form  $\omega$ , clearly  $\int_X \varphi^*\omega$  is defined. Remove compactness from all definitions of chains (calling them *long chains*), but require all maps to be proper instead, and otherwise repeat the definitions above. The set of long  $q$ -chains is the  $q$ -th *Borel–Moore chain group* of  $M$ , denoted  $C_q^{\text{BM}}(M)$ .

**Theorem 29.2.** *The Borel–Moore chain groups are abelian groups, and the boundary is a morphism of abelian groups*

$$\dots \xrightarrow{\partial} C_p^{\text{BM}}(M) \xrightarrow{\partial} C_{p+1}^{\text{BM}}(M) \xrightarrow{\partial} \dots$$

with  $\partial^2 = 0$ .

Stokes's theorem:  $\int_{\partial C} \omega = \int_C d\omega$  for any long chain  $C$  and  $C^1$  compactly supported differential form  $\omega$ .

If  $M$  is an oriented manifold, then the identity map  $\text{id}: M \rightarrow M$  has associated long chain  $[\text{id}]$ , denoted  $[M]$ , called the *fundamental cocycle*

If  $\psi: M \rightarrow N$  is a  $C^1$  proper map of manifolds with corners, to each map  $\varphi: P \rightarrow M$ , we associate the map  $\psi_*\varphi := \psi \circ \varphi$ . This descends to an abelian group morphism on long chains:

$$C_*^{\text{BM}}(M) \xrightarrow{\psi_*} C_*^{\text{BM}}(N) ,$$

commuting with  $\partial$ .

## Intersection

The intersection (i.e. fiber product) of transverse bodies will not in general be a body. Take two chains  $C, D$  represented as bodies  $\varphi: P \rightarrow M$  and  $\psi: Q \rightarrow M$ . If these maps are transverse then we define  $C \cap D$  to be the fiber product  $P \times_M Q$  of the two maps. But this is not oriented, so not a body so far. If  $M$  is oriented, we then define the orientation of  $P \times_M Q$  to be the one for which the exact sequence

$$0 \rightarrow T_{(p,q)}(P \times_M Q) \rightarrow T_p P \oplus T_q Q \rightarrow T_m M \rightarrow 0$$

is compatible with orientations, i.e. there is a positively oriented basis of  $T_{(p,q)}P \times_M Q$  which belongs to the first so many vectors in one of  $T_p P$  and also to the first so many vectors of one of  $T_q Q$ , together mapped to a positively oriented basis for  $T_m M$ . This

determines an orientation in each tangent space, smoothly varying locally, so agrees with exactly one orientation of the fiber product. So in any oriented manifold, the intersection of transverse bodies is a body. In particular, if two chains of complementary dimensions are transverse, their intersection is a finite set of points of  $M$ , with integer multiplicities.

## Chapter 30

### Fubini's theorem

We prove a version of Fubini's theorem on manifolds.

#### Fubini

Take a submersion  $\varphi: P \rightarrow Q$ . The *cross support* of a differential form  $\xi$  on  $Q$  with a differential form  $\eta$  on  $P$  is the intersection of the support of  $\xi$  with the image of the support of  $\eta$ ;  $\xi$  is *compact across*  $\eta$  if the cross support is a compact subset of  $Q$ . A *pushforward* of  $C^k$  differential form  $\eta$  on  $P$  is a  $C^k$  differential form  $\bar{\eta}$  on  $Q$  so that  $\int_Q \xi \wedge \bar{\eta} = \int_P \varphi^* \xi \wedge \eta$ , for any continuous differential form  $\xi$  on  $Q$  compact across  $\eta$ .

**Theorem 30.1** (Fubini). *Suppose that  $\varphi: P \rightarrow Q$  is a  $C^k$  submersion of oriented manifolds with corners, some  $k \geq 1$ . Suppose that  $\eta$  is a  $C^k$  differential form on  $P$ , and that  $\varphi$  is proper on the support of  $\eta$ . Then there is a unique  $C^k$  pushforward  $\varphi_* \eta$  of  $\eta$  on  $Q$ . The map  $f_*$  is linear.*

*Proof.* Uniqueness: take two differential forms  $\eta_1, \eta_2$  which agree in the preimage  $\varphi^{-1}U \subseteq P$  of an open subset  $U \subset Q$ . Suppose they each have pushforwards  $\bar{\eta}_1$  and  $\bar{\eta}_2$ . So

$$\int_Q \xi \wedge \bar{\eta}_1 = \int_Q \xi \wedge \bar{\eta}_2,$$

for any continuous form  $\xi$  on  $Q$  with compact support inside  $U$ . Take a chart in  $Q$  around some point of  $U$ , identifying an open subset of  $Q$  with an open subset of a box. Take  $\xi$  supported in the domain of that chart, so identified with a form supported in that box. Write  $\xi = b(x)dx_I$  and  $\bar{\eta}_1 - \bar{\eta}_2 = a(x)dx_J$ . Since  $\xi$  is arbitrary, and these integrals agree for all  $\xi$ , we see that  $\bar{\eta}_1 = \bar{\eta}_2$ . In particular, the pushforward is unique, if it exists, and we can then denote it  $\varphi_* \eta$  unambiguously.

Existence: first, consider the case when  $P = \mathbb{R}^{q+p}$  and  $Q = \mathbb{R}^q$  and the map is just the standard example  $\varphi(x, y) = x$ . For simplicity of notation, assume that  $\eta = a(x, y)dx_I \wedge dy_J$  and  $\xi = b(x)dx_K$ .

Suppose first that  $dy_J$  has degree too small to be a volume form on the fibers of  $\varphi$ . We want to prove that the unique pushforward is  $\varphi_* \eta = 0$ . Clearly  $\varphi^* \xi \wedge \eta$  is not a volume form, so  $0 = \int_P \varphi^* \xi \wedge \eta$ , for any  $\xi$ . So a differential form  $\bar{\eta}$  is a pushforward just when  $\int_Q \xi \wedge \bar{\eta} = 0$  for any  $\xi$ . So  $\varphi_* \eta = 0$ .

Suppose next that  $dy_J$  is a volume form on the fibers of  $\varphi$ , i.e.  $\eta = a(x, y)dx_I \wedge dy_1 \wedge dy_2 \wedge \dots \wedge dy_n$ . Let  $\bar{\eta} := A(x)dx_I$  where  $A(x)$  is the function  $A(x) = \int a(x, y)dy_1 \dots dy_n$ . By the usual Fubini theorem for functions,

$$\int_P \varphi^* \xi \wedge \eta = \int b(x)A(x)dx_K \wedge dx_I = \int_Q \xi \wedge \bar{\eta},$$

i.e.  $\varphi_*\eta = \bar{\eta}$  is the pushforward. The same reasoning holds if we constrain any of the  $x$  or  $y$  variables to lie in some interval, closed or open. More generally, the same reasoning holds if  $P$  and  $Q$  are open subsets of boxes, and  $\varphi: P \rightarrow Q$  is a linear submersion.

Consider arbitrary manifolds with corners  $P$  and  $Q$  and a  $C^k$  submersion  $\varphi: P \rightarrow Q$ . An open subset  $U \subset P$  is *ok* if every  $C^k$  differential form  $\eta$  on  $P$ , supported in  $U$ , and on whose support  $\varphi$  is proper, has a unique pushforward. We aim to prove that every open set is *ok*.

The implicit function theorem (theorem 19.2 on page 171) shows that every point of  $P$  lies in the domain of a chart on  $P$ , so that the submersion is expressed in that chart, and in some chart on  $Q$ , by a linear submersion of boxes. Hence such an open set is *ok*, i.e.  $P$  is covered in *ok* sets.

Recall that a set is *precompact* if its closure is compact. Suppose that  $\eta$  is supported in a compact set  $K_P \subseteq P$ . Each point of  $K_P$  lies in a precompact open set  $W$  whose closure lies in an *ok* set  $U$ . Cover  $K_P$  by such sets  $W$ , and take a finite subcover  $W_1, \dots, W_N$  with  $W_i \subset U_i$ . Take a smooth partition of unity on  $K_P$  subordinate to this subcover, say  $h_1, \dots, h_N$ . Since  $U_i$  is *ok*, and  $h_i$  is supported in the compact set  $\bar{W}_i \subset U_i$ ,  $h_i\eta$  admits a unique pushforward  $\bar{\eta}_i$  so that

$$\int_Q \xi \wedge \bar{\eta}_i = \int_P \varphi^* \xi \wedge h_i \eta,$$

for all continuous  $\xi$  compact across  $h_i\eta$ . Note that  $\xi$  is compact across  $\eta$  just when  $\xi$  is compact across all  $h_i\eta$ . Therefore  $\varphi_*\eta := \bar{\eta}_1 + \dots + \bar{\eta}_N$ . So every precompact open set is *ok*.

Suppose that  $U_Q \subset Q$  is precompact. We want to prove that  $U_P := \varphi^{-1}U_Q$  is *ok*. Since  $\varphi$  is proper on the support of  $\eta$ , the support of  $\eta$  intersects  $\varphi^{-1}\bar{U}_Q$  in a compact set  $K_P$ . So  $\eta$  is compactly supported, so has a pushforward. So the preimage of any precompact open set in  $Q$  is *ok*.

Cover  $Q$  in precompact open sets  $Q_\alpha$  with a choice of some compactly supported  $C^k$  function  $g_\alpha$  on  $Q$  equal to 1 on  $Q_\alpha$ . By uniqueness,  $\varphi_*(g_\alpha\eta) = \varphi_*(g_\beta\eta)$  on  $Q_\alpha \cap Q_\beta$ . Define  $\varphi_*\eta$  by  $\varphi_*\eta := \varphi_*(g_\alpha\eta)$  on  $Q_\alpha$ , unambiguously. Then

$$\int_Q \xi \wedge \varphi_*\eta = \int_P \varphi^* \xi \wedge \eta,$$

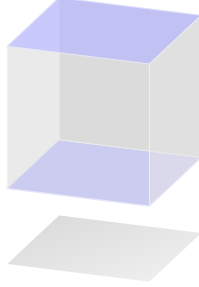
for all continuous  $\xi$  with compact support.

Take a continuous  $\xi$  compact across  $\eta$ , i.e. with compact cross support. Cover that compact cross support in precompact open sets  $Q_\alpha$ . Take a finite subcover, and a subordinate partition of unity  $g_1, \dots, g_k$ . Then

$$\int_Q (g_i\xi) \wedge \varphi_*\eta = \int_P \varphi^*(g_i\xi) \wedge \eta$$

for every  $i$ , so add up to see that  $\varphi_*\eta$  is a pushforward.  $\square$

Take a submersion  $\varphi: P \rightarrow Q$ . The *fibers* are the sets  $P_{q_0} := \varphi^{-1}\{q_0\}$ , i.e. preimages of points. Take a point  $(p, \ell) \in \partial P$ . Denote the attaching map as  $\iota: \partial P \rightarrow P$  and the regular points of  $\varphi \circ \iota$  constitute the *horizontal boundary*  $\partial P$  of  $P$  over  $Q$ . Let  $\iota := \varphi \circ \iota: \partial P \rightarrow P$  be  $\iota$  restricted to  $\partial P$ .



Pick a point  $p_0 \in P$ . Let  $q_0 := \varphi(p_0)$ . Every choice of cowall  $\ell$  for which  $(q_0, \ell) \in \partial Q$  has preimage  $(p_0, \varphi^*\ell) \in \partial P$ . These points  $(p_0, \varphi^*\ell)$  constitute the *vertical boundary* of  $P$  over  $Q$ , denoted by  $\varphi^{-1}\partial Q$  or  $\partial P$ . Define a map

$$\varphi^! : (p_0, \varphi^*\ell) \in \partial P \rightarrow (q_0, \ell) \in \partial Q.$$

**30.1** Prove that  $\partial P = \partial^+ P \sqcup \partial P$ .

Write  $(-1)^\omega$  to mean  $(-1)^k$  if  $\omega$  is a  $k$ -form.

**Corollary 30.2.**

$$\varphi_* d\eta - d\varphi_* \eta = (-1)^{\dim P + \eta} \varphi_* \iota^* \eta$$

In particular, if  $\eta$  pulls back to zero on  $\partial^+ P$ , then  $\varphi_* d\eta = d\varphi_* \eta$ .

*Proof.*

$$\begin{aligned} \int_{\partial P} \varphi^* \xi \wedge \eta &= \int_P d(\varphi^* \xi \wedge \eta), \\ &= \int_P (\varphi^* d\xi) \wedge \eta + (-1)^\xi \varphi^* \xi \wedge d\eta, \\ &= \int_Q d\xi \wedge \varphi_* \eta + (-1)^\xi \xi \wedge \varphi_* d\eta, \\ &= \int_Q d(\xi \wedge \varphi_* \eta) - (-1)^\xi \xi \wedge d(\varphi_* \eta) + (-1)^\xi \xi \wedge \varphi_* d\eta, \\ &= \int_{\partial Q} \xi \wedge \varphi_* \eta + (-1)^\xi \int_Q \xi \wedge (\varphi_* d\eta - d\varphi_* \eta). \end{aligned}$$

Since this holds for all  $\xi$ , and in particular for all  $\xi$  pulling back to zero on the boundary of  $Q$ , we see that

$$\varphi_* d\eta - d\varphi_* \eta = (-1)^{\dim P + \eta} \varphi|_{\partial P}^* \eta.$$

Since  $\xi$  pulls back to zero on  $\partial Q$ , and our map is a submersion,  $\varphi^* \xi$  pulls back to zero on the vertical walls of  $P$ .  $\square$

Here  $\varphi^-$  is a local diffeomorphism, so  $\varphi_*^-\eta$  is actually calculated as a finite sum, not really an integral.



Mapping a cube to an interval  $f(x, y, z) = x$  has  $f^-$  a submersion of each of the 4 faces  $y = 0, y = 1, z = 0, z = 1$ , so  $f_*^-\eta$  is an integral.

**30.2** Consider the map  $y = \varphi(x) = x^2, \varphi: \mathbb{R} \rightarrow [0, \infty)$ . This map is not a submersion. Prove that nonetheless, to each 1-form  $\eta = h(x) dx$  we can associate a unique 1-form  $\varphi_*\eta = h(\sqrt{y})/\sqrt{y}$  so that  $\int_{\mathbb{R}} \varphi^*\xi \wedge \eta = \int_{\mathbb{R}} \xi \wedge \varphi_*\eta$  for any continuous 0-form  $\xi = k(y)$  with compact support. Similarly prove that to any 0-form  $\eta = h(x)$  we can associate a 0-form  $\varphi_*\eta = h(\sqrt{y}) - h(-\sqrt{y})$  so that  $\int_{\mathbb{R}} \varphi^*\xi \wedge \eta = \int_{\mathbb{R}} \xi \wedge \varphi_*\eta$  for any continuous 1-form  $\xi = k(y) dy$  with compact support. However, these are not smooth at the origin, so critical points of smooth maps can cause nonsmoothness of pushforwards.

**30.3** Suppose that  $\varphi: P \rightarrow Q$  and  $\psi: Q \rightarrow R$  are submersions of manifolds with corners. Prove that  $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ .

Consider what happens to Fubini's theorem with maps worse than submersions. Map a compact domain in the plane, one unit wide, to a two unit wide interval of the line by  $\pi(x, y) = x$ . The pushdown  $\pi_*dy$  is a discontinuous function, even though  $\pi'(x, y)$  is onto at each point.



The same problem arises in a hollow square, even though the map  $\pi(x, y) = x$  is onto and  $\pi'(x, y)$  is onto.



For any submersion  $\varphi$ , if we pullback and then pushforward, we always get:  $\varphi_*\varphi^* = 0$ ; in the notation of our proof above, there are no  $dy$  1-forms in  $\varphi^*\xi$  for any  $\xi$ . So in order to get a nonzero result, it is natural to try wedging with something upstairs.

Denote the unit circle in the plane by  $S^1$  and take our usual angle variable  $\theta$  on  $S^1$ , using the usual chart of  $e^{i\theta}$ . Picture rotating the sphere around an axis. More generally, a *circle action* on manifold  $M$  (perhaps with corners) is



a smooth map  $a: S^1 \times M \rightarrow M$  so that, for any  $m \in M$ , if we write  $e^{i\theta}m$  to mean  $a(e^{i\theta}, m)$ , then  $1m = m$  and  $e^{i(\theta_0+\theta_1)}m = e^{i\theta_0}e^{i\theta_1}m$ , for any  $\theta, \theta_0, \theta_1$ . In our current notation, if  $\omega$  is a differential form on  $M$ , then its average over the circle action is

$$\frac{1}{2\pi} a_*((a^*\omega) \wedge d\theta).$$

The fibers of the submersions are circles, no boundary, so pullback and pushforward commute with  $d$ . So the averaging map takes closed forms to closed forms, and exact forms to exact forms. For example, if  $M = S^1$  is just the circle, say this time with an angle variable  $\phi$  then any 1-form on  $M$  looks like  $h(\phi)d\phi$ , with  $h(\phi)$  periodic of period  $2\pi$ .

**30.4** Check explicitly that its average is just  $c d\phi$ , where

$$c = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) d\phi.$$

**30.5** There are two common definitions of pushforward of differential forms under a submersion  $\varphi: P \rightarrow Q$ : the pushforward  $\varphi_*$  we defined above via the equation

$$\int_Q \xi \wedge \varphi_* \eta = \int_P \varphi^* \xi \wedge \eta$$

and a “mirror” pushforward  $\varphi_\star$  defined by

$$\int_Q \varphi_\star \eta \wedge \xi = \int_P \eta \wedge \varphi^* \xi$$

for any compactly supported form  $\xi$  on  $Q$  and any form  $\eta$  on  $P$  on whose support  $\varphi$  is proper. Prove that these are essentially the same:

$$\varphi_\star = (-1)^F \varphi_*$$

where  $F$  is the dimension of the fibers of  $\varphi$ .

## The Poincaré lemma

We want to write down the fundamental theorem of calculus in one variable, in terms of Fubini’s theorem.

**Lemma 30.3** (Poincaré). *Suppose that  $\varphi: P \times [0, 1] \rightarrow Q$  is continuously differentiable,  $P$  and  $Q$  manifolds with corners. Let  $\varphi_t(x) := \varphi(t, x)$ . Let  $\pi: [0, 1] \times P \rightarrow P$  be the obvious map  $(t, p) \mapsto p$ . Then on  $k$ -forms on  $Q$ ,*

$$d\pi_* \varphi^* - \pi_* \varphi^* d = (-1)^{1+k+\dim P} (\varphi_1^* - \varphi_0^*).$$

*In other words, if  $\omega$  is a continuously differentiable differential  $k$ -form on  $Q$  then*

$$d\pi_* \varphi^* \omega - \pi_* \varphi^* d\omega = (-1)^{1+k+\dim P} (\varphi_1^* - \varphi_0^*) \omega.$$

*Proof.* Let  $s_t(p) = (t, p)$ ,  $s_t: P \rightarrow [0, 1] \times P$ . Clearly  $\varphi_0 = \varphi \circ s_0$  and  $\varphi_1 = \varphi \circ s_1$ . Going back to Fubini's theorem, the map  $\varphi^\cdot$  is defined on the horizontal walls in  $[0, 1] \times P$  i.e.  $\{0\} \times P$  and  $\{1\} \times P$ . Compute

$$d\pi_*\varphi^*\omega - \pi_*\varphi^*d\omega = d\pi_*\varphi^*\omega - \pi_*d\varphi^*\omega,$$

to which we apply Fubini's theorem (theorem 30.1)

$$\begin{aligned} &= (-1)^{n+1+k} \pi_*^\cdot \varphi^*\omega, \\ &= (-1)^{n+1+k} (s_1^* - s_0^*) \varphi^*\omega. \end{aligned}$$

□

Suppose that  $X$  is a complete vector field on a manifold  $M$  with corners. Let  $\varphi$  be the flow of  $X$ , so  $\varphi_t(m)$  is the point of  $M$  where  $m$  goes under time  $t$  flow along  $X$ . Then  $\varphi_0(m) = m$ , and so by the Poincaré lemma, on any  $k$ -form  $\omega$  on  $M$ ,

$$\varphi_t^*\omega = \omega + d\eta$$

where

$$\eta = (-1)^{1+k+\dim M} \pi_*\varphi^*\omega.$$

In particular, the flow preserves every differential form, up to an exact form.

**30.6** Prove that any map  $\varphi_1: B \rightarrow B$  on a ball  $B$  is homotopic to a constant map.

**Lemma 30.4.** Take any continuously differentiable map  $\varphi_1: P \rightarrow Q$  of manifolds with corners, homotopic via a continuously differentiable homotopy to a constant map. If  $\omega$  is a closed continuously differentiable differential form, then  $\varphi_1^*\omega$  is exact.

*Proof.* Consider the expression  $\Omega = d\pi_*\varphi^*\omega - \pi_*\varphi^*d\omega$ . Lemma 30.3 on the preceding page says that  $\Omega = (-1)^{n+1+k}(\varphi_1^*\omega - \varphi_0^*\omega)$ . Since  $d\omega = 0$ , we have  $\Omega = d\pi_*\varphi^*\omega - \pi_*\varphi^*d\omega = d\pi_*\varphi^*\omega$ . But  $\varphi_0$  is a constant map so  $\varphi_0^* = 0$ , and so  $d\pi_*\varphi^*\omega = (-1)^{n+1+k}\varphi_1^*\omega$  or  $\varphi_1^*\omega = d\eta$  where  $\eta = (-1)^{n+1+k}\pi_*\varphi^*\omega$ . □

A subset  $U \subset \mathbb{R}^n$  is *star-shaped* if  $tx \in U$  for all  $x \in U$  and  $0 \leq t \leq 1$ .

**Corollary 30.5.** If  $\omega$  is a closed continuously differentiable form on a star-shaped open subset of  $\mathbb{R}^n$ , then  $\omega$  is exact.

*Proof.* Let  $\varphi(t, x) := tx$ . □

**30.7** Take the standard corner  $X = [0, \infty)^n \subset \mathbb{R}^n$ . Prove that any smooth differential form  $\omega$  on  $X$  which pulls back to zero on  $\partial X$  is closed just when  $\omega = d\eta$  for some smooth differential form  $\eta$  on  $X$  which pulls back to zero on  $\partial X$ .

**30.8** Take the standard corner  $X = [0, \infty)^n \subset \mathbb{R}^n$ . Prove that any smooth compactly supported differential form  $\omega$  on  $X$  is closed just when  $\omega = d\eta$  for some smooth compactly supported differential form  $\eta$  on  $X$ .

**Theorem 30.6.** A  $C^1$  differential form on a  $C^2$  manifold, perhaps with corners, is closed just when the manifold is covered in open sets on each of which the form is exact.

*Proof.* Every point lies in the domain of some chart. The chart identifies some small open set around that point with a star-shaped open set. □

## Convolution

For any two continuous differential forms  $\xi, \eta$  on  $\mathbb{R}^n$ , with one of them compactly supported, define  $\xi * \eta$ , the *convolution* of  $\xi$  and  $\eta$  to be

$$\xi * \eta(x) = \int_y \xi(x - y) \wedge \eta(y).$$

In other words, if we define maps  $x: \mathbb{R}^{2n} \rightarrow \mathbb{R}^n, (a, b) \mapsto a, y: \mathbb{R}^{2n} \rightarrow \mathbb{R}^n, (a, b) \mapsto b, \Delta: \mathbb{R}^{2n} \rightarrow \mathbb{R}^n, (a, b) \mapsto a - b$ , then

$$\xi * \eta = x_*((\Delta^* \xi) \wedge (y^* \eta)).$$

If  $\xi$  and  $\eta$  are both compactly supported, then  $\xi * \eta$  is compactly supported.

**30.9** Prove that convolution is commutative up to a possible minus sign.

**30.10** Suppose that  $\xi$  is a  $p$ -form. Prove that  $d(\xi * \eta) = (d\xi) * \eta + (-1)^p \xi * d\eta$ .

**30.11** If  $\xi, \eta, \zeta$  are continuous differential forms, and two of them have compact support, prove that  $\xi * (\eta * \zeta) = (\xi * \eta) * \zeta$ .

## Smoothing

Fix a degree  $p \geq 0$  and suppose that we want to approximate a continuous differential form  $\omega$  with smooth forms. Take a compactly supported  $n$ -form  $\eta$  on  $\mathbb{R}^n$ , with positive integral; rescale to arrange  $\int \eta = 1$ . So  $\eta = g(x)dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ . Consider the homotopy  $\varphi_t(x) := tx$ , and let  $\eta_t := t^p \varphi_t^* \eta$ . In other words,  $\eta_t = t^p g(tx) d(tx_1) \wedge \cdots \wedge d(tx_n) = t^{p+n} g(tx) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ . By definition,  $\int \eta_t = t^p$  for  $t > 0$ . For any continuous differential  $p$ -form  $\omega$  on  $\mathbb{R}^n$ , we let  $\omega_t = \omega * \eta_t$ .

Intuitively,  $\omega_t$  looks like  $\omega$ , but “blurry” and hence smooth.

**Lemma 30.7.** *The differential form  $\omega_t$  approaches  $\omega$  uniformly on any compact set as  $t \rightarrow \infty$ . If  $\omega$  has support  $K_\omega$  and  $\eta$  has support  $K_\eta$ , then  $\omega_t$  has support in the set of points of the form  $x + y/t$  such that  $x \in K_\omega$  and  $y \in K_\eta$ . If  $\omega$  has  $k$  continuous derivatives, then the derivatives of  $\omega_t$  approach those of  $\omega$  uniformly on any compact set.*

*Proof.* Clearly  $\omega(x) = \int_y \omega(x) \wedge \eta_t(y)$  for all  $t > 0$ . Let  $\xi_x(y) = \omega(x - y) - \omega(x)$ . Then

$$\begin{aligned} \omega_t - \omega &= \int_y (\omega(x - y) - \omega(x)) \wedge \eta_t(y), \\ &= \int_y \xi_x(y) \wedge \eta_t(y), \\ &= \int_y \xi_x(y) \wedge t^p \varphi_t^* \eta, \\ &= t^p \int_y \xi_x(y) \wedge \varphi_t^* \eta, \end{aligned}$$

and we change variable by  $y = \varphi_t(z) = tz$ :

$$= t^p \int_z ((\varphi_t^{-1})^* \xi_x)(z) \wedge \eta,$$

but it is easy to differentiate  $\varphi_t^{-1}$ , and compute how it affects  $p$ -forms:

$$\begin{aligned} &= t^p \int_z \xi_x(z/t) t^{-p} \wedge \eta, \\ &= \int_z \xi_x(z/t) \wedge \eta, \\ &= \int_z \left( \omega\left(x - \frac{z}{t}\right) - \omega(x) \right) \wedge \eta. \end{aligned}$$

Keep  $x$  inside a compact set, and as we let  $t \rightarrow 0$ , we see the slight shift from  $x$  to  $x - z/t$  becomes small enough to make  $\omega$  change only slightly. Apply the same to  $\mathcal{L}_X \omega$ , for any constant coefficient vector field  $X$ , and repeat  $k$  times.  $\square$

## Chapter 31

### Open covers

We introduce better methods to work with open covers and partitions of unity.

#### Open covers

An *open cover*  $\{U_a\}$  of a topological manifold  $M$  with corners is a collection of open sets  $U_a$  so that  $\bigcup U_a = M$ .

The disks in the plane form an open cover.

An open cover  $\{U_a\}$  is a *subcover* of an open cover  $\{V_b\}$  if each set  $U_a$  is equal to one of the sets  $V_b$ , say  $U_a = V_{b(a)}$ .

The disks in the plane form an open cover; the unit radius disks in the plane form a subcover of that open cover.

More generally,  $\{U_a\}$  is a *refinement* of  $\{V_b\}$  if each set  $U_a$  lies inside one of the sets  $V_b$ , say  $U_a \subset V_{b(a)}$ .

The unit disks in the plane form an open cover; the disks of radius  $1/2$  form a refinement, but *not* a subcover.

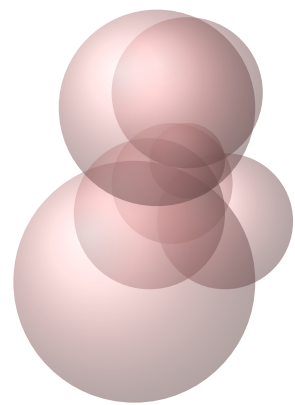
An open cover is *locally finite* if every point lies in an open set which intersects only finitely many of the  $U_a$ .

The unit disks in the plane form an open cover which is not locally finite. The disks of radius 2 centred at the integer points form a locally finite subcover.

**31.1** Prove that an open cover  $\{U_a\}$  is locally finite if and only if every compact set meets only finitely many of the sets  $U_a$ .

**Lemma 31.1.** *Take basis of open sets  $V_\beta$  on a topological manifold with corners  $M$ . Any open cover of  $M$  by open sets  $U_a$  admits a refinement by open sets from among these  $V_\beta$ . In particular, if the  $V_\beta$  are countable, then the refinement is countable.*

*Proof.* Any point  $x$  in any of the  $U_a$ , lies in an open set  $V_\beta$  lying inside  $U_a$ . As refinement: the collection of all  $V_\beta$  contained in one or more of the  $U_a$ .  $\square$



## Coordinate balls

A *coordinate ball* in an  $n$ -dimensional  $C^k$  manifold  $M$  is an open set  $U \subset M$  whose closure  $\bar{U} \subset M$  is  $C^k$  diffeomorphic to a closed ball  $\bar{B} \subset \mathbb{R}^n$ . A *coordinate ball* in an  $n$ -dimensional  $C^k$  manifold with corners  $M$  is an open set  $U \subset M$  whose closure  $\bar{U} \subset M$  is  $C^k$  diffeomorphic to a non-empty intersection  $\bar{B} \cap [0, \infty)^n$  where  $\bar{B}$  is a closed ball.

**Lemma 31.2.** *The  $C^k$  coordinate balls of a  $C^k$  manifold with corners form a basis of open sets,  $k \geq 0$ .*

*Proof.* Every point lies in the domain of a chart, and hence in a coordinate ball. We can make that coordinate ball as small as we like to get inside any smaller open set around that point.  $\square$

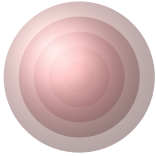
## Bases of “small” open sets

Recall that a set  $A \subset M$  in a manifold is *precompact* or *relatively compact* if its closure  $\bar{A} \subset M$  is compact.

Any coordinate ball is precompact.

**Lemma 31.3.** *Every basis of open sets of any topological manifold (perhaps with corners) admits a refinement by a countable basis of precompact open sets.*

We can even take those precompact open sets to be coordinate balls.



An exhaustion

## Exhaustions

An *exhaustion* is an open cover  $W_1, W_2, \dots$  so that  $W_i \subset W_{i+1}$  is precompact.

**Lemma 31.4.** *For any countable open cover  $U_1, U_2, \dots$  of a topological manifold by precompact open sets, there is an exhaustion  $W_1, W_2, \dots$  so that each  $W_i$  is a finite union of  $U_j$ .*

*Proof.* Let  $W_1 = U_1$ , and let  $k_1 = 1$  and  $k_2$  be the smallest positive integer greater than  $k_1$  so that  $W_2 := U_1 \cup U_2 \cup \dots \cup U_{k_2}$  covers the compact set  $\bar{W}_1$ . Inductively, let  $k_\ell$  be the smallest positive integer greater than  $k_{\ell-1}$  so that  $W_\ell := U_1 \cup U_2 \cup \dots \cup U_{k_\ell}$  covers the compact set  $\bar{W}_{\ell-1}$ .  $\square$

Take an exhaustion  $W_1, W_2, \dots$  and let  $W_0$  be the empty set. Call the set  $T_i := \bar{W}_{i+1} - W_i$  a *thin ring*, and the set  $T'_i := W_{i+2} - \bar{W}_{i-1}$  a *thick ring*. Each thin ring is compact, and lies in the associated thick ring, which is open. Each thin ring only intersects its 2 nearest neighbor thin rings. Each thick ring intersects only its 4 nearest neighbor thick rings.



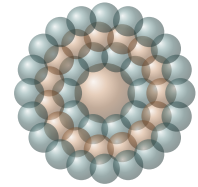
A thick ring and a thin ring

## Ripples

A *ripple* from an open cover  $\{U_a\}$  is an open cover  $V_1, V_2, \dots$  so that each  $V_i$

- a. is a precompact open set and
- b. is a union of elements of  $\{U_a\}$  and
- c. only intersects  $V_{i-1}$  and  $V_{i+1}$ .

Take the open cover of the plane by squares, centered at the integer points, and of length and width  $1 + \varepsilon$ , some  $\varepsilon > 0$ . Let  $V_j$  be the union of those squares whose center  $(x, y)$  has either  $x = \pm j$  or  $y = \pm j$ . Then  $V_0, V_1, \dots$  is a ripple.



A ripple in the plane

**Lemma 31.5.** Suppose that  $\{V_b\}$  is a basis of open sets of a topological manifold with corners  $M$ . Every open cover  $\{U_a\}$  of  $M$  has a countable and locally finite refinement by elements of  $\{V_b\}$  forming a ripple.

*Proof.* By lemma 31.3 on the preceding page, there is a countable subset  $V_1, V_2, \dots$  of  $\{V_b\}$  consisting of precompact open sets, and together refining  $\{U_a\}$ . Construct an exhaustion  $W_1, W_2, \dots$  so that each  $W_i$  is a finite union of these. Construct the thick and thin rings. Cover  $T_i$  in open sets  $V_{i1}, V_{i2}, \dots$  from among the  $\{V_j\}$ , so that the closure of each  $V_{ij}$  lies inside  $T'_i$ . Throw out all but a finite subcover  $V_{i1}, \dots, V_{iji}$ . Throw together all of these finite subcovers for all  $i$ :  $\{V_{ij}\}$ . Since we only throw in finitely many  $V_{ij}$  inside each thick ring, and only four other thick rings intersect any particular thick ring, these  $V_{ji}$  are a locally finite refinement of the open cover  $\{U_a\}$ . Each  $V_{ij}$  stays inside a thick ring, so only strikes at most its two neighboring thin rings, so a ripple.  $\square$

## Good covers

A *contraction* of a topological space  $X$  to a point  $x_0 \in X$  is a homotopy between the identity map and the constant map to  $x_0$ , i.e. a continuous map  $\varphi: X \times [0, 1] \rightarrow X$  so that  $\varphi(x, 0) = x$  and  $\varphi(x, 1) = x_0$  for any  $x \in X$ . A topological space  $X$  is *contractible* if it has a contraction. Clearly it is then connected and has trivial homotopy groups. An open cover of a topological space is *good* if every component of any finite intersection of those open sets is contractible. (One tricky issue: on a manifold, we will assume in addition that the contraction maps are at least  $C^1$ ; eventually we will see that any  $C^0$  contraction can be approximated by a  $C^k$  contraction for any  $k$ , so this won't make any difference in the long run.)

**Theorem 31.6.** Every open cover of a  $C^k$  manifold  $M$ , any  $k \geq 1$ , admits a locally finite refinement by a good cover.

We will prove the stronger theorem 31.7 on page 251.

## Convex functions

A subset  $C \subseteq \mathbb{R}^n$  is *convex* if, for any two points lying in  $C$ , the line segment connecting them lies in  $C$ . A function  $f: C \rightarrow \mathbb{R}$  from a convex subset  $C \subset \mathbb{R}^n$  is *convex* if the points of  $C \times \mathbb{R}$  lying on or above the graph of  $f$  form a convex set.

**31.2** Prove that a function  $f: C \rightarrow \mathbb{R}$  from a convex subset  $C \subset \mathbb{R}^n$  is convex just when, for any two points  $p, q \in C$ ,

$$f((1-t)p + tq) \leq (1-t)f(p) + tf(q).$$

**31.3** Suppose that  $f: C \rightarrow \mathbb{R}$  is  $C^2$ ,  $C \subset \mathbb{R}^n$  convex and open. Let

$$f''(x) = \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right)$$

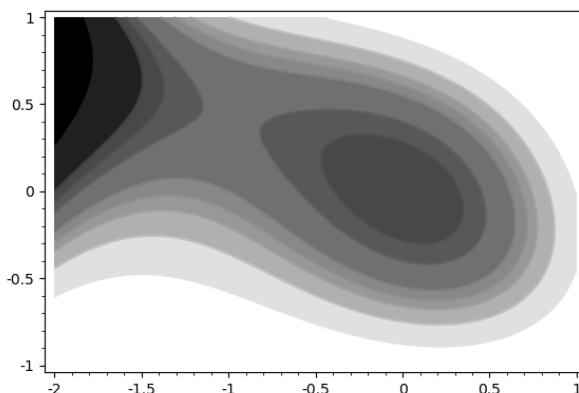
Prove that if  $f$  is convex, then none of the eigenvalues of  $f''(x)$  are negative. Prove that if all of the eigenvalues of  $f''(x)$  are positive, then  $f$  is convex.

**31.4** Suppose that  $f: C \rightarrow \mathbb{R}$  is a convex function on a convex set  $C \subset \mathbb{R}^n$ . Prove that all *sublevel sets*

$$C_{y_0} := \{x \in C \mid f(x) \leq y_0\} \subseteq C,$$

for any constant  $y_0 \in \mathbb{R}$ , are convex or empty.

A *nondegenerate minimum* of a  $C^k$  function  $f: U \rightarrow \mathbb{R}$ , some  $k \geq 2$ ,  $U \subset \mathbb{R}^n$  open, is a critical point where all of the eigenvalues of  $f''(x_0)$  are positive.



**31.5** Suppose that a  $C^k$  function  $f: U \rightarrow \mathbb{R}$ , some  $k \geq 2$ , has a nondegenerate minimum at a point  $x_0$ . Prove that, after perhaps replacing  $U$  by a smaller open set around  $x_0$ , for all sufficiently small  $\varepsilon > 0$ , the sublevel set

$$C_\varepsilon := \{x \in U \mid f(x) \leq \varepsilon\}$$

is convex, compact with nonempty interior and  $C^k$  boundary.

**31.6** Suppose that a  $C^k$  function  $f: U \rightarrow \mathbb{R}$ , some  $k \geq 2$ , has a nondegenerate minimum at a point  $x_0$ . Take a  $C^k$  diffeomorphism  $\varphi: U \rightarrow W$ , and let  $y_0 = \varphi(x_0)$ , and  $g := f \circ \varphi^{-1}$ . Prove that  $y_0$  is a nondegenerate minimum of  $g$ . Prove that, after perhaps replacing  $U$  by a smaller open set, for all small enough  $\varepsilon > 0$ , both of the sublevel sets

$$\{x \mid f(x) \leq f(x_0) + \varepsilon\}, \{y \mid g(y) \leq g(y_0) + \varepsilon\},$$

are convex, compact, with  $C^k$  boundary, and carried one to another by the diffeomorphism  $\varphi$ .



## Fine covers

A  $C^k$  domain in a manifold with corners  $M$  is the interior  $U$  of a  $C^k$  embedded submanifold with corners  $\bar{U} \subseteq M$ , of dimension equal to that of  $M$ . Because  $\bar{U}$  is embedded, if we only know the interior  $U$ , we can determine  $\bar{U}$  by taking the topological closure of  $U$ .

A  $C^k$  fine cover of a manifold with corners  $M$  is a countable collection of precompact  $C^k$  domains  $M_a$  together with  $C^k$  charts  $(U_a, \varphi_a)$ , so that

- a.  $\bar{M}_a \subset U_a$  and
- b.  $\varphi_a(\bar{M}_a \cap \bar{M}_b)$  is convex for any  $a, b$  and
- c. the boundaries of any collection of  $\bar{M}_a$  intersect transversely.

In particular if we take  $a = b$ ,  $\varphi_a(\bar{M}_a)$  is convex. Moreover any intersection of sets  $M_a$ , say for  $a \in A$ , finite or infinite, is identified by every  $\varphi_a$  with  $a \in A$  with a precompact convex domain in Euclidean space.

**Theorem 31.7.** *Every open cover of a  $C^k$  manifold  $M$ , any  $k \geq 1$ , admits a locally finite refinement by a fine cover.*

*Proof.* Select a countable locally finite refinement  $V_1, V_2, \dots$  so that each  $\bar{V}_i$  is compact and contained in the domain of a chart  $\varphi_i$  and has convex image under  $\varphi_i$ . Throw away the original open cover, and prove our result for this open cover. The problem is that that intersections  $\bar{V}_i \cap \bar{V}_j$  might not be convex in any of these charts  $\varphi_k$ . For the first half of the proof, we concentrate on fixing this problem.

Pick a point  $m_0 \in M$ . Pick a  $C^k$  function  $f$  with  $m_0$  as nondegenerate local minimum, and  $f(m_0) = 0$ ; we could let  $f$  be distance squared from  $m_0$  in one of our charts. A number  $r > 0$  presses  $f$  on an open set  $V_i$  if  $f \circ \varphi_i^{-1}$  is convex with  $C^k$  level sets on  $\varphi_i(\bar{V}_i \cap (f \leq r))$ . (We picture  $r$  as a radius, or squared radius, in a chart.) By problem 34.8 on page 268, there is some  $r$  pressing  $f$  on all  $V_i$  containing  $m_0$ . We can make  $r$  smaller to ensure that  $(f \leq r)$  lies inside every  $V_i$  containing  $m_0$ . We can also make  $r$  smaller to ensure that  $f$  is proper on  $(f \leq r)$ .

**31.7** Prove that we can pick  $r$  to press  $f$  on all  $V_i$  with  $m_0 \in \bar{V}_i$ .

**31.8** Prove that we can pick  $r$  to press  $f$  on all  $V_i$ .

Away from  $(f \leq r)$ , extend  $f$  to be defined on all of  $M$  so that  $f > r$ . Rescale to arrange that  $r = 2$ . Let  $U_{m_0} := (f < 1)$ .

Apply lemma 31.5 on page 249 to pick a countable and locally finite refinement  $U_1, U_2, \dots$  of  $V_1, V_2, \dots$  by such sets  $U_m$  for various  $m \in M$ . We have  $C^k$  functions  $f_i$  on  $M$  so that 2 presses  $f_i$  on every  $V_j$  and  $f_i$  is proper on  $(f_i \leq 2)$ . The sets  $(f_i < 1)$  form a locally finite open cover. Each  $(f_i \leq 2)$  is compact and lies inside some  $V_j$  so becomes a compact convex set in the chart  $\varphi_j$ , while  $f_i$  becomes convex. Any other  $f_k$  also becomes convex under  $\varphi_j$  on the  $\varphi_j$  image of  $V_j \cap (f_k \leq 2)$ . So  $\varphi_j$  identifies all

$$(f_i \leq \delta) \cap (f_k \leq r)$$

with compact convex sets, for any  $0 < \delta \leq 2$  and  $0 < r \leq 2$ . More generally, any intersection of sublevel sets  $(f_i \leq r_i)$  is compact, and convex in a suitable chart  $\varphi_j$ , for any  $0 < r_i \leq 2$ .

For the second half of the proof, we apply Sard's lemma to prove that we can choose constants  $r_i$  so that  $\bar{M}_i := (f_i \leq r_i)$  is a fine cover. As long as  $r_i \geq 1$ , the

interiors  $M_i$  form an open cover. We only need prove transversality of the  $f_i$  functions on the intersections of the boundaries of these sets.

For any subset  $I \subseteq \{1, 2, \dots\}$ , let

$$X_I := \bigcap_{i \in I} (1 \leq f_i \leq 2).$$

Each set  $(1 \leq f_i \leq 2)$  lies inside some  $V_i$ , and these  $V_i$  form a locally finite cover. So  $I$  must be finite for  $X_I$  to be nonempty. Hence  $X_I$  is a finite intersection of compact sets, so compact. Suppose that  $I = \{i_1, i_2, \dots, i_n\}$ , written down in order. Let  $Y_I := [1, 2]^n$  and

$$f_I := (f_{i_1}, f_{i_2}, \dots, f_{i_n}): X_I \rightarrow Y_I.$$

This map has a measure zero set of critical values (by Sard's theorem 18.5 on page 158, 19.4 on page 171). Hence every open set in  $Y_I$  contains a regular value. The critical points of  $f_I$  form a closed, hence compact, subset of  $X_I$ . The critical values of  $f_I$  are the images of the critical points, so form a compact subset of  $Y_I = [1, 2]^n$ , containing no open subset. In other words, the regular values of  $f_I$  form a dense open subset of  $Y_I$ . For any sequence positive numbers  $r_1, r_2, \dots$  so that  $1 < r_i < 2$ , let  $r_I := (r_{i_1}, \dots, r_{i_n})$ .

We want to prove that there is a sequence  $r_1, r_2, \dots$ , so that for every  $I$ , if  $X_I$  is not empty, then  $r_I$  is a regular value of  $f_I$ .

Note that if  $J \subseteq I$  and if we have a sequence of values  $r_I$ , i.e. values  $r_i$  for  $i \in I$ , then this projects to a sequence of values  $r_J$ , i.e. values  $r_i$  for  $i \in J$ .

We can order the sets  $I$  alphabetically: each lies in some  $\{1, 2, \dots, N\}$  which we think of as “letters” in an alphabet. Each letter occurs only in finitely many “words”  $I$  with  $X_I$  not empty. So the collection of nonempty sets  $X_I$  is countable. Index our sets  $I$  with  $X_I$  not empty as  $I_1, I_2, \dots$ . Reorder them if needed so that if  $I_i \subset I_j$  then  $i \leq j$ .

Pick a ball  $B_1 \subset Y_{I_1}$  of regular values. By induction, suppose that we have picked balls  $B_1, \dots, B_{N-1}$ , all of the same radius, with  $B_j \subset Y_{I_j}$  sitting inside the regular values. Moreover, suppose that if  $I_i \subseteq I_j$  then  $B_j$  projects to equal  $B_i$ .

Suppose that this next  $X_{I_N}$  intersects some  $X_{I_i}$ . So if we let  $J := I_i \cap I_N$  then  $X_J$  is not empty. So when we pick values of  $r_{I_i}$ , we will also at the same time have to pick values of  $r_J$ , hence of  $r_{I_N}$ , so that  $r_{I_i}$  and  $r_{I_N}$  are both regular values.

Since  $J \subset I_i$ ,  $J$  has already arisen in our construction as some  $J = I_k$ , say, with  $k \leq i$ . We have already picked a ball  $B_k$ , and  $B_i$  projects to this  $B_k$ . Pick a ball of regular values  $B_N \subset Y_{I_N}$  whose closure projects to lie inside  $B_k$ . Replace  $B_k$  by the image of  $B_N$ , and  $B_i$  by a ball in the preimage of the new  $B_k$ , with closure inside the old  $B_1$ , so that all of  $B_i, B_k, B_N$  have the same radius. If we pick any points  $x \in B_i$ ,  $y \in B_N$ , which project to the same point in  $B_k$ , we have a consistent choice of  $r_\ell = x_\ell$  for  $\ell \in I_i$  and  $r_\ell = y_\ell$  for  $\ell \in I_N$ . Repeat for all  $i < N$  until we have achieved our induction hypothesis.

By induction, we obtain successive choices of balls  $B_1, B_2, \dots, B_N$ . But at the  $N^{\text{th}}$  step, when we define  $B_N$ , the balls  $B_1, \dots, B_{N-1}$  get smaller finitely many times. Each time, the closure of the new  $B_1$  lies inside the old  $B_1$  from the previous step. Arrange the radii of every one of our balls to converge to zero as we go through these steps, so that each ball shrinks down successively to pick out a single point. At each step, every ball consists entirely of regular points, so as each ball shrinks to a point, that point is regular. Any two of the points project to the same point when projected to the same ball, so they give consistent values of  $r_i$  with all  $r_I$  regular. Let  $\bar{M}_i := (f_i \leq r_i)$ .  $\square$

## The classification of curves

This section can be omitted without loss of continuity.

A *curve* is a 1-dimensional  $C^k$  manifold with corners, i.e. with boundary.

**Theorem 31.8.** *Every connected  $C^k$  curve with boundary,  $k \geq 0$ , is  $C^k$ -diffeomorphic to precisely one of  $[0, 1]$ ,  $[0, \infty)$ ,  $\mathbb{R}$  or  $S^1$ .*

*Proof.* Clearly it suffices to prove the result for a connected curve. Suppose that the result is true without boundary. Take some curve  $C$  with at least one boundary point. On each boundary point, attach a half line, to make a noncompact curve without boundary, so  $C^k$  diffeomorphic to  $\mathbb{R}$ . Our curve is a connected subset of  $\mathbb{R}$ , so an interval.

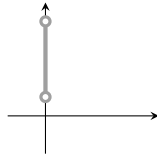
We can suppose that  $C$  has no boundary. We need a lemma before we can finish the proof.

**Lemma 31.9.** *Suppose that  $C$  is a connected  $C^k$  curve without boundary. The universal covering map is a  $C^k$  covering map  $\varphi: \mathbb{R} \rightarrow C$ .*

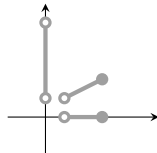
*Proof.* Pick a local  $C^k$  diffeomorphism  $\varphi: I \rightarrow C$  from a compact interval  $I \subset \mathbb{R}$ . We want to extend  $\varphi$  to have larger image. An *open interval* of  $C$  is a set sent by a chart to an open interval of  $\mathbb{R}$ ; similarly define *closed interval*.

**31.9** Prove that  $\varphi$  is a  $C^k$  diffeomorphism on any interval that maps into an open or closed interval.

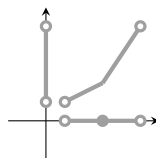
Pick a chart  $(U, \psi)$  on  $C$ ; suppose the chart has image an open interval and that  $U$  intersects  $\varphi(I)$ . We want to prove that we can extend  $\varphi$  to remain a local  $C^k$  diffeomorphism, but defined on a larger interval, so that the image of  $\varphi$  contains  $U$ . If  $U \subset \varphi(I)$ , we are done. So suppose that  $U$  does not lie inside  $\varphi(I)$  but intersects it. Then  $U$  intersects one end or the other of  $\varphi(I)$ . The chart lays out  $U$  along an open interval of the vertical axis in the plane.



In the chart,  $\varphi$  looks like an increasing or decreasing function, ending at that end point.



Extend it to continue increasing or decreasing past that end point.



If it is increasing, extend its graph to increase all the way up that vertical interval. Similarly if it is decreasing, extend its graph to decrease all the way down that vertical interval. Hence a map  $\varphi: J \rightarrow C$ , on some half open interval  $J$ , with image containing one end of  $U$ .

If  $U$  still does not lie inside  $\varphi(J)$ , repeat the construction at the other end of  $I$  to make a larger  $J$ . After these two steps, we can add any open interval  $U \subseteq C$  which intersects  $\varphi(I)$  to  $\varphi(J)$  on some open interval  $J$  containing  $I$ . Moreover,  $\varphi$  is now a  $C^k$  diffeomorphism on any interval mapping to  $U$ .

But even though we can add any open interval  $U$  to the image of  $\varphi$ , we prefer not to: we prefer to keep  $I$  compact, so it helps if we only add a closed interval of  $C$  to  $I$  to create  $J$ . That closed interval can be any closed interval intersecting  $\varphi(I)$ , since each closed interval lies in an open interval. Moreover  $J$  can be any closed interval containing  $I$  in its interior, and we can ensure that  $\varphi$  is a local  $C^k$  diffeomorphism on  $J$  and is a  $C^k$  diffeomorphism on any interval mapping to  $U$ .

Every point of  $C$  lies in some open interval  $U$  with closure  $\bar{U}$  a closed interval. Take a locally finite open cover by such sets  $U_i \subset \bar{U}_i$ . Start above with some  $C^k$  embedding  $\varphi: I_0 \rightarrow C$  of a closed interval. Inductively extend  $\varphi: I_j \rightarrow C$  as above, picking some  $U_i$  which intersects  $\varphi(I_j)$ , and adding  $\bar{U}_i$  to the image of  $\varphi$  on some  $I_{j+1}$ . Any point mapped by  $\varphi$  to a point of  $U_i$  lies in an open interval mapping onto  $U_i$ . Any open interval mapped onto  $U_i$  is mapped  $C^k$  diffeomorphically to  $U_i$ . By local finiteness of open cover, we can repeat this process to ensure that we add all  $\bar{U}_i$  for each  $U_i$  that overlaps  $\varphi(I_j)$ , before we go on to the next stage looking at those which overlap  $\varphi(I_{j+1})$ . By local finiteness, the image of  $\varphi$  eventually covers any given compact set of  $C$  if we make  $j$  large enough. We can pick each  $I_{j+1}$  as we like, as long as  $I_j$  lies in its interior. So we can arrange that these  $I_j$  cover  $\mathbb{R}$ .

**31.10** Prove that  $\varphi: \mathbb{R} \rightarrow C$  is the universal covering map.

□

We finish the proof of the theorem. If  $\varphi$  is injective, being a covering map, it is a proper injective continuous map, so a homeomorphism, and a  $C^k$  local diffeomorphism, so a  $C^k$  diffeomorphism.

Suppose that  $\varphi$  is not injective.

**31.11** Prove that  $C$  is homeomorphic to a circle.

For  $k = 0$ ,  $C$  is therefore a circle. For  $k \geq 1$ , we have classified compact curves already in theorem 6.6 on page 74. □

**31.12** Prove that every  $C^k$  curve with boundary,  $k \geq 0$  or  $k = \infty$ , admits a unique  $C^\infty$  smooth structure up to  $C^\infty$  diffeomorphism.

**31.13** Prove that any two  $C^k$  curves with boundary,  $k \geq 1$ , have zero, one or two (one reversing orientation)  $C^k$  diffeomorphisms between them, up to  $C^k$  diffeotopy. Which have one?

### Notes

The expressions “thin ring” and “thick ring” and “ripple” are not standard, and the reader is encouraged to forget them.



## Chapter 32

### Partitions of unity

#### Partitions of unity

A  $C^k$  partition of unity,  $k \geq 0$ , on a  $C^k$  manifold with corners  $M$  is a collection of  $C^k$  compactly supported functions  $f_a: M \rightarrow \mathbb{R}$  so that every point of  $M$  lies in a open set in which only finitely many of these functions  $f_a$  are nonzero, and  $\sum f_a = 1$ . The partition of unity is *subordinate* to an open cover  $\{U_b\}$  if each function  $f_a$  is supported inside one of the  $U_b$ .

**Theorem 32.1.** *Every open cover of a manifold with corners has a partition of unity subordinate to it.*

*Proof.* Take a manifold with corners  $M$  with open cover; it is enough to prove the result for any refinement of that open cover. Take a countable and locally finite refinement by coordinate balls  $U_1, U_2, \dots$ . Take a nonnegative bump function  $h$  on an open ball, and, for each  $i = 1, 2, \dots$ , pick some diffeomorphism to identify the closure of that ball with the closure of  $U_i$ , and thereby identify  $h$  with a bump function  $h_i$  on  $M$  supported inside  $U_i$ . Any compact set intersects the support of only finitely many of these functions  $h_i$ , so  $h = \sum_i h_i$  is defined (by a finite sum near each point), smooth and positive. Let  $g_i = h_i/h$ : a partition of unity subordinate to the open cover.  $\square$

**Corollary 32.2.** *On any  $C^\infty$  manifold with corners  $P$ , pick a  $C^k$  differential form  $\omega$ , for some integer  $k \geq 0$ . Pick a sequence of open sets  $U_1, U_2, \dots \subset P$  each containing the support of  $\omega$ . There is a sequence  $\omega_1, \omega_2, \dots \rightarrow \omega$  of differential forms, converging with  $k$  derivatives (uniformly on any compact set), each  $\omega_j$  smooth and compactly supported, so that the support of  $\omega_j$  lies inside  $U_j$ .*

*Proof.* For each  $U_i$ , a partition of unity subordinate to charts whose domains lie inside  $U_i$  reduces the problem to a problem in one chart, diffeomorphic to a box in  $\mathbb{R}^n$ ; apply lemma 30.7 on page 245.  $\square$

**32.1** If we can locally find continuous functions “small enough” for some purpose, can we find a global smooth function “small enough”? Suppose that  $M_a \subseteq M$  are open sets of a manifold, and every point of  $M$  lies in an open set intersecting only finitely many  $M_a$ . Take positive continuous functions  $f_a: M_a \rightarrow (0, \infty)$ . Suppose that, for each function  $f_a$  and point  $m_0 \in M$ , there is an open set  $U \subset M$  containing  $m_0$  and a continuous function  $f: U \rightarrow (0, \infty)$  so that on  $M_a \cap U$ ,  $f < f_a$ . Prove that there is a positive smooth function  $f: M \rightarrow (0, \infty)$  with  $f < f_a$  on  $M_a$  for all  $M_a$ .

**32.2\*** Suppose that  $X$  is a smooth vector field on a smooth manifold  $M$ . Prove that there is a smooth function  $f > 0$  on  $M$  so that  $fX$  is complete.

**32.3** Suppose that  $A \subset M$  is a subset of a manifold  $M$  and that  $f: A \rightarrow \mathbb{R}^n$  is a smooth map. Prove that there is smooth map  $F: U \rightarrow \mathbb{R}^n$  defined on an open set  $U$  containing  $A$  so that  $F|_A = f$ . If  $A \subset M$  is closed, prove that we can pick  $U = M$ .

**32.4** Suppose that  $M$  is a manifold with corners and that  $A \subset M$  is a closed sets and  $p_0 \in M - A$ . Prove that there is an open set  $W \subset M$  with  $p_0 \in W$ ,  $\bar{W}$  compact, and  $A \cap \bar{W}$  empty.

**32.5** Suppose that  $A, B \subset M$  are disjoint closed subsets of a  $C^k$  manifold  $M$ , perhaps with corners,  $k \geq 0$ . Prove that there is a  $C^k$  function  $f: M \rightarrow [0, 1]$  so that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

**32.6\*** Prove that every manifold  $M$  with corners bears a proper smooth positive function. Hint: take a partition of unity  $h_1, h_2, \dots$  and try  $h_1 + 2h_2 + \dots$ .

**Theorem 32.3** (Whitney). *Every smooth manifold of any dimension  $n$  admits an embedding into Euclidean space  $\mathbb{R}^{2n+1}$ .*

*Proof.* Problems 32.6, 18.13 on page 158 and lemma 31.4 on page 248.  $\square$

**32.7** Pick a smooth manifold with corners  $M$ . For each wall of  $M$ , i.e. component of the boundary, pick a real number. Prove that there is a function  $f: M \rightarrow \mathbb{R}$  so that

- $f$  is smooth in the interior of  $M$  and
- $f$  is smooth in the interior of each wall and
- $f$  approaches that real number on the interior of that wall and
- all derivatives of  $f$ , in any coordinates, vanish as we approach the interior of any wall.

**32.8\*** Compactly supported deformations of submanifolds arise from isotopies. To be precise: suppose that  $\varphi: [0, 1] \times P \rightarrow Q$  is a  $C^k$  map, some  $k \geq 2$ , with  $\varphi_t(p) := \varphi(t, p)$ , and that  $\varphi_t(p)$  is constant in  $t$  for  $p$  outside some compact subset of  $P$ , and that  $\varphi_t$  is an immersion and is injective on that compact set. Prove that there is a  $C^{k-1}$  diffeotopy  $\Phi: [0, 1] \times Q \rightarrow Q$ , with  $\Phi_t(q) := \Phi(t, q)$ , so that  $\Phi_0$  is the identity map and  $\Phi_t \circ \varphi_0 = \varphi_t$ ,  $0 \leq t \leq 1$ .

## The Whitney embedding theorem

**Theorem 32.4** (Whitney). *Take a manifold  $M$  with corners; let  $n$  be the dimension of  $M$ . Take an integer  $N \geq 2n$ , a smooth map  $\varphi: M \rightarrow \mathbb{R}^N$ , and a real number  $\varepsilon > 0$ . There is an immersion as close as we like to  $\varphi$ , i.e. for any  $\varepsilon > 0$  we can pick an immersion  $\psi: M \rightarrow \mathbb{R}^N$  so that, for any point  $p \in M$ ,  $|\psi(p) - \varphi(p)| < \varepsilon$ .*

*Proof.* We already proved this result for compact manifolds in theorem 18.3 on page 156. We can simply repeat the steps we used in that proof, choosing all open sets to be precompact and all functions having compact support. But we need an infinite sequence of steps, i.e. of precompact open sets  $U$  on whose closure which we slightly perturb our map, in place of the finite number we had there.

We ask that the open cover by these various sets called  $U$  be locally finite, so that we only alter  $\varphi$  finitely many times in each compact set. Each alteration is smooth, so they converge, after finitely many steps in any compact set, to some smooth map.  $\square$



**Theorem 32.5** (Whitney). *Take a manifold  $M$  with corners; let  $n$  be the dimension of  $M$ . Take an integer  $N \geq 2n + 1$ , a  $C^k$  map  $\varphi: M \rightarrow \mathbb{R}^N$ ,  $2 \leq k \leq \infty$ , and a real number  $\varepsilon > 0$ . There is an injective  $C^k$  immersion as close as we like to  $\varphi$ , i.e. for any  $\varepsilon > 0$  we can pick an injective immersion  $\psi: M \rightarrow \mathbb{R}^N$  so that, for any point  $p \in M$ ,  $|\psi(p) - \varphi(p)| < \varepsilon$ .*

*Proof.* We already proved this result for compact manifolds in theorem 18.3 on page 156. With the same alterations as our previous proof, we remove the requirement of compactness; note that we only get an injective immersion, which might not be an embedding since our manifold is not required to be compact.  $\square$

**Theorem 32.6** (Whitney). *Take a manifold  $M$  with corners; let  $n$  be the dimension of  $M$ . Take an integer  $N \geq 2n + 1$ , a  $C^k$  proper map  $\varphi: M \rightarrow \mathbb{R}^N$ ,  $2 \leq k \leq \infty$ , and a real number  $\varepsilon > 0$ . There is a  $C^k$  embedding as close as we like to  $\varphi$ , i.e. for any  $\varepsilon > 0$  we can pick an embedding  $\psi: M \rightarrow \mathbb{R}^N$  so that, for any point  $p \in M$ ,  $|\psi(p) - \varphi(p)| < \varepsilon$ .*

**32.9** Prove theorem 32.6.

**Theorem 32.7** (Whitney). *Take a  $C^k$  manifold  $M$  with corners,  $1 \leq k \leq \infty$ ; let  $n$  be the dimension of  $M$ . Then  $M$   $C^k$ -embeds into  $\mathbb{R}^{2n+1}$ .*

**32.10** Prove theorem 32.7.

From lemma 4.5 on page 43:

**Corollary 32.8.** *Every  $C^k$  embedded submanifold of any manifold admits a  $C^{k-1}$  tubular neighborhood.*

## Collaring

Suppose that  $M$  is a  $C^k$  manifold with corners and that  $S$  is a wall of  $M$ , i.e. a component of its boundary in the sophisticated sense of chapter 21. A *collar* is a  $C^k$  embedding  $\varphi: S \times [0, \infty) \rightarrow M$  so that  $x \in S \mapsto \varphi(x, 0) \in M$  is the boundary attaching map.

**Theorem 32.9** (The collar theorem). *Take a  $C^k$  manifold with corners,  $1 \leq k \leq \infty$ , and pick one of its walls. The following are equivalent:*

- The boundary attaching map is injective on that wall.*
- The boundary attaching map is an embedding of that wall, an embedded wall.*
- That wall has a collar.*

Recall that the boundary attaching map of the teardrop  is *not* injective, so not an embedding, and so there is no collar.

*Proof.* If a wall  $S$  has a collar, it is embedded, since it embeds as  $S \times \{0\}$  in its collar.

Call our manifold with corners  $M$  and let  $S$  be a wall of  $M$ . Near any corner point,  $S$  meets various other walls, and possibly itself.

On any manifold with corners, the boundary attaching map is a proper surjection to the closed set of naive boundary points. Suppose that  $S$  is injectively attached. Then the attached image of  $S$  is a submanifold with corners, embedded and diffeomorphic to  $S$ .

Take a chart  $(U, \varphi)$  in which  $S$  has equation  $(x^1 = 0)$ . Take a vector field

$$X = f(x) \frac{\partial}{\partial x^1}$$

for some function  $f \geq 0$  with  $f > 0$  at some point of  $S$  and  $f$  compactly supported inside  $U$ . At each point,  $S$  has only one cowall, and  $X$  is positive on that cowall at some points. Add up such vector fields using a partition of unity subordinate to a locally finite cover by charts. Multiple that vector field by a positive function to have flow defined all positive time; see problem 32.2 on page 257. The resulting complete vector field doesn't vanish at any point of  $S$ . The flow restricts to  $S \times [0, \infty)$  to a collar.  $\square$

There notion of a topological manifold with corners is unnatural, because we can homeomorphically smooth a corner, so we state the collar theorem for topological manifolds with boundary.

**Theorem 32.10** (The topological collar theorem). *Every wall of a topological manifold with boundary has a collar.*

*Proof.* Let  $M'$  be the quotient of  $M \sqcup \partial M \times [0, 1]$  by making any boundary point  $m$  equivalent to  $(m, 0)$ . Clearly  $M'$  has a collar by definition. We need only prove that  $M'$  is homeomorphic to  $M$ . Every point of  $M$  lies in the domain  $U$  of a chart  $(U, \varphi)$  for which

$$\varphi: U \rightarrow [0, \infty) \times \mathbb{R}^{n-1}$$

is a diffeomorphism to the entire half space. The map  $\varphi$  puts a collar on  $\partial U$ ,

$$\chi: (m, t) \in \partial U \times [0, \infty) \rightarrow \varphi^{-1}(\varphi(m) + te_1).$$

Cover  $M$  in the domains  $U_i$  of such charts  $(U_i, \varphi_i)$ , with associated collars  $\chi_k$ . Pick a partition of unity  $h_1, h_2, \dots$  subordinate to that cover; note that these  $h_i$  are merely assumed continuous.

Let  $f_k := h_1 + \dots + h_k$  and let  $M_k \subset M'$  be the union of  $M$  with the points  $(m, t) \in \partial M \times [0, 1]$  so that  $t \leq f_k(m)$ . So  $f_0 = 0$  and  $M_0 = M$ . Map

$$\psi_k: (m, t) \in (\partial U_k) \times [-1, 0] \mapsto \chi_k(m, -t).$$

Extend to a map

$$\psi_k: (\partial U_k) \times [-1, 1] \rightarrow M'$$

an embedding. Define  $\Psi_k: M_{k-1} \rightarrow M_k$  by making  $\Psi_k(m) = m$  for  $m \in M$  outside the image of  $\psi_k$ . Inside the embedded image of  $\psi_k$ , write each point of  $M_{k-1}$ , as  $\psi_k(m, t)$  with  $-1 \leq t \leq f_k(m)$ . Make  $\Psi_k$  take this point to the point  $\psi_k(m, t')$ , so that the map linearly matches  $t = -1$  with  $t' = -1$ , and  $t = f_{k-1}(m)$  with  $t = f_k(m)$ .

Note that  $\Psi_k$  is the identity outside of the precompact set of points where  $h_k > 0$ . Claim:  $\Psi_k$  is a homeomorphism  $M_{k-1} \rightarrow M_k$ . It is clearly injective and continuous and so an embedding on the image of  $\psi_k$ , but the identity away from it.

On every compact set, this process only changes finitely many times, as there are only finitely many nonzero functions in our partition of unity. So it converges to a homeomorphism  $M \rightarrow M'$ .  $\square$

## Chapter 33

### Star shaped sets

The reader can skip this chapter without loss of continuity. We use some elementary analysis in Euclidean space to prove that every “star shaped” open set in Euclidean space is diffeomorphic to Euclidean space. This is not an important result for us, but a nice application of the flows of vector fields and partitions of unity.

### Whitney decompositions

A *cube* is a box, all of whose side lengths are equal. Its *diameter* is the maximum distance between any two points of the cube (i.e. the distance along the diagonal between opposite vertices).

A *Whitney decomposition* of an open set  $U \subseteq \mathbb{R}^n$  is a sequence of closed cubes covering  $U$

- intersecting only along their boundaries and
- each cube touching at most  $2^n(3^n - 2^n)$  other cubes and
- any two touching cubes have side lengths in ratio of at most 4 to 1 and
- the size of each cube is comparable to the distance of any of its points to the boundary of  $U$ . To be precise, if  $d$  is the minimum distance of any point of the cube to  $\partial U$  and  $\delta$  is the diameter of the cube,

$$1 < \frac{d}{\delta} \leq 4.$$

**Theorem 33.1** (Whitney Decomposition Theorem). *Every open set in Euclidean space  $\mathbb{R}^n$ , not equal to  $\mathbb{R}^n$ , admits a Whitney decomposition.*

*Proof.* A *k-dyadic number* is an integer divided by  $2^k$ , like

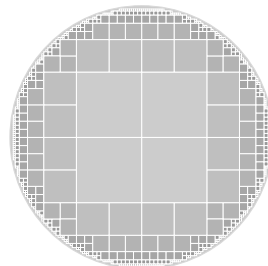
$$\dots, -3/2^k, -2/2^k, -1/2^k, 0/2^k, 1/2^k, 2/2^k, 3/2^k, \dots$$

A *k-dyadic point* in  $\mathbb{R}^n$  is one whose coordinates are *k*-dyadic numbers. A *k-dyadic cube* is one whose sides are *all* of length  $1/2^k$  and whose vertices lie on *k*-dyadic points. Each *k*-dyadic cube has maximum distance between vertices  $\delta = \sqrt{n}/2^k$ , and is uniquely the union of  $2^n$   $(k+1)$ -dyadic cubes, by cutting its sides in half.

Denote distance between a point  $x \in \mathbb{R}^n$  and the boundary  $\partial U$  as  $d_x$ . For any set  $A \subseteq \mathbb{R}^n$ , let  $d_A := \inf_{x \in A} d_x$ .

Take a strictly decreasing sequence  $\varepsilon_k > 0$  for  $k \in \mathbb{Z}$  with  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow \infty$  and  $\varepsilon_k \rightarrow \infty$  as  $k \rightarrow -\infty$ . Take an open subset  $U \subset \mathbb{R}^n$ . Every point  $x \in U$  has distance  $d_x$  to boundary between

$$\varepsilon_k < d_x \leq \varepsilon_{k-1},$$



for a unique integer  $k$ . Suppose in addition that  $\varepsilon_k > \sqrt{n}/2^k$ . Pick a  $k$ -dyadic cube  $C$  containing  $x$ . Then  $C$  lies inside  $U$ , since (by the triangle inequality)

$$\varepsilon_k - \frac{\sqrt{n}}{2^k} < d_C$$

Since  $x \in C$ ,

$$d_C \leq d_x \leq \varepsilon_{k-1}.$$

So

$$\varepsilon_k - \frac{\sqrt{n}}{2^k} < d_C \leq \varepsilon_{k-1}.$$

If we pick

$$\varepsilon_k := \frac{2\sqrt{n}}{2^k},$$

this gives

$$\frac{\sqrt{n}}{2^k} < d_C \leq \frac{4\sqrt{n}}{2^k}.$$

Note that the left hand side is the diameter  $\delta_C$ ; dividing by this,

$$1 < \frac{d_C}{\delta_C} \leq 4.$$

Collect up, for all  $k$ , each  $k$ -dyadic cube that contains a point  $x$  whose distance to  $\partial U$  satisfies  $\varepsilon_k < d \leq \varepsilon_{k-1}$  (if there are any). Throw away any  $k$ -dyadic cube in this collection if it is contained inside a  $(k-1)$ -dyadic in the collection. Take these cubes as our cover.

Suppose that two of these cubes, say  $C$  and  $D$ , intersect. The distance from  $C$  to  $\partial U$  is at worst the distance travelling all the way across  $D$  and then from  $D$  to  $\partial U$ :

$$\delta_C = \frac{\sqrt{n}}{2^{k_C}} \leq d_C \leq \delta_D + d_D \leq \delta_D + 4\delta_D \leq 5\delta_j.$$

This ratio is at most 5, but must be a power of 2, as these are dyadic cubes, so must be at most 4.

Each  $k$ -dyadic cube  $C_i$  will have neighboring cubes which are at smallest  $(k+2)$ -dyadic. So to get a bound on the number of neighbors, we only need to ask how many  $(k+2)$ -dyadic cubes sit touching a  $k$ -dyadic cube. We can rescale and so assume that  $k=0$ , i.e. count 2-dyadic cubes touching the unit cube  $[0,1]^n$ . The 2-dyadic cube neighbor intersects the unit cube along  $j$  common walls, i.e. they will touch on some face given by having some number, say  $j$ , of the  $n$  variables parameterizing the cube set to either 0 or 1, so

$$2^j \binom{n}{j}$$

choices of such face, for each  $j=1,2,\dots,n$ . The intersect is 2-dyadic cube in  $n-j$  dimensions inside the unit cube in those dimensions. There are  $4^{n-j}$  such 2-dyadic cubes. So for given  $j$ , there are

$$4^{n-j} 2^j \binom{n}{j} = 4^n 2^{-j} \binom{n}{j}.$$

Add up over all possible numbers  $j = 1, 2, \dots, n$  of intersections: at most

$$4^n \sum_{j=1}^n 2^{-j} \binom{n}{j} = 4^n \left( \left(1 + \frac{1}{2}\right)^n - 1 \right) = 6^n - 4^n$$

cube neighbors in the covering.  $\square$

**Corollary 33.2.** *Every open set  $U \subset \mathbb{R}^n$  has cover by a sequence of open cubes in  $U$ , with each point contained in at most  $2^n(3^n - 2^n)$  cubes.*

*Proof.* Pick some constant  $\varepsilon$  with  $0 < \varepsilon < \frac{1}{4}$ . Using the notation in the proof of the Whitney decomposition theorem, each cube  $C_i$  has neighbor cubes at least a quarter as large. So if we make an open cube  $U_i$  with the same center point as each  $C_i$ , but sides of size  $1 + \varepsilon$  as large,  $U_i$  only overlaps the neighboring cubes of  $C_i$ .  $\square$

### Star shaped open sets are diffeomorphic to Euclidean space

**Theorem 33.3.** *For any open set  $U \subset \mathbb{R}^n$  with nonempty boundary there is a smooth positive function  $f > 0$  on  $U$  comparable to the distance to the boundary, i.e. so that, for some constants  $0 < A < B$ ,*

$$A < \frac{f(x)}{d_x} < B,$$

where  $d_x$  is the distance of a point  $x \in U$  to the boundary of  $U$ .

*Proof.* By the Whitney decomposition theorem (theorem 33.1 on page 261), we can pick a Whitney decomposition of  $U$  by closed cubes  $C_i$ , say with maximal distance between any two points of  $C_i$  denoted  $\delta_i$ , and distance to  $\partial U$  denoted  $d_i$ . So  $\delta_i < d_i \leq 4\delta_i$ . Take the open cubes given by scaling  $C_i$  around its centre point by  $5/4$ . Each point lies in at most  $2^n(3^n - 2^n)$  of these open cubes. Take a partition of unity  $h_i$  subordinate to this open cover. Let  $f(x) := \sum d_i h_i$ , a smooth function bounded from above by  $\sum d_i$ , a sum over the (at most  $2^n(3^n - 2^n)$ ) open cubes containing  $x$ , giving the distance of the associated closed cubes from the boundary. Each of these cubes is at most  $\delta_i/4$  from  $x$  in order that  $x$  lie in the rescaled cube. So  $d_x \geq d_i - \delta_i/4 \geq 3\delta_i/4 \geq 3d_i/16$ . So

$$f(x) \leq \sum d_i \leq 2^n(3^n - 2^n) \frac{16}{3} d_x.$$

On the other hand, if  $x$  lies in  $C_i$  then  $d_x \leq d_i + \delta_i + \delta_i/4 \leq 9d_i/4$ . So

$$f(x) = \sum d_i h_i \geq \sum \frac{4d_x}{9} h_i \geq \frac{4d_x}{9}.$$

$\square$

A *centre* of a set  $A \subseteq \mathbb{R}^n$  is a point  $x_0 \in A$  so that, for any point  $x \in A$ , the line segment  $x_0x$  lies in  $A$ . A set is *star-shaped* if it has a centre, *convex* if every point is a centre.



**33.1\*** Is there a star-shaped open set in the plane which has only one centre?

The boundary of a star shaped open set might not be smooth. So it might be a surprise that:

**Theorem 33.4.** *Every star shaped open set in  $\mathbb{R}^n$  is diffeomorphic to  $\mathbb{R}^n$ .*

*Proof.* For simplicity of notation, suppose our star shaped open set  $U \subseteq \mathbb{R}^n$  has a centre at the origin. Because  $U$  is open,  $U$  contains a ball around the origin. Rescale it to arrange that it contains the closed unit ball around the origin. Denote the distance of each point  $x \in U$  to the boundary of  $U$  as  $d_x$ . Take a smooth positive function  $f > 0$  on  $U$ , comparable to distance to the boundary (theorem 33.3 on the previous page), say

$$a_0 \leq \frac{f}{d_x} \leq a_1,$$

where  $d_x$  is the distance of a point  $x \in U$  to the boundary of  $U$ . Note that  $f$  is bounded below inside the closed unit ball. Using a bump function, we can arrange that  $f$  is a positive constant inside the unit ball. We can rescale  $f$  to get that constant equal to 1. We can compose  $f$  with some smooth increasing function which behaves like  $x$  for  $0 \leq x \leq 1$  and is bounded from above. Replace  $f$  with this composition, so  $f$  is bounded from above, behaves comparable to  $d_x$  where  $d_x$  is small, and is 1 inside the unit ball.

Consider the vector field  $X(x) = f(x)x$ . Since  $f$  is bounded, the vector field is no faster than linear, so its flow exists for all time. The direction of each flow line  $x(t)$  remains always the same, as  $x(t)$  does not turn, since  $\dot{x} = X(x) = f(x)x$  has the same direction as  $x$ . Define a map  $\varphi: \mathbb{R}^n \rightarrow U$  by  $\varphi(0) = 0$  and, if  $x = ru$  with  $r > 0$  and  $u$  a unit vector, then  $\varphi(x)$  is the time  $\log r$  flow of  $X$  through  $u$ :  $\varphi(x) = e^{\log r X}(u)$ . For all  $r < 1$ ,  $\varphi(x) = x$ .

**33.2** Prove that  $\varphi$  is bijective.

**33.3** Prove that  $\varphi$  is a submersion.

□

**33.4** The boundary of a disk is a circle. The boundary of a rectangle is a continuous curve, not smooth, but still homeomorphic to a circle. (The homeomorphism is difficult to write down; let's not try.) Is there a bounded star shaped open set in the plane whose boundary is not homeomorphic to a circle?

**33.5** Suppose that  $C \subset \mathbb{R}^n$  is a star shaped. Suppose that  $\partial C$  is compact. Finally suppose that, for some centre point of  $C$ , every ray from that centre point strikes  $\partial C$  at a single point. Prove that  $\partial C$  is homeomorphic to the sphere  $S^{n-1} \subset \mathbb{R}^n$ .

## Chapter 34

### Convexity

#### Mean

A *volume form* on a manifold with corners is a nowhere zero differential form of top degree, i.e. an  $n$ -form, if the manifold has dimension  $n$ . The *mean* of a bounded continuous map  $\varphi: M \rightarrow \mathbb{R}^n$  with respect to a volume form  $dV$  with finite volume is

$$\langle \varphi \rangle := \frac{\int_M \varphi dV}{\int_M dV} \in \mathbb{R}^n.$$

Denote this as  $\langle \varphi \rangle_M$  or  $\langle \varphi \rangle_{dV}$  if we have to be more precise. The *center of mass* of a bounded and finite volume submanifold in  $\mathbb{R}^n$  is the mean of the map  $x \in M \mapsto x \in \mathbb{R}^n$ .

If

$$M = M_1 \cup M_2 \cup \cdots \cup M_k$$

is the union of manifolds with corners, of the same dimension as  $M$ , intersecting only on measure zero sets, then

$$\begin{aligned} \langle \varphi \rangle_M &= \frac{\int_M f \varphi dV}{\int_M dV}, \\ &= \frac{\sum \int_{M_i} \varphi dV}{\int_M dV}, \\ &= \sum \frac{\int_{M_i} \varphi dV}{\int_{M_i} dV} \frac{\int_{M_i} dV}{\int_M dV}, \\ &= \sum \frac{\text{Vol } M_i}{\text{Vol } M} \langle \varphi \rangle_{M_i}. \end{aligned}$$

More generally, if  $\psi: P \rightarrow Q$  is a  $k$ -sheet (as in chapter 29), and we take a volume form  $dV_Q$  of finite volume on  $Q$ , and let  $dV_P := \psi^* dV_Q/k$ , then clearly every bounded continuous map  $\varphi$  has

$$\langle \varphi \rangle_Q = \sum \frac{\text{Vol } P_i}{\text{Vol } Q} \langle \varphi \rangle_{P_i},$$

a sum over components  $P_i$  of  $P$ .

Similarly if  $U_i$  is an open cover of  $M$  and  $h_i$  a partition of unity subordinate to

that open cover, then

$$\begin{aligned}
 \langle \varphi \rangle_M &= \frac{\int_M \varphi dV}{\int_M dV}, \\
 &= \frac{\sum \int_{U_i} h_i \varphi dV}{\int_M dV}, \\
 &= \sum \frac{\int_{U_i} h_i \varphi dV}{\int_{U_i} dV} \frac{\int_{U_i} dV}{\int_M dV}, \\
 &= \sum \frac{\text{Vol } U_i}{\text{Vol } M} \langle h_i \varphi \rangle_{U_i}.
 \end{aligned}$$

**Lemma 34.1.** *If  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an affine transformation and  $\varphi: M \rightarrow \mathbb{R}^n$  is a continuous map on a manifold with corners with a volume form then*

$$\langle Af \rangle = A \langle \varphi \rangle.$$

*Proof.* Writing each point  $\varphi(x) \in \mathbb{R}^n$  as a column vector, it is enough to prove for each entry for which it is elementary.  $\square$

Similarly, if  $h: M \rightarrow M$  is any diffeomorphism preserving the volume form  $dV$ , then  $\langle \varphi \rangle = \langle \varphi \circ h \rangle$ .

### Convex sets

A *convex set*  $C \subset \mathbb{R}^n$  is a subset so that if any two points  $x, y$  lie in  $C$ , then the line segment between  $x$  and  $y$  lies in  $C$ . A *convex combination* of points  $x_1, x_2, \dots, x_k$  is a linear combination

$$a_1 x_1 + a_2 x_2 + \dots + a_k x_k$$

for which  $a_1, a_2, \dots, a_k \geq 0$  and

$$1 = a_1 + a_2 + \dots + a_k.$$

By induction, any convex combination of points of a convex set  $C$  lies in  $C$ . The *convex hull* of a set is the smallest convex set containing it, i.e. the intersection of all convex sets containing it. An *affine subspace* of  $\mathbb{R}^n$  is a translate of a vector subspace, i.e. a set

$$x_0 + V = \{x_0 + v \mid v \in V\}$$

for some  $x_0 \in \mathbb{R}^n$  and vector subspace  $V \subset \mathbb{R}^n$ . The *affine span* of a set  $S \subset \mathbb{R}^n$  is the smallest affine subspace containing it, i.e. the intersection of all affine subspaces containing it.

**34.1** Take 4 points  $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$  with no 3 in a line and no plane containing all 4. The *tetrahedron* is their convex hull. Prove that the centre of mass of the tetrahedron is the mean of its vertices:

$$\frac{v_0 + v_1 + v_2 + v_3}{4}.$$

**34.2** Find a simple convex polygon in the plane whose centre of mass is *not* the mean of its vertices.



**Lemma 34.2.** Suppose that  $C \subset \mathbb{R}^n$  is convex. Take a convex combination

$$x = a_1x_1 + a_2x_2 + \dots + a_kx_k$$

of points of  $C$ . If any one  $x_i$  of the points  $x_1, x_2, \dots, x_k$  lies in the interior of  $C$ , and the coefficient  $a_i$  is not zero, then  $x$  lies in the interior of  $C$ .

*Proof.* Wiggling  $x_i$  a little in some direction moves  $x$  in the same direction; apply the implicit function theorem.  $\square$

**Lemma 34.3.** Suppose that  $C \subset \mathbb{R}^n$  is convex. Then  $C$  contains an open subset of the affine span of  $C$ ; the largest such open set is the interior of  $C$ .

The notion of interior here is tricky; it is the interior of  $C$  as a subset of its affine span, not necessarily as a subset of  $\mathbb{R}^n$ .

If  $C$  is a compact line segment in  $\mathbb{R}^3$ , its interior in this new sense of the word is all of  $C$  except its two end points.

*Proof.* Translating, we can assume that  $C$  contains the origin, so its affine span is just its span as in linear algebra. We can replace  $\mathbb{R}^n$  by this span. Then  $C$  contains a basis of  $\mathbb{R}^n$  as well as containing 0. After perhaps a linear transformation,  $C$  contains the standard basis, and so  $C$  contains the open set of points  $x$  with positive coordinates summing to less than 1.  $\square$

**Lemma 34.4.** If  $f: M \rightarrow \mathbb{R}^n$  is a bounded continuous map of a manifold with corners, with a volume form of finite volume, then the mean of  $f$  lies in the interior of the convex hull of  $f(M)$ .

*Proof.* By translating the map, we can arrange that 0 lies in the convex hull  $C$  of the image of  $f$ . If  $f$  takes values only inside some linear subspace of  $\mathbb{R}^n$ , we can replace  $\mathbb{R}^n$  by that subspace. So we can assume that the values of  $f$  span  $\mathbb{R}^n$ . For any partition of unity  $h_i$ , each map  $h_i f$  is valued in  $C$ . If each  $h_i f$  has center of mass in the interior of the convex hull of its image, the center of mass of  $f$  is a convex combination of the centers of mass of the  $h_i f$ , so lies in the interior of the convex hull of  $f(M)$ . So we only have to prove the result locally: assume that  $\bar{M}$  is a box. Split up  $M$  into boxes  $M_k$ , meeting only along measure zero sets. Write  $\langle f \rangle$  as a convex combination of  $\langle f \rangle_{M_k}$ . It suffices to prove that at least one of these  $\langle f \rangle_{M_k}$  lies in the interior of the convex hull of  $f(M)$ , if we pick the boxes  $M_k$  suitably.

The convex hull of  $f(M)$  has nonempty interior by lemma 34.3, so there are some points  $m_k \in M$  for which some convex combination of values  $f(m_k)$  lies in the interior. By lemma 34.2, one of the points  $f(m_k)$  lies in the interior of the convex hull of  $f(M)$ . By continuity of  $f$ , if we take a sufficiently fine grid then  $\langle f \rangle_{M_k}$  is approximately equal to the value  $f(m_k)$  at the center of each box  $M_k$ . We can therefore find points  $\langle f \rangle_{M_k}$  in the interior of the convex hull of  $f(M)$ .  $\square$

**Corollary 34.5.** Take a bounded continuous map  $f: M \rightarrow \mathbb{R}^n$  of a manifold with corners, with a volume form of finite volume. Every affine diffeomorphism of  $\mathbb{R}^n$  that fixes the image  $f(M)$  also fixes the point  $\langle f \rangle$  in the interior of the convex hull of  $f(M)$ .

**34.3** Suppose that  $f: A \rightarrow C$  is a  $C^k$  map from a compact set  $A \subset \mathbb{R}^p$  to the interior of a convex set  $C \subset \mathbb{R}^q$ . Prove that  $f$  extends to  $C^k$  map  $f: \mathbb{R}^p \rightarrow C$ .

### Convex functions

A function  $f: C \rightarrow \mathbb{R}$  from a convex subset  $C \subset \mathbb{R}^n$  is *convex* if the points of  $C \times \mathbb{R}$  lying on or above the graph of  $f$  form a convex set.

**34.4** Prove that a function  $f: C \rightarrow \mathbb{R}$  from a convex subset  $C \subset \mathbb{R}^n$  is convex just when, for any two points  $p, q \in C$ ,

$$f((1-t)p + tq) \leq (1-t)f(p) + tf(q).$$

**34.5** Suppose that  $f: C \rightarrow \mathbb{R}$  is  $C^2$ ,  $C \subset \mathbb{R}^n$  convex and open. Let

$$f''(x) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

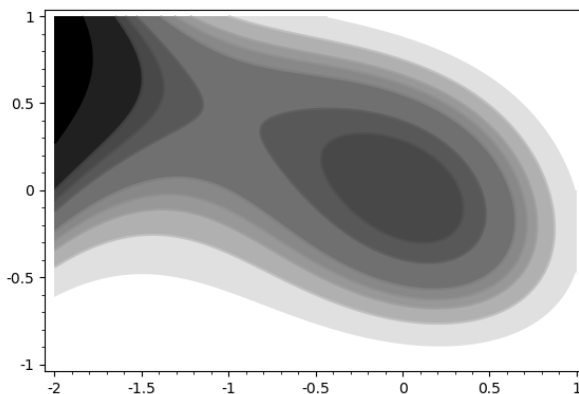
Prove that if  $f$  is convex, then the none of the eigenvalues of  $f''(x)$  are negative. Prove that if all of the eigenvalues of  $f''(x)$  are positive, then  $f$  is convex.

**34.6** Suppose that  $f: C \rightarrow \mathbb{R}$  is a convex function on a convex set  $C \subset \mathbb{R}^n$ . Prove that all *sublevel sets*

$$C_{y_0} := \{x \in C \mid f(x) \leq y_0\} \subseteq C,$$

for any constant  $y_0 \in \mathbb{R}$ , are convex or empty.

A *nondegenerate minimum* of a  $C^k$  function  $f: U \rightarrow \mathbb{R}$ , some  $k \geq 2$ ,  $U \subset \mathbb{R}^n$  open, is a critical point where all of the eigenvalues of  $f''(x_0)$  are positive.



**34.7** Suppose that a  $C^k$  function  $f: U \rightarrow \mathbb{R}$ , some  $k \geq 2$ , has a nondegenerate minimum at a point  $x_0$ . Prove that, after perhaps replacing  $U$  by a smaller open set around  $x_0$ , for all sufficiently small  $\varepsilon > 0$ , the sublevel set

$$C_\varepsilon := \{x \in U \mid f(x) \leq \varepsilon\}$$

is convex, compact with nonempty interior and  $C^k$  boundary.

**34.8** Suppose that a  $C^k$  function  $f: U \rightarrow \mathbb{R}$ , some  $k \geq 2$ , and has a nondegenerate minimum at a point  $x_0$ . Take a  $C^k$  diffeomorphism  $F: W \rightarrow U$ , and let  $y_0 = F^{-1}(x_0)$ , and  $g := f \circ F$ . Prove that  $y_0$  is a nondegenerate minimum of  $g$ . Prove that, after

perhaps replacing  $U$  by a smaller open set, for all small enough  $\varepsilon > 0$ , both of the sublevel sets

$$\{x \mid f(x) \leq f(x_0) + \varepsilon\}, \{y \mid f(F(y)) \leq f(F(y_0)) + \varepsilon\},$$

are convex, compact, with  $C^k$  boundary, and carried one to another by the diffeomorphism  $F$ .



## Chapter 35

### Convolution

We consider the problem of approximating continuous functions by smooth ones.

#### Eavesdropping

Imagine listening to a conversation through a wall. Let  $f(t)$  be the sound, i.e. the density of air approaching the wall at time  $t$ . Then the sound you hear through the wall is not  $f(t)$ , but some muffling of  $f(t)$ , because the sound bounces around inside the wall for a little while. So at time  $t$  you hear an average of values of  $f$  at times earlier than  $t$ , something like

$$f(t) + f(t-1) + f(t-2) + \dots$$

But each reverberation is partly muffled, so comes out weighted by some small factor, like

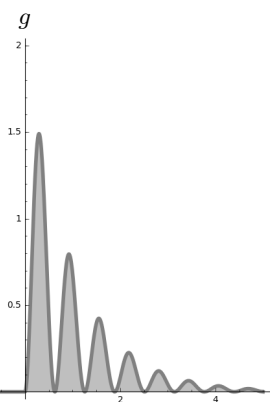
$$\lambda_0 f(t) + \lambda_1 f(t-1) + \lambda_2 f(t-2) + \dots$$

Of course, the lapses in time are not simply  $t-1, t-2, \dots$ , but occur with any difference in time, i.e.

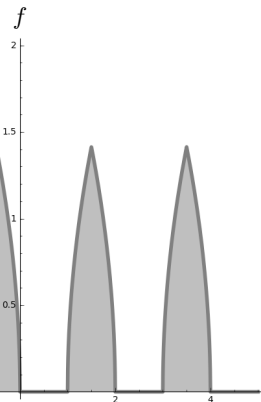
$$\int_{\mathbb{R}} g(s) f(t-s) ds.$$

The *impulse response*  $g(s)$  represents how much signal gets through at time  $t$  from an “impulse” of sound made at time  $t-s$ .

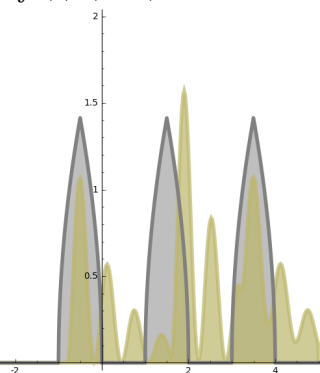
impulse response



sound emitted



sound heard  
 $\int g(s) f(t-s) ds$



We care about muffling of sounds because a slightly muffled sound is similar to the original sound, but has smoother graph. Muffling of sounds is a simple natural process for approximating rough functions with smooth ones.

If  $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$  are continuous functions, let

$$f * g := \int g(y)f(x-y) dy.$$

This  $f * g$  represents  $f$  “smeared out” by averaging against  $g$ , muffled. We will only be interested in the convolution when  $g$  has compact support, which we can assume henceforth, and which ensures that this integral converges.

**Lemma 35.1.** *If  $f$  and  $g$  are continuous functions on  $\mathbb{R}^n$  and one of them has compact support then  $f * g = g * f$*

*Proof.* Change variable by  $z = x - y$ , keeping in mind the limits of integration:

$$\begin{aligned} \int_{-\infty}^{\infty} g(y)f(x-y) dy &= \int_{\infty}^{-\infty} g(x-z)f(z) (-1)dz, \\ &= \int_{-\infty}^{\infty} f(z)g(x-z) dz. \end{aligned}$$

□

**Lemma 35.2.** *If  $f$  and  $g$  are continuous functions on  $\mathbb{R}^n$ , and one of them has compact support, then  $f * g$  is continuous.*

*Proof.* Suppose that  $g$  has compact support. Take some  $x$  and some  $\varepsilon > 0$  and some  $\delta > 0$ . By uniformity of convergence on compact sets, we can arrange by choice of  $\delta$  that for any  $(y, h)$  with  $|h| \leq \delta$ ,  $g(x+h-y)$  is within  $\varepsilon$  of  $g(x-y)$ . If  $g$  has support in some box  $X$  and we let  $X_\delta$  be the box consisting of points whose coordinates lie within  $\delta$  of those of a point of  $X$ , so  $X_\delta$  is also a box and

$$\begin{aligned} |f * g(x+h) - f * g(x)| &= \left| \int f(y)(g(x+h-y) - g(x-y)) dy \right|, \\ &\leq \varepsilon \int_{x-X_\delta} |f|. \end{aligned}$$

Making  $\delta$  smaller will only decrease the integral, and we can make  $\varepsilon$  as small as we like. □

**35.1** Prove that convolution is associative.

**Lemma 35.3.** *If  $f$  is  $C^0$  and  $g$  is  $C^k$  with compact support then  $f * g$  is  $C^k$ . In particular, if  $g$  is  $C^1$ ,  $(f * g)' = f * (g')$ .*

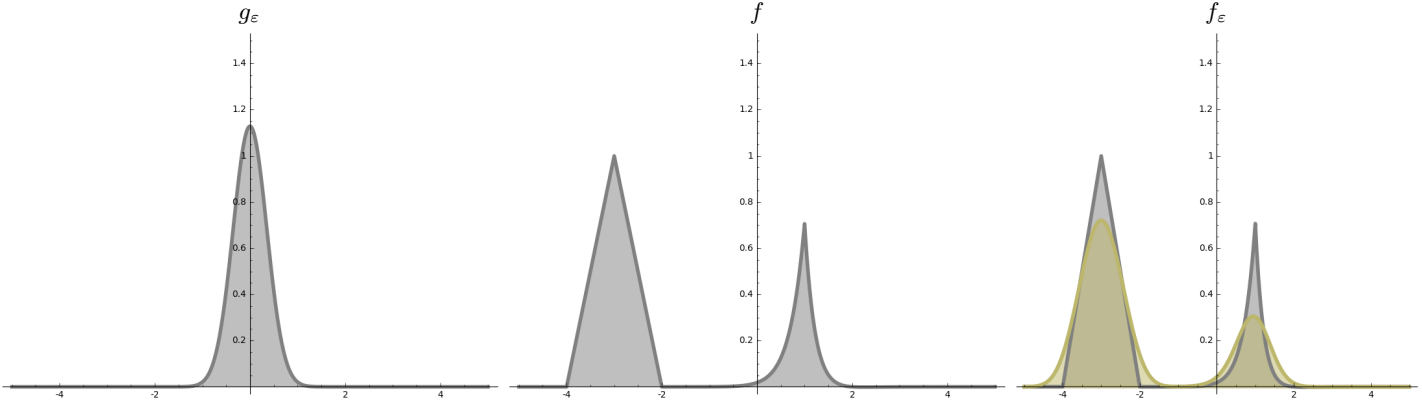
*Proof.* Differentiate under the integral sign. □

## Smoothing

A *bump function*  $g: \mathbb{R}^n \rightarrow [0, \infty)$  is  $C^\infty$  function with compact support. Rescale  $g$  by a constant to arrange that  $\int g = 1$ . For any  $\varepsilon > 0$ , let  $g_\varepsilon(x) = \varepsilon^{-n}g(x/\varepsilon)$ , so  $\int g_\varepsilon dx = 1$  too. For any continuous function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , let  $f_\varepsilon := f * g_\varepsilon$ , i.e.

$$f_\varepsilon(x) := \int g_\varepsilon(y)f(x-y) dy.$$

By lemma 35.3 on the facing page,  $f_\varepsilon$  is  $C^\infty$ .



**35.2** Prove that if  $f$  is bounded from above or below, then  $f_\varepsilon$  satisfies the same bounds: if  $f \leq C$  then  $f_\varepsilon \leq C$  with equality occurring at some point just when  $f$  is constant, equal to  $C$ , near that point, and similarly for any bound  $f \geq C$ .

**Theorem 35.4.** If  $f: \mathbb{R}^p \rightarrow \mathbb{R}^q$  is continuous then  $f_\varepsilon$  is  $C^\infty$ . If  $f$  is  $C^k$  on some open set  $U \subseteq \mathbb{R}^p$ , for some  $k \geq 0$ , then as  $\varepsilon \rightarrow 0$ , every derivative of  $f_\varepsilon$  of order up to  $k$  converges to the corresponding derivative of  $f$ , uniformly on compact sets.

*Proof.*

$$\begin{aligned} f_\varepsilon(x_1) - f_\varepsilon(x_0) &= \int g_\varepsilon(y)f(x_1-y) dy - \int g_\varepsilon(y)f(x_0-y) dy, \\ &= \int g_\varepsilon(y)(f(x_1-y) - f(x_0-y)) dy, \\ &= \int \varepsilon^{-n}g(y/\varepsilon)(f(x_1-y) - f(x_0-y)) dy, \\ &= \int g(z)(f(x_1-\varepsilon z) - f(x_0-\varepsilon z)) dz. \end{aligned}$$

By uniform continuity of  $f$  on compact sets,  $f_\varepsilon \rightarrow f$  uniformly on compact sets. Differentiate both sides in  $x$  to see that the result holds for derivatives as well.  $\square$

## Smoothing with lower bounds

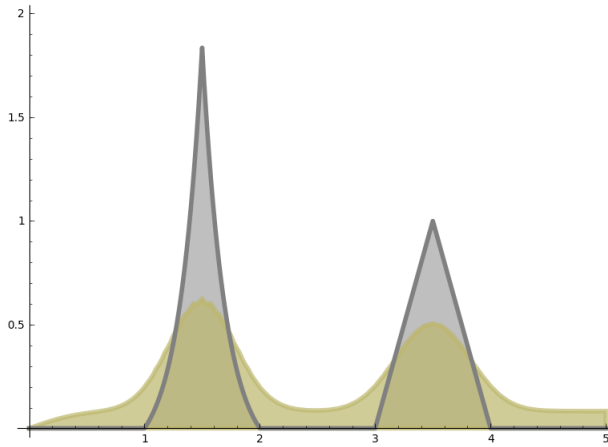
Suppose that  $f(x) \geq 0$  is a continuous function, defined for  $x \geq 0$ , with  $f(0) = 0$ . Our aim is to smooth out  $f$ , but forcing  $f(0)$  to remain 0 and forcing  $f \geq 0$ . Extend

$f$  to a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  by asking  $f$  to be odd, i.e.  $f(-x) = -f(x)$ . Take an even compactly supported  $C^\infty$  function  $g \geq 0$  so that  $g'(x) < 0$  for all  $x > 0$  close enough to zero. Rescale  $g$  by a constant if needed to ensure that  $\int g = 1$ . Let

$$g_\varepsilon(x) = \varepsilon^{-1}g(x/\varepsilon).$$

Take a odd  $C^\infty$  function  $h: \mathbb{R} \rightarrow \mathbb{R}$  with  $h(x) > 0$  for  $x > 0$  and  $h(x)$  constant for large enough  $x$ . For example, we could take  $h(x) = \int_0^x g(u) du$ . Let

$$f_\varepsilon := g_\varepsilon * ((1 - \varepsilon)f + \varepsilon h).$$



**35.3** If  $f$  is bounded above by a positive constant, prove that a suitable choice of  $h$  can ensure that  $f_\varepsilon$  satisfies the same bound. Prove that in addition, if  $f$  is  $C^k$  with a bound on its order  $k$  derivatives on some open set, then  $f_\varepsilon$  has a bound on its order  $k$  derivatives on the same open set.

**Theorem 35.5.** *If  $f: [0, \infty) \rightarrow [0, \infty)$  is continuous then  $f_\varepsilon$  is  $C^\infty$ . If  $f$  is  $C^k$ , some integer  $k \geq 0$ , on some open set  $U \subseteq (0, \infty)$ , then every derivative of  $f_\varepsilon$  of order up to  $k$  converges to the corresponding derivative of  $f$  on  $U$ , uniformly on compact sets. If  $x > 0$  then  $f_\varepsilon(x) > 0$ , while  $f_\varepsilon(0) = 0$  and  $f'_\varepsilon(0) > 0$ .*

*Proof.* Since  $g$  is even,  $g'$  is odd. So  $g'_\varepsilon$  is odd, as is  $(1 - \varepsilon)f + \varepsilon h$ . The even  $g_\varepsilon$  times the odd  $f + \varepsilon h$  is odd. Odd functions integrate to zero, while the odd  $g'_\varepsilon$  times the odd  $(1 - \varepsilon)f + \varepsilon h$  is even, and integrates to a nonnegative value, positive because  $h > 0$  and  $g'_\varepsilon(x) > 0$  for all small enough  $x$ .  $\square$

**35.4** If  $f(x) \geq 0$  is  $C^k$ , defined for  $x \geq 0$ , we could have trouble with derivatives at the origin. For example, if  $f(x) = x^2$ , extending  $f$  to be odd will not work so well. The even order derivatives at the origin of any odd smooth function vanish. Can you find a replacement for  $f_\varepsilon$  that approximates  $f$  as in the statement of the theorem, but also with all  $k$  derivatives, uniformly on compact sets, including near the origin?

We can generalise to add several irrelevant variables: suppose that  $f(x, y, z) \geq 0$  is a continuous function, defined for  $x \geq 0$  and for  $y_1, \dots, y_a \geq 0$  and for all  $z_1, \dots, z_b$ ,



and that  $f(x, y, z) = 0$  at  $x = 0$ . Extend  $f$  to a function  $f: \mathbb{R}^{1+a+b} \rightarrow \mathbb{R}$  to be odd in  $x$ , even in  $y$ . Let

$$g_\varepsilon(x, y, z) = \varepsilon^{-1-a-b} g(x/\varepsilon) g(y_1/\varepsilon) \dots g(z_b/\varepsilon).$$

Let  $h(x, y, z) = h(x)$ . Let

$$f_\varepsilon := g_\varepsilon * ((1 - \varepsilon)f + \varepsilon h).$$

A  $C^1$  function  $h(x, y, z) \geq 0$  is *tidy* if it is defined for  $x \geq 0$  and for  $y_1, \dots, y_a \geq 0$  and for all  $z_1, \dots, z_b$ , and  $h(x, y, z) = 0$  at  $x = 0$ , and  $\frac{\partial h}{\partial x} > 0$  at  $x = 0$ .

**Theorem 35.6.** *If  $f$  is continuous then  $f_\varepsilon$  is  $C^\infty$  and tidy for all  $\varepsilon > 0$ . If  $f$  is  $C^k$ , some integer  $k \geq 0$ , on some open set  $U$  on which  $x > 0$ , then every derivative of  $f_\varepsilon$  of order up to  $k$  converges to the corresponding derivative of  $f$  on  $U$ , uniformly on compact sets.*

This result still doesn't deal well with  $\frac{\partial f}{\partial x}$  along  $x = 0$  if  $f$  is  $C^1$ . Suppose that  $f$  is tidy. We can extend  $\frac{\partial f}{\partial x}$  to be even in  $x, y$ , and then define  $f(x, y, z)$  for all values of  $x, y, z$  by integration in  $x$ . Take some even compactly supported bump function  $g: \mathbb{R} \rightarrow \mathbb{R}$ . Define  $g_\varepsilon(x, y, z)$  as above from  $g$ , i.e. if  $(x, y, z) \in \mathbb{R}^n$  then

$$g_\varepsilon(x, y, z) := \varepsilon^{-n} g(x/\varepsilon, y/\varepsilon, z/\varepsilon).$$

Define

$$\begin{aligned} f_\varepsilon(x, y, z) &:= \int_0^x dx \int g_\varepsilon(\xi, \eta, \zeta) \frac{\partial f}{\partial x}(|x - \xi|, |y - \eta|, z - \zeta) d\xi d\eta d\zeta, \\ &= \int_0^x dx \int g_\varepsilon(\xi, \eta, \zeta) \frac{\partial f}{\partial \xi}(|x - \xi|, |y - \eta|, z - \zeta) d\xi d\eta d\zeta, \\ &= - \int_0^x dx \int \frac{\partial g_\varepsilon}{\partial \xi}(\xi, \eta, \zeta) f(|x - \xi|, |y - \eta|, z - \zeta) d\xi d\eta d\zeta. \end{aligned}$$

**Theorem 35.7.** *If  $f$  is tidy then  $f_\varepsilon$  is  $C^\infty$  and tidy for all sufficiently small  $\varepsilon$ . Moreover  $f'_\varepsilon \rightarrow f'$ , uniformly on compact sets.*

## Convexity

**Proposition 35.8.** *Take a continuous function  $g \geq 0$  on  $\mathbb{R}^p$  with compact support and  $\int g = 1$ , and a continuous function  $f: \mathbb{R}^p \rightarrow C$  valued in a convex set  $C \subset \mathbb{R}^q$ . Then  $f * g$  is valued in  $C$  as well. If we let  $U \subset \mathbb{R}^p$  be the set of points where  $g > 0$ , then  $f * g(x)$  lies in the interior of the convex hull of  $f(x + U)$ .*

*Proof.* Apply lemma 34.4 on page 267:  $g * f(x)$  is the mean of  $f$  in the volume form  $g(x - y) dV_y$ .  $\square$

**Corollary 35.9.** *Take a continuous function  $g \geq 0$  on  $\mathbb{R}^p$  with compact support and  $\int g = 1$ , and a continuous function  $f: [0, \infty)^p \rightarrow C$  valued in a convex set  $C \subset \mathbb{R}^q$ . Extend  $f$  to a continuous function on  $\mathbb{R}^p$ , for example by making  $f$  even in every variable. Then  $f * g$  is valued in  $C$  as well.*

**Corollary 35.10.** *Take a continuous function  $f: [0, \infty)^p \rightarrow C$  valued in a convex set  $C \subset \mathbb{R}^q$ . Then the approximation  $f_\varepsilon$  from theorem 35.4 on page 273 is valued in  $C$  as well.*



## Chapter 36

### Smoothing

We will deform continuous maps into smooth maps, by arbitrarily small perturbations.

#### Smoothing

Suppose that  $\varphi_0: P \rightarrow Q$  is a continuous map of  $C^\infty$  manifolds with corners. A *smoothing homotopy* of  $\varphi_0$  is a continuous map  $\Phi: P \times [0, 1] \rightarrow Q$ , denoted  $(p, \varepsilon) \mapsto \Phi_\varepsilon(p)$ ,  $C^\infty$  for  $t > 0$  and with  $\Phi_0 = \varphi_0$ . Usually, we denote  $\Phi_\varepsilon$  by  $\varphi_\varepsilon$  to simplify notation.

**Theorem 36.1.** *Every continuous map between manifolds with corners has a smoothing homotopy, proper if the map is proper.*

**Theorem 36.2.** *Any two smoothing homotopies are unique up to smooth homotopy, proper if the smoothing homotopy is proper.*

**36.1** Prove that every homeomorphism  $\varphi_0: \mathbb{R} \rightarrow \mathbb{R}$  belongs to a smoothing homotopy  $\varphi_\varepsilon$  so that  $\varphi_\varepsilon$  is a diffeomorphism for  $\varepsilon > 0$ . Prove that the smooth diffeomorphisms are dense, in the topology of uniform convergence on compact sets, and both smooth diffeomorphism classes mod diffeotopy and homeomorphism classes mod isotopy are contractible.

Before we can prove these theorems, we need a few lemmas.

**36.2** The Brouwer fixed point theorem: Let  $\bar{B}$  be a closed ball in  $\mathbb{R}^n$ . Suppose that  $f: \bar{B} \rightarrow \bar{B}$  is a continuous map. Use the theorems above to prove that  $f$  has a fixed point.

Let's prove that every square matrix with nonnegative entries has an eigenvector with nonnegative entries. Suppose that  $V$  is a finite dimensional real vector space and  $T: V \rightarrow V$  is a linear transformation. Suppose that  $C \subset V$  is a cone (in other words,  $tC \subset C$  for  $t \geq 0$ ). Suppose that there is a linear function  $\varphi: V \rightarrow \mathbb{R}$  so that  $\varphi > 0$  on  $C$  except at the origin. Suppose that the hyperplane  $(\varphi = 1)$  cuts the cone  $C$  in a set homeomorphic to a closed ball. We claim that  $T$  has an eigenvector in  $C$ . If  $Tx = 0$  for some nonzero  $x \in C$ , then  $T$  has a zero eigenvector in  $C$ . Otherwise, for  $x \in C - \{0\}$ ,  $Tx \neq 0$  so  $\varphi(Tx) > 0$  and  $Tx \in C$ . So the analytic map

$$x \mapsto \frac{Tx}{\varphi(Tx)}$$

has a fixed point  $x$  in  $C \cap (\varphi = 1)$ , by the Brouwer fixed point theorem. If we

let  $\lambda = \varphi(Tx)$ ,  $Tx = \lambda x$ :  $T$  has an eigenvector in the cone. In case  $V = \mathbb{R}^n$  and  $T$  is a square matrix with nonnegative entries, we take  $C$  to be the vectors with nonnegative entries, and  $\varphi(x) = x_1 + \cdots + x_n$ . Then  $(\varphi = 1) \cap C$  is the set of vectors  $x$  with all  $x_i \geq 0$  and  $x_1 + \cdots + x_n = 1$ , a simplex; we leave the reader to prove that the simplex is homeomorphic (not diffeomorphic) to a closed ball.

Take a proper continuous map  $\varphi: P \rightarrow Q$  of oriented manifolds with corners, taking attached boundary to attached boundary. Suppose that  $Q$  is connected. The *degree* of  $\varphi$  is the degree of any smooth map obtained from  $\varphi$  by a smoothing homotopy; this is well defined by the above theorems.

### Local smoothing

**Lemma 36.3.** *Suppose that  $P = \mathbb{R}^p$  or  $P = [0, \infty)^p$  and  $Q \subset \mathbb{R}^q$  is a convex set. Every continuous map  $P \rightarrow Q$  has a smoothing homotopy.*

*Proof.* Apply proposition 35.8 on page 275 and theorem 35.4 on page 273.  $\square$

We need a local notion of smoothing homotopy. If  $f_0: P \rightarrow Q$  is a continuous map of manifolds with corners, and  $U \subset P$  is an open set and  $W \subset P$  is an open set containing  $U$ , a *homotopy of  $f_0$  smoothing over  $U$  and fixed outside  $W$*  is a continuous map  $F: P \times [0, 1] \rightarrow M$ , denoted  $(p, \varepsilon) \mapsto F_\varepsilon(p)$ ,  $C^\infty$  for  $\varepsilon > 0$  and  $p \in U$ , and with  $F_0 = f_0$  and  $F_\varepsilon = f_0$  outside  $W$  for all  $\varepsilon$ . Again, we usually denote  $F_\varepsilon$  as  $f_\varepsilon$ .

**Lemma 36.4.** *Suppose that  $P = \mathbb{R}^p$  or  $P = [0, \infty)^p$  and  $Q \subset \mathbb{R}^q$  is a convex set. Take open sets  $U \subset W \subset P$  so that  $\bar{U} \subset W$ . Any continuous map  $P \rightarrow Q$  has a homotopy smoothing over  $U$  and fixed outside  $W$ .*

*Proof.* Extend  $f$ , for example to be even in every variable, and replace  $U, W, P$  with their unions with their reflections in these variables. So we can ensure that  $U, W, P \subset \mathbb{R}^p$  is open. Apply lemma 36.3 to construct a smoothing homotopy  $F_\varepsilon$  of  $f: P \rightarrow Q$ . By problem 25.7 on page 212, there is a  $C^\infty$  function  $\beta$  so that  $\beta = 1$  on  $\bar{U}$  and  $\beta = 0$  outside  $W$ . Then

$$\beta F_\varepsilon + (1 - \beta)f$$

is a homotopy smoothing over  $U$ , fixed outside  $W$ .  $\square$

**Lemma 36.5.** *Suppose that  $P \subset [0, \infty)^p$  is an open subset and  $Q \subset \mathbb{R}^q$  is a convex set. Take open sets  $U \subset W \subset P$  so that  $\bar{U} \subset W$ . Any continuous map  $f: P \rightarrow Q$  has a homotopy smoothing  $f$  over  $U$  and fixed outside  $W$ .*

*Proof.* Extend  $f$ , for example be even in every variable, and replace  $U, W, P$  with their unions with their reflections in these variables. So we can ensure that  $U, W, P \subset \mathbb{R}^p$  is open. By problem 25.7 on page 212, there is a  $C^\infty$  function  $\beta$  so that  $\beta = 1$  on  $\bar{U}$  and  $\beta = 0$  on  $\mathbb{R}^p$  outside  $W$ . Pick a point  $q_0 \in Q$ . Let  $g = \beta f + (1 - \beta)q_0$  on  $P$ , and  $q_0$  on  $\mathbb{R}^p - P$ , so  $g: \mathbb{R}^p \rightarrow Q$ . Lemma 36.4 constructs a homotopy  $g_\varepsilon$  of  $g$  smoothing over  $U$  and fixed outside  $W$ . Then

$$\beta g_\varepsilon + (1 - \beta)f$$

is a homotopy smoothing over  $U$ , fixed outside  $W$ , defined on  $P$ .  $\square$

We can “slightly enlarge” compact sets to make open sets.

**Lemma 36.6.** *Suppose that  $K \subset M$  is a compact subset of a manifold with corners  $M$  and that  $W \subset M$  is an open set containing  $K$ . There is an open set  $V \subset W$  containing  $K$  so that  $\bar{V} \subset W$  is a compact subset.*

*Proof.* Every point  $x \in K$  is a point of  $W$  so lies in a coordinate ball  $B \subset W$ . Replace  $B$  by a coordinate ball of any smaller radius (in the same chart), to make an open set  $B \subset W$  so that  $\bar{B} \subset W$  is compact. So  $K$  is covered by open sets whose closure is a compact subset of  $W$ . Take a finite subcover  $B_1, B_2, \dots, B_k$  and let  $V := B_1 \cup \dots \cup B_k$ .  $\square$

**Lemma 36.7.** *Suppose that  $P$  is a manifold with corners and  $Q \subset \mathbb{R}^q$  is a convex set. Take an open set  $U \subset P$  so that  $\bar{U}$  is a compact subset of an open set  $W \subset P$  and  $\bar{U}$  also lies in the domain of a chart. Any continuous map  $f: P \rightarrow Q$  admits a homotopy smoothing over  $U$  and fixed outside  $W$ .*

*Proof.* Take a chart  $h: V \rightarrow h(V) \subset [0, \infty)^p$  defined on an open set  $V \subset P$  so that  $\bar{U} \subset V$ . Slightly enlarge  $\bar{U}$ : by lemma 36.6 to find an open set  $V' \subset V$  so that  $\bar{U} \subset V'$  and  $\bar{V}' \subset V$  is compact. Replace  $W$  by  $W \cap V'$ , so that  $\bar{W} \subset V$  is compact. Let  $F_0(v) = f(h^{-1}(v)): h(V) \rightarrow Q$ . Apply lemma 36.5 on the facing page to construct a homotopy  $F_\varepsilon: h(V) \times X \rightarrow Q$  smoothing  $F_0$  over  $h(U)$ , fixed outside  $h(W)$ . By problem 25.7 on page 212, there is a  $C^\infty$  function  $\beta$  so that  $\beta = 1$  on  $\bar{U}$  and  $\beta = 0$  on  $P$  outside  $W$ . In particular,  $\beta = 0$  near the boundary of  $V$ . Define

$$f_\varepsilon(p) = \beta(p)F_\varepsilon(h(p)) + (1 - \beta(p))f(p)$$

on  $V$ , and  $f_\varepsilon(p) = f(p)$  on  $P - V$ .  $\square$

We need to allow our map to have some extra parameters.

**Lemma 36.8.** *Suppose that  $P$  is a manifold with corners and  $Q \subset \mathbb{R}^q$  is a convex set. Take an open set  $U \subset P$  so that  $\bar{U}$  is a compact subset of an open set  $W \subset P$  and also  $\bar{U}$  lies in the domain of a chart. For any box  $X = [0, 1]^k$ , any continuous map  $f: P \times X \rightarrow Q$  admits a homotopy smoothing over  $U \times X$  and fixed outside  $W \times X$ .*

*Proof.* Any chart  $h$  on  $U$  gives a chart  $(u, x) \mapsto (h(u), x)$  on  $U \times X$ ; apply lemma 36.7.  $\square$

**Lemma 36.9.** *Suppose that  $P, Q$  are manifolds with corners. Take an open subset  $C \subset Q$  diffeomorphic to a convex set in  $\mathbb{R}^q$ . Take an open set  $U \subset P$  so that  $\bar{U}$  lies in an open set  $W \subset P$  and also lies in the domain of a chart. For any box  $X = [0, 1]^k$ , any continuous map  $f: P \times X \rightarrow Q$  so that  $f(\bar{U} \times X) \subset C$  admits a homotopy smoothing over  $U \times X$  and fixed outside  $W \times X$ .*

*Proof.* Take a diffeomorphism  $h: C \rightarrow C' \subset \mathbb{R}^q$  where  $C'$  is a convex set. By problem 36.3 on the next page,  $f^{-1}C$  contains some open set of the form  $V \times X \subset P \times X$  so that  $\bar{U} \subset V$ . Replace  $W$  with  $W \cap V$ , so that  $h \circ f$  is defined on  $W \times X$ . Slightly enlarge  $\bar{U}$ : by lemma 36.6, find an open set  $U' \subset W$  so that  $\bar{U} \subset U'$  and  $\bar{U}' \subset W$  is compact. Replace  $W$  by  $U'$  to ensure that  $\bar{W} \subset f^{-1}C$  is compact. Apply lemma 36.8 to the map  $g_0 := h \circ f: f^{-1}C \rightarrow C'$  to build a homotopy  $g_\varepsilon$  smoothing over  $\bar{U} \times X$  and fixed outside  $W \times X$ . Let  $f_\varepsilon(p, x) = h^{-1} \circ g_\varepsilon(p, x)$  for  $p \in f^{-1}C$  and  $x \in X$ , and  $f(p)$  for  $p \in P - W$  and  $x \in X$ .  $\square$

**36.3\*** Suppose that  $U_0 \subseteq \mathbb{R}^p$  and  $U_1 \subseteq \mathbb{R}^q$  are open sets containing compact sets  $K_0 \subset U_0$ ,  $K_1 \subset U_1$ . Suppose that  $W \subseteq U_0 \times U_1$  is an open set containing  $K_0 \times K_1$ . Prove that  $W$  contains an open set of the form  $W_0 \times W_1$  so that  $K_0 \subset W_0$  and  $K_1 \subset W_1$ .

### Global smoothing

We prove theorem 36.1 on page 277:

*Proof.* Every point of  $Q$  lies in an open set  $C \subset Q$  diffeomorphic to a convex open subset  $C' \subset [0, \infty)^q$ . So every point of  $P$  lies inside a precompact open set  $U$  so that  $f|_U \subset C \subset Q$  lies in an open set  $C \subset Q$  diffeomorphic to a convex open subset  $C' \subset [0, \infty)^q$ . Such sets  $U$  cover  $P$ ; take a countable locally finite refinement  $\{U_i\}$ . Take a ripple (as in chapter 31)  $V_1, V_2, \dots$  made up from such sets  $U_i$ , say

$$V_j = U_{j1} \cup U_{j2} \cup \dots \cup U_{jk_j}.$$

Each  $V_j$  intersects only  $V_{j-1}, V_j, V_{j+1}$ . Pick a homotopy  $f_\varepsilon$  of  $f$  smoothing over  $U_1$ , fixed outside  $V_3$ . Pick a homotopy  $f_{\varepsilon_1 \varepsilon_2}$  of  $f_{\varepsilon_2}$  smoothing over  $U_1 \times X$ , fixed outside  $V_3 \times X$ , and so on. The homotopy  $f_{\varepsilon \varepsilon \dots \varepsilon}$  smooths  $f$  over  $V_1 = U_1 \cup \dots \cup U_k$ , fixing outside  $V_3$ , and we write it as  $f_\varepsilon$ . In the same way, we can smooth  $f$  over any  $V_j$ , fixing outside the overlap region

$$V_{j-1} \cup V_j \cup V_{j+1}.$$

We can simultaneously smooth over  $V_j$  and over  $V_{j+3}$ , and so on, since their overlap regions don't intersect, and the smoothing process fixes  $f$  outside each overlap region. So we can simultaneously smooth in all  $V_1, V_4, V_7, \dots$  to make one smoothing homotopy  $f_\varepsilon$ . At each step, we keep  $f_\varepsilon \bar{U}_i$  inside its associated  $C_i$ , diffeomorphic to a convex set, so we can keep applying the same smoothing procedure to it. In the same way, we can simultaneously smooth this  $f_\varepsilon$  in all  $V_2, V_5, V_8, \dots$  to make a smoothing homotopy  $f_{\varepsilon_1 \varepsilon_2}$ , and similarly for  $V_3, V_6, V_9, \dots$  to make a smoothing homotopy  $f_{\varepsilon_1 \varepsilon_2 \varepsilon_3}$ . As above, define a new smoothing homotopy  $f_\varepsilon$  to be  $f_{\varepsilon \varepsilon \varepsilon}$ . We deal with properness below.  $\square$

We prove theorem 36.2 on page 277:

*Proof.* To be precise, any two smoothing homotopies  $F_\varepsilon$  and  $G_\varepsilon$  of the same continuous map  $f$  are related by: for all  $\varepsilon_0, \varepsilon_1 > 0$ , there is a smooth homotopy  $H_\varepsilon$  between the maps  $F_{\varepsilon_0}$  and  $G_{\varepsilon_1}$ . Clearly it is enough to suppose that  $\varepsilon_0 = \varepsilon_1 = 1$ . Define a continuous homotopy

$$H_\varepsilon = \begin{cases} F_{1-2\varepsilon}, & \text{if } 0 \leq \varepsilon \leq \frac{1}{2}, \\ G_{2\varepsilon-1}, & \text{if } \frac{1}{2} \leq \varepsilon \leq 1. \end{cases}$$

Apply theorem 36.1 to  $H$  and produce a smoothing homotopy  $H_{\varepsilon_1 \varepsilon_2}$ . Let  $K_\varepsilon := H_{\varepsilon \varepsilon}$ , a smooth homotopy. Again, we deal with properness below.  $\square$

The proof of smoothing of manifolds fails if applied to a topological manifold. When we extend a map  $g$  from some open set  $U \cap V$  to  $U$ , to equal a given chart  $f$  on  $V - U$ , a  $C^0$  extension  $g$  might not be a local homeomorphism near  $U \cap \partial V$ . A  $C^1$  extension  $g$  has to have  $g' = f'$  at those points, so is a local diffeomorphism by the inverse function theorem. The *dilation* of a pair of points under a map of metric

spaces is the ratio of output distance to input distance. A *Lipschitz map* is a map of bounded dilation. A *Lipschitz transition map* is a map of open sets of a box which is Lipschitz and has Lipschitz inverse. Any topological manifold admits a maximal atlas of charts with Lipschitz transition maps, a *Lipschitz structure*, by essentially the same proof as for smooth structures.

### Better local smoothing

Above we smoothed inside a precompact open set, and fixed outside a larger open set. Now we will smooth inside an open set, and fix outside.

**Lemma 36.10.** *Suppose that  $P, Q$  are manifolds with corners. Suppose that  $f_0: P \rightarrow Q$  is a continuous map, and that  $W \subset P \times Q$  is an open set containing the graph of  $f_0$ . Then there is a smoothing homotopy  $f_\varepsilon$  of  $f_0$ , so that for every  $\varepsilon$ , the graph of  $f_\varepsilon$  lies in  $W$ . Any two such smoothing homotopies are related by a smooth homotopy.*

*Proof.* For each point  $p \in P$ , pick some open set  $C \subset Q$  containing  $f(p)$ . Making  $C$  smaller, we can ensure that  $C$  is diffeomorphic to a convex open set  $C' \subset [0, \infty)^q$ . Making  $C$  smaller again, we can ensure that there is an open set  $U \subset P$  with  $p \in U$  and  $U \times C \subset W$ . Making  $U$  smaller, arrange that  $U \subset f_0^{-1}C$  and that  $U$  is precompact. Putting together all possible such choices of  $C$  and  $U$ , the various choices of  $U$  cover  $P$ . Take a locally finite refinement  $\{U_i\}$ , with associated  $C_i \subset Q$ . Repeat our arguments above.  $\square$

**Lemma 36.11.** *Suppose that  $f_0: P \rightarrow Q$  is a continuous map of manifolds with corners, and  $U \subset P$  is an open set. There is an open set  $W \subset U \times Q$  so that any continuous map  $g: U \rightarrow Q$  whose graph lies in  $W$  extends to a continuous map  $P \rightarrow Q$  by demanding  $g = f$  outside  $U$ .*

*Proof.* As above, we suppose that we have chosen sets  $U_i$  to form an open cover of  $P$ , and each  $U_i$  is precompact, and has  $f_0(U_i) \subset C_i$  for some  $C_i \subset Q$  open and taken by some diffeomorphism  $h_i: C_i \rightarrow C'_i$  to an open convex set  $C'_i \subset [0, \infty)^q$ . We can assume further that each  $\bar{U}_i$  lies in the domain of some chart  $\phi_i$ . Suppose that  $\partial U$  intersects some  $\bar{U}_i$ . Define a metric on  $\bar{U}_i$  and a metric on  $C_i$ , both denoted  $d$ , and defined to be identified with Euclidean distance by  $\phi_i$  or by  $h_i$  respectively. Let  $W \subset U \times Q$  be the set of all points  $(p, q)$  so that, if  $U_i$  belongs to our open cover and contains  $p$  and intersects  $\partial U$ , then

$$d(p, \partial U) > d(q, f_0(p)).$$

Because each  $p$  lies in only finitely many  $U_i$ , there are only finitely many inequalities here. So  $W \subset U \times Q$  is open and contains the graph of  $f_0$  over  $U$ . Note that our inequality, in charts, cuts out a convex set, so is preserved by convolutions in local charts.

Suppose that  $g: U \rightarrow Q$  has graph in  $W$ . If  $p \rightarrow p_0 \in \partial U$ ,  $g(p) \rightarrow f_0(p_0)$  at a rate which is at worst linear in our charts.  $\square$

**Lemma 36.12.** *Suppose that  $f_0: P \rightarrow Q$  is a continuous map of manifolds with corners, and  $U \subset P$  is an open set. Then there is a homotopy  $f_\varepsilon$  of  $f_0$ , smoothing over  $U$  and fixed outside  $U$ , uniquely determined up to a homotopy which is smooth over  $U$ .*

*Proof.* Take  $W$  as in lemma 36.11 on the preceding page. Take a smoothing homotopy  $f_\varepsilon$  of  $f_0$  over  $U$  with the graph of each  $f_\varepsilon$  lying in  $W$ , as in lemma 36.10 on the previous page.  $\square$

**Corollary 36.13.** *Any continuous curve  $x_0: [a, b] \rightarrow M$  in a manifold with corners admits a smoothing homotopy  $x_\varepsilon: [a, b] \rightarrow M$  so that  $x_\varepsilon(a) = x(a)$  and  $x_\varepsilon(b) = x(b)$  for all  $\varepsilon$ , i.e. fixing end points. Any such smoothing homotopy is unique up to a smooth homotopy fixing end points.*

### Proper maps

A property of continuous maps is *stable* if, for any continuous map  $f_0: P \rightarrow Q$  with this property, there is an open set  $W \subset P \times Q$  so that, for any homotopy  $f_\varepsilon$  of  $f_0$  for which every  $f_\varepsilon$  has graph inside  $W$ , every  $f_\varepsilon$  has that property.

**Lemma 36.14.** *Properness is stable. Indeed, for any continuous proper map  $f: P \rightarrow Q$  of manifolds with corners, there is an open subset  $W \subset P \times Q$  so that if the graph of a continuous map  $g: P \rightarrow Q$  lies in  $W$  then  $g$  is a proper map.*

*Proof.* Every point  $q \in Q$  lies in a precompact open set  $U \subset Q$ . Hence we can cover  $Q$  in such open sets, and take a countable locally finite refinement  $Q_1, Q_2, \dots$ . Let  $Q'_i$  be the union of all  $Q_j$  intersecting  $\bar{Q}_i$ . So  $Q'_i$  is a finite union of  $Q_j$ , and hence  $\{Q'_i\}$  is also a locally finite open cover of  $Q$ .

Let  $P_i := f^{-1}Q_i$ , so  $\bar{P}_i = f^{-1}\bar{Q}_i$  is a compact subset of  $P$ . Let

$$W_i := ((P - \bar{P}_i) \times Q) \cup (P \times Q_i),$$

and let

$$W := \bigcup_i W_i.$$

If  $g: P \rightarrow Q$  has graph in  $W$ , then every point  $p \in \bar{P}_i$  is mapped by  $g$  to a point in  $Q_i$ . Take a compact set  $K \subset Q$ . So  $K$  is covered by finitely many  $\bar{Q}_i$ , and so  $g^{-1}K$  is a closed set covered by finitely many  $\bar{P}_i$ , so is compact.  $\square$

### Smoothing with controlled derivative

**Lemma 36.15.** *Suppose that  $f_0: P \rightarrow Q$  is a  $C^1$  map of manifolds with corners, and  $U \subset P$  is an open set. There is an open set  $W \subset U \times Q$  so that any  $C^1$  map  $g: U \rightarrow Q$  whose graph lies in  $W$  extends to a  $C^1$  map  $P \rightarrow Q$  by demanding  $g = f$  outside  $U$ .*

*Proof.* Repeat the proof of lemma 36.11 on the preceding page, but replace the line

$$d(p, \partial U) > d(q, f_0(p))$$

above with

$$d(p, \partial U)^2 > d(q, f_0(p))$$

Then  $g$  extends to  $P$  continuously by lemma 36.11 on the previous page, and the triangle inequality ensures that, in charts near points of  $\partial U$ ,  $g'$  is defined and  $g' = f'$ , so extends continuously.  $\square$



**Lemma 36.16.** *Suppose that  $f_0: P \rightarrow Q$  is a  $C^1$  map of manifolds with corners, and  $U \subset P$  is an open set. Then there is a  $C^1$  homotopy  $f_\varepsilon$  of  $f_0$ , smoothing over  $U$  and fixed outside  $U$ .*

*Proof.* Take  $W$  as in lemma 36.15 on the facing page. Take a smoothing homotopy  $f_\varepsilon$  of  $f_0$  over  $U$  with the graph of each  $f_\varepsilon$  lying in  $W$ , as in lemma 36.10 on page 281.  $\square$

### Tidy smoothing

**Lemma 36.17.** *Take a  $C^k$  function  $f: [0, \infty)^p \rightarrow \mathbb{R}^q$ . Suppose that*

$$(f, f', f'', \dots, f^{(k)})$$

*is valued in a convex set. Take the approximation  $f_\varepsilon$  from theorem 35.4 on page 273. Then, for any  $\varepsilon$ ,*

$$(f_\varepsilon, f'_\varepsilon, f''_\varepsilon, \dots, f_\varepsilon^{(k)})$$

*is valued in the same convex set.*

*Proof.* Derivatives commute with smoothing by convolution.  $\square$

Using the smoothing of theorem 35.7 on page 275, if  $f: [0, \infty)^p \rightarrow [0, \infty)^q$  is tidy, for example if  $f_i \geq 0$  and  $\frac{\partial f_i}{\partial x_j} > 0$  at  $x_j = 0$ , then the same is true of  $f_\varepsilon$  for all  $\varepsilon > 0$ .

**36.4** Suppose that  $F: P \times [0, 1] \rightarrow Q$  is a tidy map. Prove that, for any  $\varepsilon \in [0, 1]$ ,  $F_\varepsilon(p) := F(p, \varepsilon)$  is a tidy map.

The first derivative  $f'$  of a  $C^1$  map  $f: P \rightarrow Q$  associates to each point  $p \in P$  a linear map  $f'(p): T_p P \rightarrow T_{f(p)} Q$ . In the language of tensor algebra, a linear map  $T_p P \rightarrow T_q Q$  is precisely an element of the tensor product  $T_p^* P \otimes T_q Q$ . This explains the peculiar notation: let  $T^* P \boxtimes TQ$  be the set of all triples  $(p, q, \phi)$  for  $p \in P$ ,  $q \in Q$ , and  $\phi: T_p P \rightarrow T_q Q$  a linear map.

We make  $T^* P \boxtimes TQ$  into a manifold, by using any chart on  $P$  to write out coordinates  $x$  for  $p$ , and any chart on  $Q$  to write out coordinates  $y$  for  $q$ , and write out  $\phi$  as a matrix in those coordinates,  $dy = A dx$ , giving a chart  $(p, q, \phi) \mapsto (x, y, A)$  on  $T^* P \boxtimes TQ$ . We can cover  $T^* P \boxtimes TQ$  with adapted charts whose domain  $U \subset T^* P \boxtimes TQ$  is identified by the chart with an open set of points  $(x, y, A)$  so that the coordinate  $y$  and the matrix  $A$  are forced lie in convex sets. Convolution in any such chart gives a smoothing homotopy  $f_\varepsilon$  of any map  $f$  for which the graph of  $f'$  lies in  $U$ .

We don't repeat the arguments above, which are easily adapted to first derivatives, to yield:

**Theorem 36.18.** *Suppose that  $f_0: P \rightarrow Q$  is a  $C^1$  tidy map of manifolds with corners. Suppose that  $W \subset T^* P \boxtimes TQ$  is an open set containing the graph of  $f'_0$ . Suppose that  $U \subset P$  is an open set. Then there is a  $C^1$  tidy homotopy  $f_\varepsilon$  of  $f_0$ , so that for any  $\varepsilon$ ,  $f_\varepsilon$  is  $C^\infty$  on  $U$  and the graph of  $f'_\varepsilon$  lies in  $W$ .*

A property of tidy maps is *stable* if, for any tidy map  $f_0: P \rightarrow Q$  with this property, there is an open set  $W \subset T^* P \boxtimes TQ$  so that any tidy homotopy  $f_\varepsilon$  of  $f_0$  for which every  $f'_\varepsilon$  has graph inside  $W$ , every  $f_\varepsilon$  has that property. Intuitively, a stable property of tidy maps is one which is representable by inequalities on value and derivative at each point.

**36.5** Prove that submersion is stable.

**36.6** Prove that immersion is stable.

**36.7** Prove that transversality of a tidy map  $f: P \rightarrow M$  to a given tidy embedding  $g: Q \rightarrow M$  is a stable property of  $f$ .

For any finite number of stable properties, the property of having all of those properties is stable.

**Lemma 36.19.** *Diffeomorphism is stable.*

*Proof.* Properness is stable, and submersion is stable. Suppose that  $g: P \rightarrow Q$  is a  $C^1$  proper tidy map with  $g'$  onto, and hence a local diffeomorphism. By lemma 22.2 on page 188,  $g$  is a covering map, with some finite number of sheets. If  $f_\varepsilon$  is a  $C^1$  tidy homotopy of  $f$ , with the graph of every  $f_\varepsilon$  in  $W$ , then all  $f_\varepsilon$  are covering maps, and by continuity the number of sheets is independent of  $\varepsilon$ , so all  $f_\varepsilon$  are diffeomorphisms.  $\square$

**36.8** Why did we require tidyness? Give an example of a diffeomorphism which is unstable under  $C^1$  homotopy, and explain why it is stable under tidy homotopy.

## Smoothing

**Theorem 36.20** (Smoothing of manifolds). *Every  $C^1$  manifold with corners  $M$  admits a smooth structure in its  $C^1$  structure. This smooth structure is unique up to smooth diffeomorphism.*

**Theorem 36.21.** *Any  $C^1$  manifold with corners  $M$  admits at most one smooth structure inducing its  $C^1$  structure, up to smooth diffeomorphism.*

*Proof.* Take two smooth structures on the same manifold  $M$ , with the same underlying  $C^1$  structure. The identity map  $I: M \rightarrow M$  is tidy, and so admits a tidy smoothing homotopy  $I_\varepsilon$ , by theorem 36.18 on the previous page. By stability of diffeomorphism (lemma 36.19), we can arrange that all  $I_\varepsilon$  are  $C^1$  diffeomorphisms. But for  $\varepsilon > 0$  they are also  $C^\infty$ .  $\square$

**Lemma 36.22.** *Suppose that  $M$  is a  $C^1$  manifold with corners and that  $U \subset M$  is an open subset, equipped with a  $C^\infty$  structure contained in its induced  $C^1$  structure. For any open set  $V$  which is the domain of a chart for the  $C^1$  structure on  $M$ , there is a  $C^\infty$  structure on  $U \cup V$ , contained in its induced  $C^1$  structure, which induces the  $C^\infty$  structure on  $U$ .*

*Proof.* Take a  $C^1$  chart  $(V, \varphi)$ . If  $U \cap V$  is empty, use the  $C^\infty$  structure given by the  $C^\infty$  charts on  $U$  and the chart  $(V, \varphi)$ . So we can suppose that  $U \cap V$  is not empty. Let  $Q := [0, \infty)^n$ . By lemma 36.15 on page 282, there is an open set  $W \subset (U \cap V) \times Q$  containing the graph of  $\varphi$ , so that if  $\psi: U \cap V \rightarrow Q$  is a  $C^1$  map, and the graph of  $\psi$  lies inside  $W$ , then  $\psi$  admits an extension to a  $C^1$  map on  $U$ . By lemma 36.10 on page 281, there is a smoothing homotopy  $\varphi_\varepsilon$  of  $\varphi|_U$  with the graph of each  $\varphi_\varepsilon$  lying in  $W$ . Pick any one of these  $\varphi_\varepsilon$ , and call it  $\psi$ . Extend it to a  $C^1$  map  $\psi: U \rightarrow Q$ . At any  $p \in U$ ,  $\varphi'(p)$  is invertible, since  $\varphi$  is a chart. Since  $\psi$  is a  $C^1$  extension of  $\varphi$ ,  $\psi'(p) = \varphi'(p)$  is invertible on  $U - V$ , so  $\psi$  is also a chart near any point  $p \in U$ .

Restrict  $\psi$  to a small enough open set containing  $p$  so that  $\psi$  becomes a chart. But  $\psi$  is  $C^\infty$  on  $U$ , in the  $C^\infty$  structure of  $U$ . So  $\psi$  is compatible with that  $C^\infty$  structure. So  $U$  is covered in open sets on which  $\psi$  restricts to a chart, and these, together with the charts of the  $C^\infty$  structure on  $U$ , generate a  $C^\infty$  structure on  $U \cup V$ .  $\square$

We now prove theorem 36.20 on the preceding page:

*Proof.* Cover  $M$  by domains of its  $C^1$  charts, and take a countable locally finite refinement. Pick one chart, and use it as a  $C^\infty$  structure on its domain. Inductively, add one chart domain at a time by lemma 36.22 on the facing page, to construct a smooth structure. Uniqueness follows by theorem 36.21 on the preceding page.  $\square$



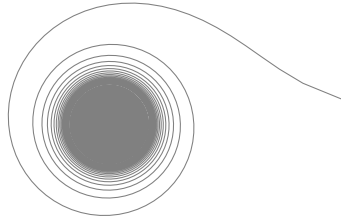
## Chapter 37

### Integrating along a path

#### Integrating along merely continuous paths

**37.1** Fix a continuous 1-form  $\omega$  on an open set in  $\mathbb{R}^n$ . For each point  $p \in \mathbb{R}^n$ , denote by  $\int_0^p$  the integral along the line segment from the origin to  $p$ . Prove that the integral  $\int_0^p \omega$  varies continuously with  $p$ , and that there is a constant  $C > 0$  so that if  $p$  is inside a small enough ball around 0, then  $\int_0^p \omega < C|p|$ .

**37.2** We can't always integrate a smooth 1-form over a continuous path. Suppose that  $\varepsilon$  is a constant with  $0 < \varepsilon < 1/2$ . Let  $C$  be the spiral path  $r(t) = t^\varepsilon$ ,  $\theta(t) = t^{-1}$ , in polar coordinates, for  $0 < t \leq 1$ , with  $x(t) = r(t) \cos \theta(t)$  and  $y(t) = r(t) \sin \theta(t)$  for  $t > 0$ , and set  $x(0) = y(0) = 0$ .



Prove that the integral of  $\omega = x dy$  over the path  $C$  is infinite.

We will see that closed 1-forms behave well when we integrate over a continuous path.

#### When is a closed form exact?

A closed  $C^k$  1-form  $\omega$  is *exact* if  $\omega = df$  for some  $C^{k+1}$  function  $f$ , called a *potential* for  $\omega$ .

The solid angle form is closed, but not exact.

On the other hand, if  $\omega = df$  for some  $f$ , then  $d\omega = 0$  since  $d^2 = 0$ : exact 1-forms are closed (on any manifold, even with corners).

Take a disk  $D$  in the plane, with center at the origin, and a 1-form  $\omega = a(x, y) dx + b(x, y) dy$  on  $D$ . To reach each point  $(x_1, y_1)$ , take the path  $(x(t), y(t))$  which starts at the origin, travels first along the  $x$ -axis from  $(0, 0)$

to  $(x_1, 0)$  by  $(x(t), y(t)) = (t, 0)$ , and then travels up the  $y$ -direction, by  $(x(t), y(t)) = (x_1, t - x_1)$  until it hits  $(x_1, y_1)$  (assuming  $y_1 > 0$  for simplicity). If  $\omega = df$  for some  $f$ , then we can add a constant to  $f$  to get  $f = 0$  at the origin, and integrate  $\omega$  along this path to find that  $f$  is given by

$$f(x_1, y_1) = \int_0^1 (x, y)^* \omega = \int_0^{x_1} a(t, 0) dt + \int_0^{y_1} b(x_1, s) ds$$

where we write  $s$  for  $t - x$ . On the other hand, even if  $\omega$  is *not* exact, let's define a function  $f$  by exactly this expression. By the fundamental theorem of calculus,  $\frac{\partial f}{\partial y}(x, y) = b(x, y)$ . Switch the roles of  $x$  and  $y$ , going along the  $y$ -axis first and then moving the  $x$  variable, and recompute the integral. If  $\omega$  is exact, then the integrals must be equal, both equal to  $f$ :

$$\int_0^{x_1} a(t, 0) dt + \int_0^{y_1} b(x_1, s) ds = \int_0^{y_1} b(0, t) dt + \int_0^{x_1} a(s, y_1) ds.$$

The same reasoning then proves  $\frac{\partial f}{\partial x}(x, y) = a(x, y)$ . On the other hand, if the integrals are equal, then we see that  $df = \omega$ . If  $\omega$  is  $C^k$  then  $f$  is  $C^{k+1}$ .

But if  $d\omega = 0$ , then these two integrals turn out the same after the switch, by Stokes's theorem: closed 1-forms in the disk are exact.

In a ball  $B \subset \mathbb{R}^n$  centred at the origin, we can integrate any  $C^k$  1-form

$$\omega = \sum a_i(x) dx_i$$

in each variable one at time

$$\begin{aligned} f(x) := & \int_0^{x_1} a_1(t, 0, \dots, 0) dt + \int_0^{x_2} a_2(x_1, t, 0, \dots, 0) dt + \dots \\ & + \int_0^{x_n} a_n(x_1, \dots, x_{n-1}, t) dt. \end{aligned}$$

If this integral turns out the same, even after we take the variables in some other order instead, then by the same reasoning,  $df = \omega$  and so  $d\omega = 0$ . On the other hand, if  $d\omega = 0$  then, by repeated applications of Stokes's theorem, swap the order of the variables:  $df = \omega$ .

Any  $C^k$  1-form on any open set in  $\mathbb{R}^n$  is locally exact, because the open set is a union of open balls.

The same works locally on any manifold, by using a chart to locally work in  $\mathbb{R}^n$ : if the integral of  $\omega$  around any embedded image of a rectangle vanishes, then  $\omega$  is locally exact, hence closed, and vice versa.

We saw that every closed form is exact on any star shaped open set in Euclidean space, by a sophisticated use of Fubini's theorem, in theorem 30.6 on page 244.

Take an open set  $U \subset \mathbb{R}^2$ , and a disk  $D_r \subset U$  of radius  $r$  with center some point  $x_0$ . Take a  $C^1$  1-form  $\omega$  on  $U$ . Write  $d\omega = h dx \wedge dy$ . The average of  $h$  over  $D_r$  is

$$\begin{aligned} \frac{\int_{D_r} h dx \wedge dy}{\pi r^2} &= \frac{\int_{D_r} d\omega}{\pi r^2}, \\ &= \frac{\int_{\partial D_r} \omega}{\pi r^2}. \end{aligned}$$

By continuity of  $h$ , this approaches  $h(x_0)$  as  $r \rightarrow 0$ . Hence  $d\omega = 0$  just when  $\int_{\partial D_r} \omega = o(r)^2$ . If the integral of a 1-form around any small loop vanishes, then that 1-form is closed, so locally exact.

The same argument works in an open set  $U \subset \mathbb{R}^n$ , if we use a disk  $D_r$  lying on some coordinate axes, for example on the  $(x_3, x_7)$ -plane instead of the  $(x, y)$ -plane:  $d\omega = 0$  just when  $\int_{\partial D_r} \omega = o(r)^2$  for any disk of radius  $r$  in any coordinate plane. If the integral of a 1-form around any small loop vanishes, then that 1-form is closed, so locally exact.

## Integrating via potentials

Consider how we can define an integral over a continuous path. Suppose that  $\omega$  is a  $C^1$  closed 1-form on a smooth manifold with corners  $M$ . Take any continuous path  $x: [0, 1] \rightarrow M$ . If the path lies inside an open set in which  $\omega$  is exact, say  $\omega = df$ , we can define  $\int_I x^* \omega$  to mean  $f(x(1)) - f(x(0))$ . The definition is unambiguous, since we could have used any potential  $f$  for  $\omega$  on any open set containing the path, and any two agree up to constant near the path.

Suppose instead the path travels outside of any open set on which a potential exists. Each point of the path lies in some open subset of  $M$  on which  $\omega$  is exact. Take finitely many such open sets  $M_0, M_1, \dots, M_k$  so that the path stays inside each on successive intervals  $I_0, I_1, \dots, I_k$  with  $\omega = df_j$  on  $M_j$ . On the overlaps of those open sets,  $\omega = df_j = df_k$ . So  $f_{j+1} - f_j$  is closed, as a smooth function on  $M_j \cap M_{j+1}$ , hence locally exact. Two successive intervals overlap on an interval on which  $f_{j+1} - f_j$  is constant, say  $c_j := f_{j+1} - f_j$  on that overlap.

Pick times  $0 = t_0, t_1, \dots, t_n = 1$  so that  $[t_j, t_{j+1}] \subset I_j$ . Define

$$\int_0^1 x^* \omega := \sum f_j(x(t_{j+1})) - f_j(t_j).$$

This is the usual integral if the path is piecewise  $C^1$ . It depends only on that various endpoints  $x(t_j)$ , so we can change the path in between those points as we like, replacing with any smooth path with the same endpoints lying in the same  $M_j$ . In particular, replacing with a smooth path, it remains unchanged if we vary path through a smooth homotopy fixing endpoints, at least as long as the path stays inside the required finite

open cover. If we change open cover, we can make a tedious check that the integral is unchanged.

Consider another approach. Define new functions  $g_j$  to replace our  $f_j$ : let  $g_j := f_j - c_0 - \cdots - c_{j-1}$ . Then  $g_0 = f_0$ , and  $g_1 = f_1 - c_0 = f_0$  and so on, so  $g_j$  agrees with its predecessor and its successor. Careful: our path could loop back on itself, so some  $g_j, g_k$  could disagree. (In a manifold of dimension three or more, we can slightly perturb any continuous path defined on a compact interval to become embedded, but in dimensions one or two, crossings are stable.) If all  $g_j$  agree where any two are defined, we can write them as one function  $g$  and we can define  $\int_0^1 x^* \omega = g(x(1)) - g(x(0))$ .

Integrate over a loop  $x(t)$  periodic of period 1. Our integral is

$$\begin{aligned} \int_0^1 x^* \omega &= f_0(x(t_1)) - f_0(x(t_0)) \\ &\quad + f_1(x(t_2)) - f_1(x(t_1)) \\ &\quad + \dots \\ &\quad + f_k(x(t_{k+1})) - f_k(x(t_k)), \\ &= f_1(x(t_1)) - c_0 - f_0(x(t_0)) \\ &\quad + f_2(x(t_2)) - c_1 - f_1(x(t_1)) \\ &\quad + \dots \\ &\quad + f_0(x(0)) - c_k - f_k(x(t_k)), \\ &= -c_0 - c_1 - \dots - c_k. \end{aligned}$$

If this vanishes then our function  $g$  is defined on some open set containing the image of our loop. So we see that if a closed form has vanishing integral around an embedded loop, then it is locally exact near that loop.

## Smoothing

Consider a different way to define an integral over a continuous path: just smooth out the path. Suppose that  $\omega$  is a  $C^1$  closed 1-form on a smooth manifold with corners  $M$ . Take any path  $x_0: [a, b] \rightarrow M$ . Take a smoothing homotopy  $x_\varepsilon: [a, b] \rightarrow M$  with fixed endpoints, which we also think of as a map  $x: [a, b] \times [0, 1] \rightarrow M$ . Because the end points of  $x_\varepsilon$  are fixed,  $x^* \omega = 0$  along  $s = a$  and  $s = b$  for  $\varepsilon > 0$ . By Stokes's theorem, for any  $\varepsilon_0 > 0$ ,

$$0 = \int_a^b \int_{\varepsilon=\varepsilon_0}^{\varepsilon=1} x^* d\omega = \int_a^b x_1^* \omega - \int_a^b x_{\varepsilon_0}^* \omega,$$

i.e.

$$\int_a^b x_\varepsilon^* \omega$$

is constant in  $\varepsilon$ . If we use a different smoothing homotopy, uniqueness of smoothing homotopy up to smooth homotopy (with fixed end points) and the same argument proves that this constant value is independent of the choice of smoothing homotopy. So we can unambiguously *define*

$$\int_a^b x_0^* \omega := \int_a^b x_\varepsilon^* \omega,$$



for any  $\varepsilon > 0$ , i.e. we can integrate closed 1-forms along merely continuous paths. This integral doesn't vary when you move the path  $x_0(t)$  by a continuous homotopy fixing end points: smooth the entire homotopy fixing end points.

## Integrals over loops

A path  $x: [0, 1] \rightarrow M$  with  $x(0) = x(1)$  is a *loop*. A continuous map  $F: C \rightarrow \mathbb{R}^n$  from a curve  $C$  is *piecewise continuously differentiable* if we can cover  $C$  by compact curves with boundary, so that any compact subset of  $C$  is covered by finitely many of those curves, and  $F$  is continuously differentiable on each of those curves: 

**Proposition 37.1.** *Take any  $C^k$  1-form  $\omega$  on a connected smooth manifold  $M$  with corners, some  $k \geq 1$ . Then the following are equivalent:*

- The 1-form  $\omega$  is exact.
- The 1-form  $\omega$  is closed, and its integral around any loop vanishes.
- For any two piecewise  $C^1$  paths  $x_1: [a_1, b_1] \rightarrow M$ , and  $x_2: [a_2, b_2] \rightarrow M$ , starting and ending at the same points,  $\int x_1^* \omega = \int x_2^* \omega$ .
- For any two  $C^1$  paths  $x_1: [a_1, b_1] \rightarrow M$ , and  $x_2: [a_2, b_2] \rightarrow M$ , starting and ending at the same points,  $\int x_1^* \omega = \int x_2^* \omega$ .
- For any smooth map  $F: C \rightarrow M$  from a circle  $C$  in the plane,  $\int_C F^* \omega = 0$ .

*Proof.* *a.* implies *b.* by smoothing the loop into a map from the circle, and Stokes's theorem  $\int_C df = \int_{\partial C} f = 0$  since  $C$  has no boundary.

*b.* implies *c.* by following one path and then the other to make a loop.

*c.* implies *d.* implies *e.*

In our examples above, we saw that *e.* implies that  $\omega$  is closed and so is locally exact. We can smooth any loop into a map from the circle. So the integral  $\int \omega$  around any loop vanishes. The integrals  $\int \omega$  along paths from a chosen point  $m_0 \in M$  therefore unambiguously determine a potential function  $f(m) = \int_{m_0}^m \omega$ , implying *a.*  $\square$

A connected manifold  $M$ , perhaps with corners, is *simply connected* if every loop starting and ending at some point  $m_0 \in M$  is homotopic through loops starting and ending at  $m_0$  to a constant map.

Balls are simply connected, by homotopy taking any loop  $x_0(t)$  starting and ending at  $p_0$  and letting

$$x_s(t) := (1 - s)x_0(t) + sp_0.$$

**37.3** Prove that  $M$  being simply connected is independent of the choice of point  $m_0$ .

**Corollary 37.2.** *On a simply connected manifold  $M$  with corners, every closed 1-form is exact.*

*Proof.* Any smooth map from the circle extends to a continuous map from the disk, which we approximate with smooth maps from the disk, hence apply Stokes's theorem to see that the integral vanishes.  $\square$

**Corollary 37.3.** *On any ball in  $\mathbb{R}^n$ , every closed 1-form is exact.*

**Corollary 37.4.** *On any manifold  $M$  with corners, every closed 1-form is locally exact: each point of  $M$  lies in an open set in which every closed 1-form is exact.*

Let  $M = \mathbb{R}^2 - 0$  be the plane punctured at a point. Let  $\tilde{M} = (0, \infty) \times \mathbb{R}$ . The universal covering map is

$$\pi: (r, \theta) \in \tilde{M} \rightarrow (x, y) = (r \cos \theta, r \sin \theta) \in M.$$

Take a closed 1-form  $\omega$ . Then  $\pi^*\omega$  is exact, because  $\tilde{M}$  is simply connected, say  $\pi^*\omega = df$ . But then  $f = f(r, \theta)$  might not be  $2\pi$ -periodic in  $\theta$ . Indeed

$$\begin{aligned} f(r, \theta + 2\pi) &= f(r, \theta) + \int_{\theta}^{\theta+2\pi} \pi^*\omega, \\ &= f(r, \theta) + \int_{C_r} \omega, \end{aligned}$$

where  $C_r$  is the circle of radius  $r$  about the origin. By Stokes's theorem,

$$\int_{C_r} \omega$$

is independent of  $r$ , so we can just let  $C = C_1$  be the unit circle and let

$$a := \int_C \omega,$$

So a closed 1-form on the punctured plane is exact just when its integral over a circle around the origin vanishes.

Recall the angle 1-form

$$\omega_0 = \frac{-y dx + x dy}{x^2 + y^2},$$

which is defined precisely on  $M$  and has

$$\int_C \omega_0 = 2\pi,$$

for any circle around the origin. Let

$$\omega' := \omega - \frac{a\omega_0}{2\pi}.$$

Then  $\int_C \omega' = 0$ , so  $\omega'$  is exact on  $M$ , say  $\omega' = df$ . Hence every closed 1-form on the punctured plane admits a unique expression as

$$\omega = df + \frac{a\omega_0}{2\pi}.$$

The same argument works for  $M$  an annulus, cutting out a closed or open disk from a closed or open disk, or for a punctured disk, or any surface diffeomorphic to such.

Start with a surface  $S$ , perhaps with corners, on which every closed 1-form is exact, for example a star shaped open set. Pick finitely many disjoint embedded closed disks  $D_i$  in  $S$ ; we can allow disks with radius zero, i.e. points, among them. Let  $M$  be  $S$  with those disks removed, i.e. holes poked. (For any disk which is not just a single point, we can remove only its interior if we like.) Arrange that each disk  $D_i$  sit in two others  $D_i \subseteq D'_i \subseteq D''_i$ , of larger radii, so that  $C_i := \partial D'_i \subset M$ , and all  $D''_i$  are disjoint. Take a closed 1-form  $\omega$ . Suppose that

$$\int_{C_i} \omega = 0$$

for all  $i$ . In the annulus  $D''_i - D_i$ , by the previous example,  $\omega = df_i$  for a function  $f_i$ . Take functions  $h_i$  equal to 1 in  $D'_i$  and to zero outside of  $D''_i$ . Let  $f = \sum h_i f_i$ , which we extend to vanish outside of the supports of the  $h_i$ , so  $f$  is defined on  $M$  and  $\omega = df$  inside  $D'_i - D_i$  for each  $i$ . We can therefore extend  $\omega - df$  to a closed 1-form  $\omega'$  on  $S$ , by setting  $\omega' = 0$  in  $D'_i$ . All closed 1-forms on  $S$  are exact, so  $\omega$  is exact.

To conclude, when we make  $m$  punctures by closed or open disks or points, the linear map

$$\omega \mapsto \left( \int_{C_1} \omega, \dots, \int_{C_m} \omega \right)$$

has kernel precisely the exact forms. For example, if  $S$  is a star shaped open set in the plane, we can translate the 1-form

$$\omega_0 = \frac{-y dx + x dy}{2\pi(x^2 + y^2)},$$

to move the origin to one of the punctures, so that we construct a differential form on  $M$  with nonzero integral around that puncture, and zero around the others. Hence the map

$$\omega \mapsto \left( \int_{C_1} \omega, \dots, \int_{C_m} \omega \right)$$

is onto, with kernel the exact 1-forms on  $M$ . A different example: if  $S$  is the sphere and  $M$  the sphere with one puncture, then these integrals around around circle around the puncture vanish, because  $M$  is contractible. So the map giving integrals around the circles might not be onto.

**37.4** On the punctured plane  $M = \mathbb{R}^2 - \{0\}$ , prove that every smooth 2-form is exact,  $k \geq 1$ .

**37.5** Prove that every smooth 2-form on the plane with  $g$  punctures is exact, for any integer  $g$ .

In three dimensional Euclidean space, pick a circle  $C$  and let  $\ell$  be the line through the center of  $C$  perpendicular to the plane containing  $C$ . Each point  $p$  not on  $\ell$  has a unique angle of rotation  $\beta$  around  $\ell$ , defined only up to  $2\pi$  integer multiples. The points  $p$  with a given angle  $\beta$  form a half plane  $H_\beta$ . If that point  $p$  is, in addition, not on  $C$ , it has an angle of rotation  $\alpha$  in that half

plane, rotated about the point where  $C$  meets  $\Pi_\beta$ , defined up to  $2\pi$  integer multiples. The angles  $\alpha, \beta$  are only defined up to  $2\pi$  integer multiples, but

$$\xi := \frac{d\alpha}{2\pi}, \eta := \frac{d\beta}{2\pi}$$

are well defined as 1-forms away from  $\ell \cup C$ . Writing each point as  $p = (x, y, z)$ , take  $C = (z = 0, x^2 + y^2 = r_1^2)$ , so  $\ell$  is the vertical axis. Since  $p$  is not on  $\ell$ , we can write  $(x, y) = re^{i\beta}$ ; the coordinates  $(r, \beta, z)$  are *cylindrical polar coordinates*. Instead we can write

$$p = (x, y, z) = ((r_1 + s \cos \alpha) \cos \beta, (r_1 + s \cos \alpha) \sin \beta, s \sin \alpha)$$

for some  $s > 0$ ;  $(\alpha, \beta, s)$  might be called *toroidal coordinates*. Expanding out,

$$\begin{aligned} \xi &= \frac{-y dx + x dy}{2\pi(x^2 + y^2)}, \\ \eta &= \frac{-z ds + s dz}{2\pi(s^2 + z^2)}, \end{aligned}$$

into which we can plug

$$s = \sqrt{(\sqrt{x^2 + y^2} - r_1)^2 + z^2}$$

to see that  $\eta$  is smooth. If we slice three dimensional Euclidean space along any half plane with boundary  $\ell$ , the angle  $\beta$  becomes defined so  $\eta$  is exact there. Similarly, the angle  $\alpha$  becomes defined outside the disk  $z = 0, x^2 + y^2 \leq r_1^2$ , with  $-\pi < \alpha < \pi$ , so  $\xi$  is exact there.

The torus sits in  $\mathbb{R}^3$  as the *anchor ring*:

$$(\sqrt{x^2 + y^2} - r_1)^2 + z^2 = r_0^2,$$

i.e.  $s = r_0$  in toroidal coordinates. Clearly  $\xi, \eta$  are closed 1-forms on the torus. They have linearly independent integrals: integrate along the paths of constant  $\alpha$  or  $\beta$ .

The compact oriented surface of genus  $g$ , as constructed in problem 4.26 on page 47, sits in 3-dimensional Euclidean space, mapping down to the plane by  $(x, y, z) \mapsto (x, y)$ , to project to a disk punctured at  $g$  holes. Recall our notation from that problem:

$$f_i = (x - x_i)^2 + (y - y_i)^2 - r_i^2$$

$f = f_0 \dots f_g$  and  $S = (f + z^2 = 0)$  is our surface. The circle  $C_i := (f_i = 0, z = 0)$  is the boundary of the closed disk  $D_i = (f_i \leq 0, z = 0)$ . Apply toroidal coordinates as above, for a single circle  $C_i$ , to generate two 1-forms defined everywhere on  $S$ , with linearly independent integrals “measuring winding” of two loops around the “doughnut hole” number  $i$ . (Careful: it takes a lot of effort to prove that  $S$  intersects a suitable vertical plane along a smooth curve on which the associated  $d\alpha$  integral is not zero. We don’t face this

problem for  $d\beta$ : the horizontal plane slices  $S$  in circles.) Each  $d\alpha$  1-form is closed outside  $D_i$ . Suppose we arrange the choices of the disks  $D_i$  so that we can slice the plane ( $z = 0$ ) along a line segment outward from the center of  $D_i$  without hitting any point of any  $D_j$  for  $j \neq 0, i$ . (For example, place the  $D_i$  symmetrically around a circle inside  $D_0$ .) This slice is the intersect of some half plane  $\Pi_\beta$  with the plane ( $z = 0$ ). Each of the  $d\beta$  1-forms becomes exact away from that half plane, and so exact near any of the other doughnut holes, i.e. the loops “measured” by the other 1-forms. Hence we have found  $2g$  closed 1-forms with linearly independent integrals.

Take a connected manifold  $M$  with chosen point  $m_0 \in M$ . Take its universal covering space  $\tilde{M}$  with chosen point  $\tilde{m}_0 \in \tilde{M}$  and universal covering map  $\pi: \tilde{M} \rightarrow M$ . Since  $\tilde{M}$  is simply connected, every closed 1-form on  $\tilde{M}$  is exact. Take a closed 1-form  $\omega$  on  $M$  and pull back to  $\tilde{M}$ , so  $\pi^*\omega = df$ . Each loop  $\gamma$ , on  $M$  starting and ending at  $m_0$  lifts to a path  $\tilde{\gamma}$  on  $\tilde{M}$ , starting at  $\tilde{m}_0$  and ending at a point which we denote  $\gamma\tilde{m}_0$ . So it suffices that  $f$  take the same values at each of those points, i.e. that

$$\int_{\gamma} \omega = 0,$$

for every loop  $\gamma$ , or equivalently for a set of generators of  $\pi_1(M)$ . So the linear map

$$\int \omega: \gamma \in \pi_1(M) \mapsto \int_{\gamma} \omega \in \mathbb{R}$$

has kernel precisely the exact forms. The integral depends only on the real homology class of the loop. Every integer homology 1-cycle is a union of loops, each consisting of a union of finitely many paths each ending where the next starts, by the classification of curves. Such a path smooths into a single smooth loop. Hence the map  $\pi_1(M) \rightarrow H_1(M, \mathbb{Z})$  is onto, and so the map  $\pi_1(M) \rightarrow H_1(M, \mathbb{R})$  is onto. So a closed 1-form on  $M$  is exact just when its integral vanishes along generators of  $H_1(M, \mathbb{R})$ .



## Chapter 38

### de Rham cohomology

We define the de Rham cohomology of a manifold, and explain how to compute it for some examples.

#### Defining de Rham cohomology

Let  $M$  be a smooth manifold of dimension  $n$ . A smooth form  $\omega$  is *exact* if  $\omega = d\eta$  for some smooth form  $\eta$ . Recall that  $d^2 = 0$ . Every exact form is a closed, because if  $\omega = d\eta$  (exact) then  $d\omega = d^2\eta = 0$  (closed). Two closed forms are *cohomologous* if they differ by an exact form; the *cohomology class*  $[\omega]$  of a closed form  $\omega$  is its equivalence class under cohomologous equivalence. The set of cohomology classes of all closed  $p$ -forms is denoted  $H_{\text{dR}}^p(M)$  and called the  *$p$ -th cohomology group* of  $M$ ; it is clearly a vector space. The sum of these groups is called the *cohomology* or *de Rham cohomology* of  $M$ .

If  $\omega$  and  $\eta$  are both closed, then  $\omega \wedge \eta$  is also closed. Moreover, if either is exact, for example if  $\omega = d\alpha$ , then  $\omega \wedge \eta = d(\alpha \wedge \eta)$ , so the wedge product of cohomologous forms is cohomologous, i.e. wedge product is defined on cohomology, making  $H_{\text{dR}}^*(M)$  an algebra. It is a *supercommutative*, in that

$$[\omega] \wedge [\eta] = (-1)^{pq} [\eta] \wedge [\omega],$$

for any  $p$ -form  $\omega$  and  $q$ -form  $\eta$ , i.e. for summands in the direct sum.

#### Simple examples and elementary results

Every 1-form on  $M = \mathbb{R}$  is closed, and looks like  $\omega = h(x)dx$  for some smooth function  $h(x)$ . But then we can always form the indefinite integral  $g(x) = \int h(x)dx$  and then  $\omega = h(x)dx = dg$ . So all 1-forms are exact, i.e. the space of closed forms equals the space of exact forms, so their quotient is  $H_{\text{dR}}^1(\mathbb{R}) = 0$ .

A function on the circle  $M = S^1$  is just a function  $h(\theta)$  which is  $2\pi$ -periodic. Keep in mind that even if a function  $h(\theta)$  is  $2\pi$ -periodic, its indefinite integral  $\int h(\theta)d\theta$  might not be: for example,  $h(\theta) = 1$  has indefinite integral  $\theta + C$ , *not* periodic. A smooth 1-form on the circle has the form  $\omega = h(\theta)d\theta$  where  $h(\theta)$  can be any smooth  $2\pi$ -periodic function. Such a 1-form is always closed, but

is exact just when  $\omega = dg$  for some  $2\pi$ -periodic function  $g(\theta)$ , i.e.  $h(\theta) d\theta = dg = g'(\theta) d\theta$ . But then integrating both sides, we see that  $g(\theta)$  is forced to be  $g(\theta) = \int h(\theta) d\theta$ . So  $\omega = h(\theta) d\theta$  is exact just when  $\int h(\theta) d\theta$  is  $2\pi$ -periodic, i.e. just when  $0 = \int_0^{2\pi} h(\theta) d\theta$ . Therefore the linear map  $\omega \mapsto \int_{S^1} \omega$  vanishes precisely on the exact forms, and this descends to a linear isomorphism  $H_{\text{dR}}^1(S^1) \cong \mathbb{R}$ .

**38.1** Prove that a closed 1-form on the cylinder  $M = \mathbb{R} \times S^1$  is exact just when its integral around the circle  $\{0\} \times S^1$  vanishes. Prove that every 2-form on the cylinder is exact.

**38.2** Generalize these observations about the circle to compute the cohomology of the torus  $S^1 \times S^1$ .

**38.3** Draw a torus as a quotient of a square, folding opposite sides together. Draw a loop around a torus, by travelling along one side of the square. Make another loop, by drawing a parallel line segment through the middle of the square. Suppose that  $\varphi$  is a diffeomorphism of the torus which fixes every point on both of those loops. Must  $\varphi$  preserve orientation? What if  $\varphi$  also preserves some point not on either of those loops?

On a ball  $M = B$  (or convex set) we saw in corollary 30.5 on page 244 that every closed form of positive degree is exact, which says precisely that  $H_{\text{dR}}^p(B) = 0$  for  $p = 1, 2, \dots$  (The proof we gave works just as well if we insist that our forms must be smooth.)

Take any connected manifold  $M$  and consider the special case of  $H_{\text{dR}}^0(M)$ . A smooth function  $f$  is exact just when  $df = 0$ , i.e.  $f$  has all derivatives vanishing, i.e.  $f$  is constant. Therefore  $H_{\text{dR}}^0(M) = \mathbb{R}$ . If  $M$  is not connected, then  $H_{\text{dR}}^0(M)$  consists in just the functions  $f$  which are constant on each component (each connected “piece”) of  $M$ , so  $H_{\text{dR}}^0(M) = \mathbb{R}^k$  if  $M$  has  $k$  components.

We now see that for  $M = B$  a ball,

$$H_{\text{dR}}^p(B) = \begin{cases} \mathbb{R}, & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We also see that for  $M = S^1$  a circle,

$$H_{\text{dR}}^p(S^1) = \begin{cases} \mathbb{R}, & \text{if } p = 0, \\ \mathbb{R}, & \text{if } p = 1. \end{cases}$$



If  $M$  is a star shaped open set in the plane punctured at  $g$  points (or perhaps allow some of those points to be embedded open or closed disks), we have seen on page 293 that the map

$$\omega \mapsto \left( \int_{C_1} \omega, \dots, \int_{C_g} \omega \right)$$

is onto, with kernel the exact 1-forms. So

$$H_{\text{dR}}^1(M) = \mathbb{R}^g.$$

If  $M$  is any manifold of dimension  $n$ , then there is only one  $(n+1)$ -form on  $M$ :  $\omega = 0$ , so therefore every  $n$ -form is closed, and every  $(n+1)$ -form is exact, i.e.  $0 = H_{\text{dR}}^{n+1}(M) = H_{\text{dR}}^{n+2}(M) = \dots$ . So we only have to compute  $H_{\text{dR}}^p(M)$  for  $p = 1, 2, \dots, n$ .

If  $M = M_0 \sqcup M_1$  is a *disjoint* union of two manifolds, then each differential form on  $M$  is a form on  $M_0$  and another one on  $M_1$ , etc., so  $H_{\text{dR}}^p(M) = H_{\text{dR}}^p(M_0) \oplus H_{\text{dR}}^p(M_1)$  for every  $p$ .

If  $f: P \rightarrow Q$  is a smooth map of manifolds, then  $f^*: \Omega^p(Q) \rightarrow \Omega^p(P)$  satisfies  $f^*d = df^*$ , so that  $f^*$  takes closed forms to closed forms, exact forms to exact forms, and so descends to a linear map  $f^*: H_{\text{dR}}^p(Q) \rightarrow H_{\text{dR}}^p(P)$ . Of course, if  $f$  is a diffeomorphism, this map is an isomorphism. The map  $x \mapsto 2x/(1+|x|^2)$  is a diffeomorphism  $\mathbb{R}^n \rightarrow B$  to the unit ball, so

$$H_{\text{dR}}^p(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Write  $H_{\text{dR}}^*(M)$  to mean  $H_{\text{dR}}^p(M)$  for some  $p$ .

## Homotopy invariance

**Lemma 38.1.** *Suppose that  $\varphi_t: P \rightarrow Q$  is a homotopy of smooth maps of manifolds. Then  $\varphi_t^*: H_{\text{dR}}^*(Q) \rightarrow H_{\text{dR}}^*(P)$  is a linear map independent of  $t$ .*

*Proof.* Suppose that  $\omega$  is a closed smooth differential form on  $Q$ . Denote by  $\pi: [0, 1] \times P \rightarrow P$  the map  $\pi(t, p) = p$ . By lemma 30.3 on page 243,  $d\pi_*\varphi^*\omega + \pi_*\varphi^*d\omega = \varphi_1^*\omega - \varphi_0^*\omega$ , but  $d\omega = 0$ , so

$$d\pi_*\varphi^*\omega = \varphi_1^*\omega - \varphi_0^*\omega.$$

The left hand side is exact, so vanishes in cohomology.  $\square$

Pick positive numbers  $r < 1 < R$  and consider the annulus  $A \subset \mathbb{R}^n$  consisting of the points  $x \in \mathbb{R}^n$  for which  $r < |x| < R$ . Consider the homotopy

$$\varphi_t(x) = \frac{x}{(1-t) + t|x|}.$$

This is defined for  $0 \leq t \leq 1$ , and at  $t = 0$  is just  $\varphi_0(x) = x$ , so  $\varphi_0^*$  is the identity on cohomology. But for  $t = 1$ , this map takes  $A$  to the unit sphere,

so maps their cohomologies. Since  $\varphi_0^* = \varphi_1^*$ , the cohomology of the annulus equals that of the unit sphere  $S^{n-1}$ . The same idea works if we take any manifold (in this case the sphere) and “thicken” it. We could of course also allow  $r = 0$  or  $R = \infty$  or both; all of these annuli are diffeomorphic.

## Rotation

We will prove that the  $n$ -dimensional sphere  $M = S^n \subset \mathbb{R}^{n+1}$  has cohomology

$$H_{\text{dR}}^p(S^n) = \begin{cases} \mathbb{R}, & \text{if } p = 0, \\ 0, & \text{if } 1 \leq p \leq n-1, \\ \mathbb{R}, & \text{if } p = n \end{cases}$$

Take a closed form  $\omega$  on the sphere  $M$ , and rotate the sphere by an angle  $\theta$  around some axis, say by a rotation map  $r_\theta: M \rightarrow M$ . Since we can connect any rotation to any other by a homotopy,  $r_\theta^*\omega$  equals  $\omega$  in cohomology. By the Poincaré lemma (lemma 30.3 on page 243)

$$r_\theta^*\omega = \omega + d\phi_\theta,$$

where

$$\phi_\theta = (-1)^{p-1+\dim M} \pi_* r^*\omega,$$

with  $r: (\theta, m) \in [0, 2\pi] \times M \rightarrow r_\theta m \in M$  our rotation map and

$$\pi: (\theta, m) \in [0, \theta] \times M \mapsto m \in M$$

the obvious projection. We have seen in example 30 on page 242 that averaging over rotations takes closed forms to closed forms. Taking averages, the average of a closed form  $\omega$  over a circle of rotations is a closed form in the same cohomology class, invariant under all of the rotations  $r_\theta$ :

$$\frac{1}{2\pi} r_* (r^* \omega \wedge d\theta) = \omega + d\left(\frac{1}{2\pi} \int_0^{2\pi} \phi_t\right).$$

**38.4** Prove that the only differential forms on the sphere invariant under all rotations are constant functions and multiples of the volume form of the sphere.

We could try to rotate a torus, but it is tricky to write down the rotations. Instead, let's put the torus into  $\mathbb{R}^4$ , by taking a circle in the plane  $S^1 \subset \mathbb{R}^2$ , and letting  $M = S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ , so that  $M$  is a torus in 4-dimensional space.

**38.5** Prove that this  $M$  is diffeomorphic to the usual torus.

We can now just rotate each of the circles. By the same argument, each cohomology class contains a closed form which is invariant under both rotations.

If we use angle coordinates  $\theta_1$  and  $\theta_2$  for the two circles, then  $\theta_1, \theta_2$  is a chart on  $M$  and in that chart, the invariant forms are

0-forms : 1, 1-forms :  $d\theta_1, d\theta_2$ , 2-forms :  $d\theta_1 \wedge d\theta_2$ .

All of the invariant forms are closed, so they give a basis in cohomology:

$$H_{\text{dR}}^p(S^1 \times S^1) = \begin{cases} \mathbb{R}, & \text{if } p = 0, \\ \mathbb{R}^2, & \text{if } p = 1, \\ \mathbb{R}, & \text{if } p = 2. \end{cases}$$

The same argument gives the cohomology of any manifold  $M = S^p \times S^q$  or  $M = S^{p_1} \times S^{p_2} \times \cdots \times S^{p_k}$ .

Recall that the circle  $S^1 \subset \mathbb{C}$  is the set of complex numbers of the form  $e^{i\theta}$  for  $\theta \in \mathbb{R}$ . A *circle action* on a manifold  $M$  is a continuously differentiable map  $r: S^1 \times M \rightarrow M$ , denoted  $e^{i\alpha}m$  instead of  $r(e^{i\alpha}, m)$ , so that  $1m = m$  for all  $\alpha \in \mathbb{R}$  and  $m \in M$  and so that

$$e^{i\alpha}(e^{i\beta}m) = e^{i(\alpha+\beta)}m$$

for all  $m \in M$ ,  $e^{i\alpha}, e^{i\beta} \in S^1$ . For example, rotation of a sphere at unit speed is a circle action.

**38.6** Show that any manifold with a circle action has a differential form in each de Rham cohomology class invariant under the circle action.

The  $n$ -dimensional torus  $T^n$  is the product  $S^1 \times S^1 \times \cdots \times S^1$  of  $n$  circles, with obvious manifold and group structures. A *torus action* on a manifold  $M$  is a continuously differentiable group action  $T^n \times M \rightarrow M$ .

**38.7** Show that any manifold with a torus action has a differential form in each de Rham cohomology class invariant under the torus action.

## Integration

Another special case is the top degree cohomology  $H_{\text{dR}}^n(M)$  of an  $n$ -dimensional manifold; we often write this as  $H_{\text{dR}}^{\text{top}}(M)$ . Since all  $n$ -forms on any  $n$ -dimensional manifold are closed,  $H_{\text{dR}}^{\text{top}}(M)$  has unusual numerator:

$$H_{\text{dR}}^{\text{top}}(M) = \Omega^n(M)/d\Omega^{n-1}(M),$$

i.e. its elements look like  $n$ -forms  $\omega$ , but up to the equivalence that  $\omega$  is equivalent to  $\omega + d\eta$  for any  $(n-1)$ -form  $\eta$  on  $M$ . In particular, if  $M$  is compact and oriented, then we can integrate  $\omega$ , and by Stokes's theorem  $\int_M d\eta = \int_{\partial M} \eta = 0$ . Therefore

$\int_M \omega = \int_M \omega + d\eta$ . So we can integrate the cohomology class of  $\omega$ , i.e. integration is a linear map

$$\int_M : H_{\text{dR}}^{\text{top}}(M) \rightarrow \mathbb{R}.$$

### Forms with compact support

If  $\omega$  has compact support, so does  $d\omega$ . The *compactly supported cohomology*  $H_c^*(M)$  is the space of closed compactly supported smooth forms, modulo the exterior derivatives of compactly supported smooth forms.

A compactly supported function  $f(x)$  on  $\mathbb{R}$  with  $df = 0$  must be  $f = \int df = 0$ : so  $H_c^0(\mathbb{R}) = 0$ .

If a manifold is compact then any compactly supported function  $f$  with  $df = 0$  is a constant on each component:  $H_c^0(M) = \mathbb{R}^c$  where  $M$  has  $c$  components.

If a manifold has  $c$  compact components and  $k$  noncompact ones, then any compactly supported function  $f$  with  $df = 0$  is a constant on each compact component, but vanishing on any noncompact component:  $H_c^0(M) = \mathbb{R}^c$ .

If  $\omega = h(x) dx$  is a continuous differential form on  $\mathbb{R}$ , we can form the indefinite integral  $g(x) = \int h(x) dx$ , so that  $\omega = dg$  is exact. Suppose that  $\omega$  has compact support. Clearly  $g(x)$  has compact support just when  $0 = \int_{\mathbb{R}} h(x)$ , and so  $\omega$  is the exterior derivative of a compactly supported 0-form  $g$  just when  $0 = \int_{\mathbb{R}} h(x) dx$ . So the map  $\omega \mapsto \int_{\mathbb{R}} \omega$  is a linear isomorphism on cohomology. Hence the compactly supported cohomology groups of  $\mathbb{R}$  are  $H_c^0(\mathbb{R}) = 0$  and  $H_c^1(\mathbb{R}) = \mathbb{R}$ .

In  $\mathbb{R}^2$ , consider a closed 1-form  $\omega = f(x, y) dx + g(x, y) dy$ . If  $\omega$  has compact support, say inside a ball  $B$ , we can pick any point  $(x_0, y_0)$  outside of  $B$  and let

$$h(x, y) = \int_{(x_0, y_0)} \omega,$$

the integral along any path. Since  $\omega$  is closed, the integral is path independent. If  $(x_1, y_1)$  is another point outside  $B$ , we can easily draw a path from  $(x_0, y_0)$  to  $(x_1, y_1)$  avoiding  $B$  entirely, so  $h = 0$  outside  $B$ , and so  $h$  has compact support. Therefore every compactly supported closed 1-form on  $\mathbb{R}^2$  is  $\omega = dh$  for some compactly supported 0-form  $h$  on  $\mathbb{R}^2$ .

Take a compactly supported 2-form  $\omega = h(x, y) dx \wedge dy$  on  $\mathbb{R}^2$ , say supported in the interior of some ball  $B$ . Clearly if  $\omega = d\eta$  for some compactly supported

form  $\eta$ , and we make  $B$  larger if needed to contain the support of  $\eta$ , then  $\int_B \omega = \int_{\partial B} \eta = 0$ . So the integral of  $\omega$  must vanish for  $\eta$  to exist. On the other hand, suppose that  $\int_B \omega = 0$ . Pick any compactly supported smooth function  $g(y)$  with  $\int_{\mathbb{R}} g(y) dy = 1$ . Let  $H(x, y) = \int_{-\infty}^y h(x, u) du$  and  $G(y) = \int_{-\infty}^y g(u) du$ . Define a 1-form

$$\xi = (G(y)H(x, \infty) - G(\infty)H(x, y))dx + \left( \int_{-\infty}^x H(u, \infty) du \right) g(y) dy.$$

and check that  $d\xi = \omega$ .

Similar very explicit tricks work in higher dimension. Write points of  $\mathbb{R}^{n+1}$  as  $(x, y)$  with  $x \in \mathbb{R}^n$  and  $y \in \mathbb{R}$ . Let  $dV_x = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$  and let  $e_i$  be the standard basis. Take a compactly supported smooth  $n$ -form  $\omega$  on  $\mathbb{R}^{n+1}$ ,

$$\omega = h(x, y) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge dy.$$

Pick a smooth function  $g(y)$  with compact support and let  $G(y) = \int g(u) du$ . Let  $H(x, y) = \int_{-\infty}^y h(x, u) du$  and

$$K_i(x, y) = \int_{-\infty}^{x_i} H(x_1, x_2, \dots, x_{i-1}, u, x_{i+1}, \dots, x_n, y) du.$$

Then let

$$\xi = (-1)^n (G(\infty)H(x, y) - G(y)H(x, \infty))dV_x + (-1)^n K_i(x)g(y)e_i \lrcorner dV_x \wedge dy.$$

It is easy to check that  $d\xi = \omega$  and that  $\xi$  has compact support just when  $0 = \int_{\mathbb{R}^n} \omega$ .

To summarise, if  $\omega$  is a top degree form supported in the interior of a box, and  $\int \omega = 0$  then  $\omega = d\eta$  for some smooth form supported in the interior of the same box. Consequently, integration of compact supported forms maps the top degree compactly supported cohomology of  $\mathbb{R}^n$  to  $\mathbb{R}$ , by a linear isomorphism. Write  $H_c^{\text{top}}(M)$  for  $H_c^n(M)$  if  $n$  is the dimension of  $M$ .

**Theorem 38.2.** *Suppose that  $M$  is a connected oriented manifold with no corners and no boundary. Integration  $\int: H_c^{\text{top}}(M) \rightarrow \mathbb{R}$  is a linear isomorphism.*

*Proof.* It suffices to prove that any top degree compactly supported form  $\omega$  whose integral vanishes is exact. Cover the support of  $\omega$  by open sets  $U_i$ , each lying in the domain of a chart. Pick a partition of unity  $f_i$  subordinate to this open cover and let  $\omega_i = f_i \omega$ . Pick a point  $p_i \in U_i$  in each open set. Near each point  $p_i$ , choose a small “bump”, i.e. a compactly supported top degree smooth differential form  $\eta_i$  with  $\int \eta_i = \int \omega_i$ , with support contained in  $U_i$ . Since  $\int_{U_i} \omega_i - \eta_i = 0$ , a compactly supported form with vanishing integral, our work above proves that  $\omega_i - \eta_i = d\xi_i$  for some compactly supported form  $\xi_i$  with support in  $U_i$ . Replacing  $\omega$  by  $\omega - \sum d\xi_i$  doesn't change the cohomology class of  $\omega$ , but replaces the  $\omega_i$  by the  $\eta_i$ , so we can assume that each  $\omega_i$  is supported as close as we like to a point  $p_i$ .

Pick distinct points  $q_i \in U_1$ . In problem 22.20, we saw that there is a smooth diffeotopy  $F_t: M \rightarrow M$  so that  $F_0$  is the identity map, and so that  $F_1(p_i) = q_i$ . We can arrange the support of each  $\omega_i$  as small as we like, to make that support so small that  $F_1$  takes it inside  $U_1$ . Therefore we can assume that all of the  $\omega_i$  are “small bumps” inside the domain of a single chart. It is enough to suppose that  $\omega$  is supported inside the domain of one chart, and so  $\omega$  is identified by the chart with some top degree form on  $\mathbb{R}^n$ . Again by our work above,  $0 = \int \omega$  just when we can find a smooth form  $\xi$  with support inside that same box so that  $\omega = d\xi$ .  $\square$

On the other hand:

**Theorem 38.3.** *Suppose that  $M$  is a connected oriented manifold with corners. Suppose that every component of  $M$  has nonempty boundary. Each top degree smooth form  $\omega$  has the form  $\omega = d\eta$  for some smooth differential form  $\eta$ , i.e.*

$$0 = H_{dR}^{\text{top}}(M).$$

*Each top degree compactly supported smooth form  $\omega$  has the form  $\omega = d\eta$  for some compactly supported smooth differential form  $\eta$ , i.e.*

$$0 = H_c^{\text{top}}(M).$$

*Proof.* Start with  $\omega$  compactly supported. Cover the support of  $\omega$  by open sets  $U_i$ , each lying in the domain of a chart valued in  $[0, \infty)^n$ . Pick a partition of unity  $f_i$  subordinate to this open cover and let  $\omega_i = f_i\omega$ . Divide up into some  $U_a$  which intersect the boundary and some  $U_i$  which don't. Pick a point  $p_i \in U_i$  in each open set. Near each point  $p_i$ , choose a small “bump”, i.e. a compactly supported top degree smooth differential form  $\eta_i$  with  $\int \eta_i = \int \omega_i$ , with support contained in  $U_i$ . Since  $\int_{U_i} \omega_i - \eta_i = 0$ , a compactly supported form with vanishing integral, our work above proves that  $\omega_i - \eta_i = d\xi_i$  for some compactly supported form  $\xi_i$  with support in  $U_i$ . By problem 30.8 on page 244, for those  $U_a$  with nonempty boundary, we can find some  $\xi_a$  supported in  $U_i$  with  $\omega_a = d\xi_a$ . Replace  $\omega$  by  $\omega - \sum d\xi_i$ , thus replacing the  $\omega_i$  by the  $\eta_i$ , so we can assume that each  $\omega_i$  is supported as close as we like to a point  $p_i$  in the interior. (If  $\omega$  is compactly supported, there are only finitely many  $\xi_i$ .) Pick distinct points  $q_i$  in the interior of one of the open sets  $U_a$ . In problem 22.20, we saw that there is a smooth diffeotopy  $F_t: M \rightarrow M$  so that  $F_0$  is the identity map, and so that  $F_1(p_i) = q_i$ . We can arrange the support of each  $\omega_i$  as small as we like, to make that support so small that  $F_1$  takes it inside  $U_a$ . Therefore we can assume that all of the  $\omega_i$  are “small bumps” inside the domain of a single chart with nonempty boundary. So we can assume that all  $\omega_i$  vanish, i.e.  $\omega = \sum d\xi_a = d \sum \xi_a$ .

Now for  $\omega$  with noncompact support, again take a partition of unity  $f_i$  subordinate to a locally finite open cover by precompact open sets. Let  $\omega_i := f_i\omega$ . Then each  $\omega_i$  has compact support, so  $\omega_i = d\eta_i$  for some compactly supported  $\eta_i$ , supported also inside the same open sets. Let  $\eta := \sum \eta_i$ , a locally finite sum, so  $\omega = d\eta$ .  $\square$

## Integration over chains

A *boundary* is a chain of the form  $A = \partial B$  for some chain  $B$ . For any closed form  $\omega$ , i.e. with  $d\omega = 0$ ,

$$\int_{\partial B} \omega = \int_B d\omega = \int_B 0 = 0.$$

So if we add a boundary to a chain, we don't alter the integrals of closed forms. A *cycle* is a chain  $A$  so that  $\partial A = 0$ . A *cochain* is a differential form. A *cocycle* is a closed cochain  $\omega$  so that  $d\omega = 0$ . For any cocycle  $\omega$  and cycle  $C$ ,  $\int_C \omega$  doesn't change if we add an exact form:

$$\int_C d\eta = \int_{\partial C} \eta = 0,$$

since  $\partial C = 0$ . Similarly,  $\int_C \omega$  doesn't change if we add a boundary to  $C$ :

$$\int_{\partial B} \omega = \int_B d\omega = 0.$$

The *homology class* of a cycle is its equivalence class, two being equivalent if they agree up to a boundary. Let  $H_p(M, \mathbb{Z})$  be the homology classes of  $p$ -dimensional cycles. For any cocycle  $\omega$  and cycle  $C$ ,  $\int_C \omega$  depends only on the homology class  $[C]$  and the cohomology class  $[\omega]$ .

Any 0-chain is a finite collection of points mapped to a manifold with corners  $M$ , with some choice of integer in front of each point. Their boundary is zero by definition. If two points lie in the same path component, connect them by a smooth path, so they form the boundary of that path. In homology, that pair of points is then trivial. So if we pick one point in each path component, then we can uniquely represent each homology class by associating an integer to each of those points:

$$H_0(M, \mathbb{Z}) = \bigoplus \mathbb{Z}$$

where the sum is over the path components of  $M$ . Similarly with rational or real coefficients.

If  $M$  has path components  $M_a$ , then

$$H_p(M, \mathbb{Z}) = \bigoplus_a H_p(M_a, \mathbb{Z})$$

for every  $p$ .

In any product  $M = P \times Q$  of manifolds with corners, a  $p$ -chain is a  $p$ -chain in each

Cycles are the kernel of  $\partial$ , and boundaries the image, so

$$H_p(M, \mathbb{Z}) = \frac{\ker \partial: C_p(M, \mathbb{Z}) \rightarrow C_{p-1}(M, \mathbb{Z})}{\text{image } \partial: C_{p+1}(M, \mathbb{Z}) \rightarrow C_p(M, \mathbb{Z})}.$$

Similarly cohomology is

$$H_{\text{dR}}^p(M) = \frac{\ker d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)}{\text{image } d: \Omega^{p-1}(M) \rightarrow \Omega^p(M)}.$$

while compactly supported cohomology is

$$H_c^p(M) = \frac{\ker d: \Omega_c^p(M) \rightarrow \Omega_c^{p+1}(M)}{\text{image } d: \Omega_c^{p-1}(M) \rightarrow \Omega_c^p(M)}.$$

Any  $C^1$  map  $\varphi: P \rightarrow Q$  has associated map  $\varphi_*$  on cycles, clearly commuting with  $\partial$ , hence descending to homology.

**Theorem 38.4.** *If two  $C^1$  maps  $\varphi, \psi: P \rightarrow Q$  of manifolds with corners are  $C^1$  homotopic, then  $\varphi_* = \psi_*$  on homology groups.*

*Proof.* Take a map  $\alpha: S \rightarrow P$  representing a chain in a manifold with corners  $P$ . Take a smooth map  $\Phi: P \times [0, 1] \rightarrow Q$  between manifolds with corners. Clearly  $\Psi: (s, t) \in S \times [0, 1] \mapsto \Phi(\alpha(s), t) \in Q$  represents a chain with

$$\partial\Psi = \Phi_1 \circ \alpha - \Phi_0 \circ \alpha + \Phi \circ \partial\alpha.$$

In particular, if  $\partial\alpha = 0$ , a cycle, then  $\Psi$  is a chain with boundary  $\Phi_{1*}\alpha - \Phi_{0*}\alpha$ .  $\square$

### Borel–Moore homology

A long chain is a *boundary* if it has the form  $A = \partial B$  for some chain  $B$ , and a *long cycle* or *Borel–Moore cycle* is a long chain  $A$  so that  $\partial A = 0$ . The  $p$ -dimensional *Borel–Moore homology* of  $M$ , denoted  $H_p^{\text{BM}}(M)$ , is the set of cycles up to equivalence, where two are equivalent if they differ by a boundary; denote the equivalence class of a chain  $C$  as  $[C]$ . For any compactly supported cocycle  $\omega$  and long cycle  $C$ ,  $\int_C \omega$  depends only on the Borel–Moore homology class  $[C]$  and the compactly supported cohomology class  $[\omega]$ :

$$H_p^{\text{BM}}(M, \mathbb{Z}) = \frac{\ker \partial: C_p^{\text{BM}}(M, \mathbb{Z}) \rightarrow C_{p-1}^{\text{BM}}(M, \mathbb{Z})}{\text{image } \partial: C_{p+1}^{\text{BM}}(M, \mathbb{Z}) \rightarrow C_p^{\text{BM}}(M, \mathbb{Z})}.$$

We see that integration pairs

$$\int: H_p(M, \mathbb{Z}) \times H_{\text{dR}}^p(M) \rightarrow \mathbb{R}$$

and

$$\int: H_p^{\text{BM}}(M, \mathbb{Z}) \times H_c^p(M) \rightarrow \mathbb{R}.$$

Borel–Moore homology maps under  $C^1$  proper maps, and is invariant under  $C^1$  proper homotopy.



## Chapter 39

### Invariant differential forms

#### Invariant inner products

**Lemma 39.1.** *For any representation  $G \rightarrow GL(V)$  of any compact Lie group  $G$ ,  $G$  preserves a positive definite inner product on  $V$ . If  $G$  acts as complex linear transformations on  $V$ , then  $G$  preserves a Hermitian inner product on  $V$ . If  $G$  acts as quaternionic linear transformations on  $V$ , then  $G$  preserves a quaternionic inner product on  $V$ .*

*Proof.* Take symmetric bilinear form, say  $q$ , on  $V$ . For each  $g \in G$ , let  $q^g(v, w) := q(g^{-1}v, g^{-1}w)$ . Pick a nonzero left invariant volume form on  $G$ , say  $\omega$ . Rescale  $\omega$  to ensure that  $\int_G \omega = 1$ , in the orientation induced by  $\omega$ . Define

$$q^G := \int_G q^g \omega_g.$$

If we set  $w = v \neq 0$ ,  $q(v, v) > 0$ , so  $q^g(v, w) = q(g^{-1}v, g^{-1}v) > 0$ , say with some minimum  $m(v)$  at some point  $g \in G$ . Hence

$$q^G(v, v) \geq m(v) \int_G \omega = m(v) > 0.$$

For any  $h \in G$ ,

$$\begin{aligned} (q^G)^h(v, w) &= \int_G q(g^{-1}h^{-1}v, g^{-1}h^{-1}w) \omega_g, \\ &= \int_G q(g^{-1}v, g^{-1}w) \omega_{h^{-1}g}, \\ &= \int_G q(g^{-1}v, g^{-1}w) \omega_g, \\ &= \int_G q^g \omega_g, \\ &= q^G(v, w). \end{aligned}$$

It follows immediately that  $q$  is  $G$ -invariant just when  $q^G = q$ . □

A (real, complex, quaternionic) representation  $G \rightarrow GL(V)$  is *irreducible* if  $V$  contains no (real, complex, quaternionic) linear subspace invariant under  $G$  except  $\{0\}$  and  $V$ . A (real, complex, quaternionic) representation  $G \rightarrow GL(V)$  is *completely reducible* if  $V$  is a direct sum of irreducible representations.

**Corollary 39.2.** *Every representation of any compact Lie group is completely reducible.*

*Proof.* Take a  $G$ -invariant inner product. Then each  $G$ -invariant subspace has a  $G$ -invariant perpendicular subspace in the inner product. □

### Invariant forms

A *constant coefficient* differential form on  $V$ , also called an *exterior form*, is one which is translation invariant. The set of exterior  $p$ -forms on  $V$  is denoted  $\Lambda^p V^*$ . The set of all exterior forms on  $V$  is denoted

$$\Lambda^* V^* = \Lambda^0 V^* \oplus \Lambda^1 V^* \oplus \cdots \oplus \Lambda^n V^*,$$

if  $V$  has real dimension  $n$ .

**Lemma 39.3.** *Take a compact Lie group  $G$  acting on a manifold  $M$ . Denote the action as  $a: (g, m) \in G \times M \rightarrow gm \in M$  and the projection to  $M$  as  $p: (g, m) \in G \times M \rightarrow m \in M$ . Pick a right invariant volume form  $\omega$  on  $G$ , with  $\int_G \omega = 1$ . Define, for any continuous differential form  $\xi$  on  $M$ , a continuous differential form  $\xi^G$  on  $M$  by*

$$\xi^G := p_*((a^* \xi) \wedge \omega).$$

The value  $\xi_m^G$  of  $\xi^G$  at each point  $m \in M$  is also an integral in the vector space  $\Lambda^* T_m^* M$ :

$$\xi_m^G = \int_{g \in G} (r_g^* \xi)_m \omega.$$

Moreover

- a.  $\xi^G$  is a  $G$ -invariant differential form on  $M$ , hence smooth on each  $G$ -orbit, and
- b.  $\xi$  is  $G$ -invariant if and only if  $\xi = \xi^G$  and
- c. if  $\xi$  is  $C^1$  then

$$d(\xi^G) = (d\xi)^G.$$

*Proof.* By compactness of  $G$ , both  $p$  and  $a$  are proper maps. By Fubini's theorem, for any continuous differential form  $\xi$  on  $M$ , say of degree  $|\xi|$ ,

$$\begin{aligned} (d\xi)^G &= p_*((a^* d\xi) \wedge \omega), \\ &= p_*((da^* \xi) \wedge \omega), \\ &= p_*\left(d((a^* \xi) \wedge \omega) + (-1)^{|\xi|}(a^* \xi) \wedge d\omega\right), \\ &= dp_*((a^* \xi) \wedge \omega), \\ &= d(\xi^G). \end{aligned}$$

By Fubini's theorem, for any continuous differential forms  $\xi, \eta$  on  $M$ ,

$$\int_M \eta \wedge p_*((a^* \xi) \wedge \omega) = \int_{G \times M} (p^* \eta) \wedge (a^* \xi) \wedge \omega.$$

Let  $G$  act on  $G \times M$  by  $h(g, m) := (gh^{-1}, hm)$ , so that  $a$  is invariant under this action:

$$a(h(g, m)) = a(gh^{-1}, hm) = gh^{-1}hm = gm = a(g, m),$$

i.e.  $ah = a$ , while  $p(h(g, m)) = p(gh^{-1}, hm) = hm = hp(g, m)$ , i.e.  $ph = hp$ . Since  $\omega$  is right invariant on  $G$ , the pullback of  $\omega$  to  $G \times M$  (also denoted  $\omega$ ) is  $G$ -invariant under this action:  $h^* \omega = \omega$  on  $G \times M$ .

**39.1** Prove that  $\xi^G$  is  $G$ -invariant.

In particular, if  $\xi = \xi^G$  then  $\xi$  is  $G$ -invariant.

Since  $a: G \times M \rightarrow M$ ,  $a^*\xi$  is a wedge product of 1-forms on  $G$  and on  $M$ . Wedge:  $a^*\xi \wedge \omega$  already has a basis of the 1-forms on  $G$  wedged into  $\omega$ , so depends only on  $a^*\xi$  modulo the 1-forms on  $G$ , i.e. at each point  $(g, m) \in G \times M$ ,  $a^*\xi \wedge \omega = r_g^*\xi \wedge \omega$ . Hence

$$\xi_m^G = p_*((a^*\xi) \wedge \omega)_m = \int_{g \in G} (g^*\xi)_m \omega.$$

In particular, if  $\xi$  is  $G$ -invariant, then  $\xi^G = \xi$ .  $\square$

**Lemma 39.4.** *Take a compact Lie group  $G$  acting on a manifold  $M$ . The inclusion  $\Omega^{*G} \rightarrow \Omega^*$  of smooth invariant differential forms into smooth differential forms commutes with exterior derivative, mapping closed to closed, exact to exact, and is an inclusion on cohomology groups. If  $G$  is connected, then the map on cohomology groups is an isomorphism.*

*Proof.* If  $\xi = \xi^G$  is invariant and also exact, say  $\xi = d\eta$ , then  $d(\eta^G) = (d\eta)^G = \xi^G = \xi$ , so  $\xi$  is invariantly exact, i.e.  $\xi = d\eta$  for some  $\eta = \eta^G$ . The inclusion  $\Omega^{*G} \rightarrow \Omega^*$  of smooth invariant differential forms into smooth differential forms commutes with exterior derivative, mapping closed to closed, exact to exact. It therefore descends to a map on cohomology. The kernel of this map is trivial: invariant forms which are closed are invariantly closed. So the map descends to an inclusion on cohomology.

In our study of the Poincaré lemma (lemma 30.3 on page 243), we saw that under the flow of any complete vector field, a differential form  $\xi$  does not change cohomology class, i.e. if  $g = e^A$  for some  $A \in \mathfrak{g}$ , then  $r_g^*\xi = \xi + d\eta$  for some differential form  $\eta$ . Composing several flows, say  $g = e^{A_1}e^{A_2}\dots e^{A_k}$ , by induction,  $r_g^*\xi = \xi + d\eta_g$  for some differential form  $\eta_g$ . This  $\eta_g$  depends smoothly on  $g$ . If  $G$  is connected, then every element of  $G$  is such a composition, so the various  $\eta_g$  have some average  $\eta$ , smooth as we average over  $g \in G$ , so  $\xi^G = \xi + d\eta$ . Hence  $\Omega^{*G} \rightarrow \Omega^*$  is surjective on cohomology.  $\square$

**39.2** Find an example of a disconnected Lie group  $G$  acting on a manifold  $M$  so that this inclusion in cohomology is not an isomorphism.

## Invariant forms on homogeneous spaces

We can parameterize the invariant differential forms on a homogeneous space using not much more than linear algebra.

**Proposition 39.5.** *Suppose that  $X$  is an  $n$ -dimensional homogeneous space of a Lie group  $G$ . Take a point  $x_0 \in X$  and let  $H := G^{x_0}$ . Recall that  $T_{x_0}X \cong \mathfrak{g}/\mathfrak{h}$ . Then, under adjoint action of  $G$ ,  $H$  acts on  $\mathfrak{g}/\mathfrak{h}$  preserving an antisymmetric multilinear form  $\omega_0 \in \Lambda^*(\mathfrak{g}/\mathfrak{h})$  just when  $X$  admits a  $G$ -invariant differential form  $\omega \in \Omega^*(X)$  with  $\omega_0$  the value of  $\omega$  on  $T_{x_0}X$ .*

*Proof.* We use the action of  $G$  to translate vectors on  $X$  from any point to the point  $x_0$ ; this translation is uniquely determined up to elements of  $G$  fixing  $x_0$ .  $\square$

On the 2-sphere, the rotations preserve the area form. If they preserve some differential form, say a 1-form, rotate by a right angle and check that at least one coefficient has changed sign. Similarly, since rotations act transitively, the rotation invariant functions are constants. So

| $p$ | $H_{\text{dR}}^p(S^2)$ |
|-----|------------------------|
| 0   | $\mathbb{R}$           |
| 0   | 0                      |
| 0   | $\mathbb{R}$           |

Rotate the sphere  $S^n$ . The differential forms invariant under rotation become constant coefficients form on the tangent plane at the north pole. Expand in wedge products of  $dx^i$ . Rotate any two of the coordinates, leaving the rest alone, to find that every term has either both of those coordinates, or neither. By induction, rotation invariant forms on the sphere are either constants or constant multiples of the volume form:

$$H_{\text{dR}}^p(S^n) = \begin{cases} \mathbb{R}, & p = 0, \\ 0, & 1 \leq p \leq n-1, \\ \mathbb{R}, & p = n. \end{cases}$$

**Corollary 39.6.** *In particular,  $H$  acts trivially on  $\Lambda^n(\mathfrak{g}/\mathfrak{h})^*$  if and only if  $X$  admits a  $G$ -invariant nonzero volume form, and so  $X$  is orientable. For instance, if  $H$  has a finite and odd number of components, then  $G$  preserves a volume form on  $X$ .*

The group  $G := \text{SO}(n+1)$  of rotations of Euclidean space  $\mathbb{R}^{n+1}$  acts on the lines through the origin, so on  $X = \mathbb{RP}^n$ . The stabiliser  $H \subset G$  is the subgroup of rotations fixing a line, i.e. the matrices of the form

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & h \end{pmatrix}$$

with  $\det h = \varepsilon = \pm 1$ . The quotient  $\mathfrak{g}/\mathfrak{h}$  consists of the antisymmetric matrices quotiented out by those vanishing in top row and leftmost column, so antisymmetric matrices

$$\begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}.$$

The matrices in  $H$  act on the quotient  $\mathfrak{g}/\mathfrak{h}$  by adjoint action

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 0 & -x^t \\ x & 0 \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & h^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -y^t \\ y & 0 \end{pmatrix}$$

where  $y = \varepsilon hx$ . The determinant of the linear map  $\varepsilon h$  is  $\varepsilon^{n+1}$ . Since  $\varepsilon = \pm 1$ , there is a rotation invariant volume form on real projective space  $\mathbb{RP}^n$  just when  $n+1$  is even, i.e.  $n$  is odd. When  $n$  is even, there is a two-to-one covering space,  $S^n \rightarrow \mathbb{RP}^n$ , which has a rotation invariant volume form. Whether  $n$

is even or odd, this  $n$ -form restricts to  $\mathfrak{g}/\mathfrak{h}$  to become  $dx_1 \wedge \cdots \wedge dx_n$ . Any rotation invariant differential form pulls back to the sphere to be rotation invariant, so

$$H_{\text{dR}}^p(\mathbb{RP}^n) = \begin{cases} \mathbb{R}, & p = 0, \\ 0, & 1 \leq p \leq n-1, \\ \mathbb{R}, & p = n \text{ and } n \text{ odd}, \\ 0, & p = n \text{ and } n \text{ even}. \end{cases}$$

**39.3\*** Which real, complex or quaternionic Grassmannians  $\text{Gr}_p V$  admit an invariant volume form under  $\text{SO}(p+q)$ ,  $\text{SU}(p+q)$ ,  $\text{Sp}(p+q)$ ? What is the top dimension cohomology group of that Grassmannian?

### Invariant forms on Euclidean space

Suppose that  $V$  is an irreducible complex or quaternionic representation of a compact Lie group  $G$ . We want to find the  $G$ -invariant exterior forms. We will think of a vector space  $V$  as a different space if a different group  $G$  acts on it. Since  $G$  is compact,  $G$  preserves some Hermitian inner product, which we write as  $\langle z, w \rangle_k$ ,  $k = \mathbb{C}$  or  $\mathbb{H}$ . The standard inner product on  $k^n$  is

$$\langle z, w \rangle_k = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n.$$

Any inner product becomes the standard one in some linear coordinates. This expression is valued in  $k$ ; its real part is the usual real inner product. Our source of invariant forms: the nonreal part of the inner product.

### Complex Euclidean space

Break into real and imaginary parts: if  $k = \mathbb{C}$ ,

$$\langle z, w \rangle_{\mathbb{C}} = \langle z, w \rangle_{\mathbb{R}} + i\omega(z, w),$$

where if  $z = x + iy$  and  $w = u + iv$  then the *Kähler form* is

$$\omega(z, w) = \sum_{\mu=1}^n (x_{\mu} v_{\mu} - y_{\mu} u_{\mu}).$$

We can write  $\omega$  as a differential form

$$\omega = \sum_{\mu} dx_{\mu} \wedge dy_{\mu} = \frac{i}{2} \sum dz_{\mu} \wedge dz_{\bar{\mu}},$$

where  $dz^{\mu} = dx^{\mu} + i dy^{\mu}$  is a complex valued 1-form and  $dz_{\bar{\mu}}$  means  $\overline{dz_{\mu}}$ . We can see that  $\omega \neq 0$  because:

**39.4** Prove that

$$\omega(z, iw) = \langle z, w \rangle_{\mathbb{R}}.$$

In particular,  $\omega(iz, z) = |z|^2 \neq 0$  for any  $z \neq 0$ .

**39.5** Prove that  $\omega^n = n!dV$  where

$$dV = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

is the standard volume form on Euclidean space.

We see that  $1 = \omega^0, \omega, \omega^2, \dots, \omega^n$  are nonzero differential forms on  $\mathbb{C}^n$  invariant under unitary linear transformations.

**39.6** Suppose that  $A$  is a commutative algebra over the field of real numbers (or complex numbers) generated by finitely many symbols  $t_i, i = 1, 2, \dots, n$ , each satisfying  $t_i^2 = 0$ . Permutations of these symbols act as algebra automorphisms. Prove that the elements of  $A$  which are invariant under all permutations of those symbols form the subalgebra of  $A$  generated by  $t_1 + \cdots + t_n$ .

**Theorem 39.7.** *The (real or complex) exterior forms on  $\mathbb{C}^n$  invariant under unitary transformations are precisely the algebra generated by the Kähler form, i.e. none of odd degree, and those of even degree, say degree  $2p$ , are  $c\omega^p$  where  $c$  is a complex number.*

*Proof.* Consider the unitary transformations which are given by diagonal matrices

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

This is unitary just when

$$\lambda_\mu = e^{2\pi i \theta_\mu}$$

for some real  $\theta_\mu, \mu = 1, 2, \dots, n$ .

Write each complex exterior form (i.e. constant coefficient complex-valued differential form) as

$$\xi = \sum_{AB} c_{A\bar{B}} dz^{A\bar{B}}$$

where we write  $dz^{1,2}$  to mean  $dz^1 \wedge dz^2$ , and so on, and  $dz^{1,\bar{2}}$  to mean  $dz^1 \wedge \overline{dz^2}$ , and so on.

Under the action of those diagonal matrices,  $\xi$  transforms to

$$\lambda_{a_1} \lambda_{a_2} \cdots \lambda_{a_p} \bar{\lambda}_{b_1} \cdots \bar{\lambda}_{b_q},$$

if

$$A = (a_1 \dots a_p)$$

and

$$B = (b_1 \dots b_q).$$

We can assume that  $a_1 < a_2 < \cdots < a_p$  and similarly for  $B$ . To make this work for any choice of  $\lambda_\mu = e^{2\pi i \theta_\mu}$ , we need these to cancel out, i.e.

$$\sum \theta_{a_\mu} - \sum \theta_{b_\nu}$$

an integer multiple of  $2\pi$  for any choice of  $\theta_1, \dots, \theta_n$ . Clearly this occurs only when all of these cancel, i.e. just when  $A = B$ . So every invariant exterior form is

$$\xi = \sum c_A dz^{A\bar{A}}.$$

So our algebra of invariant exterior forms lie in the algebra generated by the “symbols”  $t_\mu := dz^{\mu\bar{\mu}}$ ; apply problem 39.6.  $\square$

### Quaternionic Euclidean space

If  $k = \mathbb{H}$ ,

$$\langle x, y \rangle_{\mathbb{H}} = \langle x, y \rangle_{\mathbb{R}} + \omega_1(x, y)i + \omega_2(x, y)j + \omega_3(x, y)k,$$

where  $\omega_1, \omega_2, \omega_3$  are the *Kähler forms* of the quaternionic inner product space.

**39.7** Prove that

$$\langle x, y \rangle_{\mathbb{R}} = \omega_1(x, yi) = \omega_2(x, yj) = \omega_3(x, yk).$$

**39.8** Prove that

$$\begin{aligned}\omega_2(x, yi) &= \omega_3(x, y), \\ \omega_3(x, yj) &= \omega_1(x, y), \\ \omega_1(x, yk) &= \omega_2(x, y),\end{aligned}$$

for any  $x, y \in V$ .

To any finite dimensional quaternionic inner product space  $V$ , we have associated the underlying finite dimensional real inner product space  $V$  together with three 2-forms  $\omega_1, \omega_2, \omega_3$ .

**Lemma 39.8.** *The group of real linear transformations of a quaternionic inner product space  $V$  preserving the real inner product and  $\omega_1, \omega_2, \omega_3$  is precisely the group of quaternionic linear isometries  $Sp(V)$ .*

*Proof.* From the above, the real inner product and the complex structure given by right multiplication by  $i$  have Kähler form  $-\omega_1$ . So the group preserving the real inner product and  $\omega_1$  is  $U(2n)$ . Since

$$\begin{aligned}\omega_1(x, yj) &= -\omega_3(x, y), \\ \omega_1(x, yk) &= \omega_2(x, y),\end{aligned}$$

the subgroup of  $U(2n)$  preserving in addition also right multiplication by  $j$  and  $k$  preserves  $\omega_2, \omega_3$ . Conversely, the subgroup of  $U(2n)$  preserving in addition  $\omega_2, \omega_3$  preserves also

$$\omega_2(x, y) = -\langle x, yj \rangle_{\mathbb{R}}$$

for any vectors  $x, y$ , so preserves the linear transformation of right multiplication by  $j$ , and similarly for  $k$ .  $\square$

Writing each quaternionic vector in  $\mathbb{H}^n$  as  $z + wj$  where  $z, w \in \mathbb{C}^n$ , we can easily unwind the definition to explicitly express

$$\begin{aligned}\omega_1 &= \frac{i}{2}dz^{\mu\bar{\mu}} + \frac{i}{2}dw^{\mu\bar{\mu}}, \\ \omega_2 &= \frac{1}{2}dz^{\mu} \wedge dw^{\bar{\mu}} + \frac{1}{2}dz^{\bar{\mu}} \wedge dw^{\mu}, \\ \omega_3 &= \frac{i}{2}dz^{\mu} \wedge dw^{\bar{\mu}} - \frac{i}{2}dz^{\bar{\mu}} \wedge dw^{\mu}.\end{aligned}$$

### Quaternionic Kähler Euclidean space

We will run into a weaker notion than a quaternionic inner product space. Note that scaling a quaternionic vector by a unit length quaternion does *not* preserve the 2-forms  $\omega_1, \omega_2, \omega_3$  nor the quaternionic inner product. The *quaternionic Kähler group*  $\mathrm{Sp}(V)\mathrm{Sp}(1)$  of a quaternionic inner product space  $V$  is the group of all linear transformations of  $V$  of the form

$$x \mapsto Axa,$$

where  $a \in \mathbb{H}$  is a unit quaternion and  $A \in \mathrm{Sp}(V)$  is quaternionic inner product space isomorphism. If  $V = \mathbb{H}^n$  this group is denoted  $\mathrm{Sp}(n)\mathrm{Sp}(1)$ . Note that such a linear transformation is the identity just when  $(A, a) = \pm(I, 1)$ , so  $\mathrm{Sp}(V)\mathrm{Sp}(1) = (\mathrm{Sp}(V) \times \mathrm{Sp}(1)) / \pm(I, 1)$ .

**39.9** Prove that the *quaternionic-Kähler 4-form*

$$\Omega := \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$$

is preserved by  $\mathrm{Sp}(V)\mathrm{Sp}(1)$ .

**39.10** Prove that on  $V = \mathbb{H}^n$ ,  $\Omega^n = C dV$  where  $C > 0$  is a positive constant and  $dV$  is the usual Euclidean volume form; what is  $C$ ?

**Theorem 39.9.** *For any quaternionic vector space  $V$ , the group  $\mathrm{Sp}(V)\mathrm{Sp}(1)$  is precisely the group of real linear transformations preserving the quaternionic Kähler form. The differential forms  $1, \omega, \omega^2, \dots, \omega^n$  are invariant differential forms under quaternionic Kähler linear transformations of  $V$ . Conversely, every real or complex exterior form on  $V$  of a given degree  $p$  invariant under  $\mathrm{Sp}(V)\mathrm{Sp}(1)$  is a constant multiple of  $\omega^q$  if  $p = 4q$  for some  $q$ , i.e. if  $p$  is a multiple of 4, and is zero otherwise.*

*Proof.* To be invariant, Any 1-form is a unique linear combination of  $dz^\mu, dz^{\bar{\mu}}, dw^\mu, dw^{\bar{\mu}}$ . Each of these 1-forms transforms under  $\mathrm{Sp}(1)$ -action without any part being fixed. By uniqueness of the linear combination, there are no invariant 1-forms. The same argument works for any odd degree form: any odd number of wedges of  $dz^\mu, dz^{\bar{\mu}}, dw^\mu, dw^{\bar{\mu}}$  cannot be invariant. By the same argument, to have an invariant form, it must be a linear combination of invariant wedge products of these. Consider the subgroup of  $\mathrm{Sp}(V)$  acting by scaling each quaternion  $z^\mu + w^\mu j$  on the left by a complex number  $\lambda^\mu = e^{i\theta^\mu}$ . Then by the same arguments as for the unitary group, the invariant forms for such scalings are the algebra generated by any wedge with one  $\mu$  and one  $\bar{\mu}$ :

$$\begin{aligned} dz^{\mu\bar{\mu}}, \\ dz^\mu \wedge dw^{\bar{\mu}}, \\ dw^\mu \wedge dz^{\bar{\mu}}, \\ dw^{\mu\bar{\mu}}. \end{aligned}$$

Next we consider scaling on the right by some  $e^{i\theta} \in S^1 \subset \mathrm{Sp}(1)$ . The forms above then scale by

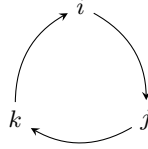
$$\begin{aligned} dz^{\mu\bar{\mu}}, \\ e^{2i\theta} dz^\mu \wedge dw^{\bar{\mu}}, \\ e^{-2i\theta} dw^\mu \wedge dz^{\bar{\mu}}, \\ dw^{\mu\bar{\mu}}. \end{aligned}$$



So the algebra of invariant forms under all of those scalings is generated by

$$dz^{\mu\bar{\mu}}, dw^{\mu\bar{\mu}}.$$

Again applying the automorphism



of the quaternions,

$$\begin{aligned} z + wj &= x + yi + (u + vi)j \mapsto x + vi + (y + ui)j, \\ &= \frac{z + \bar{z}}{2} + \frac{w - \bar{w}}{2i}i + \left( \frac{z - \bar{z}}{2i} + \frac{w + \bar{w}}{2}i \right)j, \end{aligned}$$

so

$$\begin{aligned} z &\mapsto \frac{1}{2}(z + \bar{z} + w - \bar{w}), \\ w &\mapsto \frac{i}{2}(w + \bar{w} - z + \bar{z}). \end{aligned}$$

So

$$\begin{aligned} dz^{\mu\bar{\mu}} &\mapsto \frac{1}{2}(dz^{\mu} + dz^{\bar{\mu}}) \wedge (dw^{\bar{\mu}} - dw^{\mu}), \\ dw^{\mu\bar{\mu}} &\mapsto \frac{1}{2}(dw^{\mu} + dw^{\bar{\mu}}) \wedge (dz^{\mu} - dz^{\bar{\mu}}). \end{aligned}$$

These then have no invariant components under the complex rescalings we have already looked at. So these are not in the algebra of invariant forms, but the invariant forms are in the algebra generated by these. Therefore every invariant form lies in the algebra generated by products of such, i.e.

$$dz^{\mu\bar{\mu}} \wedge dw^{\nu\bar{\nu}}.$$

But again using that same automorphism, if  $\mu \neq \nu$  then this is also not invariant. So the algebra of invariant forms lies in the algebra generated by

$$dz^{\mu\bar{\mu}} \wedge dw^{\mu\bar{\mu}}.$$

Apply problem 39.6 on page 312. □

## Cohomology of projective space

On  $\mathbb{CP}^n = G/H$  with  $G = \mathrm{U}(n+1)$  and  $H = \mathrm{U}(1) \times \mathrm{U}(n)$ , the group  $H$  acts on  $\mathfrak{g}/\mathfrak{h} = \mathbb{C}^n$  as precisely the unitary transformations. They preserve precisely the constant coefficient multiples of the Kähler form by theorem 39.7 on page 312. The  $G$ -invariant forms on  $\mathbb{CP}^n$  are precisely those given by moving those constant coefficient multiples around using  $G$ , by proposition 39.5 on

page 309. So for  $p = 0, 1, 2, \dots, n$ , the invariant  $2p$ -forms on  $\mathbb{CP}^n$  are precisely the constant coefficient multiples of  $\omega^p$ . The invariant forms of odd degree vanish. Therefore the invariant forms of even degree are closed: the Kähler form on  $\mathbb{CP}^n$  is closed. Hence

$$\begin{aligned} H_{\text{dR}}^{2p}(\mathbb{CP}^n) &= \mathbb{R}\omega^p, \\ H_{\text{dR}}^{2p+1}(\mathbb{CP}^n) &= 0. \end{aligned}$$

Consider the complex projective line  $\mathbb{CP}^1 \subset \mathbb{CP}^n$  given in the obvious way, quotienting the complex linear embedded  $\mathbb{C}^2 \subset \mathbb{C}^n$  by complex rescaling. The Kähler form restricts to the tangent plane of  $\mathbb{CP}^1$  to be invariant under the unitary transformations of  $\mathbb{CP}^n$  preserving  $\mathbb{CP}^1$ , which act transitively on  $\mathbb{CP}^1$ . So the Kähler form on  $\mathbb{CP}^n$  restricts to a positive multiple of the Kähler form, i.e. the area form, on  $\mathbb{CP}^1$ . We choose to scale the Kähler form  $\omega$  on  $\mathbb{CP}^n$  so that  $\int_{\mathbb{CP}^1} \omega = 1$ . By unitary isometry that integral is the same over all linearly embedded complex projective lines in  $\mathbb{CP}^n$ . With this choice of scaling, the Kähler form is uniquely determined, and called the *Fubini–Study form*.

On  $\mathbb{HP}^n = G/H$  with  $G = \text{Sp}(n+1)$  and  $H = \text{Sp}(n) \times \text{Sp}(1)$ , the group  $H$  acts on  $\mathfrak{g}/\mathfrak{h} = \mathbb{H}^n$  as precisely the quaternionic Kähler transformations. They preserve precisely the constant coefficient multiples of the quaternionic Kähler form by theorem 39.9 on page 314. The  $G$ -invariant forms on  $\mathbb{HP}^n$  are precisely those given by moving those constant coefficient multiples around using  $G$ , by proposition 39.5 on page 309. So for  $p = 0, 1, 2, \dots, n$ , the invariant  $4p$ -forms on  $\mathbb{HP}^n$  are precisely the constant coefficient multiples of  $\Omega^p$ . The invariant forms of all other degrees vanish. Therefore the invariant forms of degree  $0, 4, 8, \dots, 4n$  are closed: the quaternionic Kähler form on  $\mathbb{HP}^n$  is closed. Hence

$$\begin{aligned} H_{\text{dR}}^{4p}(\mathbb{HP}^n) &= \mathbb{R}\Omega^p, \\ H_{\text{dR}}^{4p+1}(\mathbb{HP}^n) &= 0, \\ H_{\text{dR}}^{4p+2}(\mathbb{HP}^n) &= 0, \\ H_{\text{dR}}^{4p+3}(\mathbb{HP}^n) &= 0. \end{aligned}$$

Similarly we can scale our choice of quaternionic Kähler form by a positive constant to arrange that  $\int_{\mathbb{HP}^1} \Omega = 1$ . With this choice of scaling, the quaternionic Kähler form is uniquely determined, and called the *Fubini–Study form*.

## Lie derivatives

If we spin a sphere around, we can imagine picking up any function, vector field, or differential form, on the sphere and carrying it with as while we spin the sphere.

Suppose that  $X$  is a continuously differentiable vector field on a manifold  $M$ , with flow  $e^{tX}$ . For any continuously differentiable vector field  $Y$ , define

$$\mathcal{L}_X Y := \left. \frac{d}{dt} \right|_{t=0} e^{-tX} Y,$$

i.e.  $\mathcal{L}_X Y$  measures how  $Y$  changes as seen by someone who follows the flow of  $X$ . (The curious minus sign is because we skip ahead to where we will be  $t$  seconds from now, grab  $Y$  at that time, and pull  $Y$  back to our current position at the current time.) Lemma 7.4 on page 81 proves that  $[XY] = \mathcal{L}_X Y$ .

**39.11** Prove that

$$\mathcal{L}_X Y = \lim_{h \rightarrow 0} \frac{Y - e_*^{hX} Y}{h}.$$

The *Lie derivative* of a continuously differentiable differential form  $\omega$  along the flow of  $X$  is

$$\mathcal{L}_X \omega := \lim_{h \rightarrow 0} \frac{e^{hX*} \omega - \omega}{h}.$$

Intuitively: the “rate” at which  $\omega$  changes along the flow of  $X$ . Before we can calculate out what the Lie derivative looks like, we need some calculation tricks.

**39.12** Prove that, for any  $C^1$  vector fields  $X, Y$  and  $C^1$  differential forms  $\alpha, \beta$ ,  $\mathcal{L}_X(\alpha + \beta) = \mathcal{L}_X \alpha + \mathcal{L}_X \beta$ ,  $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X \beta)$ , and  $\mathcal{L}_X(Y \lrcorner \alpha) = (\mathcal{L}_X Y) \lrcorner \alpha + Y \lrcorner \mathcal{L}_X \alpha$ .

The Lie derivative  $\mathcal{L}_X$  is related closely to the exterior derivative  $d$ :

**39.13** If  $f$  is a continuously differentiable function and  $X$  a continuously differentiable vector field on a manifold  $M$ , prove that  $\mathcal{L}_X f = df(X)$ .

**Lemma 39.10.**  $\mathcal{L}_X \omega = X \lrcorner d\omega + d(X \lrcorner \omega)$ , for any continuously differentiable differential form  $\omega$  and continuously differentiable vector field  $X$ . We can write this as  $\mathcal{L}_X = X \lrcorner d + dX \lrcorner$ .

*Proof.* Start with the special case that  $\omega = f$  is a function. Then the previous lemma says that  $\mathcal{L}_X f = X \lrcorner df$ , but  $X \lrcorner f = 0$  just by definition.

Now suppose that we have proven our result for some differential forms  $\omega = \alpha$  and for  $\omega = \beta$ . From problem 39.12, our result follows for  $\omega = \alpha + \beta$  and for  $\omega = \alpha \wedge \beta$ . From the equation  $d^2 = 0$  (see theorem 24.2), it is easy to check that if our theorem holds for a differential form  $\omega$ , then it holds for  $d\omega$ .

We only need to prove our result locally, i.e. our manifold can be just an open set in  $\mathbb{R}^n$ . So our theorem holds for  $\omega = x_1, \omega = x_2, \dots, \omega = x_n$ , since these are functions. It then holds for  $\omega = dx_1, \omega = dx_2, \dots, \omega = dx_n$ , since these are  $d$  applied to forms for which the theorem is checked. It then holds for all differential forms we can wedge together from these, and functions, i.e. for all differential forms  $\omega$ .  $\square$

**39.14** If  $\omega$  is a 1-form and  $X, Y$  are vector fields prove

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([XY]).$$

**39.15\*** Prove that for any  $p$ -form  $\omega$  and smooth vector fields  $X_0, X_1, \dots, X_p$  on a manifold  $M$ ,

$$\begin{aligned} d\omega(X_0, X_1, \dots, X_p) &= \sum_i (-1)^i X_i(\omega(X_0, \dots, \hat{X}_i, \dots, X_p)) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p). \end{aligned}$$

### Cohomology of homogeneous spaces

Our next problem is to understand how to compute the differential of an invariant form, on a homogeneous space, using not much more than linear algebra. If  $\mathfrak{h} \subset \mathfrak{g}$  is a subalgebra of a Lie algebra, the associated *differential* is

$$\delta: \Lambda^*(\mathfrak{g}/\mathfrak{h}) \rightarrow \Lambda^{*+1}(\mathfrak{g}/\mathfrak{h}),$$

defined, for  $\xi \in \Lambda^p(\mathfrak{g}/\mathfrak{h})$ , by

$$\delta\xi(A_0, \dots, A_p) := \sum_{i < j} (-1)^{i+j} \xi([A_i A_j], A_0, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_p).$$

This differential takes  $H$ -invariant exterior forms to one another, giving a cohomology on  $\Lambda^*(\mathfrak{g}/\mathfrak{h})^H$ , the *Lie algebra cohomology*.

**Lemma 39.11.** *Take a compact connected Lie group  $G$ , a connected homogeneous space  $X$  of  $G$ , and a point  $x_0 \in X$ . Let  $H := G^{x_0}$ . Take the Lie algebras  $\mathfrak{h} \subset \mathfrak{g}$  of  $H \subset G$ . Then the de Rham cohomology of  $X$  is the Lie algebra cohomology of  $\mathfrak{h} \subset \mathfrak{g}$ . In particular, the de Rham cohomology of  $X$  is finite dimensional:*

$$\dim H_{dR}^p(X) \leq \binom{\dim X}{p}, \quad p = 0, 1, 2, \dots, \dim X.$$

*Proof.* By lemma 39.4 on page 309, the cohomology of  $X$  is the cohomology of the  $G$ -invariant forms on  $X$ . We compute the exterior derivative of any such form as in problem 39.15 on the preceding page, using the vector fields  $\tilde{A}$  of the Lie algebra action of  $\mathfrak{g}$ . Since the invariant forms are invariant under the flows of these vector fields, the Lie derivative terms vanish, and  $d\xi$  is the  $G$ -invariant form which agrees at  $x_0$  with  $\delta\xi$ .  $\square$

### Biinvariant forms

A differential form on a Lie group  $G$  is *biinvariant* if both left and right invariant. The biinvariant forms are the forms on  $G$  invariant under the action of  $G \times G$ :

$$(g_1, g_2) \in G \times G, g \in G \mapsto g_1 g g_2^{-1} \in G.$$

The stabilizer of  $1 \in G$  is the subgroup of elements  $(g, g) \in G \times G$  for  $g \in G$ . Hence the biinvariant forms are identified with  $\Lambda^* \mathfrak{g}^G$ , i.e. alternating multilinear forms on the Lie algebra, invariant under the adjoint representation  $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ .

**Lemma 39.12.** *Every compact connected Lie group  $G$  bears a unique left invariant volume form  $\omega$  with  $\int_G \omega = 1$ , and a unique right invariant one, and these are equal, hence biinvariant.*

Note that, by biinvariance, if  $\omega \neq 0$  at some point, then  $\omega \neq 0$  at every point of  $G$ . Since  $G$  is connected, there is a unique orientation of  $G$  in which  $\omega$  has positive integrals over precompact open sets, and this is the orientation intended here.

*Proof.* By problem 12.7 on page 113, the adjoint representation  $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$  preserves a volume form on  $\mathfrak{g}$ .  $\square$

On any Lie group  $G$ , consider the map  $\psi(g) = g^{-1}$ . So  $\psi(e^A) = e^{-A}$ , and hence the derivative of  $\psi$ , also denoted  $\psi$ , is  $\psi: A \in \mathfrak{g} \rightarrow -A \in \mathfrak{g}$ . On any  $p$ -form  $\eta$ , at  $1 \in G$ ,  $\psi^*\eta = (-1)^p\eta$ . If  $\eta$  is left invariant,  $\psi^*\eta$  is right invariant, and vice versa. In particular, if  $\eta$  is biinvariant, then  $\psi^*\eta$  is as well, and so  $\psi^*\eta = (-1)^p\eta$  at every point of  $G$ .

**39.16** Prove that biinvariant differential forms are closed.

**Lemma 39.13.** *Biinvariant forms are closed. Biinvariant forms which are exact vanish, i.e. if  $\eta$  is biinvariant and  $\eta = d\xi$  for some smooth differential form  $\xi$ , then  $\eta = 0$ .*

*Proof.* We take  $X' := G$  and  $G' := G \times G$  acting on  $X$  by  $(g_1, g_2)x = g_1xg_2^{-1}$ . The stabilizer of  $1 \in X$  is the subgroup  $H' \subset G'$  of pairs  $(g, g)$  for  $g \in G$ . Compute the cohomology of  $X'$  as the Lie algebra cohomology of  $\mathfrak{h}' \subset \mathfrak{g}'$ , i.e. as the elements of  $\Lambda^*\mathfrak{g}$  invariant under the adjoint representation of  $G$ . By lemma 39.4 on page 309, if  $\eta$  is biinvariant and exact, then  $\eta = d\xi$  for some biinvariant  $\xi$ . But biinvariant forms are closed, so  $\eta = d\xi = 0$ .  $\square$

### The Cartan 3-form

A *reductive* Lie group is a Lie group  $G$  whose Lie algebra  $\mathfrak{g}$  has a positive definite inner product invariant under the adjoint representation  $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ . Denote some such inner product as  $\langle A, B \rangle$  for  $A, B \in \mathfrak{g}$ . By lemma 39.1 on page 307, compact Lie groups are reductive. The *Cartan 3-form*  $\xi$  on any reductive Lie group  $G$  is the left invariant expression defined by having, at the identity element,

$$\xi_1(A, B, C) = \frac{4}{\pi^2} \langle A, [BC] \rangle.$$

**Lemma 39.14.** *The Cartan 3-form is a 3-form*

*Proof.* At the identity

$$\xi_1(A, B, C) = \frac{4}{\pi^2} \langle A, [BC] \rangle$$

is antisymmetric in  $B$  and  $C$ . By left invariance,  $\langle \vec{A}, \vec{B} \rangle$  is constant. So

$$\begin{aligned} 0 &= \vec{C} \langle \vec{A}, \vec{B} \rangle, \\ &= \langle [\vec{C}\vec{A}], \vec{B} \rangle + \langle \vec{A}, [\vec{C}\vec{B}] \rangle, \\ &= \langle [CA], B \rangle + \langle A, [CB] \rangle, \end{aligned}$$

so  $\xi_1(A, B, C) = \xi_1(B, C, A)$ , allowing us to cyclically permute.  $\square$

**Theorem 39.15.** *If a product of spheres  $S^{d_1} \times S^{d_2} \times \cdots \times S^{d_k}$  has a Lie group structure then either (a) one of  $d_1, d_2, \dots, d_k$  is equal to 3 or (b) all of  $d_1, d_2, \dots, d_k$  are equal to 1.*

*Proof.* The Cartan 3-form is a biinvariant 3-form, as the inner product and the Lie bracket are adjoint invariant. Hence the Cartan 3-form is closed, by problem 39.16 on the preceding page. Note that if  $G$  is not abelian, then the Cartan 3-form is not zero:

$$\xi([BC], B, C) = \frac{4}{\pi^2} |[BC]|^2,$$

and so the Cartan 3-form represents a nontrivial element of cohomology. On the other hand, if  $G$  is abelian, then  $G$  has universal covering space  $\mathbb{R}^n$ , a Lie group under addition. Finally, we see that every Lie group, as an abstract smooth manifold, either has universal covering space diffeomorphic to Euclidean space or has  $0 \neq H_{\text{dR}}^3(M)$ .  $\square$

This is not an optimal result:  $S^2 \times S^3$  has no Lie group structure, but this theorem won't tell you that.

Consider the unit quaternions  $G = S^3$  as a compact Lie group. Identify the Lie algebra with the imaginary quaternions, hence with  $\mathbb{R}^3$  with cross product. The Cartan 3-form at the identity is

$$\xi(u, v, w) = \frac{4}{\pi^2} u \cdot (v \times w),$$

which we can expand out to see that it gives

$$\xi = \frac{4}{\pi^2} dx_1 \wedge dx_2 \wedge dx_3,$$

a multiple of the usual volume form on  $\mathbb{R}^3$ . Left invariance under the unit quaternions (which act as rotations on  $\mathbb{H} = \mathbb{R}^4$  in the usual Euclidean geometry) ensures that this is the usual volume form on the sphere, up to that constant factor. Note that the sphere  $S^3$  has volume  $\pi^2/2$ , so

$$\int_{S^3} \xi = 2.$$

In particular, the Cartan 3-form is nonzero in de Rham cohomology on  $S^3$ .

The quotient  $\text{SO}(3) = S^3 / \pm 1$  shows that the Cartan 3-form on  $\text{SO}(3)$  has half the integral on  $S^3$ ,

$$\int_{\text{SO}(3)} \xi = 1.$$

In particular, the Cartan 3-form is nonzero in de Rham cohomology. We also understand why there is a factor of  $4/\pi^2$  in the Cartan form.

We have seen that the only ideals in the Lie algebra  $\mathbb{R}^3$  of the unit quaternions are  $\{0\}, \mathbb{R}^3$ . Hence every Lie group morphism  $S^3 \rightarrow G$  to any Lie group  $G$  is an immersion or constant. If an immersion, the kernel lies in the center  $Z_{S^3} = \{\pm 1\}$ , so the map is either an embedding  $S^3 \rightarrow G$ , or drops to an embedding  $\text{SO}(3) \rightarrow G$ . In either case, it pulls back the Cartan 3-form to the Cartan 3-form, as long as the inner product on  $G$  pulls back to that on  $S^3$ , so

nonzero integral. For example, splitting  $\mathbb{R}^n = \mathbb{R}^3 \oplus \mathbb{R}^{n-3}$  or  $\mathbb{R}^n = \mathbb{R}^4 \oplus \mathbb{R}^{n-4}$  gives obvious morphisms  $S^3 \rightarrow \mathrm{SO}(n)$  for  $n \geq 3$ , ensuring that the Cartan 3-form on  $\mathrm{SO}(n)$  has nonzero integrals. The same works for  $\mathrm{SU}(2) \cong S^3$  and hence for any  $\mathrm{SU}(n)$  or  $\mathrm{U}(n)$ , and similarly  $S^3 = \mathrm{Sp}(1) \subset \mathrm{Sp}(n)$ . In particular, the Cartan 3-form is nonzero in de Rham cohomology on  $\mathrm{SO}(n)_{n \geq 3}$ ,  $\mathrm{O}(n)_{n \geq 3}$ ,  $\mathrm{SU}(n)_{n \geq 2}$ ,  $\mathrm{U}(n)_{n \geq 2}$  and  $\mathrm{Sp}(n)_{n \geq 1}$ .





## Chapter 40

### Homological algebra

We use algebraic methods to relate cohomology of a manifold and its open sets.

#### Homological algebra

A *differential* is a morphism  $d: A \rightarrow A$  of abelian groups so that  $d^2 = 0$ . The *cohomology* of  $d$  is the quotient  $H = \ker d / \operatorname{im} d$ .

Take  $A$  to be the set of smooth differential forms on a manifold and  $d$  the exterior derivative.

A *morphism*  $f: (A, d) \rightarrow (B, d)$  of differentials is a linear map so that  $fd = df$ .

If  $\varphi: M \rightarrow N$  is a smooth map of manifolds, then let  $f := \varphi^*: A := \Omega_N^\bullet \rightarrow B := \Omega_M^\bullet$  be the pullback of differential forms.

Each morphism induces the quotient morphism  $f: H_A^\bullet \rightarrow H_B^\bullet$ .

In most examples,  $A$  a *complex*: a direct sum of subgroups  $A^j$  for integers  $j$ , with  $A^j = 0$  for  $j < 0$ , and we write  $A$  as  $A^\bullet$  to indicate this. We then require that  $d: A^j \rightarrow A^{j+1}$  and let

$$H^j = \ker d|_{A^j} / \operatorname{im} d|_{A^{j-1}}$$

A *morphism* of complexes is a morphism of differentials  $f: A^\bullet \rightarrow B^\bullet$  so that  $fA^j \subset B^j$ .

Differential forms have degrees: take  $A^j$  to be the set of smooth  $j$ -forms on some manifold  $M$ .

A pair of morphisms  $A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet$  between complexes are *exact* if  $\operatorname{im} f = \ker g$ . A sequence of morphisms between complexes is *exact* if any two of the maps appearing in successive terms of the sequence are exact. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0.$$

**40.1** Prove that  $g$  is onto and  $f$  is 1-1.

Any short exact sequence of complexes

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$$

induces a long exact sequence in cohomology

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_A^0 & \longrightarrow & H_B^0 & \longrightarrow & H_C^0 \\ & & & & \searrow & & \\ & & H_A^1 & \longrightarrow & H_B^1 & \longrightarrow & H_C^1 \\ & & & & \searrow & & \\ & & H_A^2 & \longrightarrow & H_B^2 & \longrightarrow & H_C^2 \\ & & & & \searrow & & \\ & & H_A^3 & \longrightarrow & H_B^3 & \longrightarrow & H_C^3 \\ & & & & \searrow & & \\ & & H_A^4 & \longrightarrow & \dots\dots\dots & & \end{array}$$

The maps  $H_A^\bullet \rightarrow H_B^\bullet$  are just  $f$ , while the maps  $H_B^\bullet \rightarrow H_C^\bullet$  are  $g$ . The remaining map, with the curving arrows, is given as follows. Look at:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{j+1} & \xrightarrow{f} & B^{j+1} & \xrightarrow{g} & C^{j+1} \longrightarrow 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 & \longrightarrow & A^j & \xrightarrow{f} & B^j & \xrightarrow{g} & C^j \longrightarrow 0 \end{array}$$

Take some  $c \in C^j$  in the lower right corner satisfying  $dc = 0$ . Because  $g$  is onto,  $c = gb$  for some  $b$ . Then  $g(db) = d(gb) = dc = 0$ , so  $g(db) = 0$  and so  $db = f(a)$  for some element  $a$  in the upper right corner.

**40.2** Prove that  $da = 0$ .

**40.3** Prove that  $a$  is uniquely determined up to replacing by  $a + da'$  for some  $a' \in A$ . In particular, the cohomology class  $[a] \in H_A^\bullet$  is defined.

**40.4** Prove that  $[a]$  depends only on  $[c]$ , i.e. replace  $c$  by some  $c + dc'$  for some  $c' \in C$  and check that instead of  $a$  you come up with  $a + da'$  for some  $a' \in A$ .

We call this element  $[a] = \delta[c]$ ;  $\delta$  is the *connecting morphism*.

**40.5** Suppose we have two vector spaces  $A, B$  over a field  $k$  and a pair of linear maps  $d: A \rightarrow B$  and  $d: B \rightarrow A$  with  $d^2 = 0$  and with the image of each matching the kernel of the other. Prove that  $A$  and  $B$  are isomorphic vector spaces.

Take a complex  $A^\bullet$  of vector spaces over a field. Suppose that the cohomology groups  $H_A^\bullet$  are finite dimensional vector spaces over that field, and all but finitely many are zero dimensional. The *Euler characteristic* of  $A^\bullet$  is the alternating sum

$$\chi_{A^\bullet} := \sum_i (-1)^i \dim H_A^i.$$

Write  $H_A^{\text{ev}}, H_A^{\text{od}}$  for the direct sum of the even and the odd order cohomology

$$\begin{aligned} H_A^{\text{ev}} &:= \bigoplus H_A^{2i}, \\ H_A^{\text{od}} &:= \bigoplus H_A^{2i+1}. \end{aligned}$$

**40.6** Take a short exact sequence  $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ . Suppose that all cohomology groups in the associated long exact sequence are finite dimensional vector spaces over the same field. Find an isomorphism

$$H_A^{\text{ev}} \oplus H_B^{\text{od}} \oplus H_C^{\text{ev}} \cong H_A^{\text{od}} \oplus H_B^{\text{ev}} \oplus H_C^{\text{od}}.$$

Explain why  $\chi_{B^\bullet} = \chi_{A^\bullet} + \chi_{C^\bullet}$ .

## Homology

Sometimes we write our complexes as  $A_\bullet$  instead, with the differential increasing indices rather than decreasing; the cohomology is then referred as the *homology*. We can always reverse them by setting  $B^k := A_{-k}$ , so the direction doesn't matter.

## Bounding homology dimensions

Take any complex  $A_\bullet$  over a field, with differential denoted  $\partial$ . Recall that in this notation  $\partial$  diminishes degrees. The *Hilbert–Poincaré series* is

$$P_{A_\bullet}(t) := \sum_i \dim A_i t^i.$$

Associate to  $A_\bullet$  the complex  $H_\bullet$  whose terms are the homology groups of  $A_\bullet$  with differential 0 and  $B_\bullet$  the subcomplex of exact elements of  $A_\bullet$ , i.e.  $B_\bullet = \partial A_\bullet$ .

**Theorem 40.1.** *For any complex  $A_\bullet$  over a field, with all terms  $A_i$  finite dimensional vector spaces, and all but finitely many of dimension zero,*

$$P_{A_\bullet}(t) = P_{H_\bullet}(t) + (1+t)P_{B_\bullet}(t).$$

*Proof.* Write  $Z_\bullet$  for the kernels of  $\partial$ . Take short exact sequences

$$0 \longrightarrow B_\bullet \longrightarrow A_\bullet \longrightarrow H_\bullet \longrightarrow 0$$

$$0 \longrightarrow Z_\bullet \longrightarrow A_\bullet \longrightarrow B_\bullet[-1] \longrightarrow 0$$

$$\dim A_\bullet - \dim H_\bullet = \dim B_\bullet[-1] + \dim B_\bullet$$

Expanding out

$$\begin{aligned} P_{A_\bullet}(t) - P_{H_\bullet}(t) &= P_{B_\bullet}(t) + P_{B_\bullet[-1]}(t), \\ &= P_{B_\bullet}(t) + tP_{B_\bullet}(t), \\ &= (1+t)P_{B_\bullet}(t) \end{aligned}$$

□

### The Mayer–Vietoris sequence

On an manifold  $M$  denote by  $\Omega^\bullet$  the complex of smooth differential forms. Pick open sets  $M_0, M_1 \subset M$  covering  $M$ , i.e.  $M = M_0 \cup M_1$ . Let  $M_{01} := M_0 \cap M_1$ . Denote by  $\Omega_0^\bullet$  the complex of smooth differential forms on  $M_0$ , etc. Make a second short exact sequence  $0 \rightarrow \Omega^\bullet \rightarrow \Omega_0^\bullet \oplus \Omega_1^\bullet \rightarrow \Omega_{01}^\bullet \rightarrow 0$  by the maps  $\omega \mapsto (\omega, \omega)$  and  $(\omega_0, \omega_1) \mapsto \omega_0 - \omega_1$ . The corresponding long exact sequence is called the *Mayer–Vietoris sequence*.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_{\text{dR}}^0(M) & \longrightarrow & H_{\text{dR}}^0(M_0) \oplus H_{\text{dR}}^0(M_1) & \longrightarrow & H_{\text{dR}}^0(M_{01}) \\
 & & \searrow & & \searrow & & \searrow \\
 & & H_{\text{dR}}^1(M) & \longrightarrow & H_{\text{dR}}^1(M_0) \oplus H_{\text{dR}}^1(M_1) & \longrightarrow & H_{\text{dR}}^1(M_{01}) \\
 & & \searrow & & \searrow & & \searrow \\
 & & H_{\text{dR}}^2(M) & \longrightarrow & H_{\text{dR}}^2(M_0) \oplus H_{\text{dR}}^2(M_1) & \longrightarrow & H_{\text{dR}}^2(M_{01}) \\
 & & \searrow & & \searrow & & \searrow \\
 & & H_{\text{dR}}^3(M) & \longrightarrow & H_{\text{dR}}^3(M_0) \oplus H_{\text{dR}}^3(M_1) & \longrightarrow & H_{\text{dR}}^3(M_{01}) \\
 & & \searrow & & \searrow & & \searrow \\
 & & H_{\text{dR}}^4(M) & \longrightarrow & \dots\dots\dots & & 
 \end{array}$$

The Mayer–Vietoris sequence shows relations between the cohomologies of  $M, M_0, M_1$  and  $M_{01}$ , which can all be complicated.

**40.7** What does Mayer–Vietoris say about the cohomology of the sphere?

Take any manifold  $M$  and a point  $m_0 \in M$ . Let  $\bar{B} \subset M$  be an embedded closed ball containing  $m_0$  in its interior  $B$ . Let  $M_0 := M - \{m_0\}$  and  $M_1 := B$  so that  $M_{01}$  is the punctured ball, which is diffeomorphic to  $S^{n-1} \times \mathbb{R}$ , so has the same cohomology as  $S^{n-1}$ , i.e.

$$H_{\text{dR}}^\ell(S^{n-1}) = \begin{cases} \mathbb{R}, & \ell = 0 \text{ or } \ell = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Think of this  $S^{n-1}$  as a tiny sphere surrounding  $m_0$ ; it is called a *vanishing sphere* because it vanishes when we put  $m_0$  back into  $M_0$  to form  $M$ . The cohomology of  $B$  is that of a point.

Clearly the cohomology of  $M$  and  $M_0$  are identical on any component of  $M$  not containing  $m_0$ , so we can assume that  $M$  is connected. If  $M$  has dimension 1,  $M = \mathbb{R}$  or  $S^1$ , and then  $M_0 = \mathbb{R} \sqcup \mathbb{R}$  or  $\mathbb{R}$ ; we can see the cohomology directly. So assume that  $M$  has dimension 2 or more, and then  $M_0$  is connected.

The Mayer–Vietoris sequence is

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{\mathrm{dR}}^0(M) & \longrightarrow & H_{\mathrm{dR}}^0(M_0) \oplus \mathbb{R} & \longrightarrow & \mathbb{R} \\
& & \searrow & & \searrow & & \searrow \\
& & H_{\mathrm{dR}}^1(M) & \longrightarrow & H_{\mathrm{dR}}^1(M_0) & \longrightarrow & 0 \\
& & \searrow & & \searrow & & \searrow \\
& & H_{\mathrm{dR}}^2(M) & \longrightarrow & H_{\mathrm{dR}}^2(M_0) & \longrightarrow & 0 \\
& & \searrow & & \searrow & & \searrow \\
& & H_{\mathrm{dR}}^3(M) & \longrightarrow & H_{\mathrm{dR}}^3(M_0) & \longrightarrow & 0 \\
& & \searrow & & \searrow & & \searrow \\
& & H_{\mathrm{dR}}^4(M) & \longrightarrow & \dots\dots\dots & & 
\end{array}$$

All cohomology groups of  $M$  and  $M_0$  are isomorphic up to final two rows:

$$0 \rightarrow H_{\mathrm{dR}}^{n-1}(M) \rightarrow H_{\mathrm{dR}}^{n-1}(M_0) \rightarrow \mathbb{R} \rightarrow H_{\mathrm{dR}}^n(M) \rightarrow H_{\mathrm{dR}}^n(M_0) \rightarrow 0.$$

The map  $H_{\mathrm{dR}}^{n-1}(M_0) \rightarrow \mathbb{R}$  takes an  $(n-1)$ -form on  $M_0$  and maps it to one on the vanishing sphere, i.e. integrates it around the vanishing sphere. For example, if  $M = S^n$ , then clearly  $M_0 = \mathbb{R}^n$  and the vanishing sphere retracts to a point, so those integrals vanish. In general, those integrals vanish just when every closed  $(n-1)$ -form on  $M_0$  extends across the puncture up to an exact form.

To follow the Mayer–Vietoris sequence, take some closed  $(n-1)$ -form  $\omega_{01}$  on the punctured ball, write it as the difference  $\omega_0 - \omega_1$  of forms on  $M_0$  and the ball. The  $n$ -forms  $d\omega_0, d\omega_1$  agree on the punctured ball, since  $d\omega_{01} = 0$ , so glue together to a single  $n$ -form  $\omega$  on  $M$ , which is closed. The connecting morphism is  $\omega_{01} \mapsto \omega$ .

Suppose that  $M$  is compact and oriented.

$$\begin{aligned}
\int_{S^{n-1}} \omega_{01} &= \int_{S^{n-1}} \omega_0 - \omega_1, \\
&= \int_{S^{n-1}} \omega_0 - \int_{S^{n-1}} \omega_1, \\
&= - \int_{\partial(M-B)} \omega_0 - \int_{\partial B} \omega_1, \\
&= - \int_{M-B} d\omega_0 - \int_B d\omega_1, \\
&= - \int_M \omega.
\end{aligned}$$

Hence  $\omega_{01}$  is exact just when  $\omega$  is. The integral of some closed  $(n-1)$ -form on  $M_0$  does not vanish over the vanishing sphere just when  $H_{\mathrm{dR}}^n(M_0) = 0$ , which occurs just when  $H_{\mathrm{dR}}^{n-1}(M) \rightarrow H_{\mathrm{dR}}^{n-1}(M_0)$  is not an isomorphism, i.e. just when we cannot extend some closed  $(n-1)$ -form across the puncture, up to exact forms.

Consider the compact oriented surface of genus  $g$ , call it  $M$ . Since  $M$  is connected,  $H_{\text{dR}}^0(M) = \mathbb{R}$ , as on page 298. Since  $M$  is compact, connected and oriented,  $H_{\text{dR}}^2(M) = \mathbb{R}$  by theorem 38.2 on page 303, an isomorphism given by integrating 2-forms over  $M$ . Let  $M_0$  be the top half ( $z > 0$ ) together with a little bit of  $z \leq 0$  so that it overlaps  $M_1$ , which is just  $M_0$  upside down, along some  $M_{01}$  which is a collection of small cylindrical strips,  $g + 1$  in all. Each has the cohomology of a circle.

Each of  $M_0$  and  $M_1$  is diffeomorphic to a plane region with  $g$  holes poked in it, so we have seen on page 299 and on page 293 that the cohomology is

| $H_{\text{dR}}^\ell(X)$ | $X$            |                    |
|-------------------------|----------------|--------------------|
| $\ell$                  | $M_0$          | $M_{01}$           |
| 0                       | $\mathbb{R}$   | $\mathbb{R}^{g+1}$ |
| 1                       | $\mathbb{R}^g$ | $\mathbb{R}^{g+1}$ |
| 2                       | 0              | 0                  |

The Mayer–Vietoris sequence is

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^{g+1} \\
 & & & & & \searrow & \\
 & & & & & & H_{\text{dR}}^1(M) \longrightarrow \mathbb{R}^{2g} \longrightarrow \mathbb{R}^{g+1} \\
 & & & & & \searrow & \\
 & & & & & & \mathbb{R} \longrightarrow 0 \longrightarrow 0
 \end{array}$$

This sequence ensures that  $H_{\text{dR}}^1(M)$  is a finite dimensional real vector space. Take Euler characteristic to find  $H_{\text{dR}}^1(M) = \mathbb{R}^{2g}$ . In chapter 37, we found  $2g$  explicit closed 1-forms with linearly independent integrals on a compact oriented surface of genus  $g$ , so these must be the generators of the first cohomology.

Consider the cohomology of a punctured projective space. For  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , pick a finite dimensional  $k$ -vector space  $V$ , and a  $k$ -line  $\ell_0 \subset V$ . Let  $\mathbb{P}V$  be the associated projective space  $(V - \{0\})/(k - \{0\})$ . Let  $M := \mathbb{P}(V/\ell_0)$  and  $E = \mathbb{P}V - \{\ell_0\}$ . Let  $\pi: E \rightarrow M$  be

$$\pi(\ell) = (\ell + \ell_0)/\ell_0.$$

For any  $k$ -hyperplane  $H_0 \subset V$  not containing  $\ell_0$ , the linear isomorphism  $H_0 \rightarrow V/\ell_0$  descends to a diffeomorphism  $\mathbb{P}H_0 \rightarrow M$ . The inclusion  $H_0 \subset V$  descends to an inclusion  $\mathbb{P}H_0 \subset E \subset \mathbb{P}V$ . Write each vector  $v \in V = \ell_0 \oplus H_0$  uniquely as  $v = u + w$  with  $u \in \ell_0$  and  $w \in H_0$ . Retract  $V$  to  $H_0$  by

$$\varphi_t(v) = (1 - t)u + w.$$

This map descends to a smooth retraction

$$\varphi_t: E \rightarrow E,$$

retracting to  $\mathbb{P}H_0$ . Hence the cohomology of  $E$  is that of  $M$ , i.e. if  $p_0 \in \mathbb{P}^{n+1}$  is any point,

$$H_{\text{dR}}^\bullet(\mathbb{P}^{n+1} - p_0) = H_{\text{dR}}^\bullet(\mathbb{P}^n).$$

Start with  $\mathbb{P}^0$  being a point. Let  $d := \dim_{\mathbb{R}} k$ . Knowing that  $\mathbb{P}^{n+1}$  is a connected manifold of dimension  $d(n+1)$ , inductively, this process does not help with the real projective spaces, but we get all of the de Rham cohomology of the complex and quaternionic projective spaces

$$H_{\text{dR}}^p(\mathbb{CP}^n) = \begin{cases} \mathbb{R}, & p \text{ even,} \\ 0, & p \text{ odd.} \end{cases}$$

$$H_{\text{dR}}^p(\mathbb{HP}^n) = \begin{cases} \mathbb{R}, & p \text{ a multiple of 4,} \\ 0, & \text{otherwise.} \end{cases}$$

## Rational chains

In the definition of chain, we only allowed addition of chains with integer coefficients. A *real chain* is a formal finite sum  $\sum_i a_i C_i$  where these  $a_i$  are real numbers and the  $C_i$  are chains, but modulo the relation  $aC = (a/k)(kC)$  for any nonzero integer  $k$ . The real chains have homology the *real homology* groups  $H_\bullet(M, \mathbb{R})$ . Similarly define rational and complex homology, and define rational, real and complex Borel–Moore homology. Each real chain is a real linear combination of rational chains, i.e.

$$H_\bullet(M, \mathbb{R}) = H_\bullet(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R},$$

and the same for complex cohomology, so it suffices to understand integer and rational homology and their relations.

Recall that an element  $a_0 \in A$  of an abelian group  $A$  is a *torsion* element if  $na = 0$  for some integer  $n \neq 0$ . An integer cycle is a rational boundary just when, after suitable multiplying by denominators, it has a nonzero integer multiple which is an integer boundary. So a class vanishes in rational homology just when it is a torsion class:

$$H_\bullet(M, \mathbb{Q}) = H_\bullet(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = (H_\bullet(M, \mathbb{Z})/\text{torsion}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

## de Rham's theorem

We will prove:

**Theorem 40.2** (de Rham). *On any manifold with corners, a smooth closed differential form is exact just when its integrals over cycles are zero, and a cycle is a sum of torsion and boundary just when the integrals of all closed differential forms over it are zero.*

For the moment, we just say that a manifold with corners  $M$  is *de Rham* if this happens. (This is not standard terminology.)

**40.8** Prove that any ball or box in  $\mathbb{R}^n$  is de Rham.

Let  $\tilde{\Omega}^\bullet$  be the dual space of the space of real chains. Introduce the differential  $du$ , for any  $u \in \tilde{\Omega}^\bullet$  (i.e. any linear operation  $u$  on real chains) by  $du(C) = u(\partial C)$ . Map

$$\omega \in \Omega^\bullet \mapsto \int \omega \in \tilde{\Omega}^\bullet$$

where  $\int \omega$  means the operation  $C \mapsto \int_C \omega$ . Let  $\bar{\Omega}^\bullet = \tilde{\Omega}^\bullet / \Omega^\bullet$  be the quotient complex, i.e. the quotient spaces with quotient differential, and we see a short exact sequence  $0 \rightarrow \Omega^\bullet \rightarrow \tilde{\Omega}^\bullet \rightarrow \bar{\Omega}^\bullet \rightarrow 0$ .

**Lemma 40.3.** *For the short exact sequence*

$$0 \longrightarrow \Omega^\bullet \xrightarrow{\int} \tilde{\Omega}^\bullet \longrightarrow \bar{\Omega}^\bullet \longrightarrow 0$$

*the map  $\int$  is an isomorphism on cohomologies in the associated long exact sequence just when  $M$  is de Rham.*

*Proof.* We can write the associated long exact sequence in cohomology as

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0 & \longrightarrow & \tilde{H}^0 & \longrightarrow & \bar{H}^0 \\ & & & & \searrow & & \\ & & H^1 & \longrightarrow & \tilde{H}^1 & \longrightarrow & \bar{H}^1 \\ & & & & \searrow & & \\ & & H^2 & \longrightarrow & \tilde{H}^2 & \longrightarrow & \bar{H}^2 \\ & & & & \searrow & & \\ & & H^3 & \longrightarrow & \tilde{H}^3 & \longrightarrow & \bar{H}^3 \\ & & & & \searrow & & \\ & & H^4 & \longrightarrow & \dots\dots & & \end{array}$$

Vanishing of the cohomology of  $\bar{\Omega}^\bullet$  makes the map  $\int: \Omega^\bullet \rightarrow \tilde{\Omega}^\bullet$  an isomorphism in cohomology. The map  $\int$  is injective in cohomology just when any closed form whose integrals vanish is exact. This map  $\int$  is surjective in cohomology just when, for any cycle  $C$ , if  $\int_C \omega = 0$  for any form  $\omega$  then  $u(C) = 0$  for any linear function  $u$  vanishing on boundaries, i.e.  $C$  is a boundary.  $\square$

**Proposition 40.4.** *Suppose that  $M$  is a manifold with corners admitting an open cover by open sets  $M_0, M_1$ . Suppose that  $M_0, M_1$  and  $M_{01} := M_0 \cap M_1$  are de Rham. Then  $M$  is de Rham.*

*Proof.* Consider the short exact sequence  $0 \rightarrow \Omega^\bullet \rightarrow \tilde{\Omega}^\bullet \rightarrow \bar{\Omega}^\bullet \rightarrow 0$ . We can also replace  $\Omega$  with  $\Omega_{01}, \Omega_0, \Omega_1$ , i.e. replacing  $M$  by  $M_{01}, M_0$  or  $M_1$  respectively, replacing each object above by an object with the same name but adding, as needed, a 01, 0 or 1 subscript.

Each complex  $A = \Omega, \tilde{\Omega}, \bar{\Omega}$  in this sequence  $0 \rightarrow \Omega^\bullet \rightarrow \tilde{\Omega}^\bullet \rightarrow \bar{\Omega}^\bullet \rightarrow 0$  has its own Mayer–Vietoris sequence  $0 \rightarrow A_{01}^\bullet \rightarrow \dot{A}^\bullet \rightarrow A_0^\bullet \oplus A_1^\bullet \rightarrow 0$ , using exactly the same definitions. (The notation becomes terrible; we leave the reader to write it all out.) The long exact sequences from the various Mayer–Vietoris sequences then give a short exact sequence of long exact sequences:



$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & H_{01}^j & \longrightarrow & H^j & \longrightarrow & H_0^j \oplus H_1^j & \longrightarrow & H_{01}^{j+1} & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \tilde{H}_{01}^j & \longrightarrow & \tilde{H}^j & \longrightarrow & \tilde{H}_0^j \oplus \tilde{H}_1^j & \longrightarrow & \tilde{H}_{01}^{j+1} & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \bar{H}_{01}^j & \longrightarrow & \bar{H}^j & \longrightarrow & \bar{H}_0^j \oplus \bar{H}_1^j & \longrightarrow & \bar{H}_{01}^{j+1} & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& 0 & & 0 & & 0 & & 0
\end{array}$$

Clearly  $M_{01}$ ,  $M_0$  and  $M_1$  are all de Rham just when the  $\bar{H}$  terms with 01, 0 or 1 subscripts vanish. But then there are zeroes around all the  $\bar{H}$  term in the second column. By exactness (so injectivity and surjectivity of all of these maps), all of these  $\bar{H}$  terms vanish, so  $M$  is de Rham.  $\square$

Any sphere is de Rham: as above, we set  $M_0$  to be the complement of the south pole,  $M_1$  the complement of the north pole, and then the cohomology of  $M_{01}$  is identified with that of a sphere of lower dimension by homotopy.

#### 40.9 Why is a torus de Rham? Why is the “two-handled torus” de Rham?

Take a manifold with corners  $M$ . Take a fine cover, and form out of it a ripple, and then take  $M_0$  to be the even and  $M_1$  the odd elements of the ripple. So  $M_0$  is a union of precompact open sets  $M_{0a}$ , with disjoint closures, each a finite union of open sets from a fine cover; and the same holds for  $M_1$ . Similarly,  $M_{01} := M_0 \cap M_1$  is union of precompact open sets  $M_{01a}$ , with disjoint closures, each a finite union of open sets from a fine cover. So

$$\begin{aligned}
H_{\text{dR}}^\bullet(M_0) &= \bigoplus H_{\text{dR}}^\bullet(M_{0a}), \\
H_{\text{dR}}^\bullet(M_1) &= \bigoplus H_{\text{dR}}^\bullet(M_{1a}), \\
H_{\text{dR}}^\bullet(M_{01}) &= \bigoplus H_{\text{dR}}^\bullet(M_{01a}),
\end{aligned}$$

each a (possibly infinite) product of finite dimension vector spaces. (Keep in mind that, in such a product, a vector is precisely a choice of vector in each summand, and there is no requirement of any convergence or finiteness of the choices made.)

We now prove de Rham's theorem or, In other words, every manifold is de Rham, i.e. the pairing  $\int_C \omega$  vanishes for all  $C$  just when  $[\omega] = 0$  and for all  $[\omega]$  just when  $C = 0$ .

*Proof.* Using  $M_0, M_1, M_{01}$  as above, by proposition 40.4 on the preceding page,  $M$  is de Rham just when each of those are, and that just when each of  $M_{0a}, M_{1a}, M_{01a}$  are, to which we apply proposition 40.4 on the facing page again and induction.  $\square$

**Corollary 40.5.** *Take a manifold  $M$ , say with fundamental group  $\pi := \pi_1(M)$ . Recall that  $[\pi, \pi] \subseteq \pi$  is the subgroup generated by products  $xyx^{-1}y^{-1}$ . If the abelian group  $\pi/[\pi, \pi]$  is torsion (i.e. every element has some power which is the identity), then  $0 = H_{\text{dR}}^1(M)$ .*



The two-handled torus

*Proof.* Any 1-cycle is a sum of loops. So it suffices to prove the vanishing of all of the integrals of any closed 1-form  $\omega$  on any loop. The integrals over loops  $\alpha\beta\alpha^{-1}\beta^{-1}$  add up to zero, because the loop breaks into

$$\int_{\alpha\beta\alpha^{-1}\beta^{-1}} \omega = \int_{\alpha} \omega + \int_{\beta} \omega - \int_{\alpha} \omega - \int_{\beta} \omega = 0.$$

So integration over loops is defined on  $\pi/[\pi, \pi]$ . But if a loop  $\gamma$  satisfies  $\gamma^n = 1$  in  $\pi/[\pi, \pi]$ , then integration over it gives

$$0 = \int_{\gamma^n} \omega = n \int_{\gamma} \omega.$$

□

**Theorem 40.6.** *A closed compactly supported differential form is exact, the differential of a compactly supported differential form, just when its integrals over all long cycles vanish. A long cycle is a sum of a torsion long cycle and a boundary just when all integrals of closed compactly supported differential forms vanish over it.*

In other words, the pairing  $\int_C \omega$  vanishes for all  $C$  just when  $[\omega] = 0$  and for all  $[\omega]$  just when  $C$  is boundary plus torsion. The proof is identical.

The proof we have for de Rham's theorem actually proves a stronger result. To each smooth differential form  $\omega$  on a manifold with corners  $M$ , associate the linear map  $\int \omega$  which takes a chain  $C$  (with real coefficients) to  $\int_C \omega$ . This map  $\int \omega$  is also often denoted  $P_{\omega}$ , the *Poincaré dual* of  $\omega$ . If  $\omega$  is closed, its Poincaré dual vanishes of exact chains by Stokes's theorem, and hence  $P_{\omega}$  becomes defined on cycles  $[C]$ . Again by Stokes's theorem,  $P_{\omega}[C] = 0$  if  $\omega$  is exact so the Poincaré dual map descends to a map

$$P_{[\omega]} = \int \omega: C \in H_{\bullet}(M, \mathbb{R}) \mapsto \int_C \omega \in \mathbb{R},$$

also called the *Poincaré dual*  $P_{[\omega]}$  of  $[\omega]$ . So Poincaré duality maps

$$[\omega] \in H_{\text{dR}}^{\bullet}(M) \mapsto P_{[\omega]} = \int \omega \in H_{\bullet}(M, \mathbb{R})^{\vee}.$$

**Theorem 40.7** (Poincaré duality). *The Poincaré dual map is a linear isomorphism of real vector spaces*

$$P: H_{\text{dR}}^{\bullet}(M) \rightarrow H_{\bullet}(M, \mathbb{R})^{\vee}.$$

Similarly,

$$\omega \in H_c^{\bullet}(M) \mapsto \int \omega \in H_{\bullet}^{BM}(M, \mathbb{R})^*$$

is a linear isomorphism of real vector spaces.

*Proof.* The proof of de Rham's theorem directly identifies  $H_{\text{dR}}^{\bullet}(M)$  via the Poincaré dual map with the cohomology of  $\hat{\Omega}^{\bullet}$ , the dual space of real chains. So elements of that cohomology are linear maps on real chains, modulo the exact ones, i.e. the ones that vanish on exact chains, i.e. linear maps on the homology of chains. □

**Theorem 40.8.** [Poincaré–Lefschetz duality] Denote the dual space of any real vector space  $V$  as  $V^\vee$ . On any oriented manifold  $M$  with corners, let  $\Omega_c^\bullet$  be the compactly supported smooth differential forms, and  $\Omega_0$  the smooth differential forms pulling back to zero on  $\partial M$ . These are closed under  $d$ ; let  $H_c^\bullet$ ,  $H_0^\bullet$  be the associated cohomologies. Consider the integral

$$\alpha \in \Omega_c^\bullet, \beta \in \Omega_0^\bullet \mapsto \int_M \alpha \wedge \beta.$$

- a. If  $\alpha, \beta$  are closed, then this integral depends only on the cohomologies in  $H_c^\bullet$ ,  $H_0^\bullet$ .
- b. For any closed  $\alpha$ , the integral vanishes for all closed  $\beta$  just when  $\alpha = 0$  in  $H_c^\bullet$ , i.e.  $\alpha = d\mu$  for some  $\mu \in \Omega_c^\bullet$ .
- c. For any closed  $\beta$ , the integral vanishes for all closed  $\alpha$  just when  $\beta = 0$  in  $H_c^\bullet$ , i.e.  $\beta = d\nu$  for some  $\nu \in \Omega_0^\bullet$ .

Associate to any  $\beta \in \Omega_0^\bullet$  its Poincaré dual: the linear map

$$P_\beta: \alpha \in \Omega_c^\bullet \mapsto \int_M \alpha \wedge \beta \in \mathbb{R}.$$

If  $\beta$  is closed, its class  $[\beta] \in H_0^\bullet$  has a Poincaré dual linear map

$$P_{[\beta]}[\alpha] = P_\beta \alpha,$$

so  $P_{[\beta]} \in H_c^{\bullet\vee}$ . The Poincaré dual map  $P: H_0^\bullet \rightarrow H_c^{\bullet\vee}$  is a linear isomorphism of real vector spaces.

*Proof.* For a  $p$ -form  $\alpha$ , write  $(-1)^\alpha$  to mean  $(-1)^p$ . For  $\alpha \in \Omega_c^\bullet, \beta \in \Omega_0^\bullet$ :

$$\begin{aligned} \int_{\partial M} \alpha \wedge \beta &= \int_M d(\alpha \wedge \beta), \\ &= \int_M d\alpha \wedge \beta + (-1)^\alpha \int_M \alpha \wedge d\beta \end{aligned}$$

So if  $\beta$  is closed,

$$\int_M d\alpha \wedge \beta = \int_{\partial M} \alpha \wedge \beta$$

vanishes. Hence the integral vanishes for exact  $\alpha$ , so drops to cohomology. If  $\alpha$  is closed,

$$\int_M \alpha \wedge d\beta = (-1)^\alpha \int_{\partial M} \alpha \wedge \beta$$

vanishes. Hence the integral vanishes for exact  $\beta$ , so drops to cohomology.

On  $X = [0, \infty)^n$ , or any open subset of  $X$  containing the support of  $\alpha$ , if  $d\alpha = 0$  then  $\alpha = d\mu$ , some  $\mu \in \Omega_0^\bullet$  (see problem 30.7 on page 244).

Let  $\tilde{\Omega}^\bullet$  be the dual space of the space of compactly supported smooth differential forms (so  $\tilde{\Omega}^\bullet$  is the space of “currents”; see chapter 42). Introduce the differential  $du$ , for any  $u \in \tilde{\Omega}^\bullet$  (i.e. any linear operation  $u$  on compactly supported forms) by  $du(\alpha) = (-1)^{1+|\alpha|}u(d\alpha)$ . Map

$$\beta \in \Omega_0^\bullet \mapsto \int_M \cdot \wedge \beta \in \tilde{\Omega}^\bullet$$

where  $\int_M \cdot \wedge \beta$  means the operation  $\alpha \mapsto \int_M \alpha \wedge \beta$ . Note that

$$\int_M \alpha \wedge d\beta = (-1)^\alpha \int_{\partial M} \alpha \wedge \beta - (-1)^\alpha \int_M d\alpha \wedge \beta.$$

So if  $u = \int \cdot \wedge \beta$  then  $du = \int \cdot \wedge d\beta$ , morphism of complexes. Let  $\bar{\Omega}^\bullet = \tilde{\Omega}^\bullet / \Omega_0^\bullet$  be the quotient complex, i.e. the quotient spaces with quotient differential; a short exact sequence  $0 \rightarrow \Omega_0^\bullet \rightarrow \tilde{\Omega}^\bullet \rightarrow \bar{\Omega}^\bullet \rightarrow 0$ . The long exact sequence in cohomology gives us that the cohomology of the quotient vanishes just when the cohomologies of the other two are isomorphic. If we take a covering  $M_0, M_1, M_{01}$  as we did above when we studied de Rham cohomology, we reduce the problem to a local one, on the standard corner.  $\square$

### Relative cohomology

If  $A^\bullet$  is a complex, denote by  $A^\bullet[1]$  the complex  $B^\bullet$  so that  $B^j = A^{j+1}$ , etc. Take a morphism of complexes

$$f: A^\bullet \rightarrow B^\bullet.$$

We can see that this induces a morphism of cohomologies

$$f: H_A \rightarrow H_B,$$

but we would like to see the kernel of this morphism. Let  $B[-1]^\bullet$  be

$$B[-1]^k := B^{k-1}$$

with differential

$$d_{B[-1]} = -d_B.$$

The cohomology is independent of the sign of the differential, so

$$H_{B[-1]} = H_B[-1].$$

The *relative complex* of  $f: A^\bullet \rightarrow B^\bullet$  is  $C^\bullet = A^\bullet \oplus B[-1]^\bullet$  but with differential

$$d(a, b) = (da, fa - db).$$

The maps

$$(a, b) \in C^\bullet \mapsto a \in A^\bullet$$

and

$$b \in B[-1]^\bullet \mapsto (0, b) \in C^\bullet$$

clearly form a short exact sequence

$$0 \rightarrow B[-1]^\bullet \rightarrow C^\bullet \rightarrow A^\bullet \rightarrow 0.$$

Hence we have a long exact sequence in cohomology:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_{B[-1]}^0 & \longrightarrow & H_C^0 & \longrightarrow & H_A^0 \\
& & \searrow & & \searrow & & \searrow \\
& & H_{B[-1]}^1 & \longrightarrow & H_C^1 & \longrightarrow & H_A^1 \\
& & \searrow & & \searrow & & \searrow \\
& & H_{B[-1]}^2 & \longrightarrow & H_C^2 & \longrightarrow & H_A^2 \\
& & \searrow & & \searrow & & \searrow \\
& & H_{B[-1]}^3 & \longrightarrow & H_C^3 & \longrightarrow & H_A^3 \\
& & \searrow & & \searrow & & \searrow \\
& & H_{B[-1]}^4 & \longrightarrow & \cdots & \longrightarrow & \cdots
\end{array}$$

Shifting back indices:

$$H_{B[-1]}^k = H_B^{k-1},$$

Hence we have a long exact sequence in cohomology:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_C^0 & \longrightarrow & H_A^0 & \longrightarrow & H_B^0 \\
& & \searrow & & \searrow & & \searrow \\
& & H_C^1 & \longrightarrow & H_A^1 & \longrightarrow & H_B^1 \\
& & \searrow & & \searrow & & \searrow \\
& & H_C^2 & \longrightarrow & H_A^2 & \longrightarrow & H_B^2 \\
& & \searrow & & \searrow & & \searrow \\
& & H_C^3 & \longrightarrow & H_A^3 & \longrightarrow & H_B^3 \\
& & \searrow & & \searrow & & \searrow \\
& & H_C^4 & \longrightarrow & \cdots & \longrightarrow & \cdots
\end{array}$$

## Relative cohomology of differential forms

Take a smooth map  $\varphi: P \rightarrow Q$ . We can pull back differential forms, giving a morphism  $\varphi^*$  of the differential form complexes. Hence we can apply our construction of relative cohomology. A *relative  $k$ -cochain* is a pair  $(\xi, \eta)$  where  $\xi$  is a  $k$ -form on  $Q$  and  $\eta$  is a  $(k-1)$ -form on  $P$ . Let

$$d(\xi, \eta) := (d\xi, \varphi^*\xi - d\eta).$$

Let  $\Omega_{Q,P}^\bullet$  be the complex of relative cochains. Its cohomology is the *relative cohomology*  $H_{\text{dR}}^\bullet(Q, P)$ . Define maps

$$\begin{aligned}
(\xi, \eta) &\in \Omega_{Q,P}^\bullet \rightarrow \xi \in \Omega_Q^\bullet, \\
\eta &\in \Omega_P^\bullet[-1] \rightarrow (0, -\eta) \in \Omega_{Q,P}^\bullet,
\end{aligned}$$

We let  $\Omega_P^\bullet[-1]$  be the complex with

$$\Omega_P^k[-1] := \Omega_P^{k-1},$$

and with differential  $-d$ . Clearly the sign of the differential doesn't affect the cohomology. Obviously these give a short exact sequence

$$0 \rightarrow \Omega_P^\bullet[-1] \rightarrow \Omega_{Q,P}^\bullet \rightarrow \Omega_Q^\bullet \rightarrow 0.$$

Hence we have a long exact sequence in cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\Omega_P[-1]) & \longrightarrow & H^0(\Omega_{Q,P}) & \longrightarrow & H^0(\Omega_Q) \\ & & \searrow & & \searrow & & \searrow \\ & & H^1(\Omega_P[-1]) & \longrightarrow & H^1(\Omega_{Q,P}) & \longrightarrow & H^1(\Omega_Q) \\ & & \searrow & & \searrow & & \searrow \\ & & H^2(\Omega_P[-1]) & \longrightarrow & H^2(\Omega_{Q,P}) & \longrightarrow & H^2(\Omega_Q) \\ & & \searrow & & \searrow & & \searrow \\ & & H^3(\Omega_P[-1]) & \longrightarrow & H^3(\Omega_{Q,P}) & \longrightarrow & H^3(\Omega_Q) \\ & & \searrow & & \searrow & & \searrow \\ & & H^4(\Omega_P[-1]) & \longrightarrow & \dots\dots\dots & & \end{array}$$

Shifting back indices:

$$H^k(\Omega_P[-1]) = H^{k-1}(\Omega_P) = H_{\text{dR}}^{k-1}(P).$$

Hence we have a long exact sequence in cohomology:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{dR}}^0(Q, P) & \longrightarrow & H_{\text{dR}}^0(Q) & \longrightarrow & H_{\text{dR}}^0(P) \\ & & \searrow & & \searrow & & \searrow \\ & & H_{\text{dR}}^1(Q, P) & \longrightarrow & H_{\text{dR}}^1(Q) & \longrightarrow & H_{\text{dR}}^1(P) \\ & & \searrow & & \searrow & & \searrow \\ & & H_{\text{dR}}^2(Q, P) & \longrightarrow & H_{\text{dR}}^2(Q) & \longrightarrow & H_{\text{dR}}^2(P) \\ & & \searrow & & \searrow & & \searrow \\ & & H_{\text{dR}}^3(Q, P) & \longrightarrow & H_{\text{dR}}^3(Q) & \longrightarrow & H_{\text{dR}}^3(P) \\ & & \searrow & & \searrow & & \searrow \\ & & H_{\text{dR}}^4(Q, P) & \longrightarrow & \dots\dots\dots & & \end{array}$$

The same steps apply to compactly supported forms, using a proper map  $\varphi$ , give a long exact sequence in compact supported cohomology:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_c^0(Q, P) & \longrightarrow & H_c^0(Q) & \longrightarrow & H_c^0(P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_c^1(Q, P) & \longrightarrow & H_c^1(Q) & \longrightarrow & H_c^1(P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_c^2(Q, P) & \longrightarrow & H_c^2(Q) & \longrightarrow & H_c^2(P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_c^3(Q, P) & \longrightarrow & H_c^3(Q) & \longrightarrow & H_c^3(P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_c^4(Q, P) & \longrightarrow & \dots\dots\dots
\end{array}$$

### The relative homology sequence

Take a smooth map  $\varphi: P \rightarrow Q$ . Again we have a map  $\varphi_*$  on homology, so we follow our general theory of relative cohomology. A *relative  $k$ -chain* is a pair  $(C, D)$  where  $C$  is a  $k$ -chain on  $Q$  and  $D$  is a  $(k-1)$ -chain on  $P$ . Let

$$\partial(C, D) = (\partial C + \varphi_* D, -\partial D).$$

Its cohomology is the *relative homology*  $H_\bullet(Q, P)$  of  $\varphi$ . Let  $C_\bullet(Q, P)$  be the complex of relative chains. Let  $C_\bullet(P)[-1] := C_{\bullet-1}(P)$  but with differential  $-\partial$ . The sign of  $-\partial$  doesn't affect the cohomology of the complex, and the  $[-1]$  only shifts it. Define morphisms of complexes:

$$\begin{aligned}
C &\mapsto (C, 0), \\
(C, D) &\mapsto D.
\end{aligned}$$

Obviously these give a short exact sequence

$$0 \rightarrow C_\bullet(Q) \rightarrow C_\bullet(Q, P) \rightarrow C_\bullet(P)[-1] \rightarrow 0.$$

As above, we have a long exact sequence:

$$\begin{array}{ccccccc}
& & & & \dots & & \\
& & & & \searrow & & \\
& & H_4(P) & \longrightarrow & H_4(Q) & \longrightarrow & H_4(Q, P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_3(P) & \longrightarrow & H_3(Q) & \longrightarrow & H_3(Q, P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_2(P) & \longrightarrow & H_2(Q) & \longrightarrow & H_2(Q, P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_1(P) & \longrightarrow & H_1(Q) & \longrightarrow & H_1(Q, P) \\
& & \searrow & & \searrow & & \searrow \\
& & H_0(P) & \longrightarrow & H_0(Q) & \longrightarrow & H_0(Q, P) \longrightarrow 0
\end{array}$$

We repeat the same steps for Borel–Moore homology, to define relative Borel–Moore homology and find the exact sequence:

$$\begin{array}{ccccccc}
 & & & & \cdots & & \\
 & \swarrow & & \searrow & & \swarrow & \\
 H_4^{\text{BM}}(P) & \longrightarrow & H_4^{\text{BM}}(Q) & \longrightarrow & H_4^{\text{BM}}(Q, P) & & \\
 & \swarrow & & \searrow & & \swarrow & \\
 H_3^{\text{BM}}(P) & \longrightarrow & H_3^{\text{BM}}(Q) & \longrightarrow & H_3^{\text{BM}}(Q, P) & & \\
 & \swarrow & & \searrow & & \swarrow & \\
 H_2^{\text{BM}}(P) & \longrightarrow & H_2^{\text{BM}}(Q) & \longrightarrow & H_2^{\text{BM}}(Q, P) & & \\
 & \swarrow & & \searrow & & \swarrow & \\
 H_1^{\text{BM}}(P) & \longrightarrow & H_1^{\text{BM}}(Q) & \longrightarrow & H_1^{\text{BM}}(Q, P) & & \\
 & \swarrow & & \searrow & & \swarrow & \\
 H_0^{\text{BM}}(P) & \longrightarrow & H_0^{\text{BM}}(Q) & \longrightarrow & H_0^{\text{BM}}(Q, P) & \longrightarrow & 0
 \end{array}$$

### Relative integration

Take a  $C^1$  map  $\varphi: P \rightarrow Q$  of manifolds with corners. For any relative cochain  $(\xi, \eta)$  and relative chain  $(C, D)$ , the *relative integral* is

$$\int_{(C,D)} (\xi, \eta) := \int_C \xi + \int_D \eta.$$

**Theorem 40.9** (The relative Stokes’s theorem).

$$\int_{\partial} = \int d$$

for relative integrals, descending to pairings

$$\begin{aligned}
 H_k(Q, P) \times H^k(Q, P) &\rightarrow \mathbb{R}, \\
 H_k^{\text{BM}}(Q, P) \times H_c^k(Q, P) &\rightarrow \mathbb{R}.
 \end{aligned}$$

*Proof.*

$$\begin{aligned}
 \int_{\partial(C,D)} (\xi, \eta) &= \int_{(\partial C + \varphi_* D, -\partial D)} (\xi, \eta), \\
 &= \int_{\partial C + \varphi_* D} \xi - \int_{\partial D} \eta, \\
 &= \int_C d\xi + \int_D \varphi^* \xi - \int_D d\eta, \\
 &= \int_{(C,D)} (d\xi, \varphi^* \xi - d\eta), \\
 &= \int_{(C,D)} d(\xi, \eta).
 \end{aligned}$$



□

## Fubini

Take a fiber bundle map  $\varphi: P \rightarrow Q$ . Recall the horizontal boundary  $\partial^{\sim}P$  with attaching map  $\iota: \partial^{\sim}P \rightarrow P$  and relative cohomology short exact sequence

$$0 \rightarrow \Omega_{\partial^{\sim}P}^{\bullet}[-1] \rightarrow \Omega_{P, \partial^{\sim}P}^{\bullet} \rightarrow \Omega_P^{\bullet} \rightarrow 0.$$

The elements of  $\Omega_{P, \partial^{\sim}P}^p$  are pairs  $(\xi, \eta)$  with  $\xi$  a  $p$ -form on  $P$  and  $\eta$  a  $(p-1)$ -form on  $\partial^{\sim}P$ . The differential is  $d(\xi, \eta) = (d\xi, \iota^*\xi - d\eta)$ . So  $d(\xi, \eta) = 0$  just when  $\xi$  is closed, and exact on the horizontal boundary. By Fubini's theorem,

$$\varphi_*d\xi - d\varphi_*\xi = (-1)^{p+\dim P} \varphi_*\iota^*\xi.$$

So if  $d(\xi, \eta) = 0$  then

$$-d\varphi_*\xi = (-1)^{1+p+\dim P} \varphi_*d\eta.$$

So we can ask whether  $\varphi_*d\eta$  is exact, where here  $\varphi: \partial^{\sim}P \rightarrow Q$ . Apply Fubini again, but note that  $\partial^{\sim}\partial^{\sim}P = 0$ . We conclude that integration over the fibers maps

$$(\xi, \eta) \in \Omega_{P, \partial^{\sim}P} \rightarrow \varphi_*\xi + (-1)^{p+\dim P} \varphi_*\eta \in \Omega_Q$$

so that

$$d(\varphi_*\xi + (-1)^{p+\dim P} \varphi_*\eta) = \varphi_*d\xi + (-1)^{1+p+\dim P} \varphi_*(\iota^*\xi - d\eta),$$

a morphism of complexes giving a morphism of cohomology

$$H^{\bullet}(P, \partial^{\sim}P) \rightarrow H^{\bullet-k}(Q)$$

if  $k = \dim P - \dim Q$ .



## Chapter 41

### Čech–de Rham cohomology

We define the Čech–de Rham cohomology of a manifold, prove that it equals the de Rham cohomology, and explain how it aids computations.

#### Defining Čech–de Rham cohomology

To compute cohomology of complicated manifolds, essentially we cut them apart into pieces. Suppose we have an open cover of a manifold  $M$  by two open sets  $M_0$  and  $M_1$  and let  $M_{01} = M_0 \cap M_1$ . Take a triple  $(\omega_0, \omega_1, \omega_{01})$  of differential forms:  $\omega_0$  a  $p$ -form on  $M_0$ ,  $\omega_1$  a  $p$ -form on  $M_1$ , and  $\omega_{01}$  a  $(p-1)$ -form on  $M_{01}$ . (Careful: if  $p = 0$ , there are no  $(p-1)$ -forms, so we just let  $\omega_{01} = 0$  in some 0-dimensional vector space.) Define a map  $d$ , the *Čech–de Rham differential*, by

$$d(\omega_0, \omega_1, \omega_{01}) = (d\omega_0, d\omega_1, \omega_0 - \omega_1 - d\omega_{01}).$$

**41.1** Check that  $d^2 = 0$ .

So of course we define a *Čech–de Rham cocycle* to be a triple  $\omega = (\omega_0, \omega_1, \omega_{01})$  so that  $d\omega = 0$ , and a *Čech–de Rham coboundary* to be a triple  $\omega = (\omega_0, \omega_1, \omega_{01})$  so that  $\omega = d\eta$  for some triple  $\eta = (\eta_0, \eta_1, \eta_{01})$ , and a *Čech–de Rham cohomology class* to be a cocycle up to addition of a coboundary, and let  $\check{H}^\bullet(M)$ , the *Čech–de Rham cohomology* of  $M$ , be the space of all such equivalence classes.

**Proposition 41.1.** *The map  $\omega \mapsto (\omega, \omega, 0)$  is a morphism of complexes and the induced map  $H_{dR}^\bullet(M) \rightarrow \check{H}^\bullet(M)$  is an isomorphism of vector spaces.*

Henceforth we can use this isomorphism to identify these spaces, so we don't need the notation  $\check{H}^\bullet(M)$  anymore and we won't use it.

*Proof.* If  $d\omega = 0$ , then  $d(\omega, \omega, 0) = (0, 0, 0)$ . If  $\omega = d\eta$ , then  $(\omega, \omega, 0) = (d\eta, d\eta, \eta - \eta - 0) = d(\eta, \eta, 0)$ . So the map drops to cohomology. We need to prove that our map is injective and surjective on cohomology.

To prove that our map is surjective on cohomology, take some triple  $(\omega_0, \omega_1, \omega_{01})$  so that  $d = 0$  on it. Take a partition of unity  $f_0, f_1$  subordinate to the open cover  $M_0, M_1$ , so  $f_0 + f_1 = 1$  and  $f_0$  is supported on  $M_0$  and  $f_1$  on  $M_1$ . Let  $\omega = f_0\omega_0 + f_1\omega_1 - df_0 \wedge \omega_{01}$ . First check that this  $\omega$  is closed:

$$d\omega = df_0 \wedge \omega_0 + df_1 \wedge \omega_1 + df_0 \wedge d\omega_{01},$$

and now use  $f_1 = 1 - f_0$ :

$$\begin{aligned}
 &= df_0 \wedge \omega_0 + d(1 - f_0) \wedge \omega_1 + df_0 \wedge d\omega_{01}, \\
 &= df_0 \wedge (\omega_0 - \omega_1 + d\omega_{01}), \\
 &= df_0 \wedge 0, \\
 &= 0.
 \end{aligned}$$

Next, check that  $\omega$  maps to  $(\omega_0, \omega_1, \omega_{01})$ , at least in cohomology, so up to  $d$  of something. In other words, we need to check that  $(\omega, \omega, 0) = (\omega_0, \omega_1, \omega_{01}) + d(\eta_0, \eta_1, \eta_{01})$  for some choice of  $\eta_0, \eta_1, \eta_{01}$ , i.e. that  $\omega = \omega_0 + d\eta_0$  on  $M_0$ , and  $\omega = \omega_1 + d\eta_1$  on  $M_1$ , and  $\eta_1 - \eta_0 = d\eta_{01}$  on  $M_0 \cap M_1$ .

First, on  $M_0 - M_1$ ,  $f_0 = 1, f_1 = 0$ , so  $\omega = \omega_0$ . Similarly, on  $M_1 - M_0$ ,  $f_0 = 0, f_1 = 1$ , so  $\omega = \omega_1$ . Thirdly, on  $M_0 \cap M_1$ , since  $d(\omega_0, \omega_1, \omega_{01}) = 0$ , we know that  $\omega_1 = \omega_0 + d\omega_{01}$ , and so

$$\begin{aligned}
 \omega &= f_0\omega_0 + f_1\omega_1 - df_0 \wedge \omega_{01}, \\
 &= f_0\omega_0 + f_1(\omega_0 + d\omega_{01}) - df_0 \wedge \omega_{01}, \\
 &= f_0\omega_0 + f_1\omega_0 + f_1d\omega_{01} - df_0 \wedge \omega_{01}, \\
 &= \omega_0 + f_1d\omega_{01} - df_0 \wedge \omega_{01}, \\
 &= \omega_0 + f_1d\omega_{01} - d(1 - f_1)\omega_{01}, \\
 &= \omega_0 + f_1d\omega_{01} + df_1 \wedge \omega_{01}, \\
 &= \omega_0 + d(f_1\omega_{01}).
 \end{aligned}$$

Therefore on  $M_0$ ,  $\omega - \omega_0$  is everywhere exact, being either zero or  $d$  of  $f_1\omega_{01}$ . Swapping indices, the same analogue holds on  $M_1$ :  $\omega - \omega_1 = -d(f_0\omega_{01})$ . So  $\omega_1 - \omega_0 = d((f_1 - f_0)\omega_{01})$  on  $M_0 \cap M_1$  exact, so that our map is surjective on cohomology.

To prove that our map is injective on cohomology, take some closed differential form  $\omega$  and suppose that our map sends it to zero in Čech–de Rham cohomology. (We need to show that  $\omega$  was zero in cohomology already.) That means that  $(\omega, \omega, 0)$  is zero in Čech–de Rham cohomology, in other words that  $(\omega, \omega, 0) = d(\eta_0, \eta_1, \eta_{01})$  for some differential forms  $\eta_0, \eta_1, \eta_{01}$ . Spelling out what  $d$  is, we find  $\omega = d\eta_0$  on  $M_0$ , and  $\omega = d\eta_1$  on  $M_1$ , and  $\eta_1 = \eta_0 + d\eta_{01}$  on  $M_{01}$  for some  $\eta_0, \eta_1, \eta_{01}$ . Note that  $\eta_{01}$  is defined on  $M_{01}$ , but that since, everywhere outside  $M_1$ ,  $f_1 = 0$ , we can extend  $f_1\eta_{01}$  to be zero outside  $M_1$ . So we let  $\xi = \eta_0 - d(f_1\eta_{01}) = \eta_1 + d(f_0\eta_{01})$ , and check that these two expressions for  $\xi$  are actually equal on  $M_{01}$ :

$$\begin{aligned}
 (\eta_0 - d(f_1\eta_{01})) - (\eta_1 + d(f_0\eta_{01})) &= \eta_0 - \eta_1 - d((f_1 + f_0)\eta_{01}), \\
 &= \eta_0 - \eta_1 - d\eta_{01}, \\
 &= 0.
 \end{aligned}$$

So our map is injective. □

It will take some effort before we can compute cohomology using Čech–de Rham cocycles.

### A short exact sequence

Let  $\Omega^\bullet = \Omega^\bullet(M)$ ,  $\Omega_0^\bullet = \Omega^\bullet(M_0)$ ,  $\Omega_1^\bullet = \Omega^\bullet(M_1)$ ,  $\Omega_{01}^\bullet = \Omega^\bullet(M_{01})$ ,  $\check{\Omega}^\bullet = \Omega_0^\bullet \oplus \Omega_1^\bullet \oplus \Omega_{01}^{\bullet-1}$  and define a short exact sequence by

$$0 \rightarrow \Omega^\bullet \rightarrow \check{\Omega}^\bullet \rightarrow \Omega_{01}^\bullet \oplus \Omega_{01}^{\bullet-1} \rightarrow 0,$$

with maps  $\omega \mapsto (\omega, \omega, 0)$ , and

$$(\omega_0, \omega_1, \omega_{01}) \mapsto (\omega_0 - \omega_1, \omega_{01})$$

and differentials  $d: \Omega^\bullet \rightarrow \Omega^\bullet$  just the usual exterior derivative, and  $d: \check{\Omega}^\bullet \rightarrow \check{\Omega}^\bullet$  the map

$$d(\omega_0, \omega_1, \omega_{01}) = (d\omega_0, d\omega_1, \omega_0 - \omega_1 - d\omega_{01}),$$

and

$$d: \Omega_{01}^\bullet \oplus \Omega_{01}^{\bullet-1} \rightarrow \Omega_{01}^\bullet \oplus \Omega_{01}^{\bullet-1}$$

the map

$$d(\alpha_{01}, \beta_{01}) = (d\alpha_{01}, \alpha_{01} - d\beta_{01}).$$

**41.2** Prove that  $\Omega_{01}^\bullet \oplus \Omega_{01}^{\bullet-1}$  has trivial cohomology.

If you write out the long exact sequence in cohomology, you see that de Rham is isomorphic to Čech–de Rham. Our earlier proof of isomorphism of de Rham and Čech–de Rham cohomologies is primitive and complicated compared with the ease of applying short exact sequences.

**41.3** In the same notation, push each compactly supported Čech–de Rham-cochain  $(\omega_0, \omega_1, \omega_{01})$  forward to  $\omega_0 + \omega_1$ , a differential form on  $M$ , compactly supported. Prove that this descends to an isomorphism between the compactly supported Čech–de Rham-cohomology and the compactly supported cohomology.

## Mayer–Vietoris

Take the short exact sequence

$$0 \longrightarrow \Omega_{01} \longrightarrow \check{\Omega} \longrightarrow \Omega_0 \oplus \Omega_1 \longrightarrow 0$$

whose maps are  $\omega_{01} \mapsto (0, 0, -\omega_{01})$  and  $(\omega_0, \omega_1, \omega_{01}) \mapsto (\omega_0, \omega_1)$ . The long exact sequence in cohomology is just the Mayer–Vietoris sequence.

**41.4** By induction, find the de Rham cohomology of  $\mathbb{R}^n$  minus  $k$  points for any  $k = 0, 1, \dots$

The same exact sequence holds in compactly supported cohomology, and so the Mayer–Vietoris sequence.

**Theorem 41.2.** *If a manifold is compact (or more generally, admits a finite good cover) then its de Rham cohomology is finite dimensional.*

Its compactly supported de Rham cohomology is of course equal to its de Rham cohomology.

*Proof.* If the good cover consists of a single open set, we saw in chapter 38 on page 297 that the cohomology is zero except for  $H_{\text{dR}}^0(M) = \mathbb{R}$ . But induction, suppose that the result is true for any manifold with a good cover by fewer than  $N$  open sets, and that  $M$  has a fine cover with  $N$  open sets  $U_1, U_2, \dots, U_N$ . Let  $M_0 := U_1 \cup U_2 \cup \dots \cup U_{N-1}$  and  $M_1 := U_N$ . Then  $M_{01}$  is a finite disjoint union of contractible open sets, so has

finite dimensional cohomology. The long exact sequence in cohomology associated the short exact sequence above gives

$$\dots \longrightarrow H_{01}^\bullet \longrightarrow H^\bullet \longrightarrow H_0^\bullet \oplus H_1^\bullet \longrightarrow \dots$$

where  $H_{01}^\bullet, H^\bullet, H_0^\bullet, H_1^\bullet$  are the de Rham cohomologies of  $M_{01}, M, M_0, M_1$ . By induction on the number of open sets in a good open cover, each except perhaps  $H^\bullet$  is finite dimensional. But then  $H^\bullet \rightarrow H_0^\bullet \oplus H_1^\bullet$  has finite dimensional image, and finite dimensional kernel  $H_{01}^\bullet$ . Similar steps work with compactly supported forms.  $\square$

### Čech–de Rham integration

Picture a sphere  $M$ . Let  $C_0$  be the closed top hemisphere,  $C_1$  the closed bottom hemisphere, and  $C_{01}$  the equator, oriented as  $C_{01} = \partial C_0$ . Define  $\int_M$  on Čech–de Rham cocycles of top degree by

$$\int_M (\omega_0, \omega_1, \omega_{01}) = \int_{C_0} \omega_0 + \int_{C_1} \omega_1 + \int_{C_{01}} \omega_{01}.$$

More generally, for any manifold  $M$  with corners, suppose that we have an open cover by two open sets  $M_0, M_1$ . A *Čech–de Rham chain* of degree  $p$  is a triple  $(C_0, C_1, C_{01})$  of chains of degrees  $p, p$  and  $p-1$  in  $M_0, M_1$  and  $M_{01}$ . Its *boundary* is

$$\partial(C_0, C_1, C_{01}) = (\partial C_0 + C_{01}, \partial C_1 - C_{01}, -\partial C_{01}).$$

**41.5** Prove that  $\partial^2 = 0$ .

A *Čech–de Rham cycle* is a pair of chains  $C_0$  in  $M_0$  and  $C_1$  in  $M_1$  so that  $\partial C_0 = -\partial C_1$  and the chain  $C_{01} = \partial C_0$  is a chain in  $M_{01}$ , i.e.  $\partial = 0$ . Define

$$\int_C (\omega_0, \omega_1, \omega_{01}) = \int_{C_0} f^* \omega_0 + \int_{C_1} f^* \omega_1 + \int_{C_{01}} f^* \omega_{01}.$$

**41.6** Prove Stokes's theorem:

$$\int_{\partial C} \omega = \int_C d\omega$$

for Čech–de Rham cochains over Čech–de Rham chains.

**Lemma 41.3.** *If  $\omega = (\omega_0, \omega_1, \omega_{01})$  is any Čech–de Rham cocycle and  $C = (C_0, C_1, C_{01})$  is any Čech–de Rham cycle then  $\int_C \omega$  is independent of the choice of splitting of  $C$  into  $C_0$  and  $C_1$ , depending only on the sum  $C_0 + C_1$ . If  $\omega$  is exact, then  $0 = \int_C \omega$ . Under the isomorphism of de Rham and Čech–de Rham cohomology, the new  $\int_C$  agrees with the usual integration of differential forms over chains.*

*Proof.* First, if  $\omega$  is exact, say  $\omega = d(\eta_0, \eta_1, \eta_{01})$ , then Stokes's theorem for differential forms says that

$$\begin{aligned} \int_C \omega &= \int_{C_0} \omega_0 + \int_{C_1} \omega_1 + \int_{C_{01}} \omega_{01} \\ &= \int_{\partial C_0} \eta_0 - \int_{\partial C_0} \eta_1 + \int_{\partial C_0} \eta_0 - \eta_1 - d\eta_{01} \\ &= 0. \end{aligned}$$

Now that we know that  $\int_C$  vanishes on exact Čech–de Rham cocycles, the value of  $\int_C$  on a Čech–de Rham cocycle depends only on its cohomology class. But the Čech–de Rham cohomology is isomorphic to the de Rham cohomology, so that each Čech–de Rham cocycle is cohomologous to one of the form  $\omega = (\xi, \xi, 0)$  for some smooth differential form  $\xi$ , so that

$$\int_C \omega = \int_{C_0} \xi + \int_{C_1} \xi + \int_{C_{01}} 0 = \int_C \xi$$

just gives the usual integral as a differential form.  $\square$

### Relative Čech–de Rham cohomology

We often study groups by studying convenient quotient groups. Take a manifold  $M$  with corners and an open cover  $M_0, M_1$  of  $M$ . We want to construct cohomology, but we want to kill off all of the cocycles inside  $M_0$ , quotienting them out. A cochain *relative to  $M_0$*  is a cochain of the form  $\omega = (0, \omega_1, \omega_{01})$ . Since  $d$  takes these relative cochains to one another, we can define the *cohomology relative to  $M_0$*  to be the kernel of  $d$  on relative cochains, modulo the image of  $d$  on relative cochains, which we denote by  $H^p(M; M_0)$ . Given a chain  $C$  in  $M_1$  with boundary  $\partial C$  a chain in  $M_{01}$  and a relative cocycle  $\omega = (0, \omega_1, \omega_{01})$ , define

$$\int_C \omega = \int_C \omega_1 - \int_{\partial C} \omega_{01}.$$

Define a map  $(0, \omega_1, \omega_{01}) \mapsto \omega_1$ , taking  $H^\bullet(M; M_0) \rightarrow H_{\text{dR}}^\bullet(M_1)$ .

**41.7** Prove that every closed  $\omega = (0, \omega_1, \omega_{01})$  in the kernel of this map is cohomologous to something of the form  $\omega = (0, 0, \omega_{01})$  with  $\omega_{01}$  closed, uniquely up to replacing  $\omega_{01}$  by  $\omega_{01} + d\eta_{01}$  for any form  $\eta_{01}$ . Use this to identify an isomorphism of the kernel of our map with  $H_{\text{dR}}^\bullet(M_{01})$ . Finally, construct a “long exact sequence”

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(M; M_0) & \longrightarrow & H_{\text{dR}}^0(M_1) & \longrightarrow & H_{\text{dR}}^0(M_{01}) \\ & & \searrow & & \searrow & & \searrow \\ & & H^1(M; M_0) & \longrightarrow & H_{\text{dR}}^1(M_1) & \longrightarrow & H_{\text{dR}}^1(M_{01}) \\ & & \searrow & & \searrow & & \searrow \\ & & H^2(M; M_0) & \longrightarrow & H_{\text{dR}}^2(M_1) & \longrightarrow & H_{\text{dR}}^2(M_{01}) \\ & & \searrow & & \searrow & & \searrow \\ & & H^3(M; M_0) & \longrightarrow & H_{\text{dR}}^3(M_1) & \longrightarrow & H_{\text{dR}}^3(M_{01}) \\ & & \searrow & & \searrow & & \searrow \\ & & H^4(M; M_0) & \longrightarrow & \cdots & & \end{array}$$

of linear maps, i.e. a sequence of linear maps so that the image of each map is the kernel of the next.

Take  $M = \mathbb{R}^n$ ,  $M_0 = \mathbb{R}^n - \{0\}$ ,  $M_1 = \mathbb{R}^n$  so  $M_{01} = M_0$ . Looking at the start of the long exact sequence, we see right away that  $H^0(\mathbb{R}^n; \mathbb{R}^n - \{0\}) = 0$ . Consider the higher degree terms. The long exact sequence above has zeroes in all of the  $H_{\text{dR}}^p(M_1)$  terms for  $p > 0$ , so that it forces  $H_{\text{dR}}^p(M_{01}) = H^{p+1}(\mathbb{R}^n; M_0)$ . But  $M_{01}$  is an annulus, so has the cohomology of a sphere:  $H_{\text{dR}}^p(S^{n-1}) = H^{p+1}(\mathbb{R}^n; \mathbb{R}^n - 0)$ . So finally

$$H^p(\mathbb{R}^n; \mathbb{R}^n - 0) = \begin{cases} 0, & \text{if } p < n, \\ \mathbb{R}, & \text{if } p = n. \end{cases}$$

### The Künneth formula

**Theorem 41.4** (The Künneth formula). *Take manifolds  $P$  and  $Q$ . Let  $p: (x, y) \in P \times Q \mapsto x \in P$  and  $q: (x, y) \in P \times Q \mapsto y \in Q$ . Define a map*

$$\xi \otimes \eta \in \Omega^\bullet(P) \otimes \Omega^\bullet(Q) \mapsto (p^*\xi) \wedge (q^*\eta) \in \Omega^\bullet(P \times Q).$$

*This map descends to an isomorphism on cohomology:*

$$H_{\text{dR}}^\bullet(P) \otimes H_{\text{dR}}^\bullet(Q) = H_{\text{dR}}^\bullet(P \times Q).$$

*Proof.* Clearly the map takes tensor products of closed forms to closed forms, and takes tensor products of exact forms to exact forms, so descends to cohomology

$$\xi \otimes \eta \in H_{\text{dR}}^\bullet(P) \otimes H_{\text{dR}}^\bullet(Q) \mapsto (p^*\xi) \wedge (q^*\eta) \in H_{\text{dR}}^\bullet(P \times Q).$$

Every cycle in  $P \times Q$  is, by definition, a formal sum of maps of compact oriented manifolds with corners into  $P \times Q$ , and so a product of a map into  $P$  and a map into  $Q$ . So the integral of some such  $p^*\xi \wedge q^*\eta$  vanishes over all cycles just when  $\xi$  and  $\eta$  separately do. So the map on cohomology is injective.

Suppose that we write  $P$  as a union  $P = P_0 \cup P_1$  of open sets and let  $P_{01} := P_0 \cap P_1$ . Apply Mayer–Vietoris:

$$\begin{array}{ccccccc} \dots \rightarrow H_{\text{dR}}^\bullet(P_{01}) \otimes H_{\text{dR}}^\bullet(Q) & \rightarrow & H_{\text{dR}}^\bullet(P) \otimes H_{\text{dR}}^\bullet(Q) & \rightarrow & (H_{\text{dR}}^\bullet(P_0) \oplus H_{\text{dR}}^\bullet(P_1)) \otimes H_{\text{dR}}^\bullet(Q) & \rightarrow & \dots \\ & \downarrow & & \downarrow & & \downarrow & \\ \dots \longrightarrow H_{\text{dR}}^\bullet(P_{01} \times Q) & \longrightarrow & H_{\text{dR}}^\bullet(P \times Q) & \longrightarrow & H_{\text{dR}}^\bullet(P_0 \times Q) \oplus H_{\text{dR}}^\bullet(P_1 \times Q) & \longrightarrow & \dots \end{array}$$

As long as we can prove our result for  $P_0$ ,  $P_1$  and  $P_{01}$  separately, we will get the result for  $P$ : the isomorphisms of the first and last column easily imply an isomorphism of the middle column.

Take a good cover  $\{P_a\}$  of  $P$ , and follow the argument before theorem 40.2 on page 329 to see that we can write  $P = P_0 \cup P_1$  where  $P_0$  is a union of relatively compact open sets  $P_{0a}$ , with disjoint closures, each a finite union of open sets from a good cover; and the same holds for  $P_1$ . Similarly,  $P_{01} := P_0 \cap P_1$  is union of relatively compact open sets  $P_{01a}$ , with disjoint closures, each a finite union of open sets from a good cover. So

$$\begin{aligned} H_{\text{dR}}^\bullet(P_0) &= \bigoplus H_{\text{dR}}^\bullet(P_{0a}), \\ H_{\text{dR}}^\bullet(P_1) &= \bigoplus H_{\text{dR}}^\bullet(P_{1a}), \\ H_{\text{dR}}^\bullet(P_{01}) &= \bigoplus H_{\text{dR}}^\bullet(P_{01a}), \end{aligned}$$



each a (possibly infinite) sum of finite dimension vector spaces. The same is true if we change  $P_0, P_1, P_{01}$  to  $P_0 \times Q, P_1 \times Q, P_{01} \times Q$ . But then, we need only get the result for each  $P_{0a}, P_{1a}$  and  $P_{01a}$  separately. Apply Mayer-Vietoris repeatedly to reduce the problem down to one  $P_a$  from our good cover:  $H_{\text{dR}}^\bullet(P_a \times Q) = H_{\text{dR}}^\bullet(Q)$ , by retracting (see lemma 38.1 on page 299).  $\square$

## Spheres and Lie groups

**Theorem 41.5.** *Any connected Lie group has all of its positive degree cohomology generated by odd degree closed differential forms.*

*Proof.* Suppose that  $G$  is a connected Lie group, and let  $m: G \times G \rightarrow G$  be the multiplication  $m(g, h) := gh$ . Let  $i: g \in G \mapsto (g, 1) \in G \times G$ , and  $j: g \in G \mapsto (1, g) \in G \times G$ . Then  $m \circ i$  and  $m \circ j$  are the identity map on  $G$ , so on differential forms on  $G$ ,  $i^*m^* = 1$  is the identity map, and  $j^*m^* = 1$ . Let  $p: (g, h) \in G \times G \rightarrow g \in G$  and  $q: (g, h) \in G \times G \rightarrow h \in G$ . So  $p \circ i$  and  $q \circ j$  are the identity map as well:  $i^*p^* = 1$  and  $j^*q^* = 1$ . Also  $p \circ j(g) = 1$  for all  $g$ , a constant map, so  $j^*p^* = 0$  and  $i^*q^* = 0$ . By the Künneth formula (theorem 41.4 on the preceding page), all closed differential forms on  $G \times G$  are sums of forms  $p^*\xi, q^*\eta$  in cohomology. Take a closed differential form  $\omega$  on  $G$ , and write out

$$m^*\omega = p^*\xi + q^*\eta + \sum p^*\xi_i \wedge q^*\eta_j,$$

and then apply  $i^*$ :  $\omega = \xi$ , and  $j^*$ :  $\omega = \eta$ . So

$$m^*\omega = p^*\omega + q^*\omega + \sum p^*\xi_i \wedge q^*\eta_j$$

in cohomology.

Let  $E^\bullet$  be the quotient ring of the cohomology ring by its odd degree elements. So in  $E^\bullet$ , the odd degree elements become zero, as does the product of any element with any odd degree element. In the Künneth formula, for manifolds  $P$  and  $Q$ , the terms in the tensor product that have any odd degree factor drop, and then we drop all even degree terms which have an odd degree factor, giving a Künneth formula for the quotient:

$$E^\bullet(P \times Q) = E^\bullet(P) \otimes E^\bullet(Q).$$

Suppose that  $E^\bullet(G) \neq 0$ , i.e. the odd degree cohomology does not generate all of the cohomology. Pick some  $\omega$  of smallest degree so that  $\omega \neq 0$  in  $E^\bullet(G)$ . Then the “mixed terms”  $p^*\xi_i \wedge q^*\eta_j$  have the degree of  $\omega$ , i.e. each of  $\xi_i$  and  $\eta_j$  are forms of lower degree than  $\omega$ . Since  $\omega$  is of smallest degree with  $\omega \neq 0$  in  $E^\bullet$ , these mixed terms are zero in  $E^\bullet$ :

$$m^*\omega = p^*\omega + q^*\omega \neq 0 \in E^\bullet(G \times G).$$

Let  $k$  be the smallest positive integer for which  $\omega^k = 0$  in  $E^\bullet(G)$ . Then in  $E^\bullet(G \times G)$ ,

$$\begin{aligned} 0 &= m^*\omega^k, \\ &= (p^*\omega + q^*\omega) \wedge (p^*\omega + q^*\omega) \wedge \cdots \wedge (p^*\omega + q^*\omega), \\ &= \binom{k}{1} p^*\omega^{k-1} \wedge q^*\omega + \cdots + \binom{k}{k-1} p^*\omega \wedge q^*\omega^{k-1}. \end{aligned}$$

By the Künneth formula, this is impossible as all terms come from nonzero tensor products in different tensor product spaces. Therefore  $E^\bullet(G) = 0$ , i.e. the cohomology of  $G$  is generated in odd degrees.  $\square$

If  $M = S^n$ , then  $M$  has cohomology in degree  $n$ , so  $n$  is odd or else  $M$  cannot be a Lie group.

Since  $S^3$  is a Lie group, so is  $S^3 \times S^3$ , which has nonzero cohomology in degree 6, but generated in degree 3.

On  $S^3 \times S^4 \times S^8$  cohomology is generated precisely in degrees 3, 4, 8, not all in odd degree:  $M$  cannot be a Lie group. A product of spheres cannot be a Lie group unless all of the spheres have odd dimension.

## Notes

We have seen that  $S^1$  and  $S^3$  are Lie groups; these are the only spheres which are Lie groups.

## Chapter 42

### Currents

We define differential forms with “mild” singularities, called currents.

#### Idea

Integration over an oriented manifold is an operation that eats differential forms and splits out numbers:  $\int_M: \Omega^{\text{top}}(M) \rightarrow \mathbb{R}$ . There are other operations like this:

- a. Take a chain  $C$  and then  $\int_C: \Omega^*(M) \rightarrow \mathbb{R}$  (define integration of a  $p$ -form over a  $q$ -chain to be zero if  $p \neq q$ ).
- b. Take a compact oriented manifold with corners  $P$  and a continuously differentiable map  $f: P \rightarrow M$ , and a continuous  $k$ -form  $\xi$  on  $P$  and consider the map  $\omega \mapsto \int_P \xi \wedge f^*\omega, \Omega^*(M) \rightarrow \mathbb{R}$ .
- c. If  $\eta$  is a continuous differential form on an oriented manifold  $P$  with corners, perhaps not a compact manifold, and  $f: P \rightarrow M$  is a smooth map, proper on the support of  $\eta$ , we get an operation  $\omega \mapsto \int_P f^*\omega \wedge \eta$ , defined as long as  $\omega$  is chosen to be a compactly supported form on  $M$ .
- d. Take a vector  $v$  at a point of  $M$ , and map 1-forms  $\omega$  on  $M$  to  $\omega(v) \in \mathbb{R}$ .

All of these are operations eating forms  $\omega$  on  $M$  and giving numbers.

#### Definition

A *test form* on a manifold with corners is a smooth differential form with compact support. A sequence  $\omega_1, \omega_2, \dots$  of test forms on  $\mathbb{R}^n$  *converges* to a test form  $\omega$ , i.e.  $\omega_j \rightarrow \omega$ , if there is a compact set  $K \subset \mathbb{R}^n$  so that all of the differential forms  $\omega_j$  are supported in  $K$  and, if we write each form as  $\omega_j = \sum_I a_{jI}(x) dx_I$ , and  $\omega = \sum_I a_I(x)$ , the functions  $a_{jI}(x)$  converge uniformly to  $a_I(x)$  and their derivatives of all orders converge uniformly to the corresponding derivatives of  $a_I(x)$ . We can say this without invoking local coordinates by saying that, for any finite collection of smooth vector fields  $X = (X_1, X_2, \dots, X_k)$ , if we write  $\mathcal{L}_X$  to mean  $\mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}$ , then  $\mathcal{L}_X \omega_j \rightarrow \mathcal{L}_X \omega$ .

A *current* of dimension  $p$  (also said to be of *degree*  $n - p$ ) on a manifold  $M$  is a linear map  $u$  associating to each test  $p$ -form  $\omega$  a number, which we denote  $\int \omega \wedge u$ , so that  $u$  takes convergent sequences of forms to convergent sequences of numbers. All of our example operations are currents. If  $u$  has degree  $n - p$ , we define  $\int \omega \wedge u$  to mean zero if  $\omega$  is *not* a  $p$ -form. Write  $\int u \wedge \omega$  to mean  $(-1)^{p+q} \int \omega \wedge u$  where  $p$  is the degree of  $\omega$  and  $q$  the degree of  $u$ .

A  $p$ -chain  $C$  yields a current  $u = \int_C$  of dimension  $p$ .

On an oriented manifold  $M$ , any differential form  $\eta$  of degree  $n - p$  yields a current  $u = u_\eta$  by  $\int \omega \wedge u_\eta = \int_M \omega \wedge \eta$  of degree  $n - p$ . If a current  $u$  arises as  $u = u_\xi = u_\eta$  for two continuous forms  $\xi$  and  $\eta$ , then integrating against bumps tells us that  $\xi = \eta$ . Therefore we can just identify  $u_\eta$  with  $\eta$  itself. We think of any current as if it was a differential form with mild singularities, i.e. as if  $u$  was  $u_\eta$  for some  $\eta$ .

Define a current  $\delta$  on  $\mathbb{R}^n$  by writing any  $n$ -form  $\omega$  as  $\omega = f dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ , and then taking  $\int \omega \wedge \delta = f(0)$ .

If we take some point  $x_0 \in M$  and some vectors  $v_1, v_2, \dots, v_p \in T_{x_0}M$  and define a current  $u$  by  $\int \omega \wedge u = \omega(v_1, v_2, \dots, v_p)$ , this is called a *Dirac current* and behaves like a differential form which vanishes except at  $x_0$ , and is infinitely large at  $x_0$ .

For any compactly supported smooth function  $f: \mathbb{R} \rightarrow \mathbb{R}$ , let  $u$  be the current

$$\int f(x) dx \wedge u = \lim_{\varepsilon \rightarrow 0} \int_{|x| > \varepsilon} \frac{f(x)}{x} dx,$$

the *principal value integral* which is usually denoted

$$\text{PV} \int \frac{f(x) dx}{x}.$$

Take a sequence of points  $x_1, x_2, \dots \rightarrow 1$  with  $0 < x_j < 1$  for all  $j$ , and let

$$\int f \wedge u = \sum_j \left. \frac{d^j f}{dx^j} \right|_{x_j},$$

a current on  $(0, 1)$ .

If  $f: P \rightarrow M$  is a proper smooth map, and  $P$  is an oriented manifold with corners, the operation  $\omega \mapsto \int_P f^* \omega$  is a current.

### Convergence and order

A sequence of currents *converges* when the values of the currents converge on any given smooth compactly supported form. The *support* of a current  $u$  is the smallest closed set  $A$  so that  $\int \omega \wedge u = 0$  for any form  $\omega$  supported outside  $A$ .

**42.1** Suppose that a current  $u$  on a manifold with corners  $M$  has compact support  $A$ . Take any smooth function  $f$  equal to 1 on an open set containing  $A$  and vanishing outside some compact set containing  $A$ . Define an operation  $v$  on all smooth forms  $\omega$  by  $\int \omega \wedge v = \int f\omega \wedge u$ . Prove that  $v$  is independent of the choice of function  $f$ , and so we can treat any compactly supported current as an operation on all smooth forms.

**42.2** Say that a sequence of smooth forms, perhaps with noncompact support, converges if it converges uniformly on any compact set with any number of derivatives. Write any smooth form as a limit of a sequence of compactly supported smooth forms. Prove that a current  $u$  has compact support just when  $u$  extends to a continuous linear map taking all smooth forms  $\omega$  to numbers  $\int \omega \wedge u$ .

Suppose that our manifold  $M$  is an open set in  $\mathbb{R}^n$ . For any tuple

$$a = (a_1, a_2, \dots, a_n),$$

let  $\partial^a = \frac{\partial}{\partial x_1^{a_1}} \dots \frac{\partial}{\partial x_n^{a_n}}$  and let  $|a| = a_1 + a_2 + \dots + a_n$ . If  $\omega = \sum h_I(x) dx_i$  is a form on  $M$ , we say  $\omega$  is  $C^k$  if each  $h_I(x)$  is  $k$  times continuously differentiable, and write  $|\omega|_{C^k}$  to mean the maximum of the expressions  $|\partial^a h_I(x)|$  over  $U$ , for  $|a| \leq k$ , and call it the  $C^k$  norm of  $\omega$ . By the chain rule, the  $C^k$  norm of a differential form with compact support changes only by some bounded factor under diffeomorphism. Therefore, while the  $C^k$  norm of a differential form on a manifold is not actually well defined, the  $C^k$  norm of a compactly supported differential form is defined up to a bounded factor.

A current  $u$  on an open set  $U \subseteq \mathbb{R}^n$  has *order  $k$*  on a subset  $K \subset U$  if there is a constant  $c > 0$  so that, for any compactly supported smooth form  $\omega$  with support contained in  $U$ ,  $|\int \omega \wedge u| \leq c|\omega|_{C^k}$ . Clearly this definition extends to any current on any manifold, since it only depends on the  $C^k$  norm up to a bounded factor.

**Lemma 42.1.** *Every current has finite order on any compact set. Conversely, every linear function on test forms which has finite order is a current.*

*Proof.* Clearly a linear function of finite order is a current, because it takes suitable convergent sequences of forms to convergent sequences of numbers. Conversely, if a current  $u$  had infinite order on some compact set  $K$ , we could pick compactly supported smooth forms  $\omega_n$  supported in  $K$  so that  $\int \omega_n \wedge u / |\omega_n|_{C^k} \rightarrow \infty$ . (Careful: the  $C^k$ -norm is not really defined, but defined only up to a bounded factor independent of  $n$ .) Rescaling, we can ensure that  $\int \omega_n \wedge u = 1$  and so  $|\omega_n|_{C^k} \rightarrow 0$ . So  $\omega_n$  converge to 0 in  $K$  with all derivatives, but  $\int \omega_n \wedge u$  does not converge to 0.  $\square$

## Operations on currents

If  $f: P \rightarrow M$  is a proper smooth map from an oriented manifold with corners, we can define the current  $u = \int_P f^*$ , and naturally define  $\partial u$  to mean  $\int_{\partial P} f^*$ . By Stokes's theorem,  $\int \omega \wedge \partial u = \int_{\partial P} f^* \omega = \int_P df^* \omega = \int_P f^* d\omega = \int d\omega \wedge u$ . Therefore for any current  $u$  it is natural to define  $\partial u$  to be the current  $\int \omega \wedge \partial u = \int d\omega \wedge u$ . For any chain  $C$  if  $u = \int_C$  then  $\partial u = \int_{\partial C}$ . We can't recover a chain  $C$  from  $\int_C$ ; for example, if  $g: P \rightarrow P$  is any orientation preserving diffeomorphism, and  $f: P \rightarrow M$  is a proper smooth map, then  $\int_P f^* = \int_P g^* f^*$ .

Similarly, if  $\eta$  is a continuous  $q$ -form on an oriented manifold with corners  $X$ , and  $f: X \rightarrow M$  is a smooth map, proper on the support of  $\eta$ , define a current  $u = u_{\eta, f}$  by  $\int \omega \wedge u = \int_M f^* \omega \wedge \eta$ . A *long chain* is a linear combination of finitely many currents

of this form, generalizing the notion of chains to allow noncompact manifolds with corners. For any smooth  $p$ -form  $\omega$ ,  $d(f^*\omega \wedge \eta) = df^*\omega \wedge \eta + (-1)^p f^*\omega \wedge d\eta$ . Stokes's theorem then says that

$$\int_{\partial X} f^*\omega \wedge \eta = \int_X (df^*\omega) \wedge \eta + (-1)^p \int_X f^*\omega \wedge d\eta.$$

If  $\eta$  vanishes near  $\partial X$ , then the left hand side vanishes, so that

$$\int_X (f^*\omega) \wedge d\eta = (-1)^{p+1} \int_X \omega \wedge d\eta,$$

i.e.  $u_{d\eta, f} = (-1)^{p+1} u_\eta \circ d$ . We define an operator  $(-1)^N$  which is  $-1$  on odd degree forms and  $1$  on even degree forms, so that  $u_{d\eta, f} = -u_{\eta, f} \circ d \circ (-1)^N$ . Therefore, for any current  $u$  of degree  $p$ , it is natural to define  $du$  to be the current  $du = -u \circ d \circ (-1)^N$ , i.e.  $\int \omega \wedge du = (-1)^{p+1} \int d\omega \wedge u$ . More generally, if we have two operations  $T$  and  $T'$  on test forms, and if  $\int_M (T\omega) \wedge \eta = \int_M \omega \wedge (T'\eta)$  for any test forms  $\omega$  and  $\eta$ , define an operation  $Tu$  on currents by  $\int \omega \wedge Tu = \int (T'\omega) \wedge u$ . Define  $\partial$  on currents by  $\int \omega \wedge \partial u = \int d\omega \wedge u$ , which agrees with the usual notation if  $u = \int_C f^*$  for some chain  $C$ .

If  $\eta$  is a smooth differential form and  $u$  a current, we define  $\xi \wedge u$  by  $\int \omega \wedge (\xi \wedge u) = \int (\omega \wedge \xi) \wedge u$ , which naturally generalises the case when  $u = u_{\eta, f}$ :

$$\int \omega \wedge u_{f^*\xi \wedge \eta, f} = \int f^*\omega \wedge f^*\xi \wedge \eta = \int \omega \wedge \xi \wedge u_{\eta, f}.$$

Similarly, if  $\xi$  is a smooth form of degree  $|\xi|$  and  $u$  is a current of degree  $|u|$ , let  $u \wedge \eta = (-1)^{|\eta|+|u|} \eta \wedge u$ . In particular, for any smooth function  $f$ ,  $fu$  means  $f \wedge u$ .

**42.3** Prove that every current on  $\mathbb{R}^n$  is  $u = \sum u_I dx_I$  where  $u_I$  is a current of degree 0.

**42.4** Prove that  $d^2 = 0$  and  $\partial^2 = 0$  on currents.

If  $X$  is a smooth vector field on an oriented manifold  $M$  with corners, supposing that  $M$  has dimension  $p+q$  and  $\omega$  is  $p$ -form and  $\eta$  a  $q$ -form on  $M$ , both smooth with compact support, vanishing near  $\partial M$ , then

$$\mathcal{L}_X(\omega \wedge \eta) = X \lrcorner d(\omega \wedge \eta) + d(X \lrcorner (\omega \wedge \eta)) = d(X \lrcorner (\omega \wedge \eta)),$$

so if integrate both sides,

$$\int \mathcal{L}_X(\omega \wedge \eta) = 0.$$

But we also have

$$\mathcal{L}_X(\omega \wedge \eta) = (\mathcal{L}_X \omega) \wedge \eta + \omega \wedge (\mathcal{L}_X \eta).$$

Therefore

$$0 = \int (\mathcal{L}_X \omega) \wedge \eta + \omega \wedge (\mathcal{L}_X \eta),$$

We therefore define  $\mathcal{L}_X u$  on a current  $u$  by

$$\int \omega \wedge \mathcal{L}_X u = - \int \mathcal{L}_X \omega \wedge u.$$

**42.5** Prove that  $\mathcal{L}_X u = X \lrcorner du + d(X \lrcorner u)$  for any current  $u$  and any smooth vector field  $X$ .

**42.6** Suppose that  $u$  is a current of degree  $|u|$  and  $\xi$  is a smooth form. Prove that  $d(u \wedge \xi) = du \wedge \xi + (-1)^{|u|} u \wedge d\xi$ . Prove that for any smooth vector field  $X$ ,  $\mathcal{L}_X(u \wedge \xi) = (\mathcal{L}_X u) \wedge \xi + u \wedge \mathcal{L}_X \xi$ .

If  $\eta$  is a continuously differentiable form on a manifold  $M$  vanishing near  $\partial M$  and  $u = u_\eta$  then  $du_\eta = u_{d\eta}$  by definition, and similarly  $\mathcal{L}_X u_\eta = u_{\mathcal{L}_X \eta}$  again by definition.

**Lemma 42.2.** *Suppose that  $\eta$  is a continuous differential form. Then  $\eta$  is continuously differentiable just when, near each point  $x$ , for any smooth vector field  $X$ , the current  $\mathcal{L}_X u_\eta$  is a continuous differential form.*

*Proof.* We start with a toy example. Suppose that  $\eta$  is a 0-form, a continuous function, say  $g$  and suppose that  $X = \frac{\partial}{\partial x_1}$ . Then  $\mathcal{L}_X g = \frac{\partial g}{\partial x_1}$ . Define a function

$$h(x) = g(x) + \int_0^{x_1} \frac{\partial g}{\partial x_1}(t, x_2, x_3, \dots, x_n) dt.$$

Of course we use the derivative as a current in the integral, but by hypothesis this is a continuous function. The reader can easily check that  $h = g$  as currents.

Next consider the general case. If we have any smooth partition of unity, say  $f_a$ , then  $u_a = f_a u$  is continuous, and if each  $u_a$  satisfies our result, then  $u$  does too. So it is enough to prove our result locally, for example in a box  $M = X \subset \mathbb{R}^n$ . We expand  $u$  into  $u = \sum u_I dx_I$  and work on each  $u_I$  as above.  $\square$

## Pushing currents forward

Write  $\text{supp } \omega$  for the support of a form  $\omega$ . Suppose that  $f: P \rightarrow Q$  is a smooth map, and  $u$  is a current on  $P$ , and  $f$  is a proper map on the support of  $u$ . Any test form  $\omega_Q$  on  $Q$  pulls back to a test form  $\omega_P$  on  $P$ . Pick some compactly supported bump function  $h_P$  on  $P$  equal to 1 near the compact set  $\text{supp } u \cap \text{supp } \omega_P$ . Define  $\int \omega_Q \wedge f_* u = \int h_P \omega_P \wedge u$ . This  $f_* u$  is independent of the choice of  $h_P$ , and called the *pushforward* of  $u$  by  $f$ . Clearly  $\text{supp } f_* u \subseteq f(\text{supp } u)$ .

For example, if  $P$  is compact and oriented, then the fundamental cycle  $[P]$  pushes forward to  $f_*[P] = \int_P f^*$ , integration over  $P$ .

**42.7** Prove that  $f_* \partial = \partial f_*$  and  $f_* d = df_*$ .

**42.8** Prove that  $f_*(u \wedge f^* \xi) = (f_* u) \wedge \xi$  for any smooth form  $\xi$  on  $Q$ .

**42.9** Prove that  $(f \circ g)_* = f_* \circ g_*$ , where both sides are defined, for smooth maps  $f$  and  $g$  of manifolds.

**42.10** Suppose that  $\eta$  is a continuous differential form on  $P$  and  $f: P \rightarrow Q$  is a submersion of oriented manifolds with corners, proper on the support of  $\eta$ . Let  $u_\eta = \int_P \eta \wedge$ . Prove that  $f_* u_\eta = u_{f_* \eta}$ , where  $f_* \eta$  is the pushforward in the sense of Fubini's theorem.

**42.11** Suppose that  $f: P \rightarrow Q$  is a diffeomorphism. Prove that  $f_* u_\xi = u_{f^{-1} * \xi}$ .

Suppose that  $f_t$  is the flow of a smooth vector field  $X$ . Then we easily calculate that

$$\int \omega \wedge \mathcal{L}_X u = \lim_{h \rightarrow 0} \int \omega \wedge \left( \frac{f_{h*} u - u}{h} \right).$$

So if we define the obvious notion of convergence of currents, by convergence of their evaluations on compactly supported smooth forms, then

$$\mathcal{L}_X u = \lim_{h \rightarrow 0} \frac{u - f_{-h*} u}{h}$$

differentiating along the flow of the vector field. All currents are “smooth” in the sense that we can differentiate them as many times as well like along the flows of any smooth vector fields.

Take any two smooth compactly supported differential forms  $\omega$  and  $\eta$  and any smooth vector field  $X$  on an oriented manifold  $M$  with corners. Denote the degree of  $\omega$  by  $|\omega|$ . Then  $X \lrcorner (\omega \wedge \eta) = (X \lrcorner \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge (X \lrcorner \eta)$ . But  $X \lrcorner (\omega \wedge \eta)$  cannot be a top degree form, so its integral over  $M$  has to vanish, and therefore

$$\int (X \lrcorner \omega) \wedge \eta = (-1)^{|\omega|+1} \int \omega \wedge X \lrcorner \eta,$$

so we define  $X \lrcorner u$ , for any current  $u$  of degree  $|u|$ , by

$$\int \omega \wedge X \lrcorner u = (-1)^{|\omega|+1} \int (X \lrcorner \omega) \wedge u.$$

### Pulling currents back

If  $f: P \rightarrow Q$  is a submersion of oriented manifolds with corners, and  $v$  is a current on  $Q$ , define a current  $f^*v$  on  $P$  by  $\int f^*v \wedge \omega = \int v \wedge f_*\omega$ , for any test form  $\omega$ , where we pushforward  $\omega$  à la Fubini.

**42.12** Prove that  $f^*\partial = \partial f^*$  and  $f^*d = df^*$ .

**42.13** If  $v = u_\xi$  then prove that  $f^*v = u_{f^*\xi}$ .

**42.14** Prove that  $f^*(v \wedge \xi) = (f^*v) \wedge \xi$ .

**42.15** If  $S_Q \subseteq Q$  is a submanifold, and  $f: P \rightarrow Q$  is a submersion, the rank theorem says that  $S_P := f^{-1}S_Q$  is a submanifold. Suppose that  $P, Q$  and  $S_Q$  are oriented. Explain how to orient  $S_P$ . Use Fubini’s theorem to prove that  $f_*[S_P] = [S_Q]$ .

### Convolution

We have defined convolution only for forms on  $\mathbb{R}^n$ , because convolution makes use of adding vectors. Let  $r: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the map  $r(x) = -x$ . Given any three compactly supported differential forms  $\xi, \eta, \zeta$ , we want to prove that  $\int (\xi * \eta) \wedge \zeta = \int \xi \wedge ((r^*\eta) * \zeta)$ :

$$\begin{aligned} \int (\xi * \eta) \wedge \zeta &= \int_x (\xi * \eta)(x) \wedge \zeta(x), \\ &= \int_x \int_y \xi(x-y) \wedge \eta(y) \wedge \zeta(x), \end{aligned}$$



and we let  $z = x - y$ :

$$\begin{aligned} &= \int_z \int_x \xi(z) \wedge \eta(-(z-x)) \wedge \zeta(x), \\ &= \int \xi \wedge ((r^*\eta) * \zeta). \end{aligned}$$

We therefore define  $\eta * u$  by  $\int \omega \wedge (\eta * u) = \int (\omega * r^*\eta) \wedge u$ , for  $\eta$  a test form.

**42.16** Prove that for any constant coefficient vector field  $X$ , any current  $u$  and any test form  $\eta$ :  $\mathcal{L}_X(u * \eta) = (\mathcal{L}_X u) * \eta = u * \mathcal{L}_X \eta$ .

In particular, if  $\eta$  is any test form, then  $u * \eta$  is a smooth *form*, not just a current.

**42.17** As in lemma 30.7 on page 245, prove that every current is the limit of a sequence of test forms.



## Chapter 43

### Haar measure

#### Definition

Take a homogeneous space  $(X, G)$  and a point  $x_0 \in X$  and let  $H := G^{x_0}$ . Suppose that the identity component of  $H$  acts trivially on  $\Lambda^n T_{x_0}^* X = \Lambda^n(\mathfrak{g}/\mathfrak{h})^*$  (for example, if  $H$  is compact.) If  $H$  acts trivially on  $\Lambda^n(\mathfrak{g}/\mathfrak{h})^*$ , define  $\hat{X} := X$ . Suppose instead that  $H$  does not act trivially on  $\Lambda^n(\mathfrak{g}/\mathfrak{h})^*$ . Then  $H$  acts on  $\Lambda^n(\mathfrak{g}/\mathfrak{h})^*$  by  $\pm 1$ , with the identity component of  $H$  acting as 1. (For example, if  $H$  is compact, then either  $H$  acts trivially or by  $\pm 1$  on  $\Lambda^n(\mathfrak{g}/\mathfrak{h})^*$ .) So each component of  $H$  acts as 1 or as  $-1$ , and therefore the union of precisely half of the components of  $H$  form a subgroup  $H_0 \subset H$  acting as 1. Hence the two-to-one covering space  $\hat{X} := G/H_0 \rightarrow X = G/H$  is a homogeneous space of  $G$ , and has a  $G$ -invariant volume form. The covering space map identifies  $\hat{X}$  locally with  $X$ , giving two sheets, on each of which we have a volume form, and these disagree in sign in that local identification. Note that  $X$  is compact just when  $\hat{X}$  is, and just when  $G$  is. If  $X$  is compact, then, for each  $G$ -invariant orientation of  $\hat{X}$ , define the invariant volume form to be the one with unit integral. Otherwise, we just pick any  $G$ -invariant volume form. For any compactly supported continuous function  $f: X \rightarrow V$  valued in a finite dimensional vector space  $V$ , define  $\int_X f$  to mean  $\int_{\hat{X}} \pi^* f \omega$  where  $\pi: \hat{X} \rightarrow X$  is the  $G$ -equivariant covering map  $gH_0 \mapsto gH$ , and  $\omega$  is the invariant volume form. We say that this integration is the *Haar measure*, and  $X$  has an invariant measure, or has Haar measure. In particular, any homogeneous space with compact stabilizer has Haar measure.

#### Properties

By definition, if  $g \in G$ ,

$$\begin{aligned} \int_X g^* f &= \int_{\hat{X}} (\pi^* g^* f) \omega, \\ &= \int_{\hat{X}} (g^* \pi^* f) \omega, \\ &= \int_{\hat{X}} (\pi^* f) g^{-1*} \omega, \\ &= \int_{\hat{X}} (\pi^* f) \omega, \\ &= \int_X f. \end{aligned}$$

**Lemma 43.1.** *Suppose that  $G$  and  $G'$  are Lie groups,  $X$  is a  $G$ -homogeneous space,  $X'$  a  $G'$ -homogeneous space, both with invariant measures,  $\varphi: G \rightarrow G'$  a morphism of Lie groups, and  $\Phi: X \rightarrow X'$  is a smooth  $\varphi$ -equivariant map. For each  $x' \in X'$ ,*

let  $X_{x'} := \Phi^{-1}\{x'\} \subset X$  and let  $G^{x'}$  be the set of  $g \in G$  so that  $\varphi(g)x' = x'$ . Note that  $X_{x'}$  is  $G^{x'}$ -homogeneous. For any compactly supported smooth function  $f: X \rightarrow \mathbb{R}$ ,

$$\int_X f = \int_{x' \in X'} \int_{X_{x'}} f.$$

*Proof.* For homogeneous spaces without invariant volume form, the statement of the theorem, by definition of the measure, reduces to the same statement for the coverings with invariant volume forms. So suppose that  $X$  and  $X'$  both have invariant volume forms  $\omega, \omega'$ . Then  $\Phi^*\omega'$  vanishes on the fibers. Let  $n := \dim X$ ,  $n' := \dim X'$ ,  $p := n - n'$ . At each point  $x \in X$ , there is a  $p$ -form  $\omega''_x$  so that  $\Phi^*\omega' \wedge \omega''_x = \omega$ . Any two such  $p$ -forms agree up to a  $p$ -form which vanishes on the fiber  $X_{x'}$  where  $x' := \Phi(x)$ . The resulting form on  $X_{x'}$  is therefore uniquely determined, and hence invariant. Use a partition of unity to prove there is a smooth form  $\omega''$  on  $X$  which descends to that  $\omega''$  on each fiber  $X_{x'}$  and so that  $\omega = \omega'' \wedge \Phi^*\omega'$ . Then the result is Fubini's theorem (theorem 30.1 on page 239).  $\square$

**Corollary 43.2.** *Haar measure is invariant under automorphisms of homogeneous spaces: if  $\varphi$  is an automorphism and  $\Phi$  a  $\varphi$ -equivariant diffeomorphism, then  $\Phi$  identifies Haar measures.*

**43.1** Prove that every compact Lie group  $G$  has a biinvariant Haar measure.

Theorem 11.7 on page 107 proves that:

**Corollary 43.3.** *Every continuous automorphism of any compact Lie group preserves Haar measure.*

**43.2** Let  $i(g) := g^{-1}$  on a Lie group  $G$ ; prove that  $i^*$  preserves any biinvariant volume form.

**43.3** Take  $p, q > 0$ . Write each element  $M \in U(p+q)$  as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where  $B$  is  $p \times q$ , and so on. Pick  $Q_0 \in U(q)$ . Let  $U \subset U(p+q)$  be the subset on which  $Q_0 D$  does not have 1 as an eigenvalue. Define a map  $\varphi: U \rightarrow U(p)$  by

$$\varphi(M) = A + B(I - Q_0 D)^{-1} Q_0 C.$$

A long calculation shows that  $\varphi(M) \in U(p)$ . Prove that the Haar measure volume of any set  $X \subset U(p)$  is the same as that of  $\varphi^{-1}X \subset U(p+q)$ .

The adjoint action of  $O(3)$  acts on  $\Lambda^3(\mathfrak{so}(3))$  by 1. To see this, note that  $SO(3)$  is connected, so acts on  $\Lambda^3(\mathfrak{so}(3))$  by 1, as it must act by  $\pm 1$ , but the sign is constant on each component, and  $I \in SO(3)$  clearly acts by 1. The group  $O(3)$  has two components: the rotations, with determinant 1, constitute the identity component  $SO(3)$ , while the set of elements expressible as product of a rotation and reflection, with determinant  $-1$ , constitute the other component. But  $-I$  is in the other component, having determinant  $-1$ , and  $-I$  commutes with everything, so acts trivially in the adjoint representation:  $(-I)A(-I) = A$ .

So everything in the component of  $-I$  acts trivially too. So the left invariant volume forms on  $O(3)$  are biinvariant.

The same works for  $O(n)$  for any odd  $n$ .

For the orthogonal group  $O(2)$ , the action of a reflection is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$$

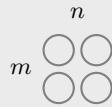
a reflection. So  $O(2)$  has no biinvariant volume form.

The same calculation works in any even dimension:  $O(n)$  has an invariant volume form just when  $n$  is odd.

It is convenient to write the elements of  $SO(2)$  as  $e^{i\theta}$  or as  $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ . Let  $r: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the operation of complex conjugation, which is a reflection. So elements of  $G := O(2)$  are either  $e^{i\theta}$  or  $re^{i\theta}$ . Elements of  $G \times G$  are expressed as  $(g_1, g_2)$  with  $g_1 = r^{k_1}e^{i\theta_1}$  and similarly for  $g_2$ , say as  $(r^k e^{i\alpha}, r^\ell e^{i\beta})$ . So the action of  $G \times G$  on  $G$  is

$$\begin{aligned} (r^k e^{i\alpha}, r^\ell e^{i\beta}) r^m e^{i\theta} &= r^k e^{i\alpha} (r^m e^{i\theta}) e^{-i\beta} r^\ell, \\ &= r^{k+m+\ell} e^{i((-1)^{m+\ell}\alpha + (-1)^\ell(\theta-\beta))}. \end{aligned}$$

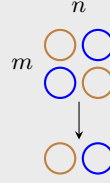
There is no  $(G \times G)$ -invariant 1-form on  $G$ , because the sign of  $\theta$  gets flipped by  $(-1)^\ell$ . Our general construction of  $\hat{G}$  is:  $\hat{G} = (G \times G)/H$  where  $H \subset G \times G$  consists of the elements  $(g, g)$  for which  $g$  acts trivially on  $\Lambda^{\text{top}}(\mathfrak{g})$ . The covering map is the quotient map of  $(g_1, g_2)H \in (G \times G)/H \mapsto g_1 g_2^{-1} \in G$ . In our case,  $e^{i\theta}$  acts trivially in the adjoint representation, and  $re^{i\theta}$  doesn't:  $H = SO(2)$ . Each element of the quotient  $\hat{G}$  is also expressed as  $(r^m e^{i\theta}, r^\ell e^{i\phi})$  but we can absorb the quantity  $\phi$  by right multiplication by  $(e^{-i\phi}, e^{-i\phi}) \in H$ , so uniquely expressed as say  $(r^m e^{i\theta}, r^n)$ . This makes  $\hat{G}$  into 4 disjoint circles:



with elements

$$\begin{pmatrix} (e^{i\theta}, 1) & (e^{i\theta}, r) \\ (re^{i\theta}, 1) & (re^{i\theta}, r) \end{pmatrix}$$

The covering map is



The  $(G, G)$ -action is

$$(r^k e^{i\alpha}, r^\ell e^{i\beta})(r^m e^{i\theta}, r^n) = (r^{k+m} e^{i\theta + (-1)^m \alpha - (-1)^n \beta}, r^{\ell+n}).$$

Hence  $d\theta$  is an invariant 1-form on  $\hat{G}$ , but not invariant under the covering map  $\hat{G} \rightarrow G$ .

## Applications

**Theorem 43.4.** *Take a representation  $V$  of a compact Lie group  $G$ . Then there is a  $G$ -invariant positive definite inner product on  $V$ . If  $V$  is complex (or quaternionic), we can assume that the inner product is Hermitian (or symplectic, i.e. quaternionic linear in one slot, quaternionic conjugate in the other).*

*Proof.* Take any inner product on  $V$ , say written  $\langle v, w \rangle$  for  $v, w \in V$ . Then let

$$\langle v, w \rangle^G := \int_G \langle gv, gw \rangle.$$

Then

$$\langle v, v \rangle^G := \int_G \langle gv, gv \rangle \geq 0$$

with equality just when  $v = 0$ . For any  $g_0 \in G$ ,

$$\langle g_0 v, g_0 w \rangle^G := \int_G \langle gg_0 v, gg_0 w \rangle = \langle v, v \rangle^G$$

by right invariance. □

**Theorem 43.5.** *Suppose that a compact Lie group  $G$  acts on Euclidean space  $\mathbb{R}^n$ ,  $\mathbb{C}^n$  or  $\mathbb{H}^n$  by affine transformations, complex affine transformations, or quaternionic affine transformations. After perhaps an affine, complex affine, or quaternionic transformation of  $\mathbb{R}^n$ ,  $\mathbb{C}^n$  or  $\mathbb{H}^n$  respectively,  $G$  acts by orthogonal, unitary or symplectic linear transformations.*

*Proof.* Take the left invariant volume form  $\omega$  on  $G$ ; use it to form our integrals. Pick a point  $x_0 \in \mathbb{R}^n$ . Take the mean  $m_0 := \langle f \rangle = \int_G f$  of the map  $f(g) = gx_0$ . Then, for any  $g_0 \in G$ ,

$$g_0 m_0 = g_0 \int_G gx_0 = \int_G (g_0 g)x_0 = \int_G gx_0,$$

by left invariance. So  $m_0$  is fixed; translate to ensure that  $m_0 = 0$ . So  $G$  acts by linear transformations, a representation, so preserves an inner product. □

**Theorem 43.6.** *If a compact Lie group  $G$  acts on a manifold  $M$  fixing a point  $m_0 \in M$ , then there are coordinates in which it acts by orthogonal linear transformations, in which  $m_0$  is identified with the origin.*

*Proof.* Each element  $g \in G$  acts on  $M$  and fixes  $m_0$ , so acts on  $T_{m_0}M$  as a linear transformation. Pick a  $G$ -invariant inner product on  $T_{m_0}M$ . Pick a chart  $(U, \varphi)$  on  $M$  near  $m_0$ . Compose  $\varphi$  with a translation to ensure that  $\varphi(m_0) = 0$ . Put the usual inner product on  $\mathbb{R}^n$ . Let  $L := \varphi'(m_0)$ . Compose  $\varphi$  with a linear isomorphism  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  to ensure that  $L$  is a linear isometry. So for each  $g \in G$ , the linear map

$$L^g := L \circ g'(m_0) \circ L^{-1}$$

is an isometry of  $\mathbb{R}^n$ .

For each  $g \in G$ , let  $U^g := gU \subset M$ , and let

$$\varphi^g := L^g \circ \varphi \circ g^{-1}.$$

Expand out to check that

$$\begin{aligned} \varphi^g(U^g) &= L^g \varphi(g^{-1}gU), \\ &= L^g \varphi(U) \end{aligned}$$

is a rotation of  $\varphi(U)$ . Any ball in  $\mathbb{R}^n$  about 0 contained in the image of  $\varphi(U)$  lies inside  $\varphi^g(U^g)$  for all  $g$ . Replace  $U$  by the inverse image of that ball, and then  $U^g = U$  for all  $g \in G$ . Note that  $\varphi^g(m_0) = 0$  for all  $g \in G$  and that

$$(\varphi^g)'(m_0) = \varphi'(m_0)$$

is constant for all  $g \in G$ .

Define a map  $\psi: U \rightarrow \mathbb{R}^n$  by

$$\psi(m) := \int_{g \in G} \varphi^g(m).$$

**43.4** Prove that  $\psi$  is a chart on some open set containing  $m_0$ .

**43.5** Prove that  $\psi^g = \psi$  for all  $g \in G$ .

Hence  $\psi \circ g = L^g \psi$ , i.e.  $\psi$  turns action by  $g \in G$  into a linear map.  $\square$

**43.6** If a compact group  $G$  acts on a manifold, prove that the set of all fixed points is an embedded submanifold. Prove that the fixed points of the identity component  $G^0 \subseteq G$  is an embedded submanifold acted on by the finite group of components  $G/G^0$ .

Write the  $n$ -dimensional torus  $T$  as

$$T := S^1 \times S^1 \times \cdots \times S^1 \subset \mathbb{R}^{2n},$$

writing each point as

$$t = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}).$$

To any sequence of integers  $k_1, k_2, \dots, k_n$ , we associate a linear torus action: each element  $t \in T$  acts on a complex number  $z$  according to

$$tz = e^{i \sum k_a \theta_a} z.$$

**Theorem 43.7.** *Suppose that a torus acts on a manifold  $M$  fixing a point  $m_0$ . Then there are coordinates on  $M$  near  $m_0$ , some complex valued, perhaps some real valued, in which the action is a product of the linear actions as above and perhaps a trivial action.*

*Proof.* By theorem 43.6 on the previous page, the action is linear, so each angle variable acts by some matrix:

$$e^{i\theta_j} x = e^{\theta_j A_j} x,$$

in some linear coordinates, where  $A_j$  is a square matrix. If  $A_j$  is not diagonalizable, its Jordan blocks exponentiate to be nonperiodic. So  $A_j$  is diagonalizable. Having period  $2\pi$ , all eigenvalues of each  $A_j$  have the form  $ik$  for integer  $k$ . All  $A_j$  commute, as the torus is abelian. So they are simultaneously diagonalizable.  $\square$



## Chapter 44

### The strong topology

First we consider what we could mean by a differential equation for maps between manifolds. Then we define and study a topology on smooth maps between manifolds.

#### The chain rule

The chain rule for partial derivatives, applied inductively, shows us that if you change variables, the derivatives of some order in new variables at a given point can be computed from the derivatives of up to that order in the old variables at the corresponding point.

For a function  $y = y(x)$  of one real variable, write  $y_x$  to mean

$$\frac{dy}{dx}.$$

If we change variables to  $\bar{x} = \bar{x}(x)$ ,

$$y_{\bar{x}} = y_x x_{\bar{x}}.$$

Differentiate again:

$$y_{\bar{x}\bar{x}} = y_{xx} x_{\bar{x}} x_{\bar{x}} + y_x x_{\bar{x}\bar{x}},$$

So we can compute the second derivatives in  $\bar{x}$  at a point from the first and second derivatives in  $x$  at the corresponding point. If we change the choice of function  $y(x)$ , the  $x_{\bar{x}}, x_{\bar{x}\bar{x}}$  terms stay the same: a universal expression for computing derivatives in new variables in terms of old:

$$\begin{pmatrix} y_{\bar{x}} \\ y_{\bar{x}\bar{x}} \end{pmatrix} = \begin{pmatrix} x_{\bar{x}} & 0 \\ x_{\bar{x}\bar{x}} & x_{\bar{x}} x_{\bar{x}} \end{pmatrix} \begin{pmatrix} y_x \\ y_{xx} \end{pmatrix}.$$

The same computation works, with more notation, in any number of variables, and for any number of derivatives.

**44.1** If we now allow a change of  $y$  variables, say to  $\bar{y} = \bar{y}(y)$ , compute out how to find second derivatives  $\bar{y}_{\bar{x}\bar{x}}$  in terms of  $y_x, y_{xx}$ .

## Jets

Take manifolds  $P, Q$  and open sets  $U, W \subset P$  and a point  $p_0 \in U \cap W$ . Two  $C^k$  maps  $f: U \rightarrow Q$  and  $g: W \rightarrow Q$  agree to order  $k$  at  $p_0$  if  $f(p_0) = g(p_0)$  and  $f$  and  $g$  have the same Taylor series to order  $k$  at  $p_0$  in some charts defined near  $p_0$  and near  $f(p_0)$ . By the chain rule, agreement is independent of the choice of charts. The  $k$ -jet of  $f$  at  $p_0$  is the equivalence class of  $f$ , denoted  $f_{p_0}^{(k)}$ .

## Jet bundles

The jet bundle  $J_{P,Q}^k$  is the set of all  $k$ -jets  $f_{p_0}^{(k)}$  of all maps  $f: U \rightarrow Q$  from any open set  $U \subset P$ .

Given coordinates  $x^1, \dots, x^p$  on an open set in  $P$ ,  $y^1, \dots, y^q$  on an open set in  $Q$ , any map  $f$ , if defined in some open subset of those charts, becomes some functions  $y = y(x)$ .

Take  $P = Q = \mathbb{R}$  and  $k = 1$ . Take coordinates  $x$  on  $P$ ,  $y$  on  $Q$ . Each Taylor series expansion of a map  $f: U_P \rightarrow U_Q$  between open sets  $U_P \subseteq P$  and  $U_Q \subseteq Q$  has derivative say  $y_x$  at some point  $x$  where  $f$  has value  $y$ ; so we have coordinates  $x, y, y_x$  on  $J_{\mathbb{R},\mathbb{R}}^1$ . If we change coordinates from  $x$  on  $P$  to, say  $\bar{x}$ , with transition map  $\bar{x} = \bar{x}(x)$ , and similarly on  $Q$ , with transition map  $\bar{y} = \bar{y}(y)$ , then we change coordinates  $x, y, y_x$  on  $J_{\mathbb{R},\mathbb{R}}^1$  to  $\bar{x}, \bar{y}, \bar{y}_{\bar{x}}$  with transition map

$$\begin{aligned}\bar{x} &= \bar{x}(x), \\ \bar{y} &= \bar{y}(y), \\ \bar{y}_{\bar{x}} &= \frac{d\bar{y}}{dy} y_x \frac{dx}{d\bar{x}}.\end{aligned}$$

Intuitively, this  $y_x$  variable is intended to represent the derivative of some function  $y = y(x)$ , but we can just think of these transition maps as giving us an abstract 3-dimensional manifold  $J_{\mathbb{R},\mathbb{R}}^1$ .

We could extend this notation  $x, y, y_x$  to many functions of many variables, and to derivatives of all orders. But this is too messy. Instead we will write out a Taylor polynomial as an abstract polynomial function of some given degree. In order to write the Taylor expansion of  $y = f(x)$ , first we need to specify a point  $x = x_0$  about which to expand. We also need some variables in which to carry out the expansion. Naturally we call these variables  $x$ , but we agree to use  $x_0$  to represent the point about which to expand. Expand  $y = f(x)$  in a Taylor series about a given point  $x = x_0$ :

$$y = f(x) = F(x - x_0) + \dots$$

where  $F$  is a polynomial of order at most  $k$ , and  $\dots$  vanishes at order  $k$  at  $x = x_0$ . We can identify any polynomial with its coefficients. Map

$$f_{p_0}^{(k)} \mapsto (x_0, F(x))$$

where  $x_0 = x(p_0)$  is the expression of the point  $p_0$  in the coordinates  $x^1, \dots, x^p$  on  $P$  and  $F(x)$  is our Taylor polynomial, written in terms of some abstract  $x$  variables; note,

that we identify  $F$  here with the collection of its coefficients written down in some order. This map is a chart for  $J_{P,Q}^k$ . The chain rule clearly produces a (complicated!) transition map between such charts.

In our notation above, we had coordinates  $x, y, y_x$ . To fit with our new notation, write these coordinates as  $x_0, y_0, y_{x,0}$ . Write out each function's Taylor expansion as

$$y(x) = y_0 + y_{x,0} \cdot (x - x_0) + \dots$$

Note that need an additional variable  $x$  here. We have Taylor polynomial

$$F(x - x_0) = y_0 + y_{x,0}(x - x_0).$$

So in our new coordinates, our point is  $(x_0, F(x))$  where  $F(x)$  is the linear function taking value  $y_0$  at  $x = 0$  and with slope  $y_{x,0}$ :  $F(x) = y_0 + y_{x,0}x$ .

**44.2** What is the dimension of  $J_{P,Q}^k$  for manifolds  $P, Q$  of dimensions  $p, q$ ?

### The strong topology

To each open set  $U \subset J_{P,Q}^k$  we associate the *strong open set* of all smooth maps  $f: P \rightarrow Q$  so that  $f^{(k)}$  is valued in  $U$ . The *strong topology* (also known as the *Whitney topology* or the *fine topology*) is the topology on smooth maps  $P \rightarrow Q$  generated by the strong neighborhoods.

Let's examine the strong topology in charts. A *locally finite atlas* is an atlas of charts with domains  $U_i \subset P$ , so that any point lies in open set intersecting only finitely many chart domains. Every manifold with corners has a locally finite atlas, as we have seen in chapter 31. Take a smooth map  $f_0: P \rightarrow Q$  for some  $k = 1, 2, \dots$  or  $k = \infty$ . The *strong neighborhood* of  $f_0$  associated to a choice of:

- a locally finite atlas with domains  $U_i$  and
- a compact set  $K_i \subset U_i$  (perhaps empty) in each chart domain and
- an atlas of charts on  $Q$ , with domains  $W_i \subset Q$ , so that  $f_0(K_i) \subset W_i$  and
- numbers  $\varepsilon_i > 0$  ("error tolerances") and
- nonnegative integers  $\ell_i$  ("orders of derivatives"), all bounded by some  $\ell$

is the set of all smooth maps  $f: P \rightarrow Q$  so that, for every  $i$ ,  $f(K_i) \subset W_i$  and, in the coordinates associated to the charts,  $f$  has all derivatives of order  $\ell_i$  within  $\varepsilon_i$  of those of  $f_0$ .

**Lemma 44.1.** *The strong topology is generated by the strong neighborhoods.*

*Proof.* The open sets  $U_i$  have finite intersections so we only impose finitely many conditions on  $f$  around each point. Each condition is an open constraint on the derivatives of some finite order, so the jets satisfying it form an open subset of the  $\ell$ -jets. Hence the strong neighborhoods are strong open sets.

On the other hand, given any open set of  $\ell$ -jets, which includes the  $\ell$ -jets of some map, we can locally write down some open sets inside it consisting of jets satisfying, in our charts, some inequalities forcing them close enough to the  $\ell$ -jets of that map.

So in suitable charts, with suitable compact sets as above, we can force a strong neighborhood to lie inside a given strong open set.  $\square$

**Proposition 44.2.** *The smooth maps between manifolds form a Baire space in the strong topology. In other words, any countable intersection of dense open sets is dense.*

*Proof.* Take a set

$$A = U_1 \cap U_2 \cap \dots$$

where each  $U_j$  is a dense open set. We want to prove that  $A$  is dense. Pick a smooth map  $f_0: P \rightarrow Q$ . It is enough to show that  $f_0$  lies in the closure of  $A$ , i.e. that every strong neighborhood of  $f_0$  intersects  $A$ .

Given a set  $S$  of  $k$ -jets, let  $S'$  be the set of all smooth maps  $f$  whose  $k$ -jet belongs to  $S$ . So if  $S$  is open, then  $S'$  is a strong neighborhood.

Take a strong neighborhood  $W'_1$  around  $f_0$ . We want to prove that there is some  $f \in A \cap W'_1$ , i.e. some such  $f$  lying simultaneously in all of  $W'_1, U_1, U_2, \dots$ . Since  $U_1$  is dense, there is some  $f_1$  in  $U_1 \cap W'_1$ . Pick a strong neighborhood  $W'_2 \subset W'_1 \cap U_1$  around  $f_1$ . Since  $U_2$  is dense, there is some  $f_2$  in  $U_2 \cap W'_2$ . Pick a strong neighborhood  $W'_3 \subset W'_2 \cap U_2$  around  $f_2$ . Proceed by induction. We need to show that, by suitable choices of these sets  $W'_1, W'_2, \dots$  of jets, we can ensure that there is some  $f$  in all  $W'_i$ .

Each  $W'_i$  is not empty by definition of a strong neighborhood. We can ask that each  $W_i$  consists of even higher order jets than  $W_{i-1}$  did, say some order  $k_i$ ,  $k_1 < k_2 < \dots$ . We can also ask that not only  $W_i$  but even  $\bar{W}_i$  lies inside the  $k_i$ -jets whose associated  $k_{i-1}$ -jets lie in  $W_{i-1}$ . We can even ask that the  $k$ -jets of a given order  $k$  get more tightly restricted at each stage  $i$ , their expressions in our charts being bound by successively tighter inequalities, so that at any given order of jets, successive functions in those sets converge as a Cauchy sequence. So at each step, we are imposing more stringent requirements, forcing more derivatives to satisfy some estimates on each compact set  $K_j$ , and also forcing any given number of derivatives to converge. The limit exists, because at each  $k$ -jet order, we have a Cauchy sequence. The limit is smooth, since we controlled all of its derivatives.  $\square$

We say “the generic map” has some property to mean that the set of maps with that property is a countable intersection of dense open sets, hence dense. We will prove that various properties of maps are generic.

**Lemma 44.3.** *The homotopy class of any map is open in the strong topology.*

*Proof.* Return to our definition of a strong neighborhood around a map  $f_0$ . We can take our locally finite atlas of charts  $(U_i, \varphi_i)$  and compact sets  $K_i$  so that each  $U_i, K_i, W_i$  has convex image in its associated chart. Take a map  $f_1$  in the associated strong neighborhood. We need only prove that  $f_0$  and  $f_1$  are all in the same homotopy class, through a homotopy which does not leave the strong neighborhood.

Over  $U_0$ , we use the convexity to homotope  $f_1$  to  $f_0$ , fixing  $f_1$  outside of some precompact open set around  $K_0$ . Call the resulting map  $f_{1/2}$ . Make this homotopy take time  $0 \leq t \leq 1/2$ . Similarly for each of the other compact sets. On any compact set, eventually the resulting homotopy is fixed, as the cover is locally finite.  $\square$

## Chapter 45

### Wiggling into transversality

We try to wiggle a map of manifolds so that it avoids, as best it can, satisfying a given differential equation.

#### Jets and polynomials

Consider a smooth real-valued function  $y(x)$  of a real variable  $x$ . In general, if we know the 2-jet at some point, say at  $x = 0$ :

$$y(x) = 7 + 3x + 2x^2 + \dots,$$

this gives *no* information about the 2-jet at any other point, say at  $x = 1$ . But in the special case where we assume the function is a quadratic polynomial, we can recover the 2-jet everywhere from the 2-jet at any one point:

$$\begin{aligned} y(x) &= 7 + 3x + 2x^2, \\ &= 7 + 3(x - 1 + 1) + 2(x - 1 + 1)^2, \\ &= 7 + 3(x - 1) + 3 + 2((x - 1)^2 + 2(x - 1) + 1^2), \\ &= 7 + 3(x - 1) + 3 + 2(x - 1)^2 + 4(x - 1) + 2, \\ &= 7 + 3 + 2 + (3 + 4)(x - 1) + 2(x - 1)^2, \\ &= 12 + 7(x - 1) + 2(x - 1)^2. \end{aligned}$$

By the same computation, at each point  $x = x_0$ , we find the Taylor series expansion of  $y(x) = 7 + 3x + 2x^2$  at any point  $x = x_0$  explicitly from the expansion at  $x = 0$ :

$$y(x) = (7 + 3x_0 + 2x_0^2) + (3 + 4x_0)(x - x_0) + 2(x - x_0)^2.$$

Write out any quadratic polynomial as

$$y(x) = a + b(x - x_0) + c(x - x_0)^2.$$

The 2-jet space has coordinates  $x_0, a, b, c$ . The Taylor coefficients of our particular polynomial at any point  $x = x_0$  are

$$(a, b, c) = (a(x_0), b(x_0), c(x_0)) = (7 + 3x_0 + 2x_0^2, 3 + 4x_0, 2).$$

So

$$x_0 \mapsto (x_0, a(x_0), b(x_0), c(x_0))$$

is a curve in the 2-jet space, parameterized by  $x_0$ . Let's vary the choice of the polynomial, not using  $7 + 3x + 2x^2$ , but instead using any quadratic polynomial, say

$$y(x) = a_0 + b_0x + c_0x^2.$$

Calculate

$$y(x) = (a_0 + b_0x_0 + c_0x_0^2) + (b_0 + 2x_0c_0)(x - x_0) + c_0(x - x_0)^2,$$

which traces out a curve

$$\begin{aligned} (x_0, a, b, c) &= (x_0, a(x_0), b(x_0), c(x_0)), \\ &= (x_0, a_0 + b_0x_0 + c_0x_0^2, b_0 + 2c_0x_0, c_0), \\ &= (x_0, y(x_0), y'(x_0), y''(x_0)) \end{aligned}$$

in the 2-jet space. These curves foliate the 2-jet space, because each quadratic polynomial is uniquely determined by its coefficients at any one point  $x = x_0$ .

This foliation is a submersion: the 2-jet space, call it  $J$  for simplicity, is mapped to the vector space  $V$  of quadratic polynomial functions by a map  $J \rightarrow V$ , taking each  $k$ -jet

$$y(x) = a + b(x - x_0) + c(x - x_0)^2 + \dots$$

to the polynomial function

$$\begin{aligned} p(x) &= a + b(x - x_0) + c(x - x_0)^2, \\ &= a + b(x - x_0) + c(x - x_0)^2, \\ &= (a - bx_0 + cx_0^2) + (b - 2cx_0)x + cx^2 \in V. \end{aligned}$$

In coordinates:  $J$  has coordinates  $x_0, a, b, c$ , and the polynomial space  $V$  has coordinates  $a_0, b_0, c_0$ , writing each polynomial as

$$p(x) = a_0 + b_0x + c_0x^2,$$

and the map is

$$(x_0, a, b, c) \in J \mapsto (a_0, b_0, c_0) = (a - bx_0 + cx_0^2, b - 2cx_0, c) \in V,$$

as the reader can check.

Similarly, in any number of  $x$  or  $y$  variables, the  $k$ -jet space of maps  $y = y(x)$ , call it  $J$  for simplicity, is mapped to the vector space  $V$  of polynomial functions of degree at most  $k$ , say  $J \rightarrow V$ , taking each  $k$ -jet

$$y(x) = \sum a_\alpha (x - x_0)^\alpha + \dots$$

to the polynomial function

$$p(x) = \sum_{|\alpha| \leq k} a_\alpha (x - x_0)^\alpha \in V.$$

This map is a submersion, with fibers given by the choice of point  $x_0$ . Careful: this submersion is defined by using coordinates, so we are not telling a story about manifolds.

## Local wiggling

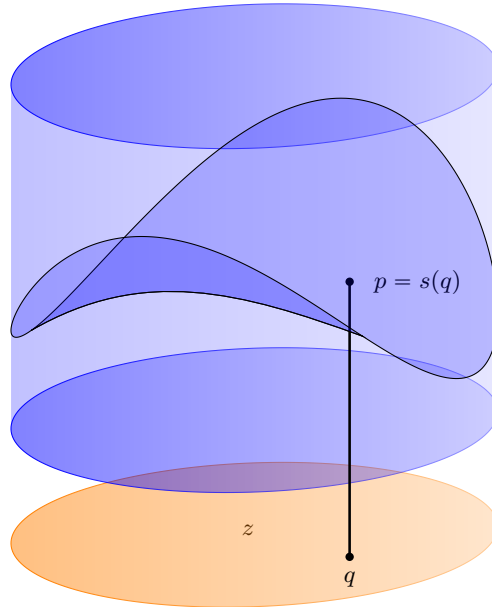
Take  $S \subset \mathbb{R}^s$  and  $T \subset \mathbb{R}^t$  open subsets of the standard box in Euclidean space. For any integer  $r \geq 0$ , a *differential relation* of order  $r$  on smooth maps  $S \rightarrow T$  is a subset  $A$  of the  $r$ -jets of such maps. The notion of a differential relation generalizes the notion of a differential equation.

**Lemma 45.1.** *For any differential relation of some order  $r$ , the generic smooth map  $S \rightarrow T$  has  $r$ -jet transverse to that differential relation. Indeed in the space of polynomial maps of degree  $r$ , the maps which are not transverse form a set of measure zero.*

*Proof.* Take a function  $y = y(x)$  taking  $S \rightarrow T$ . Slightly perturbing the function, we can assume that  $y(x)$  takes values in the interior of  $T$ , not on the boundary, so we can assume that  $T$  is open in  $\mathbb{R}^t$ . For simplicity of notation, we can suppose that the origin lies in  $S$ . We take the  $r$ -jet space, which we denote as  $J$ . Denote the differential relation as  $A \subseteq J$ . Denote the space of polynomial maps of degree at most  $r$  as  $V$ . Our map  $J \rightarrow V$  as above is a submersion. The generic fiber is  $A$ -transverse by lemma 20.3 on page 181.  $\square$

## Jet bundles

Take a smooth submersion  $\pi: P \rightarrow Q$ . Picture it as a “family” of manifolds  $P_q := \pi^{-1}\{q\}$  parameterized by the points of a manifold  $Q$ . A *section* is a smooth map  $s: Q \rightarrow P$  so that  $s(q) \in P_q$  for all  $q \in Q$ .



Take a smooth manifold  $M$ . The map  $\pi: TM \rightarrow M$ , taking a vector  $v \in T_{m_0}M$  to its “foot”  $m_0$ , is a smooth submersion. A section  $s: M \rightarrow TM$  picks out a vector  $v = s(m)$  at each point  $m$ , i.e. a section is a vector field.

Let  $Q := S^2$  be the sphere. Let  $P \subset TS^2$  be the set of all nonzero tangent vectors. The map  $\pi: P \rightarrow Q$ , taking a vector to its foot, is again a smooth submersion. A section  $s: M \rightarrow TM$  picks out a nonzero vector  $v = s(m)$  at each point  $m$ , i.e. a section is a nowhere zero vector field. By the hairy ball theorem (theorem 22.7 on page 193), there is no global nowhere vanishing vector field on the sphere, i.e. this smooth submersion has no section.

A *local section* is a smooth map  $s: U \rightarrow P$ , defined on an open subset  $U \subseteq Q$ , so that  $\pi(s(q)) = q$  for all  $q \in U$ . Every submersion admits local sections, by the implicit function theorem. Let  $P^{(r)}$  be the set of all  $r$ -jets of local sections, the  *$r$ -jet bundle*.

Take local coordinates  $x^i$  near some point of  $Q$ , and pullback to  $P$ . Pick some functions  $y^a$  to give local coordinates  $x^i, y^a$  on  $P$ . Each local section is a map  $y = y(x)$ . We can write the Taylor expansion of order  $r$  at a point  $x = x_0$  as

$$y = F(x - x_0) + \dots$$

for a polynomial  $F(x)$  of degree at most  $r$  in some abstract variables  $x^i$ . On  $P^{(r)}$ , take  $(x_0, F(x))$  as coordinates (i.e., to be more precise, we are taking the coordinates of the point  $x_0$  and the coefficients of the polynomial  $F(x)$  as our coordinates). The map  $P^{(r)} \rightarrow P$  taking  $s_{p_0}^{(k)} \mapsto s(p_0)$  is expressed as  $(x_0, F(x)) \mapsto (x_0, F(0))$ , so is a smooth submersion.

## Wiggling into transversality

A *differential relation* on a submersion  $P \rightarrow Q$  is a subset  $A \subseteq P^{(r)}$ . The notion of a differential relation generalizes the notion of a differential equation.

**Theorem 45.2** (Thom transversality). *Take a differential relation on a smooth submersion. The generic smooth section has jet transverse to that differential relation. In particular, if there is a continuous section, then there is a smooth section with jet transverse to that differential relation.*

*Proof.* In chapter 36 on page 277, we see how to smooth any continuous section, so we can assume we have a smooth section.

Pick a smooth submersion  $\pi: P \rightarrow Q$  and a differential relation  $A \subset P^{(r)}$ . Transversality to  $A$  is precisely transversality to each embedded submanifold in a covering of  $A$  by a sequence of embedded submanifolds. Because smooth sections form a Baire space, we can assume that  $A$  is an embedded submanifold.

For each compact set  $K \subset Q$ , let  $U_K$  be the set of all smooth sections  $s$ , defined on all of  $Q$ , so that  $s^{(r)}$  is transverse to  $A$  at every point of  $K$ . Transversality of  $s^{(r)}$  is a condition on  $s^{(r+1)}$ , as  $s^{(r+1)}$  determines the first derivatives of  $s^{(r)}$  in coordinates on  $P^{(r+1)}$ . Note that  $U_K$  is open in the strong topology, as it consists of the sections whose jets lie in an open subset of the jet bundle  $P^{(r+1)}$ .



Suppose that we can prove that  $U_{K_i}$  is dense in the smooth sections, in the strong topology, for some locally finite cover of  $Q$  by a sequence of compact sets  $K_1, K_2, \dots$ . The intersection

$$\bigcap U_{K_i} = U_{\bigcup K_i} = U_Q$$

is dense by proposition 44.2 on page 366: the generic section lies in  $U_Q$ .

So we need to prove that the sections whose  $r$ -jets are  $A$ -transverse over  $K_i$  enter every strong neighborhood. We can take these  $K_i$  to lie inside chart domains. Consider just one  $K_i$ , call it  $K$ . Take  $K$  to lie inside the domain of some coordinates  $x^i$ , with coordinates  $x^i, y^a$  on  $P$ . We can assume that our coordinate charts have image a convex set in Euclidean space. It follows that our coordinate charts on  $P^{(r)}$  also have convex image.

Pick a section; we need to ensure that the sections with  $A$ -transverse  $r$ -jet enter every strong neighborhood of that section. Suppose that our section is  $y = y(x)$ . We can change coordinates to arrange that the section is  $y = 0$ . We need to construct smooth sections approaching 0 in  $C^r$  over  $K$ , while their  $r$ -jets are  $A$ -transverse over  $K$ .

Suppose that we can find local sections  $y = y(x)$  whose  $r$ -jets are  $A$ -transverse over  $K$ . But suppose that they are only defined in our chart domain, not globally defined on  $P$ . Suppose that they approach 0 in  $C^r$  over  $K$ . Take a bump function  $h(x)$ , equal to 1 on  $K$ , and to zero outside some compact set around  $K$ . Then  $y = h(x)y(x)$  is  $A$ -transverse on  $K$ . By convexity of our chart images, this  $y = h(x)y(x)$  is a local section. Because it vanishes outside  $K$ , it extends to all of  $Q$  to agree with our given section.

So it suffices to prove that some sequence  $y = y_i(x)$  of local sections, only defined in our chart domain, approaches 0 in  $C^r$  over  $K_Q$  while their  $r$ -jets remain  $A$ -transverse. This is precisely lemma 45.1 on page 369.  $\square$

**Theorem 45.3.** *Take manifolds  $P, M$  and an embedded submanifold  $A \subset M$ . Any smooth map  $P \rightarrow M$  can be approximated as closely as we like, uniformly on compact sets with any number of derivatives in coordinates, by maps transverse to  $A$ .*

*Proof.* Careful: we take as submersion, not the manifold  $P$ , but the product  $P \times M$ , and apply Thom transversality.  $\square$

## Morse functions

Given a smooth function  $f: M \rightarrow \mathbb{R}$  on a manifold, the second derivatives are complicated: when we change coordinates,

$$f_{yy} = f_{xx}x_yx_y + f_{xy}x_{yy}.$$

This simplifies at a critical point, i.e. if  $f_x = 0$ :

$$f_{yy} = f_{xx}x_yx_y.$$

Expanding out the notation:

$$\frac{\partial^2 f}{\partial y^i \partial y^j} = \frac{\partial x^k}{\partial y^i} \frac{\partial^2 f}{\partial x^k \partial x^\ell} \frac{\partial x^\ell}{\partial y^j}$$

So the operation  $f''(m_0)$  given by

$$v, w \mapsto f''(m_0)(v, w) := \frac{\partial^2 f}{\partial x^i \partial x^j} v^i w^j$$

on tangent vectors

$$v = v^i \frac{\partial}{\partial x^i}, w = w^i \frac{\partial}{\partial x^i}$$

is easily seen to be defined independent of coordinates, *but* only at critical points. So  $f''(m_0)$  is a symmetric bilinear form on  $T_{m_0}M$ . A *null vector* of  $f''(m_0)$  is a tangent vector  $v$  so that  $f''(m_0)(v, w) = 0$  for all  $w$ . The set of null vectors is a linear subspace of the tangent space  $T_{m_0}M$  at each critical point, the *null space*. A *degenerate* critical point is one for which this null space is not zero. In other words, a degenerate critical point, in coordinates, is a point where

$$0 = \frac{\partial f}{\partial x^i}$$

for all  $i$ , and the matrix

$$\frac{\partial^2 f}{\partial x^i \partial x^j}$$

has zero determinant. A *Morse function* on a smooth manifold with corners  $M$  is a smooth function  $f: M \rightarrow \mathbb{R}$  so that  $f$  has only a discrete set of critical points, none on the boundary of  $M$ , and all nondegenerate.

**Theorem 45.4.** *On every manifold with corners, the generic smooth function is a Morse function.*

*Proof.* For this problem, we take as submersion  $P := M \times \mathbb{R}$ , and as base  $M$ . Let  $A \subset J_{M, \mathbb{R}}^2$  be the set of 2-jets  $f_{m_0}^{(2)}$  of functions which, in any coordinates, have degenerate critical point or critical point on the boundary. Check that  $A$  has codimension exceeding the dimension of  $M$ ; apply Thom transversality.  $\square$

Take an embedded surface  $M \subset \mathbb{R}^3$ . Perturb the last coordinate function  $z$  to a Morse function. Imagine water filling up space to some height  $z = a$ ; the sublevel set  $M_a := \{z \leq a\}$  consists of the submerged points. As the water gradually gets deeper, choosing one value  $a$  between any two successive critical values, the surface is constructed as a sequence of gluings of compact manifolds with boundary, stacked vertically on top of each other. As the water reaches a critical point, the various components glue together or pull apart.

**45.1** Suppose that  $S$  is a compact immersed hypersurface in a manifold  $M$  and that  $M$  has a foliation by hypersurfaces. Prove that in any open set of  $M$  containing  $S$ , there is a compact immersed hypersurface  $S'$  homotopic to  $S$  so that the leaves of the foliation intersect  $S'$  transversely, i.e. cut out a foliation on  $S'$ , except at finitely many points. Can you control how the leaves strike those points?

### Immersing to Euclidean space

**Theorem 45.5** (Whitney). *If  $P$  and  $Q$  are manifolds with corners and  $\dim Q \geq 2 \dim P$ , then the generic smooth map  $P \rightarrow Q$  is an immersion.*

*Proof.* For this problem, we take as submersion  $P \times Q$  mapping to  $P$  by  $(x, y) \mapsto x$ . (For the reader who has read chapter 36 on page 277, note that  $J_{P,Q}^1 = T^*P \boxtimes TQ$ .) Let  $A \subset J_{P,Q}^1$  be the jets which, in coordinates  $x^i, y^a, y_i^a$ , have  $y_i^a$  not of full rank. Check that  $A$  is of codimension exceeding the dimension of  $P$ .  $\square$

Note that the set of maps  $P \rightarrow Q$  is not empty: we can take any point  $q_0 \in Q$  and consider the constant map  $\varphi(p) = q_0$ . This theorem says that we can approximate the constant map by an immersion.

**Corollary 45.6.** *Every manifold with corners admits an immersion into any manifold with corners of twice its dimension, and in particular into Euclidean space of twice its dimension.*



## Chapter 46

### Transversality and homology

#### Intersection

Take a compact oriented manifold with corners  $P$  and an oriented manifold with corners  $Q$ , and smooth maps  $\varphi: P \rightarrow M$  and  $\psi: Q \rightarrow M$  to a manifold with corners  $M$ . Inductively, we can wiggle boundaries of one or the other a little to be transverse to the other map and all of its boundary components, and all their boundary components, and so on. Then patch together to make the map itself smooth again, to “catch up”: this is possible because we can wiggle by a sequence of wiggles, each in some chart, each with compact support. Wiggle the map on the interior to become transverse to the other map.

So any two chains in any manifold with corners are transverse, up to wiggling one of them, so without changing their homology class. Hence in an oriented manifold with corners, using chains over  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$  or  $\mathbb{C}$ , we can define their intersection, intersecting the bodies and multiplying the coefficients. More precisely, we can define the intersection of a chain with a Borel–Moore chain. Hence an oriented manifold with corners has a product in homology, which is clearly associative. On a compact oriented manifold, the fundamental class is clearly the identity in homology.

If two homology classes have complementary dimension, their intersection is a finite set of points. In an oriented connected manifold, without changing homology class, we can take those points to be a single point. So in a connected oriented manifold, complementary dimensional homology classes have an *intersection number* in the same ring as their coefficients.

Suppose that  $M_0, M_1 \subseteq M$  are domains with corners, i.e. open sets whose closures  $\bar{M}_0, \bar{M}_1 \subseteq M$  are smooth tidy embedded submanifolds with corners. We can suppose also that their walls intersect transversely, so that  $\bar{M}_{01} := \bar{M}_0 \cap \bar{M}_1$  is a domain with corners, with interior  $M_{01}$ .

**Lemma 46.1.** *Every chain in  $M$  (perhaps long, with whatever coefficients) is a sum of chains in  $M_0$  and  $M_1$  (perhaps long, with those same coefficients).*

*Proof.* Take a chain  $A \subset M$ , perhaps long. Suppose that  $\psi$  is an orientation preserving diffeomorphism of  $M$ , pushing the boundary of  $\bar{M}_{01}$  into the interior, and equal to the identity outside a compact set containing the intersection of  $A$  with  $\partial M_{01}$ . We can pick  $\psi$  so that  $A$  is transverse to the boundaries of the domains  $\psi M_0, \psi M_1, \psi M_{01}$ . So we can then take the preimages of those domains under the maps which constitute the chain  $A$ , splitting  $A$  as needed into chains mapping into  $M_0, M_1, M_{01}$ ,  $A = A_0 + A_1 - A_{01}$ , which we can write as  $A = (A_0 - A_{01}) + A_1$ . We need only prove that there is such a diffeomorphism  $\psi$ .

Take a point on the boundary of  $M_{01}$  lying in the image of the chain  $A$ . In local coordinates, we can arrange that  $M_{01}$  is the standard corner. Pick generic hypersurfaces, to cut out a corner inside  $M_{01}$ . We can make that corner be as close

as we like to our given point of  $M_{01}$ , with the same number of walls meeting, with nearly the same tangent spaces. In some new coordinates, we can arrange that the new corner is just a translate of the old one, so  $\psi$  can be chosen the identity away from some compact set, a translation near our given point, and  $\psi$  achieves the required transversality near that point. This  $\psi$  only works locally near our point. In the strong topology, we can find maps  $\psi$  which achieve global transversality by proposition 44.2 on page 366.  $\square$

So if we let  $C, C_0, C_1, C_{01}$  be the chains in  $M, M_0, M_1, M_{01}$  (perhaps long, defined with whatever coefficients), map  $C_{01} \rightarrow C_0 \oplus C_1$  by  $A \mapsto (A, A)$ , and  $C_0 \oplus C_1 \rightarrow C$  by  $(A_0, A_1) \mapsto A_0 - A_1$ , then we have an exact sequence

$$0 \rightarrow C_{01} \rightarrow C_0 \oplus C_1 \rightarrow C \rightarrow 0.$$

The associated exact sequence in homology is the *Mayer–Vietoris* sequence in homology:

$$\begin{array}{ccccccc}
 & & & & \cdots & & \\
 & \searrow & & & \nearrow & & \\
 H_4(M_{01}, \mathbb{Z}) & \longrightarrow & H_4(M_0, \mathbb{Z}) \oplus H_4(M_1, \mathbb{Z}) & \longrightarrow & H_4(M, \mathbb{Z}) & & \\
 & \searrow & & & \nearrow & & \\
 H_3(M_{01}, \mathbb{Z}) & \longrightarrow & H_3(M_0, \mathbb{Z}) \oplus H_3(M_1, \mathbb{Z}) & \longrightarrow & H_3(M, \mathbb{Z}) & & \\
 & \searrow & & & \nearrow & & \\
 H_2(M_{01}, \mathbb{Z}) & \longrightarrow & H_2(M_0, \mathbb{Z}) \oplus H_2(M_1, \mathbb{Z}) & \longrightarrow & H_2(M, \mathbb{Z}) & & \\
 & \searrow & & & \nearrow & & \\
 H_1(M_{01}, \mathbb{Z}) & \longrightarrow & H_1(M_0, \mathbb{Z}) \oplus H_1(M_1, \mathbb{Z}) & \longrightarrow & H_1(M, \mathbb{Z}) & & \\
 & \searrow & & & \nearrow & & \\
 H_0(M_{01}, \mathbb{Z}) & \longrightarrow & H_0(M_0, \mathbb{Z}) \oplus H_0(M_1, \mathbb{Z}) & \longrightarrow & H_0(M, \mathbb{Z}) & \longrightarrow & 0
 \end{array}$$

The same for Borel–Moore homology.

**46.1** Prove that any manifold with corners has countably generated homology and Borel–Moore homology.

**Corollary 46.2.** *There is a countable set of cycles representing every homology class, so that any two are transverse smooth maps of compact oriented manifolds with corners. There is a countable set of long cycles representing every Borel–Moore homology class, so that any two are transverse proper maps of compact oriented manifolds with corners.*

*Proof.* We can get generate the homology of degree at most half the dimension of  $M$  by immersed smooth chains, by theorem 45.5 on page 372. We then go back to those of degree half the dimension, putting them into a sequence, and arranging by Thom transversality that each is transverse to all of the earlier ones. Those of higher degree, we make transverse to those of lower degree by Thom transversality, and again we can perturb them to be transverse to one another.  $\square$

**Theorem 46.3.** *Cover a manifold in domains with corners, intersecting transversely. The Čech–de Rham homology of that cover is isomorphic to the homology of the manifold. This holds with any coefficients, and also in Borel–Moore variants.*

*Proof.* Map

$$(C_0, C_1, C_{01}) \mapsto C_0 + C_1,$$

taking Čech–de Rham chains to chains in  $M$ .

**46.2** Prove that this map commutes with  $\partial$ .

By lemma 46.1 on page 375, the map is surjective on chains. The kernel consists in chains  $(C_0, C_1, C_{01})$  for which  $C_0 + C_1 = 0$  in  $M$ . So  $C_1 = -C_0$  as represented by maps, which therefore, up to equivalence as chains, map into  $M_{01}$ :  $(C_0, C_1, C_{01}) = (C, -C, C_{01})$  for some chain  $C$  in  $M_{01}$ . Such a chain is a cycle just when

$$0 = \partial(C, -C, C_{01}) = (\partial C - C_{01}, -\partial C + C_{01}, -\partial C_{01}),$$

so just when  $C_{01} = \partial C$ . It bounds some element of that kernel just when

$$(C, -C, \partial C) = \partial(B, -B, B_{01}) = (\partial B - B_{01}, -\partial B + B_{01}, -\partial B_{01}),$$

for some  $(B, -B, \partial B_{01})$ . But we can set  $B = 0$  and  $B_{01} = -C$ , so the cohomology of this kernel is trivial. Hence the exact sequence in chains gives an isomorphism in homology from Čech–de Rham homology to usual homology. The same for Borel–Moore homology.  $\square$



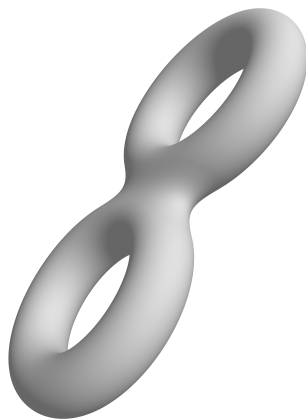


## Chapter 47

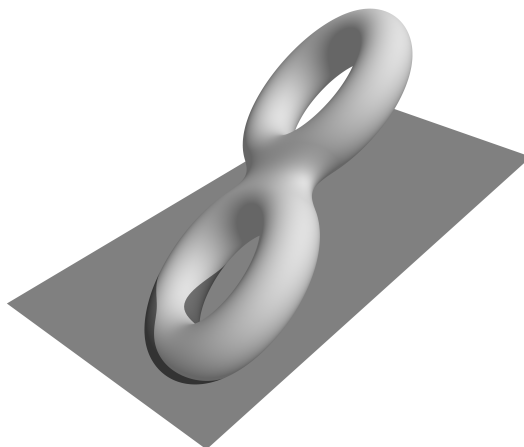
### Morse theory I: the drowning surface

#### Surfaces

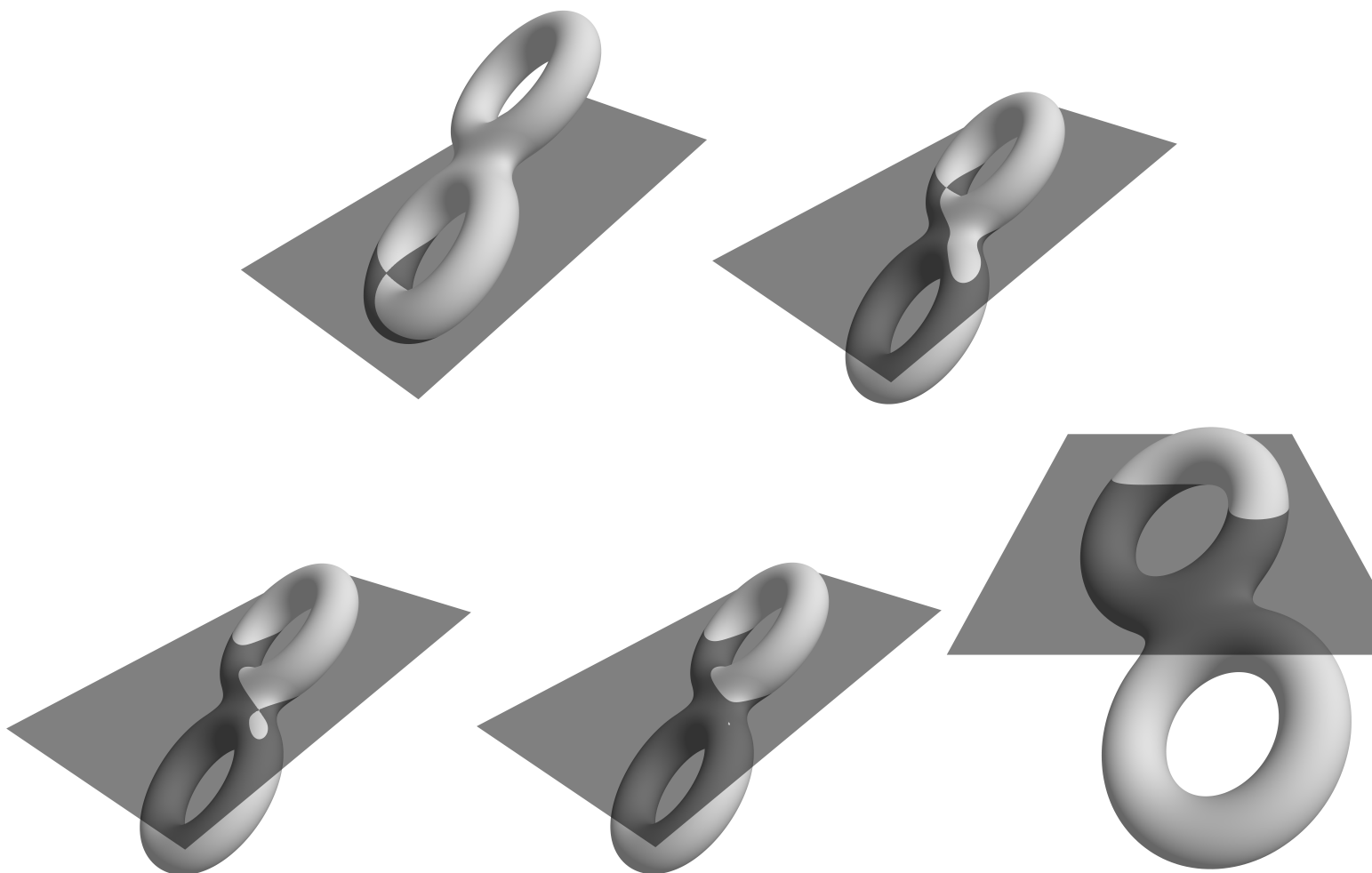
Consider a compact surface embedded in 3-dimensional Euclidean space:



Imagine still water slowly rises, filling up space, covering part of the surface.



The height of the water is a level set of the function  $(x, y, z) \mapsto z$ . By Sard's theorem, for all but a measure zero set of times, as the water rises, the surface of the water intersects our surface along a smooth curve. But at some critical moments, the intersection is *not* a smooth curve: at the first moment the water touches the surface, it touches at a single point. At the last moment, when water finally engulfs the surface, the intersection is just the top point. But there are some other critical moments:



And these are *not all*: find some more. At these moments, the height of water finds itself reaching a critical value of function  $z$  on the surface. In our pictures, we can see mostly saddle points, and one local minimum. A fish sees only the portion of the surface below water level, cut off by the water, the *submerged surface*. As the water rises toward the first saddle point, two pieces of the submerged surface draw together and join: a *handle* is attached. As the water rises toward a local maximum, a hole in the submerged surface is filled in. As the water reaches a local minimum, such as the global minimum in our example, a new piece of submerged surface is “born”: to the fish, it looks like a new surface submerging underwater. We hope to track these topological transitions, to put the whole surface together.

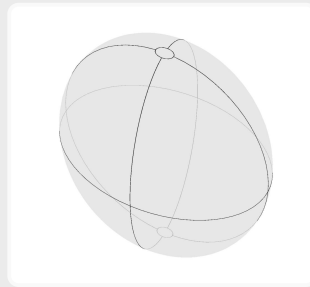
### Examples of Morse functions

Recall the discussion of Morse functions on page 371: a critical point of a function  $f: M \rightarrow \mathbb{R}$  on a manifold with corners  $M$  is nondegenerate if  $f''$  has no nonzero null

vectors.

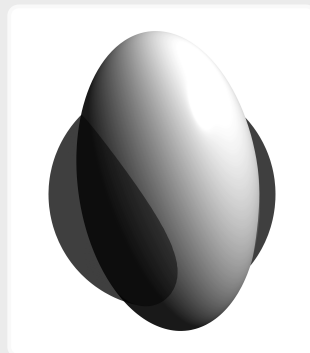
Every function on any zero dimensional manifold is Morse.

Picture an ellipsoid, with three different axis lengths, around the origin as the graph of a function on the sphere: we go outward or inward from a point of the sphere until we hit a point of the ellipsoid.

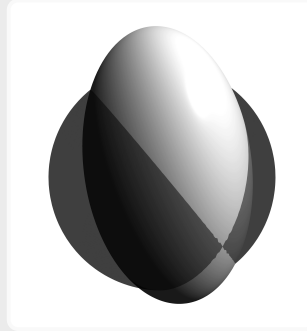


You can spot the two maxima, the two minima, and the two saddle points. The function is a Morse function on the sphere and drops to a Morse function on the real projective plane, but now with one maximum, one minimum, and one critical point.

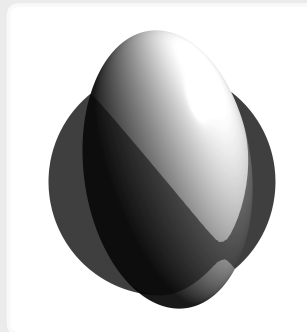
A constant function appears in this picture as a sphere, so the sublevel sets of our function are the points lying on the ellipsoid inside each sphere. On the sphere, we start with a pair of closed disks (at least, they are diffeomorphic to closed disks):



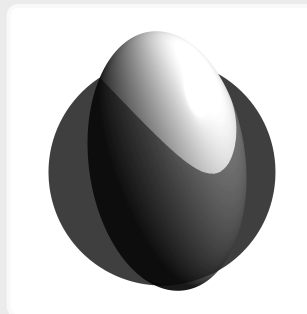
On the real projective plane, we start with a single one of the disks, as opposite points are identified. As we look at sublevel sets ( $f \leq c$ ), increasing the value of  $c$ , at the moment that the sphere of radius  $c$  hits the middle axis length:



these two disks overlap at a point. On the real projective plane, one disk overlaps itself at opposite points. As we slightly increase  $c$ , on the real projective plane, an interval of the boundary of the disk gets glued to another interval.



Follow the boundary with your finger around the previous picture and note the direction: the two intervals getting glued together are oriented the same way. This is peculiar, since gluing the top and bottom hemispheres of a sphere together, we glue along circles with opposite orientations. Its significance is that the gluing is being carried out with a twist, as if we glued together opposite sides of a disk with a belt, but the belts twists upside down as we travel along it. Finally, on the sphere, there are two open disks missing from the sublevel set, gradually growing smaller.



On the real projective plane, that is just one disk, attached along the connected boundary of the sublevel set.

**47.1** Prove that the function  $f = xy$  on the unit sphere in  $\mathbb{R}^3$  is Morse and drops to a Morse function on the real projective plane.

**47.2** Suppose that  $M \subseteq \mathbb{R}^{n+1}$  is an embedded hypersurface. Prove that there is a linear function on  $\mathbb{R}^{n+1}$  which restricts to  $M$  to be a Morse function. So we can rotate  $M$  to arrange that  $(x_1, \dots, x_n, y) \mapsto y$  is Morse. Prove that with an invertible linear transformation, we can arrange that all of the coordinate functions  $x = (x_1, \dots, x_{n+1}) \mapsto x_i$  are Morse functions.

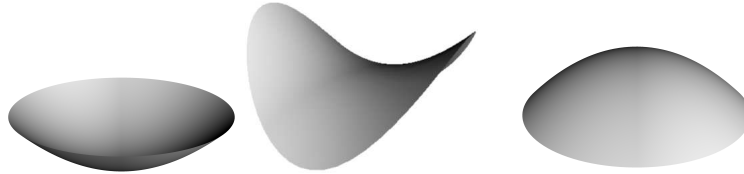
### Morse charts

**47.3\*** Prove *Hadamard's lemma*: if  $f$  is a  $C^k$  function defined in a ball in  $\mathbb{R}^n$  centered at the origin,  $1 \leq k \leq \infty$ , prove that there are  $C^{k-1}$  functions  $f_i$  defined in the same ball so that

$$f(x) = f(0) + x_1 f_1(x) + \dots + x_n f_n(x).$$

**47.4** Prove that the set of nondegenerate critical points of a smooth function is discrete.

The *index*  $\lambda = \lambda_{f, m_0}$  of a function  $f$  at a nondegenerate critical point  $m_0$  is the dimension of the largest linear subspace of  $T_{m_0} M$  on which  $f''(m_0)$  is negative definite. Indices 0, 1, 2:



Similarly define the *coindex* replacing negative by positive, i.e. the index of  $-f$ . If we compute  $f$  in coordinates in which  $m_0$  is the origin,

$$f(x) = f(0) + \frac{1}{2} f''(0)(x, x) + \dots$$

A *Morse chart* is a chart in which

$$f(x, y) = c - |x|^2 + |y|^2$$

for some constant  $c$ .

**Theorem 47.1** (Morse's lemma). *Every nondegenerate critical point of a smooth function lies in a Morse chart.*

*Proof.* The result is local, so we can work in chart, so assume our manifold is an open subset of Euclidean space. It is enough to find a chart in which  $f$  is quadratic:

$$f = f(0) + \frac{1}{2} f''(0)(x, x).$$

From linear algebra, we know that we can then choose our coordinates to diagonalize  $f''(0)$ , arranging that its diagonal entries are all  $\pm 1$ . By Hadamard's lemma,

$$f(x) = \sum_i x_i f_i(x).$$

Since 0 is critical,  $f_i(0) = 0$ , so apply Hadamard's lemma again to find

$$f(x) = \frac{1}{2} \sum_{i,j} x_i x_j f_{ij}(x).$$

If we replace  $f_{ij}$  by  $(f_{ij} + f_{ji})/2$ , we can assume that  $f_{ij} = f_{ji}$ . The diagonal entries  $f_{ij}(x)$  approach  $f''(0) = (f_{ij}(0))$  diagonal with  $\pm 1$  on the diagonal. Working in a small enough open set, the diagonal entries  $f_{ii}(x)$  are all nonzero. Let

$$y_1 := \sqrt{|f_{11}(x)|} \left( x_1 + \frac{1}{2f_{11}(x)} \sum_{j>1} f_{1j}(x) x_j \right).$$

Then, if  $\pm$  is the sign of  $f_{11}(0)$ ,

$$\pm y_1^2 = f_{11}x_1^2 + f_{12}x_1x_2 + \cdots + f_{1n}x_1x_n + \cdots = f(x) + \dots$$

where these  $\dots$  are bilinear in  $x_2, \dots, x_n$ . We have sorted one term  $y_1$  for our new coordinates. Inductively, we get the rest sorted:  $f$  is quadratic.  $\square$

## Excellent Morse functions

A Morse function is *excellent* if each critical value is taken on at a unique critical point.

The Morse function  $xy$  on the sphere is not excellent, but descends to an excellent Morse function on the real projective plane.

**Corollary 47.2.** *Excellent Morse functions are dense in the strong topology.*

*Proof.* Pick a Morse function. The critical points are discrete by the Morse lemma (theorem 47.1 on the preceding page). Label them by positive integers. Take bump functions, each equal to one near some critical point and vanishing near all others, and having compact support. Each bump function  $h$  has differential vanishing near the critical point, so only nonzero on a compact set where  $df \neq 0$ . If we scale that bump function to be very small, we can make its differential much smaller than that of  $f$  on that compact set. So  $f + h$  stays Morse, with the same critical points as  $f$ . We can build such a bump around each critical point, with any small value we like near the critical point. We can scale each bump to get it within the error tolerances of some given strong neighborhood. At the same time we can arrange that the critical values are all different.  $\square$

A Morse function is *ordered* if the critical values of all local minima are less than those of any critical points which are not local minima, and likewise the critical values of all local maxima are less than those of any critical points which are not local maxima.

**47.5** Prove that, given any Morse function  $f$  on any compact manifold, there is an ordered Morse function  $f + g$  where  $g$  vanishes except in Morse charts around local maxima and local minima of  $f$ , which are also local maxima and local minima, respectively, of  $f + g$ . Prove that every compact manifold admits an ordered excellent Morse function.

## Cylinders

**Lemma 47.3.** *Suppose that  $f: M \rightarrow \mathbb{R}$  is a smooth function, without critical points, on a manifold with corners  $M$ . There is a vector field  $X$  so that along the flow of  $X$ ,  $f$  increases at unit rate.*

*Proof.* Locally, pick coordinates  $x^1, \dots, x^n$  in which  $f = x^1$  and set  $X = \frac{\partial}{\partial x^1}$ . Use a partition of unity to glue.  $\square$

**Lemma 47.4.** *Suppose that  $M$  is a manifold and  $f: M \rightarrow \mathbb{R}$  is a smooth proper function without critical points. Take a level set  $H := (f = c_0)$ , any constant  $c_0 \in \mathbb{R}$ . Then  $M$  is diffeomorphic to  $H \times \mathbb{R}$  by a diffeomorphism  $\varphi: H \times \mathbb{R} \rightarrow M$  so that if  $m = \varphi(x, t)$  then  $f(m) = t$ .*

*Proof.* This argument is the same as that of Hoyo's theorem (theorem 51.6 on page 417, which also proves our lemma immediately), or even that of Ehressmann's theorem (theorem 51.4 on page 416), but we can give it directly. Pick a vector field  $X$  along whose flow  $f$  increases at unit rate, by lemma 47.3. The flow permutes the level sets of  $f$ , and is defined from one level set to all nearby ones by compactness.  $\square$

Given a Morse function  $f$  on a manifold  $M$ , denote by

$$\begin{aligned} M^a &= \{m \in M \mid f(m) \leq a\}, \\ M_a^b &= \{m \in M \mid a \leq f(m) \leq b\}, \\ M^{-a} &= \{m \in M \mid f(m) = a\}. \end{aligned}$$

**Theorem 47.5.** *Suppose that  $M$  is a compact manifold with boundary and  $f: M \rightarrow \mathbb{R}$  is a smooth function without critical points, equal to a constant  $f = a$  on some boundary components of  $M$  and to another constant  $f = b$  on all other boundary components. Suppose that  $a < b$ . Then  $M$  is a product  $M = [a, b] \times S$  where  $S$  is a compact manifold without boundary.*

*Proof.* Pick a vector field  $X$  so that  $Xf = 1$ , by lemma 47.3. The flow permutes the level sets of  $f$ , and is defined from one level set to all nearby ones by compactness. At  $f = b$ , all flow lines end. But no flow line ends earlier, since it lies in an interior point where  $X \neq 0$  or at a point where  $f = a$ , where  $X$  points interior to  $M$ , so has some positive amount of time left to flow. Similarly backwards in time, flow lines all begin at  $f = a$ . Let  $S := M^{-a}$ . Map

$$\varphi: (t, m) \in [a, b] \times S \mapsto e^{(t-a)X} m \in M.$$

Returning to our picture of each flow line, we see that  $\varphi$  is onto. At every point of  $[a, b] \times S$ ,  $\varphi'$  maps  $\partial_t$  to  $X$ . Note that  $X$  is not tangent to any level set. Along  $t = a$ ,  $m \mapsto \varphi(a, m)$  is the identity map to the level set  $S$ . So  $v \mapsto \varphi'(a, m)(0, v)$  maps onto  $T_m S$ . But then adding in  $\varphi'(a, m)\partial_t = X$ ,  $\varphi'(a, m)$  is onto. So  $\varphi$  is a local

diffeomorphism near  $t = a$  from some open set in  $[a, b] \times S$  to some open set in  $M$ . Interchanging  $\partial_t$  and  $X$ ,  $\varphi$  moves each level set of  $t$  to the associated level set of  $f$ , by the flow of  $X$ , a local diffeomorphism preserving  $X$  and permuting tangent spaces of the level sets. So  $\varphi'(t, m)$  is onto for any point  $(t, m) \in [a, b] \times S$ . So  $\varphi$  is a local diffeomorphism everywhere. If  $\varphi(t_0, m_0) = \varphi(t_1, m_1)$ , then applying  $f$ ,  $t_0 = t_1$ . But then applying  $e^{a-t_0}Y$ ,  $m_0 = m_1$ . So  $\varphi$  is a diffeomorphism.  $\square$

**Theorem 47.6.** *Suppose that  $M$  is a smooth manifold without boundary and  $f: M \rightarrow \mathbb{R}$  is a smooth proper function. Suppose that there are no critical points on  $M_a^b$ . Then  $M^a$  is diffeomorphic to  $M^b$  by a map which is the time one flow of a complete vector field on  $M$ .*

*Proof.* Take a slightly smaller value  $a' < a$  and a slightly larger value  $b' > b$ , so that  $M_{a'}^{b'}$  still has no critical points; this we can do because  $f$  is proper. Construct the vector field  $X$  as in theorem 47.5 on the previous page, on  $M_{a'}^{b'}$ , and then multiply by a suitable bump function to get it to have compact support inside  $M_{a'}^{b'}$  but still unchanged on  $M_a^b$ . The time  $b - a$  flow of  $Y$  takes  $M^a$  to  $M^b$  as required.  $\square$

## Pregradients

A *pregradient vector field* for a Morse function  $f$  is a complete vector field  $X$  so that  $Xf > 0$  at every regular point and, near each critical point  $m_0$  of  $f$ , there is a Morse chart, identifying our manifold with Euclidean space, in which  $X$  is the gradient of  $f$  in Euclidean space.

**47.6** Prove that every Morse function has a pregradient vector field.

**47.7** Prove that any two pregradient vector fields for the same Morse function, with the same choices of Morse charts near each critical point, live in a smooth connected 1-parameter family of pregradient vector fields.

A *unit rate vector field* is a vector field, defined only at the regular points of  $f$ , of the form

$$Y = \frac{1}{Xf}X.$$

Note that  $Yf = 1$ .

## Euclidean space

**Theorem 47.7.** *Suppose that  $M$  is a manifold and  $f: M \rightarrow [0, \infty)$  is a proper Morse function with one critical point. Then  $M$  is diffeomorphic to  $\mathbb{R}^n$  by a diffeomorphism identifying  $f$  with  $f(x) = c_0 + |x|^2$  where  $c_0 := \min f$ .*

*Proof.* Since  $f$  is proper,  $f$  has a minimum on every component, hence  $M$  is connected, with one critical point, its minimum, say at  $m_0$ . Replace  $f$  by  $f - f(m_0)$  to arrange that  $f(m_0) = 0$ . Take a Morse chart  $(U, \varphi)$  near  $m_0$ . Let  $X := 2x^i \partial_{x^i}$  on  $\mathbb{R}^n$ , i.e. the gradient of  $f$  in the usual Euclidean metric of the Morse chart, so  $Xf = 4f$ . Extend  $X$  by partition of unity to all of  $M$  so that  $Xf = 4f$ , so  $f \circ e^{tX} = e^{4t}f$ . Since  $f$  is proper, it has no maximum. Since the sublevel sets  $(f \leq a)$  are compact, we get some



positive flow time on each for which  $X$  keeps going, so  $X$  is complete. On our Morse chart,

$$\varphi(e^{tX}m) = e^{4t}\varphi(m),$$

Extend  $\varphi$  by

$$\varphi(m) := e^{4t}\varphi(e^{-tX}m)$$

where, to be specific, we can let  $t$  be anything large enough negative to force  $e^{-tX}m$  into the small ball, say

$$t := \frac{1}{4} \log((1 + \delta)f(m)/\varepsilon),$$

for some fixed  $\delta > 0$ . By construction,  $\varphi_*X = Y$ . If  $x := \varphi(m)$ , then  $f(m) = |x|^2$ , so  $\varphi$  is onto. Using flows of  $X$  and  $Y$  backwards in time to again flow any point inside the ball of radius  $\varepsilon$ , we find that  $\varphi$  is a local diffeomorphism. If  $\varphi(m_1) = \varphi(m_2)$ , both have the same value of  $|x|^2$ , i.e. lie on the same sphere, so flow backward to the ball of radius  $\varepsilon$  together, and so  $\varphi$  is injective. So  $\varphi$  is a diffeomorphism.  $\square$

## Milnor spheres

Take a diffeomorphism  $\varphi: S^{n-1} \rightarrow S^{n-1}$ . For any  $x \in \mathbb{R}^n$ , say

$$x = (x_1, \dots, x_{n-1}, x_n),$$

define  $\bar{x}$  to mean

$$\bar{x} = (x_1, \dots, x_{n-1}, -x_n).$$

(Equivalently, we can take  $x \mapsto \bar{x}$  to be an linear isometric reflection map.) Glue two copies of  $\mathbb{R}^n$  together by the map

$$x \mapsto y = \frac{\varphi(\bar{x}/|x|)}{|x|}$$

defined away from  $x = 0$  and  $y = 0$ . The resulting smooth manifold is a *Milnor sphere*  $S_\varphi^n$ .

If we take  $\varphi(x) = x$  then  $y = \bar{x}/|x|^2$  is the transition map of the sphere in Ptolemaic projection on page 26.

**47.8\*** Consider the identity map on the first copy of  $\mathbb{R}^n$ . Prove that it extends to a unique homeomorphism of any two Milnor spheres. Hence prove that every Milnor sphere is homeomorphic to the sphere. When is the map a diffeomorphism?

**Theorem 47.8** (Reeb). *Suppose that  $M$  is a compact manifold of positive dimension. The following are equivalent:*

- *The manifold  $M$  bears a Morse function with no more than two critical points.*
- *The manifold  $M$  is a Milnor sphere.*

*Every Milnor sphere is homeomorphic to the sphere of the same dimension. Any two Milnor spheres  $S_\varphi^n, S_\psi^n$  with  $\varphi$  smoothly isotopic to  $\psi$  are diffeomorphic. If  $x \mapsto \bar{x}$  is any linear reflection,  $S_{\bar{\varphi}}^n$  is diffeomorphic to  $S_\varphi^n$ . So Milnor spheres depend, up to diffeomorphism, only on diffeotopy classes of orientation preserving diffeomorphisms of spheres. Every Milnor sphere of dimension 0, 1 or 2 is diffeomorphic to the standard sphere of the same dimension.*

*Proof.* Suppose that  $M$  has a Morse function  $f$  with no more than two critical points.

**47.9** Prove that  $M$  is connected and  $f$  has exactly one local maximum, one local minimum, and no other critical points.

Rescale to arrange that the minimum of  $f$  is 0 and the maximum is 1, occurring at points  $m_0, m_1$  respectively, the *south* and *north* poles. Let  $M_0 := M - \{m_1\}$  and  $M_1 := M - \{m_0\}$ . On  $M_0$ ,  $f/(1-f)$  is a proper Morse function with one critical point, so  $M_0$  is diffeomorphic to Euclidean space by theorem 47.7 on page 386. So  $M$  is a one point compactification of the ball [13], so homeomorphic to the sphere  $S^n$ . By the same reasoning  $M_1$  is also diffeomorphic to Euclidean space. We can make better use of theorem 47.7 on page 386: we arrange  $M_0 = \mathbb{R}_x^n$  and  $f/(1-f) = |x|^2$ . Similarly, using  $1/f - 1$  instead of  $f/(1-f)$ , we make  $M_1 = \mathbb{R}_y^n$  and  $1/f - 1 = |y|^2$ . We then calculate that  $|x||y| = 1$  at corresponding points of  $M_{01} = M_0 - \{m_0\} = \mathbb{R}^n - \{0\}$ . So if  $x = ru$  with  $|u| = 1$ ,

$$y = \frac{\varphi_r(u)}{r}$$

for some smooth map

$$\varphi: (0, \infty)_r \times S^{n-1}_u \rightarrow S^{n-1}, (r, u) \mapsto \varphi_r(u),$$

an diffeotopy of sphere diffeomorphisms.

We can alter the choice of this gluing by any diffeomorphism of the Euclidean spaces  $M_0, M_1$ , i.e.  $G \circ \varphi \circ F$  for  $F, G: B \rightarrow B$  diffeomorphisms, without affecting the diffeomorphism type.

**47.10\*** Prove that we can arrange that  $\varphi_r$  is constant in  $r$ , a single diffeomorphism  $S^{n-1} \rightarrow S^{n-1}$  preserving orientation.

The zero dimensional sphere  $S^0$  is a pair of points, so its diffeomorphisms are either the identity or the antipodal map. Hence if  $M$  has dimension 1,  $M$  is the circle, and  $x, y$  are Ptolemaic projections.

The diffeomorphisms of the circle that preserve orientation are smoothly isotopic to the identity map, and  $x, y$  are Ptolemaic projections. Hence if  $M$  has dimension 2,  $M$  is the 2-sphere.

If  $M = S^n_\varphi$  is a Milnor sphere, let

$$f = \frac{|x|^2}{1 + |x|^2} = \frac{1}{1 + |y|^2}.$$

**47.11** Prove that  $f$  is a Morse function on  $M$  with two critical points:  $x = 0$  and  $y = 0$ .

□

## Notes

The term *Milnor sphere* is not standard. An *exotic sphere* is a manifold homeomorphic but not diffeomorphic to a sphere. It is not clear from our definition of Milnor spheres that there might not be some diffeomorphism between a Milnor sphere and the usual sphere, perhaps far from the homeomorphism we wrote down. In other words, our definition does not make clear whether Milnor spheres are exotic. On the other hand, the definition of exotic sphere does not make clear whether they should arise as Milnor spheres.

## Chapter 48

### Morse theory II: attaching handles

#### Birth and death

**Theorem 48.1.** *Suppose that  $M$  is a compact manifold and  $f: M \rightarrow \mathbb{R}$  is a Morse function with a critical point  $m_0 \in M$ . Take regular values  $a < b$  so that  $m_0$  is the unique critical point in  $M_a^b$  and  $f$  is proper on  $M_a^b$ . Let  $n$  denote the dimension of  $M$  and  $\bar{B} \subset \mathbb{R}^n$  some closed ball. If  $m_0$  is a local minimum then  $M^b$  is diffeomorphic to  $M^a \sqcup \bar{B}$  (a birth). If  $m_0$  is a local maximum then  $M^a$  is diffeomorphic to  $M^b \sqcup \bar{B}$  (a death).*

*Proof.* The case of a local maximum follows from that of a local minimum by replacing  $f$  by  $-f$ , so consider a local minimum. Without loss of generality, suppose  $a = -1$ ,  $b = 1$  and  $f(m_0) = 0$ . In a Morse chart near  $m_0$ , our Morse function  $f$  is, up to constant,  $f(x) = |x|^2$  with  $m_0$  the origin. The points where  $f < 0$  are outside of the Morse chart. The flow lines of any pregradient  $X$  repel in directions pointing out from  $m_0$ , so points outside the Morse chart never enter the Morse chart. In particular, points where  $f < 0$  never enter the Morse chart.

Let  $Y$  be the unit rate vector field associated to  $X$ :  $Y = (Xf)^{-1}X$ . The flow lines of  $Y$  are reparameterized flow lines of  $X$ , so also repel from our Morse chart, and so flow lines of  $Y$  outside the Morse chart never enter. So  $Y$  flows  $M^a$  diffeomorphically to an open subset of  $M^b$ : the complement of the domain of the Morse chart. Since  $M^a$  is compact, this open subset is both open and closed, a union of components of  $M^b$ . Inside the Morse chart,  $M^b$  intersects the Morse chart in a closed ball.  $\square$

**48.1** Suppose that  $M$  is a compact manifold and  $f: M \rightarrow \mathbb{R}$  is a Morse function, perhaps not excellent. Take regular values  $a < b$  so that  $f$  is proper on  $M_a^b$  and every critical point of  $f$  inside  $M_a^b$  is a local maximum or local minimum. How can  $M^a$  and  $M^b$  be related?

#### Ascending and descending cells

With a choice of pregradient vector field, a point  $m \in M$  *ascends* to a critical point  $m_0 \in M$  if  $e^{tX}m \rightarrow m_0$  as  $t \rightarrow \infty$ , and *descends* to  $m_0$  if  $e^{tX}m \rightarrow m_0$  as  $t \rightarrow -\infty$ . The *ascending cell* to a critical point is the set of points ascending to it; similarly the *descending cell* is the set of points descending to it.

**Lemma 48.2.** *Suppose that  $f$  is a proper Morse function bounded from below. Then every point ascends to a unique critical point. Suppose that  $f$  is a proper Morse function bounded from above. Then every point lies descends to a unique critical point.*

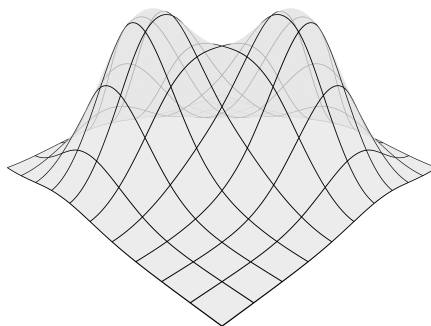
*Proof.* Suppose that  $f$  is bounded from above and proper. Take a point  $m_0$  not ascending to a critical point. In particular,  $m_0$  is not a critical point, so has a flow line which is not just a point. Consider the values  $f(m)$  for points  $m$  on that flow line. Because  $f$  is bounded from below, these values have a supremum. Because  $f$  is proper, any sequence of points  $m_1, m_2, \dots$  for which  $f(m_1), f(m_2), \dots$  approaches that supremum has a convergent subsequence, say converging to some point  $m'_0$ . Take that convergent subsequence  $m_1, m_2, \dots$  for which  $f(m_1), f(m_2), \dots$  increases strictly, so  $m_j = e^{t_j X} m_0$  for some  $t_1 < t_2 < \dots$

If the limit  $m'_0$  is a critical point then take a Morse chart around  $m'_0$ . The points  $m_1, m_2, \dots$  eventually all lie in that Morse chart and approach its critical point, all on a single flow line, so ascend to  $m'_0$ .

So we can assume that  $m'_0$  is not critical, so  $X(m'_0) \neq 0$ . In a flow box around  $m'_0$ , we see that  $m'_0$  lies on the same flow line as all but finitely many of  $m_1, m_2, \dots$ . Since  $Xf > 0$ , we can flow along  $X$  from  $m'_0$  to points of the same flow line with larger values of  $f$ , a contradiction.  $\square$

**48.2** Prove that the ascending and descending cells are diffeomorphic to  $\mathbb{R}^\lambda, \mathbb{R}^{n-\lambda}$ , transverse, and intersect only at their critical point.

### The mountain pass lemma



**Theorem 48.3** (The mountain pass lemma). *Suppose that a proper Morse function on a connected manifold has two or more local maxima and is bounded from above. Then it has a critical point of coindex 1, whose ascending cell locally separates the ascending cells of two critical points. The number of critical points of coindex 1 is at least one less than the number of local maxima.*

Similar results hold if we replace *max* with *min* and *coindex* with *index*, but we state it this way to make the picture look like a mountain pass. Similar results also hold for a much larger class of functions than proper Morse functions, but we don't need the general mountain pass theory.

*Proof.* Each local maximum has an ascending cell which is an open set. Different local maxima have disjoint ascending cells. If the only critical points are local maxima, these disjoint open sets cover. But the manifold is connected, so there is only one such open set, i.e. only one local maximum.

Suppose that there is more than one local maximum. The boundary of the ascending cell of a local maximum is invariant under the flow of the pregradient. Every flow line lies in an ascending cell. But no point of that boundary lies inside the ascending cell of any local maximum, since such an ascending cell is open. So these boundary flow lines ascend to critical points which are not local maxima. Watch a boundary flow line enter some Morse chart. So do all nearby flow lines. If the ascending cell does not separate as a hypersurface, then all of the flow lines nearby belong to the ascending cell of the same local maximum. If this occurs for all flow lines on that boundary, then all nearby flow lines ascend to the same local maximum. But then the closure of that ascending cell is both open and closed, so is the entire manifold: only one local maximum.

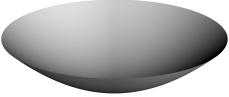
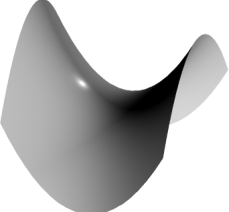
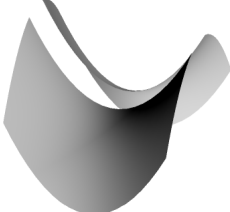
Neighboring local maximum, i.e. those whose ascending cells have intersecting closures, have an ascending cell hypersurface on that intersection. By connectivity, a third local maximum has ascending cell closure intersecting at least one of these, and so on.  $\square$

**48.3** Prove that a compact manifold is a disjoint union of finitely many Milnor spheres just when it bears a Morse function whose critical points are all either local minima or local maxima.

## Handles

The  $n$ -dimensional  $\lambda$ -handle is the manifold with corners  $H^\lambda := \bar{B}^\lambda \times \bar{B}^{n-\lambda}$ . The *rim* of the handle is

$$H_o^\lambda = (\bar{B}^\lambda - \{0\}) \times \bar{B}^{n-\lambda}.$$

| $\lambda$ | handle                                                                              | rim                                                                                  |
|-----------|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|
| 0         |  | empty                                                                                |
| 1         |  |  |
| 2         |  |  |

Given an  $n$ -dimensional manifold with boundary  $M$ , a  $\lambda$ -handle attached to  $M$  is

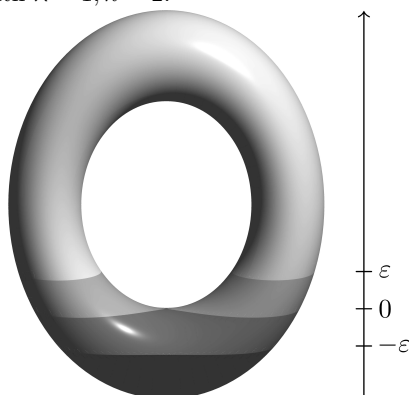
- an  $n$ -dimensional manifold with boundary  $M'$  and
- a tidy embedding  $M \rightarrow M'$  and
- a tidy embedding  $H^\lambda \rightarrow M'$  of an  $n$ -dimensional  $\lambda$ -handle

so that the images of the two embeddings overlap on the image of the rim.

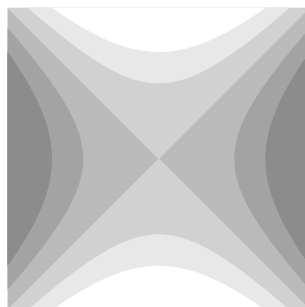
**48.4** Prove that adding a  $\lambda$ -handle can increase the de Rham cohomology by one dimensional, but only in degree  $\lambda$ .

### Attaching handles

Consider a Morse chart, say for example with  $f(x, y) = -|x|^2 + |y|^2$ , which we draw in a simple example with  $\lambda = 1, n = 2$ .

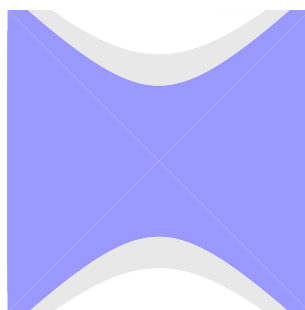


We can draw regions in the plane where the function takes values in various intervals.



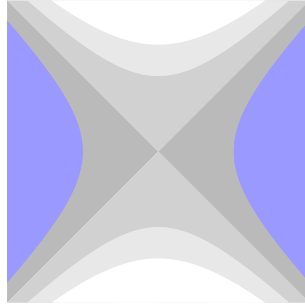
As the water rises, for small  $\varepsilon > 0$ , the sublevel set  $M^\varepsilon$  is:

$$|y|^2 \leq |x|^2 + \varepsilon :$$

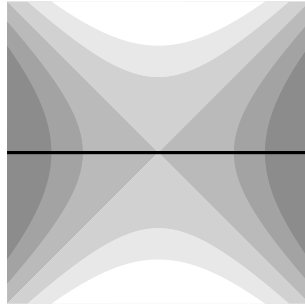


When the water was lower down:  $M^{-\varepsilon}$  is

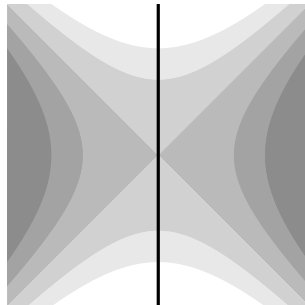
$$|x|^2 \geq |y|^2 + \varepsilon :$$



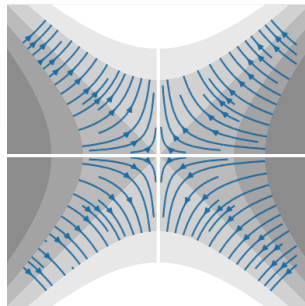
The ascending cell:



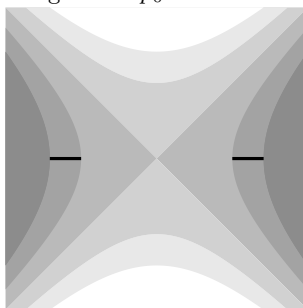
and the descending cell:



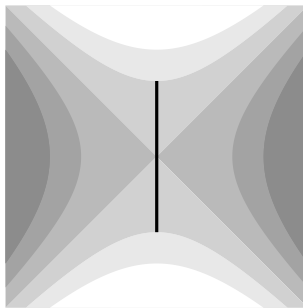
If we cut out those cells, the water flows (i.e.  $Y$  flows) the other points of  $M^{-\varepsilon}$  up to the points of  $M^{\varepsilon}$ , in time  $2\varepsilon$ .



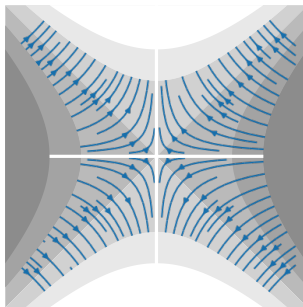
We don't need to cut the entire ascending cell out. The *ascending annulus*  $A$  is the set of points of the ascending cell of  $p_0$  on which  $-2\varepsilon \leq f \leq -\varepsilon$ :



Similarly the *descending disk*  $D$  is the set of points of the descending cell on which  $f \leq \varepsilon$ :



If we cut out the ascending annulus and the descending disk, the flow is a diffeomorphism:



The reader can generalize the definitions to an arbitrary proper Morse function and arbitrary regular sublevel sets  $M^a \subset M^b$ .

**Theorem 48.4.** *Take a proper Morse function  $f: M \rightarrow \mathbb{R}$  on a manifold  $M$  without boundary. Pick a pregradient field. Suppose that there are no critical points of  $f$  with  $a \leq f < b$ . Then  $M^b$  is diffeomorphic to  $M^a - A$  with a  $\lambda$ -handle attached for each critical point on the level set  $f = b$ , where  $A$  is an ascending annulus of that pregradient field.*

*Proof.* For simplicity, suppose that  $p_0$  is the unique critical point in  $M_a^b$ ; the general case is just more notation since the critical points are isolated. Let  $Y$  be the unit rate vector field associated to the pregradient field  $X$ . Clearly we can take  $a$  and  $b$



as close to  $f(p_0)$  as we like, so without loss of generality,  $a = -\varepsilon$ ,  $f(p_0) = 0$ ,  $b = \varepsilon$ . Through any point  $m_0 \in M$ , the flow of  $Y$  is defined for some positive time  $t$ , the time being the difference  $t = f(m_1) - f(m_0)$  where  $m_1$  is the unique critical point whose ascending cell contains  $m_0$ . Similarly for negative time. Since we are only concerned about the critical point  $p_0$ , we multiply  $Y$  by a compactly supported function on  $M$ , equal to 1 on  $M_{-\varepsilon}^\varepsilon$ , so that  $Y$  is complete and permutes level sets between  $f = -\varepsilon$  and  $f = \varepsilon$ .

The ascending annulus  $A$  is precisely the set of points of  $M^{-\varepsilon}$  which, flowing by  $Y$ , reach  $p_0$  in some time  $t$ , with  $\varepsilon \leq t \leq 2\varepsilon$ . These are precisely the points of  $M^{-\varepsilon}$  on which the flow of  $Y$  through is not defined for such times.

The descending disk  $D$  is precisely the set of points of  $M^\varepsilon$  which descend to  $p_0$ , so not on any flow line from  $M^{-\varepsilon}$ .

The flow of  $Y$  for time  $t = 2\varepsilon$  is a diffeomorphism

$$M^{-\varepsilon} - A \rightarrow M^\varepsilon - D.$$

but we have to cut out the ascending annulus to avoid it colliding with the origin.

Note that so far, we have taken *no* charts, and are working globally, although the picture of a Morse chart helps to see what we are doing.

In our Morse chart

$$X = -2x_i \partial_{x_i} + 2y_i \partial_{y_i}$$

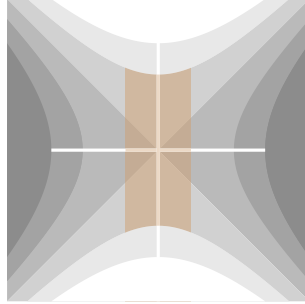
and  $Y = X/4r^2$  with  $r^2 = |x|^2 + |y|^2$ . Taking  $\varepsilon$  small enough, the ascending annulus and descending disk lie inside the chart:

$$A = \{ (x, 0) \mid \varepsilon \leq |x|^2 \leq 2\varepsilon \},$$

$$D = \{ (0, y) \mid 0 \leq |y|^2 \leq \varepsilon \}.$$

So  $A$  is a closed annulus of dimension  $\lambda$ , and  $D$  is a closed ball, of dimension  $n - \lambda$ , justifying their names.

Map a handle  $H^\lambda \rightarrow M^\varepsilon$  to cover the descending disk  $D$ :



To be precise, we use the map

$$\varphi: (x, u) \in H^\lambda = B^\lambda \times \bar{B}^{n-\lambda} \mapsto (x, \sqrt{|x|^2 + \varepsilon}) \in M^\varepsilon$$

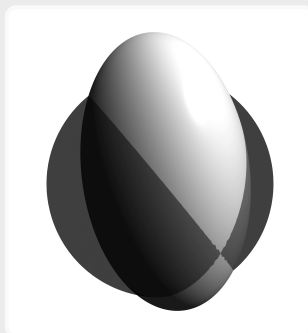
which is a diffeomorphism to its image, an open subset of  $M^\varepsilon$ , slightly thickening  $D$  in the horizontal directions in our picture.  $\square$

Cutting out the ascending annulus has essentially no effect on  $M^\varepsilon$ , since we can flow along our vector field backwards in time, to some time earlier than  $-\varepsilon$ , taking all of  $M^\varepsilon$  diffeomorphically a subset of  $M^\varepsilon$ .

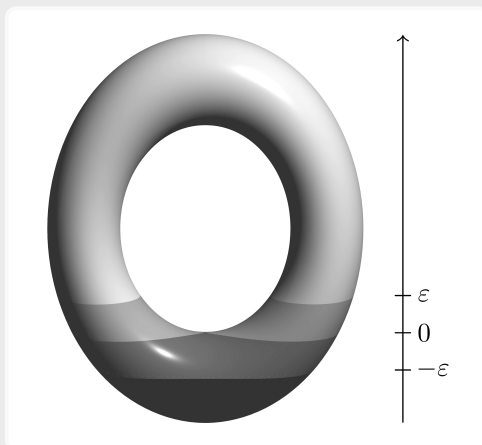
**48.5** Prove that cutting out the ascending annulus has no effect on homotopy, homology and de Rham cohomology groups.

**48.6** Prove that if a compact manifold has a proper Morse function with no critical points of index 1 then it is simply connected. Hint: Seifert–Van Kampen.

The real projective plane has a single 1-handle attached:



The torus has two 1-handles; let's look at the lower one:

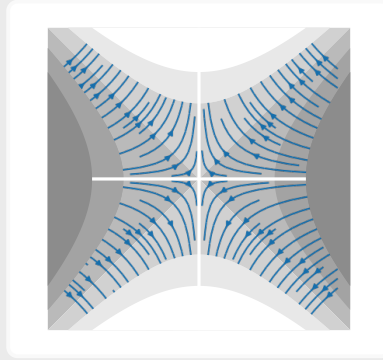


The pictures in a Morse chart are identical. Globally, each is a 1-handle attached to a disk; in each picture, the circle boundary of the disk pinches together at a point. But globally, they are different: on the real projective plane, after we attach the 1-handle, our surface below the water is compact and smooth but still with one boundary component. (Refer to the components of regular level sets of  $f$  as *circles*, because they are compact curves, so diffeomorphic to circles.) On the torus, after we attach the 1-handle, our surface below the water is compact with two boundary circles. The submerged portion of the torus remains orientable all the time during its bath. As the circle boundary splits into two, the orientation stays the same; at the instant of the handle attachment, the pattern of orientations at  $(f = 0)$  is consistent:

each incoming strand goes out again across the critical point, in the same direction.

Think of our picture of the real projective plane: we match points of the ellipsoid which lie inside our dark sphere with their opposite points. Draw an orientation of the circle ( $f = -\varepsilon$ ) on the real projective plane. On our ellipsoid picture, we draw antipodally matched orientations on that circle's two antipodally matched preimages. As we reach the handle attachment, the pattern of orientations has two strands coming in at the critical point, pointing directly at one another, and the two others going out directly away from one another: inconsistent.

In the Morse chart picture,



imagine we reflect the right half of the picture about the horizontal, leaving the left half alone. After some such reflection, we can glue together two pieces of surface with opposite orientation: a *twisted handle*. For small enough  $\varepsilon > 0$ ,  $M_{-2\varepsilon}^-$  is orientable. After a twisted handle,  $M_{-\varepsilon}^\varepsilon$  is not orientable. The number of boundary circles ( $f = -\varepsilon$ ) is the same as that of ( $f = \varepsilon$ ).

On the other hand, an untwisted handle on a surface leaves  $M_{-\varepsilon}^\varepsilon$  orientable, since it matches orientations at the gluing. It also either adds or subtracts one boundary circle.

Take distinct real constants  $c_0, c_1, \dots, c_n$ . Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ; let  $d = \dim_{\mathbb{R}} k = 1, 2$  or  $4$ . Write elements of  $V = k^{n+1}$  as

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

The function

$$f: x \in k^{n+1} \mapsto c_0|x_0|^2 + c_1|x_1|^2 + \dots + c_n|x_n|^2 \in \mathbb{R}$$

restricts to the unit sphere around the origin  $S^{dn+d-1} \subset k^{n+1}$  to be invariant under rescaling by unit scalars:

$$x \mapsto xa$$

for  $a \in k$  with  $|a| = 1$ . Therefore  $f$  descends to the quotient space  $\mathbb{P}^n = \mathbb{P}_k^n$ .

The open subset of  $S^{dn+d-1}$  on which  $x_0 \neq 0$  maps to an open subset of  $\mathbb{P}^n$ . On that open set, we can scale  $x \mapsto xa$  uniquely to get  $x_0 > 0$  positive real. Then the remaining  $x_1, \dots, x_n$  are coordinates on  $\mathbb{P}^n$ . It is convenient to let

$$y_j := |x_0| x_j x_0^{-1}.$$

These  $y = (y_1, \dots, y_n)$  in the unit ball in  $k^n$  form coordinates on our open set in  $\mathbb{P}^n$ . In these coordinates,

$$f = c_0 + (c_1 - c_0)|y_1|^2 + (c_2 - c_0)|y_2|^2 + \dots + (c_n - c_0)|y_n|^2$$

has a critical point precisely where  $y = 0$ . At this point, the index  $\lambda$  is given by:  $\lambda/d$  is the number of  $c_j$  so that  $c_j < c_0$ . So  $f$  is an excellent Morse function with indices  $0, d, 2d, \dots, (n-1)d, nd$ . If we write  $\smile$  to indicate that we cut out some ascending annulus when we glue,

$$\mathbb{P}^n = H^0 \smile_{H^d_0} H^d \smile_{H^{2d}_0} \dots \smile_{H^{(n-1)d}_0} H^{(n-1)d} \smile_{H^{nd}_0} H^{nd}.$$

Pretending that  $\smile = \cup$  gives the right homology, homotopy and cohomology.

Each regular level set of a Morse function  $f$  has a unique flow line passing through it. So if we have a regular value  $b$  nestled between successive critical values  $a < b < c$ , the regular level set ( $f = b$ ) is naturally identified with the topological space of flow lines that pass somewhere in the manifold ( $a < f < c$ ). When we cross a critical value, the sublevel set gets a handle attached, while the level set loses the inner sphere from the ascending annulus and gains in place of it the boundary of the descending disk. So if  $c_0 < c_1 < c_2$  are successive critical values, then the space of flow lines passing through  $c_0 < f < c_1$  transforms into the space of flow lines passing through  $c_1 < f < c_2$  by removing a sphere  $S^{\lambda-1}$  and gluing in a sphere  $S^{n-\lambda-1}$  in its place.

We find a Morse function on the rotation group  $\mathrm{SO}(n)$ . Consider the symmetric bilinear form

$$\langle A, B \rangle = \mathrm{tr}(AB) = A_{ij}B_{ji}.$$

where  $\mathrm{tr}$  is the trace of a square matrix.

**48.7** Prove that, in this inner product, symmetric matrices are orthogonal to antisymmetric, and that the inner product is positive definite on symmetric matrices, negative definite on antisymmetric.

Take an  $n \times n$  symmetric matrix  $A$  with  $n$  distinct eigenvalues. Consider the function

$$f: g \in \mathrm{SO}(n) \mapsto f(g) = \langle A, g \rangle.$$

**48.8** Prove that the critical points of  $f$  are the matrices  $g \in \mathrm{SO}(n)$  preserving or reflecting every eigenline of  $A$ . So  $f$  has exactly  $2^{n-1}$  critical points.

Suppose that  $A$  is diagonal with nonzero eigenvalues  $\lambda_i$  and  $g$  is diagonal with diagonal entries  $\varepsilon_i = \pm 1$ . We compute the second derivative using a matrix exponential: for antisymmetric  $B$ ,

$$\begin{aligned} \left. \frac{d^2}{dt^2} \right|_{t=0} f(ge^{tB}) &= \left. \frac{d^2}{dt^2} \right|_{t=0} \text{tr}(Ag(I + tB + t^2B/2 + \dots)), \\ &= \text{tr}(AgB^2), \\ &= A_{ij}g_{jk}B_{k\ell}B_{\ell i}, \\ &= \lambda_i \delta_{ij} \varepsilon_j \delta_{jk} B_{k\ell} B_{\ell i}, \\ &= \lambda_i \varepsilon_i \sum_{\ell} B_{i\ell}^2. \end{aligned}$$

Keep in mind that an antisymmetric matrix  $B$  does not have all of its entries free; we can only pick  $B_{i\ell}$  for  $\ell = 1, 2, \dots, i-1$ . So we write it as

$$\varepsilon_i \lambda_i \left( \sum_{\ell < i} B_{i\ell}^2 + \sum_{i < \ell} B_{\ell i}^2 \right).$$

Careful: there is a sum here, and terms with various signs, so their could be cancellations. Indeed every  $B_{ij}$  with  $i < j$  appears once with  $\varepsilon_i \lambda_i$  and once with  $\varepsilon_j \lambda_j$ . To avoid cancellations, pick each  $\lambda_i > 0$  large enough to avoid cancelling out any terms containing  $\lambda_1, \dots, \lambda_{i-1}$ . Since the expression is a sum of squares, every critical point is nondegenerate: the function  $f$  is Morse. The cohomology of  $\text{SO}(n)$  is generated by at most  $2^{n-1}$  differential forms.

## The Morse inequalities

Picture attaching a handle via a proper Morse function as above. The *handle core* is  $\bar{B}^\lambda \times \{0\}$ . The handle retracts to its core, so topologically they are essentially the same. Each handle core has its boundary sitting inside the rim:  $S^{\lambda-1} \times \{0\} \cong S^{\lambda-1}$ . Since the rim sits in the image of  $M^{-\varepsilon} - A$ , so does the handle core boundary. Each handle core is a chain in our manifold, sitting in some  $M^\varepsilon$ , and its boundary sits in  $M^{-\varepsilon}$ . But  $M^{-\varepsilon}$  retracts down to the next handle attachment. So the components of that boundary retract into other handle cores further down.

Suppose now that  $M$  is compact with a Morse function and a pregradient. The handle cores and their boundaries generate a finite dimensional subcomplex of the chain complex, say  $M_\bullet \subset C_\bullet$ . The complex  $M_\bullet$  has Hilbert–Poincaré series

$$P_{M_\bullet}(t) = \sum t^\lambda$$

a sum over critical points. By Poincaré duality and the Mayer–Vietoris theorem, the real homology  $H_\bullet := H_\bullet(M, \mathbb{R})$  is generated by the handles.

**Theorem 48.5** (The Morse inequalities). *For any compact manifold  $M$  equipped with a Morse function and a pregradient,*

$$P_{M_\bullet}(t) = P_{H_\bullet}(t) + (1+t)P_{B_\bullet}(t).$$

*Proof.* Apply theorem 40.1 on page 325. □

**Theorem 48.6** (The lacunary principle). *Take a Morse function  $f$  on a compact manifold. Suppose that the indices of any two critical points of  $f$  differ by two or more. Then*

$$P_{M_\bullet}(t) = P_{H_\bullet}(t)$$

where  $H_\bullet := H_\bullet(M, \mathbb{R})$ .

*Proof.* Suppose that  $M_{i-1} = 0$ . The short exact sequences

$$0 \longrightarrow B_\bullet \longrightarrow M_\bullet \longrightarrow H_\bullet \longrightarrow 0$$

applied to  $M_{i-1}$  forces  $B_{i-1} = 0$ . This applies to

$$0 \longrightarrow Z_\bullet \longrightarrow M_\bullet \longrightarrow B_\bullet[-1] \longrightarrow 0$$

to give an isomorphism  $Z_i \rightarrow M_i$ , so  $B_i = 0$ . We have  $B_{i-1} = 0$  and  $B_i = 0$ . Applied to all  $i$ ,  $B_\bullet = 0$ .  $\square$

The complex and quaternionic projective spaces, we have seen, have a unique homology generator in each dimension  $0, d, 2d, \dots, nd$  if  $d = 2, 4$  respectively. Since there is a single local minimum, each projective space is connected. Since there are no 1-handles, it is also simply connected. These are the indices of the Morse function we wrote. These are lacunary, so the lacunary principle correctly predicts the dimensions of the homology groups. Indeed the boundary operators are trivial so the homology is freely generated:

$$H_p(\mathbb{CP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

$$H_p(\mathbb{HP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } p = 0, 4, 8, \dots, 4n, \\ 0, & \text{otherwise.} \end{cases}$$

## Notes

Our definition of handle attachment is not quite the standard one. The term *rim* is not standard. The ascending cell is more often called the *stable manifold*, but ascending cell is surely a better name, and becoming more popular. Similarly the descending cell is more often called the *unstable manifold*.

## Chapter 49

### Homotopy groups

We use differential geometric methods to look at homotopy theory.

Recall, from any textbook on algebraic topology, the homotopy groups of topological spaces [8]. For any topological space  $X$  with marked point  $x_0$ , and any  $k \geq 1$ , let  $\pi_k(X, x_0)$  be the set of all continuous maps  $[0, 1]^k \rightarrow X$  taking every boundary point of  $[0, 1]^k$  to  $x_0$ , modulo homotopy through such maps. Each  $\pi_k(X, x_0)$  has a distinguished point: the constant map.

For any vector space  $X$ ,  $0 = \pi_k(X, x_0)$  for all  $k$ : rescale any map continuously to become a constant map to the origin.

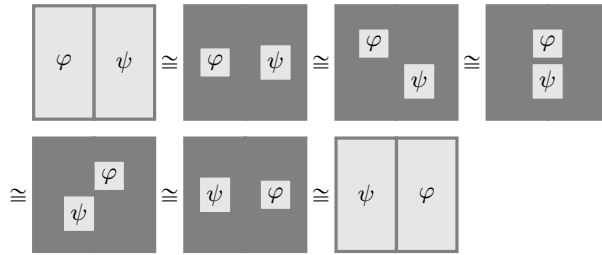
There is a special case:  $k = 0$ , for which we redefine  $\pi_0(X, x_0)$  to be the set of all path components of  $X$ , with a special chosen path component: that of  $x_0$ .

For  $k = 1, 2, \dots$ , each  $\pi_k(X, x_0)$  is a group, with multiplication being that we push two cubes  $[0, 1]^k$  beside one another, squish in half to get one cube  $[0, 1]^k$ , and make a new map from the two old ones by mapping the squished together cube's left half and right half using the given maps. (For  $k = 0$ ,  $\pi_0(X, x_0)$  is usually not equipped with any natural group structure; it is just a set.)

**Theorem 49.1.** *The groups  $\pi_k(X, x_0)$  are abelian for  $k \geq 2$ .*

Therefore we will henceforth refer to the group operation as addition for these groups, and use additive notation.

*Proof.* One can write out a long and complete proof using a picture like



where the “thick dark margins” are thick regions of the box in which the map is constant, mapping to  $x_0$ .  $\square$

If  $\varphi: X \rightarrow Y$  is a continuous map, every map  $\psi: S^k \rightarrow X$  gives a map  $\varphi \circ \psi: S^k \rightarrow Y$ , taking homotopies to homotopies, so a group morphism

$$\pi_k(X, x_0) \rightarrow \pi_k(Y, y_0)$$

where  $y_0 := \varphi(x_0)$ .

**49.1** For any topological spaces  $X, Y$  and points  $x_0 \in X, y_0 \in Y$ , describe an isomorphism of groups

$$\pi_k(X \times Y, (x_0, y_0)) = \pi_k(X, x_0) \times \pi_k(Y, y_0).$$

**Theorem 49.2.** If  $X \rightarrow Y$  is a covering space,

$$\pi_k(X, x_0) \rightarrow \pi_k(Y, y_0)$$

is an isomorphism for  $k \geq 2$ .

*Proof.* The sphere  $S^k$  is simply connected for  $k \geq 2$ , so every continuous map  $S^k \rightarrow Y$  taking the north pole to  $y_0$  lifts uniquely to a continuous map  $S^k \rightarrow X$  taking the north pole to  $x_0$ .  $\square$

The map  $\varphi: x \in S^n \rightarrow \pm x \in \mathbb{R}P^n$  is a two-to-one covering map. Hence

$$\pi_k(S^n, x_0) \cong \pi_k(\mathbb{R}P^n, \pm x_0)$$

for  $k \geq 2$ .

The map  $x \in \mathbb{R} \mapsto e^{2\pi i x} \in S^1$  is a covering map, so for  $k \geq 2$ :

$$\pi_k(S^1) = \pi_k(\mathbb{R}) = 0.$$

**49.2** Apply the van Kampen theorem (see [13]) to prove that the connected sum  $M = M_0 \# M_1$  of any two connected manifolds  $M_0, M_1$  of dimension 3 or more has fundamental group

$$\pi_1(M) = \pi_1(M_0) * \pi_1(M_1)$$

the free product of the fundamental groups. If  $M_0$  and  $M_1$  are surfaces, prove that the fundamental group is the amalgamated product

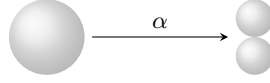
$$\pi_1(M) = \pi_1(M_0) *_{S^1} \pi_1(M_1)$$

where  $S^1$  is the circle sitting inside the punctured surfaces as the “waist” of the tube used to glue them together.

### Pinching a waist on a ball

We leave the reader to map  $[0, 1]^k \rightarrow S^k$  taking the boundary to the north pole, with a homeomorphism of the interior, so that each homotopy group  $\pi_k(X, x_0)$  can also be defined to be homotopy classes of maps  $S^k \rightarrow X$  sending the north pole to  $x_0$ , modulo homotopy through such maps. This picture gives a different description of multiplication in homotopy groups. Take two spheres  $S^+, S^- \subset \mathbb{R}^{k+1}$ , tangent at the origin. Picture a smooth map  $\alpha: S^k \rightarrow \mathbb{R}^{k+1}$ , taking some neighborhood of the equator to the origin, the upper hemisphere to  $S^+$ , and the lower hemisphere to  $S^-$ .





We leave the reader to see why such a map  $\alpha$  exists; we can fix any one such map. Use linear isomorphisms as maps  $\beta_{\pm}: S_{\pm} \rightarrow S^k$  so that the origin maps to the north pole. Let  $n_0 \in S^k$  be the north pole. Take two homotopy classes. Each can be represented by a smooth map, say  $\varphi, \psi: S^k \rightarrow M$  with  $\varphi(n_0) = \psi(n_0) = m_0$ . Define  $\varphi * \psi: S^k \rightarrow M$  by

$$\varphi * \psi(x) = \begin{cases} \varphi(\beta_+(\alpha(x))), & \text{if } x \text{ is in the upper hemisphere,} \\ \psi(\beta_-(\alpha(x))), & \text{if } x \text{ is in the lower hemisphere.} \end{cases}$$

In any manifold  $M$  with corners, suppose we change the choice of base point  $m_0$ , say to some point  $m_1$  in the same path component, both in the interior of  $M$ . By lemma 22.4 on page 192, there is a diffeomorphism  $M \rightarrow M$  taking  $m_0$  to  $m_1$ . So

$$\pi_k(M, m_0) \cong \pi_k(M, m_1).$$

**49.3** Prove that the same is true even if one or both of  $m_0$  and  $m_1$  lie on the boundary of  $M$ .

We often write  $\pi_k(M, m_0)$  as  $\pi_k(M)$ , even though this is not strictly speaking well defined.

## Smoothing

We can smooth out any continuous map of connected manifolds by a smoothing homotopy (theorem 36.1 on page 277). The transitivity of the diffeotopy group of a connected manifold (lemma 22.4 on page 192) allows us to arrange that the smooth map takes a given point of one manifold to a given point of the other. Hence we can represent homotopy groups by smooth maps, unique up to smooth homotopy.

## Low order homotopy groups of spheres

Recall that a subset  $S \subseteq M$  of a manifold with corners  $M$  is of *dimension at most  $k$*  if  $S$  lies in the union of a sequence of  $C^1$  embedded submanifolds of dimension at most  $k$ . A set is of *dimension  $k$*  if it is of dimension at most  $k$  but not at most  $k - 1$ . A set  $S \subseteq M$  is of *codimension  $k$*  if it is of dimension  $\dim M - k$ .

**Lemma 49.3.** *If  $M$  is a connected manifold and  $S \subset M$  is a subset then*

$$\pi_k(M, m_0) = \pi_k(M - S, m_0)$$

*for  $k + \dim S < \dim M$  and any point  $m_0 \in M - S$ .*

*Proof.* We represent the homotopy class by a smooth map. Smooth maps in the same homotopy class form an open set in the strong topology by lemma 44.3 on page 366. The generic such smooth map is transverse to  $S$  by Thom transversality: theorem 45.2 on page 370. By the dimension constraint, transversality is precisely not intersecting. So the generic smooth map representing the homotopy class does not intersect  $S$ .  $\square$

For a sphere  $S^n$  of dimension  $n$ , let  $x_0$  be the north pole. Note that  $S^n - \{p\}$  is diffeomorphic to  $\mathbb{R}^n$  for any point  $p$ . Take  $p$  to be a point which is not the north pole: by lemma 49.3 on the previous page, the homotopy groups of a sphere  $S^n$  in dimensions  $0, 1, \dots, n-1$  vanish.

### Low order homotopy groups of Grassmannians

Take  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take a vector space  $V$  over  $k$  of dimension  $p+q$ . Write each  $p$ -dimensional  $k$ -linear subspace  $U \subseteq V$  as the kernel of some full rank linear map  $A: V \rightarrow W$ , where  $W$  can be any vector space over  $k$  of dimension  $q$ .

If  $p = q = 1$ ,  $V = k^2$ ,  $W = k$ , write

$$A = (a_1 \ a_2),$$

and the subspace  $U$  represented by  $A$  is  $0 = a_1x_1 + a_2x_2$ .

Elementary linear algebra: any linear map  $B: V \rightarrow W$  with the same kernel as  $A$  is  $B = gA$  for a unique  $g \in \text{GL}(W)$ . In bases for  $V, W$ ,  $A$  becomes a  $q \times (p+q)$  matrix. An open dense subset of  $\text{Gr}_p V$  consists of those subspaces which can be written by taking the first  $q$  columns of  $A$  to be linearly independent. Then we can pick  $g$  to ensure that, replacing  $A$  by  $B = gA$ , we get  $A$  to have the form  $A = (I \ C)$  where  $C$  can be any  $q \times p$  matrix.

For  $p = q = 1$ ,  $A = (a_1 \ a_2)$ , we see that if  $a_1 \neq 0$  we can write the subspace

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid a_1x_1 + a_2x_2 = 0 \right\}$$

as

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 + a_1^{-1}a_2x_2 = 0 \right\}.$$

For  $p = q = 2$ ,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

if the first two columns are linearly independent, call them  $h$ :

$$h = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then replace  $A$  by  $B = gA$  where  $g = h^{-1}$ . Then  $A$  and  $B$  have the same kernel and

$$B = \begin{pmatrix} 1 & 0 & b_{13} & b_{14} \\ 0 & 1 & b_{23} & b_{24} \end{pmatrix}$$

The kernel  $U$  of  $B$  consists of the vectors

$$\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$$

so that

$$\begin{aligned} 0 &= x_1 + b_{13}y_1 + b_{14}y_2, \\ 0 &= x_2 + b_{23}y_1 + b_{24}y_2 = 0 \end{aligned}$$

which we can solve for  $x$  in terms of  $y$ .

There is a subset  $S \subset \text{Gr}_p V$  of the Grassmannian consisting of linear subspaces  $U \subset V$  which are *not* the kernels of matrices of the form  $A = (I \ C)$ , i.e. not the kernels of matrices with their first  $q$  columns linearly independent.

In familiar terms of linear algebra, the matrix  $g$  taking  $A$  to  $gA$  can carry out each of the steps of Gauss–Jordan elimination, i.e. bring  $A$  into reduced row echelon form, so that each row starts with more zeroes than any earlier row, and the first nonzero entry in any row (the *pivot entry*) is a 1.

If  $1 = p = q$ , the only such subspace is

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_2 = 0 \right\}.$$

For  $p = q = 2$ , linear dependences can arise like

$$A = \begin{pmatrix} 0 & 1 & 0 & a_{14} \\ 0 & 0 & 1 & a_{24} \end{pmatrix}$$

or

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 1 & a_{24} \end{pmatrix}$$

and so on, the possible reduced row echelon forms.

We want to examine that subset  $S$  more clearly, i.e. the possible reduced row echelon forms of matrices more clearly.

Any  $p$ -dimensional linear subspace  $U \subset V$  is the kernel of some matrix  $A$  of full rank, so  $q$  columns of  $A$  are linearly independent. Once we are given  $q$  linearly independent columns of  $A$ , we can replace  $A$  by  $B = gA$  for a unique  $g \in \text{GL}(W)$  so that those linearly independent columns become the columns of the identity matrix, i.e. the standard basis vectors. Given any matrix  $A$ , mark off the first column which is not zero, and then the first column after that one which is linearly independent of that one, and so on. We build up a basis of those marked columns. We can suppose that these columns are standard basis vectors, without altering the kernel of  $A$ .

Consider which matrices  $A$  can have some particular chosen columns turn out to be the marked ones. It is convenient to write the chosen linearly independent columns

in terms of the sizes of the gaps between them; say  $A$  has  $k_0$  zero columns and then the first standard basis vector, then  $k_1$  more columns, then the second standard basis vector, and so on. (In other words, the pivot in the first row is in column  $1 + k_0$ .) To avoid linear independence, the  $k_1$  columns after the first standard basis vector have to be multiples of it. In general, after each standard basis vector  $e_\ell$ ,  $A$  has  $k_\ell$  columns with all nonzero entries in rows  $1, \dots, \ell$ . Define  $k_q$  by

$$k_0 + k_1 + \dots + k_q = p.$$

We say that  $K = (k_0, \dots, k_q)$  is the *partition* of  $A$ . There are  $q$  standard basis vectors in  $A$ , and  $p$  other columns. Those other columns are “gaps” of sizes  $k_0, k_1, \dots, k_q$  between the standard basis vectors.

Let  $S_K$  be the subset of the Grassmannian, consisting of kernels of matrices  $A$  with given partition. Counting “gap entries”,  $S_K \subset \text{Gr}_p V$  is parameterized by choice of

$$0k_0 + 1k_1 + 2k_2 + \dots + qk_q$$

matrix entries. This is largest when we make all  $k_i = 0$  except  $k_q = p$ : this  $S_K$  is the open dense set above. The next largest is when we make all  $k_i = 0$  except  $k_{q-1} = 1$  and  $k_q = p - 1$ . Then  $S_K$  is parameterized by  $q - 1 + (p - 1)q = pq - 1$  matrix entries.

So finally,  $S \subset \text{Gr}_p V$  is the union of embedded submanifolds  $S_K \subset \text{Gr}_p V$  of dimensions  $d(pq - 1)$  if  $d$  is the dimension of  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$  as a real vector space. So  $S$  has codimension  $d$ , and  $\text{Gr}_p V - S = k^{p \times q}$ .

**Corollary 49.4.** *Complex Grassmannians are simply connected. In other words, for any finite dimensional complex vector space  $V$  and any integer  $p$  with  $0 \leq p \leq \dim_{\mathbb{C}} V$ ,*

$$\pi_1(\text{Gr}_p V) = 0.$$

**Corollary 49.5.** *Quaternionic Grassmannians are 3-simply connected, i.e. for any finite dimensional quaternionic vector space  $V$  and any integer  $p$  with  $0 \leq p \leq \dim_{\mathbb{H}} V$ ,*

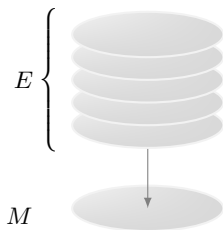
$$\pi_j(\text{Gr}_p V) = 0, j = 1, 2, 3.$$

## Chapter 50

### Fiber bundles

#### Fiber bundles

Any covering map  $\varphi: E \rightarrow M$  has the particularity that any two points of  $M$  lie in open sets above which the map “looks the same”, a picture of some “sheets”, which we draw as a stack of disks.



We want to consider more generally maps  $\varphi: E \rightarrow M$ , called *fiber bundles*, which have the property that any two points of  $M$  lie in open sets in which the map “looks the same”. This is not the same property as in the constant rank theorem, where any two points of  $E$  lie in open sets in which the map “looks the same”.

Let us make a precise definition. The standard example of a fiber bundle: a *trivial fiber bundle* is a map  $\pi_0: M \times F \rightarrow M$  where  $M$  and  $F$  are manifolds and  $\pi_0(m, f) = m$ ;  $M$  is the *base* of the fiber bundle, and  $F$  is the *fiber*. A *trivialization* of a map  $\pi: E \rightarrow M$  of manifolds is a diffeomorphism  $t: E \rightarrow M \times F$  so that the diagram

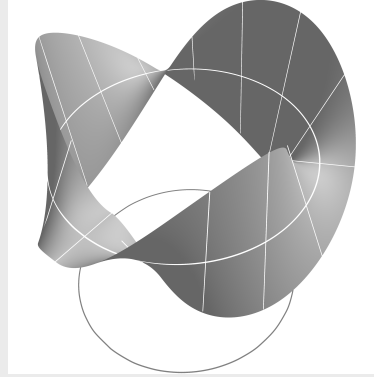
$$\begin{array}{ccc} E & \xrightarrow{t} & M \times F \\ \pi \searrow & & \swarrow \pi_0 \\ & M & \end{array}$$

commutes. A *fiber bundle map* with *typical fiber*  $F$  is a map  $\pi: E \rightarrow M$  of manifolds so that there is a covering of the *base*  $M$  by open sets  $M_a$  and a trivialization of each of the maps  $\pi_a: E_a \rightarrow M_a$ , where  $E_a := \pi^{-1}M_a$ , and  $\pi_a := \pi|_{E_a}$ . Locally,  $E \rightarrow M$  “looks like” a trivial fiber bundle map  $M_a \times F \rightarrow M_a$ . We often leave the map  $\pi$  understood, and say that  $E$  is a fiber bundle over  $M$ .

We think of a fiber bundle as a family of manifolds  $E_m := \pi^{-1}\{m\}$ , parameterized by the points  $m \in M$  of a manifold.

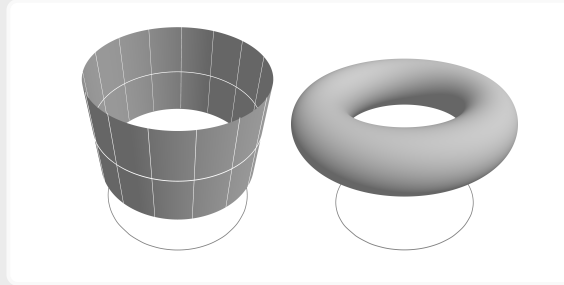
We often denote a fiber bundle by  $F \rightarrow E \rightarrow M$  to remind us that the typical fiber is  $F$ , which we picture as mapped to  $E$  as a fiber. This traditional notation is nonsense, since the typical fiber doesn’t really have a distinguished map to  $E$ .

Picture the Möbius strip (or the cylinder) as a strip of paper wound around a circle. Let  $E$  be the Möbius strip, let  $M$  be the circle, and let the map  $\pi: E \rightarrow M$  take each point along the Möbius strip and tell you at which point of the circle that bit of the Möbius strip gets wrapped to. As “typical fiber”  $F$  we can take any closed interval.



**50.1** Make a precise definition of the Möbius strip  $E$ , and the map  $\pi: E \rightarrow M$ .

The cylinder and the torus are fiber bundles in the same manner, but they are trivial bundles:



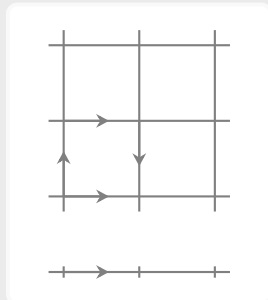
The Klein bottle is the quotient  $\mathbb{R}^2/\Gamma$  by the rigid motions  $\Gamma$  of the plane generated by the two transformations

$$(x, y) \mapsto (x, y + 1)$$

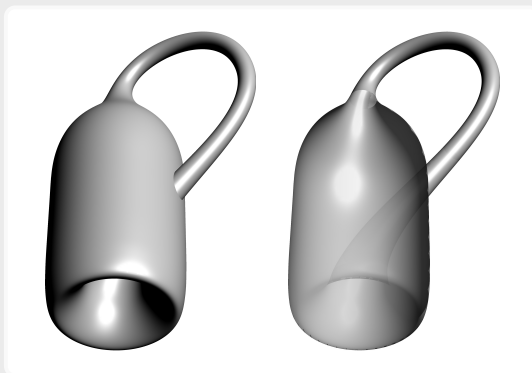
and

$$(x, y) \mapsto (x + 1, -y).$$

gluing together opposite sides of squares like:



The map  $(x, y) \mapsto x$  matches up these rigid motions with  $x \mapsto x$  and  $x \mapsto x + 1$ , so the Klein bottle is a fiber bundle over the circle  $\mathbb{R}/\mathbb{Z}$ , with circle fibers. In our picture:



the circle fibers go “up and around” the bottle.

**50.2\*** Suppose that  $\pi: E \rightarrow M$  has typical fiber  $F$  and also has typical fiber  $G$ , for some manifolds with  $F$  and  $G$ . Prove that  $F$  and  $G$  are diffeomorphic.

**50.3** Define the Hopf fibration  $S^1 \rightarrow S^3 \rightarrow S^2$  by treating  $S^3$  as the unit sphere inside  $\mathbb{C}^2$ , and  $S^2 = \mathbb{CP}^1$  as the complex projective line, and let

$$(z, w) \in S^3 \mapsto \pi(z, w) = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathbb{CP}^1,$$

i.e. map each point to the line from the origin passing through it. Prove this is a fiber bundle by constructing explicit local trivializations.

Note that  $\pi_1(S^3) = 0$  while  $\pi_1(S^2 \times S^1) = \mathbb{Z}$ , so  $S^3$  is not homeomorphic to the total space  $S^2 \times S^1$  of the trivial bundle. So the Hopf fibration  $\pi$  is not a trivial bundle

Take  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ,  $V$  a finite dimensional vector space over  $k$ . Recall the notation of on page 20: let  $E := V - \{0\}$ ,  $M := \mathbb{P}V$ , and  $\pi: x \in V - \{0\} = E \rightarrow [x] \in \mathbb{P}V = M$ , taking each nonzero vector to the line through it and the origin.

Take an injective affine map  $\iota: k^n \rightarrow V$  not striking the origin, say with image an affine hyperplane  $A \subset V$ . So  $\iota: k^n \rightarrow A$  is an affine isomorphism, i.e. an affine map with an affine inverse. We can write  $A$  as the translate  $A = b + A_0$  of a unique linear hyperplane  $A_0 \subset V$ , the translate of  $A$  through the origin. Let  $k^\times := k - \{0\}$ . Extend  $\iota$  to a map  $\iota: k^n \times k^\times \rightarrow V - A_0$  by  $\iota(x, c) = c\iota(x)$ , a bijection.

Let  $E_A := V - A_0 \subset E$  and

$$M_A := \pi(A) = \pi(\iota(k^n)) = \pi(V - A_0) = \pi(E_A) \subset M.$$

Let  $t: E_A \rightarrow M_A \times k^\times$ , defined by  $t(x) = ([x], c)$  where  $j^{-1}(x) = (y, c)$ , say. From the definition of the charts on page 20, we can check that  $t$  is a diffeomorphism, so a trivialization of  $\pi$  over  $M_A$ .

A *morphism* of fiber bundles  $\pi: E \rightarrow M$  and  $\pi': E' \rightarrow M'$  is a commutative diagram of smooth maps

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\varphi} & M' \end{array}$$

A *morphism over  $M$*  is one for which  $M = M'$  and  $M \rightarrow M'$  is the identity map, i.e.

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ \pi \searrow & & \swarrow \pi' \\ & M & \end{array}$$

The *fiber product* of two bundles  $\pi: E \rightarrow M$  and  $\pi': E' \rightarrow M$  over the same base is

$$E \times_M E' := \{ (m, p, p') \mid m \in M, p \in E, p' \in E', \pi(p) = m = \pi'(p') \}.$$

Theorem 20.2 on page 180 proves that the fiber product is a smooth manifold and that the maps  $(m, p, p') \mapsto m, p, p'$  are all smooth.

**50.4** Prove that the fiber product is a fiber bundle over  $E, E'$  and  $M$ , with the obvious bundle maps.

**50.5** Suppose that  $\pi: E \rightarrow M$  is a smooth proper fiber bundle of oriented manifolds with corners. Suppose that the boundary of  $E$  horizontal over  $M$  is empty. Prove that the pushdown map  $\pi_*$  of Fubini's theorem takes closed forms to closed forms, giving a map on cohomology

$$\pi_*: H_{\text{dR}}^*(E) \rightarrow H_{\text{dR}}^{*-k}(M)$$

where  $k$  is the dimension of the fibers.



## Principal bundles

If  $G$  is a Lie group, a *principal  $G$ -bundle* is a smooth submersion  $\varphi: P \rightarrow M$  and a free action of  $G$  on  $P$  (it is a *left* principal bundle if the  $G$ -action is left, a *right* principal bundle if the  $G$ -action is right) so that the  $G$ -orbits are precisely the fibers of  $\varphi$ . A *morphism* of principal  $G$ -bundles  $\pi: P \rightarrow M$  and  $\pi': P' \rightarrow M'$  (with the same group  $G$ ) is a commutative diagram of smooth maps

$$\begin{array}{ccc} P & \xrightarrow{\Phi} & P' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\varphi} & M' \end{array}$$

so that  $\Phi$  is  $G$ -equivariant.

**50.6** Prove that the map  $\varphi(g, m) = (m, gm)$ ,  $\varphi: G \times M \rightarrow M \times M$  has image equal to the incidence relation, and is an immersion just when the stabiliser subgroup  $G^{m_0}$  of every point  $m_0 \in M$  has dimension zero. Prove that it is an injective immersion just when the stabiliser subgroup of every point  $m_0 \in M$  is trivial.

**50.7\*** A Lie group  $G$  acts *properly* on a manifold  $M$  if the map  $\varphi: (g, m) \in G \times M \mapsto (m, gm) \in M \times M$  is a proper map, i.e. the inverse image of any compact set is compact. Prove that a Lie group  $G$  action on a manifold  $M$  is free and proper if and only if the quotient map  $M \rightarrow G \backslash M$  is a principal  $G$ -bundle.

**50.8** To check if an action is proper: prove that any action of a Lie group  $G$  on a manifold  $M$  is proper if and only if, for any convergent sequence  $x_1, x_2, \dots \rightarrow x$  in  $M$  and any sequence  $g_1, g_2, \dots$  in  $G$ , if  $g_1 x_1, g_2 x_2, \dots$  converges to some  $y \in M$ , then after perhaps replacing by a subsequence,  $g_1, g_2, \dots$  converges to some  $g \in G$ .

Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ ; let  $d$  be the real dimension of  $k$ . Recall the submersion

$$\varphi: \begin{pmatrix} a \\ b \end{pmatrix} \in k^2 \rightarrow \begin{pmatrix} 2a\bar{b} \\ |a|^2 - |b|^2 \end{pmatrix} \in k \oplus \mathbb{R}.$$

Write the unit elements of  $k$  as  $S^{d-1} \subset k$ . In problem 8.35 on page 90 and subsequent problems, we checked that this map maps to the unit sphere and is invariant under

$$g \in S^{d-1}, \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2d-1} \mapsto \begin{pmatrix} a\bar{g} \\ b\bar{g} \end{pmatrix}$$

By problem 14.13 on page 132,  $S^d$  is the quotient space  $S^d = S^{d-1} \backslash S^{2d-1}$ . Since  $S^{d-1}$  is a compact Lie group acting smoothly,  $\varphi: S^{2d-1} \rightarrow S^d$  is a principal  $S^{d-1}$ -bundle.

Suppose that  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Let  $G := k^\times$  and  $M := k^{n+1} - \{0\}$ . The quotient is  $\bar{M} := G \backslash M = \mathbb{P}^n$  is projective space. Claim:  $x \in k^{n+1} - \{0\} \mapsto [x] \in \mathbb{P}^n$  is a principal  $k^\times$ -bundle. Free: rescaling a nonzero vector cannot fix the vector unless the scalar is 1. Proper: take a sequence of vectors  $x_i \in k^{n+1} - \{0\}$ , and scalars  $c_i \in k^\times$ , so that  $x_i \rightarrow x \in k^{n+1} - \{0\}$  is convergent, and  $c_i x_i \rightarrow y$

is convergent. Since  $x \neq 0$ , some entry of the vector  $x$  is not zero, and is a limit of corresponding entries of the vectors  $x_1, x_2, \dots$ . Rescaling to get that entry to be 1, we see that  $c_1, c_2, \dots$  converges.

**50.9** Every Lie group  $G$  acts on itself on the right, a free and proper action. So every subgroup  $H \subseteq G$  also acts on  $G$  freely on the right. Prove that the action is proper just when  $H \subseteq G$  is closed, and then the quotient is the homogeneous space  $X := G/H$ , hence  $g \in G \mapsto gx_0 \in X$  is a principal right  $H$ -bundle.

### Clutching

Pick a manifold with corners  $M$  and two open sets  $M_0, M_1 \subseteq M$ , and let  $M_{01} := M_0 \cap M_1$ . A *clutching function* is a smooth map  $\varphi: M_{01} \times F \rightarrow F$  giving a diffeomorphism  $F \rightarrow F$  at each point of  $M_{01}$ , for some manifold with corners  $F$ . Define a fiber bundle  $E \rightarrow M$  by quotienting  $(m, f) \in M_0 \times F$  to  $(m, \varphi(m, f)) \in M_1 \times F$ . In the context of principal bundles, for a Lie group  $G$ , a *clutching function* is a smooth map  $\varphi: M_{01} \rightarrow G$ . We then glue together the trivial principal  $G$ -bundles  $M_0 \times G$  and  $M_1 \times G$  by identifying

$$(m_0, g_0) \text{ with } (m_1, g_1) = (m_0, g_0 \varphi(m_0))$$

so that the identification preserves the left  $G$ -action

$$g(m_0, g_0) = (m_0, gg_0)$$

on the trivial bundles  $M_0 \times G$  and  $M_1 \times G$ ; the quotient by the identification is a principal  $G$ -bundle. (Similarly one can define clutching functions to produce right principal bundles.)

**50.10** On  $M := S^d$ , let  $M_0$  be the complement of the south pole, and  $M_1$  the complement of the north pole. Find two local trivializations and their clutching function for the principal bundle  $S^{2d-1} \rightarrow S^d$  described above.

Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , let  $M := S^d$ ,  $M_0$  the complement of the south pole, and  $M_1$  the complement of the north pole. In the charts given in the solution of the last problem,  $M_{01} := M_0 \cap M_1$  is identified with  $k - \{0\}$ , with transition map  $x \mapsto 1/x$ . Consider the fiber bundle clutching function

$$\varphi(x, g) = \left( \frac{1}{x}, \frac{x^p}{|x|^p} g \frac{x^q}{|x|^q} \right),$$

for integers  $p, q$ . Note that for  $k = \mathbb{R}$  or  $\mathbb{C}$ ,  $k$  is commutative: if  $\ell = p - q$ ,

$$\varphi(x, g) = \left( \frac{1}{x}, g \frac{x^\ell}{|x|^\ell} \right),$$

so these are principal bundles, with principal bundle clutching map

$$\varphi(x) = \frac{x^\ell}{|x|^\ell}.$$

For  $k = \mathbb{H}$ , similarly if  $p = 0$  or  $q = 0$  we find principal right or left  $S^3$ -bundles, but otherwise only fiber bundles  $E_{p,q} \rightarrow S^4$  with fibers diffeomorphic to  $S^3$ . (Surprisingly, many of these  $E_{p,q}$  turn out to be homeomorphic but not diffeomorphic to  $S^7$ .) On  $S^4$ , if one of these  $S^3$ -bundles reduced to a principal bundle with a smaller structure group, some homotopy would reduce to a clutching function to a Lie subgroup of  $S^3$ , hence would have degree zero; so these bundles can't be reduced unless  $p = q = 0$ .

## Homogeneous spaces as fiber bundles

**Theorem 50.1.** *Every principal  $G$ -bundle is a fiber bundle.*

*Proof.* Suppose that the  $G$ -action on our principal bundle  $P \rightarrow M$  is a right action. The  $G$ -orbits are precisely the fibers, so embedded submanifolds. We can make coordinates on  $P$  and  $M$  in which the map  $\pi: P \rightarrow M$  is a linear projection  $\pi(x, y) = x$ . The map  $s(x) := (x, 0)$  (defined on some open set) is a smooth *section*, i.e. a smooth map so that  $\pi \circ s$  is the identity.

Cover  $M$  in open sets  $M_a \subseteq M$  on each of which there is a smooth section  $s_a$  of  $\pi$  on  $M_a$ , i.e. if we let  $P_a := \pi^{-1}M_a$  and  $\pi_a := \pi|_{P_a}$ , then  $s_a: M_a \rightarrow P_a$  and  $\pi_a(s_a(m)) = m$  for all  $m \in M_a$ . The map  $h_a: M_a \times G \rightarrow P_a$ ,  $h_a(m, g) = s_a(m)g$  is a smooth bijection. Compose with  $\pi_a$  to get  $\pi_a \circ h_a(m, g) = m$ . So if we fix a choice of  $g \in G$ , and let  $i_g(m) := (m, g)$ , then  $\pi_{a*}h_{a*}i_{g*} = 1$ . Similarly, if fix a choice of  $m \in M_a$  and let  $i_m(g) = (m, g)$ , then  $h_{a*}i_{m*}\tilde{A} = \tilde{A}$ . Since the action of  $G$  is free, every point has trivial stabilizer, so for nonzero  $A$ ,  $\tilde{A}$  is nowhere zero on  $P$ . So  $h_{a*}i_{m*} = 1$ . Therefore  $h_{a*}$  has full rank, i.e. is a local diffeomorphism. Being a smooth bijection,  $h_a$  is a diffeomorphism. So the inverse  $t_a := h_a^{-1}: P_a \rightarrow M_a \times G$  is a local trivialization of  $\pi_a: P_a \rightarrow M_a$ .  $\square$

**Corollary 50.2.** *Suppose that  $X$  is a homogeneous space of a Lie group  $G$ . Pick a point  $x_0 \in X$ , and let  $H := G^{x_0}$  be the stabilizer of  $x_0$ . Then  $H$  acts on  $G$  on the right, by right translation. The map  $g \in G \mapsto gx_0 \in X$  is a principal right  $H$ -bundle.*

**Lemma 50.3.** *Suppose that  $G$  and  $G'$  are Lie groups,  $\phi: G \rightarrow G'$  a morphism of Lie groups,  $X$  a manifold with an action of  $G$ ,  $X'$  a  $G'$ -homogeneous space, and  $\Phi: X \rightarrow X'$  a smooth  $\phi$ -equivariant map, i.e.  $\Phi(gx) = \phi(g)\Phi(x)$  for all  $g \in G$  and  $x \in X$ . Then  $\Phi$  is a fiber bundle map.*

*Proof.* Get  $G$  to act on  $X'$  by  $gx' := \phi(g)x'$ , so it suffices to assume that  $G' = G$ . The map  $\Phi$  is onto, because it is  $G$ -equivariant and  $G$  acts transitively on  $X'$ . The vector fields  $\tilde{A}$  are taken to one another by  $\Phi_*$ , and they span each tangent space of  $X'$ , so  $\Phi$  is a submersion.

Pick  $x_0 \in X$ . Let  $x'_0 := \Phi(x_0)$ ,  $H' := G^{x'_0}$ . Let  $\mathfrak{h}' \subset \mathfrak{g}$  be the Lie algebra of  $H'$ . Take a linear subspace  $V \subset \mathfrak{g}$  complementary to  $\mathfrak{h}'$ . The map

$$A \in V \mapsto e^A x'_0 \in X'$$

is smooth, with differential at  $A = 0$  given by the vector fields  $\tilde{A}$  for  $A \in V$ . So this map is a diffeomorphism of some open subset  $U \subset V$  around 0 to some open subset

$X'_U \subset X'$ . Let  $X_U := \Phi^{-1}X'_U$ . Consider the fiber  $X_{x'_0} := \Phi^{-1}\{x'_0\} \subseteq X$ . Each  $g \in G$  takes  $X_{x'_0} \rightarrow X_{gx'_0}$ , preserves fibers. Define a map

$$\psi: (A, x) \in U \times X_{x'_0} \mapsto e^A x \in X_U.$$

For fixed  $A$ ,  $\psi$  takes  $X_{x'_0} \rightarrow X_{gx'_0}$ , a diffeomorphism. Note that  $\psi'(0, x)(\tilde{A}, 0) = \tilde{A}$ , and that  $\psi'(0, x)(0, v) = (0, v)$  for  $v \in T_x X_{x'_0}$ . Hence  $\psi$  is a local diffeomorphism. If  $\psi(A, x) = \psi(B, y)$ , i.e.  $e^A x = e^B y$ , apply  $\Phi$  to get  $e^A x'_0 = e^B x'_0$ , i.e.  $A = B$  so  $x = y$ . Hence  $\psi$  is injective. Take any point  $x \in X_U$ , so  $\Phi(x) = x' \in X'_U$ , which we write as  $x' = e^A x'_0$  for a unique and smoothly determined  $A \in U$ . So if  $y = e^{-A} x$  then  $y \in X_{x'_0}$  so  $x = \psi(A, y)$ . Hence  $\psi$  is a diffeomorphism  $\psi: U \times X_{x'_0} \rightarrow X_U$ .

So for each  $g \in G$ ,  $g \circ \psi: U \times X_{x'_0} \rightarrow gX_U$  is a diffeomorphism. We want to use the  $(g \circ \psi)^{-1}$  maps as trivializations. Their transition maps are

$$(A_2, x_2) = \psi^{-1}(g_2^{-1}g_1 e^{A_1} x_1).$$

The transition map is a diffeomorphism, smoothly mapping each fiber  $X_{x'_0}$  to itself, as each of  $e^{A_1}$ ,  $g_1$ ,  $g_2^{-1}$ ,  $\psi^{-1}$  is fiber preserving.  $\square$

Take a finite dimensional vector space  $V$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Let  $X := V - \{0\}$ ,  $X' := \mathbb{P}V$  the set of  $k$ -lines through the origin in  $V$ . For each nonzero vector  $v \in V$ , let  $\Phi(v)$  be the span of  $v$ :  $\Phi: X \rightarrow X'$ . Let  $G := G' := \text{GL}(V)$ , the group of  $k$ -linear transformations. Then  $\Phi$  is a fiber bundle.

The Hopf fibrations are fiber bundles.

## ■ Topology of fiber bundles

### Fiber bundles from families of vector fields

**Theorem 51.1.** *If  $\Gamma$  is a set of vector fields  $X \in \Gamma$  on a manifold  $P$ , each of which has a corresponding vector field  $\bar{X} = f_*X \in \bar{\Gamma}$  on a manifold  $Q$ , for some smooth map  $f: P \rightarrow Q$  then  $f$  takes  $\Gamma$ -orbits into  $\bar{\Gamma}$ -orbits. On each orbit,  $f$  has constant rank. If the vector fields in  $\Gamma$  are complete, then  $f: P \rightarrow Q$  is a fiber bundle.*

*Proof.* By restricting to an orbit in  $P$ , we may assume that there is only one orbit. The map  $f$  is invariant under the flows of the vector fields, so has constant rank.

Henceforth, suppose that the vector fields are complete. Given a path

$$e^{t_1 X_1} \dots e^{t_k X_k} m_0$$

down in  $Q$ , we can always lift it to one in  $P$ , so  $f$  is onto. It might not be true that  $f_*\langle\Gamma\rangle = \langle\bar{\Gamma}\rangle$ , but nonetheless we can still push around vector fields, because the pushing upstairs in  $P$  corresponds to pushing downstairs in  $Q$ . So without loss of generality, both  $\Gamma$  and  $\bar{\Gamma}$  are closed under “pushing around”.

As in the above proof, for each point  $q \in Q$ , construct a distinguished parameterization

$$(t_1, \dots, t_k) \mapsto e^{t_1 \bar{X}_1} \dots e^{t_k \bar{X}_k} q.$$

Since  $f_*X_k = \bar{X}_k$ ,  $f$  is a surjective submersion on distinguished sets. Let  $U_Q \subset Q$  be the associated distinguished set; on  $U_Q$  these  $t_j$  are now coordinates. Let  $U_P = f^{-1}U_Q \subset P$ . Let  $Z$  be the fiber of  $f: P \rightarrow Q$  above the origin of the distinguished chart. Map

$$u_P \in U_P \mapsto (u_P, z) \in U_Q \times Y$$

by  $u_Q = f(u_P)$  and

$$z = e^{-t_k X_k} \dots e^{-t_1 X_1} u_P.$$

Clearly this gives  $P$  the local structure of a product. The transition maps have a similar form, composing various flows, so  $P \rightarrow Q$  is a fiber bundle.  $\square$

Keep in mind that all vector fields on compact manifolds are complete. Even though the orbits might not be compact, our theorem says that the orbits upstairs will fiber over the orbits downstairs.

**Corollary 51.2.** *In particular, if the vector fields span the tangent spaces of  $Q$ , and  $P$  and  $Q$  have equal dimension, then the map is a covering map.*

Take  $Q = \mathbb{R}_{x,y}^2$ , and  $P \subset Q$  a pair of disjoint disks, say those of unit radius around two points of the  $x$ -axis which are distantly separated. As the family  $\Gamma$  up in the disks, take the translation vector fields  $\partial_x, \partial_y$  along coordinate

axes in the right disk, and in the left, the pair of vector fields  $\partial_x, 0$ . Obviously these are not complete. As the family  $\bar{F}$ , take the translation vector field  $\partial_x$ , and a vector field  $f(x, y)\partial_y$  which vanishes in the left disk, and nowhere outside of closure of the left disk, and equals  $\partial_y$  in the right disk. The orbits in  $Q$  are all two dimensional, while those in  $P$  are one dimensional in the left disk, and two dimensional in the right.

Take  $E \rightarrow M$  any fiber bundle, and pick a plane field on  $E$  complementary to the tangent spaces of the fibers. Every vector field on  $M$  lifts to a unique vector field on  $E$  tangent to the plane field. Suppose that the fibers of  $E \rightarrow M$  are compact. Lifting all complete vector fields, we get a family of complete vector fields on  $E$ . Their orbits are connected and fiber over  $M$ .

### Connections

Take a submersion  $\varphi: P \rightarrow Q$  of manifolds. A *submersion connection* for  $\varphi$  is a plane field  $H \subset TP$  so that, at each point  $p \in P$ , the linear map

$$\varphi'(p)|_{H_p}: H_p \rightarrow T_q Q$$

is a linear isomorphism, where  $q := \varphi(p)$ . To each vector field  $X$  on  $Q$ , its *lift* is the vector field  $\hat{X}$  on  $P$  tangent to  $H$  so that  $\varphi_* \hat{X} = X$ .

**Lemma 51.3.** *Every smooth submersion has a smooth submersion connection.*

*Proof.* Let  $V_p$  be the kernel of  $\varphi'(p)$ ; vectors in  $V$  are *vertical* for  $\varphi$ . Each submersion connection splits up  $T_p P = H_p \oplus V_p$ , and so there is a unique linear projection mapping  $\Pi_p: T_p P \rightarrow V_p$  which is the identity on  $V_p$  and vanishes on  $H_p$ . Moreover,  $\Pi_p$  varies smoothly with  $p$ , i.e. in coordinates  $\Pi_p$  is given by a matrix, with entries smooth functions.

By the implicit function theorem, each point  $p_0 \in P$  lies in an open set  $U \subset P$  in which the map is, in some coordinates, a projection map  $\varphi(x, y) = x$ . Write each point  $p \in U$  in coordinates as  $p = (x, y)$ , say with  $x = (x^1, \dots, x^n)$ . So  $n$  is the dimension of  $Q$ . Let  $H_p \subset T_p P$  be the span of  $\partial_{x^1}, \dots, \partial_{x^n}$ . So every submersion locally has a submersion connection.

We have thus covered  $P$  in open sets  $P_a$ , and on each found some submersion connection  $H_a$ .

We have two submersion connections,  $H_a$  and  $H_b$  on the open set  $P_{ab} := P_a \cap P_b$ . So they have projection maps  $\Pi_a, \Pi_b$ . Take a partition of unity  $f_a$  subordinate to the open cover of open sets  $P_a$ . Let  $\Pi := \sum f_a \Pi_a$ . Note that  $\Pi$  is the identity on the vertical vectors, because all  $\Pi_a$  are. So  $\Pi$  has full rank, and the kernel  $H$  of  $\Pi$  is a submersion connection.  $\square$

**Theorem 51.4** (Ehressmann). *Any smooth proper submersion  $\varphi: P \rightarrow Q$  of manifolds to a connected manifold  $Q$  is a fiber bundle mapping.*

Recall that a continuous map is proper just when it is closed (i.e. images of closed sets are closed) and has compact fibers.

*Proof.* Take a fiber bundle connection  $H$ . Let  $\bar{\Gamma}$  be the set of all smooth compactly supported vector fields on  $Q$ . Let  $\Gamma$  be the set of all  $\hat{X}$  vector fields, for all smooth compactly supported vector fields on  $Q$ . Since  $P$  has compact fibers,  $\varphi: P \rightarrow Q$  is a proper map, so all vector fields in  $\Gamma$  have compact support. Apply theorem 51.1 on page 415.  $\square$

## Corners

Take a smooth submersion  $\varphi: P \rightarrow Q$  of manifolds with corners. A *submersion connection* for  $\varphi$  is a smooth plane field  $H \subset TP$  so that  $H$  is submersion connection for the interior of  $P$  and also for the horizontal boundary of  $P$  over  $Q$ .

The trivial fiber bundle  $P = Q \times F$  has the *trivial* submersion connection

$$H = TQ \oplus 0 \subset TP = TQ \times TF.$$

**Theorem 51.5.** *Every submersion of manifolds with corners admits a submersion connection.*

The proof is as for submersions of manifolds without corners: locally it is obvious, and globally we can use a partition of unity.

A *horizontal path* is a  $C^0$  and piecewise  $C^1$  path  $p(t) \in E$  so that  $p'(t) \in H_{p(t)}$  where  $p'(t)$  is defined. Given a  $C^0$  and piecewise  $C^1$  path  $m(t) \in M$ , a *lift* of that path is a horizontal path  $p(t) \in E$  so that  $\pi(p(t)) = m(t)$ . A submersion connection is *complete* if, for every  $C^0$  and piecewise  $C^1$  path  $m(t) \in M$ ,  $a \leq t \leq b$ , and point  $p_0 \in E$  so that  $\pi(p_0) = m(a)$ , there is a lift  $p(t)$  of  $m(t)$  so that  $p(a) = p_0$ .

The trivial submersion connection is complete.

To each vector field  $v$  on  $Q$ , its *lift* is the vector field  $\hat{v}$  on  $P$  tangent to  $H$  so that  $\varphi_*\hat{v} = v$ . Hence the lift of  $v$  is tangent to the horizontal boundary. Note that each flow line of the lifted vector field is the lift of a flow line. In particular, if  $H$  is complete, then all complete vector fields have complete lifts.

A smooth section  $s$  of  $\pi$  is *horizontal* if  $s'(m)T_m M \subset H_{s(m)}$  for all  $m \in U$ . If we lift a path in  $M$  up to  $E$ , so that it touches the graph of a horizontal section, then it remains inside the horizontal section. A subset of  $E$  is *horizontal* if it is a union of images of horizontal sections. The lift of a path which touches a horizontal set cannot leave that set.

The *vertical bundle* of a submersion  $\pi: E \rightarrow M$  is the subset  $V_E := \ker \pi' \subset TE$ ; the *vertical spaces* are  $V_{E,p_0} := \ker \pi'(p_0) \subset T_{p_0}E$ . A *connection 1-form* of a submersion is a linear map

$$\vartheta_{p_0}: T_{p_0}E \rightarrow V_{E,p_0}$$

on each tangent space, smoothly varying with  $p_0 \in E$ , so that  $\vartheta$  is the identity map on  $V_E$ . Each connection 1-form determines a connection  $H = \ker \vartheta$ , and conversely every connection is uniquely determined by a connection 1-form.

**Theorem 51.6 (Hoyo).** *A submersion admits a complete connection if and only if it is a fiber bundle.*

*Proof.* The proof of theorem 51.4 on page 416 applies to prove that every smooth submersion with a complete connection is a fiber bundle.

Consider a fiber bundle  $\pi: E \rightarrow M$  with typical fiber  $F$ . We want to construct a complete connection on  $E$ . If the fibers are compact, theorem 51.4 on page 416 applies to prove completeness.

In order to lift a  $C^0$  and piecewise  $C^1$  path, it is enough to lift each piece, so we need only lift  $C^1$  paths.

Suppose that we cover  $M$  in open sets  $M_a$  so that  $H$  is complete on every  $E|_{M_a}$ . Splitting any path into pieces covered by these  $M_a$ , the path is defined on a compact interval, so is finite subcover, and we can lift one piece at a time. So  $H$  is then complete.

Take an open set  $U \subset M$  over which  $E|_U \rightarrow U$  is a trivial fiber bundle. The trivial submersion connection is complete over  $U$ . The difficulty we face is that we must glue complete submersion connections together, while somehow preserving completeness.

A smooth function  $h$  on  $E$  is *disconnecting* if, for any compact set  $K \subset M$ ,  $h$  is proper on  $\pi^{-1}K$ , and  $h$  has horizontal level sets  $(h = c_i)$  for infinitely many constants  $c_i$  arbitrarily large positive and negative. The lift of a path touching those horizontal level sets stays on them. The lift of a path not touching those horizontal level sets stays between them.

Suppose that  $E \rightarrow M$  has a connection  $H$  and a disconnecting function  $h$ . Take a  $C^1$  path  $m(t) \in M$ ,  $a \leq t \leq b$ , and point  $p_0 \in E$  with  $\pi(p_0) = m(a)$ . The existence and uniqueness theorem ensures that there is some time  $\varepsilon > 0$  so that a lift  $p(t)$  exists on  $a \leq t \leq a + \varepsilon$  with  $p(a) = p_0$ . The set of such  $\varepsilon$  is open, again by existence and uniqueness. The lifted path  $p(t)$  stays between the level sets. Also  $m(t) \in M$  stays in a compact set, by continuity of  $m(t)$ . So  $p(t)$  stays in a compact set. The existence and uniqueness theorem gives a uniform lower bound on the amount of time that a solution to a differential equation survives on a compact set. So  $p(t)$  survives for at least that much longer. By induction,  $p(t)$  survives, giving a lifted path, for all  $a \leq t \leq b$ .

Take a proper smooth nonnegative function  $h: F \rightarrow \mathbb{R}$ ; problem 32.6 on page 258 ensures one exists. Cover  $M$  in open sets  $M_a$  so that each  $M_a$  has compact closure containing in an open set  $M'_a \subset M$  over which  $E_a := E|_{M'_a} \rightarrow M'_a$  is a trivial bundle. So it admits a trivial connection  $H_a$ . In particular, every smooth proper function on  $F$  is disconnecting as a function on  $E_a \cong M'_a \times F$ .

Summing up, we can cover  $M$  in open sets  $M_a$  so that  $\bar{M}_a$  lies inside an open set  $M'_a$  on which there is a connection  $H_a$  for  $E_a = E|_{M'_a}$  with a disconnecting function  $h_a$ . We can refine our  $M_a$  to form a locally finite open cover,  $a = 1, 2, \dots$

Over  $M_1$ , take some constant  $c_1$ , and consider the level set  $(h_1 = c_1) \subset E_a$ . Let  $T_1 = (h_1 = c_1) \cap \pi^{-1}\bar{M}_1$ , a compact set. Over  $M_2$ , take some constant  $c_2$  large enough that  $T_2 := (h_2 = c_2) \cap \pi^{-1}\bar{M}_2$  does not intersect  $T_1$ . Repeating by induction, we construct compact sets  $T_a \subset E_a$  so that  $\pi: T_a \rightarrow \bar{M}_a$  is onto, and each  $T_a$  has interior horizontal over  $M_a$ , and no two intersect.

Write these  $T_a$  as  $T_a^1$ . Repeat the construction, making sets  $T_a^2$ . At each step, each  $\bar{M}_a$  overlaps only finitely many  $\bar{M}_b$ . By making the constants in  $(h_a = \text{constant})$  in constructing  $T_a^2$  large enough,  $T_a^2$  does not intersect any  $T_b^1$ . Repeat, constructing  $T_a^i$  for all  $i, a = 1, 2, \dots$ . Making constants large enough, we can ensure that each  $T_a^i$  lies on a level set of  $h_a$  of arbitrarily large values of  $h_a$  as we increase  $i$ . Changing notation, let

$$T_a := \bigcup_i T_a^i,$$



which is by definition the union of level sets of a disconnecting function on the trivial bundle  $E_a$ .

Take a partition of unity  $f_a$  subordinate to the open cover

$$E_a - \bigcup_{b \neq a} T_b.$$

Take the connection 1-forms  $\vartheta_a$  of the connections  $H_a$ . Let

$$\vartheta := \sum_a f_a \vartheta_a.$$

Where  $\vartheta_a$  is not defined, i.e. outside of  $E_a$ ,  $f_a = 0$ , so interpret  $f_a \vartheta_a$  there to mean zero. On vertical vectors, clearly  $\vartheta$  is the identity. So  $\vartheta$  is a connection 1-form, say of a connection  $H$ .

Each  $f_a$  is positive only inside

$$E_a - \bigcup_{b \neq a} T_b,$$

so is zero at any points of  $T_b$  for  $b \neq a$ . So at points of any  $T_b$ , the only nonzero  $f_a$  is  $f_b$ . Being a partition of unity,  $f_b = 1$  on  $T_b$ . So  $H = H_b$  on  $T_b$ , so  $T_b$  remains horizontal. Hence  $h_b$  continues to disconnect  $H$  over  $M_b$ , so  $H$  is complete.  $\square$

## The local picture of a fiber bundle

We want to slightly generalize the notion of clutching. Take a fiber bundle  $\pi: E \rightarrow M$  over a manifold with corners  $M$ . Cover  $M$  in open sets  $M_a \subset M$ , on which the fiber bundle is trivial, given by a trivialization

$$t_a: E_a := \pi^{-1}M_a \rightarrow M_a \times F.$$

Define  $M_{ab} = M_{ba} := M_a \cap M_b$ ,

$$M_{abc} := M_a \cap M_b \cap M_c,$$

and so on, and similarly  $E_{ab} := E_a \cap E_b$  and so on. On each overlap  $M_{ab}$ , we have two trivializations, related by an isomorphism of trivial bundles.

$$t_b \circ t_a^{-1}: M_{ab} \times F \rightarrow M_{ba} \times F$$

which we write as

$$(m, x) \mapsto t_b(t_a^{-1}(m, x)) = (m, \tau_{ab}(m, x)),$$

where

$$\tau_{ab}: M_{ab} \times F \rightarrow F.$$

is the *transition map*. Write this map as

$$\tau_{ab,m}x := \tau_{ab}(m, x)$$

then

$$\tau_{ab,m}: F \rightarrow F$$

is a diffeomorphism, for any  $m \in M_{ab}$ . Clearly

$$\tau_{ba,m} = \tau_{ab,m}^{-1}.$$

A 1-cocycle  $\tau = (\tau_{ab})$  for a cover of a manifold  $M$  by open sets  $M_a \subseteq M$  and a manifold  $F$  is a collection of smooth maps

$$\tau_{ab}: M \times F \rightarrow F,$$

with notation  $\tau_{ab,m}x := \tau_{ab}(m, x)$ , so that  $\tau_{aa,m}$  is the identity map  $F \rightarrow F$  for any  $a$  and

$$\tau_{ca,m} \circ \tau_{bc,m} \circ \tau_{ab,m}$$

is the identity map  $F \rightarrow F$ , for any  $m \in M_{abc} = M_a \cap M_b \cap M_c$ .

**51.1** Prove that the transition maps of any trivializations form a 1-cocycle.

**Theorem 51.7.** *The transition maps of any fiber bundle form a 1-cocycle. Every 1-cocycle consists of the transition maps of a fiber bundle, unique up to isomorphism.*

*Proof.* Let  $E_a := M_a \times F$ . On the disjoint union  $\bigsqcup E_a$ , we have a map

$$\pi: \bigsqcup_a E_a \rightarrow M,$$

taking  $(m, x) \mapsto x$ . Declare  $(m, x) \in E_a$  to be equivalent to  $(m', x') \in E_{a'}$ , denoted  $(m, x) \sim (m', x')$ , if  $m = m'$  and

$$x' = \tau_{aa',m}(x).$$

Define  $E$  to be the quotient

$$E := \bigsqcup_a E_a / \sim.$$

There are obvious maps  $E_a \rightarrow E$  taking each point  $(m, x)$  to its equivalence class. Recall that in the quotient topology, a subset  $U \subset E$  is open just when its preimage in  $\bigsqcup E_a$  is open. Hence the open sets in  $E$  have preimages in  $\bigsqcup E_a$  precisely those open sets which are invariant under transition maps. Any open set in  $M$  has preimage in  $\bigsqcup E_a$  invariant under transition maps, so  $\pi: E \rightarrow M$  is continuous. Since  $E_a \rightarrow M_a$  is onto,  $E \rightarrow M$  is onto.

Given an open subset  $U_a \subset E_a$ , let  $W \subset \bigsqcup E_a$  be the smallest set containing  $U_a$  and invariant under transition maps; write  $W$  as a union of  $W_b = W \cap E_b$ . We want to prove that  $W_a = U_a$ . This is precisely the requirement that  $E_a \rightarrow E$  be an embedding. Note that we build  $W$  by successively taking images of  $U_a$  under transition maps. Starting at some point  $p_0 \in U_a$ , say  $p_0 = (m_0, x_0)$ , we map by a sequence of transition maps. The danger is that we might come back again to  $E_a$  via some sequence of transition maps to land outside of  $U_a$ . Otherwise, clearly  $W_a = U_a$ .

Take a sequence of transition maps applied to this point  $p_0$ , and landing back in  $E_a$ . If the sequence has length zero, i.e. we don't move  $p_0$ , then clearly we stay in  $U_a$ . If the sequence has length one, we have a transition map

$$(q, x) \mapsto (q, \tau_{aa,q}(x)) = (q, x).$$

If the sequence has length two, note that plugging  $c = a$  into the cocycle condition forces each transition map  $\tau_{ab,q}$  to be inverted by  $\tau_{ba,q} = \tau_{ab,q}^{-1}$ . So a sequence of length two goes from  $E_a$  to some  $E_b$  and back again, and has identity transition.

If we have some longer sequence, picture it as a polygon with some  $E_b$  drawn in each vertex, starting and ending at the given  $E_a$ . The point  $p_0 = (m_0, x)$  lies above the point  $m_0 \in M$ , and this stays the case after every transition map. So all of the vertices  $E_b = M_b \times F$  have  $m_0 \in M_b$ , and so  $m_0$  lies in their intersection. Pick three subsequent edges of that polygon. Draw an edge between the first and third of those vertices. The cocycle condition precisely ensures that we can replace the two sides of that polygon by that edge we have drawn. Hence we have a smaller polygon.

So all  $E_a \rightarrow E$  are embedded open sets. In particular,  $E$  is Hausdorff and locally homeomorphic to the standard corner in Euclidean space. Projecting  $E_a \rightarrow M_a$  extends to  $\pi: E \rightarrow M$ , so  $E_a$  embeds into  $E$  as  $\pi^{-1}M_a$ . We use the smooth structures on each  $E_a$  to make  $E$  smooth; transition maps are identified with the various  $(q, x) \mapsto (q, \tau_{ab,q}(x))$  by our embeddings  $E_a \rightarrow E$ .  $\square$

## Isomorphism

Take a cover of a manifold with corners  $M$  by open sets  $M_a \subseteq M$ , and a manifold with corners  $F$ . A 0-cocyle  $f = (f_a)$  is a collection of smooth maps

$$f_a: M_a \times F \rightarrow F,$$

so that, if we write  $f_{a,m}x = f_a(m, x)$ , then each  $f_{a,m}: F \rightarrow F$  is a diffeomorphism, for every  $m \in M_a$ . The 0-cocyle acts on every 1-cocycle by

$$\tau_{ab,m}^f = f_{b,m} \circ \tau_{ab,m} \circ f_{a,m}^{-1}.$$

**51.2** Prove that  $\tau^f$  is a 1-cocycle.

**Lemma 51.8.** *Any 0-cocyle acts on any 1-cocycle to form a 1-cocycle. The associated fiber bundle is isomorphic.*

*Proof.* We let  $E'_a = E_a := M_a \times F$ . We have fiber bundles

$$\begin{array}{ccc} \bigsqcup_a E_a & & \bigsqcup_a E'_a \\ & \searrow \quad \swarrow & \\ & \bigsqcup_a M_a. & \end{array}$$

Map

$$(m, x) \in E_a \mapsto \varphi(m, x) := (m, f_{a,m}x) \in E'_a.$$

a smooth map on the fiber bundles:

$$\begin{array}{ccc} \bigsqcup_a E_a & \xrightarrow{\varphi} & \bigsqcup_a E'_a \\ & \searrow \quad \swarrow & \\ & \bigsqcup_a M_a. & \end{array}$$

Consider the equivalence relations

$$(m, x) \in E_a \sim (m, \tau_{ab,m}x) \in E_b$$

and

$$(m, x) \in E'_a \sim (m, \tau'_{ab,m}x) \in E'_b$$

where  $\tau' := \tau^f$ . Equivalent points map to equivalent points under  $\varphi$ . So the map descends to a map on the quotient bundles:  $E$  and  $E'$ .  $\square$

**Theorem 51.9.** *Two fiber bundles over the same manifold with the same typical fiber are isomorphic if and only if the manifold can be covered by open sets on which both are trivial and their associated transition maps agree up to the action of a 0-cocycle.*

*Proof.* Suppose that you and I each have a fiber bundle, with diffeomorphic typical fibers, on the same manifold. Cover that manifold in open sets on which your fiber bundle is trivial, and I will do the same. Replacing those open sets by their intersections, we can simultaneously trivialize over the same open sets  $M_a \subset M$ .

Suppose our bundles are  $\pi: E \rightarrow M$  and  $\pi': E' \rightarrow M$ . Let  $E_a := \pi^{-1}M_a$  and  $E'_a := \pi'^{-1}M_a$ . You have local trivializations  $t_a: E_a \rightarrow M_a \times F$  and transition maps

$$\tau_{ab}: M_{ab} \times F \rightarrow F$$

given by

$$t_b t_a^{-1}(m, x) = (m, \tau_{ab}(m, x)).$$

Write  $\tau_{ab,m}x := \tau_{ab}(m, x)$ . I have local trivializations  $t'_a: E'_a \rightarrow M_a \times F$  and transition maps

$$\tau'_{ab}: M_{ab} \times F \rightarrow F$$

given by

$$t'_b t'^{-1}_a(m, x) = (m, \tau'_{ab,m}x).$$

Suppose that we find an isomorphism between our bundles,  $\varphi: E \rightarrow E'$ . On each  $E_a$ , we have a diffeomorphism

$$E_a \xrightarrow{\varphi} E'_a \xrightarrow{t'^{-1}_a} M_a \times F$$

projecting to the identity map on  $M_a$ . Hence this diffeomorphism agrees with  $t_a$  up to some map  $f_a$ , i.e.

$$t'_a \circ \varphi \circ t_a^{-1}(m, x) = (m, f_{a,m}x).$$

giving a 0-cocycle.

**51.3** Prove that  $\tau' = \tau^f$ .

□

### Bundles pull back

Suppose that  $\varphi: M \rightarrow M'$  is a  $C^k$  map and that  $\pi': E \rightarrow M'$  is a  $C^k$  fiber bundle. Let

$$\varphi^*E := \{ (m, m', p) \in M \times M' \times E \mid \varphi(m) = m' = \pi'(p) \}.$$

Let  $\pi: (m, m', e) \in \varphi^*E \rightarrow b \in B$ . Call  $\pi$  the *pullback fiber bundle mapping* and  $\varphi^*E$  the *pullback fiber bundle*. Let  $\Phi: \varphi^*E \rightarrow E$ ,  $\Phi(m, m', p) = p$ .

**Theorem 51.10.** *The pullback fiber bundle is a fiber bundle, giving a fiber bundle morphism*

$$\begin{array}{ccc} \varphi^*E & \xrightarrow{\Phi} & E \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\varphi} & M' \end{array}$$

*Proof.* As  $\varphi$  and  $\pi'$  are transverse,  $\varphi^*E$  is a manifold and  $\pi$  is a smooth map, by theorem 20.2 on page 180. Cover  $M'$  in open sets  $M'_a$  on which  $E \rightarrow M'$  is trivial, say by trivializations  $t'_a: E_a \rightarrow M_a \times F$ . Let  $M_a := \varphi^{-1}M'_a$ . Let  $t_a(m, m', p) = (m, x)$  if  $t'_a(p) = (m', x)$ , defined on the open sets  $\varphi^{-1}E_a \subset \varphi^*E$ . Check that

$$t_a^{-1}(m, x) = (m, \varphi(m), (t'_a)^{-1}(\varphi(m), x)),$$

so  $t_a$  is smoothly invertible, hence a diffeomorphism.  $\square$

### Homotopy of fiber bundles

Take a fiber bundle  $E \rightarrow [0, 1] \times M$ . Take the map  $\varphi_t: M \rightarrow [0, 1] \times M$  given by  $\varphi_t(m) = (t, m)$ , and consider the pullback fiber bundles  $\varphi_0^*E, \varphi_1^*E \rightarrow M$ . This fiber bundle is a *fiber bundle homotopy* between fiber bundles  $E' \rightarrow M$  and  $E'' \rightarrow M$  (on the same base  $M$  and with the same typical fiber) if there are fiber bundle isomorphisms

$$E' \cong \varphi_0^*E, E'' \cong \varphi_1^*E;$$

$E'$  is then *homotopic* to  $E''$ .

**Theorem 51.11.** *Fiber bundles are homotopic if and only if they are isomorphic.*

*Proof.* Take a complete connection on a fiber bundle  $E \rightarrow [0, 1]_t \times M$ . Let  $\hat{v}$  be the lift of the vector field  $v = \partial_t$ . So the flow of  $v$  takes  $\{t_0\} \times M$  to

$$e^{tv} \{t_0\} \times M = \{t_0 + t\} \times M.$$

The same is true for  $\hat{v}$ :

$$e^{\hat{v}}: E' \rightarrow E''$$

is a fiber bundle isomorphism.  $\square$

Take a fiber bundle  $E \rightarrow \bar{B}$  over the closed unit ball  $\bar{B} \subset \mathbb{R}^n$ . Consider the map  $\varphi: \bar{B} \times [0, 1] \rightarrow \bar{B}$  by  $\varphi(u, t) = tu$ . Then  $\varphi^*E$  is a homotopy of fiber bundles between the trivial bundle  $\varphi_0^*E \cong \bar{B} \times E_0$ , and the bundle  $\varphi_1^*E = E$ . Hence every fiber bundle over the closed ball is trivial.

The same proof works for the open ball, and for all of Euclidean space, and for any star shaped open set in Euclidean space, or for a cube.

A *circle bundle* is a fiber bundle with typical fiber a circle. On the sphere  $M := S^2$ , denote the north pole as  $n_0$ . Cover with  $M_0 := S^2 - \{n_0\}$  and  $M_1 := S^2 - \{-n_0\}$ . Clearly  $M_0, M_1$  are diffeomorphic to Euclidean space so any fiber bundle on either of them is trivial. So a fiber bundle on the sphere is trivial if and only if its transition map on  $M_{01}$  is homotopic to the trivial transition map  $(m, x) \mapsto x$  away from the poles. Clearly  $S^2 - \{\text{poles}\} \cong \mathbb{R} \times S^1$ , so we can smoothly retract down to the equator: we only need to construct the homotopy near the equator  $S^1$ . So the fiber bundles on  $S^2$  are classified by homotopy classes of maps  $\tau = \tau_{01}: S^1 \times F \rightarrow F$  through homotopies which

ensure that  $y \mapsto \psi(x, y)$  is a diffeomorphism for all  $x \in S^1$ , and up to a 0-cocycle. Using trigonometry, we can take an angle function  $\theta: \mathbb{R}^2 \rightarrow \mathbb{R}$ , so that

$$e^{2\pi i \theta(\alpha, \beta)} = \tau(e^{2\pi i \alpha}, e^{2\pi i \beta}).$$

Since  $\tau_m$  is a diffeomorphism on  $S^1 \rightarrow S^1$  (for any  $m \in S^1$ ), it has degree  $\pm 1$ , i.e.

$$\theta(\alpha, \beta + 1) = \theta(\alpha, \beta) \pm 1.$$

We can take a 0-cocycle, conjugating  $e^{i\beta} \mapsto e^{-i\beta}$  on  $M_1$ , the identity on  $M_0$ , which will flip the sign of that degree. So we can assume the degree is 1:

$$\theta(\alpha, \beta + 1) = \theta(\alpha, \beta) + 1.$$

Also  $\theta$  is increasing in  $\beta$ , since it is a diffeomorphism. So

$$\frac{\partial \theta}{\partial \beta} > 0$$

is a positive function with unit integral. We can make a homotopy through such functions to deform  $\theta$  so as to get

$$\frac{\partial \theta}{\partial \beta} = 1,$$

without changing  $\theta(\alpha, 0)$ , hence the same winding. So after homotopy,

$$\theta(\alpha, \beta) = \theta(\alpha, 0) + \beta.$$

This function  $\theta(\alpha, 0)$  can be anything which increases by an integer as  $\alpha$  increases by 1, so we can arrange by homotopy that it become linear, say

$$\theta(\alpha, \beta) = n\alpha + \beta.$$

Hence the integer  $n$  determines the circle bundle  $E \rightarrow S^2$ . Reversing our steps, we have a construction and classification of circle bundles on the sphere. The trivial circle bundle has degree zero. The Hopf fibration has degree one.

**51.4** Classify circle bundles over the circle.

**51.5** Classify circle bundles over the real projective plane.

## Chapter 52

### Homotopy groups of fiber bundles

We see that every fiber bundle has a long exact sequence in homotopy. We use this to compute many examples of homotopy groups.

#### Degree

If  $M$  is a connected manifold with corners of dimension  $n$ , then any element of  $\pi_\ell(M, m_0)$  is represented by a smooth map  $\varphi: S^\ell \rightarrow M$ , which has a degree  $\deg \varphi \in \mathbb{Z}$ . Any homotopy can be approximated by a smooth homotopy, so the degree is constant, hence an invariant of the homotopy class

$$\deg: \pi_{\dim M}(M) \rightarrow \mathbb{Z}.$$

**Lemma 52.1.**  $\deg: \pi_{\dim M}(M) \rightarrow \mathbb{Z}$  is a group morphism.

*Proof.* Any regular point of both  $\varphi$  and  $\psi$  is regular for  $\varphi * \psi$ , and with degree given by counting sheets of both maps, i.e.

$$\deg(\varphi * \psi) = \deg \varphi + \deg \psi.$$

□

**52.1** Suppose that  $M$  is a connected manifold with corners and  $\partial M$  is not empty. Prove that  $\pi_{\dim M}(M) = \{0\}$ .

**Lemma 52.2.**

$$\deg: \pi_n(S^n) \rightarrow \mathbb{Z}$$

is an isomorphism, i.e. the degree of a continuous map  $S^n \rightarrow S^n$  determines the map up to homotopy, for any integer  $n \geq 1$ .

*Proof.* Since the degree map is a group morphism, we need only prove that it is onto and that its kernel is trivial. We first prove that it is onto. Start with  $n = 1$  and consider the maps  $z \mapsto z^d$  of the complex plane; each induces a map of degree  $d$  on the unit circle. By induction, for any degree  $d$ , and a given integer  $n$ , suppose we have a smooth map  $\varphi: S^n \rightarrow S^n$  of degree  $d$ . Write the points of the sphere  $S^{n+1}$  as  $(x, y)$  for  $x \in \mathbb{R}$  and  $y \in \mathbb{R}^{n+1}$ . We would like to let

$$\Phi(x, y) = (x, |y|\varphi(y/|y|)).$$

But this map might not be smooth near the poles.

**52.2** Smooth it and check that  $\Phi$  and  $\varphi$  have equal degree.

Pick a homotopy class of maps  $S^n \rightarrow S^n$  mapping north pole to north pole. Pick a smooth map representing that homotopy class. Pick a regular value, not at the north pole. For simplicity of notation, we can arrange by a diffeomorphism that this regular value is the south pole. Take a closed disk around the south pole, small enough that it is evenly covered by the map. By diffeomorphism, we can assume that this disk is the lower hemisphere.

Let  $\bar{H}_-$  be the closed lower hemisphere,  $\bar{H}_+$  the closed upper hemisphere,  $H_-$ ,  $H_+$  their interiors. So  $\varphi^{-1}\bar{H}_-$  is a collection of diffeomorphic copies of  $\bar{H}_-$ . Take one, say  $\bar{D}$ . We can replace  $\bar{H}_-$  above in our argument by a smaller disk  $\bar{D}'$  around a regular point, to arrange that  $\bar{D}$  is very close to a single point, which we can arrange to be the south pole. Take our favourite chart, say by orthogonal linear projection to the tangent plane at the south pole. For small enough disks  $\bar{D}'$ ,  $\bar{D}$  is convex in that chart. It is not hard to prove then that there is a diffeomorphism of the sphere taking  $\bar{D}$  to a closed disk on the sphere. So we can assume that  $\bar{D}$  is closed disk. We can arrange, by diffeomorphism of the sphere, that  $\bar{D} = \bar{H}_-$ , i.e.  $\varphi: \bar{H}_- \rightarrow \bar{H}_-$  is a diffeomorphism.

**52.3** Prove that  $\varphi$  is homotopic to a smooth map evenly covering  $H_-$  but taking  $\bar{H}_+$  to the north pole.

Looking our map  $\alpha$  above, we see that  $\varphi$  is, up to homotopy, the sum of two maps with lower numbers of preimages of the points of  $H_-$ . By induction, we can “pop off bubbles” until we get a point with no preimage. Popping the sphere at that point, it becomes an open disk, contractible, so has trivial homotopy groups.  $\square$

## Homotopy groups of fiber bundles

Once we pick a point  $m_0 \in M$  in the base of a fiber bundle, we can identify the typical fiber  $F$  with the fiber  $E_{m_0}$ , and we will always do so.

Recall that a map  $S^\ell \rightarrow M$  taking north pole to some point  $m_0$  is trivial in  $\pi_\ell(M, m_0)$ , i.e. can be deformed through such maps to a constant map, just when it extends to a map  $\bar{B}^{\ell+1} \rightarrow M$  from the closed unit ball in  $\mathbb{R}^{\ell+1}$ , taking the north pole to  $m_0$ .

**Theorem 52.3.** *If  $\pi: E \rightarrow M$  is a fiber bundle, with marked points  $p_0 \in E$ ,  $m_0 = \pi(p_0) \in M$  and with typical fiber  $F = E_{m_0}$ , also sharing the marked point  $p_0 \in F$ , then the obvious maps*

$$\pi_\ell(F) \rightarrow \pi_\ell(E) \rightarrow \pi_\ell(M)$$

*fit together into an exact sequence*



$$\begin{array}{c}
\cdots \\
\downarrow \\
\pi_4(F) \longrightarrow \pi_4(E) \longrightarrow \pi_4(M) \\
\downarrow \\
\pi_3(F) \longrightarrow \pi_3(E) \longrightarrow \pi_3(M) \\
\downarrow \\
\pi_2(F) \longrightarrow \pi_2(E) \longrightarrow \pi_2(M) \\
\downarrow \\
\pi_1(F) \longrightarrow \pi_1(E) \longrightarrow \pi_1(M) \\
\downarrow \\
\pi_0(F) \longrightarrow \pi_0(E) \longrightarrow \pi_0(M) \longrightarrow 0
\end{array}$$

The idea of the proof is simple. Consider the bottom row. The “kernel” of

$$\pi_0(F) \rightarrow \pi_0(E)$$

consists of the components of  $F$  which map to the  $p_0$ -component of  $E$ , i.e. which become connected by a path inside  $E$ . This path is defined up to homotopy fixing its initial point, and fixing the path component in  $F$  of its final point. But this path goes around in  $E$ , starting and ending in  $F$ . So its image in  $M$  starts and ends at  $m_0$ , a loop, defined up to homotopy fixing its end points: an element of  $\pi_1(M)$ .

Reverse this story: given an element of  $\pi_1(M)$ , lift it up to a path in  $E$  starting at  $p_0$ . Such a lift exists, and is unique up to homotopy fixing the initial point, and fixing the terminal point up to homotopy in the fiber. Hence mapping  $\pi_1(M) \rightarrow \pi_0(F)$ , striking precisely the kernel of  $\pi_0(F) \rightarrow \pi_0(E)$ .

Consider any other row: the kernel of

$$\pi_\ell(F) \rightarrow \pi_\ell(E)$$

consists of the maps  $S^\ell \rightarrow F$  taking north pole to  $p_0$ , up to homotopy, which extend to maps  $\bar{B}^{\ell+1} \rightarrow E$  taking north pole to  $p_0$ . The boundary of the ball is taken to  $F$ , so when we project to  $M$ , the boundary of the ball goes entirely to  $m_0$ . Hence the projection into  $M$  is a map  $\bar{B}^{\ell+1} \rightarrow M$  with  $S^\ell \rightarrow \{m_0\}$ , and so we can quotient  $\bar{B}^{\ell+1}/S^\ell = S^{\ell+1}$ , a map of the sphere of one higher dimension, an element of  $\pi_{\ell+1}(M)$ . Similarly, we reverse the story to give a morphism of groups  $\pi_{\ell+1}(M) \rightarrow \pi_\ell(F)$ , striking precisely the kernel of  $\pi_\ell(F) \rightarrow \pi_\ell(E)$ .

## Homotopy groups of the Hopf fibration

We will drop all of the arrows in our exact sequences when they get long, to save paper. Recall the Hopf fibration: if  $k, K = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , and  $k \subset K$ , suppose that  $V_K$  is a finite dimensional  $K$ -vector space, and let  $V_k$  be the same vector space  $V_K$ , but thought of as a  $k$ -vector space. The *Hopf fibration* is  $\pi: \mathbb{P}V_k \rightarrow \mathbb{P}V_K$ , defined by taking the  $k$ -span of a vector  $v$  to its  $K$ -span. The fiber consists of all  $k$ -lines through the origin which lie in the same  $K$ -line, so a typical fiber is  $\mathbb{P}L_k$ , where  $L$  is a  $K$ -line,

i.e.  $L = Ke_0$  for any nonzero vector  $e_0$ . If  $\dim_K V = n$  and  $\dim_K K = d$ , then we have the exact sequence

| $\pi_\ell(X)$ | $X$                  |                       |                      |
|---------------|----------------------|-----------------------|----------------------|
| $\ell$        | $\mathbb{P}_k^{d-1}$ | $\mathbb{P}_k^{dn-1}$ | $\mathbb{P}_K^{n-1}$ |
| $\vdots$      | $\vdots$             | $\vdots$              | $\vdots$             |
| 4             | ?                    | ?                     | ?                    |
| 3             | ?                    | ?                     | ?                    |
| 2             | ?                    | ?                     | ?                    |
| 1             | ?                    | ?                     | ?                    |
| 0             | 0                    | 0                     | 0                    |

### The Hopf invariant

Consider the simplest case: our bundle is  $\mathbb{RP}^1 \rightarrow \mathbb{RP}^3 \rightarrow \mathbb{CP}^1$ , corresponding to  $n = 2$ ,  $k = \mathbb{R}$ ,  $K = \mathbb{C}$ ,  $d = 2$ . Recall  $\mathbb{RP}^1 = S^1$ ,  $\mathbb{RP}^3 = S^3/\pm 1$  and  $\mathbb{CP}^1 = S^2$ . We know many of these homotopy groups:

| $\pi_\ell(X)$ | $X$          |                          |              |
|---------------|--------------|--------------------------|--------------|
| $\ell$        | $S^1$        | $\mathbb{RP}^3$          | $S^2$        |
| $\vdots$      | $\vdots$     | $\vdots$                 | $\vdots$     |
| 4             | 0            | ?                        | ?            |
| 3             | 0            | $\mathbb{Z}$             | ?            |
| 2             | 0            | 0                        | $\mathbb{Z}$ |
| 1             | $\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0            |
| 0             | 0            | 0                        | 0            |

From the top row

$$\pi_4(S^3) = \pi_4(S^2),$$

neither of which we know. The pattern is clear on upward:

$$\pi_\ell(S^3) = \pi_\ell(S^2),$$

for  $\ell \geq 4$ . The next row gives

$$\mathbb{Z} = \pi_3(S^2),$$

which is very surprising: under the Hopf map  $S^3 \rightarrow S^2$ , taking a real unit vector to the complex line it spans in  $\mathbb{CP}^1 = S^2$ , the 3-sphere wraps around the 2-sphere, a homotopically nontrivial map.

The homotopy classes of maps  $S^3 \xrightarrow{\varphi} S^2$  are classified by an integer invariant, the *Hopf invariant*, so that every such map is homotopic to a composition

$$S^3 \xrightarrow{\psi} S^3 \xrightarrow{\text{Hopf}} S^2$$

with the Hopf map. The Hopf invariant of  $S^3 \xrightarrow{\varphi} S^2$  is the degree of  $S^3 \xrightarrow{\psi} S^3$ .

## Homotopy groups of complex projective spaces

Next we try  $n \geq 3$ , plugging in what we know:

| $\pi_\ell(X)$ | $X$             |                          |                     |
|---------------|-----------------|--------------------------|---------------------|
| $\ell$        | $\mathbb{RP}^1$ | $\mathbb{RP}^{2n-1}$     | $\mathbb{CP}^{n-1}$ |
| $\vdots$      | $\vdots$        | $\vdots$                 | $\vdots$            |
| $2n$          | 0               | ?                        | ?                   |
| $2n-1$        | 0               | $\mathbb{Z}$             | ?                   |
| $2n-2$        | 0               | 0                        | ?                   |
| $2n-3$        | 0               | 0                        | ?                   |
| $\vdots$      | $\vdots$        | $\vdots$                 | $\vdots$            |
| 2             | 0               | 0                        | ?                   |
| 1             | $\mathbb{Z}$    | $\mathbb{Z}/2\mathbb{Z}$ | 0                   |
| 0             | 0               | 0                        | 0                   |

Many low order homotopy groups of complex projective spaces are clearly trivial. Change notation: write  $n$  where we had  $n-1$ , so we can study  $\mathbb{CP}^n$ :

$$\pi_{2n+1}(\mathbb{CP}^n) = \mathbb{Z}$$

and, while we can't identify either side, we see that

$$\pi_{2n+2}(\mathbb{CP}^n) = \pi_{2n+2}(S^{2n+1}).$$

Lower down,

$$\pi_\ell(\mathbb{CP}^n) = 0, \text{ for } 3 \leq \ell \leq 2n.$$

Finally,

$$\pi_2(\mathbb{CP}^n)$$

injects into  $\mathbb{Z}$  with quotient  $\mathbb{Z}/2\mathbb{Z}$ , so injects as the even integers. Take a 2-sphere in  $\mathbb{CP}^n$ ; picture it as a closed disk with its boundary pulled together to a point. It lifts to a closed disk in  $S^{2n+1}$  with the boundary lifting to a loop in  $\mathbb{RP}^1$ , winding an even number of times around. We see that we can achieve any even number of windings of that loop. That winding number determines the 2-sphere down in  $\mathbb{CP}^2$  up to homotopy.

More generally, if we map any sphere  $S^\ell \rightarrow \mathbb{CP}^n$ , we can lift to a map  $\bar{B}^\ell \rightarrow S^{2n+1}$ , which restricts to a map on the boundary sphere  $S^{\ell-1} \rightarrow S^1$ . If  $\ell-1 \geq 2$ , this is nullhomotopic, so we can lift our  $S^\ell \rightarrow \mathbb{CP}^n$  to a map  $S^\ell \rightarrow S^{2n+1}$ , i.e.  $\pi_\ell(S^{2n+1}) \rightarrow \pi_\ell(\mathbb{CP}^n)$  is onto for  $\ell \geq 3$ . In particular, if  $3 \leq \ell \leq 2n$ , we recover our result above:  $\pi_\ell(\mathbb{CP}^n) = 0$ . When  $\ell = 2n+1 \geq 3$ , we find again a surjective map

$$\mathbb{Z} = \pi_{2n+1}(S^{2n+1}) \rightarrow \pi_{2n+1}(\mathbb{CP}^n),$$

taking a map of some given degree to its composition with the Hopf fibration.

Summing up,

| $\ell$   | $\pi_\ell(\mathbb{CP}^n)$ |
|----------|---------------------------|
| 0        | 0                         |
| 1        | 0                         |
| 2        | $\mathbb{Z}$              |
| 3        | 0                         |
| 4        | 0                         |
| $\vdots$ | $\vdots$                  |
| $2n$     | 0                         |
| $2n+1$   | $\mathbb{Z}$              |
| $2n+2$   | $\pi_{2n+2}(S^{2n+1})$    |

### The homotopy groups of the quaternionic projective spaces

For the quaternionic projective spaces, we have two sets of Hopf fibrations which we can work with:

$$\cdots \rightarrow \pi_\ell(\mathbb{RP}^3) \rightarrow \pi_\ell(\mathbb{RP}^{4n+3}) \rightarrow \pi_\ell(\mathbb{HP}^n) \rightarrow \cdots$$

and

$$\cdots \rightarrow \pi_\ell(\mathbb{CP}^1) \rightarrow \pi_\ell(\mathbb{CP}^{2n+1}) \rightarrow \pi_\ell(\mathbb{HP}^n) \rightarrow \cdots$$

The Hopf fibration

$$\mathbb{RP}^3 \rightarrow \mathbb{RP}^{4n+3} \rightarrow \mathbb{HP}^n$$

gives

| $\pi_\ell(X)$ | $X$                      |                          |                 |
|---------------|--------------------------|--------------------------|-----------------|
| $\ell$        | $S^3$                    | $\mathbb{RP}^{4n+3}$     | $\mathbb{HP}^n$ |
| $\vdots$      | $\vdots$                 | $\vdots$                 | $\vdots$        |
| $4n+3$        | $?$                      | $\mathbb{Z}$             | $?$             |
| $4n+2$        | $?$                      | $0$                      | $?$             |
| $4n+1$        | $?$                      | $0$                      | $?$             |
| $4n$          | $?$                      | $0$                      | $?$             |
| $\vdots$      | $\vdots$                 | $\vdots$                 | $\vdots$        |
| $4$           | $?$                      | $0$                      | $?$             |
| $3$           | $\mathbb{Z}$             | $0$                      | $0$             |
| $2$           | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $0$             |
| $0$           | $0$                      | $0$                      | $0$             |

The Hopf fibration

$$\mathbb{CP}^1 \rightarrow \mathbb{CP}^{2n+1} \rightarrow \mathbb{HP}^n$$

gives

| $\pi_\ell(X)$ | $X$          |                      |                 |
|---------------|--------------|----------------------|-----------------|
| $\ell$        | $S^2$        | $\mathbb{CP}^{2n+1}$ | $\mathbb{HP}^n$ |
| $\vdots$      | $\vdots$     | $\vdots$             | $\vdots$        |
| $4n+3$        | $?$          | $\mathbb{Z}$         | $?$             |
| $4n+2$        | $?$          | $0$                  | $?$             |
| $4n+1$        | $?$          | $0$                  | $?$             |
| $4n$          | $?$          | $0$                  | $?$             |
| $\vdots$      | $\vdots$     | $\vdots$             | $\vdots$        |
| $4$           | $?$          | $0$                  | $?$             |
| $3$           | $\mathbb{Z}$ | $0$                  | $0$             |
| $2$           | $\mathbb{Z}$ | $\mathbb{Z}$         | $0$             |
| $0$           | $0$          | $0$                  | $0$             |

Summing up,

| $\ell$   | $\pi_\ell(\mathbb{HP}^n)$ |
|----------|---------------------------|
| $0$      | $0$                       |
| $1$      | $0$                       |
| $2$      | $0$                       |
| $3$      | $0$                       |
| $4$      | $\mathbb{Z}$              |
| $5$      | $\pi_4(S^2)$              |
| $6$      | $\pi_5(S^2)$              |
| $\vdots$ | $\vdots$                  |
| $4n+2$   | $\pi_{4n+1}(S^2)$         |

### The homotopy groups of the rotation groups

The group  $\mathrm{SO}(n+1)$  acts as rotations on  $\mathbb{R}^{n+1}$ , rotating the unit sphere  $S^n \subset \mathbb{R}^{n+1}$ . The stabilizer of a point is  $\mathrm{SO}(n)$ . Hence  $\mathrm{SO}(n) \rightarrow \mathrm{SO}(n+1) \rightarrow S^n$ . For  $n=1$ , it is trivial. For  $n=2$ , this is the Hopf fibration again. For  $n \geq 2$ ,

| $\pi_\ell(X)$ | $X$              |                    |              |
|---------------|------------------|--------------------|--------------|
| $\ell$        | $\mathrm{SO}(n)$ | $\mathrm{SO}(n+1)$ | $S^n$        |
| $\vdots$      | $\vdots$         | $\vdots$           | $\vdots$     |
| $n+1$         | $?$              | $?$                | $?$          |
| $n$           | $?$              | $?$                | $\mathbb{Z}$ |
| $n-1$         | $?$              | $?$                | $0$          |
| $\vdots$      | $\vdots$         | $\vdots$           | $\vdots$     |
| $1$           | $?$              | $?$                | $0$          |
| $0$           | $0$              | $0$                | $0$          |

Homotopy groups of low dimension in rotation groups descend by induction downward:

$$\pi_\ell(\mathrm{SO}(n+1)) = \pi_\ell(\mathrm{SO}(n))$$

for  $0 \leq \ell \leq n-2$ . For example,

$$\pi_1(\mathrm{SO}(3)) = \pi_1(\mathbb{RP}^3) = \pi_1(S^3/\pm 1) = \{\pm 1\} = \mathbb{Z}/2\mathbb{Z},$$

so

$$\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/2\mathbb{Z}$$

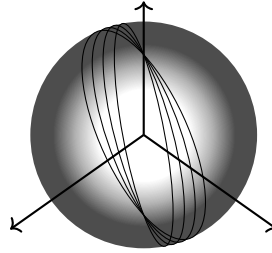
for all  $n \geq 3$ , generated, by induction, by the loop

$$e^{i\theta} \in S^1 \mapsto \begin{pmatrix} \rho(\theta) & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

where

$$\rho(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Note that in the plane, this loop has a winding number around the origin, so generates fundamental group  $\mathbb{Z}$ . But in 3 or more dimensions, we can rotate that loop



and after rotation by  $\pi$ , it turns backwards, so it is homotopic to its inverse loop.

Similarly,

$$\pi_2(\mathrm{SO}(2)) = \pi_2(\mathbb{R}/\mathbb{Z}) = 0,$$

so

$$\pi_2(\mathrm{SO}(n)) = 0$$

for all  $n \geq 2$ . Note that

$$\pi_3(\mathrm{SO}(3)) = \pi_3(S^3/\pm 1) = \mathbb{Z},$$

while

$$\pi_3(\mathrm{SO}(4)) = \pi_3(S^3 \times S^3/\pm 1) = \mathbb{Z}^2.$$

Our exact sequence with  $n = 4$  gives

$$\mathbb{Z} \rightarrow \mathbb{Z}^2 \rightarrow \pi_3(\mathrm{SO}(5)) \rightarrow 0.$$

Tensoring with  $\mathbb{Q}$  shows that  $\pi_3(\mathrm{SO}(5)) = \mathbb{Z} \oplus T$  where  $T$  is a finite sum of abelian groups. Consider the rotation

$$\rho(t) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos t & -\sin t & 0 & 0 \\ 0 & \sin t & \cos t & 0 & 0 \\ 0 & 0 & 0 & \cos t & -\sin t \\ 0 & 0 & 0 & \sin t & \cos t \end{pmatrix} \in \mathrm{SO}(5).$$

Note that  $\rho(0) = I$ . Write each element of  $\mathbb{R}^5$  as

$$\begin{pmatrix} x \\ y \end{pmatrix}.$$

with  $x \in \mathbb{H}$ ,  $y \in \mathbb{R}$ . Then

$$\rho(\pi) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ -y \end{pmatrix}.$$

For quaternions  $q, r \in S^3$ , we include  $\text{SO}(4) \subseteq \text{SO}(5)$  by

$$q, r \mapsto \rho_{q,r} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} qx\bar{r} \\ y \end{pmatrix}.$$

So

$$\rho(\pi)\rho_{q,r}\rho(\pi)^{-1} = \begin{pmatrix} rx\bar{q} \\ y \end{pmatrix}.$$

Hence  $\rho(t)$  is a homotopy of each  $S^3 \subset \text{SO}(4)$  into the other, i.e. the image of  $\pi_\ell(\text{SO}(4)) \rightarrow \pi_\ell(\text{SO}(5))$  is precisely the image of  $\pi_\ell(S^3) \rightarrow \pi_\ell(\text{SO}(5))$ . In particular, for  $k = 3$ ,

$$\pi_3(S^3) \rightarrow \pi_3(\text{SO}(5)) \rightarrow 0$$

we see that  $\pi_3(\text{SO}(5))$  is generated by a single element, so

$$\pi_3(\text{SO}(5)) = \mathbb{Z}.$$

By induction

$$\pi_3(\text{SO}(n)) = \mathbb{Z}$$

for  $n \geq 5$ . Writing vectors in  $\mathbb{R}^n$  as

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

with  $x \in \mathbb{H}$  and  $y \in \mathbb{R}^{n-4}$ , and mapping

$$q \in S^3 \mapsto \rho(q) \in \text{SO}(n),$$

by

$$\rho(q) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} qx \\ y \end{pmatrix}$$

gives the generator of  $\pi_3(\text{SO}(n))$ .

## The homotopy groups of the unitary groups

The exact sequence

$$1 \longrightarrow \text{SU}(n) \longrightarrow \text{U}(n) \xrightarrow{\det} S^1 \longrightarrow 1$$

of Lie groups makes  $U(n) \rightarrow S^1$  a fiber bundle over  $S^1$  with fiber  $SU(n)$ . Hence an exact sequence of homotopy groups:

| $\pi_\ell(X)$ | $X$      |          |              |
|---------------|----------|----------|--------------|
|               | $\ell$   | $SU(n)$  | $U(n)$ $S^1$ |
| $\vdots$      | $\vdots$ | $\vdots$ | $\vdots$     |
| $k$           | $?$      | $?$      | $0$          |
| $\vdots$      | $\vdots$ | $\vdots$ | $\vdots$     |
| $2$           | $?$      | $?$      | $0$          |
| $1$           | $?$      | $?$      | $\mathbb{Z}$ |
| $0$           | $0$      | $0$      | $0$          |

So  $SU(n) \subseteq U(n)$  share all homotopy groups except the fundamental group.

The group  $SU(n+1)$  acts as rotations on  $\mathbb{C}^{n+1}$ , rotating the unit sphere  $S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ . Let us find the stabilizer of a point: the matrices which fix the first unit vector in the standard basis are

$$g = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$$

with  $b$   $1 \times n$  and  $d$   $n \times n$ . To have  $\det g = 1$ , we need precisely that  $\det d = 1$ . To have  $g$  unitary, we need precisely that  $b = 0$  and  $d^*d = I$ , i.e.  $d \in SU(n)$  and

$$g = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$$

for any  $d \in SU(n)$ . Hence a fiber bundle  $SU(n) \rightarrow SU(n+1) \rightarrow S^{2n+1}$ . For  $n = 1$ ,  $SU(2) = S^3$ , and  $SU(1) = 1$ , this exact sequence is trivial. For  $n = 2$ ,  $SU(2) = S^3$ ,

| $\pi_\ell(X)$ | $X$          |               |          |
|---------------|--------------|---------------|----------|
|               | $\ell$       | $S^3$ $SU(3)$ | $S^5$    |
| $\vdots$      | $\vdots$     | $\vdots$      | $\vdots$ |
| $4$           | $?$          | $?$           | $0$      |
| $3$           | $\mathbb{Z}$ | $?$           | $0$      |
| $2$           | $0$          | $?$           | $0$      |
| $1$           | $0$          | $?$           | $0$      |
| $0$           | $0$          | $?$           | $0$      |



For  $n \geq 3$ ,

| $\pi_\ell(X)$ | $X$      |           |              |
|---------------|----------|-----------|--------------|
| $\ell$        | $SU(n)$  | $SU(n+1)$ | $S^{2n+1}$   |
| $\vdots$      | $\vdots$ | $\vdots$  | $\vdots$     |
| $2n+2$        | $?$      | $?$       | $?$          |
| $2n+1$        | $?$      | $?$       | $\mathbb{Z}$ |
| $2n$          | $?$      | $?$       | $0$          |
| $\vdots$      | $\vdots$ | $\vdots$  | $\vdots$     |
| $1$           | $?$      | $?$       | $0$          |
| $0$           | $0$      | $0$       | $0$          |

Homotopy groups of low dimension in rotation groups descend by induction downward:

$$\pi_\ell(SU(n+1)) = \pi_\ell(SU(n))$$

for  $0 \leq \ell \leq 2n-1$ . For example,

$$\pi_1(SU(3)) = \pi_1(SU(2)) = \pi_1(S^3) = 0,$$

and so

$$\pi_1(SU(n)) = 0$$

for all  $n \geq 1$ . We can therefore conclude that

$$\pi_\ell(U(n)) = \begin{cases} \mathbb{Z}, & \text{if } \ell = 1, \\ \pi_\ell(SU(n)), & \text{otherwise.} \end{cases}$$

for all  $n \geq 1$ , with generator of  $\pi_1(U(n))$  being the loop

$$e^{i\theta} \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & I \end{pmatrix}$$

or any other loop with determinant winding once around the unit circle. Similarly,

$$\pi_2(SU(4)) = \pi_2(SU(3)) = 0,$$

so

$$\pi_2(SU(n)) = 0$$

for all  $n \geq 2$ . Similarly,

$$\pi_3(SU(4)) = \pi_3(SU(3)) = \mathbb{Z},$$

so

$$\pi_3(SU(n)) = \mathbb{Z}$$

for all  $n \geq 3$ .

**52.4** Prove that

$$\pi_\ell(\mathrm{Sp}(n+1)) = \pi_\ell(\mathrm{Sp}(n))$$

for  $0 \leq \ell \leq 4n-4$ .

### The homotopy groups of the complex Grassmannians

We have seen that the complex Grassmannian  $X = \text{Gr}_p V$  where  $V = \mathbb{C}^{p+q}$  is simply connected (corollary 49.4 on page 406) and is acted on transitively by the Lie group  $G = \text{U}(p+q)$ . The stabilizer of the linear subspace  $\mathbb{C}^p \oplus 0$  is the set of matrices

$$g = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

so that  $a \in \text{U}(p)$  and  $d \in \text{U}(q)$ , a fiber bundle

$$\text{U}(p) \times \text{U}(q) \rightarrow \text{U}(p+q) \rightarrow \text{Gr}_p \mathbb{C}^{p+q},$$

giving an exact sequence of homotopy groups (assuming  $p, q > 0$ ):

| $\pi_\ell(X)$ | $X$                              |                 |                                |
|---------------|----------------------------------|-----------------|--------------------------------|
| $\ell$        | $\text{U}(p) \times \text{U}(q)$ | $\text{U}(p+q)$ | $\text{Gr}_p \mathbb{C}^{p+q}$ |
| $\vdots$      | $\vdots$                         | $\vdots$        | $\vdots$                       |
| 3             | $\mathbb{Z} \oplus \mathbb{Z}$   | $\mathbb{Z}$    | ?                              |
| 2             | 0                                | 0               | ?                              |
| 1             | $\mathbb{Z} \oplus \mathbb{Z}$   | $\mathbb{Z}$    | 0                              |
| 0             | 0                                | 0               | 0                              |

The map on fundamental groups is given by taking determinants winding around the unit circle, so is just of winding numbers, i.e. if

$$a(e^{i\theta}) = \begin{pmatrix} e^{ik\theta} & 0 \\ 0 & I \end{pmatrix}, d(e^{i\theta}) = \begin{pmatrix} e^{i\ell\theta} & 0 \\ 0 & I \end{pmatrix}$$

we map these to  $\text{U}(p+q)$  to give

$$\begin{aligned} \begin{pmatrix} a(\theta) & 0 \\ 0 & d(\theta) \end{pmatrix} &= \begin{pmatrix} e^{ik\theta} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & e^{i\ell\theta} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \\ &= \begin{pmatrix} e^{-i\ell\theta} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & e^{i\ell\theta} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \begin{pmatrix} e^{i(k+\ell)\theta} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \end{aligned}$$

factoring into a loop in  $\text{SU}(p+q)$  (hence trivial) and a loop in our usual picture of the fundamental group of  $\text{U}(p+q)$ . Hence the map on fundamental groups is

$$(a, d) \in \mathbb{Z} \oplus \mathbb{Z} \mapsto a + d \in \mathbb{Z},$$

so that

$$\pi_2(\text{Gr}_p \mathbb{C}^{p+q}) = \mathbb{Z}.$$

Take a smooth map  $S^2 \rightarrow \text{Gr}_p \mathbb{C}^{p+q}$ . Take a map  $\bar{D}^2 \rightarrow S^2$  taking the closed disk to the sphere by a diffeomorphism away from the boundary, mapping the boundary to the north pole. Lift the composition  $\bar{D}^2 \rightarrow S^2 \rightarrow \text{Gr}_p \mathbb{C}^{p+q}$  to a map to  $\text{U}(p+q)$ .

Then the induced map on the boundary circle  $S^1 = \partial D \rightarrow U(p) \times U(q)$  induces an element of the fundamental group of  $U(p) \times U(q)$ , say as a loop

$$e^{i\theta} \mapsto \begin{pmatrix} a(e^{i\theta}) & 0 \\ 0 & d(e^{i\theta}) \end{pmatrix}.$$

Up to homotopy, we can arrange that this loop is one as above. It will then have  $a(e^{i\theta})d(e^{i\theta}) = 1$ . Conversely, any such loop maps to something trivial in  $U(p+q)$ , and it given by such a boundary circle map. We can make an explicit representative of the  $\pi_2(\text{Gr}_p\mathbb{C}^{p+q})$ : think of  $\bar{D}^2$  as the unit disk in the complex plane, and map

$$z \mapsto g(z) := \begin{pmatrix} z & 0 & \sqrt{1-|z|^2} & 0 \\ 0 & I & 0 & 0 \\ -\sqrt{1-|z|^2} & 0 & \bar{z} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \in U(p+q).$$

This maps to a sphere in the  $\text{Gr}_p\mathbb{C}^{p+q}$ : the set of  $p$ -planes of the form

$$g(z) \begin{pmatrix} u \\ v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} zu \\ v \\ -\sqrt{1-|z|^2}u \\ 0 \end{pmatrix},$$

where  $(u, v) \in \mathbb{C} \oplus \mathbb{C}^{p-1} = \mathbb{C}^p$ .

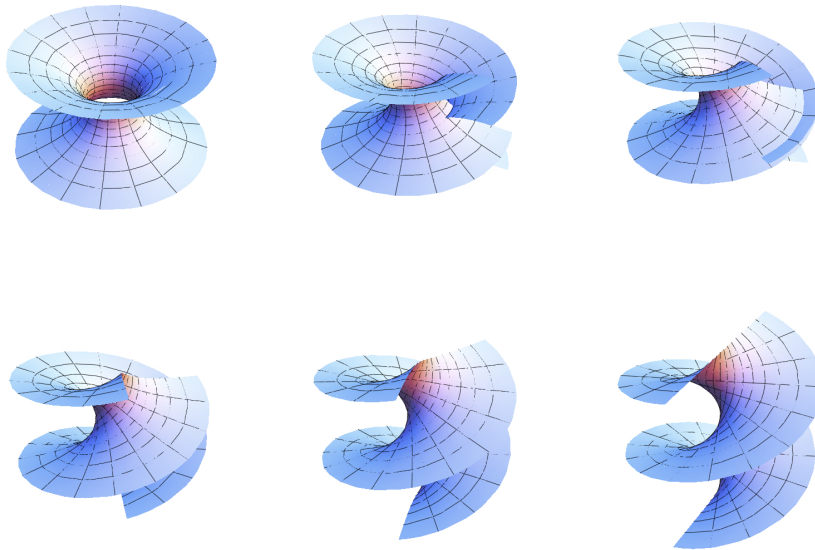


## Chapter 53

### Riemannian metrics

#### Definition

Imagine bending a surface without stretching:



In other words, deform the surface, preserving the lengths of all curves on the surface. Besides the lengths of all curves, what else stays the same? On a surface  $M$  inside Euclidean space  $\mathbb{R}^3$ , take a curve  $C$  parameterised by a map

$$x: I \rightarrow M,$$

where  $I \subset \mathbb{R}$  is an interval of the real number line. The length of  $C$  is

$$\begin{aligned} \text{length } C &= \int_I |\dot{x}(t)| \, dt, \\ &= \int_I \sqrt{\langle \dot{x}, \dot{x} \rangle} \, dt. \end{aligned}$$

The geometric structure that we use here, which is *not* available to use on some abstract manifold  $M$ , but *is* available on a submanifold of Euclidean space, is the

inner product  $\langle \cdot, \cdot \rangle$ . Note that we only calculate the inner product on the tangent vector  $\dot{x}$ . In particular, if a diffeomorphism  $\varphi: M_1 \rightarrow M_2$  between two surfaces  $M_1$  and  $M_2$  in Euclidean space preserves inner products on tangent vectors, then  $\varphi$  preserves lengths of curves. For example, picture a flat piece of paper, and the same piece of paper smoothly bent. We will see that, conversely, a bijective map  $\varphi: M_1 \rightarrow M_2$  between two surfaces  $M_1$  and  $M_2$  in Euclidean space preserves lengths of curves just exactly when  $\varphi$  is a diffeomorphism preserving inner products on tangent vectors.

To generalise this idea, a *Riemannian manifold* is a manifold  $M$  (not necessarily sitting inside Euclidean space; perhaps just an abstract manifold) equipped with a choice of inner product, either denoted  $g$  or simply  $\langle \cdot, \cdot \rangle$ , on every tangent space, and called the *metric* or *Riemannian metric*, so that if, in some open subset of  $M$ , we take a pair  $X, Y$  of smooth tangent vector fields, then the function  $g(X, Y)$  is smooth. Warning: the Riemannian metric is *not* a metric in the sense of the theory of metric spaces (i.e. a Riemannian metric is not a distance function); we will consider the relation of a Riemannian metric to a distance function more carefully soon. The *length* of a curve  $x(t)$  on  $M$  is

$$\int \sqrt{g(\dot{x}(t), \dot{x}(t))} dt.$$

A *Riemannian isometry* of Riemannian manifolds is a diffeomorphism identifying the Riemannian metrics. The *isometry group* of a Riemannian manifold is the group of Riemannian isometries of the Riemannian manifold to itself, sometimes called *rigid motions*. For example, the isometry group of the sphere in Euclidean space consists of the orthogonal matrices: any orthogonal transformation preserves the inner product, and preserves the sphere, and acts smoothly, so preserves the tangent spaces of the sphere. We will see soon why these are the only isometries of the sphere.

In 3-dimensional Euclidean space, with coordinates  $x_1, x_2, x_3$ , write  $dx_1 dx_3$  to mean the operation which associates to vector fields

$$X = \sum X_i(x) \frac{\partial}{\partial x_i}, Y = \sum Y_i(x) \frac{\partial}{\partial x_i},$$

the function

$$dx_1 dx_3(X, Y) = X_1(x) Y_3(x).$$

The *standard flat* or *Euclidean* metric is

$$g = dx_1^2 + dx_2^2 + dx_3^2$$

on  $\mathbb{R}^3$  or any open subset of  $\mathbb{R}^3$ , with obvious generalization to  $\mathbb{R}^n$ .

Pick a constant  $r > 0$ . Let  $S$  be the sphere of radius  $r$  in  $\mathbb{R}^3$  and let  $n \in S$  be the north pole  $n = (0, 0, r)$ . Let  $\kappa = 1/r^2$ . Map  $y \in S - \{n\} \mapsto x \in \mathbb{R}^2$  by

$$x = \left( \frac{y_1}{r - y_3}, \frac{y_2}{r - y_3} \right).$$

A long computation shows that the Euclidean metric on tangent vectors to

the sphere (away from the north pole) becomes the metric

$$g = \frac{dx_1^2 + dx_2^2}{\left(1 + \frac{\kappa}{4}(x_1^2 + x_2^2)\right)^2}.$$

**53.1** Generalize to a sphere in  $\mathbb{R}^{n+1}$ .

Formally, we could ask if there are “spheres of negative curvature”. The equation  $|y| = r$  doesn’t make sense with imaginary  $r$ , i.e. negative  $\kappa$ . Nevertheless, the metric in our previous example makes sense with any value of  $\kappa$ . If  $\kappa < 0$ , let  $M$  be the set of points in  $x \in \mathbb{R}^2$  for which

$$x_1^2 + x_2^2 < \frac{4}{|\kappa|}.$$

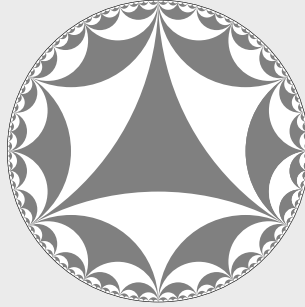
The metric

$$g = \frac{dx_1^2 + dx_2^2}{\left(1 + \frac{\kappa}{4}(x_1^2 + x_2^2)\right)^2}.$$

is then defined on  $M$ . In particular, if  $\kappa = -1$ , this metric is the *hyperbolic metric* and the manifold  $M$  (an open Euclidean ball of Euclidean radius 4) with this metric is the *hyperbolic plane*. Again, the reader can easily generalise to any number of variables to define *hyperbolic space*. Note that as we approach the “boundary”, i.e. as

$$x_1^2 + x_2^2 \rightarrow \frac{4}{|\kappa|},$$

the distances become very distorted and very small changes in the  $x_i$  variables make for very large changes in distance.



In this drawing of the hyperbolic plane, the “triangles” of the same colour are all of the same shape (i.e. carried to one another by Riemannian isometries). We will see that no surface in  $\mathbb{R}^3$  is carried by a Riemannian isometry to the hyperbolic plane.

If  $P$  and  $Q$  are Riemannian manifolds, then  $P \times Q$  is as well with metric  $g = g_P + g_Q$ .

The reader will notice that we have abandoned Einstein's notation.

**53.2** Prove that any Riemannian metric  $g$ , in any local coordinates  $x_1, x_2, \dots, x_n$ , has the form

$$g = \sum_{ij} g_{ij}(x) dx_i dx_j,$$

for a unique symmetric positive definite matrix  $g_{ij}(x)$  at each point, smoothly varying. Conversely, any such expression is a Riemannian metric where it is defined.

Rather than working with vector fields, for any vectors  $v, w$  at a point  $m_0 \in M$  of a Riemannian manifold  $M$ , define  $g(v, w)$  to mean  $g(V, W)$  at  $m_0$ , where  $V, W$  are vector fields with  $V(m_0) = v$  and  $W(m_0) = w$ .

Any submanifold  $P \subset Q$  of a manifold  $Q$  with Riemannian metric  $g_Q$  has *induced*  $g_P$ , also called the *pullback metric* defined by  $g_P(v, w) := g_Q(v, w)$ , the same metric, but only applied to the tangent vectors to  $P$ .

If  $f: P \rightarrow Q$  is an immersion of manifolds and  $Q$  bears a Riemannian metric  $g_Q$ , then  $P$  bears the *pullback metric*

$$g_P(v, w) := g_Q(f'(p)v, f'(p)w)$$

for any  $v, w \in T_p P$ . We can also denote the pullback metric as  $f^*g$ .

Riemannian geometry is the study of Riemannian manifolds. The term *metric* has two meanings in Riemannian geometry:

- a. the inner product  $g$  and
- b. the distance between points, defined to be the infimum length of path connecting the points, and well defined as long as the Riemannian manifold is connected.

These two meanings are not the same, but they are closely related, as we will see.

**Lemma 53.1.** *Every manifold admits a Riemannian metric.*

*Proof.* Every manifold admits a Riemannian metric near each point: use a chart to identify locally with Euclidean space. Take a manifold  $M$  and an open cover  $\{M_a\}$  of  $M$  by open sets  $M_a$  each equipped with a Riemannian metric  $g_a$ . Pick a partition of unity  $h_a: M \rightarrow \mathbb{R}$  subordinate to the open cover. Let  $g := \sum_a h_a g_a$ .  $\square$

**53.3** Suppose that  $\Gamma$  is a finite group acting by diffeomorphisms on a manifold  $M$ . Prove that  $M$  has a  $\Gamma$ -invariant Riemannian metric. If  $\Gamma$  acts freely on  $M$ , prove that  $M/\Gamma$  is a manifold with smooth quotient map  $M \rightarrow M/\Gamma$ .

The *distance* between two points  $p, q \in M$  of a Riemannian manifold is the infimum of lengths of paths between the points.



**Lemma 53.2.** Take a point  $p$  of the manifold  $M$  and a chart identifying  $p$  with the origin. The chart identifies the metric  $g$  near  $p$  with an inner product on  $\mathbb{R}^n$  near the origin, i.e. some positive definite symmetric matrix

$$(g_{ij}) = (g_{ij}(x))$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Suppose that  $\lambda$  is the smallest eigenvalue of this matrix for  $x \in \bar{B}$ , and that  $A$  is the largest. Let  $B_r$  be the Euclidean ball of radius  $r$  in the chart, and  $B'_r$  the Riemannian ball, about some point. If  $B'_r$  sits inside the domain of our coordinates, then  $B_{r/\sqrt{\lambda}} \subset B'_r$ . If  $B_r$  sits inside the domain of our coordinates, then  $B'_{r\sqrt{\lambda}} \subset B_r$ .

*Proof.* In terms of the Euclidean norm  $|v|$ ,

$$\lambda|v|^2 \leq g(v, v) = \sum_{ij} g_{ij}(x) v_i v_j \leq A|v|^2,$$

for any vector  $v$  and point  $x \in \bar{B}$ . Take square roots and integrate: any path  $x(t)$  in  $\bar{B}$  of Euclidean length  $\ell$  and Riemannian length  $\ell'$  has  $\ell\sqrt{\lambda} \leq \ell' \leq \ell\sqrt{A}$ .  $\square$

**Lemma 53.3.** For any connected Riemannian manifold  $M$ , the distance is a metric, and the open sets of the manifold are precisely the unions of open balls of the metric.

*Proof.* By lemma 53.2, any open subset in the sense of the Riemannian metric contains, around each of its points, an open subset of the manifold, and vice versa. Moreover, the Riemannian distance between any two points of the manifold is positive. Hence the Riemannian distance is a metric, being obviously symmetric and obviously satisfying the triangle inequality.  $\square$

**Corollary 53.4.** Every manifold is metrizable.

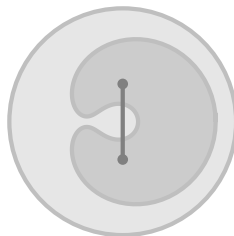
**Corollary 53.5.** If a group  $\Gamma$  acts freely by diffeomorphisms on a Riemannian manifold  $M$  preserving the Riemannian metric, with discrete orbits, then the quotient  $M/\Gamma$  inherits a unique Riemannian metric locally identified by the quotient map with the metric on  $M$ .

The torus, Klein bottle, real projective space, and the hexagonal torus, are all Riemannian manifolds under the usual quotient operations.

**53.4\*** Suppose that  $M$  is a Riemannian manifold. Prove that, for any point  $p \in M$ , and any chart around that point, there is a smooth function  $C(r)$  with  $C(r) \rightarrow 1$  as  $r \rightarrow 0$ , so that, if the closed ball  $\bar{B}_R$  about  $p$  of radius  $R$  lies in the domain of our chart, then, for any

$$r < \frac{R}{C(r)(1 + C(r))},$$

- a. any two points of  $\bar{B}_r$  are connected by a smooth path of length at most  $2rC(r)$  and
- b. any path of length  $2rC(r)$  or less between two points of  $\bar{B}_r$  lies in  $B_R$ .



If  $f: M \rightarrow N$  is a smooth map of Riemannian manifolds, then at each point  $m \in M$ , define  $\lambda(m)$  to be the maximum of

$$\frac{g_N(w, w)}{g_M(v, v)}$$

among vectors  $v \in T_m M$  with  $w = f'(m)v$ , the *infinitesimal dilation* of  $f$  at  $m$ . The *infinitesimal dilation* of  $f$  is the supremum  $\sup_{m \in M} \lambda(m)$ .

**53.5** Suppose that  $f: M \rightarrow N$  is a smooth map of Riemannian manifolds. Prove that the infinitesimal dilation of  $f$  is the dilation of  $f$  as a map of metric spaces.

Take a smooth map  $f: M \rightarrow N$  of manifolds. An open set  $U_N \subset N$  is *evenly covered* if  $f^{-1}U_N$  is a union of disjoint open sets

$$f^{-1}U_Y = \bigcup_a U_a$$

called *sheets*, so that each map

$$f|_{U_a}: U_a \rightarrow U_Y$$

is a diffeomorphism. A *Riemannian covering map* is a covering map  $f: M \rightarrow N$  for which  $N$  is equipped with a Riemannian metric and  $M$  is equipped with the pullback metric.

**53.6** Prove that, for any covering map  $M \rightarrow N$  of Riemannian manifolds,  $M$  is complete if and only if  $N$  is complete.

**Lemma 53.6.** *If  $G$  is compact Lie group acting smoothly on a manifold  $M$ , and  $g$  is a smooth Riemannian metric on  $M$ , define  $g^G$  to be*

$$g^G(u, v) := \int_{a \in G} g(a_* u, a_* v),$$

*integrated in the Haar measure. Then  $g^G$  is a smooth  $G$ -invariant Riemannian metric.*

Smoothness is just differentiation under the integral sign.

**53.7** Prove that every compact Lie group has a biinvariant Riemannian metric, i.e. invariant under both left and right multiplication.

**53.8\*** Prove that every immersed Lie subgroup of any compact Lie group has a biinvariant Riemannian metric.

**53.9\*** Prove that  $\mathrm{SL}(2, \mathbb{R})$  has no biinvariant Riemannian metric, hence admits no immersion as a subgroup of any compact Lie group.

## Notes

A deep theorem of Nash and Günter says that every smooth Riemannian manifold  $M$  of dimension  $n \geq 2$  admits a smooth isometric embedding into a Euclidean space  $\mathbb{R}^s$  of sufficiently high dimension  $s$ , i.e. so that the metric on  $M$  is induced by the embedding  $M \rightarrow \mathbb{R}^s$ , where we can take  $s$  to be any integer greater than or equal to

$$\frac{n(n+3)}{2} + \begin{cases} n, & \text{if } n \geq 5, \\ 5, & \text{if } 2 \leq n \leq 4. \end{cases}$$

It remains mysterious (as of the year 2022) whether some lower value of  $s$  might suffice. Any connected Riemannian manifold  $M$  of dimension 0 is isometric to point, and of dimension 1 is isometric to an interval of the real number line or to a circle. Any surface with Riemannian metric isometrically embeds into  $\mathbb{R}^4$ , but many do not isometrically embed into  $\mathbb{R}^3$ ; it is not known (as of the year 2022) whether there are smooth surfaces with smooth Riemannian metric which do not locally isometrically embed into  $\mathbb{R}^3$ .

Take a compact connected manifold with a Riemannian metric. Geodesics out of a point minimize distance up to some cut point. The tangent vectors which exponentiate to geodesics going up to the cut point form a bounded star shaped open set in the tangent space, hence homeomorphic to Euclidean space. The manifold is thus a compactification of Euclidean space.



## Chapter 54

### The Levi-Civita connection

*I admire the elegance of your method of computation; it must be nice to ride through these fields upon the horse of true mathematics while the like of us have to make our way laboriously on foot.*

— Albert Einstein

*Spaghetti and Levi-Civita. [asked what he liked most about Italy]*

— Albert Einstein

### The covariant derivative

Picture a surface  $M \subset \mathbb{R}^3$ . At a point of our surface  $M$ , draw the tangent plane to the surface, and the normal line: every vector in  $\mathbb{R}^3$  splits uniquely into a sum of tangent and normal parts. Draw a curve  $x(t)$  along  $M$ . At each point of  $x(t)$ , draw a vector  $v(t)$  tangent to  $M$  at  $x(t)$ . Note that the vector  $v(t)$  is not necessarily tangent to the curve drawn by  $x(t)$ ; it can “swing around” in any direction tangent to  $M$ . We can ask how quickly  $v(t)$  changes as we move along  $x(t)$ . If we ignore the surface  $M$ , the answer is just

$$\dot{v}(t) := \left. \frac{d}{dt} \right|_{t=0} v(t).$$

But if we picture ourselves living inside  $M$ , only able to measure the lengths and angles of *tangent* vectors, then we can *not* measure  $\dot{v}(t)$ , since it might not be tangent. The projection of  $\dot{v}(t)$  to the tangent plane  $T_{x(t)}M$  is called the *covariant derivative* of  $v(t)$ . We will see that we *can* measure the covariant derivative.

**54.1** Let  $M \subset \mathbb{R}^3$  be the unit sphere around the origin, let

$$x(t) = (\cos t, \sin t, 0),$$

and let

$$v(t) = (t^2 \sin t, -t^2 \cos t, t^2).$$

Prove that  $v(t)$  is tangent to  $M$  at each point  $x(t)$ . Calculate  $\nabla_{\dot{x}} v$ .

**54.2** A vector  $v(t)$  tangent to a surface in  $\mathbb{R}^3$ , and carried along a curve  $x(t)$  on that surface, *undergoes parallel transport* if  $\nabla_{\dot{x}}v = 0$ , i.e. if  $v$  is tangent while  $\dot{v}$  is normal to the surface at all times  $t$ . Compute the parallel transport of a vector around a line of latitude (i.e. parallel to the equator) on the Earth.

Similarly we can define, for any tangent vector fields  $X$  and  $Y$  on a surface  $M \subset \mathbb{R}^3$ , the *covariant derivative*  $\nabla_X Y$  at each point  $x \in M$  to be the projection of  $Y'(x)X(x)$  to  $T_x M$ . In other words,  $\nabla_X Y$  is the value of  $\nabla_{\dot{x}}Y$  along each flow line  $x(t)$  of the vector field  $X$ .

The covariant derivative has some obvious properties: for any smooth tangent vector fields  $X, Y, Z$  on  $M$  and  $C^1$  function  $f: M \rightarrow \mathbb{R}$ :

- a.  $\nabla_{fX} Y = f \nabla_X Y$ ,
- b.  $\nabla_X fY = df(X)Y + f \nabla_X Y$ ,

In the flow line description these are obvious.

The definition of covariant derivative for a surface  $M \subset \mathbb{R}^3$  generalises trivially to any submanifold  $M \subset \mathbb{R}^n$  of any Euclidean space, with the induced Riemannian metric. But in fact we can generalise the definition to all Riemannian manifolds all at once. An *affine connection* on a manifold  $M$  is an operation  $\nabla$  which associates to any two vector fields  $X, Y$  a vector field  $\nabla_X Y$  so that, for any  $C^1$  function  $f: M \rightarrow \mathbb{R}$ ,

- a.  $\nabla_{fX} Y = f \nabla_X Y$ ,
- b.  $\nabla_X fY = df(X)Y + f \nabla_X Y$ .

A *connection map* on a manifold  $M$  associates to every smooth metric  $g$  on  $M$  an affine connection  $\nabla^g$  on  $M$ . We do not assume that the resulting connection  $\nabla^g$  is continuous, or that the connection map is a continuous map. A connection map is *first order* if  $\nabla^g$  depends at each point only on  $g$  and its first derivatives, i.e. if two metrics  $g, h$  agree to first order in some coordinates at a point  $m_0 \in M$ , then

$$\nabla_X^g Y(m_0) = \nabla_X^h Y(m_0)$$

for any smooth vector fields  $X, Y$  defined near  $m_0$ . Similarly define connection maps of any order. We will prove:

**Theorem 54.1** (Robert Bryant). *No manifold has a zero order connection map. Every manifold has a unique first order connection map, and a unique second order connection map, equal to the first order one.*

We will make Bryant's connection map explicit. To prove Bryant's theorem, we need to find some first order properties of an affine connection which we can use to determine it.

### The Levi-Civita connection

An affine connection  $\nabla$  is *compatible* with a Riemannian metric  $g$  if  $Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$ . The covariant derivative on any surface is compatible with the induced metric, which we see again by looking at flow lines.

The covariant derivative on a surface also has a less obvious property:

**Lemma 54.2.** *For any vector fields  $X, Y$  on a surface  $M \subset \mathbb{R}^3$ ,*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

*Proof.* Clearly it is enough to prove this result near each point of our surface  $M$ . Therefore we can assume that  $M$  is an embedded surface, and we can extend  $X$  and  $Y$  to become vector fields defined nearby in  $\mathbb{R}^3$ . At each point  $x \in \mathbb{R}^3$  near  $M$ , we compute

$$[X, Y] = Y'(x)X(x) - X'(x)Y(x).$$

Therefore if  $P$  is the projection to  $T_x M$  at some point  $x \in M$ ,

$$P[X, Y] = P(Y'(x)X(x) - X'(x)Y(x)) = \nabla_X Y - \nabla_Y X.$$

By problem 7.15 on page 81, since the flows of  $X$  and  $Y$  are tangent to  $M$ ,  $[X, Y]$  is also a tangent vector to  $M$ , so  $P[X, Y] = [X, Y]$ .  $\square$

The *torsion* of an affine connection is the expression

$$\nabla_X Y - \nabla_Y X - [X, Y].$$

Some motivation: the expression  $\nabla_X Y$  “differentiates”  $Y$  along  $X$ ; we expect the bracket  $\nabla_X Y - \nabla_Y X$  to behave something like a Lie bracket of vector fields  $[X, Y] = \mathcal{L}_X Y - \mathcal{L}_Y X$ ; the error in this expectation is the torsion, and surprisingly there are affine connections with nonzero torsion (although we don’t want to ever think about them).

**54.3** Prove that at each point  $x$ , the value of the torsion  $t(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$  of a connection  $\nabla$  depends only on the value  $X(x), Y(x)$ , not on the derivatives of  $X$  and  $Y$  at  $x$  (even though it might appear to).

We will soon prove:

**Theorem 54.3** (The fundamental lemma of Riemannian geometry I). *Every Riemannian manifold admits a unique torsion-free affine connection compatible with the Riemannian metric, called the Levi-Civita connection.*

In particular, the Levi-Civita connection of any surface  $M \subset \mathbb{R}^3$  must be as described above, so determined by the angles and lengths of tangent vectors. We will see that the Levi-Civita connection is determined by the value and first derivative of the metric, so the connection map of Bryant’s theorem is the one associating the each metric its Levi-Civita connection.

**54.4** Consider a Riemannian metric  $g_0$  and a positive constant  $c > 0$ , and let  $g := c g_0$ . Prove that  $g_0$  and  $g$  have the same Levi-Civita connection.

## The Levi-Civita connection 1-forms

Recall that an *orthonormal coframing*  $\xi_1, \xi_2, \dots, \xi_n$  on an  $n$ -dimensional Riemannian manifold  $M$  is a collection of 1-forms so that

$$\xi_1^2 + \xi_2^2 + \dots + \xi_n^2 = g.$$

**54.5** Prove that there are local orthonormal coframings near any point of any Riemannian manifold.

An *orthonormal framing* on a Riemannian manifold  $M$  is a collection of vector fields  $e_1, e_2, \dots, e_n$  defined on some open subset of  $M$ , so that they form an orthonormal basis in each tangent space. In other words, the dual basis of 1-forms is an orthonormal basis. Warning: we will henceforth use  $e_1, e_2, \dots, e_n$  to denote the orthonormal framing dual to an orthonormal coframing  $\xi_1, \xi_2, \dots, \xi_n$ , *not* the standard basis of  $\mathbb{R}^n$ .

**54.6** Prove that given a Riemannian metric  $g$  and some 1-forms  $\xi_1, \xi_2, \dots, \xi_n$ , then

$$g = \xi_1^2 + \xi_2^2 + \dots + \xi_n^2$$

just when  $\xi_1, \xi_2, \dots, \xi_n$  is an orthonormal coframing.

**54.7** If you have some orthonormal coframing  $\eta_1, \eta_2, \dots, \eta_n$  and I have some other one, say  $\xi_1, \xi_2, \dots, \xi_n$  on a Riemannian manifold  $M$ , prove that they are related by:

$$\xi_i = h_{ij} \eta_j,$$

where

$$h = (h_{ij})$$

is a uniquely determined smooth map  $h: M \rightarrow \mathbb{R}^{n \times n}$  valued in the group of orthogonal matrices.

Take a coframing  $\xi_1, \xi_2, \dots, \xi_n$  on a manifold  $M$ . Since these  $\xi_1, \xi_2, \dots, \xi_n$  are a basis of the 1-forms, every differential form admits a unique expansion in them. In particular, the 2-forms  $d\xi_i$  admit a unique expansion (in Einstein notation):

$$d\xi_i = -c_{ijk} \xi_j \wedge \xi_k$$

for some functions  $c_{ijk}$  on  $M$ ; the minus sign is for future convenience.

**Lemma 54.4.** Suppose that  $a_{ijk}$  are numbers,  $i, j, k = 1, 2, \dots, n$ , and that  $a_{ijk}$  is symmetric in two indices and antisymmetric in the other two. The  $a_{ijk} = 0$  for all  $i, j, k$ .

*Proof.* Every permutation of some letters is generated by pairwise permutations of neighboring letters. So the permutation group on 3 letters  $i, j, k$  is generated by permuting  $i, j$  and permuting  $j, k$ . Every permutation of  $i, j, k$ , written as a product of those two, has a defined sign change, a morphism of groups to the group  $\{\pm 1\}$ . The sign depends only on the conjugacy class of each element, because  $\{\pm 1\}$  is abelian. But any two transpositions are conjugate, so determine the same sign.  $\square$

**Lemma 54.5.** Consider the vector space  $V$  for which each element  $a$  of  $V$  is a collection  $a = (a_{ijk})$  of numbers  $a_{ijk}$ , defined for  $i, j, k = 1, 2, \dots, n$ . Let  $B \subset V$  be those antisymmetric in  $i, j$ , and let  $C \subset V$  be those antisymmetric in  $j, k$ . Then the linear map  $b \in B \mapsto c \in C$  given by

$$c_{ijk} = \frac{1}{2}(b_{ijk} - b_{ikj})$$

is a linear isomorphism with inverse

$$b_{ijk} = c_{ijk} + c_{jki} + c_{kji}.$$

*Proof.* The kernel of  $b \mapsto c$  is zero by lemma 54.4, and the source and target vector spaces have equal dimension. The reader can check the inverse.  $\square$



**Lemma 54.6** (The fundamental lemma of Riemannian geometry I). *For any smooth orthonormal coframing  $\xi_1, \xi_2, \dots, \xi_n$  on a Riemannian manifold, there is a unique collection of smooth 1-forms  $\eta_{ij} = -\eta_{ji}$  so that*

$$d\xi_i = -\eta_{ij} \wedge \xi_j.$$

*Proof.* Write  $d\xi_i = -c_{ijk}\xi_j \wedge \xi_k$  for unique  $c_{ijk}$  antisymmetric in  $j, k$ . Apply our (constant) linear isomorphism at each point to write  $d\xi_i = -b_{ijk}\xi_j \wedge \xi_k$  for unique  $b_{ijk}$  antisymmetric in  $i, j$ . Let  $\eta_{ij} := b_{ijk}\xi_k$ . Then it is clear that  $\eta_{ji} = -\eta_{ij}$ . Conversely, expand any such  $\eta_{ij}$  into  $\eta_{ij} = b_{ijk}\xi_k$  and apply the linear isomorphism to see that you return back to the same  $c_{ijk}$ .  $\square$

Recall that the hyperbolic plane is the upper half plane

$$M = \{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}$$

with the metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$

Our orthonormal coframing

$$\xi_1 = \frac{dx}{y}, \quad \xi_2 = \frac{dy}{y}$$

has Levi-Civita 1-forms  $\eta_{12} = -\xi_1$ .

Consider a Riemannian metric of the form

$$g = f^2(dx_1^2 + dx_2^2 + \dots + dx_n^2),$$

where we let  $r = |x|$  be the Euclidean distance from the origin, and let  $f = f(r) > 0$  be a positive function of that distance. The 1-forms

$$\xi_i := f dx_i$$

are an orthonormal coframing. The reader can check that

$$d\xi_i = \frac{f'(r)}{r} x_j dx_j \wedge dx_i.$$

We also adopt *Dirac notation* in which any time we write  $\delta_{ij}$  we mean

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Antisymmetrise in  $j, k$  to get

$$d\xi_i = -\frac{f'}{2rf^2} (x_j \delta_{ik} - x_k \delta_{ij}) \xi_j \wedge \xi_k.$$

Let

$$c_{ijk} = \frac{f'}{2rf^2} (x_j \delta_{ik} - x_k \delta_{ij}).$$

Our linear isomorphism gives

$$\begin{aligned} b_{ijk} &= c_{ijk} + c_{jki} + c_{kji}, \\ &= \frac{f'}{rf^2}(x_j\delta_{ik} - x_i\delta_{jk}). \end{aligned}$$

So finally the Levi-Civita connection forms in this coframing are

$$\eta_{ij} = \frac{f'}{rf^2}(x_j\xi_i - x_i\xi_j).$$

**54.8** Take an orthonormal coframing  $\xi_1, \dots, \xi_n$ , with Levi-Civita connection forms  $\eta_{ij}$  and with dual orthonormal framing  $e_1, \dots, e_n$ . Suppose that we expand out  $\eta_{ij} = b_{ijk}\xi_k$ . Recall that we let

$$c_{ijk} = \frac{1}{2}(b_{ijk} - b_{ikj}).$$

Prove that, for any  $i, j$ :

$$[e_i e_j] = -2c_{kij}e_k.$$

We prove theorem 54.3 on page 449:

*Proof.* Take an orthonormal coframing  $\xi_1, \dots, \xi_n$ , with Levi-Civita connection forms  $\eta_{ij}$  and with dual orthonormal framing  $e_1, \dots, e_n$ . Take any affine connection  $\nabla$ . Define functions, traditionally denoted  $\Gamma_{ij}^k$ , by

$$\nabla_{e_i} e_j = \Gamma_{ij}^k e_k.$$

Take any two vector fields  $X, Y$  and expand them in the framing as  $X = X_i e_i$ ,  $Y = Y_j e_j$ . From the definition of affine connection,

$$\begin{aligned} \nabla_X Y &= \nabla_X (Y_j e_j), \\ &= dY_j(X) e_j + Y_j \nabla_X e_j, \\ &= dY_j(X) e_j + Y_j X_i \nabla_{e_i} e_j, \\ &= dY_j(X) e_j + Y_j X_i \Gamma_{ij}^k e_k, \\ &= (dY_j(X) + Y_k X_i \Gamma_{ik}^j) e_j. \end{aligned}$$

So the connection is completely determined by these  $\Gamma_{ij}^k$  functions, and, reversing steps, any functions  $\Gamma_{ij}^k$  define a connection by the above formula.

If we take the affine connection  $\nabla$  for which  $\Gamma_{ij}^k := b_{kji}$ , then a short calculation shows that this connection is torsion-free and compatibility with the metric.

Suppose that we had another torsion free connection  $\bar{\nabla}$  compatible with the metric. Define functions  $\bar{\Gamma}_{ij}^k$  by

$$\bar{\nabla}_{e_i} e_j - \nabla_{e_i} e_j = \bar{\Gamma}_{ij}^k e_k.$$

In other words,

$$\bar{\nabla}_{e_i} e_j = (b_{kji} + \bar{\Gamma}_{ij}^k) e_k.$$

The torsion of  $\bar{\nabla}$  differs from the (vanishing) torsion of  $\nabla$  by

$$\begin{aligned} \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] &= (\bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]) - (\nabla_X Y - \nabla_Y X - [X, Y]), \\ &= (\bar{\nabla} - \nabla)_X Y - (\bar{\nabla} - \nabla)_Y X \end{aligned}$$

for any vector fields  $X, Y$ . But this expression, computed at some point, depends only on the values of  $X$  and  $Y$  at that point, so it is enough if we take  $X = e_i$  and  $Y = e_j$  for arbitrary  $i, j$ : the resulting torsion is

$$\bar{\nabla}_{e_i} e_j - \bar{\nabla}_{e_j} e_i - [e_i, e_j] = (\Gamma_{ij}^k - \Gamma_{ji}^k) e_k.$$

Hence the affine connection  $\bar{\nabla}$  will be torsion free just when  $0 = \Gamma_{ij}^k - \Gamma_{ji}^k$ . Incompatibility with the metric is measured by

$$\begin{aligned} 0 &= e_i \langle e_j, e_k \rangle - \langle \bar{\nabla}_{e_i} e_j, e_k \rangle - \langle e_j, \bar{\nabla}_{e_i} e_k \rangle, \\ &= (e_i \langle e_j, e_k \rangle - \langle \bar{\nabla}_{e_i} e_j, e_k \rangle - \langle e_j, \bar{\nabla}_{e_i} e_k \rangle) + (e_i \langle e_j, e_k \rangle - \langle \nabla_{e_i} e_j, e_k \rangle - \langle e_j, \nabla_{e_i} e_k \rangle), \\ &= -\Gamma_{ij}^k - \Gamma_{ik}^j. \end{aligned}$$

Hence  $\bar{\nabla}$  is torsion-free and compatible with the metric if and only if  $\Gamma_{ij}^k$  is symmetric in  $ij$  and antisymmetric in  $jk$ . We leave the reader to check (using linear algebra) that this is impossible unless  $\Gamma_{ij}^k = 0$  for all  $i, j, k$ .  $\square$

### Bryant's theorem

**Lemma 54.7.** *Consider the vector space  $V$  for which each element  $a$  of  $V$  is a collection  $a = (a_{ijk})$  of numbers  $a_{ijk}$ , defined for  $i, j, k = 1, 2, \dots, n$ . Let  $B \subset V$  be those symmetric in  $j, k$ , and let  $C \subset V$  be those symmetric in  $i, j$ . Then the linear map  $b \in B \mapsto c \in C$  given by*

$$c_{ijk} = \frac{1}{2}(b_{ijk} + b_{jik})$$

*is a linear isomorphism.*

*Proof.* The linear map  $b \mapsto c$  has kernel consisting of the  $b$  with  $b_{ijk}$  symmetric in  $j, k$  and antisymmetric in  $i, j$ , hence zero by lemma 54.4 on page 450. This linear map is injective, and the source and target vector spaces have the same dimension.  $\square$

The Kronecker delta symbol is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The Euclidean metric on Euclidean space  $\mathbb{R}^n$  is  $g = g_{ij} dx_i dx_j$  with  $g_{ij} = \delta_{ij}$ .

**Lemma 54.8.** *Any  $C^2$  Riemannian metric, near any point, in some coordinates, has the form*

$$g = (\delta_{ij} + O(x)^2) dx_i dx_j.$$

*Proof.* Take some coordinates  $x_i$  vanishing at the given point. By Gram-Schmidt orthogonalization theorem, after a linear change of coordinates, we can get  $g_{ij} = \delta_{ij}$  at that point. Expand out

$$g = g_{ij} dx_i dx_j$$

in a Taylor series

$$g_{ij} = \delta_{ij} + c_{ijk} x_k + \dots$$

Define coordinates  $y_i$  by  $x_i = y_i - \frac{1}{4}b_{ijk}y_jy_k$  for some constants  $b_{ijk} = b_{ikj}$ ; the implicit function theorem implies that there are such coordinates. Plug in and find

$$g = h_{ij}dy_idy_j,$$

where

$$h_{ij} = \delta_{ij} + (c_{ijk} - \frac{1}{2}(b_{ijk} + b_{jik}))y_j + \dots$$

Apply lemma 54.7 on the preceding page to pick  $b_{ijk}$ .  $\square$

We prove Bryant's theorem (theorem 54.1 on page 448): the existence and uniqueness of a first order connection map.

*Proof.* The Levi-Civita connection  $\nabla$  is a first order connection map, as we have seen in computing its expression. If some other connection  $\nabla'$  also arises from a first order connection map, consider the difference

$$t(v, Y) := \nabla_v Y - \nabla'_v Y.$$

We have only to prove that  $t = 0$  at any one arbitrary point  $m_0$ .

Expand out to check that  $t(v, fY) = ft(v, Y)$  for any smooth function  $f$ . For any basis  $e_1, \dots, e_n$  of vector fields near some point  $m_0$ , define functions

$$t_{ij}^k$$

by

$$t(e_i, e_j) = t_{ij}^k e_k.$$

Expand out to find that, if  $v = v^i e_i$  and  $Y = Y^j e_j$ , then

$$t(v, Y) = v^i Y^j t_{ij}^k e_k,$$

so that  $t(v, Y)$  depends only on the vector  $Y(m_0)$  and  $t$  is a linear map

$$t: T_{m_0}M \times T_{m_0}M \rightarrow T_{m_0}M.$$

By lemma 54.8 on the preceding page, we can assume that  $g$  is the Euclidean metric on Euclidean space. Our problem reduces to finding, for the Euclidean metric, all such  $t$ . Under any orthogonal linear transformation  $r$  of Euclidean space, the Euclidean metric is preserved, and so both connections are preserved, and so  $t$  is preserved:  $t(rx, ry) = rt(x, y)$  for any vectors  $x, y$ . In particular, if  $r = -I$ , then  $t(-x, -y) = -t(x, y)$  but by linearity in  $x$  and  $y$ ,  $t(-x, -y) = t(x, y)$ . So  $t = 0$ .

Consider a second order connection map. We can make our metric into a Euclidean metric at first order, but at second order we will find quadratic terms in the Taylor expansion. These are invariant under reflection, but our  $t(x, y)$  is not, unless  $t = 0$ .  $\square$

**54.9** Prove that there is no zero order connection map on any manifold.

There is a unique connection map of any finite order on any curve. On any manifold of dimension 2 or more, there are infinitely many connection maps of any order three or more.

## Chapter 55

### Geodesics

#### Acceleration

A vector field  $v$  along a path  $x: I \rightarrow M$ , with  $I \subset \mathbb{R}$  an interval, is a map associating to each  $t \in I$  a tangent vector  $v(t) \in T_{x(t)}M$ .

**Lemma 55.1.** *Associated to any affine connection  $\nabla$ , and  $C^k$  path  $x(t)$ , there is an operator  $\nabla_{\dot{x}}$  taking any  $C^k$  vector field  $v$  along  $x$ , and determining a  $C^{k-1}$  vector field  $\nabla_{\dot{x}}v$  along  $x$ , so that*

- a.  $\nabla_{\dot{x}}(v + w)(t) = \nabla_{\dot{x}}v(t) + \nabla_{\dot{x}}w(t)$  for any  $v$  and  $w$  vector fields along  $x$  and
- b.  $\nabla_{\dot{x}}(fv)(t) = \left(\frac{d}{dt}f(x(t))\right)v(t) + f(x(t))\nabla_{\dot{x}}v(t)$  and
- c. if  $t_0 \in I$  and  $m_0 := x(t_0)$  and  $X$  is a vector field with  $X(m_0) = \dot{x}(t_0) \in T_{m_0}M$  and  $Y$  is a vector field with  $Y(x(t)) = v(t)$  for all  $t$  near  $t_0$ , then

$$\nabla_{\dot{x}}v(t) = \nabla_X Y(m_0).$$

*Proof.* Take a local framing  $e_1, \dots, e_n$ , and define functions  $\Gamma_{ij}^k$  by  $\nabla_{e_i}e_j = \Gamma_{ij}^k e_k$ . Define functions  $\dot{x}_i(t)$  by  $\dot{x}(t) = \dot{x}_i(t)e_i(x(t))$  and functions  $v_i(t)$  by  $v(t) = v_i(t)e_i(x(t))$ . Then our requirements are precisely that

$$\nabla_{\dot{x}}v = \frac{dv_i}{dt}e_i + v_j(t)\dot{x}_i(t)\nabla_{e_i}e_j.$$

This determines  $\nabla_{\dot{x}}$  directly from  $\nabla$ , so uniqueness is assured. But this expression clearly satisfies our requirements, so existence is assured.  $\square$

A geodesic of an affine connection  $\nabla$  is a  $C^2$  path  $x(t)$  for which  $\nabla_{\dot{x}}\dot{x} = 0$ .

**Lemma 55.2.** *Each affine connection has a geodesic passing through any given point  $m_0 \in M$  at a given time  $t_0$ , with any given initial velocity in  $T_{m_0}M$ . Any two such agree on the largest interval of time on which they are both defined.*

*Proof.* The equation, in any local framing, is

$$0 = \nabla_{\dot{x}}\dot{x} = \frac{d\dot{x}_i}{dt}e_i + \dot{x}_i(t)\dot{x}_j(t)\nabla_{e_i}e_j.$$

In particular, if our local framing is  $e_i = \partial_{x_i}$  for some local coordinates  $x_1, \dots, x_n$ , then this is

$$0 = \frac{d^2x_k}{dt^2} + \dot{x}_i(t)\dot{x}_j(t)\Gamma_{ij}^k,$$

to which we apply the existence and uniqueness theorem for ordinary differential equations.  $\square$

In Euclidean space, the geodesic passing through a point  $x_0$  with velocity  $v_0$  is precisely the linear map  $t \mapsto x_0 + tv_0$ . To see this, note that any rotation or reflection around  $x_0$  fixing  $v_0$  preserves the Euclidean metric, and so preserves the geodesic. But only the line through  $x_0$  in the direction  $v_0$  is invariant. So the geodesic heads along that line, at constant speed.

On the unit sphere  $S^2$  about the origin in Euclidean space  $S^2 \subset \mathbb{R}^3$ , the geodesic passing through a point  $x_0$  with velocity  $v_0$  is precisely the circle

$$t \mapsto x_0 \cos(s_0 t) + \frac{v_0}{s_0} \sin(s_0 t),$$

where  $s_0 = |v_0|$ . To see this, note that reflection fixing  $x_0$  and  $v_0$  preserves the Euclidean metric, and so preserves the Levi-Civita connection, and so preserves the geodesic. But only the circle through  $x_0$  in the direction  $v_0$  is invariant. So the geodesic heads along that circle, at constant speed.

Similarly, we can easily find the geodesics of a product of a Euclidean space with any number of spheres of any radii and of any dimensions.

The geodesic equation is expressed purely in terms of local information about the Riemannian metric, so geodesics are matched up by any local isometry. For example, the flat disk in the Euclidean plane has as geodesics precisely the line segments that lie in the disk.

The two-to-one map  $S^n \rightarrow \mathbb{RP}^n$  is a local isometry, so matches up geodesics.

Take a torus  $M = \mathbb{R}^n / \Lambda$ , where  $\Lambda \subset \mathbb{R}^n$  is a discrete subgroup spanning  $\mathbb{R}^n$ . The Euclidean metric on  $\mathbb{R}^n$  descends to a Riemannian metric on the torus, by corollary 53.5 on page 443. The quotient map  $\mathbb{R}^n \rightarrow M$  is a covering map, so a local isometry, so matches up geodesics.

**55.1** Suppose that  $x(t)$  is a geodesic of nonzero velocity and let  $y(t) = x(\tau(t))$  for some smooth function  $\tau(t)$ , 1 function of 1 variable. Prove that  $y(t)$  is also a geodesic just when  $\tau(t)$  is a linear function of  $t$ .

### Minimal geodesics

Picture an earthquake at the north pole; two tectonic plates slide over one another, and the shock wave travels the globe. By symmetry, the front of the shock wave moves in an expanding circle of latitude, in all directions away from the north pole, until the circle reaches the girth of the planet, and then

contracts toward the south pole. A bird flying away from the earthquake at the same pace as the shock wave will never overtake it, and only keep pace with it by flying due south along a line of longitude. Hence the bird travels away from the north pole fastest by moving straight south along longitude lines: the shortest paths from the north pole are lines of longitude. The function  $f$  giving distance from the north pole has level sets tracking the motion of the shock wave; we will see that  $\xi = df$  is a *calibrating* 1-form.

On a Riemannian manifold, a *calibrating* 1-form is a 1-form  $\xi$  so that  $|\xi| = 1$  and  $d\xi = 0$ . If  $s$  is a smooth function satisfying the *eikonal equation*:  $|ds| = 1$  then  $\xi = ds$  is a calibrating 1-form.

A *minimal geodesic* is a geodesic which is the shortest piecewise continuously differentiable path between any two of its points. A *geodesic foliation* is a foliation by oriented geodesics, and so has a uniquely defined unit tangent vector field  $X$  pointing in the positive direction along each geodesic.

**Theorem 55.3.** *Suppose that  $\xi$  is a calibrating 1-form on a Riemannian manifold  $M$ . Let  $X$  be the vector field so that  $|X| = 1$  and  $\xi(X) = 1$ . The flow lines of  $X$  are geodesics of  $M$ , traversed at unit speed, giving a geodesic foliation. The directions perpendicular to  $X$  form a foliation by hypersurfaces. For any one of these hypersurfaces, the geodesics of the foliation are locally just precisely those which are perpendicular to that hypersurface. Every point of  $M$  lies in an open set in which  $\xi = ds$  for some smooth function  $s$  satisfying the eikonal equation. In any such open set, the hypersurfaces are the level sets of  $s$  and every geodesic of the foliation is minimal.*

*Proof.* Since  $d\xi = 0$ , locally  $\xi = ds$  for some function  $s$  by the Poincaré lemma (theorem 30.6 on page 244). Since  $|\xi| = 1$ ,  $|ds| = 1$ , the eikonal equation. The hypersurfaces on which  $s$  is constant are just those on which  $\xi = 0$ . The vector field  $X$  is by definition perpendicular to those hypersurfaces. The 1-form  $\xi$  calibrates curves travelling along the flow lines of  $X$  between those hypersurfaces. We want to see that these flow lines are minimal length curves between those hypersurfaces. Suppose that  $\xi_1, \xi_2, \dots, \xi_n$  is an orthonormal coframing and  $\xi_1 = \xi$ . In the dual basis  $e_1, e_2, \dots, e_n$ , our curves are the flow lines of  $e_1$ . Take a flow line  $C$  of  $e_1$ , say starting at some point  $p_0$  and ending at some point  $p_1$ . As we move along  $C$ ,  $e_1 s = ds(e_1) = \xi(e_1) = 1$ , so  $s$  increases at unit rate along  $C$ . So the change in  $s$  between starting point and ending point of our curve  $C$  is the length of  $C$ :

$$\text{length } C = s(p_1) - s(p_0).$$

If we move in any other direction,  $e_2 s = ds(e_2) = \xi_1(e_2) = 0$ , etc., so as we move along a curve  $C'$  with the same endpoints, but moving in any other direction than  $e_1$ , at unit speed,  $s$  increases at less than unit rate. Hence the length of  $C'$  is

$$\text{length } C' = \int dt \geq \int ds = s(p_1) - s(p_0) = \text{length}_C$$

with equality if and only if  $C'$  has velocity  $e_1$  all the time, i.e.  $C' = C$ .

Since  $ds = \xi_1 \neq 0$ , the sets  $s = \text{constant}$  are smooth hypersurfaces. Along the flow of  $e_2$ , as we saw,  $e_2 s = 0$ , so  $s$  is constant:  $e_2$  flows along the  $s = \text{constant}$  hypersurfaces.

We need to prove that the flow lines of  $e_1$  are geodesics, i.e. that  $\nabla_{e_1} e_1 = 0$ . Again note that  $e_1 s = 1$  while  $e_j s = 0$  for  $j > 1$ . But then  $[e_i, e_j]s = 0$  for any  $i, j$ . So  $[e_i, e_j]$  is a linear combination of  $e_2, e_3, \dots, e_n$ , with no  $e_1$ . In particular  $\langle e_1, [e_i, e_j] \rangle = 0$  for all  $i, j$ . By torsion-freeness,

$$\langle e_1, \nabla_{e_i} e_j \rangle = \langle e_1, \nabla_{e_j} e_i \rangle$$

for any  $i, j$ . Using compatibility with the metric to get

$$\begin{aligned} \langle \nabla_{e_1} e_1, e_j \rangle &= -\langle e_1, \nabla_{e_1} e_j \rangle, \\ &= -\langle e_1, \nabla_{e_j} e_1 + [e_1, e_j] \rangle, \\ &= -\langle e_1, \nabla_{e_j} e_1 \rangle, \\ &= \langle \nabla_{e_j} e_1, e_1 \rangle, \end{aligned}$$

and compare the last line with the line above it to see that this vanishes for every  $j$ , so  $\nabla_{e_1} e_1 = 0$ .  $\square$

**Lemma 55.4.** *Every geodesic foliation perpendicular to the leaves of a foliation of hypersurfaces arises, perhaps after replacing the manifold with a two-to-one covering space, from a unique calibrating 1-form  $\xi$ , i.e. has hypersurfaces tangent to the directions in which  $\xi = 0$ , and geodesics tangent to the dual vector field to  $\xi$ .*

*Proof.* Take a covering space to ensure that there is a choice of unit vector field  $e_1$  perpendicular to the leaves of the foliation, hence tangent to the geodesics. Locally extend to an orthonormal framing  $e_1, e_2, \dots, e_n$ . If we pick any  $j, k \geq 2$ , then  $[e_j, e_k]$  is a linear combination of  $e_2, \dots, e_n$ , by the Frobenius theorem (theorem 13.3 on page 123), so  $0 = \langle e_1, [e_j, e_k] \rangle$ .

Using the torsion-freeness and compatibility of the Levi-Civita connection,

$$\begin{aligned} \langle [e_1, e_j], e_1 \rangle &= \langle \nabla_{e_1} e_j - \nabla_{e_j} e_1, e_1 \rangle, \\ &= \langle \nabla_{e_1} e_j, e_1 \rangle - \langle \nabla_{e_j} e_1, e_1 \rangle, \\ &= -\langle e_j, \nabla_{e_1} e_1 \rangle - \langle \nabla_{e_j} e_1, e_1 \rangle, \\ &= -0 - \langle \nabla_{e_j} e_1, e_1 \rangle, \\ &= -\frac{1}{2} e_j \langle e_1, e_1 \rangle, \\ &= 0. \end{aligned}$$

Therefore all brackets  $[e_i, e_j]$  are perpendicular to  $e_1$ , so  $0 = \xi_1([e_i, e_j])$  for all  $i, j$ .

Two proofs:

- By problem 39.15 on page 317,  $d\xi_1 = 0$ .
- By the Frobenius theorem, we can pick local coordinates  $x_1, x_2, \dots, x_n$  so that our hypersurfaces are level sets of  $x_1$ . The vector fields

$$\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$$

are tangent to the hypersurfaces, so they are linearly combinations of  $e_2, \dots, e_n$  and vice versa. In these coordinates,

$$e_1 = \sum_i f_{1i} \frac{\partial}{\partial x_i}, e_i = \sum_{ij} f_{ij} \frac{\partial}{\partial x_j},$$



where  $f_{11} \neq 0$ , and the matrix  $f_{ij}$  is invertible as these are linearly independent vector fields. Compute brackets, and check that the  $\frac{\partial}{\partial x_1}$  coefficient of  $[e_1, e_k]$  is

$$-\sum_j f_{ij} \frac{\partial f_{11}}{\partial x_j}.$$

But  $[e_1, e_i]$  has no  $e_1$  in it, i.e.

$$\frac{\partial f_{11}}{\partial x_j} = 0$$

for all  $j = 2, 3, \dots, n$ , so  $f_{11} = f_{11}(x_1)$ . Let

$$s = \int \frac{dx_1}{f_{11}(x_1)}.$$

□

In the Euclidean plane  $dx^2 + dy^2$  has orthonormal coframing  $\xi_1 = dx$ ,  $\xi_2 = dy$ , so horizontal and vertical lines are minimal geodesics.

In the Euclidean plane  $dx^2 + dy^2 = dr^2 + (r d\theta)^2$  has orthonormal coframing  $\xi_1 = dr$ ,  $\xi_2 = r d\theta$ , so the rays of constant  $\theta$  are minimal geodesics.

Take the metric from chapter 53:

$$g = \frac{dx_1^2 + dx_2^2}{\left(1 + \frac{\kappa}{4}(x_1^2 + x_2^2)\right)^2}.$$

Again let  $x_1 = r \cos \theta$  and  $x_2 = r \sin \theta$ . Then

$$\begin{aligned} \xi_1 &= \frac{dr}{1 + \frac{\kappa r^2}{4}}, \\ \xi_2 &= \frac{r d\theta}{1 + \frac{\kappa r^2}{4}} \end{aligned}$$

is an orthonormal coframing. So again the rays of constant  $\theta$  are minimal geodesics. A tough integral: the length of any interval of each of these geodesics is:

$$\int \xi_1 = \begin{cases} \frac{2 \arctan\left(\frac{1}{2} \sqrt{\kappa} r\right)}{\sqrt{\kappa}}, & \text{if } \kappa > 0, \\ r, & \text{if } \kappa = 0, \\ \frac{2 \operatorname{arctanh}\left(\frac{1}{2} \sqrt{|\kappa|} r\right)}{\sqrt{|\kappa|}}, & \text{if } \kappa < 0. \end{cases}$$

On a surface of revolution  $M \subset \mathbb{R}^3$ :

$$M := \{ (x, f(x) \cos \theta, f(x) \sin \theta) \mid x \in \mathbb{R}, 0 \leq \theta \leq 2\pi \},$$

the metric on  $\mathbb{R}^3$  pulls back to

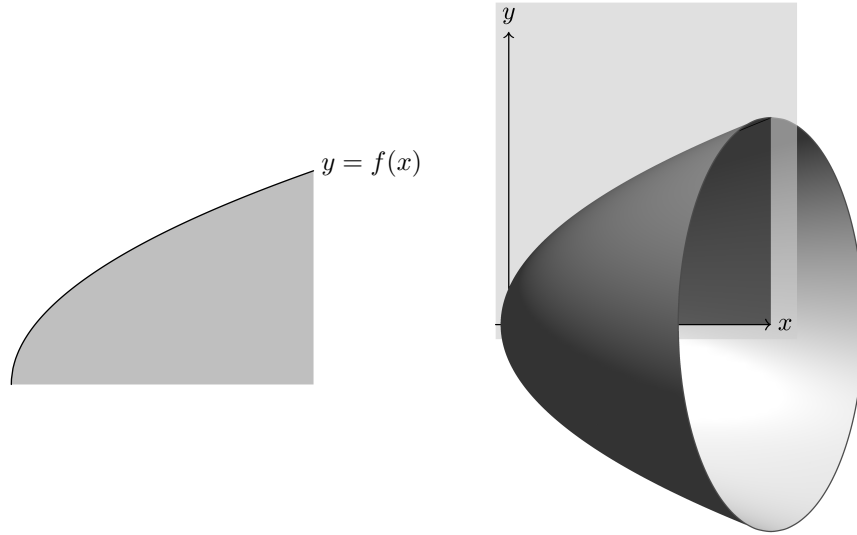
$$\begin{aligned} dx^2 + dy^2 + dz^2 &= dx^2 + [d(f(x) \cos \theta)]^2 + [d(f(x) \sin \theta)]^2, \\ &= (1 + f'(x)^2) dx^2 + f(x)^2 d\theta^2. \end{aligned}$$

The 1-forms

$$\begin{aligned} \xi_1 &= \sqrt{1 + f'(x)^2} dx, \\ \xi_2 &= f(x) d\theta \end{aligned}$$

are an orthonormal coframing, and  $\xi_1$  is calibrating, calibrating the curves on which  $\xi_2 = 0$ , i.e. the curves of constant angle  $\theta$ : the meridians are minimal geodesics. The length of any interval of any one of these geodesics is

$$\int \sqrt{1 + f'(x)^2} dx.$$



### Example: the hyperbolic plane

The hyperbolic plane is the upper half plane  $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$  with the metric

$$g = \frac{dx^2 + dy^2}{y^2}.$$

By reflection symmetry  $(x, y) \mapsto (2x_0 - x, y)$  of the metric about any vertical line, every vertical line is a geodesic of the hyperbolic plane metric: the geodesic can never

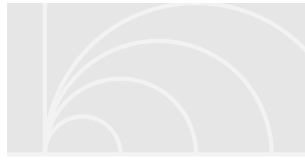
turn left or right without breaking the symmetry. In the orthonormal coframing

$$\begin{aligned}\xi_1 &= \frac{dx}{y}, \\ \xi_2 &= \frac{dy}{y} = d \log y,\end{aligned}$$

clearly  $\xi_2$  is closed, so the vertical lines are minimal geodesics. The length of such a path, between a point  $iy_0$  and a point  $iy_1$ , is then

$$\int_{y_0}^{y_1} \frac{dy}{y} = \log \frac{y_1}{y_0}.$$

We will soon see that the geodesics are: any vertical half line starting at the horizontal axis, or any semicircle meeting the horizontal axis at right angles.



A *linear fractional transformation* is a map

$$f(z) = \frac{az + b}{cz + d}$$

for any complex numbers  $a, b, c, d$  with  $ad - bc \neq 0$ . A linear fractional transformation  $z \mapsto z + b$  is a *translation*, while  $z \mapsto az$  is a *dilation* and  $z \mapsto -1/z$  is *inversion*.

**55.2** To each  $2 \times 2$  matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

associate the linear fractional transformation

$$f_A(z) := \frac{az + b}{cz + d}.$$

Define the *quotient map*  $\rho$  on pairs of complex numbers by

$$\rho \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{z_1}{z_2}.$$

This is defined just when  $z_2 \neq 0$ , and is clearly onto. Check that

$$\rho \circ A = f_A \circ \rho.$$

Use this fact to prove that

$$f_A \circ f_B = f_{AB}$$

for any  $2 \times 2$  matrices  $A, B$ .

Any linear fractional can be written

$$f(z) = \frac{az + b}{cz + d} = \frac{-1}{c^2(z + d/c)} + \frac{a}{c},$$

as a composition of a translation, a dilation, an inversion and a translation. If  $c = 0$ , so that this is not defined, then

$$f(z) = (a/d)z + (b/d)$$

is a composition of dilation and translation.

**55.3** Check that  $a, b, c, d$  are all real (perhaps after rescaling) and  $ad - bc > 0$  just when  $f$  takes the upper half plane  $y > 0$  to itself.

**55.4** Use polar coordinates to check that inversion is an isometry of the hyperbolic plane.

**Lemma 55.5.** *The hyperbolic plane metric is invariant any linear fractional transformation*

$$f(z) = \frac{az + b}{cz + d}$$

for any real  $a, b, c, d \in \mathbb{R}$  with  $ad - bc > 0$ .

*Proof.* Under translations of the variable  $x$ ,  $dx$  is unchanged, so  $g$  is unchanged. Under dilations,  $x \mapsto ax$ ,  $y \mapsto ay$ ,  $dx \mapsto a dx$ ,  $dy \mapsto a dy$ , so the ratio

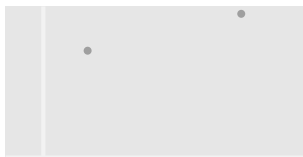
$$g = \frac{dx^2 + dy^2}{y^2}$$

has numerator and denominator both scaling by  $a^2$ . So it suffices to check inversion, which is problem 55.4.  $\square$

**55.5** How does inversion affect vertical lines? How does inversion affect semicircles perpendicular to the real axis?

**Theorem 55.6.** *The geodesics in the hyperbolic plane are precisely the vertical half lines starting at the horizontal axis and the upper halves of circles centered on the real axis. All of them are minimal. There is a unique one through any two points.*

*Proof.* Take two points  $z, w$  in the hyperbolic plane. If  $z$  and  $w$  lie on the same vertical line, the only geodesic between them must be that line, because it is the only curve invariant under the reflection in that vertical line, and the reflection preserves the metric. If  $z$  and  $w$  do not lie on the same vertical line,



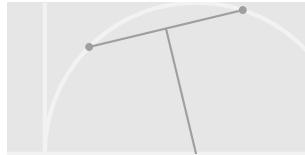
then the Euclidean line segment from  $z$  to  $w$  is not vertical,



so its perpendicular bisector is not horizontal, and hits the real axis at some point  $c_0$ .



The points  $z, w$  are the same distance from any point on their perpendicular bisector. So the circle centered at  $c_0$  with suitable radius strikes  $z$  and  $w$ .



Translate to get the leftmost point of the circle to lie at the origin. Invert to get a vertical line, reducing to the previous case.  $\square$

**55.6** Prove that every geodesic extends to both directions to have infinite length. In particular, the real axis is infinitely far away.

**55.7** Prove that the expression

$$\frac{|z - w|}{|z - \bar{w}|}$$

is invariant under simultaneous translation of  $z, w$  by any real constant, under simultaneous dilation of  $z, w$  by any nonzero real constant, and also under simultaneous inversion of  $z, w$ . Prove that the distance from  $z$  to  $w$  in the hyperbolic plane is

$$d(z, w) = 2 \tanh^{-1} \left( \frac{|z - w|}{|z - \bar{w}|} \right).$$

**55.8** Suppose that  $C$  is a geodesic of the hyperbolic plane and that  $z$  is a point of the hyperbolic plane not lying on  $C$ . Prove that there are infinitely many geodesics of the hyperbolic plane passing through  $z$  and not intersecting  $C$ . Prove that the hyperbolic plane is not isometric to the Euclidean plane, as a metric space or as a Riemannian manifold.

### Geodesics are locally minimal

Take a Riemannian manifold  $M$ , and a point  $m_0 \in M$  and a tangent vector  $v_0 \in T_{m_0}M$ . The *exponential map* is the map associating to any  $(x_0, v_0) \in TM$  the point  $\exp_{x_0} v_0$  defined to be  $x(1)$  if  $x(t)$  is the geodesic with  $x(0) = x_0$  and  $\dot{x}(0) = v_0$ . Such a value  $x(1)$  might not be defined. This is certainly defined if  $v_0 = 0$  to be  $\exp_{x_0} 0 = x_0$ . But for some values of  $v_0$ , the geodesic  $x(t)$  might run off the manifold before  $t$  gets to  $t = 1$ . Therefore the exponential map is only defined in some neighborhood of the origin in each tangent space.

**Proposition 55.7.** *Suppose that  $x_0 \in M$  is a point in a Riemannian manifold. The exponential map is a local diffeomorphism from an open set*

$$\exp_{x_0} : \text{open} \subset T_{x_0}M \rightarrow M.$$

*Proof.* Take a vector  $v_0 \in T_{x_0}M$ . Let  $x(t)$  be the geodesic with  $x(0) = x_0$  and  $\dot{x}(0) = v_0$ . The path  $\exp_{x_0} tv_0$  is precisely  $x(t)$ , by reparameterizing the geodesic. Therefore its velocity at  $t = 0$  is  $v_0$ . Hence  $\exp'_{x_0}(0)v_0 = v_0$ : the linear map

$$\exp'_{x_0} : T_0T_{x_0}M = T_{x_0}M \rightarrow T_{x_0}M$$

is the identity map. □

**55.9** Suppose that  $\Gamma$  is a finite group acting by diffeomorphisms on a manifold  $M$  and fixing a point  $m_0$ . Prove that there are coordinates on  $M$  near  $m_0$  in which  $\Gamma$  acts by orthogonal linear transformations.

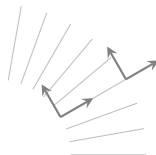
**Lemma 55.8.** *Take a Riemannian manifold  $M$ . Take two perpendicular vectors  $v_0, w_0 \in T_{x_0}M$ . Map a square to a surface in  $M$  by*

$$(s, t) \in [0, s_0] \times [0, t_0] \mapsto \exp_{x_0}(sv_0 + tw_0) \in M,$$

*called a rectangle. The rectangle has right angled corners. In particular, the vectors*

$$\exp'_{x_0}(v_0)v_0, \exp'_{x_0}(v_0)w_0$$

*are perpendicular.*



*Proof.* For the corner at  $x_0$ , the result is clear: the corner has tangent vectors  $v_0, w_0$ . It is enough to prove the result for  $v_0, w_0$  both nonzero. We can break up our rectangle into smaller rectangles, and patch together, so it is enough to prove the result for very small rectangles. We can therefore assume that our rectangle is embedded into  $M$  as a surface with corners. Take local coordinates in  $M$  in which  $s, t$  are among the coordinates. So there are commuting vector fields  $S, T$  so that the flow of  $S$  moves through the parameter  $s$ , and the flow of  $T$  moves through the parameter  $t$ , and  $S, T$  are tangent to the rectangle. Since each geodesic has constant speed, say unit speed,  $1 = \langle S, S \rangle$  all along the rectangle. Hence  $0 = T\langle S, S \rangle$  all along the rectangle. Thus at every point of the rectangle,

$$0 = \langle \nabla_T S, S \rangle + \langle S, \nabla_T S \rangle = 2\langle \nabla_T S, S \rangle$$

and

$$\begin{aligned} S\langle T, S \rangle &= \langle \nabla_S T, S \rangle + \langle T, \nabla_S S \rangle, \\ &= \langle \nabla_S T, S \rangle, \\ &= -\langle \nabla_T S, S \rangle, \\ &= 0. \end{aligned}$$

But  $\langle S, T \rangle = 0$  at the origin  $(s, t) = (0, 0)$  of the rectangle, so remains zero along the flow of  $T$ . By the same argument, with  $S, T$  reversed,  $\langle S, T \rangle$  is constant every on the rectangle.  $\square$

**Corollary 55.9.** *Take a hypersurface in a Riemannian manifold. The geodesics normal to the hypersurface form a geodesic foliation of some open set containing the hypersurface. Any such open set is foliated by hypersurfaces perpendicular to the geodesics. There is a calibrating 1-form on that open set, calibrating those geodesics. Locally the hypersurfaces are the level sets of a solution  $s$  of the eikonal equation  $|ds| = 1$ .*

*Proof.* Draw a curve in the hypersurface, and then flow that curve along the exponential map to some distance  $s$  along the normal geodesics. The resulting curve is perpendicular to the normal geodesics by lemma 55.8 on the facing page. The same trick works for not just curves, but also flows of vector fields along the hypersurface. The perpendicular directions to the geodesics form a foliation by the Frobenius theorem (theorem 13.3 on page 123). Apply theorem 55.3 on page 457.  $\square$

**Theorem 55.10.** *All geodesics are locally minimal. Every piecewise continuously differentiable path which is locally of shortest length between its points is a geodesic.*

*Proof.* Pick a point  $x_0$  of a Riemannian manifold  $M$ . Define a vector field  $S(v) = v/|v|$  on  $T_{x_0}M - \{0\}$ . Pick any vector field  $T$  on  $T_{x_0}M$  whose flow is a rotation in  $T_{x_0}M$ . Note that  $T(0) = 0$ , because the flow of  $T$  rotates around the origin. Define a function  $r: T_{x_0}M \rightarrow \mathbb{R}$  by  $r(v_0) = |v_0|$ . This function is smooth except where  $v_0 = 0$ . Note that  $S$  is the gradient of  $r(v_0) = |v_0|$ . It is invariant under rotations, so  $T$  and  $S$  commute.

Define vector fields  $S, T$  and a function  $r$  on  $M$  near  $x_0$ , by using the exponential map at  $x_0$  to identify with these  $S, T, r$ . As we flow along a geodesic  $x(t) := \exp_{x_0} tv_0$ , our speed is constant, always equal to  $|v_0|$ . The quantity  $r$  is increasing at constant rate, i.e. at rate  $|v_0|$ . In particular, taking  $v_0$  to be a unit length vector, we see that

the function  $r$  increases at unit rate along unit speed geodesics from  $x_0$ . The vector field  $S$  is the velocity field of the unit speed geodesics emerging from  $x_0$ .

Note that

$$\begin{aligned}
 S\langle S, T \rangle &= \langle \nabla_S S, T \rangle + \langle S, \nabla_S T \rangle, \\
 &= \langle 0, T \rangle + \langle S, \nabla_S T \rangle, \\
 &= \langle S, \nabla_S T \rangle, \\
 &= \langle S, \nabla_T S + [S, T] \rangle, \\
 &= \frac{1}{2} T \langle S, S \rangle, \\
 &= \frac{1}{2} T 1, \\
 &= 0.
 \end{aligned}$$

Hence  $\langle S, T \rangle$  remains constant along the flow lines of  $S$ , the geodesics, and goes to zero as we approach  $x_0$ , so  $S$  and  $T$  are perpendicular. We can pick any vector field  $T$  which rotates  $T_{m_0} M$  around the origin. In particular, such vector fields  $T$  span the tangent spaces of the level sets of  $r$ . Hence  $S$  points perpendicularly to the level sets of  $r$ , and  $r$  increases at unit rate along the flow of  $S$ , so  $dr$  calibrates the flow lines of  $S$ , the geodesics through  $x_0$ . Therefore the shortest path from  $x_0$  to any nearby point is the geodesic.  $\square$

### Geodesic convexity

A subset  $S \subseteq M$  of a Riemannian manifold  $M$  is *convex* if any two points of  $S$  are connected by a unique geodesic that lies inside  $S$ , and this geodesic is minimal in  $M$ . The *convexity radius* of a point  $m_0 \in M$  is the radius of the largest convex ball about  $m_0$ . The *convexity radius* of a subset of  $M$  is the infimum of the convexity radii of the points of that subset. A convex subset  $S \subseteq M$  of a Riemannian manifold  $M$  is *strongly convex* if there is a smooth map  $v: S \times S \rightarrow TM$  so that, for any two points  $p, q \in S$ , the unique geodesic from  $p$  to  $q$  inside  $S$  is  $t \mapsto \exp_p tv(p, q)$ . The *strong convexity radius* of a point  $m_0 \in M$  is the radius of the largest strongly convex ball about  $m_0$ . The *strong convexity radius* of a subset of  $M$  is the infimum of the strong convexity radii of the points of that subset.

**55.10** Prove that every smooth map to any strongly convex set is smoothly null homotopic.

A function  $f$  on a Riemannian manifold  $M$  is *convex* if  $f(x(t))$  is a convex function, for all geodesics  $x(t)$ , i.e. the graph of  $f(x(t))$ , on any interval  $t_0 \leq t \leq t_1$ , lies on or under the straight line between its end points. Note that a convex function achieves its minimum in the interior of the interval, and its maximum at one or both of the end points. It is *strictly convex* if smooth with second derivative bounded below by a positive constant.

**Lemma 55.11.** *On any Riemannian manifold  $M$ , each point lies in an open set  $U$  for which there is a positive number  $r > 0$ , so that on the ball of radius  $r$  around any point  $m_0 \in U$ , the function measuring squared distance to  $m_0$  in  $M$  is strictly convex.*



*Proof.* Take  $r$  smaller than the injectivity radius near  $m_0$ . The exponential map is a diffeomorphism of the Euclidean  $\bar{B}_r \subset T_{m_0}M \cong \mathbb{R}^n$  to a compact closed ball; identify  $\bar{B}_r$  with  $M$ , and so put the metric of  $M$  on that  $\bar{B}_r$ , with  $m_0$  at the origin. Let  $b_{ij}(x) := \delta_{ij} - x_\ell \Gamma_{ij}^\ell$ . Note that at  $x = 0$ ,  $b_{ij}(0) = \delta_{ij}$  is positive definite. So if we shrink  $r$ , we can ensure that  $b_{ij}(x)$  is positive definite throughout  $\bar{B}_r$ . On any geodesic  $x(t)$  in  $\bar{B}_r$ ,

$$\begin{aligned} \frac{d^2}{dt^2} \frac{1}{2} |x|^2 &= \frac{1}{2} \frac{d^2}{dt^2} x_i x_i, \\ &= \frac{d}{dt} x_i \dot{x}_i, \\ &= \dot{x}_i \dot{x}_i + x_i \ddot{x}_i, \\ &= |\dot{x}|^2 - x_i \Gamma_{jk}^i \dot{x}_j \dot{x}_k, \\ &= b_{ij}(x) \dot{x}_i \dot{x}_j, \end{aligned}$$

a positive definite quadratic form in  $\dot{x}$  for all  $x_0$ .  $\square$

**Theorem 55.12.** *The strong convexity radius of any compact subset of any Riemannian manifold is positive.*

*Proof.* On any ball of radius less than half the injectivity radius, there is a unique minimal geodesic between any two of its points, and this geodesic depends smoothly on the choice of the points. But that geodesic might extend outside that ball, so we don't see whether the ball is convex.

Take a point  $m_0$ , and a closed ball about  $m_0$  of radius less than the injectivity radius of  $m_0$ , hence compact, and then take the infimum over all  $m$  in that ball of the injectivity radius at  $m$ , and let  $r$  be less than a third of that infimum, and consider the closed ball  $\bar{B}_r$  of radius  $r$  around  $m_0$ . Any two points  $p, q$  in that ball are within one another's injectivity radii. So there is a unique minimal geodesic between  $p$  and  $q$ . Going from  $p$  to  $q$  via 0 has length at most  $2r$ , so the minimal geodesic has length at most  $2r$ , so it never leaves the ball of radius  $3r$  around  $m_0$ . Moreover that geodesic has velocity at  $p$ , and at  $q$ , depending smoothly on  $p$  and  $q$ . But that geodesic might extend outside  $\bar{B}_r$ , so we don't see whether  $\bar{B}_r$  is convex.

By lemma 55.11 on the preceding page, if we set  $r$  small enough, then on any geodesic in  $M$ , starting and ending at two distinct points of  $\bar{B}_{3r}$ , the Euclidean distance to the origin is a convex function, i.e. has no interior maxima. So the geodesic does not leave  $\bar{B}_r$ .  $\square$

**55.11** What is the strong convexity radius of the unit circle? What are the convex sets? How can two convex sets intersect?

**55.12** What is the strong convexity radius of the torus  $\mathbb{R}^2/\mathbb{Z}^2$ ? What are the convex sets? How can two convex sets intersect?

## Recalling metric spaces and their maps

This section requires familiarity with the Hopf–Rinow theorem from metric spaces [11]. The *injectivity radius* of a point  $m_0$  of a Riemannian manifold  $M$  is the largest number  $r > 0$  so that the exponential map takes a ball of radius  $r$  about the origin in  $T_{m_0}M$  diffeomorphically to a ball of radius  $r$  about  $m_0$  in  $M$ . The injectivity radius

is positive, from our proof of theorem 55.10. The injectivity radius can't jump to become suddenly smaller when we take a limit of points, as the geodesic equation varies smoothly. The *injectivity radius* of a subset of  $M$  is the infimum of the injectivity radii of the points of that subset; the injectivity radius of any compact set is thus positive. In the proof of theorem 55.10, we see that any two points whose distance  $r$  is less than the larger of their injectivity radii are connected by a unique minimal geodesic, and this geodesic stays inside the ball of radius  $r$  around either of the points.

**55.13** Prove that on any compact Riemannian manifold  $M$ , the exponential map is defined throughout  $TM$ .

**Theorem 55.13.** *Any map of Riemannian manifolds is an isometry of metric spaces if and only if it is a diffeomorphism preserving the Riemannian metric.*

*Proof.* Take a point  $p_0 \in P$  in a Riemannian manifold and any orthonormal basis  $e_1, e_2, \dots, e_n$  of  $T_{p_0}P$ . Pick a positive number  $t$  less than the convexity radius of  $p_0$ . Let  $p_i := \exp_{p_0} te_i$ . Let  $f_i: P \rightarrow \mathbb{R}$  be the function  $f_i(p) = d(p, p_i)$ . Since  $p_0$  lies in the injectivity radius ball of all of  $p_1, p_2, \dots, p_n$ , and is not equal to any of them, the function  $f_i$  is smooth near  $p_0$  with differential  $df_i(p_0)$  dual to  $e_i$ . Therefore the functions  $f_i$  have linearly independent differentials near  $p_0$ , so the continuous map  $f = (f_1, f_2, \dots, f_n): P \rightarrow \mathbb{R}^n$  is a diffeomorphism near  $p_0$ . An isometry of metric spaces matches up shortest paths, i.e. minimal geodesics, and injectivity and convexity radii, and therefore matches up such functions.  $\square$

**Theorem 55.14.** *Every connected Riemannian manifold is a length space, i.e. for any two points  $m_0, m_1 \in M$ , the infimum of lengths of rectifiable paths from  $m_0$  to  $m_1$  is the infimum length of piecewise continuously differentiable paths.*

*Proof.* Start working in a geodesically convex compact set. The distance between any two points is exactly the length of the unique minimal geodesic in the set which connects them. By the triangle inequality, rectifiable paths (assume arc length parameterised) can only be longer, and approach the geodesic uniformly as they get closer to the infimum length. Globally, for any two points, we can take any rectifiable path (assume arc length parameterised) and use the triangle inequality to make a path which is no longer, and piecewise geodesic.  $\square$

**Theorem 55.15** (Hopf–Rinow II). *On any connected Riemannian manifold  $M$  the following are equivalent:*

- The Riemannian manifold  $M$  is complete in its metric, i.e. Cauchy sequences converge.*
- The Riemannian manifold  $M$  is proper in its metric, i.e. all balls are compact.*
- The exponential map on  $M$  has flow defined for all time through all points.*
- Every geodesic on  $M$  defined on an open interval extends uniquely to a geodesic defined on a closed interval.*
- There is a point  $m_0 \in M$  so that every minimal geodesic  $x: [a, b) \rightarrow M$  with  $x(a) = m_0$  extends uniquely to a continuous path  $x: [a, b] \rightarrow M$ .*

*In particular, if  $M$  is a compact Riemannian manifold, then all of these hold.*

*Proof.* Every geodesic on  $M$ , parameterised by arc length, is defined on a maximal open interval, since the geodesics are the solutions of an ordinary differential equation. By the Hopf–Rinow theorem for metric spaces,  $M$  is complete just when all geodesics extend to be defined on closed intervals. They then extend further to be defined on larger closed intervals, since they continue to solve their differential equation. So if  $M$  is complete, geodesics are defined for all time. But if geodesics are defined for all time, they extend, so again by the Hopf–Rinow theorem,  $M$  is complete.  $\square$

Consider the metric

$$g = f^2(dx_1^2 + dx_2^2 + \cdots + dx_n^2),$$

on the Euclidean ball of radius  $R$  in  $\mathbb{R}^n$ , where  $f = f(r)$  is a smooth positive function depending only on  $r := |x|$ . Clearly the radial lines outward from the origin are geodesics, by reflection symmetry in any hyperplane through the origin. The distance along these geodesics from the origin is the time taken to travel along them at unit rate:

$$t = \int f(r) dr,$$

so the total length of these geodesics is infinite:

$$\int_0^R f(r) dr = \infty$$

if and only if this metric is complete, since the minimal geodesics from the origin extend to be defined on for all time  $t$ .

Suppose that  $M = \mathbb{R}^2$  is the plane with a Riemannian metric, say

$$g_M = e(x, y)dx^2 + 2f(x, y)dx dy + g(x, y) dy^2,$$

for some functions  $e(x, y), f(x, y), g(x, y)$  periodic of period 1 in both  $x$  and  $y$ . Let  $N = \mathbb{R}^2/\mathbb{Z}^2$  be the torus, with the metric corresponding metric  $g_N$ . Then  $M \rightarrow N$  is a Riemannian covering map, so  $M$  is a complete Riemannian manifold.

**55.14** Suppose that  $g$  and  $h$  are Riemannian metrics on the same manifold  $M$ , and equal outside of some compact set. Prove that  $g$  is complete if and only if  $h$  is complete.

**55.15** Suppose that  $M$  is the upper half plane with a Riemannian metric, say

$$g_M = e(x, y)dx^2 + 2f(x, y)dx dy + g(x, y) dy^2,$$

for some functions  $e(x, y), f(x, y), g(x, y)$ . Suppose that the matrix

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

has eigenvalues  $\lambda_1, \lambda_2 \geq 1/y^2$ . Prove that  $M$  is a complete Riemannian manifold.

**55.16** Suppose that  $M$  is the upper half plane with a Riemannian metric, say

$$g_M = e(x, y)dx^2 + g(y)dy^2,$$

for some functions  $e(x, y), g(y)$ . Suppose that

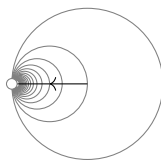
$$\int_0^1 \sqrt{g(y)} dy = \int_1^\infty \sqrt{g(y)} dy = \infty.$$

Prove that  $M$  is a complete Riemannian manifold.

Two Riemannian metrics  $g, g'$  are *conformal* if  $g' = \lambda g$  for some positive function  $\lambda$ . Geometrically, this is precisely the condition that  $g$  and  $g'$  give the same measurements of angles between any two curves.

**Proposition 55.16.** *Every manifold admits a complete Riemannian metric. Indeed, given any Riemannian metric, there is a complete one conformal to it.*

*Proof.* Take a Riemannian metric  $g$  on a manifold  $M$ . The *compactness radius*  $r_x$  of a point  $x \in M$  is the supremum distance  $r > 0$  (possibly  $+\infty$ ) so that the ball of radius  $r$  in  $M$  is precompact, i.e. has compact closure. The compactness radius  $r_x$ , thought of as a function of  $x$ , may fail to be smooth or even to be continuous, so we approach it carefully. By compactness of the sphere, at each point  $x_0$ , there is a direction in which the unit speed geodesic from  $x_0$  survives precisely for those times  $t$  with  $0 \leq t < r_{x_0}$  and not for time  $t = r_{x_0}$ . Follow that geodesic, say  $s \mapsto x(s)$ , at unit rate:  $r_{x(s)} = \ell - s$  where  $\ell$  is the length of the geodesic.



By the triangle inequality, if we head in any other direction, then  $r_{x(s)}$  goes up with at most unit speed. So along any curve  $x(s)$  parameterised at unit speed,

$$r_{x(0)} - s \leq r_{x(s)} \leq r_{x(0)} + s.$$

If a geodesic from  $x$  survives for some time  $r > 0$  and is still minimal, then we can look at a tube in  $M$  close enough to it, and see that all nearby geodesics, in nearly the same direction, survive for almost as long. So for  $x$  near  $x_0$ ,  $r_x$  is not much smaller than  $r_{x_0}$ , but might be much bigger, say  $r_x \geq r_{x_0}/2$  for all  $x$  near  $x_0$ . Take any bump function  $f \geq 0$  so that, near  $x_0$ ,  $f(x) \geq 2/r_{x_0}$  and so that  $f = 0$  outside some small neighborhood of  $x_0$  in which  $r_x \geq r_{x_0}/2$ . Such neighborhoods, each equipped with such a bump function, form an open cover. Take a locally finite refinement, with associated bump functions  $f_i$ . Let  $f = \sum f_i$ . Then  $g' = f^2 g$  is a metric conformal to  $g$ .

A curve parameterised as  $s \mapsto x(s)$  at unit  $g$ -speed with  $g$ -length  $\ell$  has  $g'$ -length

$$\ell' = \int f(x(s)) ds \geq \int \frac{ds}{r_{x(s)}} \geq \int \frac{ds}{s + r_{x_0}} = \log(1 + \ell/r_{x_0}).$$

If we fix a constant  $\varepsilon$  with  $0 < \varepsilon < 1$ , then the compact  $g$ -ball of radius  $(1 - \varepsilon)r_{x_0}$  contains the  $g'$ -ball of radius  $\log(2 - \varepsilon)$ , hence both are compact. If the compactness radius  $r'_{x_0}$  of  $g'$  is finite at a point  $x_0$ , it decays as

$$r'_{x(s)} = r'_{x(0)} - s$$

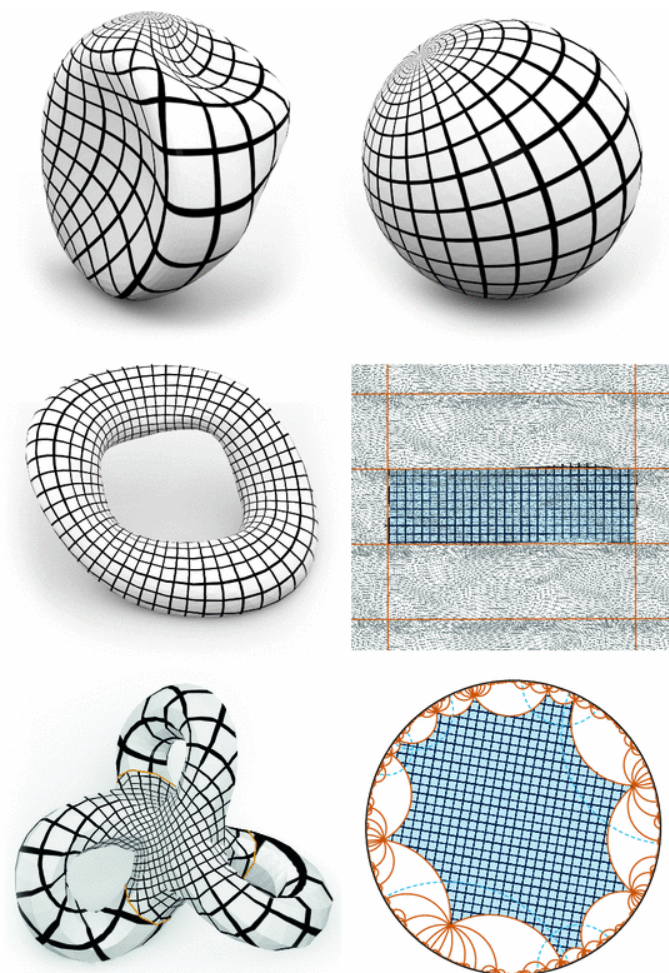
in some direction, so can't be always at least  $\log(2 - \varepsilon)$ . Hence  $g'$  has infinite compactness radius. By the Hopf–Rinow theorem,  $g'$  is complete.  $\square$

**Corollary 55.17.** *Every manifold admits a metric which induces its topology, i.e. is a metrizable topological space.*



## Chapter 56

### Riemann surfaces



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Discrete Conformal Maps: Boundary Value Problems,

Circle Domains, Fuchsian and Schottky Uniformization

Alexander I. Bobenko, Editor

**Advances in Discrete Differential Geometry**

## Conformality

A *conformal map* of Riemannian manifolds is a diffeomorphism which preserves angles between curves. Clearly the composition of conformal maps is conformal, and the inverse of any conformal map is conformal. By definition, the angle between two curves is defined to be the angle between their tangent lines.

**56.1** Prove that an orientation preserving linear map is conformal just when it is a composition of a rotation and a rescaling.

In  $\mathbb{R}^2$ , with its Euclidean geometry, such a linear map is:

$$r_0 \begin{pmatrix} \cos \theta_0 & -\sin \theta_0 \\ \sin \theta_0 & \cos \theta_0 \end{pmatrix}.$$

So in any complex vector space of complex dimension 1, a linear map is conformal just when it is

$$z \mapsto r_0 e^{i\theta_0} z,$$

i.e. multiplication by a nonzero complex number.

**56.2** Prove that orientation preserving conformal maps  $f$  between open subsets of the plane  $\mathbb{R}^2 = \mathbb{C}$  are precisely biholomorphisms.

The sphere, with its usual metric, is locally conformal to the plane: puncture at a point, and then apply the Ptolemaic projection map from the puncture to the plane tangent to the sphere opposite the puncture. Elementary Euclidean geometry allows us to check that this map is conformal. This observation makes the sphere into a Riemann surface, because these conformal maps agree up to local biholomorphism. Moreover, using only north and south pole, we check that these conformal maps are precisely the maps  $w = 1/z$  and  $z = 1/w$  used to define the Riemann sphere.

A *conformal structure* is a Riemannian metric defined only up to conformality.

An *isothermic chart* on an oriented surface with Riemannian metric is an orientation preserving conformal map to the complex plane. A maximal atlas of isothermic charts is a complex structure, giving the structure of a Riemann surface.

The *hyperbolic plane* is the upper half plane

$$\mathbb{CH}^1 := \{ z = x + iy \in \mathbb{C} \mid y > 0 \},$$

with metric

$$g' := \frac{dx^2 + dy^2}{y^2},$$

and the usual orientation. Giving  $\mathbb{C}$  the Euclidean metric  $g := dx^2 + dy^2$ , the inclusion map  $z \in \mathbb{CH}^1 \mapsto z \in \mathbb{C}$  is conformal, so the hyperbolic plane is a Riemann surface.



All surfaces of constant curvature are locally isometric to the plane, the sphere or the hyperbolic plane, up to constant scaling. Therefore all constant curvature surfaces have a canonical Riemann surface structure, given by those local isometries.

## Conformal change of metric

We can picture a Riemannian metric  $g = g_{ij}dx^i dx^j$  on a surface as a family of ellipses: at each point  $x$  we draw the ellipse of vectors  $v$  for which  $g(v, v) = 1$ . In this picture, two metrics are *conformal*, i.e. have the same angle measurements, just when the ellipses have the same shape, but are rescaled. In other words, a metric  $g'$  is conformal to a metric  $g$  just when  $g' = hg$  for some positive function  $h$ . It is convenient to write the conformal factor  $h$  as  $e^{-2u}$ .

For example, the hyperbolic plane has metric

$$g' := \frac{dx^2 + dy^2}{y^2},$$

conformal to the Euclidean metric  $g = dx^2 + dy^2$  by the conformal factor

$$e^{-2u} = \frac{1}{y^2},$$

i.e.  $u = \log y$ .

**Corollary 56.1.** *If a Riemannian metric  $g$  can become real analytic in some coordinates, then it must become real analytic in any isothermic chart.*

## Almost complex manifolds

An *almost complex structure* on a manifold  $M$  is a smooth map  $J$  of the tangent vectors of  $M$  which is a linear isomorphism of each tangent space and satisfies  $J^2 = -I$  on each tangent space. Given an almost complex structure on a surface  $M$ , a complex-valued 1-form  $\omega$  on some open subset of  $M$  is *complex linear* if  $\omega$  is complex linear on each tangent space, i.e.  $\omega(Jv) = \sqrt{-1}\omega(v)$  for any tangent vector  $v$ . Similarly  $\omega$  is *conjugate linear* if  $\omega(Jv) = -\sqrt{-1}\omega(v)$ . Taking any complex-valued 1-form  $\omega$ , we can split it into complex linear and conjugate linear parts as

$$\begin{aligned}\omega'(v) &:= \frac{1}{2}\omega(v) - \frac{\sqrt{-1}}{2}\omega(Jv), \\ \omega''(v) &:= \frac{1}{2}\omega(v) + \frac{\sqrt{-1}}{2}\omega(Jv),\end{aligned}$$

so that  $\omega = \omega' + \omega''$ . A *coframing* for an almost complex structure on a surface is a nowhere zero complex linear 1-form. In particular, if we pick  $\omega$  to be a complex linear isomorphism at some point, then  $\omega'$  is a coframing nearby. Hence local coframings exist. Conversely, an almost complex structure  $J$  arises uniquely from any complex 1-form  $\omega$  which has linearly independent real and imaginary parts: the almost complex structure for which  $\omega$  is a complex linear isomorphism on each tangent space, and then  $\omega$  is a coframing for  $J$ . So we can locally specify an almost complex structure by a coframing, and any two coframings  $\omega_0, \omega_1$  agree up to complex linear multiple:  $\omega_1 = f\omega_0$  for some complex valued smooth function  $f \neq 0$ .

## Oriented conformal structures as almost complex structures

Take an oriented surface  $M$  with Riemannian metric  $g$ . Applying Gram–Schmidt to any local choice of linearly independent 1-forms, we obtain an orthonormal basis of 1-forms  $\omega_1, \omega_2$ , orthonormal in the sense that

$$g = \omega_1^2 + \omega_2^2.$$

We can swap the order of the two of them if needed, to get a positively oriented basis. Let  $\omega := \omega_1 + i\omega_2$ . If we change the choice of orthonormal oriented basis, we do so by rotation, say by an angle  $\theta$ , a smoothly varying function, and so change  $\omega$  to  $e^{i\theta}\omega$ . If we rescale the metric to a conformal metric  $g' = e^{-2u}g$ , we rescale  $\omega$  to  $\omega' = e^{-u}\omega$ .

**Theorem 56.2.** *An orientation and conformal structure on a surface determine an almost complex structure. Every almost complex structure determines an orientation and a conformal structure.*

## Complex manifolds

A function  $f: U \rightarrow \mathbb{C}$  on an open set  $U \subseteq \mathbb{C}^n$  is *holomorphic* if, near each point of  $U$ , it is expressed as the sum of a complex coefficient power series expansion converging in some open set. A map  $\varphi: U \rightarrow W$  of open sets  $U \subseteq \mathbb{C}^p$ ,  $W \subseteq \mathbb{C}^q$  is *holomorphic* if its constituent functions  $\varphi = (\varphi^1, \dots, \varphi^q)$  are. The terms *holomorphic* and *complex analytic* are synonyms. A *complex manifold*  $M$  of complex dimension  $n$  is a real manifold of real dimension  $2n$  equipped with an atlas of charts valued in complex Euclidean space (its *holomorphic charts*), maximal subject to having holomorphic transition maps. A continuous map  $\varphi: P \rightarrow Q$  between complex manifolds is *holomorphic* when its expression in those charts is holomorphic. A *biholomorphism* is a holomorphic map with a holomorphic inverse map. A *Riemann surface* is a connected complex manifold of complex dimension 1, so of real dimension 2: the charts of a Riemann surface are valued in  $\mathbb{C}$ . A *complex curve* is a compact Riemann surface.

**Theorem 56.3.** *Every Riemann surface bears a canonical almost complex structure, and a canonical orientation and conformal structure: the one for which the holomorphic charts are local isomorphisms of almost complex structure, i.e. are conformal and orientation preserving.*

## Derivatives in charts

Every real analytic function on  $\mathbb{C} = \mathbb{R}^2$  has a unique convergent Taylor series

$$f(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}\frac{x^2}{2} + a_{11}xy + a_{02}\frac{y^2}{2} + \dots$$

Write

$$\begin{aligned} z &= x + iy, \\ \bar{z} &= x - iy, \end{aligned}$$

or equivalently,

$$\begin{aligned}x &= \frac{1}{2}(z + \bar{z}), \\y &= \frac{i}{2}(\bar{z} - z).\end{aligned}$$

Plug in to the series expansion: every *real analytic* complex-valued function  $f(x, y)$  has a unique convergent Taylor series expansion

$$f(x, y) = b_{00} + b_{10}z + b_{01}\bar{z} + b_{20}\frac{z^2}{2} + b_{11}z\bar{z} + b_{02}\frac{\bar{z}^2}{2} + \dots$$

The function is holomorphic just when there are no  $\bar{z}$  terms.

Recall that  $\frac{\partial x}{\partial x} = 1$  and  $\frac{\partial y}{\partial x} = 0$ . It is convenient notation to introduce formal symbols  $\frac{\partial}{\partial z}$  and  $\frac{\partial}{\partial \bar{z}}$  so that  $\frac{\partial z}{\partial z} = 1$  and  $\frac{\partial \bar{z}}{\partial z} = 0$ , as if  $z$  and  $\bar{z}$  are independent variables. (A physicist once apologized to me for doing this.) More generally, they must differentiate formal Taylor series in  $z, \bar{z}$  as if  $z$  and  $\bar{z}$  were independent variables. Check that the only possible operators that can solve such equations are

$$\begin{aligned}\frac{\partial}{\partial z} &:= \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \\ \frac{\partial}{\partial \bar{z}} &:= \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).\end{aligned}$$

## The Beltrami equation

A map from an open subset of  $M$  to an open subset of the complex plane is conformal just when it pulls back  $dx, dy$  to rotation and a dilation applied to  $\omega_1, \omega_2$ . In other words, just when  $dx + i dy$  is identified with  $\omega$  up to a complex scalar.

Take any chart (maybe not conformal) defined near some point of  $M$ , taking that point to the origin. In that chart, say with coordinates  $z = x + iy$ ,  $\omega = a dz + b d\bar{z}$ , say. Make a linear change of variables to arrange that the coordinates  $z = x + iy$  are conformal at the origin. So  $a = 1$  and  $b = 0$  at the origin. So  $a$  is near 1 and  $b$  near 0 when we stay near the origin. Let  $\mu = b/a$  and write  $\omega = a dz + b d\bar{z} = a(dz + \mu d\bar{z})$ . Writing  $u_z$  to mean

$$u_z := \frac{\partial u}{\partial z},$$

any complex-valued function  $u$  has differential

$$\begin{aligned}du &= u_z dz + u_{\bar{z}} d\bar{z}, \\ &= u_z(dz + \mu d\bar{z}) + (u_{\bar{z}} - \mu u_z) d\bar{z}.\end{aligned}$$

So  $u$  is locally conformal just when  $du$  is complex linear and not zero, just when

$$u_{\bar{z}} = \mu u_z,$$

the *Beltrami equation*.

**56.3** Prove that the rank of this equation, as a pair of real linear equations in first derivatives, drops exactly when  $|\mu| = 1$ .

A *Beltrami equation* is an equation

$$u_{\bar{z}} = \mu u_z,$$

in the complex plane, written in the usual Euclidean derivatives, for some smooth complex-valued function  $\mu$  with  $|\mu| < 1$ . Conversely, given any Beltrami equation in the complex plane, let

$$\omega := dz + \mu d\bar{z}.$$

Then  $g = |\omega|^2$  is a Riemannian metric, with this Beltrami equation as its equation of conformal maps, and  $\omega$  is a coframing of the associated almost complex structure.

**Theorem 56.4.** *Take an oriented surface  $M$  with a Riemannian metric  $g$ . Suppose that  $u$  is a conformal map from an open subset of  $M$  to an open subset of the plane. Then  $u$  satisfies the associated Beltrami equation. Conversely, if  $u$  is a smooth complex valued solution of the associated Beltrami equation and  $du \neq 0$  at some point  $m_0 \in M$ , then  $u$  is a conformal map of an open subset of  $M$  near  $m_0$ .*

Our aim is to prove:

**Theorem 56.5** (The local uniformisation theorem (Korn–Lichtenstein)). *Every Beltrami equation has a local smooth solution  $u$ , near any point  $m_0$ , so that  $du \neq 0$  at  $m_0$ .*

This proof is due to [5].

**Corollary 56.6.** *Every Riemannian metric on any surface is locally conformal to a flat metric, i.e. the metric admits isothermic coordinates.*

**Corollary 56.7.** *Every conformal structure on any surface is locally flat.*

**Corollary 56.8.** *Every almost complex structure on any surface arises from a complex structure, i.e. from a Riemann surface.*

## The Poincaré lemma

A *locally integrable* differential form is a form expressed in each local chart with  $L^1$  coefficients, changing from chart to chart by the usual transition map expression. Clearly we can integrate a locally integrable volume form over a domain with compact closure, using bump functions to reduce to coordinates, just as we did for continuous differential forms.

**Theorem 56.9** (The Cauchy integral theorem with remainder). *Suppose that  $f: \bar{U} \rightarrow \mathbb{C}$  is a  $C^1$  complex valued function on a bounded domain  $U$  with  $C^1$  boundary with corners. Then*

$$f(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{f(w) dw}{w - z} + \frac{1}{2\pi i} \int_U f_{\bar{w}}(w) \frac{dw \wedge d\bar{w}}{w - z}.$$

**56.4** Using Stokes theorem on the differential form

$$\omega = \frac{1}{2\pi i} \frac{f(w) dw}{w - z},$$

prove that  $\omega$  is locally integrable and hence prove the existence of the integrals and prove the theorem.

**Lemma 56.10** (Poincaré). *If  $f: \bar{U} \rightarrow \mathbb{C}$  is a  $C^k$  function,  $k \geq 1$ , on a bounded domain  $U$  with  $C^1$  boundary with corners, let*

$$u(z) := \frac{1}{2\pi i} \int_D f(w) \frac{dw \wedge d\bar{w}}{w - z}.$$

*Then  $u: D \rightarrow \mathbb{C}$  is a  $C^k$  function with  $\bar{\partial}u = f d\bar{z}$ .*

*Proof.* The proof is almost the same as the Cauchy integral theorem with remainder. Pick a disk  $D \subseteq U$  around a point  $z_0 \in U$ . Suppose first that  $f = 0$  throughout  $D$ . The integral is smooth in  $z$  by differentiating under the integral sign and holomorphic in  $z$ , so  $\bar{\partial}u = 0 = f d\bar{z}$  for  $z \in U$ .

Now suppose instead that  $f(z) = 0$  in  $\bar{U}$  outside of  $D$ . Extend  $f$  to be zero in the complex plane outside of  $D$ . The integral:

$$\begin{aligned} u(z) &= \frac{1}{2\pi i} \int_U f(w) \frac{dw \wedge d\bar{w}}{w - z}, \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} f(w) \frac{dw \wedge d\bar{w}}{w - z}, \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} f(z + w) \frac{dw \wedge d\bar{w}}{w}, \end{aligned}$$

writing  $w = z + re^{i\theta}$ ,

$$= -\frac{1}{\pi} \int_{\mathbb{C}} f(z + re^{i\theta}) e^{-i\theta} dr \wedge d\theta,$$

from which we see that the differential form is locally integrable, and the integral is  $C^k$  in  $z$ . Differentiate in  $\bar{z}$  and reverse the steps:

$$\begin{aligned} u_{\bar{z}} &= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial f}{\partial \bar{z}}(z + re^{i\theta}) e^{-i\theta} dr \wedge d\theta, \\ &= \frac{1}{2\pi i} \int_U f_{\bar{w}}(w) \frac{dw \wedge d\bar{w}}{w - z}, \end{aligned}$$

Since  $f$  vanishes on the boundary of  $D$ , we can apply the Cauchy integral theorem with remainder:

$$u_{\bar{z}} = f.$$

For a general  $C^k$  function  $f$ , take a bump function  $h$  equal to 1 near some point  $z_0$ , vanishing outside some disk  $D$ , and write  $f = (1 - h)f + hf$ , to split into a sum of two functions, each of which suits one of the hypotheses above.  $\square$

## $L^2$ methods

The *Fourier transform* of an  $L^1$  function  $f$  on the plane  $\mathbb{R}^2 = \mathbb{C}$  is

$$\hat{f}(\zeta) = \int_{\mathbb{R}^2} f(x + iy) e^{-2\pi i(x\xi + y\eta)}$$

for  $\zeta = \xi + i\eta \in \mathbb{C}$ . The Fourier transform is thereby defined on  $L^2 \cap L^1$ , and isometric in the  $L^2$  norm; see [7] p. 17 or [15] p. 16. Since  $L^1 \cap L^2 \subset L^2$  is dense,  $f \mapsto \hat{f}$  extends uniquely to an isometry of  $L^2$ .

We easily check that, in the sense of distributions,

$$\begin{aligned}\hat{f}_z &= i\pi\zeta\hat{f}, \\ \hat{f}_{\bar{z}} &= i\pi\bar{\zeta}\hat{f}.\end{aligned}$$

The *Beurling transform*  $L$  is the operator so that

$$\widehat{Lf} = (\bar{\zeta}/\zeta)\hat{f}.$$

Since  $\zeta/\bar{\zeta}$  has norm 1 as a complex number, it doesn't alter  $L^2$  norms, so  $L$  is unitary. Clearly

$$L\frac{\partial}{\partial z} = \frac{\partial}{\partial \bar{z}}.$$

**Theorem 56.11.** *Take a  $C^1$  bounded domain  $U \subset \mathbb{C}$  with corners. To every  $f \in L^2(U)$ , there is a unique  $u \in L^2(\mathbb{C})$  so that*

- a.  $u_{\bar{z}} = f$  and
- b. both  $u_z$  and  $u_{\bar{z}}$  are also in  $L^2(\mathbb{C})$ .

*Proof.* By the Poincaré lemma, we solve

$$u_{\bar{z}} = f$$

in the complex plane, for any smooth compactly supported function  $f$  by

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} f(w) \frac{dw \wedge d\bar{w}}{w - z},$$

an integral in the entire complex plane. Then  $u(z) = O(z^{-1})$  as  $z \rightarrow \infty$ . Also note that this is a convolution:

$$u(z) = \frac{1}{\pi z} * f(z).$$

The Hausdorff–Young inequality ([7] p. 10, [15] p. 178) says that if  $f$  is supported on some domain  $U \subset \mathbb{C}$  then

$$|u|_{L^2} \leq |f|_{L^2} \left| \frac{1}{\pi z} \right|_{L^1(U)}.$$

Check that, if  $\Delta_{r_0} \subseteq U \subseteq \Delta_{r_1}$  are disks around the origin contained in and containing  $U$ , then

$$2r_0 \leq \left| \frac{1}{\pi z} \right|_{L^1(U)} \leq 2r_1.$$

So once we fix a compact domain  $U$  for  $f$  to be supported on, the map  $f \mapsto u$  is bounded and so extends to a linear map

$$f \in L^2(U) \mapsto u \in L^2(\mathbb{C}).$$

Since  $f$  has compact support,  $u$  is holomorphic outside of  $U$ . Note that

$$u_z = Lu_{\bar{z}} = Lf \in L^2(\mathbb{C}).$$

□

Returning to solving a Beltrami equation: since this problem is local, we can assume that  $\mu$  is smooth with compact support and  $|\mu| < 1$  at every point of the complex plane. Moreover, we can assume that  $\mu = 0$  at any one chosen point.

**Theorem 56.12.** *Pick any smooth complex-valued function  $\mu: \mathbb{C} \rightarrow \mathbb{C}$  with compact support and with  $|\mu| < 1$ . There is a unique solution  $u = u_\mu$  to the Beltrami equation*

$$u_{\bar{z}} = \mu u_z,$$

*of the form  $u(z) = z + \varphi(z)$  where  $\varphi \in L^2$  has  $L^2$  distributional derivatives.*

*Proof.* Let  $c := \max |\mu|$ , so  $0 \leq |\mu| \leq c < 1$ . We expect that  $I - \mu L$  has an inverse on  $L^2$  given by geometric series

$$(I - \mu L)^{-1} = I + \mu L + (\mu L)^2 + \dots$$

Let's check that this is correct. On each function  $f$ , the norm of  $(\mu L)^k f$  is at most

$$\left| (\mu L)^k f \right|_{L^2} \leq c^k |f|_{L^2}.$$

Our geometric series converges since  $L^2$  is a Hilbert space, so complete. Plugging in the series to

$$(I - \mu L)(I + \mu L + (\mu L)^2 + \dots)$$

and expanding term by term, we find that it is indeed an inverse. Let  $f := (I - \mu L)^{-1} \mu$ , an  $L^2$  function on the plane. Then  $(I - \mu L)f = \mu$ , i.e.  $f = \mu L f + \mu$ . Since  $\mu$  has compact support, say in some domain  $U \subset \mathbb{C}$ , for any point outside of  $U$ , this equation gives  $f = 0$ . So  $f \in L^2(U)$ . Let  $\varphi$  be the  $L^2$  function with

$$\varphi_{\bar{z}} = f.$$

Then plug into  $f = \mu L f + \mu$  to see that

$$(z + \varphi)_{\bar{z}} = \varphi_{\bar{z}} = f = \mu L f + \mu = \mu \varphi_z + \mu = \mu(z + \varphi)_z.$$

□

## Smoothness

So far, we have a solution  $u(z) = z + \varphi(z)$  to the Beltrami equation, unique, with  $\varphi, \varphi_z, \varphi_{\bar{z}} \in L^2$ . We want to prove that  $u$  is smooth. Since our problem is local, we can multiply  $\mu$  by a bump function, equal to 1 near 0 and to 0 not far away. Since we have  $\mu(0) = 0$ , we can safely assume that  $\mu$  is uniformly as small as we like; we will suppose that  $|\mu| < 1/3$ .

### Distant peaks

Take a continuous function  $y = h(x) \geq 0$  defined for  $x \geq 0$ , with  $h(x) \rightarrow 0$  as  $x \rightarrow \infty$ , but not rapidly, i.e. more slowly than some rational function  $1/x^k$ . For each  $n$ , from among those  $x$  for which  $h(x) \geq n/x^k$ , pick a point  $x = x_n$  at which  $h(x)$  is largest: call  $x_n$  a *distant peak*. As we make  $n$  larger, in any compact interval of values of  $x$ , eventually we make  $n/x^k$  larger than  $h(x)$ , so we have to find a more distant “distant peak”. By definition,  $x_n^k h(x_n) \geq n \rightarrow \infty$ , so at distant peaks,  $h$  decays slowly, and, roughly speaking, nowhere more slowly.

**Lemma 56.13.** *Take a continuous function  $h(x) \geq 0$  defined for  $x \geq 0$ , with  $h(x) \rightarrow 0$  as  $x \rightarrow \infty$ , but not rapidly. Pick any function  $r: \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$  with  $r(x) = o(x)$  as  $x \rightarrow \infty$ . For some integer  $k > 0$ , there are distant peaks  $x_n \rightarrow \infty$  where*

$$1 = \lim_{n \rightarrow \infty} \sup_{x > x_n - r(x_n)} \frac{h(x)}{h(x_n)}.$$

*Proof.* If  $h(x) \geq n/x^k$ , then  $h(x) \leq h(x_n)$ , by definition of a distant peak. If  $h(x) < n/x^k$  then

$$h(x) < \frac{n}{x^k} = \frac{n}{x_n^k} \frac{x_n^k}{x^k} = h(x_n) \frac{x_n^k}{x^k}.$$

If

$$x \geq x_n - r(x_n),$$

then

$$\frac{x_n^k}{x^k} < \left( \frac{x_n}{x_n - r(x_n)} \right)^k.$$

So finally, for any  $x$  with  $x \geq x_n - r(x_n)$ ,

$$\frac{h(x)}{h(x_n)} \leq \left( \frac{x_n}{x_n - r(x_n)} \right)^k = \left( \frac{1}{1 - \frac{r(x_n)}{x_n}} \right)^k \rightarrow 1.$$

□

Holding down the Fourier transform

**Lemma 56.14.** *Suppose that  $f: \mathbb{C} \rightarrow \mathbb{C}$  is continuous and  $f$  decays but not rapidly. Roughly: we can find distant peaks  $\zeta_n$  of  $|f|$ . Precisely: pick any function  $r: \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$  with  $r(x) = o(x)$  as  $x \rightarrow \infty$ . For some integer  $k > 0$ , there are points  $\zeta_n \rightarrow \infty$  with  $|\zeta_n^k f(\zeta_n)| \rightarrow \infty$  and*

$$1 = \lim_{n \rightarrow \infty} \sup_{|\zeta| \geq |\zeta_n| - r(|\zeta_n|)} \frac{|f(\zeta)|}{|f(\zeta_n)|}.$$

*Proof.* Let

$$h(x) = \sup_{|\zeta|=x} |f(\zeta)|.$$

□

**Theorem 56.15.** *Pick any smooth complex-valued function  $\mu: \mathbb{C} \rightarrow \mathbb{C}$  with compact support and with  $|\mu| < \frac{1}{3}$ . Then the solution  $u = u_\mu$  to the Beltrami equation*

$$u_{\bar{z}} = \mu u_z,$$

*defined in theorem 56.12 on the preceding page is a smooth diffeomorphism.*

*Proof.* Fourier transform the equation  $f = \mu Lf + \mu$  to get

$$\hat{f} = \hat{\mu} * (\bar{\zeta}/\zeta) \hat{f} + \hat{\mu}.$$

Recall that a *Schwartz function* or *test function* is a smooth function all of whose partial derivatives decay rapidly. The Fourier transform of any Schwartz function is Schwartz [7] p. 15, [15] p. 20. Since  $\mu$  is smooth and compactly supported, it is



Schwartz, so  $\hat{\mu}$  is Schwartz. Convolution with  $\hat{\mu}$  is smoothing, so  $\hat{f}$  is smooth. Take  $L^1$  norm:

$$\|\hat{f}\|_{L^1} = \|\hat{\mu}\|_{L^1} \|\hat{f}\|_{L^1} + \|\hat{\mu}\|_{L^1},$$

or

$$\|\hat{f}\|_{L^1} = \frac{\|\hat{\mu}\|_{L^1}}{1 - \|\hat{\mu}\|_{L^1}} = \frac{\|\mu\|_{L^\infty}}{1 - \|\mu\|_{L^\infty}} < \frac{1}{2}.$$

Again by the Hausdorff–Young inequality [15] Theorem 1.1 (a) p. 2,

$$\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1} < \frac{1}{2}.$$

Hence

$$\|\varphi_{\bar{z}}\|_{L^\infty} < \frac{1}{2}.$$

Note then that

$$\|\varphi_z\|_{L^\infty} \leq \left\| \widehat{\frac{\partial \varphi}{\partial z}} \right\|_{L^1} = \left\| (\bar{\zeta}/\zeta) \hat{f} \right\|_{L^1} = \|\hat{f}\|_{L^1} < \frac{1}{2}.$$

Hence writing  $u = z + \varphi$ ,

$$\|u_z\|_{L^\infty} = \|1 + \varphi_z\|_{L^\infty} > \frac{1}{2}$$

while

$$\|u_{\bar{z}}\|_{L^\infty} = \|\varphi_{\bar{z}}\|_{L^\infty} < \frac{1}{2}.$$

It suffices to prove that  $f = \varphi_z$  is smooth, so we can see that  $\varphi$  is smooth from  $u = \frac{1}{\pi z} * f$ . The Fourier transform of differentiation is multiplication by a linear function. So to prove that  $f$  is smooth is to prove that  $\hat{f}$  decays even when multiplied by any polynomial function, i.e. decays rapidly. So suppose that  $\hat{f}$  fails to decay rapidly.

Apply the lemma above, we find the “worst behaviour” of  $\hat{f}$ , i.e. the “distant peaks”, occurring at some points  $\zeta_n$ . We can pick  $r: \mathbb{R}^{>0} \rightarrow \mathbb{R}^{>0}$  as long as  $r(x)/x \rightarrow 0$  as  $x \rightarrow \infty$  and, in addition, we add the assumption that  $r(x)$  is bounded below by some positive power of  $x$ , as  $x \rightarrow \infty$ .

Let

$$\mu_n = \begin{cases} \mu, & \text{if } |\zeta| < r(|\zeta_n|), \\ 0, & \text{otherwise.} \end{cases}$$

Note that since  $|\mu| < 1/3$ , we have  $|\mu_n| < 1/3$ , and so

$$\|\hat{\mu}_n\|_{L^1} \leq \|\hat{\mu}\|_{L^1} = \|\mu\|_{L^\infty} < \frac{1}{3}.$$

The equation

$$\hat{f} = \hat{\mu} * (\bar{\zeta}/\zeta) \hat{f} + \hat{\mu}$$

tells us that

$$\hat{f} = \hat{\mu}_n * (\bar{\zeta}/\zeta) \hat{f} + (\hat{\mu} - \hat{\mu}_n) * (\bar{\zeta}/\zeta) \hat{f} + \hat{\mu}.$$

The first term is convolved only inside a particular disk, so is at worst

$$\left| \hat{\mu}_n * (\bar{\zeta}/\zeta) \hat{f} \right| \leq \frac{1}{3} \sup_{|\zeta - \zeta_n| \leq r(|\zeta_n|)} |\hat{f}(\zeta)|.$$

Plugging in our “worst points”  $\zeta = \zeta_n$ :

$$\left| \hat{\mu}_n * (\bar{\zeta}/\zeta) \hat{f} \right| \leq \frac{1}{3} |\hat{f}(\zeta_n)| (1 + o(1)).$$

Return to the equation

$$\hat{f} = \hat{\mu}_n * (\bar{\zeta}/\zeta) \hat{f} + (\hat{\mu} - \hat{\mu}_n) * (\bar{\zeta}/\zeta) \hat{f} + \hat{\mu}.$$

The third term decays rapidly. For the second term,

$$|\hat{\mu} - \hat{\mu}_n|_{L^1} = \int_{|\zeta - \zeta_n| > r(|\zeta_n|)} |\hat{\mu}|$$

decays more rapidly than any power of  $|\zeta_n|$ , as  $\hat{\mu}$  is Schwartz. So finally,

$$|\hat{f}(\zeta_n)| \leq \frac{1}{3} |\hat{f}(\zeta_n)| (1 + o(1)),$$

which forces  $\hat{f}(\zeta_n) \rightarrow 0$ , a contradiction to our hypothesis that  $\hat{f}$  does not decay rapidly.  $\square$

Theorem 56.5 on page 478 follows.

## Notes

We will never revisit almost complex manifolds, Beltrami equations, distant peaks, Schwartz functions or Fourier transforms.

## Chapter 57

### Compact Riemann surfaces

We will make use of a little complex analysis to prove some results about compact Riemann surfaces and use those to smooth topological surfaces.

#### Meromorphic functions

**57.1** Prove that every holomorphic function on any compact Riemann surface is constant.

A *meromorphic function* on a Riemann surface  $C$  is a function  $f$  defined on the complement of a discrete set of points so that, near each point of  $C$  there are holomorphic functions  $g, h$ ,  $h$  nonzero except on discrete set, with  $f = g/h$  where defined. Two meromorphic functions are identified if they agree where both are defined. Recall from complex analysis that any holomorphic function vanishes only to finite order [1] p. 125 theorem 8. Any meromorphic function  $f = g/h$  also has a finite order of vanishing, perhaps negative: the difference in degree of  $g$  and  $h$ . We can scale both  $g$  and  $h$  to increase or decrease orders of both by the same integer. A *pole* is a point of negative order of vanishing.

On a Riemann surface, a meromorphic function  $f$  locally written as  $f = g/h$  has an associated map to  $\mathbb{P}^1$ :  $[f : 1] = [g : h]$ . We recall that we have already identified  $\mathbb{P}^1$  with  $\mathbb{C} \cup \{\infty\}$ . If we change the choice of representation of  $f$  as a ratio  $g/h$ , we don't change  $[f : 1]$ , which is defined even at points where  $f$  is not defined, since some choice of  $g, h$  is still available, extending the map uniquely to those points. So nonconstant meromorphic functions of a Riemann surface are precisely holomorphic maps to  $\mathbb{P}^1$ . The degree of a value of a meromorphic function  $f$  is its degree as a map to  $\mathbb{P}^1$ , which clearly the order of vanishing of  $f(p) - f(p_0)$  as  $p \rightarrow p_0$  for any point  $p_0$ .

**Theorem 57.1.** *The meromorphic functions on the Riemann sphere are precisely the rational functions.*

*Proof.* Recall the Laurent series expansion: every meromorphic function on the complex plane is uniquely

$$f(z) = p(1/z) + h(z)$$

where  $p$  is a polynomial and  $h(z)$  is holomorphic [1] p. 182. But then  $f(z)$  is also meromorphic in  $w = 1/z$  so  $f(z)$  has finitely many positive and finitely many negative degree terms.  $\square$

The meromorphic functions on an elliptic curve  $C = \mathbb{C}/\Lambda$ ,  $\Lambda \subset \mathbb{C}$  any lattice, are precisely the elliptic functions, hence rational functions of the Weierstrass function  $\wp(z)$  and its derivative  $\wp'(z)$  [1] p. 255.

## Holomorphic forms

A *holomorphic (or meromorphic) 1-form* is a complex valued 1-form  $\omega$  so that, in some chart near each point,  $\omega = f(z)dz$  for some holomorphic (or meromorphic) function  $f(z)$ . Note that  $d\omega = f'(z)dz \wedge dz = 0$ : holomorphic 1-forms are closed. The *residue* of  $\omega$  at a point  $p$  is its integral around the boundary of an embedded closed disk not intersecting any pole of  $\omega$  other than  $p$ . By Stokes's theorem, the residue does not depend on the disk, since we can rewrite it as an integral around some tiny disk in any chart near  $p$ .

**Lemma 57.2.** *The sum of residues of any meromorphic 1-form on any compact Riemann surface vanishes.*

**57.2** Prove it

**Lemma 57.3.** *Every nonconstant meromorphic function on any compact Riemann surface has degree zero as a map to the projective line.*

*Proof.* Take the sum of residues of  $df/f$ . □

Otherwise expressed: the number of zeroes is equal to the number of poles, counted with multiplicity. But we already knew that: that number is the degree.

## The Dolbeault differential

On a Riemann surface, we have seen that the notion of complex linear 1-form is well defined, as under any holomorphic change of variable  $Z = Z(z)$ , with inverse  $z = z(Z)$ , each differential form

$$\omega = f(z) dz,$$

becomes

$$\omega = f(z(Z)) \frac{dz}{dZ} dZ,$$

where  $dz/dZ$  is the complex analytic derivative. Similarly, a conjugate linear form

$$\omega = f(z) d\bar{z},$$

becomes

$$\omega = f(z(Z)) \overline{\left(\frac{dz}{dZ}\right)} d\bar{Z}.$$

A complex linear 1-form is also called a  $(1, 0)$ -form, and similarly a conjugate linear 1-form is a  $(0, 1)$ -form. (The reader can guess how this notation extends for complex manifolds of all dimensions, but we will stick with Riemann surfaces.) Any 2-form can be uniquely written a

$$\omega = f(z) dz \wedge d\bar{z},$$

so thought of as a  $(1, 1)$ -form. Similarly complex valued functions are called  $(0, 0)$ -forms.

Since we can split forms into parts like this, we can see how forms of some  $(p, q)$ -type split when we take  $d$ . We define an operator  $\partial$  by the recipe:  $\partial$  on a  $(p, q)$ -form means first take  $d$  and then take the  $(p + 1, q)$  part of the result.

If

$$f(z) = z\bar{z} + \bar{z}^2 + z^3,$$

a  $(0, 0)$ -form, then

$$df = z d\bar{z} + \bar{z} dz + 2\bar{z} d\bar{z} + 3z^2 dz.$$

is a mix of  $(1, 0)$  and  $(0, 1)$ . So  $\partial f$  is the  $(1, 0)$ -part of  $df$ , i.e. the  $dz$  bits:

$$\partial f = \bar{z} dz + 3z^2 dz = (\bar{z} + 3z^2) dz.$$

Similarly the *Dolbeault differential*  $\bar{\partial}$  of a  $(p, q)$ -form is the  $(p, q + 1)$  of  $d$  of that form.

**57.3** Prove that  $\partial^2 = 0$  and that  $\bar{\partial}^2 = 0$ . (Complex conjugation in any coordinates takes one to the other.)

**57.4\*** Prove that  $\bar{\partial}f = 0$  for a continuously differentiable function  $f$  just when  $f$  is holomorphic.

Since we are interested in holomorphic functions, i.e. functions  $f(z)$  with  $\bar{\partial}f = 0$ , we naturally study  $\bar{\partial}$ , and never study  $\partial$ .

Consider the holomorphic complex linear 1-forms  $\omega$ , locally of the form  $\omega = f(z) dz$  for some holomorphic function  $f(z)$ . Each holomorphic 1-form has a cohomology class in  $H_{\text{dR}}^1(C, \mathbb{C})$ .

**57.5** Prove that a holomorphic 1-form is uniquely determined by its de Rham cohomology class.

**Lemma 57.4** (Dolbeault–Grothendieck). *On the open disk, or the complex plane, every smooth  $(0, 1)$ -form is  $\bar{\partial}$ -exact.*

*Proof.* Take a smooth complex-valued function  $f$  on our disk or complex plane. Take a sequence of disks  $D_1 \subset D_2 \subset \dots$  exhausting our disk or complex plane. By the Poincaré lemma, if we set

$$u(z) := \frac{1}{2\pi i} \int_{D_i} f(w) \frac{dw \wedge d\bar{w}}{w - z}.$$

then  $u: D_i \rightarrow \mathbb{C}$  has  $\bar{\partial}u = f d\bar{z}$  on  $D_i$ . The problem is that  $f$  has to be defined on the closed disk to get  $u$  defined on the open disk.

Write  $u$  as  $u_i$  to indicate the number of each disk. Maybe  $u_{i+1}$  is close to  $u_i$  where both are defined, i.e. on  $D_i$ . On  $D_i$ ,  $\bar{\partial}(u_{i+1} - u_i) = 0$ , so  $u_{i+1} - u_i$  is a holomorphic function, which we expand in a Taylor series in  $D_i$ . After finitely many terms in that

series, by absolute convergence, the remaining terms stay small inside the closed disk  $\bar{D}_{i-1}$ , say smaller than some tiny number  $\varepsilon_i > 0$ . We can subtract those finitely terms of that Taylor series from  $u_{i+1}$ , without changing  $\bar{\partial}u_{i+1} = f$ . So  $u_{i+1} - u_i$  is smaller than  $\varepsilon_i$  on  $\bar{D}_{i-1}$ . Hence the sequence converges uniformly on compact sets, so the limit  $u$  is defined at each point of  $D$ .

On any chosen closed disk  $\bar{D}_i$ , all  $u_j$  for  $j > i$  agree with  $u_i$  up to a holomorphic function, uniformly controlled throughout the disk by a sum of various  $\varepsilon_i + \cdots + \varepsilon_j$ . Uniform convergence of complex Taylor series implies smooth convergence.  $\square$

## The Riemann sphere

**Theorem 57.5.** *Up to biholomorphism, there is precisely one compact Riemann surface  $C$  on which every smooth  $\bar{\partial}$ -closed 1-form is  $\bar{\partial}$ -exact: the Riemann sphere, i.e. the complex projective line.*

*Proof.* Pick a holomorphic chart on  $C$  and a smooth bump function  $h$  equal to 1 near the origin of that chart, equal to zero outside a compact subset of that chart. Let

$$\omega := \frac{1}{z} \bar{\partial}h.$$

Note that  $\omega$  vanishes near  $z = 0$ , since  $h = 1$  is constant near  $z = 0$ , so  $\bar{\partial}h = 0$  near  $z = 0$ . So  $\omega$  is a smooth  $(0, 1)$ -form on our Riemann surface, so  $\omega = \bar{\partial}g$  some complex-valued function  $g$  on our Riemann surface. The function  $F := h/z - g$  is  $\bar{\partial}$ -closed, so holomorphic away from the origin of our chart and having a pole there like  $1/z$ . There are no other poles, since  $F$  is smooth away from  $z = 0$ . The function  $F$  has exactly one zero, counted with multiplicity, somewhere on  $C$ , by lemma 57.3 on page 486. In particular,  $F$  is a local biholomorphism near the zero, since it has multiplicity 1. Applying the same ideas to  $F - c$  for any constant  $c$ ,  $F$  takes on every value precisely once, counted with multiplicity, i.e. no critical points. Hence  $F: C \rightarrow \mathbb{P}^1$  is a local biholomorphism. Since  $C$  is compact,  $F$  is a covering map. Since  $C$  is connected and  $\mathbb{P}^1$  is connected and simply connected,  $F$  is a biholomorphism.  $\square$

## Higher genus

**Theorem 57.6.** *For any point of any compact Riemann surface, there is a meromorphic function with a pole at that point.*

*Proof.* Take a chart where our point is the origin. Suppose  $h$  is a bump function supported in a compact neighborhood of the origin in the chart, and equal to 1 in some neighborhood of the origin. Let

$$\omega := \frac{f(z)}{z^p} \bar{\partial}h,$$

where  $f(z)$  is any holomorphic function defined in a neighborhood of the support of  $h$ . So  $\omega$  is a  $(0, 1)$ -form on our Riemann surface, and  $\bar{\partial}\omega = 0$ , so  $\omega = \bar{\partial}g$  for some smooth function  $g$ . The function  $F = hf(z)/z^p - g$  is  $\bar{\partial}$ -closed, so holomorphic away from the origin of our chart. But if  $f(z)$  has order of vanishing less than  $p$  at  $z = 0$ ,  $F$  has a pole at  $z = 0$ .

Restrict to the space of polynomials  $f(z)$  of degree at most  $p - 1$ , so that our smooth function  $g$  cannot cancel the pole, i.e.  $F = hf(z)/z^p - g$  has a pole. If we make  $p$  larger than the dimension of the de Rham cohomology  $H_{\text{dR}}^1(C) \otimes \mathbb{C}$  we guarantee the existence of a nonzero meromorphic function  $F$ .  $\square$

**Corollary 57.7.** *For any point of any compact Riemann surface, there is a meromorphic function with a simple pole at that point, i.e. order of vanishing  $-1$ .*

*Proof.* By picking an addition choice of pole, using the same mechanism, near some other point of the same chart, make it a single pole.  $\square$

Take reciprocal:

**Corollary 57.8.** *For any point of any compact Riemann surface, there is a meromorphic function with a simple zero at that point, i.e. order of vanishing  $1$ .*

**57.6** Prove that for any two points of a compact Riemann surface and any two complex numbers, there is a meromorphic function on the Riemann surface taking the one value at the one point, the other at the other.

**Theorem 57.9.** *Any meromorphic functions  $f_0, f_1, \dots, f_n$ , not all everywhere zero, on a compact Riemann surface  $C$  define a holomorphic map*

$$p \in C \mapsto \varphi(p) := \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

*Proof.* Wherever all  $f_i$  are holomorphic, and one is not zero, then the map is clearly defined. Where all have a zero or pole, take the least order of vanishing  $d$ , and write

$$\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} z^{-d} f_0 \\ z^{-d} f_1 \\ \vdots \\ z^{-d} f_n \end{bmatrix}.$$

$\square$

**57.7** When does the image of the map lie in a projective subspace?

**Theorem 57.10.** *Every compact Riemann surface admits a holomorphic embedding into complex projective space.*

*Proof.* We pick some meromorphic functions  $f_1, \dots, f_n$  so that the set of points where  $f_i$  is holomorphic, nonzero and an immersion is some open set  $C_i \subset C$  of our Riemann surface. By compactness, finitely many such will ensure that these  $C_i$  cover  $C$ . So our map as above runs into trouble only where one of these functions has a pole. There, we have to rescale by some  $z^{-d}$ , which might render none of our functions an immersion. We then pick another one, immersing near that point, and write it as  $z^{-d}$  times a meromorphic function, which we add to the list. Only finitely many additions need be made. So our map is a holomorphic immersion, but perhaps not an embedding.

If our map fails to be injective, say taking the same value at some points  $p, q \in C$ , by our same recipe we can produce a meromorphic function taking on distinct values at those points, and add it to our list. But we might find still some more points where this occurs. Consider the set of points  $(p, q) \in C \times C$  for which  $\varphi(p) = \varphi(q)$ . This set is closed so compact. Once we add another function  $f_i$  to our list, killing some pair  $(p, q)$ , it also kills all nearby pairs, an open set in  $C \times C$  covering that point. By compactness, adding finitely many functions to the list makes our map injective, holomorphic, immersive, on a compact manifold, so an embedding.  $\square$

Take a nonconstant meromorphic function  $f$  on a compact Riemann surface  $C$ , thought of as a map to the Riemann sphere. Take a linear function on that sphere, a height function:

$$z \mapsto \frac{|z|^2}{1 + |z|^2}.$$

The level sets slice up the sphere into oriented surfaces, and the map  $f$  is oriented preserving. So the function

$$\frac{|f|^2}{1 + |f|^2}$$

is a real-valued Morse function on  $C$  with no twisted handles.



## Chapter 58

### Vector bundles

We introduce the language of vector bundles, with a few easy theorems. In a picture, a vector bundle is a “smoothly varying” family of vector spaces, like the tangent planes of a surface, or the normal lines of a surface.

#### Definition

A *section* of a map  $f: P \rightarrow Q$  is a map  $g: Q \rightarrow P$  so that  $f \circ g = \text{id}$ . A *real, complex or quaternionic vector bundle* of rank  $r$  on a manifold  $M$  is a manifold  $V$  with a smooth map  $\pi: V \rightarrow M$  so that

- each fiber  $V_m = \pi^{-1}(m)$  is equipped with the structure of a real, complex or quaternionic vector space and
- adding smooth sections, using these vector space structures, yields a smooth section and
- scaling a smooth section by a smooth function, using these vector space structures, yields a smooth section and
- each point  $m_0 \in M$  lies in an open set  $U \subset M$  on which there are smooth sections  $e_1, e_2, \dots, e_r$  whose values  $e_1(m), e_2(m), \dots, e_r(m)$  are a basis of the vector space  $V_m$  at each  $m \in U$ , a *basis of local sections*.

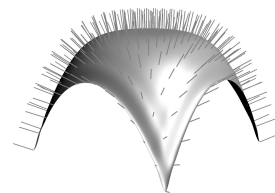
**58.1** Prove that every vector bundle is a fiber bundle.

The tangent bundle  $TM$  of any manifold  $M$  is a vector bundle over  $M$ .

The *cotangent bundle*  $T^*M$  of any manifold  $M$  is the set of pairs  $(m, \xi)$  so that  $m \in M$  and  $\xi$  is the value of a 1-form at  $m$ , i.e.  $\xi$  eats vectors and gives numbers, i.e.  $\xi \in T_m^*M$  belongs to the dual vector space.

If  $P \subset Q$  is a submanifold, the *normal bundle* is the set of pairs  $(p, v)$  so that  $p \in P$  and  $v \in T_p Q / T_p P$  is a normal vector to  $P$  at  $p$ . This is an important example: picture any vector bundle as if it were the normal bundle.

The *zero section* is the section that associates to each point  $m \in M$  the vector  $0 \in V_m$ . The zero section embeds  $M$  as a submanifold of  $V$ , so  $V$  is the



normal bundle of  $M$  (i.e. of the zero section) inside  $V$ : any vector bundle is the normal bundle of some embedding.

For  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , take a  $k$ -vector space  $V_0$ . The *model trivial vector bundle* on a manifold  $M$  is  $V := M \times V_0$ , with obvious map  $M \times V_0 \rightarrow M$  and obvious vector space structure on the fibers.

The *Möbius strip* is the complex vector bundle on the circle with

$$V = \{ (z, w) \mid z, w \in \mathbb{C}, |z| = 1, w \perp \pm\sqrt{z} \}$$

and with map  $\pi: (z, w) \in V \mapsto z \in M$  to the circle  $M$ . Intuitively, as we turn  $z$  around the circle, turn  $w$  too, but when  $z$  completes a rotation,  $w$  only completes half a rotation.

Let  $M$  be the circle and  $V_0$  a vector space over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take a quotient of the trivial bundle on  $\mathbb{R}$  by action of an element  $g \in \mathrm{GL}(r, k)$ :

$$(\theta, v) \sim (\theta + 2\pi, gv).$$

The quotient  $V_g \rightarrow M$  is a vector bundle on the circle.

The quotient  $V_{-1} \rightarrow M$  is the Möbius strip over the circle, where  $-1$  means the  $1 \times 1$  matrix and the vector space  $V_0$  is  $V_0 = \mathbb{R}$ .

For the reader familiar with covering spaces: take a manifold  $M$  with universal covering space  $\tilde{M} \rightarrow M$ , and fundamental group  $\pi_1(M)$ . By corollary 3.2 on page 25,  $\tilde{M}$  is a manifold and  $\tilde{M} \rightarrow M$  is a smooth local diffeomorphism. Take a finite dimensional vector space  $V_0$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Denote by  $\mathrm{GL}(V_0)$  the group of all invertible  $k$ -linear maps  $V_0 \rightarrow V_0$ . Take any representation of  $\pi_1(M)$ , i.e. a group morphism

$$\rho: \pi_1(M) \rightarrow \mathrm{GL}(V_0).$$

Get  $\pi_1(M)$  to act as usual on  $\tilde{M}$  and to act via the representation on  $V_0$ , and take the product action on  $\tilde{M} \times V_0$ . As is well known in the theory of covering spaces [13], the quotient space

$$V := (\tilde{M} \times V_0) / \pi_1(M)$$

has universal covering space  $\tilde{V} = \tilde{M} \times V_0$  and fundamental group

$$\pi_1(V) = \pi_1(M).$$

The action of  $\pi_1(V)$  on  $\tilde{V}$  acts via  $k$ -linear transformations of the fibers, so that the fibers of  $V \rightarrow M$  are equipped with  $k$ -vector space structure. Over

any evenly covered open subset of  $M$ , we obtain an evenly covered open subset of  $V$ , and produce a basis of local smooth sections out of any choice of basis of  $V_0$ .

## Morphisms

A *vector bundle morphism*  $f: V \rightarrow W$  is a smooth map  $f$  so that  $f: V_m \rightarrow W_{f(m)}$  is a linear map for every  $m \in M$ . A *vector bundle isomorphism* is a vector bundle morphism with an inverse vector bundle morphism. A vector bundle is *trivial* if it is isomorphic to the model trivial vector bundle. A *subbundle*  $V \subset W$  is a vector bundle  $V$  which is a subset of a vector bundle  $W$  so that the inclusion map is smooth and linear on each fiber.

**58.2** Take a vector bundle morphism  $f: V \rightarrow W$ . Suppose that  $f: V_m \rightarrow W_{f(m)}$  is an invertible linear map, for every  $m \in M$ . Prove that  $f$  is a vector bundle isomorphism.

Over the manifold  $M = \mathbb{R}$ , take the trivial vector bundles  $V = W = M \times \mathbb{R}$  and the vector morphism  $f: V \rightarrow W$  given by  $f(x, u) = (x, x^2 u)$ . Then the kernel of  $f$  at each point  $x \in M$  is 0, except at  $x = 0$ , where the kernel jumps to  $\mathbb{R}$ . So the kernel of the linear map  $f_x: V_x \rightarrow W_x$  can jump up in dimension at a closed set of points  $x \in M$ , and the union of the kernels might not constitute a vector bundle. Similarly, the image of the linear map  $f_x: V_x \rightarrow W_x$  can jump down in dimension at that closed set of points  $x \in M$ , and the union of the images might not constitute a vector bundle.

**58.3** Suppose that  $f: V \rightarrow W$  is a morphism of vector bundles and the associated linear maps  $f_m: V_m \rightarrow W_m$  on fibers have constant rank. Prove that the kernel  $K := \{v \in V \mid f(v) = 0\}$  is a vector subbundle and that the image  $f(V) \subset W$  is a vector subbundle.

**58.4** Suppose that  $f: V \rightarrow W$  is a morphism of vector bundles and the associated linear maps  $f_m: V_m \rightarrow W_m$  on fibers are surjective. Prove that for any smooth section  $w$  of  $W$  there is a smooth section  $v$  of  $V$  so that  $w = f \circ v$ , i.e. surjectivity on fibers is equivalent to surjectivity on sections.

**58.5** Give an example of a morphism  $f: V \rightarrow W$  of vector bundles on a manifold  $M$  which is injective on local sections but not injective on fibers, i.e. if  $f \circ v = 0$  for some smooth local section  $v$  of  $V$ , then  $v = 0$ , but  $f(v_0) = 0$  for some nonzero vector  $v_0 \in V_{m_0}$  at some point  $m_0 \in M$ .

If  $V \rightarrow M$  and  $W \rightarrow M$  are vector bundles, let  $V \oplus W \rightarrow M, V^* \rightarrow M, V \otimes W \rightarrow M$  be the vector bundles whose fibers are  $(V \oplus W)_m = V_m \oplus W_m$ ,  $(V^*)_m = (V_m)^*$  and (if  $k = \mathbb{R}$  or  $\mathbb{C}$ )  $(V \otimes W)_m = V_m \otimes W_m$ , and so on for all of the standard constructions of linear algebra. We will soon see that these are vector bundles.

Consider the rank 1 vector bundle  $V_{-1} \rightarrow M$  on the circle  $M$  (the Möbius strip, as defined on page 492), i.e. the quotient

$$(\theta, v) \sim (\theta + 2\pi, -v).$$

The sum  $V_{-1} \oplus V_{-1} \rightarrow M$  admits a basis of global sections

$$\begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}, \begin{pmatrix} -\sin 2\theta \\ \cos 2\theta \end{pmatrix}$$

and is therefore trivial.

Denote the *symmetric product* of vector spaces, and hence of vector bundles, as  $\text{Sym}^k(V)$ , and the *exterior product* as  $\Lambda^k V$ . We will soon see that these are vector bundles.

If  $\varphi: P \rightarrow Q$  is a smooth map of manifolds with corners, and  $V \rightarrow Q$  is a vector bundle,  $\varphi^*V$  is the vector bundle whose fibers are  $\varphi^*V_p = V_q$  where  $q = \varphi(p)$ . If  $P \subset M$  is a submanifold,  $V|_P \rightarrow P$  is the pullback vector bundle for the inclusion map  $\varphi(p) = p$ .

**58.6** Prove that the pullback has a unique structure of vector bundle for which  $V \rightarrow \varphi^*V$  is a morphism is vector bundles.

### Transition maps

If  $V \rightarrow M$  is a vector bundle, we write a basis of local sections as  $e$ , rather than  $e_1, e_2, \dots, e_k$ . Given two bases of local sections, say  $e$  and  $\underline{e}$ , there is a unique invertible matrix  $g = g(m)$  so that  $\underline{e}_j = \sum_i g_{ij} e_i$ , where  $g$  maps the open subset of  $M$  over which both bases are defined, to  $\text{GL}(k, \mathbb{R})$ ; denote the new basis in terms of the old as  $\underline{e} = eg$ . So if we cover  $M$  by open sets  $M_a \subset M$ , and let  $M_{ab} = M_a \cap M_b$  and so on, and over each open set we pick a basis  $e_a$  of local sections, then  $e_b = e_a g_{ab}$  for a smooth map  $g_{ab}: M_{ab} \rightarrow \text{GL}(k, \mathbb{R})$ . These  $g_{ab}$  are the *transition maps* of this choice of bases of local sections. They satisfy the three laws:

- a.  $g_{aa} = \text{id}$  on  $M_a$ ,
- b.  $g_{ba} = g_{ab}^{-1}$  on  $M_{ab}$ ,
- c.  $g_{ab}g_{bc}g_{ca} = \text{id}$  on  $M_{abc}$ .

**Lemma 58.1.** Suppose that  $M$  is a manifold, covered by a countable collection of open sets  $M_a$ , and let  $M_{ab} = M_a \cap M_b$  and so on. On each  $M_{ab}$ , pick any smooth map  $g_{ab}: M_{ab} \rightarrow \text{GL}(k, \mathbb{R})$  satisfying these three laws. Then there is a vector bundle  $V \rightarrow M$  with these transition maps, determined uniquely up to isomorphism.

The proof is the same as for fiber bundles.

## Constructions

If  $g_{ab}$  are the transition maps of  $V$ , then  ${}^*g_{ab}^{-1}$ , the inverse transposes, are the transition maps of  $V^*$ . If  $g_{ab}$  and  $h_{ab}$  are the transition maps of vector bundles  $V$  and  $W$ , then

$$\begin{pmatrix} g_{ab} & 0 \\ 0 & h_{ab} \end{pmatrix}$$

are the transition maps of  $V \oplus W$ ,

$$g_{ab} \otimes h_{ab}(v \otimes w) = (g_{ab}v) \otimes (h_{ab}w)$$

(if  $k = \mathbb{R}$  or  $\mathbb{C}$ ) are the transition maps of  $V \otimes W$ , and so on. Various bundles arise as subbundles of tensor products  $V \otimes \cdots \otimes V$

- The  $p$ -th *symmetric power*  $S^p V$  has fibers the exterior powers the symmetric powers  $S^p V_m$ .
- The  $p$ -th *exterior power*  $\Lambda^p V$  has fibers the exterior powers  $\Lambda^p V_m$ .
- If  $V$  has rank  $p$ ,  $\Lambda^p V$  is also denoted  $\det V$ , the *determinant line bundle*.
- All of these real vector bundle constructions have obvious complex vector bundle analogues.
- The quaternion analogues exist for sums of vector bundles, but there is no easy way to make sense of quaternionic tensor products.
- If  $\varphi: V \rightarrow W$  is a vector bundle morphism of constant rank, then its kernel and image and cokernel are vector subbundles of  $V, W$  and  $W$  respectively.
- Recall the complexification  $U_{\mathbb{C}}$  of a real vector space is  $U \otimes_{\mathbb{R}} \mathbb{C}$ , whose vectors are pairs  $(u, v)$  of vectors  $u, v \in U$ , make into a complex vector space by asking that  $i(u, v) = (-v, u)$ ; we can write  $(u, v)$  formally as  $u + iv$ . If  $V \rightarrow M$  is a vector bundle,  $V_{\mathbb{C}}$ , also denoted  $V \otimes_{\mathbb{R}} \mathbb{C}$ , is the vector bundle whose fibers are the complexifications  $V_m \otimes_{\mathbb{R}} \mathbb{C}$ .
- Similarly we can quaternionize.
- If  $V \rightarrow M$  is a real vector bundle, and  $J: V \rightarrow V$  is an isomorphism taking each fiber to itself, with  $J^2$  the identity, turn  $V$  into a complex vector bundle by defining  $iv$  to mean  $Jv$ . Similarly, if we have such maps  $I, J, K$  satisfying the quaternion identities, we can make  $V$  into a quaternion vector bundle.

## Clutching

Pick a manifold with corners  $M$  and two open sets  $M_0, M_1 \subseteq M$ , and let  $M_{01} := M_0 \cap M_1$ . Take vector bundles  $V_0 \rightarrow M_0, V_1 \rightarrow M_1$ , of equal rank, say  $r$ , and a smooth vector bundle isomorphism  $\varphi$  over  $M_{01}$ , a *clutching isomorphism*. Define a new vector bundle  $V$  by quotienting  $v_1 \in V_1$  to  $v_0 \in V_0$  if  $v_1 = \varphi v_0$ . In particular, if  $V_0$  and  $V_1$  are both trivial, with chosen bases of sections  $e_i$  and  $\underline{e}_i$  respectively, then  $\varphi e_i = g_i^j \underline{e}_j$  for unique map  $g = (g_i^j): M_{01} \rightarrow \text{GL}(r, k)$ , the *clutching function*.

### Clutching around a point

Consider the special case of clutching around some points. Take a discrete set of points  $\Delta \subset M$ ; write the points as  $p_a$ . Take a chart  $\varphi_a$  taking  $p_a$  to the origin for each  $p_a \in \Delta$ . For each point  $p_a$ , let  $M_a \subset M$  be the  $\varphi_a$ -preimage of some ball centered at the origin  $0 = \varphi_a(p_a)$ , small enough so that  $M_a, M_b$  don't intersect for  $a \neq b$ . Let

$$M_0 := M - \Delta,$$

and  $M_1$  the union of these  $M_a$ . The only nonempty overlaps are  $M_{0a}$ ; their union is  $M_{01}$ . All triple overlaps with distinct indices are empty. Let  $n$  be the dimension of  $M$ . Take a map  $c_a: S^{n-1} \rightarrow \text{GL}(k, r)$  for each  $p_a \in \Delta$ , a *clutching function* around  $p_a$ . Make transition maps  $g_{0a}$  for  $a > 0$  by, in the chart  $\varphi_a$ ,

$$g_{0a}(x) := c_a(x/|x|).$$

No triple overlaps, so we don't have to check the complicated law for triple overlaps. So these are the transition maps of a vector bundle  $V \rightarrow M$  of rank  $r$ . This vector bundle is *clutched* around these points, but trivial away from them. If we make the  $M_a$  a bit smaller, so that their closures don't overlap, we can move them around (by problem 22.21 on page 192) so the precise choice of points  $p_a$  and open sets  $M_a$  is irrelevant.

**58.7\*** Prove that every vector bundle obtained by clutching around a discrete set of points, finitely many on each connected component of the manifold, is isomorphic to a vector bundle obtained by clutching around one point on each component.

### Line bundles over the projective line

Recall the projective line. Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . The *projective line*, denoted  $\mathbb{P}^1$ , is the set of all lines through the origin in  $k^2$ . Denote the span of a nonzero vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in k^2$$

as

$$\begin{bmatrix} x \\ y \end{bmatrix}.$$

Define an open set  $M_0 = (x \neq 0)$  and an open set  $M_1 = (y \neq 0)$ . Pick an integer  $p$ . On the overlap, we define a clutching map

$$g = (y/x)^p \in k - \{0\}.$$

This is the clutching map of a line bundle called  $\mathcal{O}(p)$ .

Recall the we identified projective lines with spheres

$$\mathbb{RP}^1 = S^1, \mathbb{CP}^1 = S^2, \mathbb{HP}^1 = S^4.$$

by

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{a}{1-b} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\bar{a}}{1+b} \end{bmatrix}.$$

The clutching function of  $\mathcal{O}(p)$  is then

$$\left(\frac{y}{x}\right)^p = \left(\frac{\bar{a}}{1-b}\right)^p.$$

## Embedding in a trivial bundle

**Theorem 58.2.** *Every rank  $p$   $k$ -vector bundle,  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , on an  $n$ -dimensional manifold is a  $k$ -subbundle of a trivial bundle of rank  $p + q$ , as long as  $q \geq 0$  and*

$$\frac{n}{d} - 1 < q,$$

where  $d = \dim_{\mathbb{R}} k$ .

*Proof.* Take any integer  $q \geq 0$ , let  $\underline{k}^{p+q} \rightarrow M$  be the trivial  $k$ -vector bundle of rank  $p + q$ . So  $V^* \otimes_k \underline{k}^{p+q}$  is the vector bundle whose fibers are  $k$ -linear maps from fibers of  $V$  to  $k^{p+q}$ . (Careful: for  $k = \mathbb{H}$ , there is no reasonable notion of tensor product, so we just use this as a notation for the real vector bundle whose fibers are those  $k$ -linear maps.) Let  $A \subset V^* \otimes_k \underline{k}^{p+q}$  be the determinantal variety, i.e. the  $k$ -linear maps which are not injective. So  $A$  is the union of the submanifolds  $A_r$  of  $k$ -linear maps of rank  $r$ ,  $r = 0, 1, 2, \dots, p - 1$ . Note that  $A_r$  is a manifold of real dimension

$$n + dr(2p + q - r),$$

as we have seen previously. As a quadratic function of  $r$ , this dimension peaks at  $r = p + q/2$ . Since  $q \geq 0$ , and  $0 \leq r \leq p - 1$ , we don't reach the peak, so  $A_r$  increases in dimension with  $r$ , so

$$\begin{aligned} \dim M + \dim A &= \dim M + \dim A_{p-1}, \\ &= n + n + d(p-1)(2p + q - (p-1)), \\ &= 2n + d(p-1)(p + q + 1). \end{aligned}$$

We can apply Thom transversality to the wiggle the zero section of  $V^* \otimes_k \underline{k}^q$  away from  $A$  just when

$$\dim M + \dim A < \dim V^* \otimes_k \underline{k}^q,$$

i.e.

$$2n + d(p-1)(p + q + 1) < n + dp(p + q),$$

which we can simplify to

$$n < d(q + 1).$$

The Thom transversality theorem perturbs the zero section into an injective linear map of each fiber, i.e. an injective  $k$ -linear vector bundle injection  $V \rightarrow \underline{k}^{p+q}$ .  $\square$

## Regular zeroes

Given a vector bundle  $V \rightarrow M$ , each local section  $s$  of  $V$  vanishing at a point  $m \in M$  has *differential* denoted  $ds_m \in T_m^* M \otimes V_m$  defined as follows. Take a basis of local sections  $e_1, e_2, \dots, e_k$  of  $V$  near  $m$ . Then write  $s$  in terms of that basis as

$$s = \sum_i s^i e_i$$

for some smooth functions  $s^1, s^2, \dots, s^k$ . Define

$$ds_m := ds_m^i \otimes e_i(m).$$

We leave the reader to check that differential is defined at a zero  $m$  of  $s$  independent of the choice of basis of local sections. Note that the differential  $ds_m$  is *not* defined at a point where  $s \neq 0$ . A section  $s$  has a *regular zero* at  $m$  if  $s(m) = 0$  and  $ds_m: T_m M \rightarrow V_m$  is surjective. By the implicit function theorem, near a regular zero of any section of a vector bundle, the zeroes of the section form a smooth submanifold of dimension equal to the dimension of the kernel of  $ds_m$ .

By Thom transversality, the generic section of a real vector bundle  $V \rightarrow M$  is transverse to the zero section, so has zeroes forming a submanifold of codimension equal to the rank of the vector bundle  $V$ .

**58.8** Prove that the set  $A$  of 1-jets of sections which are irregular zeroes has codimension at least equal to the dimension of the base manifold.

If the rank of a real vector bundle  $V \rightarrow M$  exceeds the dimension of the manifold  $M$ , there is a global nonzero section.

Given sections  $s_1, \dots, s_p$  of various vector bundles  $V_1, \dots, V_p \rightarrow M$ , treat them as a single section  $s_1 \oplus \dots \oplus s_p$  of  $V_1 \oplus \dots \oplus V_p$ . The generic section of that sum bundle is a collection of sections of each bundle, whose zeroes intersect in smooth submanifolds.

### Index of a zero

A vector bundle is *oriented* if each fiber is an oriented vector space, and near each point there is a positively oriented basis of local sections.

**58.9** Take a real vector bundle  $V \rightarrow M$  of rank  $r$ . Let

$$\text{Or } V := (\Lambda^r V^*)/\mathbb{R}^\times$$

under the rescaling action. Explain how to identify (local) orientations of  $V$  with (local) sections of the 2-fold covering space  $\text{Or } V \rightarrow M$ . Prove that a real is orientable (i.e. admits an orientation) if and only if  $\Lambda^r V^* \rightarrow M$  is a trivial vector bundle.

Take a rank  $n$  oriented real vector bundle  $V \rightarrow M$  on an oriented  $n$ -dimensional manifold with corners  $M$ . Take a section  $s$  and an isolated zero  $s(m_0) = 0$ . Pick a positively oriented basis of local sections  $e_1, \dots, e_n$  for  $V$ . In that basis  $s = s^i e_i$ , with  $s^i(m_0) = 0$  but  $s^i$  not all zero at every point near  $m_0$ . Define a map

$$\varphi: S^{n-1} \rightarrow S^{n-1}$$

by taking a positively oriented chart in which  $m_0$  is the origin, with the unit ball in the image of that chart, and

$$\varphi(x) = \frac{(s^1(x), \dots, s^n(x))}{\sqrt{(s^1)^2 + \dots + (s^n)^2}}$$



for  $x$  on the unit sphere bounding that unit ball. The *index* of the zero  $m_0$  is the degree of  $\varphi$ . Clearly this integer varies continuously with the data we have picked, and therefore is independent of that data. Note that any two orientation preserving embeddings of the closed ball are smoothly isotopic, by problem 22.21 on page 192, so we can vary continuously from one to the other. Hence the index is well defined, independent of any choices made except the section and the zero, and is constant if they vary continuously.

**58.10** Prove that the generic smooth section has all zeroes isolated and of index  $\pm 1$ .

At some zero, consider the invariant  $ds$ . In some positively oriented chart, write  $s^i(x) = a_j^i x^j + \dots$ , so  $ds = (a_j^i)$ , in the bases  $\partial_i$  and  $e_j$ . The rank of  $ds$  is an invariant of the zero. Suppose that  $ds$  is invertible, a *nondegenerate* zero. Changing chart and basis of local sections by orientation preserving linear maps allow us to then make  $ds = I$  or

$$ds = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}.$$

So  $ds$  is an orientation preserving linear isomorphism just when we can arrange  $ds = I$  and our map  $\varphi$ , after rescaling the chart, becomes as close as we like to the identity map, so index 1. By the same reasoning, if  $ds$  is an orientation reversing linear isomorphism, then index  $-1$ .

## Complex line bundles on surfaces

**Theorem 58.3.** *Take an oriented surface and a complex line bundle. Then the line bundle is obtained by clutching.*

*If the surface is compact, the line bundle is obtained by clutching around a single point, and the degree of the clutching map  $S^1 \rightarrow S^1$  determines the line bundle up to isomorphism. That degree is equal to the sum of indices of any smooth section with finitely many zeroes, not necessarily regular zeroes.*

The complex line bundles on the plane  $\mathbb{R}^2$  are trivial, so it is still possible, despite this theorem, that two clutchings of different degree could determine the same line bundle.

*Proof.* Take a section  $s_1$  with discrete set of zeroes. Let  $s_2 := is_1$ . Let  $M_0$  be the open set where they are both nonzero, and hence linearly independent over the real numbers. Let  $M_1$  be an open set containing the zeroes of  $s_1$  and on which the vector bundle is trivial; it is trivial in some open set around each zero, and we can make those open sets smaller so as not to overlap, trivializing in each one separately. So our vector bundle is obtained by clutching. We can make  $M_1$  out of small disks (i.e. open sets diffeomorphic to disks). These disks overlap with  $M_0$  on punctured disks. We can retract the punctured balls to a circle in each. The clutching is given precisely by the degree of the clutching map in each circle.

If the number of zeroes is finite, for example if  $M$  is compact, we can arrange by diffeomorphism preserving orientation that these disks all lie in any given open set by lemma 22.4 on page 192. Pick that open set to be a disk in which the vector bundle is trivial. So the clutching occurs over a single disk. The complex line bundle

is determined by the degree of the clutching, which equals the sum of the indices of the section  $s$ .

By Thom transversality, we can perturb our section  $s_1$  slightly, to have regular zeroes, still inside those same disks, so the degree is the sum of the indices.  $\square$

**58.11** Prove that quaternionic line bundles are trivial on manifolds with corners of dimension 0, 1, 2, 3.

**58.12** Prove that quaternionic line bundles on compact oriented 4-dimensional manifolds with corners are determined up to isomorphism by the degree of a clutching map around a single point.

### Metrics

A *vector bundle metric* on a real, complex or quaternionic vector bundle  $V \rightarrow M$  is a positive definite inner product on each fiber  $V_m$ , real or Hermitian or quaternionic Hermitian, so that on smooth sections  $v, w$ , the inner product  $\langle v, w \rangle$  is smooth. *Warning:* a vector bundle metric does *not* determine a metric in the sense of the theory of metric spaces; so the terminology is dangerous but standard.

**58.13** Use a partition of unity to ensure that every vector bundle admits a metric.

**58.14** Prove that every vector subbundle of a vector bundle admits a complement vector subbundle. Hint: pick a metric, and take the perpendicular space.

**58.15** Prove that the automorphism group of a real vector bundle acts transitively on the metrics on that vector bundle.

A *line bundle* is a rank 1 vector bundle.

**Proposition 58.4.** *Every real vector bundle splits into a direct sum of (1) trivial real line bundles and of (2) some real vector bundles whose ranks sum to at most the dimension of the base manifold.*

*Proof.* We saw above, by Thom transversality, that every real vector bundle of rank exceeding the dimension of the base manifold has a global nonzero section. This section spans a real line subbundle. Take a complement. Apply induction.  $\square$

**58.16** Apply the same method as proposition 58.4 to complex vector bundles on a real manifold, and quaternionic vector bundles on a real manifold.

If  $V$  is a rank 2 oriented real vector bundle with metric, then each fiber has a natural action of the group of rotations  $S^1$  of the plane: take a linear isometry of the fiber with the oriented plane, preserving orientation, and turn by a given angle. The oriented linear isometry is unique up to rotations, and these commute, so the definition is unambiguous. Let complex scalars act on  $V$ , by letting  $re^{i\theta}$  act as rotation by angle  $\theta$  and scaling by real number  $r$ . Clearly  $V$  becomes a complex line bundle. For example, take an oriented surface with a Riemannian metric: its tangent bundle is a complex line bundle.

## Group actions and vector bundles

If a group  $G$  acts on two manifold with corners  $P, Q$ , we defined  $P \times^G Q$  to be the quotient by the action  $g(p, q) = (gp, gq)$ .

Take a manifold with corners  $M$  acted on by a Lie group  $G$ , with quotient  $\bar{M} := G \backslash M$ . Take a  $G$ -representation  $V$ . Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take a  $k$ -vector space  $V$  and a  $k$ -linear representation  $\rho: G \rightarrow \text{GL}(V)$ . Then  $G$  acts on  $M \times V$  by  $g(m, v) = (gm, gv)$ . Map  $(m, v) \in M \times V \mapsto m \in M$  and form the quotient map

$$\begin{array}{ccc} M \times V & \longrightarrow & M \times^G V \\ \downarrow & & \downarrow \\ M & \longrightarrow & \bar{M} \end{array}$$

**Theorem 58.5.** *If the  $\rightarrow$  are submersions then the  $\downarrow$  are vector bundles. To be precise, if  $M \rightarrow \bar{M}$  is a smooth submersion and  $M \times V \rightarrow M \times^G V$  is a smooth submersion, then  $M \times^G V \rightarrow \bar{M}$  is a vector bundle of rank equal to the dimension of  $V$ .*

*Proof.* At any given point  $m_0 \in M$ , add by  $(m_0, v) + (m_0, w) = (m_0, v + w)$ . When we move to another point on the same orbit,

$$g(m_0, v) + g(m_0, w) = (gm_0, gv) + (gm_0, gw) = g((m_0, v) + (m_0, w)),$$

so that addition commutes with the group action, so that we can add entire  $G$ -orbits through  $M \times V$  by this recipe. Subtraction similarly, and scaling by elements of  $k$ . So the set of all  $G$ -orbits in  $M \times V$  over a given  $G$ -orbit in  $M$  is a vector space. In other words, each fiber of  $M \times^G V \rightarrow \bar{M}$  is a vector space. The dimension of this vector space is that of  $V$ , since the  $G$ -orbit is determined by its element  $(m_0, v)$ , and  $v$  is arbitrary. So  $M \times V \rightarrow M \times^G V$  is a linear isomorphism from fibers over  $M$  to fibers over  $\bar{M}$ .

Since  $M \rightarrow \bar{M}$  is a smooth submersion, we can cover  $\bar{M}$  in open sets  $\bar{M}_a$ , with preimages  $M_a \subseteq M$  covering  $M$ , on each of which we have some section  $s_a: \bar{M}_a \rightarrow M_a$ . Pick any basis  $e_1, \dots, e_r$  of  $V$ . Define  $e_{aj}(\bar{m}) = G(s_a(\bar{m}), e_j)$ . Our sections are projections to  $M \times^G V$  of sections upstairs in  $M \times V$ . The quotient map  $M \times V \rightarrow M \times^G V$  is a smooth submersion, so adding and scaling are smooth.  $\square$

**Theorem 58.6.** *Suppose that  $P \rightarrow M$  is a principal left (right)  $G$ -bundle. Take a  $G$ -representation  $V$ . Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take a  $k$ -vector space  $V$  and a  $k$ -linear representation  $\rho: G \rightarrow \text{GL}(V)$ . The quotient  $P \times^G V \rightarrow M$  is a smooth  $k$ -vector bundle, the associated vector bundle of the representation.*

*Proof.* We give the proof for a left bundle; the obvious change of notation deals with a right bundle. Take the incidence relation, say  $R_P$  for  $G$  acting on  $P$ , and  $R_{P \times V}$  for  $G$  acting on  $P \times V$ . We can define a smooth map  $\gamma: (p, gp) \in R_P \mapsto g \in G$  since the bundle is locally trivial, and this is clear for trivial principal  $G$ -bundles. But then  $R_{P \times V} \subset P \times V \times P \times V$  consists of exactly the points

$$(p, v, p', \gamma(p, p')v),$$

so is a smooth closed embedded submanifold of  $P \times V \times P \times V$ . By theorem 14.6 on page 130, the quotient map  $q: P \times V \rightarrow P \times^G V$  is a smooth submersion of manifolds for a unique smooth structure on  $P \times^G V$ .  $\square$

### Homogeneous vector bundles

Take a homogeneous space  $(X, G)$  and a point  $x_0 \in X$ . Let  $H := G^{x_0}$  be the  $G$ -stabilizer of  $x_0$ . Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Take a  $k$ -vector space  $V$  and a  $k$ -linear representation  $\rho: H \rightarrow \text{GL}(V)$  of  $H$ . Since we know that  $g \in G \mapsto gx_0 \in X$  is a principal right  $H$ -bundle, the associated vector bundle  $G \times^H V$  is indeed a vector bundle, often called a *homogeneous vector bundle*, although strictly speaking it is not a homogeneous space if  $V$  has positive dimension. The left action of  $G$  on  $G \times V$ , by left translation on  $G$ , trivial on  $V$ , commutes with the right  $H$ -action, so descends to an action by vector bundle automorphisms on  $G \times^H V$ .

For  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$  and a  $k$ -vector space  $V$ , the Grassmannian  $X = \text{Gr}_p V$ , i.e. the set of all  $p$ -dimensional  $k$ -linear subspaces of  $V$  is a homogeneous space  $(X, G)$  for  $G = \text{GL}(V)$  the general linear group.

The stabilizer  $H = G^{P_0} \subset G$  of a  $p$ -dimensional linear subspace  $P_0 \subset V$  consists of the linear transformations of  $V$  which preserve  $P_0$ . Clearly  $P_0$  is itself a representation of  $H$ . Hence there is an associated vector bundle, the *universal bundle*  $U \rightarrow X$  (also called the *universal subbundle*, defined as  $U = G \times^H P_0$ ).

**58.17** Explain how to identify each fiber  $U_P$  of the universal bundle over each  $P \in X$  with  $P$  itself. So  $U$  is the bundle whose fiber over  $P$  is  $P$ .

The trivial bundle  $\underline{V} \rightarrow X$  on the Grassmannian is the bundle whose fiber at every point is  $V$ . The *universal quotient bundle*  $Q \rightarrow X$  is  $V/U$ .

**Theorem 58.7.** *Every rank  $p$   $k$ -vector bundle  $V \rightarrow M$ ,  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , on an  $n$ -dimensional manifold is pulled back from the universal bundle*

$$\begin{array}{ccc} W & \longrightarrow & U \\ \downarrow & & \downarrow \\ M & \longrightarrow & \text{Gr}_p k^{p+q} \end{array}$$

by some smooth map  $M \rightarrow \text{Gr}_p k^{p+q}$ , as long as

$$\frac{n}{d} - 1 < q$$

where  $d := \dim_{\mathbb{R}} k$ .

*Proof.* Embed  $V$  into the trivial bundle  $\underline{k}^{p+q}$ , so each fiber  $V_m$  sits as a linear subspace  $V_m \subset k^{p+q}$ . Let  $\varphi(m) := V_m$ , so  $U_{\varphi(m)} = V_m$ , and hence  $\varphi^* U_m = V_m$ .  $\square$

By this theorem, complex line bundles (i.e. complex vector bundles of rank 1) on the sphere  $S^2$  are pulled back from maps  $S^2 \rightarrow \text{Gr}_1 \mathbb{C}^2 = \mathbb{CP}^1 = S^2$ . Homotopy classes of such maps are classified by degree, so every complex line bundle on the sphere arises from pulling back the universal line bundle by a map  $S^2 \rightarrow S^2$  of some degree, say winding around the equator some number

of times. By looking at clutching around the equator, we see that different degree maps give different line bundles, not isomorphic, so a classification of complex line bundles on  $S^2$ .

The homotopy groups  $\pi_\ell(\mathbb{CP}^n)$  vanish for  $\ell = 3, 4, \dots, 2n$ , so all complex line bundles on  $S^\ell$  are trivial. For  $\ell = 2n + 1$ ,  $\pi_{2n+1}(\mathbb{CP}^n) = \mathbb{Z}$ , so the complex line bundles on  $S^{2n+1}$  are classified by an integer invariant.

A *homotopy* between two vector bundles  $V_0 \rightarrow M$  and  $V_1 \rightarrow M$  is a vector bundle  $V \rightarrow [0, 1] \times M$  with vector bundle isomorphisms

$$V|_{\{0\} \times M} \cong V_0, \quad V|_{\{1\} \times M} \cong V_1.$$

**Theorem 58.8.** *Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Let  $d := \dim_{\mathbb{R}} k$ . Take a manifold with corners  $M$ , say of dimension  $n$ . Suppose that  $q > \frac{n+1}{d} - 1$ . The map  $\varphi \mapsto \varphi^*U$  taking homotopy classes of maps  $M \rightarrow \text{Gr}_p k^q$  to homotopy classes of vector bundles is a bijection.*

*Proof.* Take any homotopy of vector bundles on  $M$ . Apply theorem 58.7 on the preceding page to  $[0, 1] \times M$  to realize the homotopy of vector bundles as the pullback  $\Phi^*U$  of a smooth map  $\Phi: [0, 1] \times M \rightarrow \text{Gr}_p k^{p+q}$ , i.e. a homotopy of maps to the Grassmannian.  $\square$



## Chapter 59

### Connections

#### Einstein convention

From here on, we adopt *Einstein notation*: if we write an expression such as  $a_j^i b_k^j$ , where some index  $j$  arises twice in a single term, once up and once down, then we mean  $\sum_j a_j^i b_k^j$ . This saves writing a great many summation signs.

#### Definition

In a trivial bundle  $V = M \times \mathbb{R}^r$ , a local section  $s$  is a map  $s(m) = (m, f(m))$ , so canonically identified with a function  $f(m)$ , i.e., a function  $f$  on an open subset of  $M$ . If we have a vector field  $X$  on  $M$  and a function  $f$  on  $M$  we can differentiate to compute  $\mathcal{L}_X f$ . A *connection* on a vector bundle is a linear differential operator which imitates  $\mathcal{L}_X$  on sections. To be precise, a connection on  $V \rightarrow M$  is a linear map  $\nabla$  taking each smooth section  $s$  of  $V$  to a smooth section  $\nabla s$  of  $T^*M \otimes V$ , so that  $\nabla(fs) = df \otimes s + f\nabla s$  for any continuously differentiable function  $f$  defined on the same open set where  $s$  is defined. For any vector field  $X$  on  $M$ , let  $\nabla_X s := X \lrcorner \nabla s$ .

Given a basis of local sections  $e_1, e_2, \dots, e_r$  of  $V$ , the various  $\nabla e_i$  are sections of  $T^*M \otimes V$ . In the basis, we can expand these into  $\nabla e_i = \gamma_i^j \otimes e_j$  for a matrix of 1-forms  $\gamma = (\gamma_i^j)$ , called the *connection 1-forms* of the basis. Conversely, given any 1-forms  $\gamma_i^j$ , defined on an open set where we have a basis of local sections, define a connection by writing each local section as  $s = s^i e_i$  (adopting the usual convention that components of a section are written as functions  $s^1, s^2, \dots, s^k$ ), and then let

$$\nabla s = \left( ds^i + \sum_j \gamma_j^i s^j \right) e_i.$$

In particular, at any zero of  $s$ ,  $\nabla s = ds$ . If we choose  $\gamma = 0$ , we have the *standard flat connection*

$$\nabla s = ds^i e_i.$$

**Lemma 59.1.** *Every vector bundle  $V \rightarrow M$  on any manifold  $M$  admits a connection.*

*Proof.* As above, we can locally construct connections from a basis of local sections and an arbitrary choice of connection 1-forms. Take an open cover of  $M$  by open sets  $M_a \subset M$ , on each of which we have a connection  $\nabla^a$  defined. Replace the open cover by a locally finite refinement and take a partition of unity  $h_a$ . Let

$$\nabla := h_a \nabla^a s.$$

□

Given connections on vector bundles  $V$  and  $W$ , both denoted  $\nabla$ , define a connection on  $V \oplus W$  by  $\nabla(s, t) = (\nabla s, \nabla t)$  and (if  $V$  and  $W$  are real or complex) on  $V \otimes W$  by  $\nabla(s \otimes t) = (\nabla s) \otimes t + s \otimes \nabla t$ .

Given a smooth map  $\varphi: P \rightarrow Q$  and a vector bundle  $V \rightarrow Q$  with connection  $\nabla$ , the *pullback connection*  $\varphi^*\nabla$  is the connection on  $\varphi^*V \rightarrow P$  given by

$$(\varphi^*\nabla)(\varphi^*s) = \varphi^*(\nabla s).$$

To see that the pullback connection exists, note that its connection 1-forms must be the pullback 1-forms, and then use them to determine a connection.

**59.1** Prove that there is a unique pullback connection of any connection by any smooth map of manifolds.

Define the dual basis  $e^1, e^2, \dots, e^n$  of  $V^*$  by

$$e^i(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then define the dual connection by  $\nabla e^i = -\gamma_j^i e^j$ .

**59.2** Prove that the dual connection is well defined independent of choice of basis of local sections. Hint:  $\nabla t(s) + t(\nabla s) = d(t(s))$  if  $s$  is a section of  $V$  and  $t$  a section of  $V^*$ .

## Parallel transport

A smooth section  $s$  of a vector bundle  $V \rightarrow M$  is *parallel* for a connection  $\nabla$  if  $\nabla s = 0$ .

Suppose that  $I \subset \mathbb{R}$  is an interval (closed, open, half-open, finite, or infinite) and  $V \rightarrow I$  is a vector bundle. Take a basis of local sections  $e_1, e_2, \dots, e_k$  so that every section is  $s = s^i e_i$  and

$$\nabla s = (ds^i + \gamma_j^i s_j) e_i.$$

Take a variable  $t \in \mathbb{R}$ . Write the connection 1-forms as

$$\gamma_j^i = a_j^i(t) dt.$$

A parallel section  $s$  throughout the domain of our basis of local sections is just precisely a solution of the linear system

$$\frac{ds^i}{dt} + a_j^i(t) s^j(t) = 0,$$

of ordinary differential equations. Conversely, every system of linear ordinary differential equations of first order has this form, so is the equation of parallel transport of a connection. By global existence and uniqueness for linear systems of ordinary differential equations, there is a unique smooth solution  $s(t)$  on any interval where our basis of local sections is defined.



**Lemma 59.2.** *If  $I \subset \mathbb{R}$  is an interval and  $V \rightarrow I$  is a vector bundle with connection then for any point  $t_0 \in I$  and any vector  $v_0 \in V_{t_0}$ , there is a unique parallel section  $s$  of  $V$  so that  $s(t_0) = v_0$ . For any  $t_1 \in I$ , the value of  $s(t_1)$  depends linearly on the choice of  $v_0$ , as a linear isomorphism, depending smoothly on  $t_1$ .*

*Proof.* Cover  $I$  in open intervals  $I_a$  on each of which we have a basis of local sections of  $V$ . If  $x$  lies in any  $I_a$ , we can define  $s$  on  $I_a$  as above by existence and uniqueness of solutions of linear ordinary differential equations. If  $I_a$  does not contain  $x$  but intersects some  $I_b$  which contains  $x$ , we extend  $s$  to  $I_a \cup I_b$  by taking the solution on  $I_a$  which agrees with the one on  $I_b$  at some point, and noting that on  $I_a \cap I_b$  it must agree with  $s$ . Therefore  $s$  extends to any finite union of  $I_a$  intervals, and any two such extensions agree where both are defined. In particular,  $s$  extends uniquely to any compact subinterval of  $I$ , and thereby extends uniquely to all of  $I$ . The smooth dependence follows from the smooth dependence in the existence and uniqueness of solutions of ordinary differential equations.  $\square$

Take a vector bundle  $V \rightarrow M$  with connection and a smooth map  $x: I \rightarrow M$  from an interval  $I \subset \mathbb{R}$ . Take two points  $t_0, t_1 \in I$  and the associated points  $m_0 := x(t_0)$  and  $m_1 := x(t_1)$  in  $M$ . The *parallel transport* of an element  $v_0 \in V_{m_0}$  along  $x$  is the value  $v_1 := s(t_1)$  of the parallel section  $s$  of  $x^*V \rightarrow I$ .

**Corollary 59.3.** *For any integer  $n \geq 0$ , every vector bundle on  $\mathbb{R}^n$  is trivial.*

*Proof.* Take a vector bundle  $V \rightarrow \mathbb{R}^n$ . Choose a connection. Given any point  $x \in \mathbb{R}^n$ , let  $\varphi_x: V_x \rightarrow V_0$  be the parallel transport along the path  $s_x(t) = (1-t)x$  along the interval  $[0, 1]$ , and thereby define an isomorphism  $\varphi: V \rightarrow M \times V_0$ .  $\square$

**Corollary 59.4.** *If a manifold with corners  $M$  admits an embedding into Euclidean space as a star shaped set, then every vector bundle on  $M$  is trivial.*

*Proof.* The same proof as for corollary 59.3.  $\square$

**Corollary 59.5.** *Suppose that  $\varphi, \psi: P \rightarrow Q$  are two smooth maps and there is a smooth homotopy between them. Then there is an isomorphism of vector bundles  $\varphi^*V \rightarrow \psi^*V$  for any vector bundle  $V \rightarrow Q$ .*

If  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the identity map and  $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the constant map, we see the triviality of vector bundles on  $\mathbb{R}^n$ .

**Theorem 59.6.** *Let  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . Let  $d := \dim_{\mathbb{R}} k$ . Take a manifold with corners  $M$ , say of dimension  $n$ . Suppose that  $q > \frac{n+1}{d} + p - 1$ . The map  $\varphi \mapsto V_\varphi$  taking homotopy classes of maps  $M \rightarrow \text{Gr}_p k^q$  to isomorphism classes of vector bundles is a bijection.*

**59.3** Suppose that  $P \subset \mathbb{R}^{n \times n}$  is the set of orthogonal matrices (the *orthogonal group*) and that  $Q \subset \mathbb{R}^{n \times n}$  is the set of invertible matrices (the *general linear group*). Use the solutions of problem 4.31 on page 48 and problem 8.10 on page 86 to prove that the identity map  $\varphi: q \in Q \mapsto q \in Q$  admits a smooth homotopy to a smooth map  $\psi: Q \rightarrow P$ . Use this to prove that  $\psi^*$  is an isomorphism on the set of vector bundles modulo isomorphism; roughly speaking, the two manifolds have the same vector bundles.

Let  $M$  be the circle and  $\varphi: \theta \in \mathbb{R} \rightarrow e^{i\theta} \in M$ . Given a vector bundle  $V \rightarrow M$  and a connection  $\nabla$ , the bundle  $\varphi^*V \rightarrow \mathbb{R}$  admits a basis of global parallel sections  $e_1, e_2, \dots, e_r$ . Fibers are canonically identified:  $\varphi^*V_\theta = V_{\varphi(\theta)} = \varphi^*V_{\theta+2\pi}$ . Under this identification, the  $e_1, e_2, \dots, e_r$  transform to some  $\underline{e} = eg$ , some element  $g \in \text{GL}(r, k)$ , giving a map  $\psi: \mathbb{R} \rightarrow \text{GL}(r, k)$ . Since  $\underline{e}$  is also parallel, being the same local basis on the circle  $M$ , we find that  $0 = \nabla \underline{e} = (\nabla e)g + e dg = e dg$ , i.e.  $g$  is constant, the *holonomy* of the connection  $\nabla$ . We have an isomorphism of vector bundles

$$V = \varphi^*V/\sim = V_g,$$

where  $v \in \varphi^*V_\theta \sim gv \in \varphi^*V_{\theta+2\pi}$  giving the bundle  $V_g$  we saw previously. Therefore every vector bundle on the circle, with connection, is a quotient of the trivial bundle on  $\mathbb{R}$  by action of an element  $g \in \text{GL}(r, k)$ :

$$(\theta, v) \sim (\theta + 2\pi, gv).$$

If we write this vector bundle with connection as  $V^g \rightarrow M$  with connection  $\nabla^g$ , then clearly under conjugation in  $\text{GL}(r, k)$ ,  $V^{hgh^{-1}} \cong V^g$  are isomorphic vector bundles with connection by  $(\theta, v) \mapsto (\theta, hv)$ , for any  $h \in \text{GL}(k, \mathbb{R})$ . On the other hand, the holonomy of a vector bundle with connection on the circle is clearly uniquely determined up to such conjugation, i.e. up to choice of basis of any one fiber. Therefore we have the classification of vector bundles with connection on the circle, up to isomorphism preserving the connection: the quotient of  $\text{GL}(r, k)$  by its conjugation action on itself. Note that this space is not Hausdorff.

Let  $M$  be the circle and  $V_0$  a  $k$ -vector space,  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ . On the manifold  $P := M \times \text{GL}(V_0)$ , consider the vector bundle  $V \rightarrow P$  given as the quotient

$$(\theta, g, v) \sim (\theta + 2\pi, g, gv)$$

for any

$$(\theta, g, v) \in \mathbb{R} \times \text{GL}(V_0) \times V_0.$$

This vector bundle pulls back to the fiber over each  $g$  in  $\text{GL}(V_0)$  to become the vector  $V_g \rightarrow M$  we saw in the previous example. If two linear transformations  $g_0, g_1$  in  $\text{GL}(V_0)$  are connected by a smooth path in  $\text{GL}(V_0)$ , then there is a smooth homotopy identifying the vector bundles  $V_{g_0} \rightarrow M$  and  $V_{g_1} \rightarrow M$ . For  $k = \mathbb{R}$ , any two linear transformations  $g_0, g_1 \in \text{GL}(V_0)$  are connected by a smooth path just they have the same sign of their determinant. So there are precisely two isomorphism classes of real rank  $N$  vector bundles on the circle, isomorphic to the trivial bundle and isomorphic to the sum of a trivial bundle and a Möbius strip. There is one isomorphism class of complex or quaternionic rank  $N$  vector bundle: trivial.

The vector bundles on the punctured plane  $\mathbb{R}^2 - \{0\}$  restrict to the circle. As in problem 59.3 on page 507, this restriction identifies the set of vector bundles modulo isomorphism. From our last example, we see the identification of the vector bundles on the punctured plane.

The vector bundles on the sphere  $M := S^n \subset \mathbb{R}^{n+1}$  are trivial away from the north pole  $m_0$ , because the punctured sphere is diffeomorphic to Euclidean space. Similarly, the vector bundles are trivial away from the south pole  $m_1$ . If we let  $M_{01} \subset M$  be the sphere punctured at the north and south pole, then the two trivial vector bundles on  $M_0 = M - \{m_0\}$  and  $M_1 = M - \{m_1\}$  glue together along  $M_{01}$  to form our vector bundle on  $M$ , by some isomorphism over  $M_{01}$ , i.e. a transition map  $M_{01} \rightarrow \text{GL}(V_0)$ . If we deform the transition map by a smooth homotopy, we don't change the vector bundle. The trivialization of the bundle away from the south pole is only defined up to vector bundle isomorphism, i.e. up to changing by a smooth map  $M_0 \rightarrow \text{GL}(V_0)$ . Similarly for the south pole.

By Ptolemaic projection, identify  $M_{01} = \mathbb{R}^n - \{0\} = \mathbb{R} \times S^{n-1}$ . Clearly we can use a smooth homotopy to make our transition map constant in the radial direction, so the classification of vector bundles on the sphere  $S^n$  is a quotient of the classification of smooth maps  $g: S^{n-1} \rightarrow \text{GL}(V_0)$  up to smooth homotopy and up to  $g \sim bga^{-1}$  for smooth maps  $a, b: S^{n-1} \rightarrow \text{GL}(V_0)$  that extend across the north pole or the south pole smoothly. A smooth map  $g: S^{n-1} \rightarrow \text{GL}(V_0)$  extends continuously, hence smoothly (by smoothing), from the equator to  $M_0$  just when it extends to the upper hemisphere smoothly, i.e. extends from  $S^{n-1}$  to the closed ball, i.e. just when  $g$  is trivial in homotopy. So if  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , the  $k$ -vector bundles on  $S^n$  of rank  $N$  are classified by homotopy groups

$$\pi_{n-1}(\text{GL}(N, k)) = \begin{cases} \pi_{n-1}(\text{O}(N)), & \text{if } k = \mathbb{R}, \\ \pi_{n-1}(\text{U}(N)), & \text{if } k = \mathbb{C}, \\ \pi_{n-1}(\text{Sp}(N)), & \text{if } k = \mathbb{H}. \end{cases}$$

As a simpler example, consider rank 1 real vector bundles on spheres. As above, they are given by smooth homotopy classes of transition maps  $S^{n-1} \rightarrow \text{GL}(1, \mathbb{R}) = \mathbb{R}^\times$ . We can retract  $\mathbb{R}^\times$  down to  $\pm 1$ , and smooth maps  $S^{n-1} \rightarrow \pm 1$  are just locally constant maps, i.e. 1 or  $-1$  constant. But each of these extends as a constant map across the whole of  $S^n$  if  $n \geq 2$ . So every rank 1 vector bundle on  $S^n$  is trivial if  $n \geq 2$ , and if  $n = 1$ , we get the Möbius strip as the total space of the only nontrivial line bundle on the circle.

## Product neighborhoods

As a simple example of the power of the concept of trivial vector bundle, we look for product neighborhoods. A *product neighborhood* of an embedding  $\varphi: P \rightarrow Q$  is an

embedding  $\Phi: P \times \bar{B} \rightarrow Q$  so that  $\bar{B}$  is the closed unit ball in Euclidean space around the origin with  $\dim P + \dim \bar{B} = \dim Q$  and  $\Phi(x, 0) = \varphi(x)$  for all  $x \in P$ .

**Theorem 59.7** (Product neighborhoods). *Suppose that  $\varphi: P \rightarrow Q$  is an injective immersion of a compact manifold with corners into the interior of a manifold with corners. Then  $\varphi$  has trivial normal bundle just when  $\varphi$  admits a product neighborhood. If  $Q$  is oriented and  $P$  has lower dimension than  $Q$ , we can arrange that the product neighborhood is an orientation preserving diffeomorphism.*

*Proof.* Note that  $\varphi$  is an injective immersion of a compact manifold with corners hence an embedding. Suppose that  $P$  has dimension  $p$  and  $Q$  has dimension  $p + q$ .

**59.4** If there is a product neighborhood of  $\varphi$ , prove that  $\varphi$  has trivial normal bundle.

Suppose that we have a basis of sections of the normal bundle. Let  $X^n$  be the standard  $n$ -dimensional corner. Each point  $p_0 \in P$  has image  $q_0 = \varphi(p_0)$  in the domain of a chart  $(U, \psi)$  “flattening”  $\psi(P)$ , i.e. so that

$$\psi(P \cap U) = X^p \cap \psi(U) \subset X^{p+q}.$$

Our sections of the normal bundle are mapped by  $\psi_*$  to sections of the normal bundle of  $X^p$  in  $X^{p+q}$ , i.e. to vectors in  $\{0\} \times \mathbb{R}^q$ , which we identify with vector fields in  $\psi(U) \cap X^{p+q}$ , and hence identify by  $\psi_*^{-1}$  with vector fields on  $U$ .

So we can cover  $\varphi(P) \subset Q$  in open sets  $Q_a \subseteq Q$ , on each of which there are vector fields  $v_{a1}, \dots, v_{aq}$  which, at each point of  $P \cap Q_a$ , map to our basis sections in the normal bundle.

With a partition of unity, we can therefore construct global vector fields  $v_1, \dots, v_q$  on  $Q$ , which at each point of  $P$  map to our given sections in the normal bundle. By problem 32.2 on page 257, we can scale each of them by a positive function to make them complete. Define

$$\Phi: (x, y) \in P \times \mathbb{R}^q \mapsto e^{y_1 v_1 + \dots + y_q v_q} \varphi(x) \in Q,$$

a smooth map and a local diffeomorphism near  $\varphi$ .

**59.5** Prove that  $\Phi$  restricts to some product  $P \times \bar{B}$  to be a local diffeomorphism for some closed ball  $\bar{B}$  around the origin.

**59.6** Prove that  $\Phi$  restricts to some product  $P \times \bar{B}$  to be a diffeomorphism for some closed ball  $\bar{B}$  around the origin.

To fix the orientation, we might need to change the sign on one of the sections of the normal bundle. □

**Theorem 59.8** (Palais’s extension theorem). *Take a manifold  $M$ , say of dimension  $n$ , an open set  $U \subseteq M$  and an injective immersion  $\varphi: \bar{B} \rightarrow U$  of a closed ball  $\bar{B} \subset \mathbb{R}^k$  for some  $k < n$ . Let  $\bar{B}'$  be the ball in  $\mathbb{R}^n$  with same center as  $B$  and with  $B \times \{0\} = \bar{B}' \cap \mathbb{R}^k \times \{0\}$ . There is an embedding  $\Phi: \bar{B}' \rightarrow M$  so that  $\Phi(x, 0) = \varphi(x)$  for any  $x \in B$ . If  $M$  is oriented we can arrange that  $\Phi$  is orientation preserving.*

**59.7** Prove Palais’s extension theorem.

## Metric compatible connections

Suppose that  $V \rightarrow M$  is a vector bundle (real, complex or quaternionic) with an inner product. A connection  $\nabla$  for  $V$  is *compatible* with that inner product if parallel transport preserves inner products. Any basis of local sections  $e_1, \dots, e_r$  has inner products

$$g_{i\bar{j}} := \langle e_i, e_j \rangle.$$

Writing  $t^{\bar{j}}$  to mean the conjugate of  $t^j$ . Let

$$\gamma_{i\bar{j}} := \gamma_i^k g_{k\bar{j}}$$

and

$$\gamma_{\bar{j}i} := \overline{\gamma_{j\bar{i}}}.$$

be the conjugate of  $\gamma_{j\bar{i}}$ . So the inner product of local sections  $s, t$  defined along a curve is,

$$\begin{aligned} 0 &= d\langle s, t \rangle, \\ &= d(g_{i\bar{j}} s^i t^{\bar{j}}), \\ &= dg_{i\bar{j}} s^i t^{\bar{j}} + g_{i\bar{j}} ds^i t^{\bar{j}} + g_{i\bar{j}} s^i dt^{\bar{j}}, \\ &= dg_{i\bar{j}} s^i t^{\bar{j}} + g_{i\bar{j}} (-\gamma_k^i s^k) t^{\bar{j}} + g_{i\bar{j}} s^i (-\gamma_{\bar{k}}^{\bar{j}} t^{\bar{k}}), \\ &= (dg_{i\bar{j}} - \gamma_{i\bar{j}} - \gamma_{\bar{j}i}) s^i t^{\bar{j}}. \end{aligned}$$

Hence a connection is compatible for a metric just when

$$dg_{i\bar{j}} = \gamma_{i\bar{j}} + \gamma_{\bar{j}i}.$$

A basis of local sections  $e_1, \dots, e_r$  is *orthonormal* if its inner products are

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Gram-Schmidt algorithm produces an orthonormal basis from any basis of local sections.

**Lemma 59.9.** *Every metric admits a compatible connection.*

*Proof.* Cover our manifold in open sets  $M_a$  on which we have orthonormal bases of local sections. On each  $M_a$ , pick any 1-forms  $\gamma_a = (\gamma_j^i)$  for which the associated

$$\gamma_{i\bar{j}} := \gamma_i^k \delta_{k\bar{j}}$$

are skew-Hermitian and construct the associated connection

$$\nabla^a s = (ds^i + \gamma_j^i s^j) e_i.$$

Then each connection preserves the metric along its parallel transports. Hence on the overlaps, each connection has skew-symmetric connection 1-forms in any orthonormal basis.

Take a partition of unity, consisting of real valued functions  $h_a$ , and let

$$\nabla = h_a \nabla_a.$$

On each open set, in its orthonormal basis,  $\nabla^a = d + \gamma^a$  for a skew-symmetric  $\gamma^a$ , and so

$$\nabla = d + \gamma = d + h_a \gamma^a,$$

has skew-symmetric  $\gamma = h_a \gamma^a$ . □

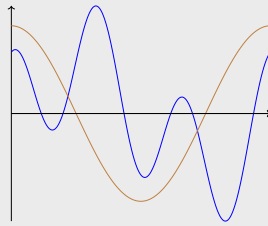
## Chapter 60

### Characteristic classes

#### The Möbius strip

A big idea: we can distinguish different vector bundles on the same manifold by the “typical geometry” of the zero locus of a “typical” section of each vector bundle.

Take the trivial bundle on the circle: sections are just precisely periodic smooth functions.



Count *multiplicity* of a zero as being  $k$  at a point  $\theta = \theta_0$  if

$$f(\theta) = c(\theta - \theta_0)^k + o(\theta - \theta_0)^{k+1},$$

for some  $c \neq 0$  as  $\theta \rightarrow \theta_0$ . A zero has *infinite order* if

$$f(\theta) = o(\theta - \theta_0)^k,$$

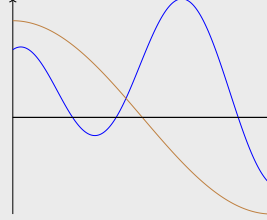
for every positive integer  $k$ . A *simple zero* is one with multiplicity one.

**Lemma 60.1.** *If a smooth section of the trivial line bundle over the circle has no infinite order zeroes then the sum of the multiplicities of the zeroes is even. Any even sum can occur.*

*Proof.* Each zero that changes sign has odd order. We need an even total number of sign changes to get back to some nonzero value where we started.  $\square$

Sard’s theorem: every zero of finite order  $k$  becomes at most  $k$  simple zeroes, nearby, after we add a small enough nonzero constant. For example, a triple zero can become just one simple zero.

Recall that the Möbius strip is the set of pairs  $(\theta, y)$  of angle and real number  $y$  modulo the equivalence that  $(\theta, y) \sim (\theta + 2\pi, -y)$ . A section of the Möbius strip is a map  $y = f(\theta)$  so that  $f(\theta + 2\pi) = -f(\theta)$ .



Any finite sum

$$y = f(\theta) = \cos\left(\frac{2k+1}{2}\theta\right)$$

or

$$y = f(\theta) = \sin\left(\frac{2k+1}{2}\theta\right)$$

is a section, as are any linear combinations of these. The simplest example is

$$y = f(\theta) = \cos\left(\frac{\theta}{2}\right),$$

which has a zero at precisely at one point,  $\theta = \pi$ . The other sections we wrote down have more zeroes. Clearly we have found a section  $y = \cos(\frac{\theta}{2})$  of the Möbius strip with a single zero, and it is a simple zero.

**Lemma 60.2.** *If a smooth section of the nontrivial line bundle over the circle (i.e. the Möbius strip) has no infinite order zeroes then the sum of the multiplicities of the zeroes is odd. Any odd sum can occur.*

*Proof.* Each zero that changes sign has odd order. We need an odd total number of sign changes to get a single sign reversal on some nonzero value where we started.  $\square$

In complex analysis, we count the zeroes minus poles of a complex analytic function  $f(z)$  inside a domain  $D$  of the complex plane with

$$\int_{\partial D} \frac{df}{2\pi i f},$$

if  $f$  has no zeroes or poles on  $\partial D$  and a finite set of zeroes and poles inside  $D$ , each of finite order.

Consider how we prove this; recall our discussion on page on page 221. In terms of complex valued 1-forms, a function  $f(z)$  is complex analytic just when

$$df = f'(z)dz$$

for some complex valued smooth function  $f'(z)$ , its complex derivative.

In the complex plane with coordinate  $w$ , consider the differential form

$$\xi = \frac{dw}{2\pi i w}.$$



Check that  $d\xi = 0$ . We claim that

$$\int_C \xi = 1$$

around the circle  $C = \partial\Delta$  which is the boundary of a disk  $\Delta$  around the origin. To see this, parameterize the disk as  $w = r_0 e^{i\theta}$ , for  $r_0$  the radius of the disk, and compute

$$\int_C \xi = \int_0^{2\pi} \frac{r_0 e^{i\theta} i d\theta}{2\pi r_0 e^{i\theta}} = 1.$$

By Stokes's theorem,  $\int_{\partial\Delta} \xi = 1$  around the boundary of any bounded domain containing the origin.

Writing  $w = f(z)$ , the differential form

$$\omega = \omega_f := f^* \xi = \frac{df}{2\pi i f}$$

is closed, as  $d\omega = f^* d\xi = 0$ . Take a circle in the  $z$ -plane, which avoids any zeroes or poles of  $f$ . Then  $f$  wraps that circle some integer number of times around the origin in the  $w$ -plane, and clearly  $\int \omega$  counts that number. Lets prove that carefully.

Suppose that  $f$  has only discrete zeroes and poles inside  $D$ . Take some closed disks  $\Delta_j$  around the zeroes and poles of  $f$ , each containing only one zero or pole. Let  $D' := D - \bigcup \Delta_j$ . We can't apply Stokes's theorem to  $D$ , because of the zeroes and poles of  $f$ , where  $\omega$  is not smooth. But we can apply Stokes's theorem on  $D'$ :

$$\int_{\partial D'} \omega = \int_{D'} d\omega = 0.$$

But  $\partial D' = \partial D - \sum \partial \Delta_j$ , so

$$\int_{\partial D} \omega = \sum \int_{\partial \Delta_j} \omega.$$

This reduces our problem to each zero or pole separately, with domain some possibly small disk  $\Delta$ . We can assume our zero or pole is at the origin. In complex analysis, we learn that  $f(z)$  has a zero of order  $k$ , or pole of order  $-k$ , at the origin just when  $f(z) = z^k g(z)$  where  $g(z)$  is analytic with  $g(0) \neq 0$ . Check that  $\omega_f = k\omega_z + \omega_g$ . Make the disk so small that  $g$  has no zeroes or poles in it, so  $\omega_g$  is smooth across the disk, and closed:

$$\int_{\partial \Delta} \omega_g = \int_{\Delta} d\omega_g = 0.$$

Note that  $\omega_z$  is just  $\xi$  written in the  $z$ -plane.

These two examples are very different: the vector bundles on the circle are distinguished by their being a different number of zeroes modulo 2 of the generic section. For a domain in the complex plane, there are also poles, and the number of zeroes minus poles is an integer, not just defined modulo 2, taking on arbitrary values, and

the vector bundle is the trivial complex line bundle.

By Ptolemaic projection, we identify the sphere punctured at the north pole with the complex plane parameterized by a complex coordinate  $z$ . Similarly, we identify the same sphere punctured at the south pole with the complex plane parameterized by a complex coordinate  $w = 1/z$ . Any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

becomes

$$p(1/w) = a_0 + \frac{a_1}{w} + \cdots + \frac{a_n}{w^n}.$$

But then  $w^n p(1/w)$  becomes the same polynomial in reverse

$$w^n p(1/w) = a_0 w^n + a_1 w^{n-1} + \cdots + a_n.$$

So clutching by the map  $w \mapsto w^n \in \text{GL}(1, \mathbb{C})$  on the overlap  $z \neq 0$  defines a complex line bundle traditionally called  $\mathcal{O}(n)$  over the Riemann sphere  $\mathbb{CP}^1$ . Every polynomial of degree  $n$   $p(z)$  defines a section. There are  $n$  zeroes to any such polynomial, with multiplicity, typically all distinct. So when we count zeroes of sections, we find the integer  $n$ , which is the degree of the clutching function. The degree of a section, measured as the sum of indices of its zeroes, we will see is also  $n$ .

By the hairy ball theorem (theorem 22.7 on page 193), the tangent bundle of the sphere admits nowhere vanishing sections just on odd-dimensional spheres. On any sphere, consider the vector field  $x \mapsto X(x) = Ax$  where  $A$  is any antisymmetric matrix. This vector field is tangent to the sphere, as the reader can check. In an odd-dimensional Euclidean space  $\mathbb{R}^{2n+1}$ ,  $A$  has a kernel of dimension at most one; we can pick some  $A$  with kernel of dimension one. Then  $X(x) = Ax$  has two zeroes on the sphere.

**60.1** Find the index of that vector field on the sphere at each of its two zeroes.

Take an oriented vector bundle on a compact oriented manifold. If a section is transverse to the zero section, as a submanifold of the vector bundle, then their intersection is a chain, i.e. the zeroes of that section form a chain on the manifold. The homology class of that chain is independent of the section, as any two transverse sections are homologous by a transverse homotopy. If we don't have orientations, we can still intersect, but only in homology with  $\mathbb{Z}/2\mathbb{Z}$  coefficients, so that we don't need an orientation.

For an oriented real vector bundle of rank  $n$  on a compact oriented  $n$ -dimensional manifold, take a section transverse to the zero section. We have computed that the index is the equal to  $\pm 1$ , depending on whether the linearization  $ds$  is orientation preserving or reversing. Hence the zero locus in homology of the generic section is a sum of oriented points, and the index is the intersection number. Since we can perturb any section to be nondegenerate, without altering the index (which we compute in some sphere away from the small perturbation) we find that the index is the intersection number for any smooth section with a finite set of zeroes, even without assuming the section transverse to the zero section.

For a compact oriented manifold with corners, we will have to ensure that there are no zeroes on the boundary, or some similar condition, to be able to count.

## Curvature

Take a vector bundle. If you use a basis of local sections  $e_1, e_2, \dots, e_r$  and I use  $\underline{e}_1, \underline{e}_2, \dots, \underline{e}_r$ , then there is a unique invertible matrix-valued function on the overlap of the domains of our bases, say  $g = (g_j^i)$ , so that

$$\underline{e}_j = g_j^i e_i.$$

Therefore

$$\nabla \underline{e}_j = (dg_j^i + g_j^k \gamma_k^i) e_i.$$

Finally, this tells us that the connection 1-forms of  $e$  and  $\underline{e}$  are related by

$$\underline{\gamma}_j^k g_k^i = dg_j^i + g_j^k \gamma_k^i$$

We can write this symbolically as  $\underline{e} = eg$  and

$$g\underline{\gamma} = dg + \gamma g.$$

We don't like the  $dg$  term here; it says that in order to compute the connection 1-forms in the new basis, you need to compute the derivative of the change of basis. First solve for the  $dg$  evil term:

$$dg = g\underline{\gamma} - \gamma g.$$

Next, differentiate both sides, and plug back in this equation wherever you hit a  $dg$ :

$$0 = g(d\underline{\gamma} + \underline{\gamma} \wedge \underline{\gamma}) - (d\gamma + \gamma \wedge \gamma)g,$$

by noting that  $\gamma g \wedge \underline{\gamma} = \gamma \wedge g\underline{\gamma}$ . The *curvature form*  $\Omega := d\gamma + \gamma \wedge \gamma$  therefore transforms from one basis  $e$  of local sections to another  $\underline{e}$  by

$$g\underline{\Omega} = \Omega g.$$

The essential fact is that there is no dependence on  $dg$ , so that we can calculate the curvature form at a point, and it will transform into another basis by a linear transformation. In particular, vanishing of curvature form is a condition independent of basis.

Differentiating the equation  $\Omega = d\gamma + \gamma \wedge \gamma$ , the curvature form  $\Omega$  satisfies the *Bianchi identity*

$$d\Omega = \Omega \wedge \gamma - \gamma \wedge \Omega.$$

Let us get rid of the dependence of  $\Omega$  on the choice of basis of local sections. Define, for any two tangent vectors  $u, v \in T_m M$  and any local section  $s = s^i e_i$ , a local section  $R(u, v)s$  by

$$R(u, v)s = s^j \Omega_j^i(u, v) e_i.$$

Call this operation  $R$  the *curvature*; the curvature of a connection  $\nabla$  on a vector bundle  $V \rightarrow M$  is a 2-form on  $M$  valued in  $V^* \otimes V$ . Check that if we change basis of local sections from  $e$  to  $\underline{e} = eg$ , we change curvature to

$$\underline{R}(u, v)s = R(u, v)s,$$

so the curvature is defined independently of the choice of basis of local sections. We will often write  $R$  also as  $\Omega$  to save letters.

Similarly, the expression  $\gamma$  depends on choice of basis of local sections. If we have two connections  $\nabla_0, \nabla_1$ , we would like to define their difference to be  $\gamma_1 - \gamma_0$ , from a basis of local sections. Given a local section  $s$ , expand out to see that

$$\nabla_1 s - \nabla_0 s = s^j (\gamma_{1j}^k - \gamma_{0j}^k) e_k$$

so we can define the difference form of the two connections, independent of basis of local sections, as the form  $\nabla_1 - \nabla_0$ , a 1-form valued in  $V^* \otimes V$ , so that

$$\nabla_1 s - \nabla_0 s = (\nabla_1 - \nabla_0)s$$

for any local section  $s$ . In any basis of local sections, this difference is  $\gamma_1 - \gamma_0$ .

For the reader who has read chapter 53 on page 439, the expression

$$\nabla_v \xi_i = -(v \lrcorner \eta_{ij}) \xi_j$$

determines a connection on the cotangent bundle, dual to the connection

$$\nabla_v e_i = (v \lrcorner \eta_{ij}) e_j,$$

on the tangent bundle, called the *Levi-Civita connection*.

**60.2** Prove that the connections are defined (as connections in the sense of this chapter) independently of the choice of orthonormal coframing.

Recall that, for a connection compatible with a metric, we defined

$$\gamma_{i\bar{j}} := \gamma_i^k \delta_{k\bar{j}}$$

We analogously define

$$\Omega_{i\bar{j}} := \Omega_i^k \delta_{k\bar{j}}.$$

**Lemma 60.3.** *The curvature of any connection compatible with a metric is a 2-form valued in skew symmetric linear maps.*

By skew symmetric we mean of course skew Hermitian for a complex vector bundle, and similarly for a quaternionic one.

*Proof.* Since  $\gamma$  is skew-symmetric, so is  $d\gamma$  and  $\gamma \wedge \gamma$ , and so  $\Omega$ . □

## Lie brackets and wedge products

We will find it convenient to define, for 1-forms,

$$[\alpha\beta]_j^i = \alpha_k^i \wedge \beta_j^k + \beta_k^i \wedge \alpha_j^k,$$

and in particular  $[\alpha\beta] = [\beta\alpha]$ . For a 2-form  $\Omega$  and a 1-form  $\omega$  valued in square matrices, define

$$[\Omega\omega]_j^i = \Omega_k^i \wedge \gamma_j^k - \gamma_k^i \wedge \Omega_j^k.$$

More generally, the Lie bracket of matrix-valued differential forms of degrees  $p$  and  $q$ , denoted

$$[\alpha\beta](v_1, v_2, \dots, v_{p+q})$$

is defined to mean

$$\frac{1}{p!q!} \sum_{\pi} (-1)^{\pi} [\alpha(v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(p)}), \beta(v_{\pi(p+1)}, v_{\pi(p+2)}, \dots, v_{\pi(p+q)})].$$

In this notation, the Bianchi identity is

$$d\Omega = [\Omega\gamma].$$

### Characteristic forms of vector bundles

Take  $P(A)$  a homogeneous polynomial function of a square matrix variable  $A$ , invariant under change of basis  $A \mapsto g^{-1}Ag$ . Denote its associated symmetric multilinear form by the same symbol  $P$ . Denote by  $P(A, B)$  the expression

$$P(A, B) = P(A, A, \dots, A, B).$$

Differentiate the condition that  $P$  is invariant under change of basis to discover that, as a function of entries  $A_1, A_2, \dots, A_k$ ,

$$0 = P([BA_1], A_2, \dots, A_k) + P(A_1, [BA_2], \dots, A_k) + P(A_1, A_2, \dots, [BA_k]).$$

Writing out explicitly the definition of the wedge product, one then sees that

$$0 = \sum_i P(\Omega_1, \dots, [\Omega_i\gamma], \dots, \Omega_k),$$

if all of the  $\Omega_i$  are even degree differential forms and  $\gamma$  is a 1-form, all valued in square matrices of the same size.

If we replace our basis of local sections by  $\underline{e} = eg$ , then  $\underline{\Omega} = g^{-1}\Omega g$ , so the differential form  $P(\Omega)$ , the *characteristic form*, is independent of the choice of basis of local sections, depending only on the connection, so we write it as  $P(\nabla)$ .

**Lemma 6o.4.** *For any connection  $\nabla$  on any vector bundle  $V \rightarrow M$  and any invariant polynomial  $P$ , the form  $P(\nabla)$  is closed.*

*Proof.* Compute

$$\begin{aligned} dP(\Omega) &= \sum_i P(\Omega, \dots, d_i\Omega, \dots, \Omega), \\ &= \sum_i P(\Omega, \dots, [\Omega\gamma], \dots, \Omega), \\ &= 0. \end{aligned}$$

□

### Comparing two connections

Take two vector bundles  $V_0 \rightarrow M$  and  $V_1 \rightarrow M$  with connections  $\nabla_0$  and  $\nabla_1$ . Suppose that there is a homotopy between them, i.e. a vector bundle  $V \rightarrow [0, 1] \times M$  with vector bundle isomorphisms

$$V|_{\{0\} \times M} \cong V_0, \quad V|_{\{1\} \times M} \cong V_1.$$

Pick a connection  $\dot{\nabla}$  on  $V$  so that it pulls back to  $\nabla_0$  on  $V_0$  and to  $\nabla_1$  on  $V_1$ .

**60.3** Prove that such a connection  $\dot{\nabla}$  exists.

Call  $\dot{\nabla}$  a *homotopy* between the connections  $\nabla_0$  and  $\nabla_1$ . Parallel transport via  $\dot{\nabla}$  identifies them, so  $V_0 \cong V_1$ : vector bundles are homotopic if and only if isomorphic. Let  $\pi: (t, m) \in [0, 1] \times M \mapsto m \in M$ .

If  $V_0 = V_1$  is a vector bundle with two different connections  $\nabla_0$  and  $\nabla_1$ , we can let  $V := \pi^*V_0$  and define a connection  $\dot{\nabla}$  on  $V$  by

$$\dot{\nabla}s = \frac{\partial s}{\partial t} dt + (1-t)\nabla_0 s + t\nabla_1 s,$$

giving the *standard homotopy* between  $\nabla_0, \nabla_1$ .

**60.4** Calculate the curvature  $\dot{\Omega}$  of  $\dot{\nabla}$  of the standard homotopy.

The *Bott difference form* of a homotopy of connections is

$$\dot{P}(\dot{\nabla}) := \pi_* P(\dot{\nabla}).$$

By Fubini's theorem (theorem 30.1 on page 239), with a suitable orientation of  $[0, 1] \times M$ ,

$$d\dot{P}(\dot{\nabla}) = P(\nabla_1) - P(\nabla_0).$$

Therefore the cohomology class  $[P(\nabla_0)]$  in de Rham cohomology, written as  $P(V)$  and called a *characteristic class*, depends only on the vector bundle  $V$  up to homotopy, and is independent of the connection. For example, a trivial vector bundle has  $P(V) = 0$  for all  $P$  of positive degree.

**60.5** Suppose that  $P(A) = \text{tr } A$ . For the standard homotopy, express  $\dot{P}(\dot{\nabla})$  as a differential form in terms of some basis of local sections.

Note that the Bott difference form can depend on the choice of homotopy between the connections. If  $\dot{\nabla}_0$  and  $\dot{\nabla}_1$  are two connections on  $V = \pi^*V_0$ , both of which are homotopies between the connections  $\nabla_0$  and  $\nabla_1$ , and we write the associated Bott difference forms as

$$\dot{P}(\dot{\nabla}_0) := \pi_* P(\dot{\nabla}_0), \quad \dot{P}(\dot{\nabla}_1) := \pi_* P(\dot{\nabla}_1),$$

then we can make a homotopy connection  $\ddot{\nabla}$  between them and clearly

$$\dot{P}(\dot{\nabla}_1) - \dot{P}(\dot{\nabla}_0) = d\ddot{P}(\ddot{\nabla}).$$

for a homotopy  $\check{\nabla}$  between them. So the Bott difference form depends on the homotopy between connections only by an exact form. Suppose that  $P$  has degree  $d$  as a homogeneous polynomial. For any pair of connections  $\nabla_0, \nabla_1$ , let

$$P(\nabla_0, \nabla_1) \in \Omega^{2d-1}(M)/d\Omega^{2d-2}(M)$$

be the Bott difference  $\dot{P}(\dot{\nabla})$  for any homotopy between them (which might not be closed), but only defined up to adding an exact differential form. Note that by pasting paths together,

$$P(\nabla_0, \nabla_1) + P(\nabla_1, \nabla_2) = P(\nabla_0, \nabla_2).$$

Vector bundles are isomorphic just when they are homotopic. So  $P(\nabla_0, \nabla_1)$  is defined for any pair of vector bundles  $V_0, V_1$  equipped with connections, as long as there is some isomorphism between  $V_0$  and  $V_1$ , even if we don't know how to exhibit that isomorphism.

In particular, if we take a homotopy  $\check{\nabla}$  from a connection  $\nabla_0$  to another connection  $\nabla_1$ , and if  $P(\nabla_0) = P(\nabla_1)$ , then  $P(\nabla_0, \nabla_1)$  is closed and its cohomology class is an invariant of the pair  $\nabla_0, \nabla_1$  of connections on the pair of vector bundles  $V_0, V_1$ .

## Čech–de Rham cohomology

Take a vector bundle  $V \rightarrow M$  on a manifold  $M$  and cover  $M$  by two open sets  $M_0, M_1 \subset M$ . Let  $V_0$  be the pullback of  $V \rightarrow M$  to  $M_0$  and let  $V_1$  be the pullback of  $V \rightarrow M$  to  $M_1$ . Denote the pair of vector bundles as  $V_\bullet$ . Take a connection  $\nabla_0$  on  $V_0$  and a connection  $\nabla_1$  on  $V_1$ . Denote the pair of connections as  $\nabla_\bullet$ . Take an invariant polynomial  $P$  of  $r \times r$  matrices. Take the Bott difference form  $P(\nabla_0, \nabla_1)$  associated to some choice of homotopy between the connections  $\nabla_0$  and  $\nabla_1$  defined on  $M_0 \cap M_1$ . Define

$$P(\nabla_\bullet) := (P(\nabla_0), P(\nabla_1), P(\nabla_0, \nabla_1)).$$

From above, we see that  $P(\nabla_\bullet)$  is closed in Čech–de Rham cohomology. If we vary  $\nabla_0$  and  $\nabla_1$  through connections, the same reasoning as above shows that the Čech–de Rham class

$$P(V_\bullet) := [P(\nabla_\bullet)]$$

is independent of the choice of connections  $\nabla_\bullet$ . From proposition 41.1 on page 341, the map

$$[\omega] \in H_{\text{dR}}^*(M) \mapsto (\omega, \omega, 0) \in \check{H}^*(M)$$

is an isomorphism on cohomology. In our case, we can extend  $\nabla_0$  to a connection defined on all of  $V$ , and apply this to the characteristic class  $P(\nabla_0)$  to see that the characteristic classes computed in Čech–de Rham cohomology are the same as those in de Rham cohomology.

Suppose that  $V_0, V_1$  are trivial. Take bases of local sections  $e_i$  on  $M_0$  and  $\underline{e}_i$  on  $M_1$ . Take the associated standard connections  $\nabla, \underline{\nabla}$ . On the overlap  $M_{01} = M_0 \cap M_1$ , these bases agree up to  $\underline{e}_i = g_i^j e_j$ . In other words,  $V$  is clutched over  $M_{01}$ . Their connections forms are both 0 in those respective

bases, and curvature forms are 0. Compute the connection  $\nabla$  in the  $\underline{e}$  basis:

$$\begin{aligned}\nabla \underline{e} &= \nabla g e, \\ &= dg e, \\ &= dg g^{-1} \underline{e}.\end{aligned}$$

So the connection 1-forms of  $\nabla$  in the basis  $\underline{e}_i$  are not zero: they are  $\gamma_0 = dg g^{-1}$ , which we denote as  $\gamma$  to save ink. We write  $\gamma^2$  to mean  $\gamma \wedge \gamma$ . The difference connection in the basis  $e_i$  is

$$\dot{\gamma} = (1-t)\gamma_0 + t\gamma_1 = (1-t)\gamma.$$

The curvature is

$$\begin{aligned}\dot{\Omega} &= d\dot{\gamma} + \dot{\gamma} \wedge \dot{\gamma}, \\ &= -dt \wedge \gamma + (1-t)d\gamma + (1-t)^2\gamma^2, \\ &= -dt \wedge \gamma + (1-t)(\Omega - \gamma^2) + (1-t)^2\gamma^2, \\ &= -dt \wedge \gamma - t(1-t)\gamma^2.\end{aligned}$$

The characteristic form  $P(\dot{\nabla})$  has this plugging in to  $P$  in every slot. Any pair of  $dt$  wedge to zero; expand out to have one term with no  $dt$ , and a sum of terms with one  $dt$  in each possible slot. Suppose that  $P$  is homogeneous of degree  $\ell$ . By symmetry, we can put that  $dt$  in the first slot,  $\ell$  times. When we push down with  $\pi_*$ , we integrate out, so we need exactly one  $dt$  form.

## Chern classes

The *Chern forms* of a connection  $\nabla$  are the differential forms  $c_j(\nabla)$  defined from the polynomials

$$\det\left(I + \frac{\sqrt{-1}}{2\pi} A\right) = 1 + c_1(A) + \cdots + c_r(A),$$

where  $r$  is the rank of the vector bundle and  $A$  is any  $r \times r$  matrix with complex entries. The classes  $c_j(V)$  are the *Chern classes*. The sum

$$c(V) := 1 + c_1(V) + \cdots + c_r(V)$$

is the *total Chern class*.

The *Chern character forms*  $\text{ch}_j(\nabla)$  are the characteristic forms associated to the invariant polynomials

$$\text{tr } e^A = \text{ch}_0(A) + \text{ch}_1(A) + \cdots$$

with  $\text{ch}_j(A)$  homogeneous of degree  $j$ . The associated characteristic classes are the *Chern characters*  $\text{ch}_j(V)$  and the sum

$$\text{ch}(V) = \text{ch}_0(V) + \text{ch}_1(V) + \cdots$$

is the *total Chern character*.



### Properties of characteristic classes

If  $V = V_0 \oplus V_1$  is a sum, connections  $\nabla_0$  on  $V_0$  and  $\nabla_1$  on  $V_1$  impose a connection  $\nabla = \nabla_0 + \nabla_1$  on  $V$ , with connection form

$$\Omega = \begin{pmatrix} \Omega_0 & 0 \\ 0 & \Omega_1 \end{pmatrix}.$$

Therefore taking determinant

$$c(V_0 \oplus V_1) = c(V_0)c(V_1).$$

More generally, if we have an exact sequence  $0 \rightarrow V_0 \rightarrow V \rightarrow V_1 \rightarrow 0$  of vector bundle maps, then we can pick a smoothly varying inner product on  $V$ , and an orthogonal complement to  $V_0 \subset V$ , which is therefore isomorphic to  $V_1$ , giving a vector bundle isomorphism  $V = V_0 \oplus V_1$ . So for an exact sequence  $0 \rightarrow V_0 \rightarrow V \rightarrow V_1 \rightarrow 0$  of vector bundle maps,

$$c(V) = c(V_0)c(V_1).$$

If the connection is unitary, then  $\gamma_{i\bar{j}} + \gamma_{\bar{j}i} = 0$  is skew-Hermitian. It follows then that the curvature  $\Omega$  is also a skew-Hermitian matrix of 2-forms. For any real coefficient polynomial  $P$ ,  $P(\Omega)$  is thus a real-valued differential form, so  $P(V)$  is a real cohomology class. In particular, Chern classes are real cohomology classes

$$c_j \in H_{\text{dR}}^{2j}(M).$$

### The Pfaffian

Take an even dimensional real vector space  $V$ , say of dimension  $2n$ . Pick a 2-form  $\omega$ , translation invariant, i.e. of constant coefficients. Pick a basis  $e_1, \dots, e_{2n}$  of  $V$ . Let  $\omega^1, \dots, \omega^{2n}$  be the dual basis. Define a number  $c$  by

$$\frac{\omega^n}{n!} = c\omega^1 \wedge \dots \wedge \omega^{2n}.$$

Expand out  $\omega$  in our basis, say as

$$\omega = a_{ij}\omega^i \wedge \omega^j,$$

and let  $A = (a_{ij})$ . Clearly  $c$  is a polynomial in the entries  $a_{ij}$ , which we denote  $\text{Pf } A$ . To be precise, just expanding out wedge products,

$$\text{Pf } A = \frac{1}{n!2^n} \sum_{\pi} (-1)^{\pi} A_{\pi(1)\pi(2)} A_{\pi(3)\pi(4)} \dots A_{\pi(2n-1)\pi(2n)}$$

with the sum over permutations  $\pi$  of  $1, 2, \dots, 2n$ ,  $(-1)^{\pi}$  the sign of the permutation.

Change basis by  $\omega^i = b_j^i \xi^j$ ; let  $B = (b_j^i)$ . So

$$\omega^1 \wedge \dots \wedge \omega^{2n} = \det(B) \xi^1 \wedge \dots \wedge \xi^{2n},$$

and

$$\omega = c_{ij} \xi^i \wedge \xi_j,$$

where  $C = (c_{ij})$  is  $C = BA^tB$ . In the new basis:

$$\begin{aligned}\frac{\omega^n}{n!} &= \text{Pf}(BA^tB)\xi^1 \wedge \cdots \wedge \xi^{2n}, \\ &= \text{Pf}(A)\det(B)\xi^1 \wedge \cdots \wedge \xi^{2n}.\end{aligned}$$

So the Pfaffian polynomial in matrix entries transforms as

$$\text{Pf}(BA^tB) = \text{Pf}(A)\det(B),$$

for any invertible matrix  $B$ . In particular,  $\text{Pf}$  is invariant under  $A \mapsto BA^tB$  for all  $B$  of unit determinant, and so  $\text{Pf}$  is invariant under conjugation of  $A$  by rotations  $B$ , since then  $B^t = B^{-1}$ .

In  $\mathbb{R}^2$ , every skew-symmetric matrix is

$$A = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

which has determinant  $\det A = \theta^2$  the square of a polynomial  $\text{Pf } A = \theta$  in the entries.

In  $\mathbb{R}^4$ , as we will see, every skew-symmetric matrix  $A$  can be brought by rotation to the form

$$A = \begin{pmatrix} 0 & \theta_1 & & \\ -\theta_1 & 0 & & \\ & & 0 & \theta_2 \\ & & -\theta_2 & 0 \end{pmatrix}$$

(0 in blank entries) which has determinant  $\det A = \theta_1^2\theta_2^2$  the square of a polynomial  $\text{Pf } A = \theta_1\theta_2$  in the entries. The same is true in any even dimension.

Computing Pfaffians is not so easy using this polynomial expression.

A huge calculation shows that if  $A$  is a  $4 \times 4$  skew-symmetric matrix, then

$$\text{Pf } A = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23},$$

and that this squares to the determinant.

### The Pfaffian on skew symmetric linear maps

Pick an oriented finite dimensional real vector space  $V$  of even dimension, with a positive definite inner product. Pick a skew symmetric linear transformation  $A: V \rightarrow V$ . The Pfaffian of  $A$  is defined as above, in any orthonormal oriented basis. The eigenvalues of any skew-symmetric  $A$  are imaginary. Being a normal operator,  $A$  is unitarily diagonalizable. Split  $V_{\mathbb{C}}$  into eigenspaces, over eigenvalues  $i\theta_1, \dots, i\theta_n$

counted with multiplicity. Any complex eigenvector

$$w = \frac{u + iv}{\sqrt{2}}$$

with eigenvalue  $i\theta$  has a corresponding complex eigenvector

$$\bar{w} = \frac{u - iv}{\sqrt{2}}$$

with eigenvalue  $-i\theta$  because  $A$  is real. The real span of  $u, v$  is acted on by our  $2 \times 2$  real skew-symmetric block as above. We have a real orthonormal basis  $u_1, w_1, \dots, u_n, w_n$  in which  $A$  is expressed as  $2 \times 2$  skew-symmetric blocks. By our expression above,  $\text{Pf } A$  is the product of these  $\pm\theta_i$ ; there is a problem as to which sign we pick. Since  $V$  is oriented, we orient the span of each  $u_j, w_j$  any way we like, as long as the orientations together reproduce that of  $V$ . If they don't, we swap sign of any odd collection of  $w_j$  vectors. But with this done, we have the signs of the  $w_j$ , and hence of the  $\theta_j$ , defined up to changing an even collection of them, so giving the same result for

$$\text{Pf } A = \theta_1 \dots \theta_n.$$

## Partitions

A *partition* of the numbers  $1, 2, \dots, 2n$  is a choice of division of these numbers into pairs. For example, we could choose to partition  $1, 2, 3, 4, 5, 6$  into

$$\{4, 1\}, \{2, 5\}, \{6, 3\}.$$

This is the same partition if we write the pairs down in a different order, like

$$\{2, 5\}, \{6, 3\}, \{4, 1\},$$

or if we write the numbers inside each pair down in a different order, like

$$\{1, 4\}, \{5, 2\}, \{6, 3\}.$$

It isn't really important that the objects partitioned be numbers. Of course, you can't partition an odd number of objects into pairs. Each permutation  $p$  of the numbers  $1, 2, 3, \dots, 2n$  has an *associated partition*

$$\{p(1), p(2)\}, \{p(3), p(4)\}, \dots, \{p(2n-1), p(2n)\}.$$

For example, the permutation  $3, 1, 4, 6, 5, 2$  has associated partition

$$\{3, 1\}, \{4, 6\}, \{5, 2\}.$$

Clearly two different permutations  $p$  and  $q$  could have the same associated partition, i.e. we could first transpose various of the pairs of the partition of  $p$ , keeping each pair in order, and then transpose entries within each pair, but not across different pairs. Consequently, there are  $n!2^n$  different permutations associating the same partition:  $n!$  ways to permute pairs, and  $2^n$  ways to swap the order within each pair. When you permute a pair, like changing  $3, 1, 4, 6, 5, 2$  to  $4, 6, 3, 1, 5, 2$ , this is the effect of a pair of transpositions (one to permute 3 and 4 and another to permute 5 and 6), so has no effect on signs. Therefore if two permutations have the same partition, the

root cause of any difference in sign must be from transpositions inside each pair. For example, while it is complicated to find the signs of the permutations 3, 1, 4, 6, 5, 2 and of 4, 6, 1, 3, 5, 2, it is easy to see that these signs must be different.

On the other hand, we can write each partition in “alphabetical order”, like for example rewriting

$$\{4, 1\}, \{2, 5\}, \{6, 3\}$$

as

$$\{1, 4\}, \{2, 5\}, \{3, 6\}$$

so that we put each pair in order, and then order the pairs among one another by their lowest elements. This in turn determines a permutation, called the *natural permutation* of the partition, given by putting the elements in that order; in our example this is the permutation

$$1, 4, 2, 5, 3, 6.$$

Define the sign of a partition  $P$  to be sign of its natural permutation. Watch out: if we start with a permutation  $p$ , like 6, 2, 4, 1, 3, 5, then the associated partition  $P$  is

$$\{6, 2\}, \{4, 1\}, \{3, 5\}.$$

This is the same partition as

$$\{1, 4\}, \{2, 6\}, \{3, 5\}$$

(just written in alphabetical order). The natural permutation  $q$  of  $P$  is therefore

$$1, 4, 2, 6, 3, 5,$$

so the original permutation  $p$  is not the natural permutation of its associated partition.

**6o.6** How many partitions are there of the numbers  $1, 2, \dots, 2n$ ?

**6o.7** Write down all of the partitions of

- (a) 1, 2;
- (b) 1, 2, 3, 4;
- (c) 1, 2, 3, 4, 5, 6.

**Lemma 6o.5.** For any  $2n \times 2n$  skew-symmetric matrix  $A$ ,

$$\text{Pf } A = \sum_P (-1)^P A_{p(1)p(2)} A_{p(3)p(4)} \cdots A_{p(2n-1)p(2n)},$$

where the sum is over partitions  $P$  and the permutation  $p$  is the natural permutation of the partition  $P$ . In particular,  $\text{Pf } A$  is an integer coefficient polynomial of the entries of the matrix  $A$ .

*Proof.* Each permutation  $p$  has an associated partition  $P$ . So we can write the Pfaffian as a sum

$$\text{Pf } A = \frac{1}{n!2^n} \sum_P \sum_p (-1)^P A_{p(1)p(2)} A_{p(3)p(4)} \cdots A_{p(2n-1)p(2n)}$$

where the first sum is over all partitions  $P$ , and the second over all permutations  $p$  which have  $P$  as their associated partition. But if two permutations  $p$  and  $q$  both have the same associated partition

$$\{p(1), p(2)\}, \{p(3), p(4)\}, \dots, \{p(2n-1), p(2n)\},$$

then  $p$  and  $q$  give the same pairs of indices in the expression  $A_{p(1)p(2)}A_{p(3)p(4)} \dots A_{p(2n-1)p(2n)}$ . Perhaps some of the indices in these pairs might be reversed. For example, we might have partition  $P$  being

$$\{1, 5\}, \{2, 6\}, \{3, 4\},$$

and permutations  $p$  being 1, 5, 2, 6, 3, 4 and  $q$  being 5, 1, 2, 6, 3, 4. The contribution to the sum coming from  $p$  is

$$(-1)^p A_{15} A_{26} A_{34},$$

while that from  $q$  is

$$(-1)^q A_{51} A_{26} A_{34}.$$

But then  $A_{51} = -A_{15}$ , a sign change which is perfectly offset by the sign  $(-1)^q$ : each transposition of pairs gives a sign change from  $(-1)^p$ .

So put together, we find that for any two permutations  $p$  and  $q$  with the same partition, their contributions are the same:

$$(-1)^q A_{q(1)q(2)} A_{q(3)q(4)} \dots A_{q(2n-1)q(2n)} = (-1)^p A_{p(1)p(2)} A_{p(3)p(4)} \dots A_{p(2n-1)p(2n)}.$$

Therefore the  $n!2^n$  permutations with associated partition  $P$  all contribute the same amount as the natural permutation of  $P$ .  $\square$

## The Euler class

For a real vector bundle  $V$ , we can complexify and generate Chern classes as above of  $V_{\mathbb{C}}$ . But we restrict to orthonormal bases which are also real bases of  $V$ . Along  $V$ ,  $\gamma_{i\bar{j}}$  is now a skew-symmetric real matrix, as is  $\Omega$ , so  $c_j(\Omega) = 0$  for  $j$  odd. The *Pontryagin forms are the forms*

$$p_j(\Omega) = (-1)^j c_{2j}(\Omega).$$

Suppose that  $V \rightarrow M$  is an oriented real vector bundle of rank  $2n$  on a manifold  $M$  of dimension  $2n$ . Pick a metric on  $V$  and a compatible connection  $\nabla$  with curvature  $\Omega$ . The *Euler form* is

$$e(\nabla) := \text{Pf}(\Omega/2\pi).$$

We can repeat our proof above that the resulting class is closed, now using only invariance under rotations, since the curvature is a 2-form valued in skew symmetric linear transformations. The *Euler class*  $e(V)$  is the de Rham cohomology class of the Euler form.



## Hints

**1.1.** Take charts  $(U, \varphi)$  and  $(W, \psi)$ . Take any point  $x_0$  in the image of  $\psi$ , say  $x_0 = \psi(m_0)$ . We need to prove that  $\varphi \circ \psi^{-1}$ , if defined near  $x_0$ , is smooth near  $x_0$ . In other words, we need to prove that if  $\varphi$  is defined near  $m_0$ , then  $\varphi \circ \psi^{-1}$  is smooth there. We can assume that  $\varphi$  is defined near  $m_0$  as is  $\psi$ .

Suppose that  $(U, \varphi)$  and  $(W, \psi)$  are both compatible with the charts  $(V_a, \chi_a)$  of some atlas. Because the  $V_a$  cover  $M$ ,  $m_0 \in V_a$  for some  $a$ . So then  $\varphi \circ \chi_a^{-1}$  and  $\psi \circ \chi_a^{-1}$  are defined near the point  $y_0 := \chi_a(m_0)$  in Euclidean space. The transition maps

$$\varphi \circ \chi_a^{-1}, \psi \circ \chi_a^{-1}$$

are smooth where defined. So the composition of the former with the inverse of the later is smooth where defined:

$$(\varphi \circ \chi_a^{-1}) \circ (\psi \circ \chi_a^{-1})^{-1} = \varphi \circ \psi^{-1}.$$

**1.3.** Take two distinct parallel planes  $P, Q \subset \mathbb{R}^3$ . If a line  $\ell$  is not parallel to either, it hits both, say at  $p \in P$  and  $q \in Q$ . Conversely, take any points  $p \in P$  and  $q \in Q$ . Since  $P$  and  $Q$  are distinct parallel planes, they are disjoint, so  $p \neq q$ , and so there is a unique line  $\ell = pq$  through  $p$  and  $q$ . You pick disjoint parallel planes  $P$  and  $Q$  and I pick disjoint parallel planes  $\bar{P}$  and  $\bar{Q}$ , not parallel to yours. Each line  $\ell$  not parallel to any of our 4 planes strikes  $P$  and  $Q$  at points  $p$  and  $q$  and  $\bar{P}$  and  $\bar{Q}$  at points  $\bar{p}$  and  $\bar{q}$ . We need to see that  $(p, q) \mapsto (\bar{p}, \bar{q})$  is smooth. It then will follow that the inverse is smooth, by just changing notation as to which planes get bars on top of their names. We can carry out a translation and a linear transformation so that  $P$  is the plane  $x = 0$ ,  $Q$  is  $x = 1$ ,  $\bar{P}$  is  $y = 0$  and  $\bar{Q}$  is  $y = 1$ . It is enough to check after such transformations, as the smoothness will not be affected. Suppose that  $p = (0, y_0, z_0)$  and  $q = (1, y_1, z_1)$ . Our line  $\ell = pq = \bar{p}\bar{q}$  is not parallel to  $\bar{P}$ , i.e.  $y \neq Y$ . The points of  $\ell$  are the points  $(1-t)p + tq$  for any  $t \in \mathbb{R}$ , i.e. points

$$(1-t)(0, y_0, z_0) + t(1, y_1, z_1) = (t, (1-t)y_0 + ty_1, (1-t)z_0 + tz_1).$$

Such a point reaches  $y = 0$  at  $0 = (1-t)y_0 + ty_1 = y_0 + t(y_1 - y_0)$ , i.e.

$$t = -\frac{y_0}{y_1 - y_0},$$

so at

$$\bar{p} = \frac{1}{y_1 - y_0}(-y_0, 0, z_0y_1 - y_0z_1).$$

and similarly a point reaches  $y = 1$  at  $1 = (1-t)y_0 + ty_1$ , and we find

$$\bar{q} = \frac{1}{y_1 - y_0}(1 - y_0, 1, z_0y_1 - y_0z_1 + z_1 - z_0).$$

Clearly these are smooth where defined, i.e. away from the lines  $\ell$  with  $y_0 = y_1$ , i.e. away from the lines parallel to  $y = 0$ .

**1.4.** If  $p \neq 0$ , there is a unique line  $\ell$  through both  $p$  and the origin: the span of  $p$  as a vector. So one chart  $(U, \varphi)$  is defined on the set  $U$  of points  $(p, \ell) \in M$  with  $p \neq 0$ , and is defined by  $(p, \ell) \mapsto \varphi(p, \ell) := p$ . On the other hand, if  $p = 0$ , this is not defined, as any line  $\ell$  through the origin passes through both  $p = 0$  and through  $0$ . So we need some more charts. Consider the plane  $P$  with equation  $x_1 = 1$ . Each line  $\ell$  not parallel to  $P$  contains a unique point  $q = (1, x_2, \dots, x_n) \in P$ , and as above we use  $(x_2, \dots, x_n)$  to say which line  $\ell$  is. But then the points of  $\ell$  are precisely the points  $p = tq$  for any real number  $t$ . So we use the values of  $(t, x_2, \dots, x_n)$  as a chart: let  $U \subset M$  be the set of all  $(p, \ell)$  so that  $\ell$  is not parallel to  $P$ , and let  $\varphi(p, \ell) = (t, x_2, \dots, x_n)$  as we have defined it. We similarly define planes  $P_i$  where  $x_i = 1$  and charts  $\varphi_i(p, \ell) = (x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ . Check that

$$\varphi_i \circ \varphi(x) = \left( \frac{x_1}{x_i}, \frac{x_2}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right)$$

and that

$$\varphi_i \circ \varphi_j^{-1}(x) = \frac{1}{x_j} (x_1, x_2, \dots, x_{i-1}, x_i x_j^2, x_{i+1}, \dots, x_n).$$

**1.5.** If we write

$$p = \begin{pmatrix} a \\ b \end{pmatrix}$$

then

$$x = \varpi(p) = \frac{a}{1-b}.$$

The inverse map is

$$p = \varpi^{-1}(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{1+|x|^2} \begin{pmatrix} 2x \\ -2 \end{pmatrix}.$$

**1.6.** Just change all signs of the last entry in each vector:

$$y = \varpi(p) = \frac{a}{1+b}.$$

The inverse map is

$$p = \varpi^{-1}(y) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{1}{1+|y|^2} \begin{pmatrix} 2y \\ 2 \end{pmatrix}.$$

**1.7.**

$$\varpi_s \circ \varpi_n^{-1}(y) = \frac{y}{|y|^2},$$

defined as a map  $\mathbb{R}^n - \{0\} \rightarrow \mathbb{R}^n - \{0\}$ .

**1.8.** One direction is clear: if  $V$  is open, then  $\varphi_a(U_a \cap V)$  is open for any chart  $(U_a, \varphi_a)$ . So we need to prove the other direction: we assume that there are some charts  $(U_a, \varphi_a)$  from the atlas so that  $\varphi_a(U_a \cap V)$  is open and every point of  $V$  lies in some  $U_a$ . Take any chart  $(U, \varphi)$  from the atlas. We need to prove that  $\varphi(U \cap V)$



is open in Euclidean space. It suffices to prove that  $\varphi(U \cap V \cap U_a)$  is open, for every  $U_a$ , since the union of open sets in Euclidean space is open and  $\varphi$  is a bijection. But

$$\varphi(U \cap V \cap U_a) = \varphi \circ \varphi_a^{-1} \circ \varphi_a(U \cap V \cap U_a),$$

is the image of the open set  $\varphi_a(U \cap V \cap U_a)$  in Euclidean space under the transition map  $\varphi \circ \varphi_a^{-1}$ .

**1.10.** Take a set  $M$  with an atlas consisting of charts  $(U_a, \varphi_a)$  where  $a$  varies over some collection  $a \in A$ . An open set, by definition, is a subset  $W \subset M$  so that, for any  $(U_a, \varphi_a)$  from the atlas,  $\varphi_a(W \cap U_a) \subset \mathbb{R}^n$  is open. We need to prove that

- a.  $M$  is open and
- b. the empty set is open and
- c. the union of any collection of open sets is open and
- d. the intersection of any finite collection of open sets is open.

Take  $W = M$ , we find  $\varphi_a(W \cap U_a) = \varphi(U_a)$  is open (by definition of a chart: every chart has open image).

Take  $W$  empty:  $\varphi_a(\emptyset \cap U_a) = \varphi_a(\emptyset) = \emptyset$  is empty, so open.

Take some collection  $W_b$ ,  $b \in B$ , of open sets and let  $W$  be their union. For any map,

$$\varphi_a(W \cap U_a) = \bigcap_b \varphi_a(W_b \cap U_a)$$

is a union of open sets in  $\mathbb{R}^n$ , hence open. So  $W$  is open. Similarly for intersections, but for this we need to use that  $\varphi_a$  is bijective, for every  $a \in A$ .

**1.26.** Any line is homeomorphic to the real number line. Any point  $x_0$  of the real number line lies in an open interval  $(a, b)$ , an open set so, if we puncture the interval,  $(a, b) - \{x_0\}$  consists of two connected components. Take our pair of lines; arrange (for simplicity of notation) by linear isomorphism that they are the horizontal and vertical axes. Let  $X$  be their union. The open sets of the induced topology on  $X$  are, by definition, the intersections  $X \cap U$  with open sets  $U$  of the plane. Any open set in the plane around the origin contains a ball around the origin. So any open set of  $X$  around the origin contains the union of two intervals, one vertical, one horizontal, around the origin. Punctured at the origin, this open set contains at least 4 connected components. Numbers of components are preserved under any homeomorphism.

**1.27.** See [13].

**1.28.** Take a point  $x_0$  in the intersection of all open sets containing a given compact set  $K$ . Suppose that  $x_0$  is not in  $K$ . For each point  $x \in K$ , take an open set  $U_x$  containing  $x_0$  and an open set  $V_x$  containing  $x$  so that  $U_x, V_x$  are disjoint. Take a finite subcover of  $K$  by the open sets  $V_x$ , say  $V_1, \dots, V_n$ . Then  $K$  lies inside  $V_1 \cup \dots \cup V_n$  but  $x_0$  doesn't.

**1.29.** The proof needs the theory of one point compactification [13]: they are one point compactifications of locally compact Hausdorff spaces, so they are homeomorphic.

**1.30.** Each point of  $X$  lies in the domain of a chart; intersect that chart domain with the open complement of  $Y$ , so the domain does not intersect  $Y$ . Our point is taken by the chart to a point in the image of the chart. That image point lies inside some closed ball in the image of the chart. The inverse image of that closed ball is a compact set, whose interior contains our point. So  $X$  is covered in open sets with compact closure not touching  $Y$ . Take a finite subcover, and take the union.

**1.31.** Every point lies inside the domain of some chart, having some image point. The inverse of the chart applied to a closed ball around the image point yields a compact set with open interior covering our point. Take all such open sets; an open cover by open sets with compact closure. Take as well some countable basis. The basis contains some open set inside each of our previous open sets. Every subset of a countable set is countable.

**1.32.** A *chart ball* is an open set whose closure is identified by some chart with a closed ball. Every point  $m_0 \in M$  of any manifold lies in the domain of some chart  $(V, \psi)$ . Because  $V \subseteq \mathbb{R}^n$  is open,  $V$  contains a ball  $B$  centered at  $\psi(m_0)$ . Take a perhaps smaller ball  $B$ , so that its closure  $\bar{B}$  still lies in  $V$ . The set  $\psi^{-1}B$  is a chart ball. So every point of any manifold lies inside a chart ball. Every compact set  $X \subseteq M$  therefore has a finite cover by chart balls; choose a covering by chart balls, each with its own chart, with union an open set  $U_1$  containing  $X$ . Since there are finitely many chart balls, their union  $U_1$  still has compact closure  $\bar{U}_1$ .

Each chart ball has some radius as measured in the chart. Fix those charts, but now allow yourself to replace each chart ball with a collection of chart balls, of half the radius (as measured in that associated chart). We can still pick finitely many to cover all of  $X$ . We still have just as many charts, but each now has finitely many chart balls in it. Their union is again an open set  $U_2$  covering  $X$ , with compact closure  $\bar{U}_2 \subset U_1$ . Continuing, eventually the chart balls get small radius around the points of  $X$ , as “seen” in the charts. Take a point outside  $X$  but in one of the chart balls. That point has image, in that chart, at some positive distance from the chart image of the points of  $X$ . So it doesn’t lie in the open set  $U_k$  at some stage  $k$ .

**1.36.** Write any  $p \times q$  matrix as

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

of sizes

$$\begin{pmatrix} r \times r & r \times (q-r) \\ (p-r) \times r & (p-r) \times (q-r) \end{pmatrix}.$$

Suppose that  $X$  has rank at least  $r$ ; an open set of matrices have rank at least  $r$ . After permutation of rows and columns, we can assume that  $X$  has  $A$  invertible. An open set of matrices  $X$  have  $A$  invertible: those with  $\det A \neq 0$ . The rank of  $X$  is unchanged when we multiply by any invertible matrix, so multiply on the right by

$$\begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & I \end{pmatrix}$$

and on the left by

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix}$$

to get

$$\begin{pmatrix} I & 0 \\ 0 & D - CA^{-1}B \end{pmatrix}$$

So any matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with  $\det A \neq 0$  has rank  $r$  precisely just when  $D = CA^{-1}B$ . Let  $U \subset k^{p \times q}$  be the open set of matrices with upper left  $r \times r$  corner invertible. Define a map

$$\varphi: X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U \rightarrow \begin{pmatrix} A & B \\ C & D - CA^{-1}B \end{pmatrix} \in k^{p \times q}.$$

This “flattens out” the determinantal variety to have the equation  $D = 0$  on  $U$ .

Any  $p \times q$  matrix  $X$  has some invertible  $r \times r$  submatrix. We can permute the rows and columns of  $X$  to get that submatrix to the upper left corner, say by permutation matrices  $P$  and  $Q$ , so  $PXQ^{-1} \in U$ . So the open sets  $P^{-1}UQ$  cover the matrices of rank  $r$  or more, and on each we can “flatten out” using  $X \mapsto \varphi(PXQ^{-1})$ . Each  $p \times q$  matrix with entries in  $k$  has  $d pq$  real numbers arising in the various real and imaginary parts of its entries, hence a factor of  $d$  in the dimension count.

**1.37.** Pick a point  $p_1$  of the hypersurface. Pick a chart near  $p_1$  in which the hypersurface is a linear subspace. Make the domain of the chart smaller if needed to arrange that the image of the chart is a ball, divided in two solid open hemispheres  $U, V$  by the hypersurface. Our manifold  $M$  is connected, so path connected. Any point of the complement is thus connected by a path to the hypersurface. Let  $p_0$  be the first point of the hypersurface which our path reaches. We can then stop the path just before it strikes  $p_0$ , at some point  $q_0$ , and still inside some chart in which the hypersurface is a linear hypersurface. Pick any other point  $p_1$  of the hypersurface, and connect  $p_0$  to  $p_1$  by a continuous path. Cover the path in charts, in each of which the hypersurface is linear, each chart having image a ball in Euclidean space. One chart at a time, lift or lower the path just above or just below the linear hypersurface to connect, piece at a time, to produce a path between  $q_0$  and some point  $q_1$  near  $p_1$ . Then  $q_1$  lies in  $U$  or in  $V$ : at most two path components.

**1.39.** Take a topological manifold  $M$  in the broad sense. So  $M$  is Hausdorff, locally homeomorphic to  $\mathbb{R}^n$ . We need to prove that there is a countable basis of open sets if and only if  $M$  has a countable set of components. If the set of components is uncountable, there is no countable basis of open sets, since we would need at least one of those open sets to sit inside each component. Suppose that the set of components is countable. Pick a metric on  $M$  inducing the topology. Every connected, locally compact metric space is a countable union of compact sets; see, for example, the discussion of compactness radius in [11]. So each component of  $M$  is a countable union of compact sets, and so therefore is  $M$ , say  $M = K_1 \cup K_2 \cup \dots$ . Pick any open cover of  $M$  by sets, each locally homeomorphic to an open set in Euclidean space. For each  $K_i$  pick a finite subcover. Throw those together to a countable open cover of  $M$  by sets, each homeomorphic to Euclidean space. Each has a countable basis of open sets, as Euclidean space does.

**2.1.** Suppose that  $M$  is our set with smooth structure.

Take a countable basis of open sets  $U_1, U_2, \dots \subseteq M$ . For each  $U_i$  which lies in the domain of some chart  $(U, \varphi)$ , we pick such a chart and let  $\varphi_i := \varphi|_{U_i}$ . Every point lies in an open set  $U_i$  which is in the domain of some chart, so in the domain of the chart  $(U_i, \varphi_i)$ .

On the other hand, take a countable set of charts  $(U_i, \varphi_i)$  whose domains cover  $M$ . Take a countable basis  $V_j$  open sets of Euclidean space. Let  $U_{ij} := \varphi_i^{-1}(V_j)$ . So  $U_{ij}$  lies in the domain of  $\varphi_i$  and is identified by  $\varphi_i$  with an open set  $\varphi(U_i) \cap V_j$  in Euclidean space. By problem 1.8 on page 8,  $U_{ij} \subseteq M$  is an open set.

Take a point  $m_0 \in M$  and an open set  $W \subset M$  containing  $m_0$ . The point  $m_0$  lies in some  $U_i$ , since the domains  $U_i$  of the charts cover  $M$ . The open set  $\varphi_i(W \cap U_i)$

contains  $\varphi(m_0)$ . The  $V_j$  are a basis of open sets, so some  $V_j$  lies inside  $\varphi(W \cap U_i)$  and contains  $\varphi(m_0)$ . So  $m_0$  lies in some  $U_{ij}$  which lies inside  $W$ . So the  $U_{ij}$  form a basis of open sets.

**2.3.** We computed the transition map between the north pole Ptolemaic projection  $\varpi_n$  and the south pole one  $\varpi_s$  in problem 1.7 on page 7, and we saw that it is smooth. Since there are two charts  $\varpi_n, \varpi_s$ , the sphere has a countable basis of open sets. Any pair of points of the sphere lie in the domain of  $\varpi_n$ , unless one or both are  $s$ , and vice versa. So both points lie in the domain of the same chart, unless one is  $n$  and the other  $s$ . Let  $U$  be the spherical cap consisting of points  $x$  for which  $x_{n+1} > 0$ , and  $V$  be the spherical cap consisting of points  $x$  for which  $x_{n+1} < 0$ . These are open in both charts, so open. They don't overlap, so they house off the north and south poles.

**2.4.** Let  $U \subset M$  be the set of pairs  $(p, \ell)$  for which  $p \neq 0$ . Then  $\ell$  is uniquely determined as the line through 0 and  $p$ . So we map  $f: (p, \ell) \in U \mapsto p \in \mathbb{R}^2$ , and take  $(U, f)$  as one chart. Next, we need to worry about what happens when  $p = 0$ . But the domain of a chart should be an open set, and we can't imagine that setting  $p$  to be 0 would give us an open subset of  $M$ . So instead we will pick some conditions on  $(p, \ell)$  which we guess should give open sets in a reasonable topology on  $M$ . Let  $V \subset M$  be the set of pairs  $(p, \ell)$  for which  $\ell$  is not vertical. We can specify  $\ell$  by its slope, say  $m$ , and then  $p$  is on  $\ell = (y = mx)$ , so  $p = (x, mx)$ ; specify  $p$  by its horizontal coordinate  $x$ . So let  $g: (p, \ell) = ((x, mx), \{y = mx\}) \in V \mapsto (x, m)$ . Let  $W \subset M$  be the set of pairs  $(p, \ell)$  for which  $\ell$  is not horizontal. We can specify  $\ell$  by  $\ell = \{x = ny\}$ , say, and then  $p$  is on  $\ell = (x = ny)$ , so  $p = (ny, y)$ ; specify  $p$  by its vertical coordinate  $y$ . So let  $h: (p, \ell) = ((ny, y), \{x = ny\}) \in W \mapsto (y, n)$ . The transition map  $f \circ g^{-1}$  is  $(x, m) \mapsto (x, mx)$ . The transition map  $f \circ h^{-1}$  is  $(y, n) \mapsto (ny, y)$ . The transition map  $g \circ f^{-1}$  is  $(x, y) \mapsto (x, y/x)$ . The transition map  $h \circ f^{-1}$  is  $(x, y) \mapsto (x/y, y)$ . The transition map  $g \circ h^{-1}$  is  $(y, n) \mapsto (ny, 1/n)$ . The transition map  $h \circ g^{-1}$  is  $(x, m) \mapsto (mx, 1/m)$ . For any pair  $(p_1, \ell_1), (p_2, \ell_2)$ , both lie in the domain of one of these, except if  $p_1 = p_2 = (0, 0)$  and  $\ell_1, \ell_2$  are the axes. So  $M$  is Hausdorff as long as we can house off these last two. To do that, take  $V_0 \subset V$  the pairs  $(p, \ell)$  for which  $\ell$  has slope between -1 and 1, and similarly define an open set  $W_0 \subset W$ , houses.

Alternatively, you could keep track of the direction of the line  $\ell$ , say as an angle  $\theta$ . Careful that  $-\theta$  and  $\theta$  give the same line. Let  $U$  be the set of pairs  $(p, \ell)$  for which  $\ell$  is not horizontal and  $V$  the set of pairs  $(p, \ell)$  for which  $\ell$  is not vertical. Let  $f: (p, \ell) \in U \mapsto (t, \theta)$  if  $\ell$  is the span of the vector  $u = (\cos \theta, \sin \theta)$  and  $p = tu$ , for  $0 < \theta < \pi$ . Let  $g: (p, \ell) \in V \mapsto (t, \theta)$  if  $\ell$  is the span of the vector  $u = (\cos \theta, \sin \theta)$  and  $p = tu$ , for  $-\pi/2 < \theta < \pi/2$ . The transition maps are

$$f \circ g^{-1}(t, \theta) = \begin{cases} (t, \theta), & \text{if } 0 < \theta < \pi/2, \\ (-t, \pi + \theta), & \text{if } -\pi/2 < \theta < 0. \end{cases}$$

and

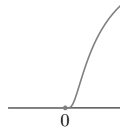
$$g \circ f^{-1}(t, \theta) = \begin{cases} (t, \theta), & \text{if } 0 < \theta < \pi/2, \\ (-t, -\pi + \theta), & \text{if } \pi/2 < \theta < \pi. \end{cases}$$

If we have two points of  $M$ , say  $(p_0, \ell_0), (p_1, \ell_1)$ , which are not in the same one of these two charts, then (after perhaps changing the choice of which is labelled with which subscript)  $\ell_0$  is the  $x$ -axis and  $\ell_1$  is the  $y$ -axis. To deal with this, we can use a third chart, say defined as  $h(p, \ell) = (t, \theta)$  if  $\ell$  is the span of  $u = (\cos \theta, \sin \theta)$  and  $p = tu$ , for  $\pi/4 < \theta < \pi + \pi/4$ .

**2.5.** Path connected sets are connected, as we know from previous experience in topology. On the other hand, in a topological manifold  $M$ , pick a point  $m_0 \in M$ , and consider the set  $U \subset M$  of points path connected to  $m_0$ . Every point of  $U$  lies in an open set of  $M$  identified by some chart with an open ball in  $\mathbb{R}^n$ , so lies in  $U$ . Hence  $U \subseteq M$  is open. But if a sequence of points of  $U$  converges in  $M$ , then all but finitely elements of that sequence lie in a compact subset of  $M$  identified by a chart with a closed ball in  $\mathbb{R}^n$ , and the limit lies inside that compact set, so path connected to all but finitely points of the sequence, so lies in  $U$ .

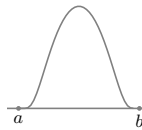
**3.1.** Take a closed ball  $\bar{B}$  centered at  $x_0$ , sitting inside an open ball  $B'$  centered at  $x_0$ , with  $\bar{B}' \subseteq U$ . We want a bump function  $h$  with  $h = 1$  on  $\bar{B}$ , declining inside  $B' - \bar{B}$ , and  $h = 0$  outside of  $B'$ . Consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$



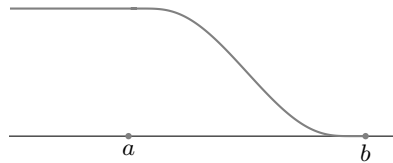
Differentiating any number of times clearly  $f^{(n)}(x)$  is some rational function of  $x$  times  $e^{-1/x}$ . Let  $u := 1/x$ , so  $f^{(n)}$  is a rational function of  $u$  divided by  $e^u$ . Send  $u \rightarrow +\infty$  to see  $f^{(n)}(x) \rightarrow 0$  as  $u \rightarrow +\infty$ , i.e. as  $x \rightarrow 0^+$ . For any  $0 < a < b$  let

$$g(x) = f(x-a)f(b-x).$$



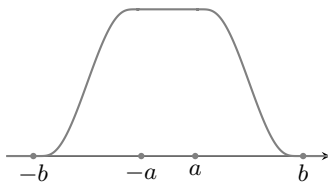
Let

$$i(x) = \frac{\int_x^b g(x) dx}{\int_a^b g(x) dx}.$$



Let

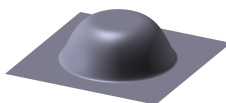
$$j(x) = h(x)h(-x).$$



For  $x \in \mathbb{R}^n$ , let

$$k(x) = j(|x|).$$

Set  $a$  to be the radius of the smaller ball, and  $b$  the radius of the larger ball.



Finally set

$$h(x) = k(x - x_0).$$

**3.4.** For simplicity, we consider the case  $k = \mathbb{R}$ ,  $K = \mathbb{C}$  and  $V_K = \mathbb{C}^3$ , so our Hopf fibration is  $\mathbb{RP}^5 \rightarrow \mathbb{CP}^2$ . Following Einstein, we write the components of vectors using superscripts, instead of subscripts. Take a vector

$$z = \begin{pmatrix} z^0 \\ z^1 \\ z^2 \end{pmatrix} \in \mathbb{C}^3.$$

Its span, in chart 0, is represented as

$$\begin{pmatrix} 1 \\ z^1/z^0 \\ z^2/z^0 \end{pmatrix}.$$

We write real and imaginary parts  $z^0 = x^0 + iy^0$  and so on, and then the vector  $z$  corresponds to a real vector

$$\begin{pmatrix} x^0 \\ y^0 \\ x^1 \\ y^1 \\ x^2 \\ y^2 \end{pmatrix} \in \mathbb{R}^6.$$

In chart 0, the span of this vector is represented as

$$\begin{pmatrix} 1 \\ y^0/x^0 \\ x^1/x^0 \\ y^1/x^0 \\ x^2/x^0 \\ y^2/x^0 \end{pmatrix}.$$

So in these charts we map

$$\begin{pmatrix} 1 \\ y^0 \\ x^1 \\ y^1 \\ x^2 \\ y^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ (x^1 + iy^1)(1 + y^0)^{-1} \\ (x^2 + iy^2)(1 + y^0)^{-1} \end{pmatrix},$$

which is clearly smooth. The map is described in the other charts similarly.

**3.7.** It is easy to check that  $X$  is Hausdorff, since the evenly covered open sets in  $Y$  are. Compose the charts of  $Y$  with  $f$  to produce charts on  $X$  with smooth transition maps. Take a countable basis of open sets on  $Y$ , and then locally, on each sheet of  $X \rightarrow Y$ , invert  $X \rightarrow Y$  and look at the image of that open set. This gives a basis of open sets of  $X$ , by definition of a covering map.

**3.11.** Take a finite dimensional real vector space  $V$ , say of dimension  $n$ . The set  $O_V$  of all linear isomorphisms  $V \rightarrow \mathbb{R}^n$  is a manifold of dimension  $n^2$ , identified by any choice of basis of  $V$  with  $GL(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ , the set of invertible  $n \times n$  matrices. Hence  $O_V$  is a manifold with two components; each component of  $O_V$  is an *orientation* of  $V$ . If  $o$  is one orientation, write the other orientation as  $-o$ .

Suppose that  $M$  is a  $n$ -dimensional manifold. Let  $\hat{M}$  be the set of pairs  $(m, o)$  where  $m \in M$  and  $o$  is an orientation of  $T_m M$ . Pick some vector fields  $v_1, \dots, v_n$  defined near  $m$ . After perhaps changing the sign of one of the vectors, we can get them to be positively oriented in  $o$ , i.e. taken to the standard basis by one of the linear isomorphisms in  $o$ . But then the same holds nearby for some orientation  $o$  at each point where these vector fields are linearly independent. In particular, if we take  $v_i = \partial_{x^i}$  in some chart, we find a local orientation. Clearly, every local orientation induces an orientation of each tangent space, and conversely. Let  $\pi: (m, o) \in \hat{M} \mapsto m \in M$ , a 2 to 1 map. For any chart  $(U, x)$  on  $M$ , we can take the linear map  $dx: T_m M \rightarrow \mathbb{R}^n$ . Define an orientation  $o(m)$  on  $T_m M$  to be the orientation associated to  $dx$ , i.e. the component of  $dx$  in  $O_{T_m M}$ . Let  $\hat{U}$  be the image of  $m \in U \mapsto (m, o(m)) \in \hat{M}$ . Let  $\hat{x}(m, o(m)) = x(m)$  for  $(m, o(m)) \in \hat{U}$ .

Each chart  $(\hat{U}, \hat{x})$  is a bijection, because it has the same image as  $(U, x)$  and the map  $(m, o(m)) \in \hat{U} \rightarrow m \in U$  is a bijection. The transition maps are just exactly those of the charts of  $M$ , hence smooth, so these charts induce a smooth structure.

We need to check that  $\hat{M}$  is Hausdorff. Take two points  $(m_1, o_1), (m_2, o_2) \in \hat{M}$ . If  $m_1 \neq m_2$ , take disjoint open sets  $U_1, U_2 \subset M$  so that  $m_1 \in U_1$  and  $m_2 \in U_2$ . Make the open sets smaller if needed to ensure that there are charts  $(U_1, f_1)$  and  $(U_2, f_2)$ . Then  $\hat{U}_1$  and  $\hat{U}_2$  are disjoint. If  $m_1 = m_2$ , but  $o_1 \neq o_2$ , then  $o_2 = -o_1$ , so we denote these two points as  $(m_0, o)$  and  $(m_0, -o)$ . Take a chart  $(U, f)$  on  $M$  with  $m_0 \in U$ . We can assume, changing notation if needed, that  $o$  is the orientation of  $df_{m_0}$ . Pick any  $n \times n$  reflection matrix  $g$  and then  $-o$  is the orientation of  $d(gf)_{m_0}$ . At every  $m \in U$ ,  $d(gf)_m = g df_m$  has opposite orientation to  $df_m$ . Hence

$$\begin{aligned}\tilde{U} &:= \{ (m, df_m) \mid m \in U \}, \\ -\tilde{U} &:= \{ (m, d(gf)_m) \mid m \in U \}\end{aligned}$$

are disjoint open subsets of  $\hat{M}$  containing  $(m_0, o)$  and  $(m_0, -o)$  respectively.

Take countable atlas of charts  $(U_i, \varphi_i)$  of  $M$ . These, and their compositions with any reflection matrix, determine a countable atlas of charts of  $\hat{M}$ , say  $(\tilde{U}_i, \tilde{\varphi}_i)$  and  $(-\tilde{U}_i, -\tilde{\varphi}_i)$ . So  $\pi: \hat{M} \rightarrow M$  is a smooth two-to-one covering map. Each  $U_i$  has preimage  $\pi^{-1}U_i = -\tilde{U}_i \sqcup \tilde{U}_i$ . The map  $-$  identifies  $-\tilde{U}_i \rightarrow \tilde{U}_i$ , so  $\pi$  is a covering map.

**3.12.** A triangle is determined by its vertices, but its vertices have no preferred choice of ordering. A *labelled triangle* is a triangle with a choice of ordering of its vertices, say as  $p, q, r$ . A labelled triangle is thus identified by just any triple  $(p, q, r)$  of points of the plane  $p, q, r \in \mathbb{R}^2$ , as long as they are distinct and no one lies on the line segment between the other two. More precisely,  $|p - q|, |p - r|, |q - r| > 0$

and  $|p - q| + |q - r| > |p - r|$  as well as the same with any permutation of  $p, q, r$ . So the set of labelled triangles is identified with the nonempty open set of  $(p, q, r) \in \mathbb{R}^6$  satisfying these inequalities. The triangles are the quotient of the labelled triangles by the permutations of  $p, q, r$ , a finite group. Since the vertices  $p, q, r$  of a labelled triangle are distinct, the permutations act freely. For each labelled triangle  $\Delta_0 = (p_0, q_0, r_0)$  and positive real number  $d > 0$ , let  $U_d(\Delta_0)$  be the set of points  $(p, q, r)$  so that  $|p - p_0|, |q - q_0|, |r - r_0| < d$ . Clearly  $U_d$  is an open set invariant under the permutations of the vertices. Any two distinct labelled triangles  $\Delta_0 \neq \Delta_1$  have some vertices different. Those vertices are separated by some distance  $2d$ , and lie in disjoint disks of half that distance. Hence the triangles lie in disjoint open sets  $U_d(\Delta_0), U_d(\Delta_1)$  invariant under the action of the permutations of the vertices.

**3.17.** Take two points  $p_0, q_0 \in M$  in different  $\Gamma$ -orbits. Recall that open sets  $U, V$  “house” points  $p_0, q_0$  “from each other” if  $p_0 \in U$  and  $q_0 \in V$  and  $U$  and  $V$  do not intersect. For each  $g \in \Gamma$ , pick some open sets  $U_g$  and  $V_g$  housing  $gp_0$  and  $q_0$  from each other. Let  $V := \bigcap_g V_g$  and  $U := \bigcap_g g^{-1}U_g$ . Then  $gU$  and  $V$  house  $gp_0$  from  $q_0$ . So  $hgU$  and  $hV$  house  $hgp_0$  from  $hq_0$ . Renaming  $hg$  to be  $g$ ,  $gU$  and  $hV$  house  $gp_0$  from  $hq_0$ , for all  $g, h \in \Gamma$ . So finally

$$\bigcap_g gU, \bigcap_g V$$

are  $\Gamma$ -invariant “houses”.

**3.19.** The group  $\Gamma$  of diagonal matrices with entries  $\pm 1$ , i.e.  $\{\pm 1\}^n$ , acts by multiplying on the right of  $P$ , changing the signs of the eigenvectors. Clearly the unit length eigenvectors are uniquely determined up to  $\pm 1$ , since each eigenspace is 1-dimensional. So  $f$  is a covering map, and  $\Gamma$  acts smoothly, freely, transitively on the fibers of  $f$ , and without any fixed points.

The reader might be curious to see what might happen if we allow a single multiple eigenvalue. Consider an example of Rellich:

$$\begin{aligned} x(t) &:= \begin{pmatrix} \cos \frac{1}{t} \\ \sin \frac{1}{t} \end{pmatrix}, \\ y(t) &:= \begin{pmatrix} -\sin \frac{1}{t} \\ \cos \frac{1}{t} \end{pmatrix}, \\ \lambda_1(t) &:= e^{-\frac{1}{t^2}}, \\ \lambda_2(t) &:= -e^{-\frac{1}{t^2}}, \\ g(t) &:= \begin{pmatrix} x(t) & y(t) \end{pmatrix} \begin{pmatrix} \lambda_+(t) & 0 \\ 0 & \lambda_-(t) \end{pmatrix} \begin{pmatrix} x(t) & y(t) \end{pmatrix}^{-1} \\ &= e^{-\frac{1}{t^2}} \begin{pmatrix} \cos \frac{2}{t} & \sin \frac{2}{t} \\ \sin \frac{2}{t} & -\cos \frac{2}{t} \end{pmatrix}. \end{aligned}$$

Then the directions of the eigenvectors  $x(t)$  and  $y(t)$  rotate so wildly as  $t \rightarrow 0$  that we cannot continuously pick a nonzero eigenvector near  $t = 0$ .



**3.20.** First consider just a single gluing  $\bar{Q}$ . There is a unique map making commute

$$\begin{array}{ccc} & P & \\ \swarrow & \Downarrow & \searrow \\ \bar{Q} & Q & \bar{Q} \\ \swarrow & & \searrow \\ \bar{Q} & \xrightarrow{\quad} & \bar{Q} \end{array}$$

But the identity map makes this commute, so this map is the identity: there is only one “automorphism” of a gluing trivial on the underlying gluing plan. Take two gluings  $\bar{Q}_0, \bar{Q}_1$ , and consider the map  $\bar{Q}_0 \rightarrow \bar{Q}_1$  for which

$$\begin{array}{ccc} & P & \\ \swarrow & \Downarrow & \searrow \\ \bar{Q}_0 & Q & \bar{Q}_1 \\ \swarrow & & \searrow \\ \bar{Q}_0 & \xrightarrow{\quad} & \bar{Q}_1 \end{array}$$

commutes and the corresponding one  $\bar{Q}_1 \rightarrow \bar{Q}_0$ . Their composition  $\bar{Q}_0 \rightarrow \bar{Q}_0$  makes commutative

$$\begin{array}{ccc} & P & \\ \swarrow & \Downarrow & \searrow \\ \bar{Q}_0 & Q & \bar{Q}_0 \\ \swarrow & & \searrow \\ \bar{Q}_0 & \xrightarrow{\quad} & \bar{Q}_0 \end{array}$$

so is the identity map. Similarly for composition with labels 0, 1 swapped. So both maps are diffeomorphisms.

**3.21.** An open set in  $U_{\bar{Q}} \subseteq \bar{Q}$  has preimage  $U_P \cup U_Q$  consisting of an open set  $U_P \subseteq P$  and an invariant open set  $U_Q \subseteq Q$ , so that  $U_P$  is the preimage of  $U_Q$  under either map. Conversely, if  $U_Q \subseteq Q$  is invariant open, let  $U_{\bar{Q}} := \pi U_Q$ . Then for each  $\bar{q} \in U_{\bar{Q}}$  there is some  $q \in U_Q$  mapping to it, and for any such  $q$  anything equivalent is also in  $U_Q$ . So  $\pi^{-1}U_{\bar{Q}} = U_P \cup U_Q$  where  $U_P$  is the inverse image of  $U_Q$  by either map. Therefore  $U_{\bar{Q}} \subseteq \bar{Q}$  is open.

**3.22.** Write the maps  $P \rightrightarrows Q$  as  $\varphi_0, \varphi_1$ . Take an open set  $U \subseteq Q$ . Let

$$\begin{aligned} U_0 &:= U, \\ U_1 &:= U_0 \cup \varphi_1 \varphi_0^{-1} U_0, \\ U_2 &:= U_1 \cup \varphi_0 \varphi_1^{-1} U_1, \\ &\vdots \\ U' &= U_0 \cup U_1 \cup \dots \end{aligned}$$

Because both maps are open maps, each  $U_i$  is open, so  $U'$  is open. So  $U'$  is the smallest invariant open set containing  $U$ .

Take any countable basis of open sets  $U_i$ . Every invariant open set  $U_Q \subseteq Q$  is a union of some of the  $U_i$ . Being invariant, and containing some  $U_i$ ,  $U_Q$  also contains the associated  $U'_i$ , so is a union of some  $U'_i$ . Hence the invariant open sets have a countable basis, and so  $\bar{Q}$  has a countable basis.

**3.23.** Take a double point  $q_0, q_1 \in Q$ . Suppose their images in  $\bar{Q}$  are housed off. Then  $q_0, q_1$  are housed off by invariant open sets, say  $U_0, U_1$ . Denote the maps  $P \rightrightarrows Q$  as  $\varphi_0, \varphi_1$ . Up to reordering,  $\varphi_0^{-1}U_0$  overlaps  $\varphi_1^{-1}U_1$ , say at some point  $p$ . Then  $\varphi_0(p) \in U_0$ , and  $U_0$  is invariant so  $\varphi_1(p) \in U_0$  and  $\varphi_1(p) \in U_1$ , so  $U_0, U_1$  overlap.

**3.24.** House them off by some disjoint open sets  $U_0, U_1$  in  $Q$ . We make these sets invariant by adding to them

$$\varphi_1\varphi_0^{-1}U_0 \sqcup \varphi_0\varphi_1^{-1}U_0, \varphi_1\varphi_0^{-1}U_1 \sqcup \varphi_0\varphi_1^{-1}U_1.$$

If, for example,  $U_0$  intersects  $\varphi_1\varphi_0^{-1}U_1$ , then  $\varphi_1^{-1}U_0$  overlaps  $\varphi_0^{-1}U_1$ . Since  $\varphi_0, \varphi_1$  have disjoint images, after reordering indices,  $U_0$  overlaps one of

$$\varphi_1\varphi_0^{-1}U_1, \varphi_0\varphi_1^{-1}U_1$$

so

$$\varphi_1^{-1}U_0 \text{ overlaps } \varphi_0^{-1}U_1$$

or

$$\varphi_0^{-1}U_0 \text{ overlaps } \varphi_1^{-1}U_1.$$

Since this persists for all  $U_0, U_1$  open sets containing  $q_0, q_1$ , one of these conditions occurs for a collection of open sets  $U_0$  containing sets lying inside any open set around  $q_0$ , and similarly for  $q_1$ . If the condition is

$$\varphi_0^{-1}U_0 \text{ overlaps } \varphi_1^{-1}U_1$$

then clearly this is a double point. But if instead it is

$$\varphi_1^{-1}U_0 \text{ overlaps } \varphi_0^{-1}U_1$$

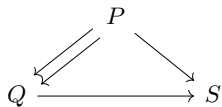
then  $q_0 \in \partial P_1$  and  $q_1 \in \partial P_0$  so we can swap indices. Hence a double point.

**3.25.** Take a point  $q_0 \in Q$ . Since  $q_0, q_0$  is not a double point, we can pick open sets  $U_0, U_1$  both containing  $q_0$  so that  $\varphi^{-1}U_0$  does not overlap  $\varphi_1^{-1}U_1$ . Replace them both by their intersection  $U := U_0 \cap U_1$ . So  $\varphi_0^{-1}U$  does not overlap  $\varphi_1^{-1}U$ . So  $U$  does not overlap  $\varphi_0\varphi_1^{-1}U$  or  $\varphi_1\varphi_0^{-1}U$ . So  $U$  intersects each equivalence class at most once, a *slice*. So  $\pi: U \rightarrow \pi U$  is a bijective continuous map to an open set. Making  $U$  smaller, we can make  $U$  have compact closure, so  $\bar{Q}$  is locally compact. Take a compact subset  $\bar{K} \subseteq \bar{Q}$ . Its preimage  $K \subseteq Q$  consists of a union of equivalence classes. Take any open cover of  $K$  by open sets  $U_a \subseteq Q$ . To show that there is a finite subcover, it suffices to prove the same for some refinement, so we can make these  $U_a$  smaller so that they slice. But then their images  $\bar{U}_a = \pi U_a \subseteq \bar{Q}$  form an open cover of  $\bar{K}$ , so have a finite subcover. Hence  $\pi$  is a proper map to a locally compact Hausdorff space, bijective on all open slices, and so a homeomorphism on each open slice. Take a chart  $(U, \varphi)$  for which  $\pi$  is a homeomorphism on  $U$ . Then

$$\left( \pi U, \varphi \circ (\pi|_U)^{-1} \right)$$

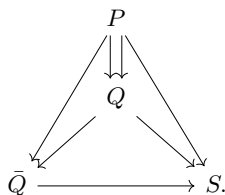
is a chart on  $\bar{Q}$ . Such charts cover  $\bar{Q}$  and their transition maps are those of  $Q$ .

**3.26.** Take a manifold  $S$  and smooth maps  $P \rightarrow S$  and  $Q \rightarrow S$ . Suppose that



commutes.

Define  $\bar{Q} \rightarrow S$  by taking each point  $\bar{q} \in \bar{Q}$ , writing it as  $\bar{q} = \pi(q)$  and applying  $Q \rightarrow S$  to  $q$ . This is well defined because equivalent points map to the same point of  $S$ . It is the unique map for which the diagram



commutes. On each open slice,  $\pi$  is a local homeomorphism, so we can locally invert and composing with  $Q \rightarrow S$ , so  $\bar{Q} \rightarrow S$  is smooth.

**4.2.** Let

$$F(x, y, z) = \begin{pmatrix} x^2 + y^2 - z \\ x^2 + z^2 - 1 \end{pmatrix}.$$

So  $C$  is the set where  $F = 0$ . Calculate

$$\frac{\partial F}{\partial(y, z)}(x, y, z) = \begin{pmatrix} 2y & -1 \\ 0 & 2z \end{pmatrix}.$$

This matrix is invertible except where  $y = 0$  or  $z = 0$ . If  $z = 0$ , on the paraboloid,  $x = y = 0$ . If  $y = 0$ , on the paraboloid  $z = x^2$ , and so we find  $z = (\sqrt{5} - 1)/2$  and  $x = \pm\sqrt{z}$ . So the implicit function theorem says that near every point of  $C$ , except perhaps at

$$(x, y, z) = \left( \pm\sqrt{\frac{\sqrt{5}-1}{2}}, 0, \frac{\sqrt{5}-1}{2} \right),$$

the curve  $C$  can be somehow expressed as the graph  $y = y(x), z = z(x)$  of smooth functions.

**4.5.** First, we will prove that  $\Phi'(p_0)$  is well defined on the velocities of paths. Secondly, we can compose paths in Euclidean space with  $\varphi^{-1}$  to see that every tangent vector is the velocity of a path. (It suffices to show that the velocities of paths span the tangent vectors.) We have to see that  $p'(t_0)$  determines  $(\Phi \circ p)'(t_0)$ . Suppose that we have two paths  $p(t)$  and  $\tilde{p}(t)$  with  $p_0 = p(t_0) = \tilde{p}(t_0)$  and with  $p'(t_0) = \tilde{p}'(t_0)$ . Let  $q(t) := \Phi(p(t))$  and  $\tilde{q}(t) := \Phi(\tilde{p}(t))$ . We need to prove that  $q'(t_0) = \tilde{q}'(t_0)$ . Take a chart  $(U, \varphi)$  on  $P$  near  $p_0$  and a chart  $(W, \psi)$  on  $Q$  near  $q_0 := \Phi(p_0)$ . Let  $x_0 := \varphi(p_0)$ ,  $x(t) = \varphi(p(t))$ ,  $\tilde{x}(t) = \varphi(\tilde{p}(t))$ . Let  $y_0 := \psi(q_0)$ ,  $y(t) = \psi(q(t))$ ,  $\tilde{y}(t) = \psi(\tilde{q}(t))$ . So  $p'(t_0) = \tilde{p}'(t_0)$  has a unique representative  $(\varphi, v)$  in the chart  $\varphi$ , given by

$$v := x'(t_0) = \tilde{x}'(t_0).$$

Similarly  $q'(t_0)$  and  $\tilde{q}'(t_{\bar{0}})$  have unique representatives  $(\psi, w)$  and  $(\psi, \tilde{w})$  in the chart  $\psi$ , given by

$$w := y'(t_0)$$

and

$$\tilde{w} = \tilde{y}'(t_{\bar{0}}).$$

We have to check precisely that  $\tilde{w} = w$ .

$$\begin{aligned}\tilde{w} &= \tilde{y}'(t_{\bar{0}}), \\ &= (\psi \circ \tilde{q})'(t_{\bar{0}}), \\ &= (\psi \circ \Phi \circ \tilde{p})'(t_{\bar{0}}),\end{aligned}$$

Plug  $\varphi^{-1} \circ \varphi$  in:

$$= (\psi \circ \Phi \circ \varphi^{-1} \circ \varphi \circ \tilde{p})'(t_{\bar{0}}),$$

Apply chain rule on  $\psi \circ \Phi \circ \varphi^{-1}$ :

$$\begin{aligned}&= (\psi \circ \Phi \circ \varphi^{-1})'(x_0)(\varphi \circ \tilde{p})'(t_{\bar{0}}), \\ &= (\psi \circ \Phi \circ \varphi^{-1})'(x_0)\tilde{x}'(t_{\bar{0}}), \\ &= (\psi \circ \Phi \circ \varphi^{-1})'(x_0)x'(t_0), \\ &= (\psi \circ \Phi \circ \varphi^{-1})'(x_0)(\varphi \circ p)'(t_0), \\ &= (\psi \circ \Phi \circ \varphi^{-1} \circ \varphi \circ p)'(t_0), \\ &= (\psi \circ \Phi \circ p)'(t_0), \\ &= (\psi \circ q)'(t_0).\end{aligned}$$

**4.7.** Take a path  $m(t)$  with  $m(0) = m_0$ , and map it to its equivalence class in  $T_{m_0}M$ . Since  $M \subset \mathbb{R}^n$  is an embedded submanifold, there are coordinates  $u_1, \dots, u_n$  on  $\mathbb{R}^n$  near  $m_0$  in which  $M$  is the set of points where  $0 = u_{p+1} = \dots = u_n$ . Let  $\varphi$  be the chart  $\varphi = (u_1, \dots, u_p)$  on  $M$  near  $m_0$ . Write the standard coordinates of  $\mathbb{R}^n$  as  $x_1, \dots, x_n$ . Hence we map  $m(t)$  to its equivalence class in  $T_{m_0}M$ , which has representative  $(\varphi, v)$  where

$$v = \left. \frac{d}{dt} \right|_{t=0} \varphi(x(t)) = \varphi'(x_0)x'(0).$$

Write the equivalence class of  $(\varphi, v)$  as  $[\varphi, v]$ . Define a linear map

$$m'(0) \in T'_{m_0}M \mapsto [\varphi, \varphi'(m_0)m'(0)] \in T_{m_0}M.$$

The kernel consists in the vectors  $m'(0)$  so that  $\varphi'(m_0)m'(0) = 0$ , i.e. with  $u'_j(m_0)m'(0) = 0$  for all  $j = 1, 2, \dots, p$ . But  $u_j = 0$  on  $M$  for  $j = p+1, \dots, n$ , so  $u'_j(m_0)m'(0) = 0$  for all  $j = 1, \dots, n$ . Since  $u_j$  are coordinates,  $m'(0) = 0$ . So our linear map is a linear isomorphism.

**4.9.** Let

$$f_i(x, y) := \int_0^1 \left. \frac{\partial f}{\partial x_i} \right|_{(tx, y)} dt.$$

Then

$$\begin{aligned} f(x, y) &= f(0, y) + \int_0^1 \frac{d}{dt} f(tx, y) dt, \\ &= 0 + \int_0^1 \sum_i \left. \frac{\partial f}{\partial x_i} \right|_{(tx, y)} x_i dt, \\ &= f_1(x, y)x_1 + f_2(x, y)x_2 + \cdots + f_p(x, y)x_p. \end{aligned}$$

**4.12.** Locally we can split up the variables  $\mathbb{R}^n$  into two collections, say  $x$  and  $y$ , so that  $S$  is the graph of  $y = y(x)$ . The tangent space at a point  $(x_0, y_0)$  is the set of vectors  $(\dot{x}, \dot{y})$  for which

$$\dot{y} = \frac{\partial y}{\partial x} \dot{x}.$$

So the normal space is the set of vectors perpendicular to those, i.e. vectors  $v = (u, w)$  so that

$$0 = (u, w) \cdot (\dot{x}, \dot{y}) = u\dot{x} + w \frac{\partial y}{\partial x} \cdot \dot{x} = (u + w \frac{\partial y}{\partial x}) \dot{x}.$$

In other words,

$$u = -w \frac{\partial y}{\partial x},$$

and we can pick  $w$  as we like. So the equations of points  $(x, y), (u, w)$  in the normal bundle are

$$y = y(x), u = -w \frac{\partial y}{\partial x}.$$

**4.13.** Some open neighborhood  $U$  is tubular, so diffeomorphic to some  $U_\rho$ , say by a map  $\psi: U_\rho \rightarrow U$ . Map  $\varphi_t(u) = (x, (1-t)v)$  if  $(x, v) = \psi^{-1}(u)$ , for  $u \in U$ .

**4.16.** Looking at the equations, we see that on the intersection,  $y \geq 0$  and  $z \geq 0$ . Let

$$F(x, y, z) := (x^2 + y^2 - z, x^2 + z^2 - y).$$

Then

$$F'(x, y, z) = \begin{pmatrix} 2x & 2y & -1 \\ 2x & -1 & 2z \end{pmatrix}.$$

Looking at  $2 \times 2$  determinants, we can see that the matrix has full rank except at points where either  $x = 0$  or  $y = -1/2$  or  $z = -1/2$ . But  $y \geq 0$  and  $z \geq 0$ , so  $x = 0$  on the intersection. On  $x = 0$ , the rank drops only when  $4yz = 1$ . But then  $x = 0$  and  $x^2 + y^2 = z$  and  $x^2 + z^2 = y$  forces  $y^4 = y$ . Since  $4yz = 1$ ,  $y \neq 0$ , so  $y^4 = y$  gives  $y^3 = 1$  so  $y = 1$ . The same argument gives  $z = 1$ . But  $4yz = 1$ , a contradiction. Therefore  $F'$  has rank 2 at every point of the intersection, so the intersection, being the zero locus of  $F$ , is a manifold. Since  $F'$  has rank 2, the set where  $F = 0$  is a manifold of dimension  $3 - 2 = 1$ .

**4.18.**

$$\varphi'(x, y, z) = 2 \begin{pmatrix} x & y & z \\ x - a & y & 0 \end{pmatrix}.$$

Suppose that  $x^2 + y^2 + z^2 = 1$  and  $(x - a)^2 + y^2 = r^2$ . At any point of  $x^2 + y^2 + z^2 = 1$ , we find the first row is not zero. The implicit function theorem fails to apply just

where  $\varphi'$  is not onto, i.e. has rank less than 2, so just where the second row is a multiple of the first one, say

$$(x - a \quad y \quad 0) = \lambda(x \quad y \quad z)$$

for some constant  $\lambda$ . If  $\lambda = 0$ , then  $x = a$  and  $y = 0$ . But  $(x - a)^2 + y^2 = r^2$ , forcing  $r = 0$ , a contradiction. So  $\lambda \neq 0$ . But then

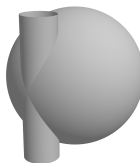
$$(x - a \quad y \quad 0) = \lambda(x \quad y \quad z)$$

forces  $z = 0$ , and either  $y = 0$  or  $\lambda = 1$ .

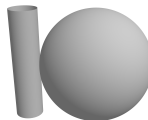
If  $y = 0$ , then  $(x, y, z) = (\pm 1, 0, 0)$  and  $a \pm r = \pm 1$ . Clearly  $a + r \geq 0$ .

If  $\lambda = 1$ , then  $x - a = x$  and  $z = 0$ , so  $x^2 + y^2 = 1$  and  $a = 0$  so  $x^2 + y^2 = r^2$ . So  $r = 1$ .

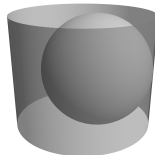
- a.  $(x, y, z) = (1, 0, 0)$ ,  $a + r = 1$ , sphere intersecting cylinder in a “figure 8” shaped curve, smooth except at that one point where the sphere and cylinder are tangent.



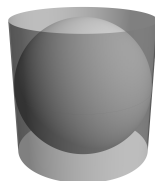
- b.  $(x, y, z) = (1, 0, 0)$ ,  $a - r = 1$ , sphere outside cylinder, touching at one point.



- c.  $(x, y, z) = (-1, 0, 0)$ ,  $a - r = -1$ , sphere inside cylinder, touching at one point.



- d.  $(x, y, z) = (x, y, 0)$ ,  $a = 0$ ,  $r = 1$ , sphere inside cylinder, intersecting along the entire equator, which is smooth, but the implicit function theorem doesn't know that.



**4.19.**  $(\sqrt{x^2 + y^2} - r_1)^2 + z^2 = r_0^2$

**4.20.** Let

$$\varphi(x, y, z) := (x^2 + y^2 - z, x^2 + z^2 - 4).$$

Then

$$\varphi'(x, y, z) = \begin{pmatrix} 2x & 2y & -1 \\ 2x & 0 & 2z \end{pmatrix}.$$

Looking at  $2 \times 2$  determinants, we can see that the matrix has full rank except at points where either  $x = 0$  or  $y = 0$ . If  $x = 0$ , we still have full rank unless either  $y = 0$  or  $z = 0$ . If  $y = 0$ , we still have full rank unless either  $x = 0$  or  $z = -\frac{1}{2}$ . There are 3 cases of rank 1:

- a.  $x = 0$  and  $y = 0$ ,
- b.  $x = 0$  and  $z = 0$ ,
- c.  $y = 0, z = -\frac{1}{2}$ .

The first is impossible at any point of the intersection since being on the paraboloid implies  $z = 0$ , but being on the cylinder implies  $x^2 + z^2 = 4$ . The second is impossible at any point of the intersection since being on the cylinder implies  $x^2 + z^2 = 4$ . The third is impossible at any point of the intersection since being on the paraboloid implies  $z \geq 0$ . Therefore  $\varphi'$  has rank 2 at every point of the intersection, so the intersection, being the zero locus of  $\varphi$ , is a manifold. Since  $\varphi'$  has rank 2, the set where  $\varphi = 0$  is a manifold of dimension  $3 - 2 = 1$ .

**4.21.** Take parameterisation  $\Phi(x) = (x, \varphi(x))$ , with left inverse  $\Psi(x, y) = x$ .

**4.23.** Let  $F(x, \lambda, t) := (A(t)x - \lambda x, x_1 - 1, t)$ ,  $F: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ . If  $F'(x, \lambda, t)$  is invertible, then  $F$  is locally invertible. Check that

$$F'(x, \lambda, t)(y, \mu, s) = (A(t)y + A'(t)sx - \lambda y - \mu x, y_1, s).$$

The kernel of  $F'(x, \lambda, t)$  is given by the equations on  $(y, \mu, s)$ :

$$(A(t) - \lambda I)y = -A'(t)sx + \mu x, y_1 = 0, s = 0.$$

The only points where  $F = 0$  are those with  $t = 0$ ,  $A(0)x = \lambda x$ ,  $x_1 = 1$ , so  $t = 0$  and  $x$  is a  $\lambda$ -eigenvector. Take the particular choice of  $x$  and  $\lambda = \lambda_0$  given in the problem. Look at the equations of the kernel of  $F'(x, \lambda, t)$  at that  $(x, \lambda_0, t)$  point:

$$(A_0 - \lambda_0 I)y = \mu x, y_1 = 0, s = 0.$$

But then

$$(A_0 - \lambda_0 I)^2 y = \mu(A_0 - \lambda_0 I)x = 0.$$

Since the generalized  $\lambda_0$ -eigenspace has dimension 1,  $y$  must be a  $\lambda_0$ -eigenvector. But again since the generalized  $\lambda_0$ -eigenspace has dimension 1,  $y = cx$  for some  $c$ . But  $y_1 = 0$  while  $x_1 = 1$ , so  $c = 0$ , i.e.  $y = 0$ . Our equations become  $0 = \mu x$  and  $x \neq 0$  so  $\mu = 0$ . Therefore  $(y, \mu, s) = (0, 0, 0)$ , i.e.  $F'(x, \lambda, t)(y, \mu, s) = 0$  just when  $(y, \mu, s) = (0, 0, 0)$ , so is  $F'(x, \lambda, t)$  invertible. Therefore  $F$  is invertible near  $(x, \lambda_0, 0)$ , say with inverse  $G(p, \sigma, t)$ . Let  $(x(t), \lambda(t), t) = G(0, 0, t)$ , and by definition of  $F$ ,  $F(x(t), \lambda(t), t) = (0, 0, t)$ , i.e.  $A(t)x(t) = \lambda(t)x(t)$  and  $x_1(t) = 0$ .

**4.24.** The map

$$F(\theta, y, z) = (y \sin \theta, y \cos \theta, z)$$

is a local diffeomorphism on the set where  $y > 0$ , by the rank theorem:

$$F'(\theta, y, z) = \begin{pmatrix} y \cos \theta & \sin \theta & 0 \\ -y \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so  $\det F' = y > 0$ . Therefore we can locally invert, identifying  $S$  by local diffeomorphism with the surface  $\mathbb{R} \times C$ .

**4.26.** No two  $f_i$  vanish at the same point, and  $f = 0$  just where exactly one  $f_k = 0$ . Differentiate:

$$\frac{\partial f}{\partial x} = \sum_i f_1 f_2 \cdots \hat{f}_i \cdots f_g \frac{\partial f_i}{\partial x},$$

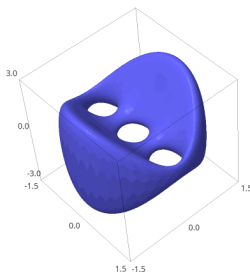
But at any point where  $f_k = 0$ , the terms with  $f_k$  in them drop out:

$$\frac{\partial f}{\partial x} = f_1 f_2 \cdots \hat{f}_k \cdots f_g \frac{\partial f_k}{\partial x},$$

and similarly

$$\frac{\partial f}{\partial y} = f_1 f_2 \cdots \hat{f}_k \cdots f_g \frac{\partial f_k}{\partial y},$$

one of which is not zero. So  $S$  is a smooth surface, compact because all  $f_i$  get as large positive we like for large enough  $x, y$ , so then  $f(x, y) + z^2 > 0$ , not a point of the surface. The surface  $S$  has  $g$  “doughnut holes”:



**4.32.** For  $x \in \mathbb{R}$  let  $F(x) = x^2$ , so  $M = \{0\}$ , but  $F'(0) = 0$ , so as a  $1 \times 1$  matrix,  $F'(0)$  has rank 0, *not* rank 1.

**4.33.** A counterexample: consider  $\varphi: (x, y) \in \mathbb{R}^2 \rightarrow z = x + x^2 y \in \mathbb{R}$ . Differentiate:

$$z_x = 1 + 2xy, z_y = x^2,$$

cannot both vanish, since  $0 = z_y = x^2$  forces  $x = 0$ , but then  $z_x = 1 + 2xy = 1 \neq 0$ . The fiber over  $z = 0$  is

$$\varphi^{-1}(0) = (z = 0) = (x = 0) \cup (xy = -1)$$

a union of a line and a hyperbola, so three components each diffeomorphic to  $\mathbb{R}$ . The fiber over  $z = z_0$  for any  $z_0 \neq 0$  is

$$y = \frac{z_0 - x}{x^2},$$



for  $x \neq 0$ , so a union of two components:  $x > 0$  and  $x < 0$ . Hence the fiber over  $z = 0$  is not even homeomorphic to the other fibers.

**4.37.** Suppose that  $\varphi$  fails to be injective on every open set  $U$  containing  $X$ . Take a sequence of nested open sets  $U_i$  whose intersection is  $X$ , each with compact closure; see problem 1.32 on page 11. For each, we have some  $p_i \neq p'_i \in U_i$  with  $q_i = \varphi(p_i) = \varphi(p'_i)$ . They lie in the compact  $\bar{U}_i$ , so after taking infinite subsequences, we can assume that  $p_i \rightarrow p$  and  $p'_i \rightarrow p'$  converge. By continuity  $q := \varphi(p) = \varphi(p')$ . Since  $p_i, p'_i \in U_i$  we find  $p, p' \in X$ . But  $\varphi$  is injective on  $X$  so  $p = p'$ . By the inverse function theorem, every point of  $X$  lies in an open subset of  $P$  in which  $\varphi$  is a diffeomorphism. So near  $p$ ,  $\varphi$  is injective, i.e.  $p_i = p'_i$  for all but finitely many  $i$ , a contradiction.

So  $\varphi$  is injective on some open set. Replace  $P$  by that open set, and so assume that  $\varphi$  is injective. Replace  $P$  by the open set of points  $p \in P$  at which  $\varphi'(p)$  is injective.

Suppose that  $P$  and  $Q$  have the same dimension. By the inverse function theorem,  $P$  is covered in open sets on which  $\varphi$  is a diffeomorphism. On these open sets,  $\varphi$  takes open sets to open sets. Writing any open set in  $P$  as a union of such,  $\varphi$  takes any open subset of  $P$  to an open subset of  $Q$ . Replace  $Q$  by  $\varphi(P)$ ; we can assume that  $\varphi$  is onto.

**5.3.** Suppose that some map  $\psi$  descends:  $\psi = \Psi \circ \varphi$ . Then for any points  $p_0, p_1$  of any fiber  $P_{q_0}$ ,

$$\psi(p_0) = \Psi(\varphi(p_0)) = \Psi(q_0) = \psi(p_1).$$

Suppose that every  $C^k$  map constant on the fibers descends to a continuous map. Let  $q_0 \in Q$  be a point on the topological boundary of the image of  $\varphi$ , but not in the image. Let  $M := Q - \{q_0\}$ ,  $\psi := \varphi: P \rightarrow M$ . Then  $\psi$  is constant on the fibers of  $\varphi$ , so descends, say to  $\Psi: Q \rightarrow M$ . On the image of  $\varphi$ ,  $\Psi$  is the identity map, so also on the closure of the image of  $\varphi$ , so  $\Psi(q_0) = q_0$ , but  $q_0$  is not in  $M$ , a contradiction.

Suppose that every continuous map constant on the fibers has at most one descent. Pick a point  $q_0$  outside the closure of the image of  $\varphi$ . Take two maps  $\Psi_0, \Psi_1$ , differing near  $q_0$ , but both constant on the image of  $\varphi$ , and let  $\psi = \Psi_0 \circ \varphi = \Psi_1 \circ \varphi$ .

Suppose that  $\varphi$  has dense image. Take a continuous map  $\psi$  with two descents  $\Psi_0, \Psi_1$ . Then  $\psi = \Psi_0 \circ \varphi = \Psi_1 \circ \varphi$ , so  $\Psi_0 = \Psi_1$  along the image of  $\varphi$ , and so everywhere on  $Q$ .

Suppose that  $\varphi$  is a surjective submersion. Uniqueness follows from dense image. Constancy on the fibers allows us to define a map, but perhaps not continuous,  $\Psi: Q \rightarrow M$  by defining  $\Psi(q)$  to be any value of  $\psi$  at any point of  $P_q$ . Pick a point  $q_0 \in Q$  and a point  $p_0 \in P_{q_0}$ . By the implicit function theorem, we can find coordinates  $x^1, \dots, x^{q+k}$  on  $P$  near  $p_0$  and  $y^1, \dots, y^q$  on  $Q$  near  $q_0$  so that

$$y = (y^1, \dots, y^q) = \varphi(x) = (x^1, \dots, x^q),$$

taking  $p_0$  and  $q_0$  to the origin. Take coordinates  $z^1, \dots, z^s$  on  $M$  near  $m_0 := \psi(p_0)$ . The map  $z = \psi(x)$  is constant on fibers, so does not depend on  $x^{q+1}, \dots, x^{q+k}$ , and so

$$\Psi(y) = \Psi(x^1, \dots, x^q) = \psi(\varphi(x)) = \psi(x^1, \dots, x^q, 0, \dots, 0)$$

is smooth.

**4.41.** Take a point  $m_0$  of  $M_k \cap Z$ . Take  $k$  functions from  $A$  whose partial derivatives are linearly independent there. By the rank theorem, change coordinates to get them to be  $x^1, \dots, x^k$  in coordinates  $x^1, \dots, x^k, y^1, \dots, y^{n-k}$ , with  $m_0$  at the origin of those coordinates. All of the functions  $f$  in  $A$  vanish at every point of  $Z$ . In particular,

$x^1, \dots, x^k$  vanish on  $Z$ , so on  $M_k \cap Z$ . So  $M_k \cap Z \subseteq Z \subseteq M' = (0 = x^1 = \dots = x^k)$ . But then  $M_k \cap Z$  is a submanifold of codimension  $k$  in  $M$  just when it equals  $M'$  near the origin, and hence equals  $Z$  near the origin.

As example, let  $M = \mathbb{R}$  and  $A = (f)$  where  $f$  is a smooth function with  $f(x) = 0$  for  $x \leq 0$ , and  $f(x) > 0$  and  $f'(x) > 0$  for all  $x > 0$ . Then  $M_0 = Z = \{x \leq 0\}$ ,  $M_1 = \{x > 0\}$ , and so  $M_0 \cap Z = \{x \leq 0\}$  is not a submanifold, while  $M_1 \cap Z$  is empty.

By problem 4.9 on page 42,  $M_k \cap Z = Z = M'$  just when the functions which vanish on  $M'$  are those of the form

$$x_1 f_1 + \dots + x_k f_k.$$

**4.42.** Since we don't yet know if  $A$  is a submanifold, we can only define a "formal tangent space": for any  $a_0 \in A$ , define  $T_{a_0}A$  to mean the set of all vectors  $v \in T_{a_0}M$  so that  $\varphi'(a_0)v = v$ . So  $T_{a_0}A \subseteq T_{a_0}M$ . Note that

$$\varphi \circ \varphi = \varphi,$$

so

$$\varphi' \circ \varphi' = \varphi',$$

For any point  $m_0 \in M$ , if we let  $a_0 := \varphi(m_0)$ , and take any tangent vector  $v \in T_{m_0}M$ ,

$$\varphi'(a_0) \circ \varphi'(m_0)v = \varphi'(m_0)v,$$

so  $\varphi'(m_0): T_{m_0}M \rightarrow T_{a_0}A$ . If  $m_0 = a_0$ , then by definition of  $T_{a_0}A$ ,

$$\varphi'(a_0): T_{a_0}A \rightarrow T_{a_0}A$$

is the identity map, so

$$\varphi'(a_0): T_{a_0}M \rightarrow T_{a_0}A$$

is a surjection. By continuity,

$$\varphi'(a_0): T_{m_0}M \rightarrow T_{a_0}A$$

is a surjection for any  $m_0$  near  $a_0$ . (To see this: in some chart, the determinant of some rows of  $\varphi'(a_0)$  is not zero, so the same is true of  $\varphi'(m_0)$  for  $m_0$  near  $a_0$  by continuity.) Therefore  $\varphi'(m_0)$  is of constant rank near  $m_0$ . So locally  $\varphi$  becomes a linear projection in some charts, and  $A$  becomes a linear subspace.

**4.43.** The tangent cone is a cone, since we can rescale the time parameter:

$$\left. \frac{d}{dt} \right|_{t=0} x(c_0 t) = c_0 x'(0).$$

**4.45.** Take a cone  $C$ . Given any  $x_0 \in C$ , let  $x(t) := tx_0$ . Clearly  $x_0 = x'(0)$  is in the tangent cone of  $C$ . So  $C$  lies in its own tangent cone.

We need to prove that every vector  $v$  in the tangent cone of  $C$  lies in  $C$ . If  $v = 0$ , clearly  $v \in C$ . So assume that  $v \neq 0$ . We can scale, so we can assume that  $|v| = 1$ . So there is some continuously differentiable  $x(t) \in C$  so that  $x(0) = 0$  and  $x'(0) = v$ , i.e.

$$\frac{x(t)}{t} \rightarrow v.$$

But  $x(t)/t \in C$  and  $C$  is closed, so  $v \in C$ .

**4.46.** Suppose that  $C$  is a cone and an embedded submanifold. An embedded submanifold is locally closed near any of its points, so  $C$  is locally closed near the origin. By rescaling,  $C$  is closed. The tangent cone of  $C$  at the origin is the tangent space. But the tangent cone of  $C$  is  $C$  so  $C$  is its own tangent space.

**5.6.** If not continuous, then not smooth. Suppose continuous. Take a small enough open subset of  $S$  so that that open subset is embedded in  $M$ . Then its preimage in  $P$  is mapped to an embedded submanifold. We can even arrange that the embedded submanifold is contained in the domain of a chart in which it is identified with a linear subspace. Clearly having image inside that linear subspace, our map is a smooth map to that linear subspace.

**5.14.** Write any quaternion as  $q = q_0 + iq_1 + jq_2 + kq_3 = (q_0 + iq_1) + j(q_2 - iq_3) = \alpha + j\beta$ , with  $\alpha, \beta \in \mathbb{C}$ . For any vector  $v \in V$ , define

$$vq := \alpha v + \beta Jv.$$

The vector space laws are all obvious except  $v(qr) = (vq)r$ , which you need to expand out to check.

On the other hand, given any quaternionic vector space  $V$ , let  $cv := vc$  for  $c \in \mathbb{C} \subset \mathbb{H}$ , and let  $Jv := vj$ .

**5.15.** If  $\varphi$  is a bijection,  $\varphi$  has an inverse map  $\varphi^{-1}$ , so  $0 = \varphi(v)$  forces  $0 = \varphi^{-1}(\varphi(v)) = v$ , i.e.  $\varphi$  has zero kernel. If  $\varphi$  has zero kernel, then adding dimensions of image and kernel, the image of  $\varphi$  is all of  $W$ . Note that  $\varphi(v_1) = \varphi(v_2)$  implies  $0 = \varphi(v_1 - v_2)$ , so  $0 = v_1 - v_2$ , so  $v_1 = v_2$ . So  $\varphi$  is a bijection, so has an inverse map  $\varphi^{-1}$ . Since  $\varphi$  identifies addition and scaling operations in  $V$  and  $W$ , so does  $\varphi^{-1}$ . So  $\varphi^{-1}$  is  $k$ -linear.

**5.19.** Compactness is easy, as the rows (or columns) are an orthonormal basis. Via the Cayley transform: see problem 2.6 on page 18); the same steps work identically for any  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ . Via the implicit function theorem: for any  $n \times n$  matrix  $A$  with entries in  $k$ , denote by  $A^*$  the adjoint, i.e. if  $k = \mathbb{R}$ ,  $A^*$  is the transpose of  $A$ , and otherwise  $A^*$  is the transpose with all entries conjugated. The reader can check that the group  $G$  (after we take a basis) is the set of matrices  $A$  with  $A^*A = I$ . Define a map  $f(A) = A^*A$ , taking  $n \times n$  matrices with entries in  $k$  to self adjoint matrices  $S = f(A)$ , i.e.  $S^* = S$ . The set of self adjoint matrices forms a vector space, and we can identify each self adjoint matrix with its diagonal and below diagonal entries:  $k^{n(n+1)/2}$ . Fix some  $A \in G$ . Expand  $f(A + tB) - f(A) = t(A^*B + B^*A) + O(t)^2$ , so  $f'(A)B = A^*B + B^*A$ . For each matrix  $B$ , let  $C := A^*B$ . This is linear isomorphism  $B \mapsto C$ . So to find the rank of  $f'(A)$ , we need only find the rank of the linear map

$$C \mapsto B = A^{*-1}C \mapsto f'(A)B = C + C^*.$$

This is a surjective linear map, so  $A$  is a regular point of  $f$ . So by the implicit function theorem, the level set  $G = (f = I)$  is a manifold of dimension  $dn(n-1)/2$  where  $d$  is the real dimension of  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ .

**6.4.** Each point has some “survival time”  $\varepsilon > 0$ : how long its flow is defined (in past and future). The flow is defined on some open set containing  $\{0\} \times \{x\}$ , so containing some set  $(-\varepsilon, \varepsilon) \times V$  where  $V \subset U$  is open. In other words, every point  $x$  lies in an open set with a common lower bound on “survival times” of all points in that open set. Take points of  $K$  whose survival times approach the infimum survival

time. A subsequence converges to a point with infimum survival time, so survival time achieves a positive minimum.

**6.5.** By problem 6.4 on page 69, at each point of the compact set, the flow line through that point is defined for time at least some  $\varepsilon > 0$ . But then we can repeatedly flow for time  $\varepsilon/2$ , as many times as we like, so for times as large as we like.

**6.6.** By problem 6.4 on page 69, all of the flow lines are defined for time at least some  $\varepsilon > 0$ . But then we can repeatedly flow for time  $\varepsilon/2$ , as many times as we like, so for times as large as we like.

**6.8.** By problem 6.5 on page 69, if a flow line stays in a compact set, it is defined for all time. So any flow line that stays inside  $c_- \leq f \leq c_+$  is defined for all time. We can replace  $c_+$  by any larger number, so assume that  $c_+ > 0$ , and similarly that  $c_- < 0$ . We can replace  $a_+, b_+$  by any larger numbers, so assume  $a_+, b_+ > 0$ , and similarly  $a_-, b_- > 0$ .

Along each flow line, as we go forward, if  $f(x) \geq c_+$ , then (for simplicity, writing  $a_+, b_+$  as  $a, b$ ):

$$\frac{df}{dt} \leq af + b,$$

so

$$\frac{1}{af + b} \frac{df}{dt} \leq 1.$$

Integrate:

$$f \leq Ce^{at}.$$

Take a point where  $c_- \leq f \leq c_+$ , which has the least “survival time” among all such points. (“Survival time” meaning positive or negative time for which the flow line is defined.) There is such a point, by compactness. So under the flow of  $X$ , the compact set  $(c_- \leq f \leq c_+)$  expands its maximum and minimum  $f$  values by at most some exponential factor, mapping still to some compact set. On that new compact set, the “survival time” is still positive, a contradiction unless all points have infinite survival time.

In  $\mathbb{R}^n$  with  $r^2 = x_1^2 + \cdots + x_n^2$ , the vector field

$$X = -e^{r^{10}}(x_1^{97}, x_2^{97}, \dots, x_n^{97}) + P,$$

(where  $P$  is any polynomial vector field) compresses  $f = r^2$ , because for large enough  $x$ ,  $X$  is dominated by the nonpolynomial part, and that points toward the origin, and  $f$  grows as we go out from the origin, so  $f'(x)X(x) < 0$ . Even though  $X$  gets very large, and can do almost anything it likes near the origin,  $X$  is complete.

**6.9.** Take a bump function  $h$  in  $\bar{B}$  vanishing near  $\partial\bar{B}$  and equal to 1 in some smaller ball containing both  $x_0$  and  $x_1$ . Let  $X$  be the vector field  $X(x) = h(x)(x_1 - x_0)$ . Since this vector field vanishes outside a compact set, its flow lines are defined for all time. Let  $\varphi(x)$  be the point that  $x$  gets to along the flow of  $X$  at time  $t = 1$ ;  $\varphi$  is a diffeomorphism, with inverse given by flowing along  $-X$  for time  $t = 1$ . Inside the smaller ball  $\varphi(x) = x + x_0$ .

**6.11.** Cover  $P$  in open sets  $P_a \subseteq P$  so that

$$\psi_a := \psi|_{P_a}$$

is a diffeomorphism to  $Q_a := \psi_a(P_a)$ . Let  $X_a := X|_{Q_a}$ . Define a vector field  $Y_a$  on  $P_a$  by  $Y_a := (\varphi_a^{-1})_* X_a$ . On any overlap  $P_{ab} := P_a \cap P_b$ ,  $\psi_a = \psi_b$ , and  $X_a = X_b$  on

$Q_{ab} = Q_a \cap Q_b$ . So  $Y_a = Y_b$  on  $P_{ab}$ . Hence we can uniquely define a smooth vector field  $Y$  on  $P$  by  $Y = Y_a$  on  $P_a$ . Moreover, for any  $p_0 \in P_a$ , if we let  $q_0 := \psi(p_0)$ ,  $\psi'(p_0)Y(p_0) = (\psi_a^* Y_a)(q_0) = X_a(q_0) = X(q_0)$ .

**6.12.** Since  $df \neq 0$  there, this equation has full rank in the components of  $X$  in any coordinates, so there are local solutions by the implicit function theorem. With a partition of unity near  $(f = c)$ , we can glue together to get one such vector field over some interval  $a_0 < f < b_0$ .

**6.13.** Take some point  $(m, 0)$  and consider the flow line through it. Suppose it is defined on some interval  $t_0 < f(m_0) + t < t_1$  with  $a < t_0 < t_1 < b$ . The set  $K := f^{-1}[t_0, t_1] \subset M$  is compact, because  $f$  is proper. By problem 6.4 on page 69, there is some minimum time  $\varepsilon > 0$  so that the flow on every point of  $K$  is defined still for time at least  $\varepsilon$  in past and future.

**6.14.** Note that  $\varphi(s, t) = s$  for any  $s \in S$  so  $\varphi'(s, t)v = v$  for  $v \in T_s S$ . Also  $\varphi'(s, t)\frac{\partial}{\partial t} = X(\varphi(s, t))$  by the equation of the flow of  $X$ . Since  $Xf = 1$ ,  $X$  is not tangent to  $S$ , so  $\varphi'(s, t)$  has full rank, so  $\varphi$  is a local diffeomorphism. If

$$\varphi(s_0, t_0) = \varphi(s_1, t_1),$$

then

$$t_0 = f(\varphi(s_0, t_0)) = f(\varphi(s_1, t_1)) = t_1.$$

Since the flow of  $X$  is a diffeomorphism,  $s_0 = s_1$ . So  $\varphi$  is injective. Flow  $X$ , for time  $c - t$ , to take  $(f = t)$  to  $(f = c) = S$ , to see that  $\varphi$  is onto.

**6.16.** Take two points  $q_0, q_1 \in Q$ . Take a piecewise smooth path in  $Q$  between them. Cut up that path into pieces, each an embedded closed interval. Each lies in a slightly larger embedded open interval. Apply theorem 6.4 on page 72 to each of those embedded open intervals.

**6.17.** Let  $Q$  be the set of all compact oriented surfaces of genus  $g$  presented as in that problem: as a closed disk  $D_0$  in the plane, and  $g$  nonoverlapping disks  $D_1, \dots, D_g$  sitting in the interior of  $D_0$ . So each point of  $Q$  consists of the centers  $(x_i, y_i)$  and radii  $r_i$  of these disks  $i = 0, \dots, g$ :

$$q = (x_0, y_0, \dots, x_g, y_g, r_0, \dots, r_g) \in Q \subset \mathbb{R}^{3(g+1)}.$$

These are arbitrary subject to

$$\begin{aligned} \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} &< r_0 - r_i, \\ \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} &> |r_i - r_j|, \end{aligned}$$

for any  $i = 1, 2, \dots, g$  and  $j = i + 1, \dots, g$ . In particular,  $Q \subset \mathbb{R}^{3(g+1)}$  is an open set. As in that problem, we let

$$f_i(x, y, q) := (x - x_i)^2 + (y - y_i)^2 - r_i^2,$$

let  $f := f_0 f_1 f_2 \dots f_g$ . But now  $f$  is really a function not just of  $x, y$  but also of these additional parameters, i.e.  $f(x, y, q)$ . Let

$$P := \{(x, y, z, q) \mid f(x, y, q) + z^2 = 0\} \subset Q \times \mathbb{R}_{x, y, z}^3.$$

So  $P$  is the choice of one of these surfaces, together with a point on that surface. The map

$$\varphi: (x, y, z, q) \in P \mapsto q \in Q$$

is proper, as its fiber  $P_q$  over a point  $q \in Q$  consists of the points  $(x, y, z)$  lying on a particular compact surface of genus  $g$ . The 1-form  $df + 2z dz$  is linearly independent of the 1-forms from  $\mathbb{R}^{3(g+1)}$ , clearly, where  $z \neq 0$ . But then where  $z = 0$ , we find that  $df$  has a nonzero  $dx$  or  $dy$  component. So  $P \subset Q \times \mathbb{R}^3$  is a smooth hypersurface, and the equation of the tangent plane of  $P \subset Q \times \mathbb{R}^3$  at any point projects by linear surjection to that of  $Q$ , i.e.  $\varphi$  is a submersion. By problem 6.16 on page 73, all fibers of  $\varphi: P \rightarrow Q$  are diffeomorphic. These fibers are the compact surfaces of genus  $g$  we constructed in problem 4.26.

**6.18.** Take a surface of genus  $g = g_0 + g_1$  as constructed in problem 4.26 on page 47. We can take the large disk  $D_0$  to be the unit disk centered at the origin. Pick a line in the plane passing through the center, say  $x = 0$ . By problem 6.17 on page 73, we can move  $g_0$  of our disks  $D_i$  to one side of that line, and  $g_1$  to the other side. We center those disks all on the line  $y = 0$  for simplicity of notation. The plane  $x = 0$  strikes the surface in a compact smooth curve, by the implicit function theorem. The equation of that curve is

$$0 = y = z^2 + (x^2 - 1) \prod ((x - x_i)^2 - r_i^2).$$

These factors  $(x - x_i)^2 - r_i^2$  vanishes on circles not intersecting  $x = 0$ , so have  $r_i < |x_i|$ .

We can deform the equation of our curve by replacing each function

$$f_i(x, y) = (x - x_i)^2 + (y - y_i)^2 - r_i^2$$

by  $t + (1 - t)f_i$ , a family of curves

$$0 = z^2 + (x^2 - 1) \prod (t + (1 - t)((x - x_i)^2 - r_i^2)).$$

Consider the set  $M$  of points  $(x, z, t)$  satisfying this equation. Map  $(x, z, t) \mapsto t$ . We want to check that this map is a submersion from  $M$  to some open interval containing  $[0, 1]$ . That is precisely checking that our equation has nonzero derivative in  $t$ . By theorem 6.4 on page 72, these are all diffeomorphic for any value of  $t$ . For  $t = 0$ , we have the curve we started with, while for  $t = 1$  we have a circle, so our curve is connected and diffeomorphic to a circle.

Theorem 6.5 on page 73 says that our surface is glued together along that circle, i.e. a connected sum of two surfaces:  $M_0 = (x < 0)$  and  $M_1 = (x > 0)$ .

Consider how we can present the surfaces of genus  $g_1$  and  $g_2$  as glued together in the plane. Consider the closed disk  $D'_0$  of radius  $1/2$  with center at  $(1/2, 0)$ . Make a diffeomorphism

$$(x, y) \in D_0 \cap (x > 0) \rightarrow (x, \eta) \in D'_0 - \{(0, 0)\},$$

by

$$\eta = y \sqrt{\frac{x}{x+1}}.$$

Let

$$g_0 := \left(x - \frac{1}{2}\right)^2 + y^2 - \frac{1}{2^2},$$

so  $D'_0 = (g_0 \leq 0)$ . Our diffeomorphism takes the disks  $D_i$  inside  $D_0$  into complicated shapes ( $g_i = 0$ ) where

$$g_i := (x - x_i)^2 + \frac{(x+1)}{x} \eta^2 - r_i^2.$$

We use the same argument again, putting in a parameter, say

$$g_i^t := (x - x_i)^2 + \left( (1-t) \frac{(x+1)}{x} + t \right) \eta^2 - r_i^2$$

which deforms the surface. Let  $g = g_0 \dots g_i$ . For each  $t$  we have a surface with equation  $g^t + z^2 = 0$  in  $\mathbb{R}^3$ . As we deform  $t$ , it deforms, and again we need to check that the left hand side of this equation has nonzero differential in  $t$ , so that we can deform the  $t$  parameter, again by theorem 6.4 on page 72.

So finally  $M_0$  is diffeomorphic to a compact oriented surface of genus  $g_0$ , punctured at one point. Similarly,  $M_1$  is diffeomorphic to a compact oriented surface of genus  $g_1$ , punctured at one point.

**6.20.** It is  $C^1$  diffeomorphic to a disjoint union of some number of circles, hence has the smooth structure of those circles. If it has another smooth structure, in that smooth structure it is a  $C^\infty$  compact curve, so  $C^\infty$  diffeomorphic to some number of circles, the same number (its number of connected components as a topological space).

**6.21.** A *basis of local sections* of the normal bundle is a collection of local sections, all on the same open set, so that they yield a basis of each normal space. Any two bases of local sections are  $C^{k-1}$  linear combinations of one another. In a local chart, there are clearly bases of local sections.

Write the immersed submanifold  $S$  as an immersion  $\varphi: S \rightarrow M$ . On a small enough open subset of  $S$ , we can find coordinates on  $S$  and also find that the immersion  $\varphi: S \rightarrow M$  is an embedding. On a perhaps smaller open set, the immersion is expressed as  $\varphi: x \mapsto (x, 0)$  where  $(x, y)$  are coordinates on  $M$ . So the vector fields

$$\frac{\partial}{\partial y^i}$$

are a basis of local sections. We associate to these charts on  $S$  and  $M$  the *associated* chart

$$\Phi: v^j \frac{\partial}{\partial y^j} \mapsto (x, v)$$

on  $\nu_S$ .

On the overlap of any two such charts, the transition map  $(x, y) \mapsto (X, Y)$  preserves  $S$ , i.e.  $Y = 0$  just when  $y = 0$ . By Hadamard's lemma,  $Y = Y(x, y) = A(x)y + O(y)^2$ , some matrix  $A(x)$ . To have a diffeomorphism, clearly  $A(x)$  is invertible. Each local section

$$v^i(x) \frac{\partial}{\partial y^i}$$

of the normal bundle is carried to

$$v^i(x) A_i^j(x) \frac{\partial}{\partial Y^j} =$$

The transition maps are thus

$$(x, v) \mapsto (X, V) = (X(x, 0), A(x)v).$$

Clearly in this  $C^{k-1}$  structure we have  $\pi C^{k-1}$  and all local sections  $C^{k-1}$ .

**6.22.** Take a tubular neighborhood of  $S \subseteq \mathbb{R}^n$

$$\varphi_{S, \mathbb{R}^n} : U_{S, \mathbb{R}^n} \rightarrow \mathbb{R}^n.$$

Take a smooth deformation retraction to  $M$

$$\psi : U_{M, \mathbb{R}^n} \times [0, 1] \rightarrow U_{M, \mathbb{R}^n}$$

of some open set  $U_{M, \mathbb{R}^n} \subset \mathbb{R}^n$  containing  $M$ . Note that  $\nu_{S, M} \subseteq \nu_{S, \mathbb{R}^n}$ . Let

$$U_{S, M} := U_{S, \mathbb{R}^n} \cap \nu_{S, M}$$

and let

$$\varphi : (x, v) \in U_{S, M} \mapsto \psi(\varphi_{S, \mathbb{R}^n}(x, v), 1) \in M.$$

Then

$$\varphi(x, 0) = \psi(\varphi_{S, \mathbb{R}^n}(x, 0), 1) = \psi(x, 1) = x$$

for  $x \in S$ . For any  $x \in M$ ,  $\psi(x) = x$ . So for any  $(x, v) \in \nu_S$ ,

$$\begin{aligned} \varphi'(x, 0)(0, v) &= \left. \frac{d}{dt} \right|_{t=0} \varphi(x, tv), \\ &= \left. \frac{d}{dt} \right|_{t=0} \psi(\varphi_{S, \mathbb{R}^n}(x, tv), 1) \\ &= \left. \frac{\partial \psi(y, 1)}{\partial y} \right|_{y=\varphi_{S, \mathbb{R}^n}(x, 0)} \left. \frac{d}{dt} \right|_{t=0} \varphi_{S, \mathbb{R}^n}(x, tv) \\ &= \left. \frac{\partial \psi(y, 1)}{\partial y} \right|_{y=x} v \\ &= \frac{\partial \psi(x, 1)}{\partial x} v, \\ &= v. \end{aligned}$$

**7.4.** Reversing  $X$  to  $-X$ , we change positive to negative time, so it is enough to deal with positive time. Let  $D$  be the domain of the flow of  $X$ . If  $(t, m) \in D$ , since  $D$  is open, all  $(t', m')$  close enough to  $(t, m)$  (in some coordinates around  $m$ , say) also lie in  $D$ . So  $(t, m') \in D$  for all  $m'$  close enough to  $m$ , say in some open set  $U_m$ . We cover  $M$  in open sets  $U_m$  with  $m \in U_m$  so that, for some  $t = t(m) > 0$ ,  $(t, m') \in D$  for all  $m' \in U_m$ . Finitely many such  $U_m$  cover a compact set  $K \subset M$  containing all points where  $X \neq Y$ . Hence there is a minimum time  $\varepsilon > 0$  so that if a point enters  $K$  then its flow is still defined for time  $\varepsilon$ . Take a point where the flow is only defined for some time less than  $\varepsilon$ . This point is not in  $K$ , and its flow line never enters  $K$  during any positive time, otherwise its flow would survive for time at least  $\varepsilon$ . So along its flow line,  $X = Y$ , and so its flow line survives for all time. Hence every flow line survives for at least time  $\varepsilon$ . But then if we flow for time  $\varepsilon$ , we still have another  $\varepsilon$  time left to flow, so we never run out of time to keep flowing.

**7.5.** Let  $M := \mathbb{R}^2$ ,

$$\begin{aligned} X &= y^2 \frac{\partial}{\partial x}, \\ Y &= x^2 \frac{\partial}{\partial y}. \end{aligned}$$



The flows are

$$\begin{aligned} e^{tX}(x_0, y_0) &= (x_0 + ty_0^2, y_0), \\ e^{tY}(x_0, y_0) &= (x_0, y_0 + tx_0^2). \end{aligned}$$

By symmetry (swapping variables  $x, y$ ), the vector field  $X + Y$  is tangent to the line  $x = y$ , and at any point of that line, the flow of  $X + Y$  is  $\dot{x} = x^2$  and  $y = x$ . So for example, through  $(x, y) = (1, 1)$ , the flow line of  $X + Y$  is

$$(x(t), y(t)) = \left( \frac{1}{1-t}, \frac{1}{1-t} \right).$$

**7.8.**  $[XY](x) = -[AB]x$ , where  $[AB] := AB - BA$ .

**7.9.**

$$\begin{aligned} [X[YZ]]f &= X[YZ]f - [YZ]Xf, \\ &= XYZf - XZYf - YZXf + ZYXf. \end{aligned}$$

Permute  $X, Y, Z$  cyclically:

$$[Y[ZX]]f = YZXf - YXZf - ZXYf + XZYf$$

and again:

$$[Z[XY]]f = ZXYf - ZYXf - XYZf + YXZf.$$

Add these up and cancel.

**7.10.** For any point  $x_0$ , with flow line  $x(t) := e^{tX}x_0$ , if we let  $y_0 := F(x_0)$ , the vector field  $F_*X$  has flow line  $y(t)$  through  $y_0$  given by  $F(x(t))$ , i.e.

$$e^{tF_*X} \circ F = F \circ e^{tX}.$$

**7.11.** If  $F$  is a diffeomorphism, the flow of  $F_*Y$  is  $e^{tF_*Y} \circ F = F \circ e^{tY}$ , by definition of pushforward. In particular, if  $F = e^{sX}$ ,

$$e^{te^{sX}Y} \circ e^{sX} = e^{sX} e^{tY}.$$

**7.12.** From the proof of lemma 7.1 on page 78 (and dropping  $\circ$  symbols for simplicity of notation):

$$fe^{sX}e^{tY} - fe^{tY}e^{sX} = -st[XY]f + O(s, t)^3,$$

so

$$fe^{sX}e^{tY}e^{-sX}e^{-tY} - f = -st[XY]fe^{-sX}e^{-tY} + O(s, t)^3,$$

since  $e^{-sX}e^{-tY}$  is bounded in  $s, t$  as  $s, t \rightarrow 0$ . Therefore, replacing  $s, t$  by  $-\sqrt{t}$ ,

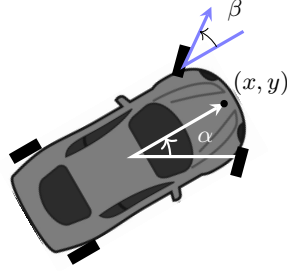
$$fe^{-\sqrt{t}X}e^{-\sqrt{t}Y}e^{\sqrt{t}X}e^{\sqrt{t}Y} - f = -t[XY]fe^{\sqrt{t}X}e^{\sqrt{t}Y} + O(t)^{3/2}.$$

Swap letters  $X$  and  $Y$ :

$$\begin{aligned} fe^{-\sqrt{t}Y}e^{-\sqrt{t}X}e^{\sqrt{t}Y}e^{\sqrt{t}X} - f &= -t[YX]fe^{\sqrt{t}Y}e^{\sqrt{t}X} + O(t)^{3/2}, \\ &= t[XY]fe^{\sqrt{t}Y}e^{\sqrt{t}X} + O(t)^{3/2}. \end{aligned}$$

Divide both sides by  $t$  and send  $t \rightarrow 0$ .

**7.14.**



We consider a front wheel drive car. Our car's configuration is determined by  $x, y, \alpha, \beta$ : centre of the front axle  $(x, y)$ , angle of the car's body  $\alpha$  measured from the  $x$ -axis, and angle of the front wheels  $\beta$  measured from the angle of the body. Let  $z = x + iy$ . So our manifold is  $M = \mathbb{C}_z \times S_\alpha^1 \times S_\beta^1$ . We turn the steering wheel as the flow of the vector field

$$X = \partial_\beta.$$

The flow is

$$e^{tX}(z, \alpha, \beta) = (z, \alpha, \beta + t).$$

Driving is harder. When we drive forward with unit speed, our front wheels point us in a particular direction, and then our axle centre moves that way:

$$\dot{z} = e^{i(\alpha+\beta)}.$$

Suppose our car has length  $\ell_0$ , which is surely constant. The rear axle has centre at point

$$z - \ell_0 e^{i\alpha}.$$

This point has velocity parallel to the direction of the back wheels. The direction of the back wheels is  $e^{i\alpha}$ . So if we rotate this velocity vector by multiplying by  $e^{-i\alpha}$ , it becomes a real number. Hence

$$\begin{aligned} \text{some real number} &= e^{-i\alpha} \frac{d}{dt}(z - \ell_0 e^{i\alpha}), \\ &= e^{-i\alpha}(\dot{z} - \ell_0 e^{i\alpha} i \dot{\alpha}), \\ &= e^{-i\alpha}(e^{i(\alpha+\beta)} - \ell_0 e^{i\alpha} i \dot{\alpha}), \\ &= (e^{i\beta} - \ell_0 i \dot{\alpha}), \end{aligned}$$

so the imaginary part vanishes, i.e.

$$\ell_0 \dot{\alpha} = \sin \beta.$$

Let's drive while holding the steering wheel fixed, so fix  $\beta$  constant. Then

$$\begin{aligned} \dot{z} &= e^{i(\alpha+\beta)}, \\ \dot{\alpha} &= \frac{\sin \beta}{\ell_0}, \\ \dot{\beta} &= 0, \end{aligned}$$

so the motion is that of the flow of the vector field

$$Y = \cos(\alpha + \beta) \partial_x + \sin(\alpha + \beta) \partial_y + \frac{\sin \beta}{\ell_0} \partial_\alpha.$$

The flow (which we don't need at all!) is

$$e^{tY}(z, \alpha, \beta) = \left( z - \frac{i\ell_0}{\sin \beta} e^{i(\alpha+\beta)} \left( e^{it(\sin \beta)/\ell_0} - 1 \right), \alpha + t \frac{\sin \beta}{\ell_0}, \beta \right).$$

The car turns in the circle of radius

$$\frac{\ell_0}{|\sin \beta|}$$

tangent to its initial direction, with  $\beta$  fixed the whole time, or in straight line if  $\beta = 0$ , i.e. with the front wheels pointing straight ahead.

Parallel parking is some complicated combination of steering, driving, steering backwards, driving backwards. If we carry out the simplest combination:

$$e^{-\sqrt{t}Y} e^{-\sqrt{t}X} e^{\sqrt{t}Y} e^{\sqrt{t}X} = e^{t[XY]} + \dots$$

i.e. that resulting motion is approximated by the bracket

$$[XY] = -\sin(\alpha + \beta)\partial_x + \cos(\alpha + \beta)\partial_y + \frac{\cos \beta}{\ell_0}\partial_\alpha.$$

For a small angle  $\beta$ , this moves the location  $(x, y)$  perpendicular to the direction  $\alpha$  of the vehicle, much like parallel parking. But this is not parallel parking, because (again for small  $\beta$ ) it moves the angle  $\alpha$  at a rate inversely proportion to the length of the car.

On the other hand, if we keep  $\beta$  close to  $\pi/2$ , so keep the wheels at nearly a right angle to the direction of the car, we get rid of the increase in  $\alpha$ , but pay a price by having the car move backwards, not like parallel parking.

Try another bracket:

$$[X[YY]] = -\frac{\sin(\alpha + 2\beta)}{\ell_0}\partial_x + \frac{\cos(\alpha + 2\beta)}{\ell_0}\partial_y.$$

For small  $\beta$ , this slides the car in the direction perpendicular to its motion, without changing its direction. If the car is very long (i.e.  $\ell_0$  is big), you don't get much motion, so it takes longer to parallel park a bus than a car.

**7.15.** Take an immersion  $\varphi: S \rightarrow M$ ; by the implicit function theorem, we can locally identify  $\varphi$  in some coordinates with the map  $s \mapsto x = (s, 0)$ ,  $s = (s^1, \dots, s^p)$  and  $x = (x^1, \dots, x^n)$ . The vector fields  $X$  and  $Y$  on  $M$  have expressions in  $x^1, \dots, x^n$  coordinates:

$$X = X^i(x) \frac{\partial}{\partial x^i}, Y = Y^i(x) \frac{\partial}{\partial x^i}.$$

At each point  $(s, 0)$ , they must be tangent to  $S$ , i.e. have no components in directions

$$\frac{\partial}{\partial x^{p+1}}, \frac{\partial}{\partial x^{p+2}}, \dots, \frac{\partial}{\partial x^n}.$$

So  $X_i(s, 0) = 0$  and  $Y_i(s, 0) = 0$  if  $i > p$ . But then

$$[XY](s, 0) = \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$

Note that all terms with  $j > p$  drop out along  $S$  as  $0 = X^j = Y^j$ . But then  $Y^i, X^i$  are only differentiated in directions tangent to  $S$ , i.e. in directions

$$\frac{\partial}{\partial x^j} = \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^p}.$$

Since  $Y^i, X^i$  vanish along  $S$  for  $i > p$ , their derivatives along  $S$  also vanish, i.e. these terms drop too, and

$$[XY](s, 0) = \sum_{i,j < p} \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

is tangent to  $S$ .

**8.2.** The group  $SU(2)$  consists of the matrices

$$\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

with  $|z|^2 + |w|^2 = 1$ .

**8.4.**

- a. For any  $x, y \in \mathbb{R}^n$  and linear map  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ , the squared length of a vector  $Ax$  differs from the length of  $x$  by

$$\begin{aligned} |Ax|^2 - |x|^2 &= \langle Ax, Ax \rangle - \langle x, x \rangle, \\ &= \langle A^t Ax, x \rangle - \langle x, x \rangle, \\ &= \langle (A^t A - I)x, x \rangle. \end{aligned}$$

The matrix  $S := A^t A - I$  is symmetric. So  $A$  preserves distance to the origin if and only if we have  $0 = \langle Sx, x \rangle$  for any vector  $x$ . So replace  $x$  by  $x + y$ , and expand to get

$$0 = \langle Sx, x \rangle + \langle Sx, y \rangle + \langle Sy, x \rangle + \langle Sy, y \rangle.$$

We know that the first and last terms vanish, and by symmetry of  $S$ , the middle terms are equal, so this says  $0 = \langle Sx, y \rangle$  for all  $x, y$ . In particular, taking  $x$  and  $y$  from among the standard basis vectors  $x = e_i, y = e_j$ , force  $S_{ij} = 0$  for all  $i, j$ , i.e.  $S = 0$ . So  $A$  preserves distance to the origin if and only if  $A$  is orthogonal. An isometry preserving the origin preserves distance to the origin. On the other hand, the distance between vectors is  $|x - y|$ , so preserving distance is precisely  $|x - y| = |Ax - Ay| = |A(x - y)|$ , i.e. preserving distance to the origin.

- b. Take an isometry  $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$  fixing the origin. By the triangle inequality,  $g$  takes line segments to line segments, and so lines to lines. Preserving distances,  $g$  takes parallel lines to parallel lines, so takes parallelograms to parallelograms, so preserves addition. On each line,  $g$  takes that line to a line, as an isometry taking the origin to the origin, so linear. So  $g$  preserves scaling, so  $g$  is linear.
- c. Translate to get the origin back into place.
- d. If it is a rotation, we can move it continuously back to the identity, say as a family  $g_t$  of transformations depending continuously on a parameter  $t$ . For every  $t$ , there is an associated orthogonal matrix  $A_t$ , depending continuously on  $t$  since  $g_t x = A_t x$  is continuous in  $t$  for any  $x$ . The determinant of any orthogonal matrix  $A$  satisfies

$$1 = \det I = \det(A^t A) = \det(A^t) \det(A) = \det(A)^2,$$

so  $\det A = \pm 1$ . But  $\det(A_0) = \det(I) = 1$ . By continuity,  $\det(A_t) = 1$  for all  $t$ : a rotation matrix.

On the other hand, given a rotation matrix  $A$ , its columns form an orthonormal basis. If  $n > 1$ , we can continuously rotate it to get its first column to become  $e_1$ . Then all its other columns are perpendicular, so lie inside the span of  $e_2, \dots, e_n$ . So we can assume that  $A$  is such a matrix to begin with. We can continue rotating until finally we are reduced to a 1-dimension span, i.e. of  $e_n$ . But then finally we can assume that  $A$  has columns  $A = (e_1 \ e_2 \ \dots \ e_{n-1} \ \pm e_n)$ . Then  $1 = \det A = \pm 1$ .

e. Follows from above.

**8.7.** To each matrix  $M$  over  $k = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , apply Gram–Schmidt to its columns (or to its rows), and let  $\varphi(M)$  be the matrix whose columns are the resulting orthonormal vectors. By explicit formula for Gram–Schmidt, the map  $\varphi$  is continuous when applied to invertible matrices: it only causes trouble, division by zero, when applied to a matrix whose columns (or rows) are not all linearly independent. We can continuously vary each column, one at a time, from its given value in  $M$  to the value assigned it by Gram–Schmidt, so this process deforms each group of invertible matrices into its group of orthogonal matrices. If we apply Gram–Schmidt to a real or complex matrix with determinant 1, its determinant stays 1.

**8.8.** Since  $K$  is orthogonal:  $K^t K = I$ , and we differentiate in  $t$  to find  $(K')^t K + K^t K' = 0$ . So  $K^t K' = -(K')^t K$ . Note that  $K^t = K^{-1}$  since  $K$  is orthogonal, so this is  $K^{-1} K' = -(K^{-1} K')^t$ .

**8.9.** First: existence. Diagonalize:  $P = KDK^t$  where  $K$  is orthogonal and  $D$  is diagonal. Suppose that  $D = E^2$ . Then clearly  $Q = KEK^t$  satisfies  $Q^2 = P$ . But we can choose  $E$  to be diagonal, with entries the square roots of the entries of  $D$ .

Second: uniqueness. Suppose that  $P = Q^2 = R^2$ . Since  $Q$  is orthogonal,  $Q$  has an orthonormal basis  $u_i$  of eigenvectors, so  $Qu_i = \lambda_i u_i$ . But then  $Pu_i = Q^2 u_i = \lambda_i^2 u_i$ , so these are also an orthonormal basis of eigenvectors of  $P$ . So the eigenspaces of  $P$  are exactly those of  $Q$ , and hence exactly those of  $R$ . The eigenvalues of  $Q$  and  $R$  on each eigenspace are both the same: the square root of the eigenvalue of  $P$ . So  $Q = R$ .

Third: smoothness. Consider the map  $\varphi(Q) = Q^2$ , defined on the space of symmetric matrices with only positive eigenvalues. We know now that this map has a unique inverse. So it is enough to show that it has a smooth local inverse. So suppose that  $\varphi'(Q)\dot{Q} = 0$  for some  $Q$  symmetric matrix (i.e. tangent vector to the space of symmetric matrices). Then  $0 = \varphi'(Q)\dot{Q} = Q\dot{Q} + \dot{Q}Q$ . So  $Q\dot{Q} = -\dot{Q}Q$ . Take a basis  $u_1, \dots, u_n$  of eigenvectors of  $Q$ , so  $Qu_i = \lambda_i u_i$ . Then  $Q\dot{Q}u_i = -\dot{Q}Qu_i = -\lambda_i \dot{Q}u_i$ . So  $\dot{Q}u_i$  is an eigenvector of  $Q$  with eigenvalue  $-\lambda_i$ . But all eigenvalues of  $Q$  are positive, so  $\dot{Q}u_i = 0$  for all  $u_i$ , so  $\dot{Q} = 0$ .

**8.10.** Write  $\mathbb{R}^{n^2}$  for the set of all  $n \times n$  matrices,  $\mathbb{R}^{n(n+1)/2}$  for the set of all symmetric matrices, and  $\mathbb{R}^{n(n-1)/2}$  for the set of all antisymmetric matrices. For any matrix  $R$ , let  $L(R)$  be the antisymmetric matrix which has the same entries as  $R$  below the diagonal. Let  $F(Q, R) = (QR, Q^t Q - I, L(R))$ ,

$$F: \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \rightarrow \mathbb{R}^{n^2} \times \mathbb{R}^{n(n+1)/2} \times \mathbb{R}^{n(n-1)/2}.$$

Show that

$$F'(Q, R)(Q_0, R_0) = (QR_0 + Q_0 R, Q^t Q_0 + Q_0 Q^t, L(R_1)),$$

for any matrices  $Q, Q_0, R, R_0$ . With a little algebra, check that the kernel of  $F'(Q, R)$  is zero, so that  $F'(Q, R)$  is a linear isomorphism. By the inverse function theorem,  $F$

has a local inverse near each point  $(A, B, C)$ . So we take some invertible matrix  $A$ , split into  $A = QR$ , and then smoothly map all nearby matrices  $A$  to  $A \mapsto F^{-1}(A, 0, 0) = (Q(A), R(A))$ . Since we have taken  $B = 0$  and  $C = 0$ ,  $Q(A)$  is orthogonal, and  $R(A)$  has zeroes under the diagonal. But then if we move  $A$  only a little bit,  $R = R(A)$  will move only a little bit, and still have positive entries. So these  $Q = Q(A)$  and  $R = R(A)$  must be the  $QR$  decomposition.

**8.11.** Hint: much as in the previous problem, let  $V_+$  be the vector space of upper triangular matrices,  $V_-$  the vector space of lower triangular matrices, and map

$$F: V_- \times V_+ \rightarrow \mathbb{R}^{n^2}, F(L, U) = LU.$$

**8.12.** One approach:  $P = \sqrt{A^t A}$  and  $K = AP^{-1}$ , and we apply results previously obtained to prove that these are smooth.

A longer approach: Let  $\mathcal{S}$  be the set of all symmetric  $n \times n$  matrices,  $\mathcal{A}$  the antisymmetric,  $\mathcal{P} \subset \mathcal{S}$  the ones with positive eigenvalues (an open set clearly). Note that  $T_P \mathcal{P} = \mathcal{S}$  since  $\mathcal{S}$  is a vector space and  $\mathcal{P} \subset \mathcal{S}$  is an open set. Map

$$\varphi: (K, P) \in O(n) \times \mathcal{P} \mapsto A = KP \in \text{GL}(n, \mathbb{R}).$$

This map is smooth.

Suppose that  $\varphi'$  vanishes on some nonzero tangent vector. Suppose we have curves  $K(t) \in O(n)$  and  $P(t) \in \mathcal{P}$  so that the curve  $(K(t), P(t))$  is tangent to that tangent vector at  $t = 0$ . Differentiate:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \varphi(K(t), P(t)) &= \left. \frac{d}{dt} \right|_{t=0} K(t)P(t), \\ &= K'P + KP'. \end{aligned}$$

This vanishes just when

$$K'P = -KP',$$

so multiply on the right by  $P^{-1}$  and on the left by  $K^{-1}$ :

$$K^{-1}K' = -P'P^{-1}.$$

By problem 8.8 on page 85, the left hand side is antisymmetric. So  $P'P^{-1}$  is antisymmetric. So  $P'P^{-1} = -P^{-1}P'$ . Multiply both sides by  $P$ :  $PP' = -P'P$ . Choose a basis in which  $P$  is diagonal, say  $Pe_i = \lambda_i e_i$ . Since  $P$  is positive definite, all eigenvalues of  $P$  are positive, so  $\lambda_i > 0$ . But then  $PP'e_i = -P'Pe_i = -\lambda_i P'e_i$ . So  $P'e_i$  is an eigenvector of  $P$  with eigenvalue  $-\lambda_i$ , forcing  $P'e_i = 0$  for all  $e_i$ , so  $P' = 0$ . But  $K' = -KP'P^{-1} = 0$ .

**8.14.** The orthogonal group splits into  $\det > 0$  and  $\det < 0$  (since orthogonal transformations are invertible). Indeed every orthogonal transformation has determinant  $\pm 1$ . Those with  $\det = 1$  are precisely the rotations  $\text{SO}(n)$ . Pick any orthogonal transformation  $g_0 \in O(n)$  with  $\det g_0 = -1$ , for example the diagonal matrix with diagonal entries  $-1, 1, 1, \dots, 1$ . Map

$$\varphi: (g, \varepsilon)\text{SO}(n) \times \{\pm 1\} \mapsto \begin{cases} g, & \text{if } \varepsilon = 1, \\ gg_0, & \text{if } \varepsilon = -1, \end{cases}$$

a smooth map with smooth inverse

$$\varphi^{-1}: g \in O(n) \mapsto (gg_0^{\det g}, \det g) = \begin{cases} (g, 1), & \text{if } \det g = 1, \\ (gg_0^{-1}, -1), & \text{if } \det g = -1. \end{cases}$$

So  $SO(n) \times \{\pm 1\}$  is diffeomorphic to  $O(n)$ .

If  $n$  is odd, we can pick  $g_0 = -I$ , and then  $\varphi$  is also a Lie group morphism, and  $\varphi^{-1}$  is too:

$$\varphi(g) = ((\det g)g, \det g).$$

So for  $n$  odd

$$O(n) \cong SO(n) \times \{\pm 1\}$$

are isomorphic Lie groups.

On the other hand for  $n$  even, we will see that they are not isomorphic Lie groups. We consider first  $n = 2$ , and then all other even  $n$ .

If  $n = 2$ , note that  $SO(2)$  is the group of rotations of the plane, each identified with an angle of rotation, so an abelian group, and  $\{\pm 1\}$  is also an abelian group, so  $SO(2) \times \{\pm 1\}$  is an abelian group. But in  $O(2)$ ,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

do not commute (complex conjugation and rotation by a right angle), so  $O(2)$  is not an abelian group.

Denote the centre of a group  $G$  as  $Z_G$ . Clearly  $Z_{G \times H} = Z_G \times Z_H$  for two groups  $G, H$ .

Consider  $n \geq 3$ . A rotation  $g_0$  belongs to  $Z_{SO(n)}$  just when it commutes with all rotations. If we rotate a vector by  $g_0$ , it rotates in some plane, a  $g_0$ -invariant plane, or else it stays put or swaps direction, staying in the same line. Suppose it rotates. Pick any rotation turning that plane into a different plane, but leaving that vector alone:  $g_0$  commutes with that rotation, so that plane is also  $g_0$ -invariant, a contradiction. So every vector stays pointing in the same line, i.e. is a  $g_0$ -eigenvector; indeed with a real eigenvalue, since otherwise  $g_0$  moves it into an  $\mathbb{R}$ -linearly independent vector.

The  $g_0$  eigenvectors are also permuted by rotations: for any rotation  $g$ , and  $g_0$ -eigenvector with eigenvalue  $\lambda$ ,  $g_0gv = gg_0v = g\lambda v = \lambda gv$ . So all vectors are  $g_0$ -eigenvectors, with the same eigenvalue, i.e.  $g_0 = \lambda I$  for some real  $\lambda = \pm 1$ . The same argument holds for  $O(n)$ :  $Z_{O(n)}$  consists of elements of the form  $g_0 = \lambda I$ ,  $\lambda = \pm 1$ . Taking determinant of  $\lambda I$ :

$$Z_{O(n)} = \{\pm I\},$$

$$Z_{SO(n)} = \begin{cases} SO(n), & n = 2, \\ \{\pm I\}, & n \geq 4 \text{ even}, \\ \{I\}, & n \geq 3 \text{ odd}. \end{cases}$$

Hence for  $n \geq 4$  even,

$$Z_{O(n)} = \{\pm I\} \not\cong Z_{SO(n) \times \{\pm 1\}} = \{\pm I\} \times \{\pm 1\}.$$

**8.16.** See problem 8.15 on page 87.

**8.17.** Consider the elements which satisfy the equation  $x^2 = 1$ . For  $\text{GL}(n, \mathbb{R})$ , these are the square matrices  $g$  so that  $(g - I)(g + I) = 0$ , i.e.  $g$  has an eigenspace of eigenvalue 1, and one of eigenvalue  $-1$ , and these can be any two complementary linear subspaces. The set of these is parameterized by arbitrary choices of those subspaces. Once we pick the dimension of the eigenspace with eigenvalue 1, we have a connected manifold of choices of that subspaces and of its complement. (We leave the reader to fill in the details here.) So the set of solutions of  $x^2 = 1$  in  $\text{GL}(n, \mathbb{R})$  is a union of  $n + 1$  disjoint submanifolds. For  $\text{SL}(n, \mathbb{R}) \times \mathbb{R}^\times$ , these are pairs  $(g, \pm 1)$  so that  $g^2 = I$ , i.e.  $g$  has an eigenspace of eigenvalue 1, and one of eigenvalue  $-1$ . So these form a union of  $2(n + 1)$  disjoint manifolds.

**8.19.** A homogeneous polynomial of degree 3 could be

- a. the zero polynomial,
- b. a triple linear factor  $(\alpha x + \beta y)^3$ ,
- c. a double linear factor times a linearly independent factor  $(\alpha x + \beta y)^2(\gamma x + \delta y)$  with  $\alpha\delta \neq \beta\gamma$ ,
- d. three distinct linear factors, any two linearly independent,
- e. a linear factor times an irreducible quadratic.

Each of these is an orbit: no change of variables can alter how the factorization works, and we can use a suitable linear change of variables to get one nonzero linear factor to be  $x$ , and so on.

**8.20.** Take any smooth function  $f$  on any open subset of  $Y$ . We need to prove that  $g, y \mapsto f(gy)$  is smooth, where defined, i.e. the action pulls back smooth functions to smooth functions. But  $f(p(gx)) = f(gp(x))$  depends smoothly on  $x$ . Near any point  $y_0 \in Y$ , there is a smooth local section  $x: \text{open} \subset Y \rightarrow \text{open} \subset X$  so that  $p(x(y)) = y$ , by the implicit function theorem. Therefore  $f(gy) = f(gp(x(y)))$  is smooth in  $(g, y)$ .

**8.25.** For any imaginary quaterion,  $\bar{u} = -u$  so if  $u$  has unit length, then  $u^2 = -1$ .

**8.26.** Take two imaginary quaternions  $u, v \in \mathbb{R}^3$ . So  $\bar{u} = -u$  and  $\bar{v} = -v$ . So  $u$  and  $v$  are perpendicular just when

$$\begin{aligned} 0 &= u\bar{v} + v\bar{u}, \\ &= u(-v) + v(-u), \\ &= -uv - vu, \\ &= -(uv + vu), \end{aligned}$$

**8.27.** Since  $u$  and  $v$  anticommute,  $u(uv) = -u(vu) = -(uv)u$ , i.e.  $u$  and  $uv$  anticommute, so are perpendicular.

**8.28.** Expand in a Taylor series, we see that  $q = \cos \theta + u \sin \theta$ . So  $q$  and  $u$  commute. Conjugate:  $\bar{q} = e^{-\theta u/2}$ , so  $q, u, \bar{q}$  all commute, so  $\rho(q)u = u$ . If  $v$  is an imaginary



quaternion perpendicular to  $u$ , then

$$\begin{aligned}
 qv\bar{q} &= (\cos \theta + u \sin \theta)v(\cos \theta - u \sin \theta), \\
 &= (v \cos \theta + uv \sin \theta)(\cos \theta - u \sin \theta), \\
 &= v \cos^2 \theta - vu \cos \theta \sin \theta + uv \cos \theta \sin \theta - uvu \sin^2 \theta, \\
 &= v \cos^2 \theta + uv \cos \theta \sin \theta + uv \cos \theta \sin \theta + u^2 v \sin^2 \theta, \\
 &= v \cos^2 \theta + uv 2 \cos \theta \sin \theta - v \sin^2 \theta, \\
 &= v(\cos^2 \theta - \sin^2 \theta) + uv 2 \cos \theta \sin \theta, \\
 &= v \cos 2\theta + uv \sin 2\theta,
 \end{aligned}$$

rotating  $v$  in the basis  $v, uv$  of the plane perpendicular to  $u$ .

**8.30.** To have  $q^n = s$ , taking lengths,  $q$  has to have length 1. Replacing both sides by  $rq^n\bar{r} = rs\bar{r}$  for some unit length  $r$ , we can rotate  $s$  to the form  $s = \cos \alpha + i \sin \alpha$ , for some  $\alpha > 0$ . Write  $q = \cos \theta + u \sin \theta$ , for some unit vector  $u$ . If  $\sin \theta = 0$ , plug into  $q^n = s$  to find a contradiction. We can suppose that  $\theta > 0$ , so  $q = e^{u\theta}$  and  $q^n = e^{nu\theta}$  forces  $\cos(n\theta) = \cos \alpha$  and  $\sin(n\theta)u = \sin \alpha i$ , so  $u = i$  and  $n\theta = \alpha$  up to  $2\pi$  multiples:  $q = e^{2\pi i \ell/n}$  for  $\ell = 0, 1, 2, \dots, n$ . Hence there are  $n$  roots to the equation  $q^n = s$  for  $s \neq \pm 1$ , i.e. the map  $q \mapsto q^n$  has degree  $n$ . Note that the story is messy when  $s = \pm 1$ , since  $q^n = \pm 1$  has a more complicated solution set.

**8.31.** Expand out the multiplication to check that there is a unique unit for multiplication: 1, so  $\varphi(1) = 1$ . Similarly for addition,  $\varphi(0) = 0$ . Preserving addition and 1,  $\varphi$  fixes every positive integer. Preserving subtraction,  $\varphi$  fixes every negative integer. Preserving equations  $qx = p$  for integers  $p, q$ ,  $\varphi$  fixes every rational number. The quaternions that commute with every quaternion are precisely the real quaternions, so the real numbers form a subring of the quaternions invariant under every automorphism of the quaternions. A real number  $a$  is not negative just when it satisfies  $a = b^2$  for some real number  $b$ . This equation is invariant under any automorphism, so the nonnegative real numbers are invariant, and hence the positive real numbers are invariant. Any inequality  $a < b$  among real numbers is equivalent to asking that  $b - a$  be positive, so inequalities among real numbers are invariant. So every interval of the real number line maps to an interval of the real number line. Every rational number is fixed, so every interval between rationals is fixed. Hence every real number is fixed, as being the unique real number lying in every interval with rational end points which contains that real number. So every automorphism is a real linear transformation. Expand out  $a^2$  to find that the solutions of the equation  $a^2 = -1$  are precisely the unit length vectors  $u = u_1i + u_2j + u_3k$  with  $|u| = 1$ , hence these are preserved by every automorphism. Every automorphism preserves the real numbers, and also preserves the unit imaginary vectors, and hence preserves their span: the imaginary quaternions. So every automorphism preserves the splitting into real and imaginary parts, so preserves quaternion conjugation, so preserves lengths as  $a\bar{a} = |a|^2$ , so is an orthogonal linear transformation fixing every point on the real axis, so an orthogonal transformation of the imaginary vectors, preserving cross product, so preserving orientation, hence a rotation. Every automorphism fixes some axis (by the spectral theorem, every rotation of an odd dimensional real vector space has a fixed axis), containing a unique vector  $u$ . Follow problem 8.28 on page 89.

**8.34.** To have  $\rho(q, r) = I$ , we need precisely  $qs\bar{r} = s$  for all  $s \in \mathbb{H}$ . Plug in  $s = 1$  to find  $r = q$ , so  $qs = sq$  for all  $q$ . Plug in  $s = i$  and expand out to find that, since  $ij = k$  and  $ji = -k$ ,  $q_2 = 0$ . Similarly,  $q_3 = 0$ . So  $q = q_0 + iq_1$ . Now plug in  $s = j$

to find  $qj = jq$ , forcing  $q_1 = 0$ :  $q = q_0 \in \mathbb{R}$ , and  $|q| = 1$ , so  $q = \pm 1$  and  $r = q$ , so  $(q, r) = (1, 1)$  or  $(q, r) = (-1, -1)$ .

**9.2.** Taking determinant, we see that  $\mathrm{GL}(n, \mathbb{R})$  and  $\mathrm{O}(n)$  are not path connected. Their subgroups  $\mathrm{GL}^+(n, \mathbb{R})$ ,  $\mathrm{SO}(n)$  consisting of their matrices with positive determinant, as we will see, are connected, ensuring that  $\mathrm{GL}(n, \mathbb{R})$  and  $\mathrm{O}(n)$  each have two components. Applying the retractions from problem 8.7 on page 85, we need only prove connectivity of  $\mathrm{GL}^+(n, \mathbb{R})$ ,  $\mathrm{SL}(n, \mathbb{C})$ ,  $\mathrm{GL}(n, \mathbb{C})$ ,  $\mathrm{GL}(n, \mathbb{H})$ .

Apply the steps of Gaussian elimination to a complex matrix  $M$ . Each step

- a. adds a multiple of a row to another row, or
- b. swaps two rows, or
- c. scales a row.

Each of these steps fits into a continuous family of matrices:

- a. Add a multiple of a row to a row, by vary that multiple continuously starting at zero, and proceeding to whatever multiple you want.
- b. Apply a rotation, replacing two rows  $u, v$  by  $\cos \theta u - \sin \theta v, \sin \theta u + \cos \theta v$ , and vary  $\theta$  continuously. This doesn't swap: it swaps up to a sign change, which is good enough to get Gaussian elimination to work.
- c. Scale by a nonzero number  $c$ , starting at  $c = 1$ , and proceeding continuously deforming  $c$  until it becomes what you want.

But Gaussian elimination takes any invertible matrix to  $I$  in finitely many steps. So  $\mathrm{GL}(n, \mathbb{C})$  is path connected. The same works for  $\mathrm{GL}(n, \mathbb{H})$ . It fails for  $\mathrm{GL}(n, \mathbb{R})$ , because we might need negative scalars. But it works by the same reasoning for the set of positive determinant matrices  $\mathrm{GL}^+(n, \mathbb{R})$ . It works for  $\mathrm{SL}(n, \mathbb{C})$ , because we just skip all of the rescalings of rows until the very end, and then we rescale one row by some nonzero constant, and the next row by the reciprocal of that constant, so that we preserve the determinant.

There are many other proofs of this result. For complex matrices, a different proof uses the spectral theorem. Suppose that  $M$  is an invertible diagonalizable complex matrix, say  $M = PDP^{-1}$  for some invertible complex matrix  $P$  and diagonal matrix  $D$ . The matrix  $M$  is invertible just when  $D$  has only nonzero complex numbers down its diagonal. Rotate each of those complex numbers to become real and positive, continuously deforming  $D$  into a real matrix with zeroes off the diagonal and positive real numbers on the diagonal. Continuously deform those real numbers to all become equal to 1, so that  $D = I$  and  $M = PIP^{-1} = I$ . So the space of diagonalizable invertible complex matrices is path connected.

Any complex matrix can be approximated by diagonalizable matrices arbitrarily closely, which is obvious in Jordan normal form. So we can move an invertible complex matrix slightly (inside any path connected open set) to get it to remain invertible and become diagonalizable and therefore the set of complex invertible matrices is path connected.

By the spectral theorem for normal matrices, each unitary matrix  $U$  is diagonalizable, i.e.  $U = PDP^{-1}$ , for a unitary matrix  $P$  and a diagonal matrix  $D$  with unit length complex number eigenvalues. Applying the same method,  $\mathrm{U}(n)$  is path connected.

Geometrically, the result is clear for rotations. Consider the group  $\mathrm{SO}(n)$  of rotation matrices. Each rotation matrix has columns forming an orthonormal basis. Rotate all of the columns by some continuous family of rotations which takes the first

column to the first standard basis vector. By induction,  $SO(n)$  is path connected. The same proof works for  $U(n), SU(n), Sp(n)$ .

Another proof for  $SO(n)$ : pick a rotation matrix in  $SO(n)$ . Using the real spectral theorem, since rotations are normal matrices, we can split  $\mathbb{R}^n$  into a direct sum of mutually orthogonal planes on which the rotation rotates each plane, say by some angle  $\theta_i$ . If  $n$  is odd, there is also some straight line perpendicular to all of those planes which is fixed by the rotation. But then we continuously deform each angle  $\theta_i$  to zero, dragging our rotation matrix into the identity matrix. The same proof works for  $U(n), SU(n), Sp(n)$ .

**9.6.** We can diagonalize any matrix belonging to any of these Lie algebras, and in the corresponding Lie group. So it suffices to check that the exponential map is onto from the diagonal ones in  $\mathfrak{g}$  to the diagonal ones in  $G$ .

**9.7.** Take any matrix  $g \in SL(2, \mathbb{R})$ . Write it as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So the condition  $1 = \det g$  is the quadratic equation  $ad - bc = 1$  in 4 variables:  $SL(2, \mathbb{R})$  is a quadratic 3-manifold in 4-dimensional Euclidean space, a hyperboloid of one sheet. The *half trace*  $\text{tr } g/2 = (a + d)/2$  has level sets striking this hyperboloid in surfaces.

If an element  $g \in SL(2, \mathbb{R})$  is diagonalizable over the real numbers, it has two eigenvalues which multiply to its determinant 1, so are reciprocals. These eigenvalues have 1-dimensional eigenspaces, unless  $g = \pm I$ , so a pair of lines in the plane. The half-trace is

$$\frac{\lambda + \frac{1}{\lambda}}{2},$$

which the reader can check, using calculus, is a function of  $\lambda$  which achieves minimum value for  $\lambda > 0$  at  $\lambda = 1$ , and maximum value for  $\lambda < 0$  at  $\lambda = -1$ , corresponding to  $g = \pm I$ .

If an element  $g \in SL(2, \mathbb{R})$  is not diagonalizable over the real numbers, but over the complex numbers, it has eigenvalues reciprocal complex numbers but also conjugates, hence on the unit circle, so  $g$  is conjugate to a rotation around the circle;  $g$  “rotates” around an ellipse. We can draw any ellipse, and rotate around by any angle. The half trace is strictly between  $-1$  and  $1$ .

If  $g \in SL(2, \mathbb{R})$  is not diagonalizable, it has a real Jordan normal form.

If its eigenvalues are positive, it is a *shear*:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

preserving height of each point, but sliding it horizontally by an amount which increases with its height, like a shear zone of rock in geology. A shear has half trace 1. Note that every shear has the form  $I + N$  where  $N$  is an  $2 \times 2$  matrix with zero trace and determinant. These matrices  $N$  form a quadratic cone  $\det N = 0$  inside the trace zero linear subspace. So the shears form a quadratic cone with cone point  $I$  in the 3-dimensional manifold  $SL(2, \mathbb{R})$ , with the elliptical rotations inside. Outside are the real diagonalizable with positive eigenvalues.

If its eigenvalues are negative, it is an *antishear*:

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

with half trace  $-1$ . These similarly form a cone with cone point  $-I$ , elliptical rotations inside. Outside are the real diagonalizable with negative eigenvalues.

Every element of  $\mathrm{SL}(2, \mathbb{R})$  is in precisely one of the sets:

| half trace | type                                                                                     |
|------------|------------------------------------------------------------------------------------------|
| 1          | $I$                                                                                      |
| $-1$       | $-I$                                                                                     |
| 1          | shear                                                                                    |
| $-1$       | antishear                                                                                |
| $(-1, 1)$  | elliptic rotation, angle not 0 or $\pi$                                                  |
| $> 1$      | linear map given by a pair of lines and positive reciprocal eigenvalues attached to each |
| $< -1$     | linear map given by a pair of lines and negative reciprocal eigenvalues attached to each |

The last three sets are open, given by condition on the half trace. Those three open sets meet along a common boundary of the first four sets: half trace  $\pm 1$ .

The elements of  $\mathfrak{sl}(2, \mathbb{R})$  are the matrices

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

The eigenvalues of such a matrix are

$$\lambda = \begin{cases} \pm\sqrt{a^2 + bc}, & \text{if } a^2 + bc > 0, \\ 0, & \text{if } a^2 + bc = 0, \\ \pm i\sqrt{|a^2 + bc|}, & \text{if } a^2 + bc < 0. \end{cases}$$

In the first case,  $A$  has two real eigenvalues  $\pm\lambda$  on two distinct lines in the plane. Conversely, taking any two lines in the plane, and a real number  $\lambda > 0$ , give one line the eigenvalue  $\lambda$  and the other the eigenvalue  $-\lambda$ , and you have a picture of  $A$ . Hence  $e^A$  is the same picture, but with reciprocal positive eigenvalues.

In the second case,  $A$  is conjugated by a linear transformation of the plane into

$$A = \lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Finally, the theory of Jordan normal form for real matrices tells us that if  $A \neq 0$ , then  $A$  is conjugated into

$$A = \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e^A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

a shear. Of course if  $A = 0$ , we have  $e^A = I$ .

The exponential map on  $\mathfrak{sl}(2, \mathbb{R})$  has image consisting precisely of all but the open set with negative eigenvalues and the antishears, but includes  $-I$ . In particular, the image is not dense, and neither closed nor open. You can hit all rotations around a given ellipse, which form a loop inside the 3-dimensional manifold  $\mathrm{SL}(2, \mathbb{R})$ . These form a sort of “waist” around the hyperboloid. All of those loops pass through  $\pm I$ ,

and the exponential map hits everything near  $I$ . But they also pass through  $-I$ , near which you find points you cannot hit: all other points of half trace  $-1$  or less.

**9.8.** Take any  $h_0 \in H$ . Then  $L_{h_0}$  is a diffeomorphism of  $U$  to  $h_0U$ ,  $H$  to  $H$ , taking  $1$  to  $h_0$ , so  $H \cap U$  is embedded in  $U$  just when  $H \cap h_0H$  is embedded in  $h_0U$ , i.e.  $H$  is embedded near each point of  $H$ , so embedded.

**9.9.** Up to isomorphism, the Lie subalgebras of  $\mathfrak{SO}(3)$  are:

- a.  $\mathfrak{o}$  and
- b.  $\mathfrak{so}(3)$  and
- c. the Lie algebra of the group of rotations of the plane.

To prove this: we saw that subalgebras are of dimension 0, 1 or 3. Take a 1-dimensional subalgebra, spanned by some nonzero  $3 \times 3$  antisymmetric matrix  $A$ . Linear algebra tells us that  $A$  has eigenvalues  $i\theta, -i\theta, 0$ ; scale to get  $\theta = 1$ . Again some linear algebra tells us that  $A$  has an eigenvector  $x + iy$  with eigenvalue  $i$ , i.e.  $A(x + iy) = -y + ix$ , i.e.  $Ax = -y$ ,  $Ay = x$ , a rotation by a right angle. Rotate to get  $x = e_1, y = e_2$ , which conjugates  $A$ .

**9.10.** Since the manifold  $\mathrm{GL}(n, k)$  is analytic, and the vector field  $\vec{A}$  on it is analytic, the flow is analytic. So for short time, each flow line is an analytic function, say  $g(t)$  with  $g(0) = g_0$ . Expand out, using the identification of the Lie algebra with matrices:

$$\begin{aligned} \frac{dg}{dt} &= \vec{A}(g(t)), \\ &= L_g A, \\ &= gA. \end{aligned}$$

Differentiate repeatedly and apply induction to find

$$\frac{d^k g}{dt^k} = gA^k.$$

The Taylor series is

$$\begin{aligned} g(t) &= g_0 + tg'(0) + \frac{t^2}{2}g''(0) + \cdots + \frac{t^k}{k!} \frac{d^k g}{dt^k} \Big|_{t=0}, \\ &= g_0(I + tA + \frac{t^2 A^2}{2} + \cdots + \frac{t^k A^k}{k!} + \cdots), \\ &= e^{tA}. \end{aligned}$$

So this Taylor series has to converge for small values of  $t$ , for any  $A$ . But for larger values of  $t$ , use a rescaled  $A$ . So  $e^A$  converges for all  $A$ .

**10.1.** Take  $A \in \mathfrak{g}$  and let  $x(t) = \exp(tA)$ . By definition,  $x(t)$  is the flow of  $\vec{A}$  through  $1 \in G$ , so  $x'(0) = A$ . Hence  $\exp'(0)A = \exp'(0)\vec{A}(0) = A$ ,  $\exp'(0): T_0\mathfrak{g} = \mathfrak{g} \rightarrow T_1G = \mathfrak{g}$ .

**10.2.** The Heisenberg Lie algebra is the collection of  $3 \times 3$  upper triangular nilpotent real matrices, i.e. matrices of the form

$$A = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}.$$

The exponential map on a matrix is just the usual matrix exponential function:  $e^A = I + A + A^2/2 + \dots$ , which here gives

$$e^A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & xz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \dots$$

where all  $\dots$  terms turn out to be zero, by induction, so

$$e^A = \begin{pmatrix} 1 & x & y + \frac{xz}{2} \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

**10.3.** We know that  $\exp$  is a diffeomorphism near 0, so the image of  $\exp$  contains a neighborhood of  $1 \in G$ . By lemma 9.1 on page 93, the subgroup generated by exponentials is a union of components. But all exponentials are connected to  $1 \in G$ , so lie in the identity component.

**10.4.** By induction,

$$\begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} A^k & A^{k-1}b \\ 0 & 0 \end{pmatrix}$$

so

$$\exp \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^A & f(A)b \\ 0 & 1 \end{pmatrix}$$

where

$$f(A) = \sum_{j=1}^{\infty} A^{j-1} j!.$$

**10.5.** Take two Lie subgroups  $H_1, H_2 \subseteq G$ , say with Lie algebras  $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g}$ . Let  $H_0 := H_1 \cap H_2$  and  $\mathfrak{h}_0 := \mathfrak{h}_1 \cap \mathfrak{h}_2$ . The exponential map identifies  $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g}$  diffeomorphically with  $H_1, H_2 \subseteq G$  near the identity, so identifies intersections with intersections. So  $H_0$  is an embedded Lie subgroup near the identity, hence an embedded Lie subgroup by problem 9.8 on page 96.

**10.6.** Suppose that  $G$  is a connected one dimensional Lie group. The Lie algebra  $\mathfrak{g}$  of  $G$  is one dimensional, spanned by some  $A \in \mathfrak{g}$ . By antisymmetry of the Lie bracket,  $0 = [AA]$ , and so the Lie algebra has Lie bracket equal to zero. The map  $t \in \mathbb{R} \mapsto tA \in \mathfrak{g}$  is a Lie algebra morphism. The map  $t \in \mathbb{R} \mapsto e^{tA}$  is a Lie group morphism, so a covering map  $\mathbb{R} \rightarrow G$ . If  $G$  is simply connected, then  $G = \mathbb{R}$ , a trivial covering, with Lie group multiplication given by addition of real numbers. Otherwise  $G = \mathbb{R}/N$  where  $N$  is some discrete subgroup of  $\mathbb{R}$ . But the only discrete subgroup, up to rescaling, is  $2\pi\mathbb{Z}$ , so  $G$  is the circle with usual multiplication.

**10.7.** Take two Lie subgroups  $H_1, H_2 \subseteq G$ , say with Lie algebras  $\mathfrak{h}_1, \mathfrak{h}_2 \subseteq \mathfrak{g}$ . Think of these subgroups as given by immersions  $\iota_1: H_1 \rightarrow G, \iota_2: H_2 \rightarrow G$ . Then we can identify  $H_0 = H_1 \cap H_2$  with the set of pairs  $(h_1, h_2) \in H_1 \times H_2$  so that  $\iota_1(h_1) = \iota_2(h_2)$ . So  $H_0 \subset H_1 \times H_2$  is a closed subgroup of  $H_1 \times H_2$ , hence a Lie subgroup with Lie algebra  $\mathfrak{h}_0 \subset \mathfrak{h}_1 \oplus \mathfrak{h}_2$  given by the equation  $\iota_1 = \iota_2$ . On the other hand, if we had some other topology or smooth structure on the intersection, smoothly mapped to  $G$  with the same Lie algebra, then matching Lie algebra gives matching exponential map, so an isomorphism of Lie groups near the identity, and a bijection, so an isomorphism.

**10.8.** Consider any intersection

$$H = \bigcap_{\alpha} H_{\alpha} \subseteq G.$$

By problem 10.7 on page 101, and induction, any finite intersection  $H_{\alpha_1} \cap \cdots \cap H_{\alpha_k}$  is a Lie subgroup, for a unique topology and smooth structure. Any such finite intersection has some dimension as a manifold, never larger than the dimension of the ambient Lie group  $G$ . So among all such dimensions, there is a least one, say the dimension of some finite intersection  $L := H_{\alpha_1} \cap \cdots \cap H_{\alpha_k}$ . Intersecting all  $H_{\alpha}$  with  $L$  only makes them smaller, but doesn't change

$$H = \bigcap_{\alpha} H_{\alpha},$$

since  $L \subseteq H$ . So we can assume that all  $H_{\alpha}$  lie inside  $L$ , with Lie algebras inside that of  $L$ . Then since  $L$  has the smallest dimension of any finite intersection, and lies inside every finite intersection, all finite intersections now have the same dimension: that of  $L$ . But all lie inside  $L$ , so have Lie algebras inside that of  $L$ , but of the same dimension as that of  $L$ . So all  $H_{\alpha}$ , and all finite intersections, have the same Lie algebra. Therefore they all have the same identity component, say  $H^0$ , and their other components have smooth structure arising from left translating their identity component. In particular  $H^0 \subseteq H$  is a subgroup. Each finite intersection lops off perhaps some components, so is just a subset of the components of  $L$ . But then  $H$  is also a subset of those components.

**10.9.** A 2-dimensional extension of  $\mathbb{R}$  is a 2-dimensional vector space, so spanned by 1 and some element  $\alpha$ , and every element is  $x + y\alpha$  for some real  $x, y$ . In particular,  $\alpha^2 = x + y\alpha$ , for some real  $x, y$ ; rewrite this as  $\alpha^2 + b\alpha + c = 0$ . If the discriminant  $b^2 - 4c$  is positive, then this polynomial already factors over the real numbers, so has no additional factors in the field extension. Replace  $\alpha$  by

$$\frac{b/2 + \alpha}{\sqrt{c - b^2/4}},$$

to arrange that  $\alpha^2 = -1$ .

**10.10.** Denote our field as  $k$  and the component of any point  $x \in k$  as  $\bar{x}$ . Suppose that  $\bar{x} = \bar{0}$ , but  $x \neq 0$ . Then  $y \mapsto xy$  takes  $1 \mapsto x$  so  $\bar{1} \rightarrow \bar{x} = \bar{0}$ . But it takes  $0 \mapsto 0$  to  $\bar{0} \rightarrow \bar{0}$ . Since  $x \neq 0$  is invertible,  $\bar{0} = \bar{1}$ . Take a path  $x(t)$  in  $k$  with  $x(0) = 0$  and  $x(1) = 1$ . Then  $\varphi_t(y) = x(t)y$  is a retraction, ensuring that  $k$  is connected and simply connected.

Under addition, our field  $k$  is a connected abelian Lie group, so a product of a finite dimensional real vector space and a torus. Pick a path  $x(t)$  with  $x(0) = 0$  and  $x(1) = 1$ . Then the map  $\varphi_t(y) = x(t)y$  is a smooth retraction, forcing trivial fundamental group, so  $k$  is isomorphic as an abelian Lie group to a real vector space. As in any abelian group, for any integer  $p > 0$ , write  $px$  to mean  $x + \cdots + x$ ,  $p$  times. Similarly if  $p < 0$ , write  $px$  to mean  $-x - x - \cdots - x$ ,  $|p|$  times. As is true in any vector space, every element  $x$  has unique scalings  $(p/q)x$ , for any integers  $p, q \in \mathbb{Z}$  with  $q \neq 0$ , so that  $q(p/q)x = px$  (Note that the equations of these scalings are defined using only the abelian group structure, so the scalings are invariants of the abelian group structure, if they happen to exist. But in some abelian groups, such scalings do not exist.) Moreover, as in any finite dimensional real vector space, there is a unique smooth extension of these scalings to a smooth scaling by real numbers. So the real

vector space structure is well defined, not just the abelian group structure. Fix  $x_0 \in k$  and consider left multiplication  $x \mapsto x_0 x$ . This map preserves the abelian Lie group structure, and is smooth, so preserves the vector space structure, i.e. is linear, and similarly right multiplication is linear. Thus  $k$  is an algebra over the real numbers, so a field extension. Since  $k$  has positive dimension, every component has positive dimension, so contains infinitely many distinct elements. We apply our result above.

**11.1.** Write the bilinear term as  $b(A, B)$ , so

$$(s + t)A = (sA) \dot{+} (tA) = (sA) + (tA) + b(sA, tA) + \dots$$

forces  $0 = b(A, A)$  for any  $A$ , so

$$\begin{aligned} 0 &= (A + B) \dot{+} (A + B) - (A \dot{+} A) - (B \dot{+} B), \\ &= A + B + b(A + B, A + B) - (A + A + b(A, A)) - (B + B + b(B, B)), \\ &= b(A + B, A + B) - b(A, A) - b(B, B), \\ &= b(A, B) + b(B, A) + b(A, A) + b(B, B) - b(A, A) - b(B, B), \\ &= b(A, B) + b(B, A). \end{aligned}$$

So  $b(A, B)$  is skew-symmetric.

**11.2.** Take an embedded subgroup  $H \subseteq G$  of a Lie group  $G$ . Every embedded submanifold is locally closed, but perhaps not closed. We need to show that  $\bar{H} = H$ ; pick some  $y \in \bar{H}$ . Since  $H \subseteq G$  is an embedded submanifold, we can find some open set  $U \subset G$  containing 1 so that  $H \cap U$  is closed in  $U$ . Let  $U^{-1} := \{u^{-1} \mid u \in U\}$ . So  $yU^{-1}$  is an open set around  $y \in \bar{H}$ , and thus contains some  $x \in H$ , say  $x = yu^{-1}$ . So  $x^{-1}y = u \in U \cap \bar{H} = U \cap H$ , so  $x^{-1}y \in H$  and  $x \in H$  so  $y \in H$ .

**11.5.** The map  $G \times G \rightarrow G$  given by  $(x, y) \mapsto xyx^{-1}y^{-1}$  is continuous and vanishes on  $A \times A$ , so vanishes on its closure, which is  $\bar{A} \times \bar{A}$ . The flow of any left invariant vector field is a one parameter subgroup, so abelian. Take its closure.

**11.8.** Since  $\text{Sp}(1)$  is the unit sphere in the quaternions, its Lie algebra  $\mathfrak{sp}(1)$  is the tangent space to that sphere at the quaternion 1, i.e.  $\mathfrak{sp}(1)$  is the collection of all imaginary quaternions. Left and right multiplications by quaternions are real linear transformations of  $\mathbb{H}$ , so left and right translation have derivatives  $L'_q(e) = L_q$ , and  $R'_q(e) = R_q$ . Hence  $\text{Ad}_q r = qrq^{-1} = q\bar{r}q$  is the representation  $\rho$  which we encountered previously:  $\text{Ad}: \text{Sp}(1) \rightarrow \text{SO}(3)$ , with kernel  $\pm 1$ .

**11.9.** The Lie algebra of  $\text{O}(3)$  is  $\mathfrak{o}(3)$ , i.e. matrices

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

Identifying each such matrix with a vector  $v = (a, b, c) \in \mathbb{R}^3$  turns the bracket into the cross-product of vectors. The adjoint representation is

$$\text{Ad}_g A = g \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} g^{-1}.$$

The reader can check by computing out explicitly that if

$$g = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



then the vector associated to this adjoint action is  $gv$ . However, for  $g$  diagonal with  $\pm 1$  down the diagonal, this doesn't work: if

$$g = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

then  $\text{Ad}_g A$  has associated vector  $\det(g)gv$ . This gives some evidence, and comes close to a proof, that the adjoint representation is isomorphic to the representation  $v \mapsto \det(g)gv$ .

**11.10.** If  $H$  is normal, then  $ghg^{-1} \in H$  for all  $h \in H$ , so if  $A \in \mathfrak{g}$  and  $B \in \mathfrak{h}$ , then  $e^{sA}e^{tB}e^{-sA} \in H$ , for all  $s, t \in \mathbb{R}$ . Differentiate at  $t = 0$ :  $\text{Ad}_{e^{sA}} B \in \mathfrak{h}$  for all  $s \in \mathbb{R}$ . Differentiate at  $s = 0$ :  $[AB] = \text{ad}_A B \in \mathfrak{h}$  for all  $A \in \mathfrak{g}$ ,  $B \in \mathfrak{h}$ .

On the other hand, if  $\mathfrak{h} \subset \mathfrak{g}$  is an ideal, then for any  $A \in \mathfrak{g}$ ,  $\text{Ad}_{e^{sA}} = e^{s\text{ad}_A} \in \text{GL}(\mathfrak{g})$ , preserves  $\mathfrak{h} \subset \mathfrak{g}$ , so if we fix  $s$  and vary  $t$ :

$$\varphi(t) = e^{sA}e^{tB}e^{-sA}$$

is a one parameter subgroup in  $G$ , with velocity at  $t = 0$  given by  $\text{Ad}_{e^{sA}} B = e^{s\text{ad}_A} B \in \mathfrak{h}$ . So

$$\varphi(t) = e^{t\text{Ad}_{e^{sA}} B} = e^{sA}e^{tB}e^{-sA}$$

is a one parameter subgroup in  $H$ . Hence  $H$  is invariant under conjugation by any one parameter subgroup of  $G$ , hence any product of finitely many one parameter subgroups of  $G$ , hence every element of  $G$ .

**12.1.** Recall that in  $G$ ,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\dots}$$

so, reversing order and applying this fact 3 times:

$$\begin{aligned} e^{-B}e^{-A}e^B e^A &= e^{-B-A+\frac{1}{2}[B,A]+\dots} e^{B+A+\frac{1}{2}[B,A]+\dots}, \\ &= e^{[B,A]+\dots} \\ &= e^{-[A,B]+\dots}. \end{aligned}$$

From problem 7.12 on page 80,

$$\left. \frac{d}{dt} \right|_{t=0} e^{-\sqrt{t}B} e^{-\sqrt{t}A} e^{\sqrt{t}B} e^{\sqrt{t}A} = \overline{[AB]}$$

while from the above,

$$e^{-\sqrt{t}B} e^{-\sqrt{t}A} e^{\sqrt{t}B} e^{\sqrt{t}A} = e^{-t[A,B]+\dots}.$$

**12.2.** By existence and uniqueness of flow lines, a vector field vanishes at a point just when its flow line through that point is constant, i.e. just when that point is a fixed point of the flow. So  $\tilde{A}(m_0) = 0$  just when  $e^{tA}m_0 = m_0$  for all  $t$ , just when  $e^{tA} \in G^{m_0}$  for all  $t$ , just when  $A \in \mathfrak{g}^{m_0}$ .

**12.3.** Recall the notion of degree from problem 3.3 on page 24: every continuous map  $f: S^1 \rightarrow S^1$  has an expression as  $f(e^{i\theta}) = e^{i\hat{f}(\theta)}$  for some continuous maps  $\hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ ,

and this map  $\hat{f}$  is uniquely determined up to shifting its domain and range by integer multiples of  $2\pi$ . The degree  $k$  of  $f$  is

$$\frac{\hat{f}(2\pi) - \hat{f}(0)}{2\pi}.$$

Two continuous maps of the circle are homotopic if and only if they have the same degree. Replace  $f$  by  $e^{i\theta} \mapsto e^{-ik\theta} f(\theta)$ , to arrange that  $f$  has degree zero. The resulting map on Lie algebras is a linear map  $f'(1) = \hat{f}: \mathbb{R} \rightarrow \mathbb{R}$ , precisely the lift of  $f$  for which  $\hat{f}(0) = 0$ . But  $\hat{f}(2\pi) = 0$ , so by linearity  $\hat{f}(\theta) = 0$  and so  $f(e^{i\theta}) = 1$ , for all  $\theta$ . We have found the weights. Take a character  $f: S^1 \rightarrow \mathbb{C}^\times$ . The group  $S^1$  is compact, so the image of  $f$  is a compact subgroup of  $\mathbb{C}^\times$ . If some element  $z \in \mathbb{C}^\times$  has  $|z| > 1$ , then the elements  $1, z, z^2, z^3, \dots$  are unbounded, so  $z$  lies in no compact subgroup. If  $|z| < 1$ , then  $z^{-1}$  has  $|z^{-1}| > 1$ , so  $z$  lies in no compact subgroup. Hence  $S^1 \subset \mathbb{C}^\times$  is the unique maximal compact subgroup, and contains all other compact subgroups. So  $f: S^1 \rightarrow S^1$ .

**12.4.** Suppose that  $\phi: S^1 \rightarrow \text{GL}(n, \mathbb{R})$  is a cocharacter. The map  $\Phi(t) := \phi(e^{2\pi i t})$  is a one parameter subgroup by lemma 11.6 on page 107, say  $\Phi(t) = e^{tA}$  for some matrix  $A$ . But  $\Phi(t) = \phi(e^{2\pi i t})$  is periodic, so  $e^{tA}$  is also periodic, with period 1. Taking Jordan normal form of  $A$ , we see that  $e^{tA}$  has period 1 just when all of its Jordan blocks do. But looking at a single Jordan block, and taking exponential (as in problem 7.1 on page 77), we see that all Jordan blocks are  $1 \times 1$ , and all eigenvalues are imaginary. After a suitable real linear change of basis, elementary linear algebra shows that  $A$  is a collection of  $2 \times 2$  real blocks like

$$\begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$$

with  $k$  an integer to get period 1.

**12.5.** Take a character  $\xi: G \rightarrow \mathbb{C}$ ; replace  $\xi$  by  $|\xi|$  if needed; we can assume that  $\xi$  is real-valued. Since  $G$  is compact and connected, the image of  $\xi$  is a compact connected subgroup of  $\mathbb{R}^\times$ , i.e.  $\{1\}$ .

**12.6.** By problem 12.5 on page 113,  $G$  has image inside the unit circle. Lift  $G \rightarrow S^1$  to a map to the universal covering group  $x \in \mathbb{R} \mapsto e^{2\pi i x} \in S^1$ , to get a Lie group morphism  $G \rightarrow \mathbb{R}$ . Since  $G$  is compact, the image is a compact subgroup of  $\mathbb{R}$  under addition, i.e.  $\{0\}$ .

**12.7.** Any two volume forms are constant multiples of one another; the representation on volume forms is into  $\mathbb{R}^\times$ ; apply problem 12.5 on page 113.

**12.9.** Any  $T$ .

**12.10.** The image of  $T$  is the set  $W := TV$  of vectors in  $\mathfrak{g}_T$  which arise as brackets of elements, and so is an invariant of  $\mathfrak{g}_T$  under isomorphism. Every element of  $\mathfrak{g}_T$  acts on  $W$  as multiples of  $T$  acting on the image of  $T$ . Take any basis  $e_1, e_2$  for  $V$ ,  $e_3$  for  $\mathbb{R}$ . We can rescale  $e_3$ , to rescale  $T$ , an isomorphism of Lie algebras. So  $T$  is perhaps not really defined, but the action of  $T$  on  $W = TV$  is defined, up to scaling. So  $W = \{0\}$ , i.e.  $T = 0$ , just when  $\mathfrak{g}_T$  is abelian. For 1-dimensional  $W$ , take  $e_1, e_2$  a basis for  $V$ , and  $e_3$  a basis for  $\mathbb{R}$ , and arrange that  $e_1 \in W$ . Rescaling  $e_3$ , we can arrange that  $Te_1 = e_1$ ,  $Te_2 = 0$  or that  $Te_1 = 0$  and  $Te_2 = e_1$ , so we can compute out all brackets. Next suppose that  $T$  has 2-dimensional image  $W$ , so  $W = V$ , and the action of  $T$  on  $V$  is determined up to scaling and choice of basis of  $V$ . So under isomorphism of Lie algebras,  $T$  remains in precisely one of the following categories:

- a.  $T = 0$ ,  $\mathfrak{g}_T$  is abelian,
- b.  $T$  has 1-dimensional image, an eigenspace of  $T$ ; we can arrange  $0 = [e_1e_2] = [e_2e_3], [e_3e_1] = e_1$ ,
- c.  $T$  has 1-dimensional image, in the kernel of  $T$ ; we can arrange  $0 = [e_1e_2] = [e_3e_1], [e_2e_3] = e_1$ ,
- d.  $T$  is a multiple of the identity on 2-dimensional  $W$ : we can arrange  $0 = [e_1e_2], [e_3e_1] = e_1, [e_3e_2] = e_2$ ,
- e.  $T$  is diagonalizable with distinct eigenvalues on  $W$ ; rescale  $e_3$  to rescale  $T$ : we can arrange  $0 = [e_1e_2], [e_3e_1] = \lambda e_1, [e_3e_2] = \lambda^{-1}e_2$ ,
- f.  $T$  has conjugate imaginary eigenvalues on  $W$ : we can arrange  $0 = [e_1e_2], [e_3e_1] = e_2, [e_3e_2] = -e_1$ ,
- g.  $T$  has conjugate complex eigenvalues on  $W$ , not real or imaginary: we can arrange  $0 = [e_1e_2], [e_3e_1] = e_1 + \alpha e_2, [e_3e_2] = e_2 - \alpha e_1$ ,
- h.  $T$  acts with an invariant 1-dimensional subspace: we can arrange  $0 = [e_1e_2], [e_3e_1] = e_1, [e_3e_2] = e_2 + \alpha e_1$ .

**13.1.** We need to take the curves of constant  $y$  to curves of constant  $v$ . If  $\frac{\partial v}{\partial x} \neq 0$  somewhere, then moving  $x$ , even with fixed  $y$ , will move  $v$ . So  $\frac{\partial v}{\partial x} = 0$  everywhere, i.e.  $v$  is locally constant in  $x$ . But then  $v = v(y)$ , since  $U$  is connected.

**13.3.** We want to prove that, if  $L \subset M$  is a leaf of a foliation, then every smooth map  $f: P \rightarrow M$ , from a manifold  $P$ , with image contained in  $L$ , is a smooth map  $f: P \rightarrow L$ . Pick a point  $p_0 \in P$  and an open set  $U \subset M$  which is the domain of an adapted chart. The map  $f$  takes  $f^{-1}U$  to the plaques of  $L$  in  $U$ . In particular, every connected open set  $W \subset f^{-1}U$  containing  $p_0$  is mapped to a single plaque.

**14.1.** Map  $\mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  by  $A \mapsto \text{ad}_A$ , a linear injection because  $\mathfrak{g}$  is semisimple. By the Jacobi identity, this map is a Lie algebra morphism, so  $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$  is a subalgebra, and  $\mathfrak{gl}(\mathfrak{g})$  is the Lie algebra of  $\text{GL}(\mathfrak{g})$ . By theorem 14.1 on page 127,  $\mathfrak{g}$  is the Lie algebra of a Lie group.

**14.2.** The multiplication map  $G \times G \rightarrow G$  is smooth, so restricts to a smooth map  $H \times H \rightarrow G$ , and this map has image in  $H$ . The problem is really that maps to  $G$  with image in an immersed submanifold might not be smooth as maps to that immersed submanifold, as we saw with figure 8s in chapter 5. Consider the plane field which associates to each point  $g \in G$  the subspace  $g\mathfrak{h}$ . The submanifolds  $gH$  are leaves, so this is a foliation. By the solution to problem 13.3 on page 119, the smooth map  $H \rightarrow H \rightarrow G$  is smooth as a map  $H \times H \rightarrow H$ .

**14.4.** In polar coordinates,

$$X = (1 - r^2) \frac{\partial}{\partial \theta}.$$

So the flow lines are  $\dot{r} = 0$ ,  $\dot{\theta} = 1 - r^2$ , i.e.

$$\begin{aligned} r(t) &= r_0, \\ \theta(t) &= \theta_0 + (1 - r_0^2)t. \end{aligned}$$

Back in rectangular coordinates, if  $(x_0, y_0) = (r_0 \cos \theta_0, r_0 \sin \theta_0)$ ,

$$\begin{aligned} x(t) &= r_0 \cos(\theta_0 + (1 - r_0^2)t), \\ y(t) &= r_0 \sin(\theta_0 + (1 - r_0^2)t). \end{aligned}$$

If  $r_0 = 0$  or  $r_0 = 1$ , these are fixed. Otherwise, the point  $(x(t), y(t))$  travels around a circle centered at the origin in time

$$t = \frac{2\pi}{1 - r_0^2}.$$

The orbits are circles, except at the origin and at points of the unit circle (where  $x^2 + y^2 = 1$ ). At those points,  $X = 0$ , so they are fixed under the flow of  $X$ . All orbits are closed.

The space  $G \backslash M$  is Hausdorff just when, for any two points of the plane, we can surround these points with disjoint  $G$ -invariant open sets  $U, W$ . Suppose that those two points lie on the unit circle. Each open set  $U, W$  contains all points sufficiently close to one of those points. So each in particular contains points on every circle about the origin, of any radius close enough to 1. Being  $G$ -invariant,  $U, W$  contain the entire circle around the origin of any radius sufficiently close to but not equal to 1. Therefore  $U$  and  $W$  overlap.

**14.5.** The incidence relation  $R$  has closure  $\bar{R}$  the set of points  $(x_0, y_0, x_1, y_1)$  with  $y_1 x_1 = y_0 x_0$ . All of the points  $(\varepsilon, 1, 1, \varepsilon)$  lie in the same orbit, for any real  $\varepsilon \neq 0$ , but not for  $\varepsilon = 0$ . So we can leave the incidence relation on a convergent sequence of points.

**14.8.** The quotient map is surjective by definition. The smooth functions on any open subset of  $G \backslash M$  are precisely the continuous functions which pull back to smooth  $G$ -invariant functions on the preimage of that open subset in  $M$ , so the smooth structure is uniquely determined. Denoting the quotient map as  $q: M \rightarrow G \backslash M$ ,  $R = \{(m, n) \in M \times M \mid q(m) = q(n)\}$  is closed, and a submanifold by the implicit function theorem, because  $q$  is a submersion.

**14.11.** See problem 5.3 on page 57.

**14.12.** Note that  $R = \{(x, y) \in P \times P \mid \varphi(x) = \varphi(y)\}$  is closed, and a submanifold by the implicit function theorem, because  $\varphi$  is a submersion. So the quotient map  $P \rightarrow \bar{P} := G \backslash P$  is a smooth submersion for a unique smooth structure on  $\bar{P}$ . The map  $P \rightarrow Q$  is smooth and constant on fibers of  $P \rightarrow \bar{P}$ , so descends to a smooth map  $\bar{P} \rightarrow Q$ , and vice versa: mutually inverse smooth maps  $\bar{P} \rightarrow Q$  and  $Q \rightarrow \bar{P}$ .

**15.1.** By problem 9.9 on page 97, up to isomorphism, the Lie subalgebras of  $\mathfrak{so}(3)$  are:

- a. 0 and
- b.  $\mathfrak{so}(3)$  and
- c. the span of

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

i.e. the Lie algebra of the group  $\mathrm{SO}(2) \subset \mathrm{SO}(3)$  fixing  $e_3$ .

So if  $H \subset \mathrm{SO}(3)$  is a connected subgroup, then

| $H$              | $X$              |
|------------------|------------------|
| $\{1\}$          | $\mathrm{SO}(3)$ |
| $\mathrm{SO}(2)$ | $S^2$            |
| $\mathrm{SO}(3)$ | point            |

If  $H$  is not connected, but has Lie algebra  $\{0\}$ , then  $H$  is discrete and closed in the compact group  $\mathrm{SO}(3)$ , so finite. Some classical geometry determines that  $H$  is precisely one of the subgroups of rotations

- a. of a regular polygon in the plane, preserving the orientation of the polygon or
- b. of a regular polygon in the plane, together with a rotation which turns the polygon upside down or
- c. of a tetrahedron or
- d. of a cube (or, equivalently, an octahedron, which has the same symmetries as a cube) or
- e. of a dodecahedron (or, equivalently, an icosahedron, which has the same symmetries as a dodecahedron).

Each of these makes an interesting 3-dimensional manifold  $X = G/H$ .

Suppose that  $H$  is not connected, but has Lie algebra containing the rotations around some axis, say forming a connected subgroup  $H^0 \subset H$ . If  $H$  contains some element  $g$  which does not preserve the span of  $e_3$ , then  $H$  also contains  $g\mathrm{SO}(2)g^{-1}$ , and so contains the  $g$ -conjugate of  $\mathfrak{so}(2)$  in its Lie algebra, i.e. its Lie algebra is not 1-dimensional. So  $H = H^0 \cup rH^0$ , where  $r$  is a rotation matrix that reverses direction of the axis, and then  $X = G/H = \mathbb{RP}^2$ .

**15.2.** Take a Lie group morphism  $\varphi: G \rightarrow G'$  with  $\varphi(H) \subset H'$ . Let  $\Phi: G \rightarrow X'$ ,  $\Phi(g) := \varphi(g)y_0$ , a smooth map since  $\varphi$  is smooth. This map satisfies  $\Phi(g_0) = \Phi(g_1)$  just when  $g_1^{-1}g_0 \in \varphi^{-1}H'$ . So  $\Phi$  is invariant under right  $G^{x_0}$ -action on  $G$ , and so drops to a smooth  $\varphi$ -equivariant map  $\Phi: X \rightarrow X'$ . If there were two  $\varphi$ -equivariant maps  $\Phi_0, \Phi_1: X \rightarrow X'$  with  $\Phi_0(x_0) = \Phi_1(x_0) = x'_0$ , then  $\Phi_0(gx_0) = gx'_0 = \Phi_1(gx_0)$  forces  $\Phi_0 = \Phi_1$ .

If  $\Phi: X \rightarrow X'$  is some  $\varphi$ -equivariant map, and  $x'_0 = \Phi(x_0)$ , then any  $g \in H$  gives  $\varphi(g)x'_0 = \varphi(g)\Phi(x_0) = \Phi(gx_0) = \Phi(x_0) = x'_0$ , so  $\varphi(g) \in H'$ . From the last paragraph,  $\Phi$  is smooth. The rank is invariant under  $G$ , so constant. At the identity,  $\varphi: \mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{g}'/\mathfrak{h}'$  is surjective, so  $\Phi: X \rightarrow X'$  is a submersion.

**15.3.** At  $m$ ,  $\tilde{A}(m) = 0$  just when the constant map  $t \mapsto m$  is a flow line of  $\tilde{A}$ , just when it is the flow line of  $\tilde{A}$  through  $m$ , just when  $e^{tA}$  fixes  $m$ , just when  $e^{tA} \in G^m$ , just when  $A \in \mathfrak{g}^m$ .

**17.4.** Pick a basis for the exact sequence. For the one vector space whose orientation is not given, we can try both possibilities. If the basis gives an odd number of sign mistakes with one orientation, it gives an even number with the other. So the uniqueness is clear for a given basis.

Any two bases for the short exact sequence (which we think of as vectors in  $B$ ) are linear combinations of one another, with coefficients forming an invertible block matrix

$$\begin{pmatrix} P & 0 \\ R & S \end{pmatrix}$$

so that we preserve the condition that the first so many lie in the image of  $A$ . Once we fix two orientations, and a basis with an even number of sign mistakes, our matrix can change only to mess up an even number of signs, i.e. an even number of the numbers

$$\det P, \det R, \det \begin{pmatrix} P & 0 \\ R & S \end{pmatrix} = \det P \det R.$$

But that fixes all three signs as positive. So the basis is uniquely determined up to a matrix which preserves all three orientations.

**18.1.** It looks impossible: you have to cover with disks, spaced out from one another so that they cover the line without “bunching up”, but that won’t allow the disks to have small radii, because each disk has to overlap with the next disk. However, if you let the radii be  $r_j = 1/j$  and the centers be  $x_1 = 0$  and  $x_{j+1} = x_j + r_j/2$ , for example, you can cover “half” the line, by the nonconvergence of the harmonic series, while the finiteness of the area sum of the disks is due to the convergence of  $\sum 1/j^2$ . Then cover the “other half” in the same way. Rescale the picture to get the total area to be as small as you like.

**18.2.** If we can cover in balls with small volume sum, we can then cover each small ball by a small box: small volume sum. The other direction: if  $A$  is a set of measure zero using box coverings, is it also a set of measure zero using ball coverings? Take a box  $X \subset \mathbb{R}^n$ . We want to cover  $X$  by balls, so that the volume sum of the balls is close to  $\text{Vol } X$ . We can increase the side lengths of  $X$ , making a bigger box with sides of rational lengths, say  $p_i/q_i$ , only slightly increasing the volume. By taking least common multiples, we can arrange that all of the  $p_i/q_i$  have the same denominator:  $p_i/q_i = p_i/q$ . Then we can chop  $X$  into a grid, with  $p_i$  subdivisions of side  $i$  of  $X$ , so into boxes whose every side has length  $1/q$ . Each of these small boxes fits into a ball of radius  $r = \sqrt{n}/2q$ . The volume of such a ball is

$$\text{Vol } B_r = (\text{Vol } B_1) r^n,$$

as we saw on page 224. The volume sum of all these balls is

$$p_1 p_2 \cdots p_n \left( \frac{\sqrt{n}}{2q} \right)^n \text{Vol } B_1 = \left( \frac{\sqrt{n}}{2} \right)^n \text{Vol } B_1 \text{Vol } X.$$

If we can cover a set by boxes with arbitrarily small volume sum, then we can cover it by balls so that we increase the volume sum by as close as we like to a constant factor of

$$\left( \frac{\sqrt{n}}{2} \right)^n \text{Vol } B_1,$$

so that we can still make the volume sum as small as we need.

**18.3.** From the solution of problem 18.2 on page 155, boxes, rotated or not, can be covered by balls, increasing the sum of volumes by only a constant factor. So if there is a covering by rotated boxes, with small volume sum, then there is a covering by balls with small volume sum.

**18.4.** Choose some  $\varepsilon > 0$  and then pick  $\varepsilon_1, \varepsilon_2, \dots \rightarrow 0$  positive so that

$$\sum_i \varepsilon_i \leq \varepsilon.$$

Pick a cover of the first set by a sequence of balls with volumes summing to less than  $\varepsilon_1$ , and so on. Put all of these balls in a sequence.

**18.5.** Clearly if  $A$  has measure zero, then  $A \cap W$  does, for any open set  $W$ . Conversely, suppose that  $A \cap W_\alpha = 0$  for all  $\alpha$ . Let  $A_j = A \cap \bar{B}_{r_j} 0$  for some sequence of radii  $r_j \rightarrow \infty$ . We need only prove that  $A_j$  has measure zero, for all  $j$ . Fix some value of  $j$ . By compactness, we can find a finite cover of  $\bar{B}_{r_j} 0$ , say by  $W_{\alpha_1}, W_{\alpha_2}, \dots, W_{\alpha_k}$ . Then

$$\begin{aligned} A_j &= (A_j \cap W_{\alpha_1}) \cup (A_j \cap W_{\alpha_2}) \cup \cdots \cup (A_j \cap W_{\alpha_k}), \\ &\subset (A \cap W_{\alpha_1}) \cup (A \cap W_{\alpha_2}) \cup \cdots \cup (A \cap W_{\alpha_k}) \end{aligned}$$

has measure zero, since it is contained in a finite union of measure zero sets.

**18.6.** Take a number  $\varepsilon > 0$ . Take a covering by a infinite sequence of balls (or boxes)  $X_1, X_2, \dots$  with volumes summing to some  $\varepsilon/(1 + \varepsilon)$ . Replace each  $X_i$  by a slightly larger open ball (or interior of a slightly larger box)  $B_i$ , say with volume  $\text{Vol } B_i \leq (1 + \varepsilon) \text{Vol } X_i$ . Take a finite subcover of these  $B_i$ , and then replace them by their closures.

**18.8.** We will prove a somewhat stronger result. We can cover  $P$  in a countable collection of chart domains mapped into regions of  $Q$  lying in chart domains. A countable union of measure zero sets has measure zero. So it is enough to prove the result in chart domains, i.e. for maps of open sets of Euclidean space.

We really just need  $\varphi$  to have some bound on how it stretches lengths, so that when we cover a measure zero by little balls, the image is covered by little balls of only some bounded factor ratio of radius. But clearly we only need such a bound locally, since we can cover by countably many open sets, with a bound on each.

The *dilation* of a map of metric spaces is the ratio of output distances to input distances. A map of metric spaces is *Lipschitz* if it has bounded dilation, *locally Lipschitz* if it has locally bounded dilation.

A continuous map of manifolds is *locally Lipschitz* if, in any local coordinates, it is locally Lipschitz. Clearly *locally Lipschitz* is well defined for maps of  $C^1$  manifolds but the chain rule. Any locally Lipschitz map between equal dimensionals manifolds has measure zero image, because of the length stretch bound. Note that if the dimensions differ, balls of similar radius might have very different volumes.

Every continuously differentiable map of open subsets of Euclidean space is locally Lipschitz. By continuity we can bound  $|\varphi'(x)|$  on  $\bar{B}$ , say  $|\varphi'(x)| < C$ . By the chain rule,  $\varphi$  moves points apart by a factor at worst  $C$ , i.e.  $\varphi$  takes each ball, say of radius  $r$ , inside a ball of radius at most  $Cr$ , i.e. locally Lipschitz.

A map  $\varphi: X \rightarrow Y$  of metric spaces is *nearly Lipschitz* if, for any  $x_0 \in U$

$$\limsup_{x \rightarrow x_0} \frac{d(\varphi(x), \varphi(x_0))}{d(x, x_0)} < \infty.$$

A continuous map of manifolds  $\varphi: P \rightarrow Q$  is *nearly Lipschitz* if it is in some chart around each point of  $P$  and around the corresponding point of  $Q$ . We will prove our result for nearly Lipschitz maps. In particular, differentiable maps are nearly Lipschitz. As above, we only need to prove the result for open sets in Euclidean space.

Suppose that  $U, W \subseteq \mathbb{R}^n$  are open sets and  $\varphi: U \rightarrow W$  is nearly Lipschitz. Let  $U_j$  be the set of points  $x_0 \in U$  at which, if  $|x - x_0| < 1/j$  then

$$|\varphi(x) - \varphi(x_0)| < j|x - x_0|.$$

So

$$U = U_1 \cup U_2 \cup \dots$$

On each  $U_j$ ,  $\varphi$  is Lipschitz, stretching the radius of a ball by at most  $j$ . Cover a measure zero subset of  $U_j$  in a sequence of small balls; their images still lie in small balls of at most  $j$  times the radius in  $W$ .

**18.9.** Suppose  $P$  has dimension  $p$  and  $Q$  dimension  $q$  with  $p < q$ . Consider the map  $\Phi(p, x) = \varphi(p)$  for  $x \in \mathbb{R}^{q-p}$ ; we need only prove this map has measure zero on  $P \times \{0\}$ , a measure zero set.

**18.10.** Note that the dimension of  $V_r$  is quadratic in  $r$ , vanishing at  $r = 0$  and  $r = n + N$ , so as a quadratic function, it has maximum in the middle: at  $r = (n + N)/2$ . But  $N \geq 2n$  so the maximum occurs at

$$\frac{n + N}{2} > n > n - 1.$$

The quadratic function's values are increasing, not hitting the maximum, for  $0 \leq r \leq n - 1$ .

**18.11.** For  $B \in V_r$ : the domain has smaller dimension than the range as long as

$$Nn - n - \dim V_r.$$

Plugging in the worst case,  $r = n - 1$ ,

$$Nn - n - \dim V_r = N - 2n + 1 > 0.$$

Apply problem 18.9 on page 156: the map has measure zero image on every  $V_r$ ,  $r = 0, 1, 2, \dots, n - 1$ , and hence on  $V$ .

**18.12.** Apply theorem 18.4 on page 157 to a constant map.

**18.15.** There is a countable set of rational numbers, so a countable set of these, say

$$p_i(x, y), q_i(x, y)$$

for  $i = 1, 2, \dots$ . Consider, for each, the map

$$\varphi_i(x, y) = (p_i(x, y), q_i(x, y)).$$

By Sard's theorem, each  $\varphi_i$  has a measure zero set of critical values. The union of these is measure zero, since it is a countable union. So there is a regular point  $(x_0, y_0)$  for all of them. So  $\varphi_i$  is a local diffeomorphism near any preimage point of  $(x_0, y_0)$ . So that preimage point is locally unique, i.e. such points form a discrete set.

With some algebra, we can do a little better: suppose that  $p(x, y) - x_0, q(x, y) - y_0$  have infinitely many common zeroes. For any finite set of those zeroes, we can construct the ideal of polynomials vanishing on them. We easily construct polynomials, with real coefficients, vanishing on those points. Taking ever larger finite sets, we build an infinite chain of ever smaller ideals of two variable real polynomials, all containing  $p(x, y) - x_0$  and  $q(x, y) - y_0$ , contradicting the Noetherian property of polynomials in two variables. So in fact each  $p(x, y), q(x, y)$  has only finitely many points  $(x, y) \in \mathbb{R}^2$  at which  $p(x, y) = x_0$  and  $q(x, y) = y_0$ .

**18.16.** Suppose that  $\dim P = p$  and  $\dim Q = p + q$ . The map  $\Phi: (x, y) \in P \times \mathbb{R}^q \rightarrow \varphi(x) \in Q$  has differential nowhere injective, since  $\Phi'(x, y)(0, v) = 0$  for all  $v \in \mathbb{R}^q$ . So every point is critical, and so the image consists only of critical values.

**18.17.**

a. Recall that the *tangent bundle* of  $M$  is

$$TM = \{ (x, v) \mid x \in M, v \in T_x M \}.$$

Let  $f: (x, v) \in TM \mapsto v \in \mathbb{R}^{p+q}$ . By dimension count and Sard's lemma, the generic point of  $TM$  is not a critical point, i.e. there is a vector  $u \in \mathbb{R}^{p+q}$  which is not in the image, i.e. nowhere tangent to  $M$ . So the linear projection map to  $u^\perp \subset \mathbb{R}^{p+q}$  has no kernel on any tangent spaces of  $M$ , i.e. the differential of that linear map has no zeroes on those tangent planes, i.e. the linear projection restricts to  $M$  to a local diffeomorphism.



- b. Let  $f: (x, y, t) \in M \times M \times \mathbb{R} \mapsto t(x - y) \in \mathbb{R}^{p+q}$ . By dimension count and Sard's lemma, there is some vector  $w \neq 0$  not in the image of  $f$ , i.e. so that the differences  $x - y$  of points  $x \neq y$  of  $M$  are nowhere parallel to the vector  $w$ . Rescale  $w$  to have unit length, call it  $u$ .

**18.19.** Submersions clearly have nonzero measure image. Take a constant rank map with nonzero measure image. By Sard's theorem, the map has a measure zero set of critical values. So there are regular values in the image. So the rank is the dimension of  $Q$  at such points, hence everywhere.

**18.20.** Take a bijective  $C^k$  map  $\varphi: P \rightarrow Q$ . If the map is an immersion, it has constant rank. Because the map is almost onto and constant rank, it is a submersion. By the rank theorem, it is locally a linear projection in some coordinates, so not a bijection unless it is a locally diffeomorphism. There is a unique inverse because  $\varphi$  is a bijection. There is therefore a unique local inverse near each point of  $Q$ . The local  $C^k$  inverses are therefore equal to the global inverse, i.e. the inverse is  $C^k$ .

**18.21.** The critical values form a measure zero set, so remove them from  $Q$  and their preimage from  $P$  (which might be all of  $P$ ). The remaining points of  $P$  are regular, so apply the implicit function theorem to replace  $\varphi$  by the map  $\varphi(x, y) = x$ . Apply theorem 18.1 on page 156.

**18.22.** One approach: the determinantal variety is the closure of the determinantal manifold from problem 1.36 on page 13, and clearly is precisely the union of the determinantal manifolds  $M_{p,q,s}$  for  $0 \leq s \leq r$ .

Another approach. Taking transpose identifies  $V_{p,q,r} \cong V_{q,p,r}$ , so we can assume that  $p \leq q$ . Every such matrix  $A$  has at most  $r$  linearly independent columns, and then the other columns lie in their span, so are linear combinations of those. Pick some  $r$  columns, say columns  $j_1, j_2, \dots, j_r$ . Then map  $\mathbb{R}^{p \times r} \rightarrow V$  by mapping each matrix  $M$  to the matrix  $A$  whose columns  $j_1, \dots, j_r$  are the columns  $1, 2, \dots, r$  of  $M$ , and whose other columns are "empty" so far. But then we want the other  $q - r$  columns of  $A$  to be linear combinations of those columns, so we need to pick, for each column  $j$  still empty some row matrix  $\xi_j$  of size  $1 \times p$  so then we set column  $j$  of  $A$  to be  $\xi_j B$ , a linear combination of those columns. So these  $\xi_j$  are rows of a matrix  $C$  of size  $(q - r) \times p$ . This map  $(B, C) \mapsto A$  covers the portion of  $V$  which consists of matrices  $A$  with columns  $j_1, j_2, \dots, j_r$  of full rank (and covers some more of  $V$  as well). Taking all possible choices of columns  $j_1, j_2, \dots, j_r$ , finitely many possibilities, we cover all of  $V$ .

**19.1.** Since any manifold is locally diffeomorphic the Euclidean space, it suffice to assume  $X \subset \mathbb{R}^n$  is a region with corners in Euclidean space. Moreover, we only need to prove the result locally, so we can pick one collection of boundary functions  $f^i$ . The functions  $f^i$  have linearly independent differentials, so the matrix

$$\left( \frac{\partial f^i}{\partial x^j} \right)$$

has full rank. After permutation of the orders of  $f^i$  and  $x^j$ , we can assume that it has nonzero entries down the diagonal. The map

$$y = f(x) = (f^1(x), \dots, f^q(x), x^{q+1}, \dots, x^n)$$

is therefore a local diffeomorphism, identifying  $X$  with an open subset of the set where  $y^1, \dots, y^q \geq 0$ . If we translate to ensure that, near some chosen point, the coordinates

$x^{q+1}, \dots, x^n$  are positive, we have identified  $X$  with an open subset of the standard corner, i.e. to have all  $y^i \geq 0$  on  $X$ .

**19.4.** It is enough to prove this for the standard corner. If  $x$  lies on a wall ( $x^i = 0$ ), we can take  $f = x^i$ ; note that many other functions would do just as well. On the other hand, any interior local minimum of any  $C^1$  function  $f$  is a critical point:  $df = 0$ . So a local minimum where  $df \neq 0$  lies on the boundary, i.e. on some wall  $x^i = 0$ .

**19.5.** Take a boundary point  $y_0$  of  $Y$ . So there is some  $C^k$  boundary function  $f$  defined near  $y_0$ . Let  $g := f \circ \varphi$ . So then  $g \geq 0$  and any point where  $g = 0$  maps to a point where  $f = 0$ , so a boundary point by problem 19.4 on page 166. So if  $\varphi(x_0) = y_0$  then  $x_0$  is a boundary point. Every boundary point of  $Y$  has some such  $f$  vanishing at it, so  $\varphi^{-1}\partial Y \subseteq \partial X$ . Reversing the roles of  $X$  and  $Y$ ,  $\varphi\partial X \subseteq \partial Y$ . Applying twice,  $\partial X \subseteq \varphi^{-1}\partial Y \subseteq \partial X$ . So  $\varphi$  is a diffeomorphism of boundary to boundary.

**19.6.** If  $f(x) = 0$  and

$$\frac{\partial f}{\partial x^i} < 0,$$

then making  $x^i$  a little larger will make  $f < 0$ .

**19.7.** Suppose that  $f$  is a  $C^k$  boundary function. Recall that by definition of a  $C^k$  function on a subset of a manifold,  $f$  extends somehow to be  $C^k$  near  $Y$  in whatever manifold  $Y$  lives in. Similarly we can extend  $\varphi$ . Let  $g := f \circ \varphi$ . Note that  $df \neq 0$  and  $\varphi'(x)$  is a linear isomorphism for any  $x \in X$ . So  $dg \neq 0$ . The hypersurface on which  $f = 0$  is diffeomorphically mapped to the one on which  $g = 0$ . Boundary is diffeomorphically mapped, so  $g = 0$  exactly on of the faces  $x^i = 0$ , say  $x^1 = 0$ . By problem 19.6 on page 166, for each variable  $x^1, \dots, x^n$ , where  $g = 0$ , if  $x^i = 0$  then

$$\frac{\partial g}{\partial x^i} \geq 0.$$

Because  $dg \neq 0$ , at least one these is not zero, so positive. If two are positive then the hypersurface  $g = 0$  intersects with  $x^1 = 0$  in a codimension 2 submanifold, by the implicit function theorem. But we know that  $g = 0$  is  $x^1 = 0$ , a hypersurface. Hence all along  $x^1 = 0$ ,  $dg$  is a positive multiple of  $dx^1$ . So  $g, x^2, \dots, x^n$  have linearly independent differentials. For any point near that hypersurface,  $g > 0$  just when  $x^1 > 0$ , and  $g = 0$  just when  $x^1 = 0$ . Replace  $x^1$  by  $g$  as a boundary function.

**19.8.** After a diffeomorphism near our chosen point, we can assume that  $g^i = x^i$ . As in the proof of problem 19.7 on page 166, we can arrange that  $df$  is a positive multiple of  $dx^1$  all along ( $x^1 = 0$ ). Let

$$u := \log(f/x^1),$$

which is finite near our chosen point by l'Hôpital's rule.

**19.9.** Roughly: it has 1-dimensional bits and 2-dimensional bits. Every manifold is only made out of pieces all of the same dimension: locally diffeomorphic to an interval or to a rectangle or to a solid rectangular box, and so on, but not to more than one of these.

**19.10.** No: for example, let  $M$  be the image of the unit square under the map  $(x, y) \mapsto (x + y, y)$ :  $\square \searrow$

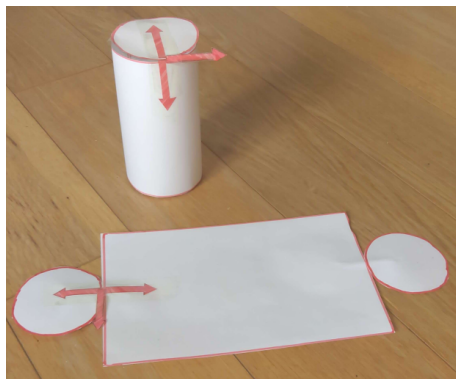
**19.12.** Just to be very precise, let's be careful about what the question is asking. Recall that a function on an open subset of the hollow cylinder is smooth just when it is locally the restriction to the hollow cylinder of a smooth function on some open set in  $\mathbb{R}^3$ . So we are looking to see if these smooth functions make a smooth structure,

making the hollow cylinder into a smooth manifold with corners. Equivalently, a map of the hollow cylinder to a manifold with corners is smooth just when it is locally the restriction to the hollow cylinder of a smooth map on some open set in  $\mathbb{R}^3$ . Similarly for maps from any manifold with corners to the hollow cylinder.

Away from the rim, clearly the hollow cylinder is a surface. So a dense open set of points of the hollow cylinder is a surface. Therefore if the hollow cylinder is a manifold with corners, it must be a two-dimensional manifold with corners, i.e. surface with corners.

The hollow cylinder has two tangent half planes at each point of the rim, representing the possible velocity vectors of curves headed toward the rim. (And another two representing the possible velocity vectors of curves headed away from the rim.) Near each point of the rim, we must have a chart identifying the hollow cylinder with an open subset of the standard corner  $X = \{(x, y) \mid x, y \geq 0\}$ . But then it has a single tangent plane or tangent half plane at every point. These are identified by the derivative of the chart, a contradiction.

Another proof: if a point moves radially inside the top, its velocity as it reaches the boundary is a horizontal nonzero vector. As another point moves around the rim, its velocity as it reaches the same point is also horizontal, linearly independent of the previous vector. As a third point moves up the side, its velocity at the instant it reaches that same point is a linearly independent vertical vector. So the tangent vectors at any point of the rim span a three dimensional vector space.



But the tangent vectors to motions along the standard corner span a two dimensional vector space. These are identified by the derivative of any chart, a contradiction.

Careful: the hollow cylinder is homeomorphic to the sphere. To see this, note that the function

$$(x, y, z) \mapsto \sqrt{x^2 + y^2 + z^2}$$

is smooth on  $\mathbb{R}^3 - \{(0, 0, 0)\}$ . So the map

$$(x, y, z) \mapsto \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

is smooth on  $\mathbb{R}^3 - \{(0, 0, 0)\}$ . In particular, it is smooth, by definition, on the hollow cylinder. It is not difficult to prove that the resulting map from hollow cylinder to unit sphere is a smooth homeomorphism, with a continuous (but not smooth) inverse; see problem 33.5 on page 264.

Careful: there are smooth paths with nowhere zero velocity which travel from a point of the top to a point of the side. Their construction is not easy. Picture a nonzero velocity vector tangent to the rim. Draw a path which travels to the rim, to have that velocity, agreeing to infinite order in derivatives with the circular motion around the rim. Then glue on a similar path going from the rim downward. However, any path from the top to the side, or vice versa, has to have velocity tangent to the rim as it reaches the rim, which is not consistent with a surface with corners.

**19.17.** This is difficult to prove. It is even difficult to prove that a curve can't have precisely three end points. We can treat each connected component of the curve separately, so assume the curve is connected. If the curve has no end points (i.e. corners), theorem 6.6 on page 74 proves it is a circle. So we can assume our curve is connected, and has at least one end point.

First, we prove that every oriented  $C^k$  curve with corners bears a nowhere vanishing  $C^{k-1}$  vector field. Each point of the curve lies in the domain of a positively oriented chart  $(U, x)$ , and there is a bump function  $f \geq 0$ , perhaps nonzero on the boundary of the curve, but vanishing outside the chart domain and not vanishing at the point. The vector field

$$f(x) \frac{\partial}{\partial x}$$

as written in the coordinate  $x$  of the chart, is nonzero at the given point. We can arrange that  $f$  is nonzero in an interval in that chart.

So we can cover the curve in open sets, on each of which there is such a vector field nonzero. Because the curve is compact, we can take a finite subcover. Add up the vector fields.

Suppose for the moment that  $C$  is oriented. We can assume  $C$  bears a nowhere zero vector field, flowing in the positively oriented direction, by adding up local bumps as in the proof of 6.6 on page 74. Pick coordinates near a boundary point. The coordinates identify  $C$  near the boundary point with an open set in  $[0, \infty)$  near 0. Glue some more curve to  $C$ , using those coordinates to glue to  $\mathbb{R}$ , making a noncompact curve  $D$  containing  $C$ . Extend the vector field in those coordinates to become smooth and constant outside some compact set. So  $D$  has no boundary, and the vector field on  $D$  is complete after it flows through that boundary point. Carry out this same procedure to all boundary points. It follows that the flow of the vector field is defined globally on  $D$ . So we can assume that  $C \subseteq D$  is compact and that  $D$  has a complete nowhere zero vector field.

Each flow line is a local diffeomorphism  $x: \mathbb{R} \rightarrow D$ , hence has open image. So the curve is a union of the flow lines through its various points, a union of connected open sets. Being a connected curve, there is only one such open set. Hence each flow line is surjective. If injective, is a bijective local diffeomorphism, so a diffeomorphism. So  $D = \mathbb{R}$  contains a compact connected set  $C$ , hence  $C$  is a closed interval.

If not injective, say  $x(t_0 + h) = x(t_0)$  for some  $t_0$  and some  $h > 0$ . As in the proof of 6.6 on page 74,  $x$  has period  $h$ , and there is a least period. But then we have a local diffeomorphism from a compact circle to a noncompact curve  $D$ .

**19.26.** Each of  $P$  and  $Q$  is a countable union of charts, so we can assume that  $P, Q \subseteq [0, \infty)^n$  are open sets. The boundary of  $Q$  has measure zero; delete it, and delete its preimage in  $P$ . If this deletes all of  $P$ , then all of the interior points of  $Q$  are regular values: they don't lie in the image of  $P$ . So we can assume that  $\partial Q$  is empty. As  $\partial Q$  is empty, there are no cowalls for  $Q$ , so all conditions of regularity are satisfied wherever  $\varphi^*$  is injective. The proof of the standard Sard theorem shows that

the points  $q \in Q$  for which  $\varphi^*(p)$  is not injective, for some  $p \in \varphi^{-1}\{q\}$ , form a set of measure zero in  $Q$ .

**19.33.** Pick a function  $h$  with  $h = 1$  near the origin and vanishing outside the unit ball around the origin. Let  $h_\delta(x) := h(x/\delta)$ , so supported in the ball of radius  $\delta$ , and let  $f := ph_\delta$ . Outside of the ball of radius  $\delta$ ,  $f = 0$ . By compact support of  $h$ , we can pick a bound on any derivative. Rescale:  $h_\delta$  has derivatives of order  $\ell$  bounded by  $O(\delta^{-\ell})$ . Our polynomial of degree  $d$  has derivatives of order  $\ell$  bounded by some  $O(\delta^{d-k})$  on the ball of radius  $\delta$ . Differentiate  $f$  to some order  $k$ : by the chain rule, on the ball of radius  $\delta$ , we differentiate  $p$  some number of times, say  $j$  times, and  $h_\delta$  some number of times, say  $k = \ell - j$  times. The product is bounded by  $O(\delta^{d-j}\delta^{-k}) = O(\delta^{d-\ell})$  for each  $j + k = \ell$ .

**19.35.** Clearly such a function  $f(x)$  depends only on  $|x|^2$ ; write  $f(x) = g(|x|^2)$  for some function  $g$ , and we need to prove that  $g$  is smooth. Writing  $g(s) = f(\sqrt{s}u)$  for any unit length vector  $u$ , it is clear that  $g$  is smooth away from  $s = 0$ , since  $f$  is. By Seeley's extension theorem 19.10 on page 174, we need only bound  $g$  and each of its derivatives at the origin, i.e. find a Taylor series for  $g$ .

We can rotate any vector  $x$  to  $-x$ , and  $f$  is rotation invariant, so  $f(-x) = f(x)$ . It follows that all odd order derivatives of  $f$  vanish. Expanding  $f$  in a Taylor series, using only even order derivatives, plug in  $x = \sqrt{s}u$  to find a Taylor series for  $g(s)$ .

**19.36.** Rotations don't preserve any vector fields other than multiples of the radial: tangent components at any point need to be invariant under rotations fixing the line through that point. So every rotation invariant vector field has the form  $Y = fX$  for some function  $f$ . By invariance,  $f$  is a invariant function, smooth away from the origin, say  $f(x) = g(|x|^2)$ . So  $g$  is smooth away from the origin. Clearly  $Y(-x) = -Y(x)$ , so if we expand  $Y$  in a Taylor series, we only get odd terms, and zero constant term. Take any unit vector  $u$  and plug in  $x = \sqrt{s}u$  to find that  $g(s)$  has a Taylor series. By Seeley's extension theorem 19.10 on page 174,  $g$  is smooth.

**20.1.** For simplicity of notation, suppose we just have one  $f$  and one  $g$  function. The  $f$  and  $g$ , when sampled at points of  $S$ , take their minimal values at  $s_0$ , say equal to zero, so  $df(s_0), dg(s_0) \in S_{M,s_0}^+$ . We need to see that they span. Suppose not; take some  $C^1$  function  $h$  defined on  $M$  near  $s_0$  and  $\geq 0$  on  $S$ , with  $h(s_0) = 0$ , and suppose that  $dh$  is linearly independent of  $df, dg$ . Extend the functions  $f, g, h$  to be  $C^1$  near the origin. Change variables to get them to be coordinate functions  $x, y, z$ . Then  $S$  is the set of points where  $x \geq 0$  and  $y = 0$ . But the coordinate function  $z$  corresponding to  $h$  varies arbitrarily near the origin.

**20.2.** Take any local coordinates  $x^1, x^2, \dots, x^n$  on  $M$  defined near some point of  $P \cap Q$ . Throw out any three of these coordinate functions and replace them by  $f, g, h$ , as long as the  $n - 3$  remaining functions  $x^i$  have differentials independent of  $df, dg, dh$ . After perhaps relabelling the coordinates, we can suppose that we threw out  $x^1, x^2, x^3$ . Rename  $f, g, h$  by the names  $x^1, x^2, x^3$ . Then  $x^1, x^2, \dots, x^n$  are local coordinates with  $f = x^1, g = x^2$  and  $h = x^3$ . Locally  $P$  and  $Q$  are just  $P = \{x^1, x^2 \geq 0\}$  and  $Q = \{x^3 \geq 0\}$ , the same example. The independence of the differentials here is the key.

**20.4.** Take  $M = [0, \infty)^2$  with coordinates written  $x, y$ . Let  $P$  be the union of two closed line segments: the segment from  $(x, y) = (1, 1)$  to  $(x, y) = (3, 3)$  and the segment from  $(x, y) = (1, 0)$  to  $(x, y) = (3, 0)$ . Let  $Q$  be the union of two closed line segments: the segment from  $(x, y) = (1, 3)$  to  $(x, y) = (3, 1)$  and the segment from

$(x, y) = (2, 0)$  to  $(x, y) = (4, 0)$ . The intersection consists of a point  $(2, 2)$  and a closed line segment from  $(x, y) = (2, 0)$  to  $(x, y) = (3, 0)$ .

**21.5.** It is clearly enough to check the orientations in open sets covering these, so we can assume that  $P \subseteq [0, \infty)^p$ ,  $Q \subseteq [0, \infty)^q$  are open sets with their standard orientations. But then it suffices to check the result for  $P = [0, \infty)^p$  and  $Q = [0, \infty)^q$ , say with variables written as  $x^1, \dots, x^p$  and  $x^{p+1}, \dots, x^{p+q}$ . So  $P \times Q = [0, \infty)^{p+q}$  with its standard orientation. Swapping any two variables changes orientation.

For any integer  $n \geq 0$ , let  $X_i^n \subset \mathbb{R}^n$  be the set of points at which  $x^i = 0$ , equipped with chart

$$x^1, x^2, \dots, \hat{x}^i, \dots, x^n.$$

The wall  $P_1 := (x^1 = 0)$  of  $P$  has orientation given by chart  $x^2, \dots, x^p$ , so  $P_1 = X_1^p$ . The wall  $P_2 := (x^2 = 0)$  of  $P$  has orientation given by first swapping variables with  $x^1$ , which changes orientation, and taking  $x^1, \hat{x}^2, \dots, x^{p+q}$  as chart, but now change back the orientation, so  $P_2 = -X_2^p$ . By induction,

$$\partial P = \bigsqcup (-1)^{i-1} X_i^p.$$

Similarly,

$$\begin{aligned} \partial P &= \bigsqcup (-1)^{i-1} X_i^p, \\ \partial Q &= \bigsqcup (-1)^{i-1} X_i^q, \\ \partial(P \times Q) &= \bigsqcup (-1)^{i-1} X_i^{p+q}. \end{aligned}$$

But if  $i \leq p$ , then

$$X_i^{p+q} = X_i^p \times Q,$$

while if  $i \geq p+1$ , then

$$X_i^{p+q} = P \times X_{i-p}^q.$$

So finally

$$\begin{aligned} \partial(P \times Q) &= \bigsqcup_i (-1)^{i-1} X_i^{p+q}, \\ &= \bigsqcup_{i \leq p} (-1)^{i-1} X_i^{p+q} \sqcup \bigsqcup_{i \geq p+1} (-1)^{i-1} X_i^{p+q}, \\ &= \bigsqcup_{i \leq p} (-1)^{i-1} X_i^p \times Q \sqcup \bigsqcup_{i \geq p+1} (-1)^{i-1} P \times X_{i-p}^q, \\ &= \left( \bigsqcup_{i \leq p} (-1)^{i-1} X_i^p \right) \times Q \sqcup P \times \left( \bigsqcup_{i \geq p+1} (-1)^{i-1} X_{i-p}^q \right), \\ &= \partial P \times Q \sqcup (-1)^p P \times \left( \bigsqcup_{i \geq p+1} (-1)^{i-p-1} X_{i-p}^q \right), \\ &= \partial P \times Q \sqcup (-1)^p P \times \left( \bigsqcup_{i \geq 1} (-1)^{i-1} X_i^q \right), \\ &= \partial P \times Q \sqcup (-1)^p P \times \partial Q. \end{aligned}$$

**22.2.** A polynomial function of one variable is proper just when it isn't constant. In many variables, it is more complicated.

**22.3.** Take a compact set in  $Q$ . Its preimage in  $P$  is also compact, because  $\varphi$  is proper. So there are finitely many critical points in that preimage. Hence there are finitely many critical values in any compact set in  $Q$ .

Take an open subset  $U$  of  $Q$  with compact closure. There are finitely many critical values in  $U$ . So the critical values are discrete in  $Q$ .

Near each of those values, take local coordinates. We can find a ball, in those local coordinates, about that critical value, in which the coordinates are defined. A punctured ball is path connected (by induction on dimension, starting in dimension 2). Any path in  $Q$  can, whenever it hits a critical value, be spliced with a path inside the ball around that critical value, not striking the critical value. So the regular points of  $Q$  form a path connected open set  $Q_{\text{reg}} \subseteq Q$ . By properness of  $\varphi$ ,  $\varphi$  is a finite covering map on  $Q_{\text{reg}}$ , so locally of constant degree, globally since  $Q_{\text{reg}}$  is path connected.

Suppose there is some point  $q_0 \in Q$  not in the image of  $\varphi$ . Take a compact subset of  $Q$  containing an open set around  $q_0$ . Its preimage is compact in  $P$ . If there are points of  $Q$  in the image of  $\varphi$  approaching  $q_0$ , there are points in  $P$  mapping to them, sitting in a compact set, so having a convergent subsequence, converging to a point mapping to  $q_0$ . Therefore  $q_0$  lies in some open set outside the image of  $\varphi$ , and hence is regular. So all regular values have empty preimages. The image lies in the critical values, i.e. all points of  $P$  are critical. But the critical points form a discrete subset of  $P$ , finitely many in any open set, a contradiction since  $P$  has positive dimension. Hence  $\varphi$  is onto.

Take  $P = Q = \mathbb{C}$ , and  $\varphi: P \rightarrow Q$  a polynomial map  $w = p(z)$ . Then  $w'(z) = p'(z)$  has finitely many roots, so a finite set of critical points. We saw that complex polynomials are proper maps. Hence  $\varphi$  is onto, i.e. we can solve  $w_0 = p(z)$  for  $z$ , for any constant  $w_0$ .

**22.4.** It is clear that  $\varphi$  admits local inverses near each point of  $P_0$ . There are only finitely many points  $p \in P_0$  over each point  $q \in Q_0$ , so these local inverses have images in  $Q$  intersecting in an open set around  $q$ , i.e. there is an open set around  $q$  consisting entirely of points lying in evenly covered open sets.

**22.6.** Take an angle function  $\Phi: \mathbb{R} \rightarrow \mathbb{R}$ , so  $\varphi(e^{i\theta}) = e^{i\Phi(\theta)}$ . Note that  $\Phi$  is uniquely determined up to adding a constant integer multiple of  $2\pi$ . Since  $e^{i0} = e^{i2\pi}$ ,  $\Phi$  is periodic up to a constant integer multiple of  $2\pi$ , i.e.  $\Phi(\theta + 2\pi) = \Phi(\theta) + 2\pi n$ , for a unique integer  $n$ . Let  $\Psi_t(\theta) = (1-t)\Phi(\theta) + tn\theta$ . Clearly  $\Psi_t(\theta + 2\pi) = \Psi_t(\theta) + 2\pi n$ , so if we let  $\psi_t(\theta) = e^{i\Psi_t(\theta)}$ , then  $\psi$  is  $2\pi$ -periodic, so descends to a unique map, also called  $\psi_t$ , on  $S^1$ , defined by  $\psi_t(e^{i\theta}) := \psi_t(\theta)$ . Using the wrong value of  $n$  makes  $\Psi_t(\theta)$  no longer stay periodic in  $\theta$  up to a constant integer multiple of  $2\pi$ .

**22.7.** Suppose that there is some  $C^0$  homotopy  $\varphi_t$ . Lift the homotopy uniquely, once we lift one point  $(s, t) = (0, 0)$ , to a homotopy  $\Phi_t(s)$  of maps of the angles, say  $\Phi: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ , by first taking an angle function for  $t \mapsto \varphi_t(1)$  and then extending to an angle function for  $s \mapsto \varphi_t(s)$  for each  $t$ . The map  $\Phi$  is as smooth as  $\varphi$ , as it is locally identified with  $\varphi_t(e^{i\theta})$  by our complicated collection of trig and inverse trig formulas. The degree

$$\frac{\Phi_t(2\pi) - \Phi_t(0)}{2\pi}$$

is continuous in  $t$ , integer valued, and so constant, equal to  $p$  at  $t = 0$  and to  $q$  at  $t = 1$ .

**22.10.** Given diffeotopies  $\Phi, \Psi$ , each starting at the identity, we can “smoothly compose”  $\Phi$  with  $(t, x) \mapsto \Psi_t(\Phi_1(x))$  to get an diffeotopy to  $\Psi_1 \circ \Phi_1$ . The diffeotopy  $\Phi_t^{-1}$  inverts.

**22.14.** Complex conjugation has degree  $-1$ , while the identity map has degree  $1$ , so they are not homotopic through smooth maps, so not diffeotopic. Clearly any diffeomorphism has degree  $1$  or  $-1$ . Composing with complex conjugation if needed, we can assume that our diffeomorphism  $\varphi: S^1 \rightarrow S^1$  has degree  $1$ . Composing with a rotation, we can assume that  $\varphi(1) = 1$ . Take an angle function, so  $\varphi(e^{i\theta}) = e^{i\Phi(\theta)}$ . Since  $\varphi$  has degree one,  $\Phi(\theta + 2\pi) = \Phi(\theta) + 2\pi$  and  $\Phi' > 0$  since  $\varphi$  preserves orientation. Write  $\Phi(\theta) = \theta + \varepsilon(\theta)$ , so  $\Phi' > 0$  means precisely that  $\varepsilon' > -1$ . Let

$$\begin{aligned}\Phi_t(\theta) &= \theta + t\varepsilon(\theta), \\ \varphi_t(\theta) &= e^{i\Phi_t(\theta)}.\end{aligned}$$

**22.16.** If the map is  $\varphi$ , take the homotopy

$$\Phi_t(x) := \begin{cases} t\varphi(x/t), & \text{if } 0 \leq |x| \leq t, \\ x, & \text{if } t \leq |x| \leq 1. \end{cases}$$

**22.17.** Compose one with the inverse of the other and apply the Alexander trick.

**22.18.** Zoom in by rescaling input and output variables reciprocally to get to a linear map. Use some linear algebra.

**22.19.** Let  $\Phi_t(x) := \varphi(tx)/t$ , for  $t \neq 0$ , and  $\Phi_0(x) = x$ :  $\varphi \xrightarrow{\Phi} \varphi'(0)$ . By composing isotopies, we reduce to the case when both  $\varphi$  and  $\psi$  are linear. Compose with a reflection as needed, to arrange that both preserve, or both reverse, orientation. We need to compose with a suitable diffeotopy  $L_t$  between the linear transformations  $L_0 = I$  and  $L_1 = \varphi^{-1} \circ \psi$ . For this we use a sequence of diffeotopy compositions corresponding to the steps in Gauss–Jordan elimination, which the reader can make precise.

**22.21.** By smoothly composing, we need only carry this out using a sequence of such isotopies. By lemma 22.4 on page 192, the group of compactly supported isotopies fixing every boundary point acts transitively. So we can assume that  $\varphi_0(0) = \varphi_1(0)$ . By definition of an embedding, we can extend  $\varphi_0, \varphi_1$  to be defined each in some open set around  $\bar{B}$ , say in some larger open ball  $B'$ . For simplicity of notation, suppose that  $B'$  has center at the origin. In  $B'$ , define a vector field  $X(x) = -x$ . Multiply by a bump function to get  $X$  to vanish away from  $\bar{B}$ . Using  $\varphi_0^{-1}$  as chart,  $X$  becomes a vector field on an open subset of  $M$ , which we can extend to vanish away from that chart domain. Flow the closed unit ball, in that chart, by that vector field until  $\bar{B}$  flows inside the image of  $\varphi_1$ . Now switch to using  $\varphi_1^{-1}$  as chart. We can throw away the manifold  $M$ , and just work in that chart, so we can assume that  $M = \bar{B}$ ,  $\varphi_1(x) = x$ , and  $\varphi_0: \bar{B} \rightarrow \bar{B}$  has image inside some small ball contained in  $\bar{B}$ . Apply problem 22.19 on page 192 to get an diffeotopy of  $\varphi_0$  to some map  $x \mapsto \varepsilon x$  as maps to  $\mathbb{R}^n$ . We can use some more flow of  $X$  to get the image of that entire homotopy to live inside  $\bar{B}$ . Rescale to get our map to become the identity map.

**23.2.** Plug in the vector fields

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^p}$$

to see that  $a_{12\dots p} = 0$ . Similarly for all of the other  $a_I$ .

**23.4.**  $2 dx \wedge dy \wedge du \wedge dv$



**24.6.** A helpful test: given a 1-form  $\omega = a dx + b dy$  on the plane, if  $\omega = df$  then

$$\omega = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

then clearly  $a = \frac{\partial f}{\partial x}$  and  $b = \frac{\partial f}{\partial y}$  so therefore

$$\frac{\partial a}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial b}{\partial x}.$$

Even without that trick, we can see:

a.  $d(x^2) = 2x dx$

b.  $d(x^2/2) = x dx$

c.  $d(x^2/2 + y^2/2) = x dx + y dy$

d.  $d(xy) = y dx + x dy$

e. But  $y dx - x dy$  is *not*  $df$  for any  $f$  since to have  $\frac{\partial f}{\partial x} = y$  integrate in  $x$  to see that we need  $f = xy + F(y)$  for some unknown  $F(y)$ . But differentiate in  $y$  to see that  $\frac{\partial f}{\partial y} = x + F'(y)$  doesn't fit.

**24.9.** No: a 1-form  $\omega$  which is not zero at some point in some manifold can't just be turned into  $dx^1$  in any chart, unless  $d\omega = 0$ , since  $d(dx^1) = 0$ . Similarly, a 1-form  $\omega$  can't be turned into  $(1 + x_2)dx^1$  in any chart unless  $d\omega \wedge \omega = 0$ .

**24.13.** Let

$$g(x, y, z) := \int_0^x f(u, y, z) du,$$

and try

$$\xi = g(x, y, z) dy \wedge dz.$$

**24.13.** Let

$$g(x) := \int_0^x f(u, x^2, \dots, x^n) du,$$

and try

$$\xi = g(x) dx^{2 \dots n}.$$

**24.15.** If  $\omega = f(x) dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$  then let

$$g(x) = \int_0^{x^1} f(x) dx^1$$

and let

$$\xi = g(x) dx^2 \wedge dx^3 \wedge \dots \wedge dx^n.$$

**24.16.** If exact, it is closed by problem 24.12 on page 206. If closed,

$$0 = d\omega = \frac{\partial f}{\partial x} dx \wedge dy \wedge dz,$$

then  $\frac{\partial f}{\partial x} = 0$  so  $f = f(y, z)$ . So  $\omega$  is a 2-form on the  $(y, z)$ -plane, and we can use our tricks from the earlier questions in this section. To be precise, let

$$g(y, z) := \int_0^y f(u, z) du,$$

and

$$\eta := g(y, z) dz.$$

Then

$$d\eta = f(y, z) dy \wedge dz = \omega.$$

**24.16.** If exact, it is closed by problem 24.12 on page 206. Say

$$\omega = f(x, y, z) dy \wedge dz + g(x, y, z) dz \wedge dx + h(x, y, z) dx \wedge dy$$

is closed:

$$0 = d\omega = (f_x + g_y + h_z) dx \wedge dy \wedge dz.$$

Let

$$\begin{aligned} G(x, y, z) &= \int_0^x g(u, y, z) du, \\ H(x, y, z) &= \int_0^x h(u, y, z) du, \\ \xi &= H(x, y, z) dy - G(x, y, z) dz. \end{aligned}$$

Then

$$\begin{aligned} d\xi &= h(x, y, z) dx \wedge dy + \left( \int_0^x h_z(u, y, z) du \right) dz \wedge dy \\ &\quad - g(x, y, z) dx \wedge dz - \left( \int_0^x g_y(u, y, z) du \right) dy \wedge dz, \\ &= h(x, y, z) dx \wedge dy + g(x, y, z) dz \wedge dx - \left( \int_0^x (g_y + h_z)(u, y, z) du \right) dy \wedge dz \\ &= h(x, y, z) dx \wedge dy + g(x, y, z) dz \wedge dx + \left( \int_0^x f_x(u, y, z) du \right) dy \wedge dz \\ &= h(x, y, z) dx \wedge dy + g(x, y, z) dz \wedge dx + f(x, y, z) dy \wedge dz. \end{aligned}$$

**25.2.** We can change  $\omega$  to be 0 away from  $F(X) \cup G(X)$ , throughout  $\mathbb{R}^n$ . As we saw in problem 24.15 on page 207  $\omega = d\xi$  for some  $(n-1)$ -form  $d\xi$ , so

$$\begin{aligned} \int_X F^* \omega &= \int_{\partial X} F^* \xi, \\ &= \int_{\partial X} G^* \xi, \\ &= \int_X G^* \omega \end{aligned}$$

by Stokes's theorem in a box.

**25.7.** Since  $A$  is closed, the complement of  $A$  is open. Since  $K$  is contained in the complement of  $A$ , every point  $x \in K$  lies in an open set  $U_x$  containing  $x$  lying in the complement of  $A$ . Use some chart defined in some open subset of  $U_x$  containing  $x$  to identify with an open subset of the standard corner. Take an open set in the image of the chart, with compact closure; its preimage in  $U_x$  also has compact closure. So we can replace  $U_x$  by that smaller open set if needed, to ensure that  $\bar{U}_x \cap A$  is empty. The open sets  $U_x$  cover  $K$ ; take a finite subcover  $U_1, \dots, U_k$ . Take a  $C^0$  partition of unity  $f_1, \dots, f_k$  subordinate to that open cover, and let  $f = f_1 + \dots + f_k$ .

**25.8.** Take a compact manifold  $M$ ; let  $n$  be its dimension. Consider all charts  $(U, \varphi)$  with image containing the origin in  $\mathbb{R}^n$ . We can cover  $M$  with such charts. Take a finite subcover  $U_1, \dots, U_N$ , and a subordinate partition of unity  $f_1, \dots, f_N$ . Define a map  $\psi_i: M \rightarrow \mathbb{R}^n$  for each open set:

$$\psi_i(x) := \begin{cases} 0, & x \text{ not in } U_i, \\ f_i(x)\varphi_i(x), & x \text{ in } U_i. \end{cases}$$

Let

$$\psi(x) := (\psi_1(x), \psi_2(x), \dots, \psi_N(x), f_1(x), \dots, f_N(x)).$$

If  $\psi(x) = \psi(y)$ , then  $f_i(x) = f_i(y)$  for all  $i$ . Since the  $f_i$  add to 1, we must have  $f_i(x) > 0$  for some  $f_i$ . Hence  $x \in U_i$ . But  $f_i(y) = f_i(x) > 0$ , so  $y \in U_i$  too. Then  $\psi_i(x) = \psi_i(y)$ ; divide off  $f_i(x)$  to get  $\varphi_i(x) = \varphi_i(y)$ . So  $x = y$ .

If  $\psi'(x) = 0$  then  $f'_i(x) = 0$  for all  $f_i$ , and so  $0 = \psi'_i(x) = f_i(x)\varphi'(x)$ . We can pick some  $f_i$  so that  $f_i(x) > 0$ , so  $x \in U_i$ , and so  $\varphi'(x) \neq 0$  since  $\varphi_i$  is a chart in  $U_i$ . So  $\psi$  is an immersion.

Our  $\psi$  is an injective immersion, mapping from a compact manifold, so proper, hence a diffeomorphism to its image.

**25.9.** If  $(u, v) = F(x, y) = (x + y^2, y)$ , the inverse map is  $(x, y) = F^{-1}(u, v) = (u - v^2, v)$ , and in particular  $F$  is a diffeomorphism. The inequalities  $u \geq v^2, v \geq 0, u + v \leq 1 + v^2$  become  $x \geq 0, y \geq 0, x + y \leq 1$ , a triangle. Most important: check that  $F^*du \wedge dv = d(x + y^2) \wedge dy = dx \wedge dy$ , so this diffeomorphism  $F$  preserves areas. Therefore the area is that of the triangle:  $\frac{1}{2}$ .

**25.11.** The area form is  $\omega = dx \wedge dy = r dr \wedge d\theta$ , so we integrate

$$\begin{aligned} A &= \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \left( \int_{r=0}^{r=\cos(2\theta)} r dr \right) d\theta, \\ &= \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \frac{r^2}{2} \Big|_{r=0}^{r=\cos(2\theta)} d\theta, \\ &= \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \frac{\cos(2\theta)^2}{2} d\theta, \\ &= \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \frac{1 + \cos(4\theta)}{4} d\theta, \\ &= \frac{\pi}{8}. \end{aligned}$$

**25.12.** For any function  $f(z)$  we find

$$F^*dX = f'(z)x\,dz + f(z)\,dx,$$

$$F^*dY = f'(z)y\,dz + f(z)\,dy,$$

$$F^*dZ = dz.$$

Wedge together to get

$$F^*(dX \wedge dY \wedge dZ) = f^2(z)dx \wedge dy \wedge dz.$$

Therefore

$$\begin{aligned} \text{Vol } B &= \int_B dX \wedge dY \wedge dZ, \\ &= \int_C f^2(z)dx \wedge dy \wedge dz, \\ &= \left( \int_{x^2+y^2 \leq 1} dx \wedge dy \right) \int_{z=-1}^{z=1} f^2(z)\,dz, \\ &= \pi \int_{z=-1}^{z=1} f^2(z)\,dz. \end{aligned}$$

In this particular problem,  $f(z) = \sqrt{1-z^2}$ , so we compute

$$\text{Vol } B = \frac{4\pi}{3}.$$

To make this calculation rigorous, we would need to worry a little about the fact that  $F$  is not a diffeomorphism on the top and bottom of the cylinder, so we would have to check that if we cut a little bit off the top and bottom of the cylinder, it won't alter our integrals much.

**25.13.**  $(x_1 - x_0)(y_1 - y_0 + y_1^3 - y_0^3)$

**25.14.** The map  $F: X \rightarrow M$  is a diffeomorphism from the box  $X = [0, 1] \times [0, 1]$ , as we can easily check by sticking in  $x = 0$  or  $x = 1$  or  $y = 0$  or  $y = 1$  to  $F$  to see where the sides go. Differentiate

$$u = x^2 - y^2,$$

$$v = 2xy$$

to get

$$du = 2x\,dx - 2y\,dy,$$

$$dv = 2y\,dx + 2x\,dy,$$

so that when we wedge together

$$du \wedge dv = 4(x^2 + y^2)\,dx \wedge dy.$$

The area of  $M$

$$\begin{aligned}
 \int_M du \wedge dv &= \int_X 4(x^2 + y^2) dx \wedge dy, \\
 &= \int_{x=0}^{x=1} \int_{y=0}^{y=1} 4(x^2 + y^2) dx dy, \\
 &= \int_{x=0}^{x=1} \int_{y=0}^{y=1} 4x^2 dy dx + \int_{x=0}^{x=1} 4y^2 dx dy, \\
 &= 4 \left. \frac{x^3}{3} \right|_{x=0}^{x=1} + 4 \left. \frac{y^3}{3} \right|_{y=0}^{y=1}, \\
 &= \frac{8}{3}.
 \end{aligned}$$

**25.16.**

$$\begin{aligned}
 dx &= \sin \alpha \cos \beta dr + r \cos \alpha \cos \beta d\alpha - r \sin \alpha \sin \beta d\beta, \\
 dy &= \sin \alpha \sin \beta dr + r \cos \alpha \sin \beta d\alpha + r \sin \alpha \cos \beta d\beta, \\
 dz &= \cos \alpha dr - r \sin \alpha d\alpha.
 \end{aligned}$$

Wedging these together, a long calculation yields

$$dx \wedge dy \wedge dz = r^2 \sin \alpha dr \wedge d\alpha \wedge d\beta.$$

Let

$$Y = \{ (r, \alpha, \beta) \mid 0 \leq r \leq 1, 0 \leq \beta \leq 2\pi, 0 \leq \alpha \leq \pi/2 \}.$$

The integral is

$$\begin{aligned}
 \int_X (1 - \sqrt{x^2 + y^2 + z^2}) dx \wedge dy \wedge dz &= \int_Y (1 - r) r^2 dr \sin \alpha d\alpha d\beta, \\
 &= \int_0^1 (1 - r) r^2 dr \int_0^{\pi/2} \sin \alpha d\alpha \int_0^{2\pi} d\beta
 \end{aligned}$$

and we let the reader fill in the remaining details.

**25.17.** Find the area of a disk slice and multiply by  $\ell$ . The water in such a slice, with origin at the center of the disk, lies below  $y = h - r$  and above  $y = -\sqrt{r^2 - x^2}$ . These two curves meet at  $x = \pm\sqrt{h(2r - h)}$ . So if the tank is less than half full, then the area of the slice is

$$A = \int_{x=-\sqrt{h(2r-h)}}^{x=\sqrt{h(2r-h)}} (h - r + \sqrt{r^2 - x^2}) dx$$

It helps to change variable to  $u = x/r$ , to let  $c$  be the constant

$$c = \sqrt{\frac{h}{r} \left( 2 - \frac{h}{r} \right)},$$

and to use the integral

$$2 \int \sqrt{1 - u^2} du = u \sqrt{1 - u^2} + \frac{\pi}{2} - \cos^{-1} u.$$

This gives

$$A = r^2 \left( \frac{\pi}{2} - \left( 1 - \frac{h}{r} \right) \sqrt{\frac{h}{r} \left( 2 - \frac{h}{r} \right)} - \cos^{-1} \sqrt{\frac{h}{r} \left( 2 - \frac{h}{r} \right)} \right).$$

Check at  $h = 0$  and  $h = r$  to be sure this is right. The volume is  $V = \ell A$ . If the tank is more than half full, then take the total volume of the whole tank  $\pi r^2 \ell$  and subtract the volume of the empty part.

**25.18.**

$$dx = \cos \theta \, dr - r \sin \theta \, d\theta,$$

$$dy = \sin \theta \, dr + r \cos \theta \, d\theta,$$

$$dz = dz.$$

i.e.  $F^* dx = \cos \theta \, dr - r \sin \theta \, d\theta$ , etc. Wedge these to find

$$dx \wedge dy \wedge dz = r^2 \, dr \wedge d\theta \wedge dz.$$

and

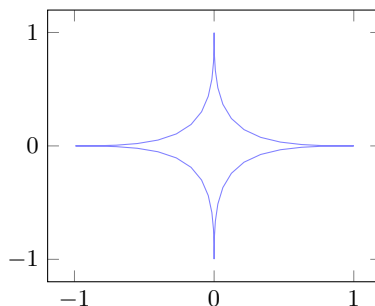
$$F^{-1}X = \{ (r, \theta, z) \mid 2 \leq r \leq 3, 0 \leq z \leq 1, \pi/4 \leq \theta \leq \pi/2 \}.$$

(You really have to draw it to find the correct  $\theta$  interval.)

$$\begin{aligned} \int_X z(x^2 + y^2) \, dx \wedge dy \wedge dz &= \int_{r=2}^{r=3} \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{z=0}^{z=1} z r^4 \, dr \wedge d\theta \wedge dz, \\ &= \int_2^3 r^4 \, dr \frac{\pi}{4} \int z \, dz \end{aligned}$$

and we leave the rest to the reader.

**26.2.** It helps to draw the curve:



Parameterise the boundary by

$$x(t) = \cos^5 t, \quad y(t) = \sin^5 t$$

and compute  $\int_{\partial} \omega$  where  $\omega = \frac{1}{2}x \, dy - \frac{1}{2}y \, dx$ .

**26.4.** 0

**26.5.** 0

**26.6.** Apply Stokes's theorem to  $\omega = -y dx + x dy$  to get

$$\int_{\partial U} \omega = \int_U d\omega = 2 \text{ Area } U.$$

Writing derivatives with respect to  $s$  as dots:

$$\begin{aligned} \int_{\partial U} \omega &= \int_{s=0}^{s=\ell} (-y\dot{x} + x\dot{y})ds, \\ &= \int_{s=0}^{s=\ell} (-y, x) \cdot (\dot{x}, \dot{y})ds, \end{aligned}$$

by the Cauchy–Schwarz inequality:

$$\begin{aligned} &\leq \int_{s=0}^{s=\ell} \sqrt{y^2 + x^2} \sqrt{\dot{x}^2 + \dot{y}^2} ds, \\ &= \int_{s=0}^{s=\ell} \sqrt{y^2 + x^2} ds, \\ &= \int_{s=0}^{s=\ell} r ds. \end{aligned}$$

Equality occurs just when  $(-y, x)$ ,  $(\dot{x}, \dot{y})$  are nonnegative multiples of one another, i.e. we are travelling perpendicularly to our current position vector, hence conserving  $x^2 + y^2$ :

$$\frac{d}{ds}(x^2 + y^2) = 2x\dot{x} + 2y\dot{y} = 0.$$

Hence we stay on a circle.

**26.10.** Note that  $\eta = df$  where  $f = x^2y$ . Therefore

$$\int_C \eta = \int_{\partial C} f = f(1, 0) - f(0, 0) = 0.$$

**26.10.** Note that  $\eta = df$  where  $f = \cos(\pi x^2 + y)$ . Therefore

$$\int_C \eta = \int_{\partial C} f = f(1, 0) - f(0, 0) = \cos \pi - \cos 0 = -1.$$

**26.13.** Draw a disk  $D_i$  around each zero  $z = z_i$  of  $p(z)$  inside  $U$ , say of radius  $\varepsilon$  small enough so that  $\bar{D}_i \subseteq U$ . Let

$$\omega = \frac{1}{2\pi i} \frac{dp}{p}.$$

Check that  $d\omega = 0$ . Let  $U_\varepsilon = U - \bigcup_i \bar{D}_i$ . Then  $\omega$  is  $C^1$  inside  $U_\varepsilon$ , but not inside any  $D_i$ . By Stokes's theorem,

$$0 = \int_{\partial U_\varepsilon} \omega = \int_{\partial U} \omega - \sum_i \int_{\partial D_i} \omega.$$

So it suffices to assume that  $U$  is just  $U = \bigcup_i D_i$ . It suffices to check the result for each  $D_i$ , so assume there is only one, say  $D$ . Translate to get  $D$  to have centre at the

origin. Write  $p(z)$  as  $p(z) = z^k q(z)$  with  $q(0) \neq 0$ . Take exterior derivative of both sides to check that

$$\omega = \frac{1}{2\pi i} \frac{dp}{p} = \frac{k}{2\pi i} \frac{dz}{z} + \frac{1}{2\pi i} \frac{dq}{q}.$$

Since  $dq/q$  is a closed form defined throughout  $\bar{D}$ ,

$$\begin{aligned} \int_{\partial D} \omega &= \frac{k}{2\pi i} \int_{\partial D} \frac{dz}{z} + \frac{1}{2\pi i} \int_{\partial D} \frac{dq}{q}, \\ &= \frac{k}{2\pi i} \int_{\partial D} \frac{dz}{z} + \frac{1}{2\pi i} \int_D d\frac{dq}{q}, \\ &= \frac{k}{2\pi i} \int_{\partial D} \frac{dz}{z} + \frac{1}{2\pi i} \int_D 0, \\ &= \frac{k}{2\pi i} \int_{\partial D} \frac{dz}{z}. \end{aligned}$$

Plug in  $z = \varepsilon e^{i\theta}$  for  $0 \leq \theta \leq 2\pi$  and compute:

$$\int_{\partial D} \omega = k.$$

**27.5.** Every closed  $C^1$  1-form on the plane is exact, but on the punctured plane the angle form  $d\theta$  is not exact.

**27.6.** Around the sphere  $S_r$  of radius  $r$  centered at the origin,

$$\begin{aligned} \int_{S_r} \Omega &= \int_{\alpha=0}^{\alpha=\pi} r \sin \alpha \int_{\beta=0}^{\beta=2\pi} d\beta, \\ &= r(-\cos \pi + \cos 0)(2\pi), \\ &= 4\pi r, \\ &= \text{area of } S_r. \end{aligned}$$

**27.7.** Every closed  $C^1$  2-form on  $\mathbb{R}^3$  is exact by problem 24.17 on page 207. On the punctured  $\mathbb{R}^3 - \{0\}$ , the solid angle form is not exact: if it were, its integral over the unit sphere  $S = (x^2 + y^2 + z^2 = 1)$  would be zero by Stokes's theorem, as the sphere has no boundary: if  $\Omega = d\xi$  then

$$4\pi = \int_S \Omega = \int_S d\xi = \int_{\partial S} \xi = 0.$$

**27.12.** Every closed  $C^1$   $(n-1)$ -form on  $\mathbb{R}^n$  is exact by the same method as problem 24.17 on page 207. On the punctured  $\mathbb{R}^n - \{0\}$ , the solid angle form is not exact: if it were, its integral over the unit sphere  $S$  would be zero by Stokes's theorem, as the sphere has no boundary: if  $\Omega = d\xi$  then

$$0 < \int_S \Omega = \int_S d\xi = \int_{\partial S} \xi = 0.$$

**27.13.** Define

$$K: \bar{B} \times \mathbb{R} \rightarrow S, K(x, t) = (1-t)x + tF(x).$$



Fix some  $x \in \bar{B}$ . Consider the function  $q(t) = |K(x, t)| - 1$ . Expand out  $q(t)$  to see exactly where  $t$  has to be so that  $q(t) = 0$ , i.e. so that  $K(x, t) \in S$ . Show that  $q(t)$  has one root  $\geq 0$  and one negative root, and that if we take the root  $t = t(x) \geq 0$ , then  $t(x)$  is continuously differentiable for  $x \in B$ .

**27.16.** If  $X$  has no zero, then  $F$  is defined on  $\bar{B}$  and  $C^2$ . On the sphere  $S = \partial\bar{B}$ ,  $F(x) = x$  for all  $x \in S$ , so  $F^*\Omega = \Omega$  and

$$\begin{aligned} 0 &< \int_S \Omega, \\ &= \int_S F^* \Omega, \\ &= \int_{\bar{B}} dF^* \Omega, \\ &= \int_{\bar{B}} F^* d\Omega, \\ &= 0, \end{aligned}$$

a contradiction, so  $X$  has a zero.

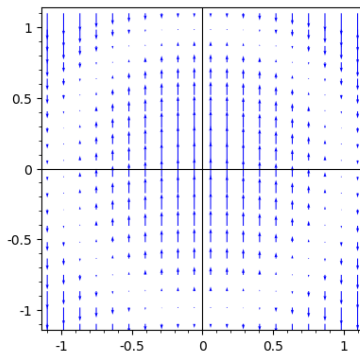
**27.17.** In the simplest case, the one-dimensional closed “ball”  $[-1, 1]$ , the vector field  $X(x) = 1 - x^2$  has zeroes at  $x = \pm 1$ . Any point  $x$  with  $-1 < x < 1$  moves forward under the flow. The flow is the map

$$F_t(x) = \frac{(1+x)e^{2t} - (1-x)}{(1+x)e^{2t} + (1-x)}.$$

Since the points  $x = \pm 1$  are fixed, the interval  $[-1, 1]$  is taken to itself, so  $F_t$  is a diffeomorphism  $[-1, 1] \rightarrow [-1, 1]$ , with inverse  $F_{-t}$ . More generally, in  $\mathbb{R}^n$ , the vector field

$$X(x) = (0, 0, \dots, 0, 1 - |x|^2)$$

has no zeroes inside the ball, and zeroes everywhere on the boundary.



By the same argument, the flow  $F_t$  of this vector field is a diffeomorphism of the closed ball, for any constant  $t$ , and pushes each interior point upward, so has no interior fixed points. Its inverse is  $F_{-t}$ . In fact, we can calculate the flow:

$$F_t(x_1, \dots, x_{n-1}, x_n) = \left( x_1, \dots, x_{n-1}, \frac{(a+x_n)e^{2t} - (a-x_n)}{(a+x_n)e^{2t} + (a-x_n)} \right)$$

where

$$a = a_t(x_1, \dots, x_{n-1}) = \sqrt{1 - x_1^2 - \dots - x_{n-1}^2}.$$

**28.5.** Recall the construction of the surface. Pick an integer  $g \geq 0$  and  $g$  nonoverlapping closed disks  $D_i$  in the plane, say with centre  $(x_i, y_i)$  and radius  $r_i > 0$ ,  $i = 1, 2, \dots, g$ . Take a closed disk  $D_0$  containing all of these  $D_i$  in its interior. Let

$$f_i(x, y) := (x - x_i)^2 + (y - y_i)^2 - r_i^2,$$

and let  $f := f_0 f_1 f_2 \dots f_g$  and

$$S := \{(x, y, z) \mid f(x, y) + z^2 = 0\}$$

is the surface of genus  $g$ . The unit normal vector is

$$\nu = \frac{(f_x, f_y, 2z)}{\sqrt{f_x^2 + f_y^2 + 4z^2}}.$$

Consider the points  $(x, y, z)$  at which  $\nu = (1, 0, 0)$ . At these points,  $z = 0$ , so  $f = 0$ , so  $f_i = 0$  for some  $i$ . Hence

$$f_x = f_{i,x} \prod_{j \neq i} f_j, \quad 0 = f_y = f_{i,y} \prod_{j \neq i} f_j.$$

So  $y = y_i$  and  $x > x_i$  so  $x = x_i + r_i$ . This can occur for  $i = 0, 1, 2, \dots, g$ . Hence there are  $g + 1$  such points.

If we write  $\nu = w/|w|$ , then calculate that as we move along a curve,  $\dot{\nu}$  is just the projection of  $\dot{w}$  to the plane perpendicular to  $w$ . For us,  $w = (f_x, f_y, 2z)$ . So at our points,  $\dot{\nu}$  is the projection of  $\dot{w}$  to  $e_1^\perp$ . Along a curve with velocity  $(0, 1, 0)$ ,

$$\dot{w} = (f_{xy}, f_{yy}, 0),$$

but  $f_{xy} = 0$ , while along a curve with velocity  $(0, 0, 1)$ ,

$$\dot{w} = (0, 0, 2).$$

So

$$\nu'(x, y, z)e_2 = f_{yy}e_2, \nu'(x, y, z)e_3 = 2e_3.$$

The Gauss map is taking the plane  $e_2, e_3$  tangent to our surface to the plane  $e_2, e_3$  tangent to the sphere, both oriented by making  $e_2, e_3$  a positively oriented basis. Hence the sign of the determinant of the Gauss map is the sign of  $f_{yy}$ . Compute

$$f_{yy} = \sum f_{i,yy} \prod_{j \neq i} f_j$$

since  $f_{i,y} = 0$  at our point. The product is positive if all  $f_j > 0$  for  $j \neq i$ , i.e. if  $i = 0$  so outside all disks, and negative if inside  $D_0$  and outside all  $D_i$ . So there are  $g$  points where the sign is negative, and 1 point where it is positive. Hence the degree of the Gauss map is  $1 - g$ .

**29.2.** Take a chain represented by a body  $\varphi: X \rightarrow M$ .



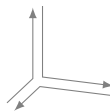
near a corner point, the boundary of  $X$  looks like the faces coming together near some corner



By our definition of the boundary, there are two copies of each of the corner points, which we indicate by drawing them a little bit apart.



But then the boundary of this consists of the various double intersections of walls, which appear as edges in our 3-dimensional picture, but each duplicated and oriented in opposite directions:



We draw them a little bit apart, but of course they coincide. The same idea works in any number of dimensions, and in any number of dimensions we only need to see the result for the standard corner, since  $X$  is locally diffeomorphic to the standard corner.

A more formal proof: Note that  $\partial X$  consists of pairs  $(x, \ell)$  where  $x \in X$  and  $\ell$  is a cowall at  $x$ . So  $\partial^2 X$  consists of triples  $(x, \ell_1, \ell_2)$  where  $x \in X$  and  $\ell_1 \neq \ell_2$  are distinct cowalls at  $x$ . The map  $(x, \ell_1, \ell_2) \mapsto (x, \ell_2, \ell_1)$  is a diffeomorphism  $\partial^2 X \rightarrow \partial^2 X$  reversing orientation, so  $\partial^2 X$  arises as a  $2 - 1$  covering of its quotient  $Y$  by this diffeomorphism, which we orient according to one of the one or two sheets of the quotient, so a degree zero sheet:  $\partial^2 X = 0Y$ .

**29.3.** A 0-body in a manifold with corners  $M$  is a finite set of points, which are “oriented”, so we keep a  $\pm$  sign attached to each, together with a map from that finite set to  $M$ . The equivalence says that if several points map to the same point of  $M$ , we can treat them as a single point, with an integer attached, given by summing the number of  $+$  points minus number of  $-$  points. The boundary  $\partial\varphi$  of a 0-chain is trivial.

**30.7.** Take the map  $\varphi(t, x) := tx$ . Apply the Poincaré lemma to this homotopy, and note that it preserves the boundary. The pushdown of a form which vanishes on some face clearly also vanishes on that face.

**30.8.** Take a point  $x_0 \in X$  outside the support of  $\omega$ . Apply the Poincaré lemma to the map  $\varphi(t, x) := (1 - t)x_0 + tx$  to construct  $\eta = (-1)^{n+1+k} \pi_* \varphi^* \omega$ . Suppose that the support of  $\omega$  lies inside the compact set of points  $x \in X$  at which  $x_1 + x_2 + \cdots + x_n \leq C$  for some constant  $C > 0$ . For any  $x$  outside that set, the line from  $x$  to  $x_0$  lies outside that set. The integral that constructs  $\eta$ , written out in coordinates, is an integral of the coefficients of  $\omega$  evaluated along that line, so vanishes at  $x$ .

**30.9.** Change variables by  $z = x - y$  so  $y = x - z$ :

$$\xi * \eta(x) = \int_z \xi(z) \wedge \eta(x - z).$$

If  $\xi$  is a  $p$ -form, write  $p$  as  $|\xi|$ , and then

$$\xi * \eta(x) = (-1)^{|\xi|+|\eta|} \int_z \eta(x - z) \wedge \xi(z) = (-1)^{|\xi|+|\eta|} \eta * \xi(x).$$

**34.5.** The function  $f((1-t)p + tq)$  has second derivative

$$\frac{d^2}{dt^2} f((1-t)p + tq) = \langle f''((1-t)p + tq)(p - q), p - q \rangle.$$

A negative eigenvalue will force a negative second derivative for some  $p, q$ , i.e. force the graph of  $f((1-t)p + tq)$  to curve above some parabola which is convex downward. Conversely, if all eigenvalues are positive, the graph of  $f((1-t)p + tq)$  curves below some parabola which is convex upward.

**34.6.** If  $p, q \in C_{y_0}$ , so  $f(p), f(q) \leq y_0$ , then by convexity,  $f((1-t)p + tq) \leq y_0$  for all  $0 \leq t \leq 1$ .

**31.5.** By problem 34.6 on page 268,  $C_\varepsilon$  is convex. Translate to arrange that  $x_0 = 0$ ;  $f(x)$  is bounded from below by  $\lambda|x|^2$ , for small enough  $x$ , where  $\lambda$  is the small eigenvalue of  $f''(x)$ . We see that  $f$  attains some value  $\varepsilon > 0$  in every direction going away from the origin, at some distance no more than  $\sqrt{\varepsilon/\lambda}$ . By Taylor expansion,

$$f'(x) = f'(0) + tf''(0)x + o(x) = tf''(0)x + o(x),$$

so  $f'(x)x \neq 0$  along the nearby points of the  $\varepsilon$ -level set, so  $f'(x) \neq 0$ , so the  $\varepsilon$ -level set is  $C^k$  by the implicit function theorem.

**34.8.**

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i \partial x_j} &= \frac{\partial^2 g \circ \varphi}{\partial x_i \partial x_j}, \\ &= \frac{\partial}{\partial x_i} \frac{\partial g \circ \varphi}{\partial x_j}, \\ &= \sum_k \frac{\partial}{\partial x_i} \frac{\partial g}{\partial y_k} \frac{\partial \varphi}{\partial x_j}, \\ &= \sum_k \frac{\partial^2 g}{\partial y_k y_\ell} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_k \frac{\partial g}{\partial y_k} \frac{\partial^2 \varphi}{\partial x_j \partial x_i}, \end{aligned}$$

but the second term vanishes at any critical point of  $g(y)$ , so

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_k \frac{\partial^2 g}{\partial y_k y_\ell} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_i}.$$

In other words, as a symmetric bilinear form

$$f''(x_0)(u, v) = g''(y_0)(\varphi'(x_0)u, \varphi'(x_0)v)$$

for any vectors  $u, v$ . So if  $g''(x_0)(v, v) > 0$  for all  $v \neq 0$ , i.e. a positive definite symmetric matrix, then

$$f''(x_0)(w, w) = g''(x)(v, v) > 0,$$

where  $v = \varphi'(y_0)w$ , for all  $w \neq 0$ .

**31.7.** If  $m_0 \in \bar{V}_i$  then  $V_i$  meets any open set around  $m_0$ . Should that open set be precompact, then only finitely many  $V_i$  meet it. So only finitely many  $\bar{V}_i$  contain  $m_0$ ; pick one such  $\bar{V}_i$ . The domain of  $\varphi_i$  contains  $\bar{V}_i$  so includes  $m_0$ . Again by problem 34.8 on page 268, we can pick  $r$  to press  $f$  on the domain of  $\varphi_i$ , and hence on  $V_i$ .

**31.7.** Since  $(f \leq r)$  is compact, it intersects only finitely many  $V_i$ . So  $(f < r)$  intersects only finitely many  $\bar{V}_i$ . Shrink  $r$  to ensure that only finitely many  $\bar{V}_i$  intersect  $(f \leq r)$ . Shrink  $r$  to arrange that  $(f \leq r)$  misses any  $\bar{V}_i$  not meeting  $m_0$ . So  $r$  presses  $f$  on all  $V_i$  with  $\bar{V}_i$  intersecting  $(f \leq r)$ . But  $r$  presses  $f$  on any other  $V_i$  tautologically.

**31.9.** Take an open interval  $U \subset C$ , say with some chart  $(U, \psi)$ , and a connected component  $J \subset \varphi^{-1}U$ . Then  $\psi \circ \varphi$  on the interval  $J$  is a local  $C^k$  diffeomorphism from an interval to an interval, so increasing or decreasing, so a  $C^k$  diffeomorphism to its image.

**31.10.** Since  $\mathbb{R}$  is simply connected, if  $\varphi$  is a covering map, it is the universal covering map. Take a point  $x_0$  in some  $U_i$ . Take a point  $t_0 \in \mathbb{R}$  with  $\varphi(t_0) = x_0$ . We have added  $\bar{U}_i$  to the image of  $\varphi$ . Moreover  $\varphi$  maps some open interval around  $t_0$  to  $U_i$  diffeomorphically. So the preimage of  $U_i$  is a collection of open intervals mapped diffeomorphically to  $U_i$ . Hence  $\varphi$  is a  $C^k$  covering map.

**31.11.** Every space with universal covering space  $\mathbb{R}$  is the circle or to  $\mathbb{R}$ ; see [13], solution to problem 4.17. One could instead argue as follows. Take any points  $t_0 < t_1$  with  $\varphi(t_0) = \varphi(t_1)$ . Take points  $s_0 < s_1$  in the interval  $[t_0, t_1]$  with  $\varphi(s_0) = \varphi(s_1)$ . The difference  $s_1 - s_0$  cannot go to zero, since  $\varphi$  is a local diffeomorphism. Replace  $t_0, t_1$  with  $s_0, s_1$  with minimal difference. The map  $\varphi: [t_0, t_1] \rightarrow C$  descends to the circle by gluing the ends together, injectively and continuously, hence a homeomorphism.)

**32.1.** Each point  $m_0$  lies in an open set  $U$  intersecting only finitely many  $M_a$ . On a perhaps smaller open set, each of the corresponding  $f_a$  is bounded below on  $U \cap M_a$  by a positive continuous function  $g_a$  on  $U$ . Since there are finitely many of these  $g_a$ , we can bound all of them below by a positive constant on any precompact open subset of  $U$  containing  $U$ ; replace  $U$  by that subset.

Such open sets form an open cover. Take a locally finite refinement  $U_1, U_2, \dots$ , so that, for each  $U_i$ , there is a constant  $c_i > 0$  for which, on each  $U_i \cap M_a$ ,  $f_a \geq c_i > 0$ . Take a partition of unity  $h_i$  subordinate to this open cover, and let  $f = \sum h_i c_i$ .

**32.2.** Suppose that  $f: M \rightarrow \mathbb{R}$  is some smooth function. Take a flow line  $x(t)$  of  $X$ , defined on some interval  $a < t < b$ , with  $x(0) = x_0$ . A reparameterization  $x(\tau(t))$  is a flow line of  $fX$  precisely when  $\tau(t)$  satisfies the ordinary differential equation

$$\frac{d\tau}{dt} = f(x(\tau)),$$

i.e. a flow line of the vector field

$$f(x(\tau))\partial_\tau$$

on the interval  $a < \tau < b$ . If  $\varepsilon > 0$  is a constant with  $f(x(t)) \leq \varepsilon$  then this vector field on  $a < \tau < b$  has flow defined on an interval of time at least  $a/\varepsilon < t < b/\varepsilon$ . So if we ensure that  $f < b$  and  $f < |a|$  on any flow line, then  $fX$  will have flow line through  $x_0$  defined for at least time  $-1 < t < 1$ .

The domain of the flow of  $X$  is open in  $M \times \mathbb{R}$ ; cover by product open sets  $M_\alpha \times (a_\alpha, b_\alpha)$ . So on  $a_\alpha < t < b_\alpha$ , the flow is defined on any point in  $M_\alpha$ . We need to ensure that on every flow line through  $M_\alpha$ ,  $f$  is always at most  $b_\alpha, |a_\alpha|$ . Let  $C_\alpha$  be some positive number less than  $1, b_\alpha, |a_\alpha|$ . We need to ensure that on every flow line through  $M_\alpha$ ,  $0 < f \leq C_\alpha$ . Apply problem 32.1 on page 257.

For each time  $t$ , if  $x(t) \in M_\alpha$  for some  $\alpha$ , then  $f \leq C_\beta \leq C_\alpha$ . So  $fX$  has flow defined and staying inside  $M_\alpha$  on an interval of time at least  $a_\alpha/C < t < b_\alpha/C$ , so on  $-1 < t < 1$ . Hence  $fX$  has flow for time at least time 1. But once we flow for time  $1/2$ , we can still flow again for time at least 1. Repeat to see that the flow is defined for all time.

**32.3.** The definition of a smooth map on a set: around each point  $a \in A$ , there is some open set  $M_a \subset M$  and a smooth map  $F_a: U_a \rightarrow \mathbb{R}^n$  so that  $F_a|_A = f$ . Let

$$U = \bigcup_{a \in A} U_a.$$

Pick a partition of unity  $\sum h_i = 1$  subordinate to the open cover  $\{U_a\}_{a \in A}$ , so each  $h_i$  is supported in some open set  $U_{a_i}$ . Extend  $F_{a_i}$  to be zero outside  $U_{a_i}$ . Let  $F = \sum h_i F_{a_i}$ . Suppose that  $A$  is closed. Let  $U_0 = M - A$ . Then  $\{U_0\} \cup \{U_a\}_{a \in A}$  is an open cover, and we can pick any smooth function  $F_0: U_0 \rightarrow \mathbb{R}^n$  and repeat the argument.

**32.4.** Let  $U = M - A$ . Apply lemma 3.1 on page 23 to construct a continuous function  $f: M \rightarrow \mathbb{R}$  so that  $f(p_0) = 1$  and so that  $f = 0$  outside a compact subset of  $U$ . Let  $W$  be the set of points  $p \in M$  at which  $f > \frac{1}{2}$ .

**32.5.** Each point  $a_0 \in A$  is not in  $B$ , and  $B$  is closed, so  $a_0$  is not the limit of points of  $B$ . So there is an open set around  $a_0$  not containing any point of  $B$ . By problem 32.4 on page 258, we can replace that open set by a smaller open set, to arrange that the closure of that open set does not contain any point of  $B$ . These open sets together with  $M - (A \cup B)$  form an open cover of  $M$ . Take a partition of unity  $\{f_i\}$  subordinate to that open cover. Consider the collection of the various  $f_i$  supported in open sets  $U_i$  from our cover for which  $\bar{U}_i \cap B$  is empty. Take the sum  $f := \sum f_i$  over that collection. At each point  $a_0 \in A$ , that sum contains all  $f_i$  with  $f_i(a_0) \neq 0$ , so is  $f(a_0) = 1$ , and  $f = 1$  near  $a_0$ . At each point  $b_0 \in B$ , that sum contains only terms vanishing near  $b_0$ , so  $f = 0$ .

**32.6.** Take partition of unity  $h_1, h_2, \dots$  subordinate to a locally finite open cover by precompact open sets  $U_1, U_2, \dots$ . Let  $f = \sum j h_j$ . At each point  $m_0 \in M$ , there are only finitely many  $j$  with  $h_j \neq 0$ , say  $j_1, j_2, \dots, j_k$ . Sort them in increasing order. Then

$$f(m_0) = \sum j h_j \geq \sum j_1 h_j = j_1 \sum h_j = j_1.$$

Any compact set  $K \subset \mathbb{R}$  lies in some interval  $[0, j_0]$ . It is enough to prove that the set of points of  $M$  where  $f \leq j_0$  is compact. If  $f(m_0) \leq j_0$  then as above we have  $j_0 \geq j_1$ , i.e.  $m_0 \in U_1 \cup \dots \cup U_{j_0}$ . So

$$f^{-1}K \subset f^{-1}[0, j_0] \subset \bar{U}_1 \cup \dots \cup \bar{U}_{j_0}.$$

**32.7.** We start with the standard corner in the plane. Consider the angle function  $-\pi < \theta < \pi$  on the plane (defined everywhere on the plane except at points  $(0, y)$  with  $y \leq 0$ ). Take some smooth increasing function  $h: [0, 1] \rightarrow [0, 1]$  so that  $h(0) = 0$  and  $h(1) = 1$  and so that all derivatives  $h', h'', \dots$  vanish both at 0 and at 1. We

can ask that  $h'(x) > 0$  except at  $x = 0, 1$ . Extend  $h$  to  $h: \mathbb{R} \rightarrow [0, 1]$  by  $h(x) = 0$  if  $x \leq 0$  and  $h(x) = 1$  if  $x \geq 1$ . Then the function  $f(x, y) = h(2\theta(x, y)/\pi)$  is smooth on standard corner, except at the origin, taking the value 0 on the wall  $y = 0$ , and 1 on  $x = 0$ .

Take constants  $a, b$ . Taking  $af + b(1 - f)$ , we can make  $f$  go to  $a$  on  $x = 0$  and  $b$  on  $y = 0$ .

In three variables, the function

$$f(x, y)f(x, z)$$

vanishes when  $y = 0$  or  $z = 0$ , and gives 1 when  $x = 0$ . Generalizing in the obvious way to any number of dimensions, we can make a function  $f$  which has required limits as we approach the interiors of the walls, with all derivatives vanishing.

Take a manifold with corners  $M$ . Cover by open sets  $M_a$ , each equipped with a chart to the standard corner. Equip each with a smooth function  $f_a$  taking required limits on the various walls. Take a partition of unity  $h_a$  subordinate to the open cover of the  $M_a$ . Then  $f = \sum h_a f_a$  does the trick.

**32.8.** Call the compact set  $K$ . On  $[0, 1] \times P$ , define the vector field

$$X(p, t) := \left(1, \frac{\partial \varphi}{\partial t}(t, p)\right).$$

Each point  $(t_0, p_0)$  flows in time  $t - t_0$  to  $(t, \varphi_t(p))$ . We can extend this vector field smoothly to  $\mathbb{R} \times P$  since  $[0, 1] \times P$  is embedded into  $\mathbb{R} \times P$ ; imitate problem 32.3 on page 258. The map  $(t, p) \mapsto (t, \varphi_t(p))$  embeds a neighborhood of  $[0, 1] \times K \subset \mathbb{R} \times P$  into  $\mathbb{R} \times Q$ , by problem 4.37 on page 49.

Multiply  $X$  by a bump function equal to one on  $[0, 1] \times K$ , vanishing outside that neighborhood. Then extend from that embedded neighborhood to all of  $\mathbb{R} \times Q$ , a compactly supported vector field, hence complete. Denote its flow on  $\{0\} \times Q$  by

$$e^{tX}(0, q) = (t, \Phi_t(q)).$$

Since the flow of  $X$  is a diffeomorphism,  $\Phi$  is a diffeotopy.

**32.9.** Apply theorem 32.5 on page 259 to construct a nearby injective immersion. Being uniformly nearby, the map is also proper. An injective proper immersion is an embedding by lemma 5.1 on page 56.

**32.10.** Take a smooth proper function  $f: M \rightarrow \mathbb{R}$  (see problem 32.6 on page 258), let  $\varphi(x) = (0, f(x))$ , and apply theorem 32.6 on page 259.

**33.2.** Note that  $\varphi$  is the identity inside the closed unit ball. It is enough to prove bijectivity on the exterior of the closed unit ball.

To prove that  $\varphi$  is one-to-one: note that  $\varphi$  moves each point  $x = ru$  always in the same direction  $u$ , i.e. along the line through 0 and  $u$ , and always outward since  $f > 0$ .

To prove that  $\varphi$  is onto: on the line segment from a point  $u$  of the closed unit ball to some point  $su \in U$ , with  $s > 1$ , there is a positive minimum distance to the boundary, so a positive minimum to  $f$ , so flow for some time takes  $u$  to  $su$ .

**33.3.** For any fixed  $r$ , the time  $\log r$  flow map  $\psi(x) = e^{\log r X} x$  is a diffeomorphism where defined, and has

$$\psi'(x)X = X(\psi(x)).$$

So

$$\begin{aligned}\varphi'(ru)u &= \left. \frac{d}{dt} \right|_{t=0} \varphi((r+t)u), \\ &= \frac{1}{r} X(\varphi(ru)), \\ &= \psi'(ru) \frac{1}{r} X.\end{aligned}$$

For  $v \perp u$ ,

$$\begin{aligned}\varphi'(ru)rv &= \left. \frac{d}{dt} \right|_{t=0} \varphi(r(\cos tu + \sin tv)), \\ &= \left. \frac{d}{dt} \right|_{t=0} e^{\log r X}(\cos tu + \sin tv), \\ &= \left. \frac{d}{dt} \right|_{t=0} \psi(r(\cos tu + \sin tv)), \\ &= \psi'(ru)rv.\end{aligned}$$

So the image of  $\varphi'(ru)$  is that of  $\psi'(ru)$ , i.e. is the entire tangent space of  $\mathbb{R}^n$ , so  $\varphi$  is a local diffeomorphism.

**33.4.** The unit open disk minus the set of points  $(x, y) = (x, 0)$  for  $\frac{1}{2} \leq x < 1$ .

**33.5.** Translate and rescale so that the unit sphere  $S^{n-1} \subset \mathbb{R}^n$  around the origin sits inside  $C$ . Let

$$\Phi: x \in \mathbb{R}^n - \{0\} \mapsto \frac{x}{|x|} \in S^{n-1}.$$

and let  $\varphi := \Phi|_A: A \rightarrow S^{n-1}$ . Suppose that  $\varphi(x) = \varphi(y)$ . Then  $x$  and  $y$  lie on the same ray through the origin, and both lie on  $A = \partial C$ . So every open set around  $x$  intersects  $C$ . So there are points  $x_i \in C$  approaching  $x$  on the line segment from 0 to  $x$ . So then the entire line segment from 0 to  $x_n$  lies in  $C$ . This holds for all  $x_n$ , so  $[0, 1)x$  lies in  $C$ , but  $[0, 1]x$  does not. Similarly for  $y$ . Since  $\varphi(x) = \varphi(y)$ ,  $x = ty$  and  $y = x/t$  for some  $t > 0$ . We can assume that  $t \neq 1$  (if not swap  $x$  with  $y$ ). So  $t = 1$  just when  $x = y$ . Suppose that  $0 < t < 1$ . Then  $[0, t]y = [0, 1]x \subset [0, 1)y \subset C$ , a contradiction.

Pick some  $x \in S^{n-1}$ ; we want to prove that  $x$  lies in the image of  $\varphi$ . For any  $0 < t$  sufficiently small  $tx \in C$ . But  $C$  is bounded, so for sufficiently large  $t$ ,  $tx$  is not in  $C$ . Since  $C$  is open,  $\mathbb{R}^n - C$  is closed, so there is some smallest  $t \geq 0$  for which  $tx$  is not in  $C$ . Clearly  $t > 0$ . Then  $tx \in A$  and so  $\varphi(tx) = x \in S^{n-1}$ . So  $\varphi$  is continuous, injective, and maps to a compact Hausdorff space (the sphere), so is a homeomorphism to its image [13] p.20, Corollary 2.6.

**34.1.** Compute no integrals. Translate to get  $v_0$  at the origin, and then apply the unique linear transformation that sends  $v_1, v_2, v_3$  to the standard basis vectors  $e_1, e_2, e_3$ ; there is a unique affine transformation that does this. By the same argument, there is also a unique affine transformation that does this to the *equilateral tetrahedron* with vertices

$$-(1, 1, 1), e_1, e_2, e_3.$$

Applying the inverse affine transformation to this one, we can assume that our tetrahedron is the equilateral tetrahedron. The centre of mass is invariant under the linear



transformations which permute  $e_1, e_2, e_3$  in any order, and also under the rotation

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

sending  $e_2 \leftrightarrow e_3$  and  $e_1 \leftrightarrow -(1, 1, 1)$ . So the centre of mass is invariant under these. Permuting  $e_1, e_2, e_3$  shows that the centre of mass lies on the line  $x = y = z$ . Invariance under this last transformation shows that the centre of mass lies at the origin: the mean of the vertices.

**34.2.** For any  $t > 0$ , take the points  $v_0 = (0, 0)$ ,  $v_1 = (1, 0)$ ,  $v_2 = (1 + t, 1 + t)$ ,  $v_3 = (0, 1)$ . Let  $Q$  be the quadrilateral which is the convex hull of  $v_0, v_1, v_2, v_3$ . Splitting  $Q$  into a square  $S$  with vertices  $v_0, v_1, (1, 1), v_3$ , and two triangles  $T_1, T_2$ :

$$\langle Q \rangle = \frac{\text{Area}(S)\langle S \rangle + \text{Area}(T_1)\langle T_1 \rangle + \text{Area}(T_2)\langle T_2 \rangle}{\text{Area } Q}.$$

Use obvious symmetry to find  $\langle S \rangle$ . The obvious analogue of problem 34.1 on page 266 for triangles gives  $\langle T_1 \rangle$  and  $\langle T_2 \rangle$ : take mean of the vertices (no integrals). Finally, the centre of mass of  $Q$  is at  $\bar{x}, \bar{y}$  where

$$\bar{x} = \bar{y} = \frac{3 + 2t}{6}.$$

But the mean of the vertices is at  $(x_m, y_m)$  where

$$x_m = y_m = \frac{2 + t}{4}.$$

These are not equal for any  $t > 0$ .

**34.3.** Replace  $\mathbb{R}^q$  by the affine span of  $C$ , and translate to arrange that the origin lies in the interior of  $C$ . By definition of  $C^k$  map, every point  $a_0 \in A$  lies in an open set  $U \subset \mathbb{R}^p$  to which  $f$  admits a local  $C^k$  extension  $f_0: U \rightarrow \mathbb{R}^q$ . Replacing  $U$  by  $f_0^{-1}C$ , we can suppose that  $f_0: U \rightarrow C$ . By compactness of  $A$ , we can take a finite cover of  $A$  by open sets  $U_1, \dots, U_n \subset \mathbb{R}^p$  on which we have  $C^k$  maps  $f_i: U_i \rightarrow C$  extending  $f$ . Take a partition of unity  $h_1, \dots, h_n$  subordinate to  $U_1, \dots, U_n$ . Let  $F_i = h_i f_i$  where defined, and  $F_i = 0$  elsewhere. Let  $F = \sum F_i$ .

**34.5.** The function  $f((1-t)p + tq)$  has second derivative

$$\frac{d^2}{dt^2} f((1-t)p + tq) = \langle f''((1-t)p + tq)(p-q), p-q \rangle.$$

A negative eigenvalue will force a negative second derivative for some  $p, q$ , i.e. force the graph of  $f((1-t)p + tq)$  to curve above some parabola which is convex downward. Conversely, if all eigenvalues are positive, the graph of  $f((1-t)p + tq)$  curves below some parabola which is convex upward.

**34.6.** If  $p, q \in C_{y_0}$ , so  $f(p), f(q) \leq y_0$ , then by convexity,  $f((1-t)p + tq) \leq y_0$  for all  $0 \leq t \leq 1$ .

**34.7.** By problem 34.6 on page 268,  $C_\varepsilon$  is convex. Translate to arrange that  $x_0 = 0$ . For  $x$  near 0,  $f(x)$  is bounded below by  $\lambda|x|^2$ , for small enough  $x$ , where  $\lambda$  is the smallest eigenvalue of  $f''(x)$ . we see that  $f$  attains some value  $\varepsilon > 0$  in every

direction going away from the origin, at some distance no more than  $\sqrt{\varepsilon/\lambda}$ . By Taylor expansion,

$$f'(x) = f'(0) + tf''(0)x + o(x) = tf''(0)x + o(x),$$

so  $f'(x)x \neq 0$  along the nearby points of the  $\varepsilon$ -level set, so  $f'(x) \neq 0$ , so the  $\varepsilon$ -level set is  $C^k$  by the implicit function theorem.

**34.8.**

$$\begin{aligned} \frac{\partial^2 g}{\partial y_i \partial y_j} &= \frac{\partial^2 f \circ F}{\partial y_i \partial y_j}, \\ &= \frac{\partial}{\partial y_i} \frac{\partial f \circ F}{\partial y_j}, \\ &= \sum_k \frac{\partial}{\partial y_i} \frac{\partial f}{\partial x_k} \frac{\partial F}{\partial y_j}, \\ &= \sum_k \frac{\partial^2 f}{\partial x_k \partial x_\ell} \frac{\partial F}{\partial y_j} \frac{\partial F}{\partial y_i} + \sum_k \frac{\partial f}{\partial x_k} \frac{\partial^2 F}{\partial y_j \partial y_i}, \end{aligned}$$

but the second term vanishes at any critical point of  $f(x)$ , so

$$\frac{\partial^2 g}{\partial y_i \partial y_j} = \sum_k \frac{\partial^2 f}{\partial x_k \partial x_\ell} \frac{\partial F}{\partial y_j} \frac{\partial F}{\partial y_i}.$$

In other words, as a symmetric bilinear form

$$g''(y_0)(u, v) = f''(x_0)(F'(y_0)u, F'(y_0)v)$$

for any vectors  $u, v$ . So if  $f''(x_0)(v, v) > 0$  for all  $v \neq 0$ , i.e. a positive definite symmetric matrix, then

$$g''(y_0)(w, w) = f''(x_0)(v, v) > 0,$$

where  $v = F'(y_0)w$ , for all  $w \neq 0$ .

**35.2.** If  $f$  is bounded below, say  $f \geq C$ , then

$$f_\varepsilon(x) - C = \int g_\varepsilon(y)(f(x-y) - C) dy$$

is the integral of a nonnegative function, so nonnegative, and zero just when  $f(x-y)$  is everywhere equal to  $C$  for  $y$  in the support of  $g_\varepsilon(y)$ .

**35.3.** Rescale  $h$  by a constant to have the same upper bound as  $f$  and apply the solution to problem 35.2 on page 273.

**35.4.** If all even derivatives of  $f$  up to order  $k$  vanish, we can define  $f_\varepsilon$  as above. Otherwise, suppose that  $f$  has a lowest order nonzero term  $f(x) = a_n x^n + \dots$ . Clearly  $a_n > 0$  since  $f \geq 0$ . Let  $h_\varepsilon = g_\varepsilon * f^{(n)}$  and let

$$f_{0\varepsilon} = \varepsilon h + \int_0^x \int_0^x \dots \int_0^x h_\varepsilon^{(n)}.$$

Let

$$f_{1\varepsilon} := g_\varepsilon * f$$

after extending  $f$  to be odd. Take some value  $x = x_0$  so that  $f^{(n)}(x) > 0$  for all  $0 \leq x \leq x_0$ . Take a  $C^\infty$  function  $\beta(x)$  with  $\beta(x) = 1$  for  $0 \leq x \leq x_0/2$  and  $\beta(x) = 0$  for  $x \geq x_0$ . Redefine

$$f_\varepsilon(x) := \beta(x)f_{0\varepsilon}(x) + (1 - \beta(x))f_{1\varepsilon}(x).$$

**36.1.** If  $\varphi_0$  is decreasing, compose with  $x \mapsto -x$ , for simplicity of notation. Take a bump function  $g(x)$  which increases for  $x < 0$  to a maximum at  $x = 0$  and then decreases for  $x > 0$ . Define  $g_\varepsilon$  as in our discussion of smoothing. Let  $\varphi_\varepsilon := \varphi_0 * g_\varepsilon$ . Note that since  $\varphi_0$  strictly increases, so does the integral in the convolution, so  $\varphi_\varepsilon$  strictly increases. Since  $\varphi$  increases, expand the integral to find

$$\varphi_\varepsilon(x) \geq \varphi(x - \varepsilon) \rightarrow \infty$$

as  $x \rightarrow \infty$ . Similarly as  $x \rightarrow -\infty$ . So  $\varphi_\varepsilon$  is proper. Differentiate  $\varphi'_\varepsilon = g'_\varepsilon * \varphi$  and expand out the integral, writing it as two integrals:  $-\infty < y \leq 0$  and  $0 \leq y \leq \infty$ . Rearrange to find

$$\varphi'_\varepsilon(x) = \int_{y=0}^{y=\infty} |g'_\varepsilon(x)| (f(x+y) - f(x-y)) dy > 0.$$

**36.2.** By theorem 36.1 on page 277, we can make a smoothing homotopy  $f_\varepsilon$  so that  $f_\varepsilon$  is  $C^\infty$ . Take any sequence of values  $\varepsilon_1, \varepsilon_2, \dots \rightarrow 0$ ; then  $f_i := f_{\varepsilon_i} \rightarrow f$  uniformly. By theorem 27.2 on page 226, every  $f_i$  has a fixed point, say  $x_i$ , so  $f_i(x_i) = x_i$ . Since the closed ball is compact, we can replace the sequence by some convergent subsequence, so we can assume that  $x_i \rightarrow x \in \bar{B}$ . Then  $f(x) = f(\lim x_i) = \lim f_i(x_i) = \lim x_i = x$ .

**36.3.** Since  $W$  is open, every point of  $W$  lies in an open ball in  $W$ . But the centre of any open ball lies at the center of a smaller open box, i.e. a product of open intervals, hence in a product  $E \times F \subseteq W$  of open sets. Fixing a point  $x_0$ , the compact set  $\{x_0\} \times K_1$  is covered by such open sets. So there is a finite subcover,  $E_1 \times F_1, \dots, E_k \times F_k \subseteq W$ . Each  $E_i$  contains that point  $x_0$ , and so their intersection does:  $x_0 \in E := E_1 \cap \dots \cap E_k$ . So  $E \times F_1, E \times F_2, \dots, E \times F_k$  are all open subsets of  $W$ , and cover  $\{x_0\} \times K_1$ . So  $F_1, F_2, \dots, F_k$  cover  $K_1$ . If we let  $F := F_1 \cup F_2 \cup \dots \cup F_k$ , then  $E \times F \subseteq W$  with  $K_1 \subset F$  and  $x_0 \in E$ .

So we can cover  $K_0$  by open sets  $E$  for which there is an open set  $F \subseteq U_1$  with  $K_1 \subset F$ . Take a finite subcover, say  $E_1, \dots, E_k$ , so that there are sets  $F_1, \dots, F_k$  with  $K_1 \subset F_i$  and  $E_i \times F_i \subseteq W$ . Then let  $F := F_1 \cap F_2 \cap \dots \cap F_k$  so that  $E_i \times F \subseteq W$  for all  $i$ . So if we let  $E := E_1 \cup E_2 \cup \dots \cup E_k$ , then  $K_0 \subset E$  and  $K_1 \subset F$  and  $E \times F \subseteq W$ .

**36.5.** Let  $W \subset T^*P \boxtimes TQ$  be the set of linear surjections.

**36.6.** Let  $W \subset T^*P \boxtimes TQ$  be the set of linear injections.

**36.7.** To each  $(p, q, \phi) \in T^*P \boxtimes TM$ , let

$$\dot{\phi} := \{\xi \in T_m^*M \mid \phi^*\xi \text{ is a coface of } P\}$$

and let

$$\phi^\perp := \{\xi \in T_m^*M \mid \phi^*\xi = 0\}.$$

Let  $W \subset T^*P \boxtimes TM$  be the set of  $(p, m, \phi)$  so that

a.  $m$  is not in  $g(Q)$  or

b.  $m$  is in  $g(Q)$  and

$$\dot{\phi} \cap g_q^+ = M_m^+$$

and

$$M_m^+ \cap \dot{\phi} \cap g_q^\perp$$

is spanned by cowalls of  $M$  at  $m$ .

**36.8.**  $x \in [0, \infty) \mapsto f_\varepsilon(x) := x + \varepsilon \in [0, \infty)$

**37.1.** Write out  $\omega$  as, say,

$$\omega = a_1(x) dx_1 + a_2(x) dx_2 + \cdots + a_n(x) dx_n$$

and let  $u = p/|p|$  and compute out that if  $x(t) = tu$  then

$$f^* \omega = \sum_i a_i(tu) u_i,$$

so that

$$\begin{aligned} \int_0^p \omega &= \int_0^{|p|} \sum_i a_i(tu) u_i dt, \\ &= \sum_i u_i \int_0^{|p|} a_i(tu) dt. \end{aligned}$$

Pick a closed ball around 0 and let  $M$  be the maximum of the  $a_i$  in that ball. For  $p$  in that ball, our integral is never larger in magnitude than  $n|p|M$ .

**37.2.** Note that  $d\omega = dx \wedge dy$ . By Stokes's theorem, the integral of  $\omega$  over the part of the path  $C$  with  $2\pi n \leq \theta \leq 2\pi(n+1)$  is the area contained inside between the path and the origin. Note that  $r$  decreases with larger  $\theta$ , spiralling inward. So this area is greater than the area of a circle of radius  $r = (2\pi(n+1))^\varepsilon$ , i.e.

$$A = \pi(2\pi(n+1))^{-2\varepsilon}.$$

Adding these up gives

$$2^{-2\varepsilon} \pi^{1-2\varepsilon} \sum_{n=1}^{\infty} (n+1)^{-2\varepsilon}.$$

By the integral test, this is infinite exactly when  $1 - 2\varepsilon > 0$ .

**37.4.** Take any smooth 2-form  $\omega$ . In polar coordinates,

$$\omega = f(r, \theta) dr \wedge d\theta$$

for some  $f$  with  $f(r, \theta + 2\pi) = f(r, \theta)$ . Let

$$g(r, \theta) := \int_{r_0}^r f(u, \theta) du.$$

Clearly  $g(r, \theta + 2\pi) = g(r, \theta)$ , so

$$\eta := g d\theta$$

is defined on  $M$ , and  $d\eta = \omega$ . The same argument works for  $M$  an annulus, cutting out a closed or open disk from a closed or open disk, or for a punctured disk, or any surface diffeomorphic to such.

**39.1.**

$$\begin{aligned}
\int_M \left( h^* (\xi^G) \right) \wedge \eta &= \int_M \xi^G \wedge h^{-1*} \eta, \\
&= \int_M (p_* ((a^* \xi) \wedge \omega)) \wedge h^{-1*} \eta, \\
&= \int_{G \times M} (a^* \xi) \wedge \omega \wedge p^* h^{-1*} \eta, \\
&= \int_{G \times M} (a^* \xi) \wedge \omega \wedge (h^{-1} p)^* \eta, \\
&= \int_{G \times M} (a^* \xi) \wedge \omega \wedge (ph^{-1})^* \eta, \\
&= \int_{G \times M} (a^* \xi) \wedge \omega \wedge h^{-1*} p^* \eta, \\
&= \int_{G \times M} (a^* \xi) \wedge \omega \wedge h^{-1*} p^* \eta, \\
&= \int_{G \times M} (h^* a^* \xi) \wedge (h^* \omega) \wedge p^* \eta, \\
&= \int_{G \times M} ((ah)^* \xi) \wedge \omega \wedge p^* \eta, \\
&= \int_{G \times M} (a^* \xi) \wedge \omega \wedge p^* \eta, \\
&= \int_M (p_* (a^* \xi) \wedge \omega) \wedge \eta, \\
&= \int_M \xi^G \wedge \eta.
\end{aligned}$$

So if  $\zeta := h^*(\xi^G) - \xi^G$  then  $\int_M \zeta \wedge \eta = 0$  for all  $\eta$ . It is easy to see that this implies that  $\zeta = 0$ . Therefore  $\xi^G$  is a  $G$ -invariant form on  $M$ .

**39.2.** Let  $M := G := \{\pm 1\}$ , so  $\Omega^{*G}$  is 1-dimensional, while  $\Omega^*$  is 2-dimensional.

**39.3.** Let  $G = \mathrm{SO}(p+q), \mathrm{U}(p+q), \mathrm{Sp}(p+q)$  for  $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$ . The stabiliser  $H \subset G$  is the subgroup fixing a  $p$ -dimensional  $k$ -linear subspace, i.e. the matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with  $\det A \det D = 1$  for  $k = \mathbb{R}$ . For  $k = \mathbb{C}, \mathbb{H}$ ,  $H = \mathrm{U}(p) \times \mathrm{U}(q)$  or  $H = \mathrm{Sp}(p) \times \mathrm{Sp}(q)$  is connected, so there is an invariant volume form. The quotient  $\mathfrak{g}/\mathfrak{h}$  consists of the skew-adjoint matrices quotiented out by those vanishing in top row and leftmost column, so antisymmetric matrices

$$\begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}.$$

The matrices in  $H$  act on the quotient  $\mathfrak{g}/\mathfrak{h}$  by adjoint action

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix} \begin{pmatrix} A^* & 0 \\ 0 & D^* \end{pmatrix} = \begin{pmatrix} 0 & -Y^* \\ Y & 0 \end{pmatrix}$$

where  $Y = DXA^*$ . Consider now  $k = \mathbb{R}$ . If  $A$  is a rotation matrix, so is  $D$ . The space of rotation matrices is connected, so we have a connected compact group of such, the identity component of  $H$ , so preserves a volume form. Every orthogonal matrix is a product of rotations and reflections, so we only need consider whether there is a volume form on  $\mathfrak{g}/\mathfrak{h} = \mathbb{R}^p \otimes \mathbb{R}^q$  invariant under reflections  $A, D$ . Take the standard basis  $\{e_j\}$  of  $\mathbb{R}^p$  and  $\{e_J\}$  of  $\mathbb{R}^q$ . Then  $\{e_j \otimes e_J\}$  is a basis of  $\mathbb{R}^p \otimes \mathbb{R}^q$ . We take  $A$  to reflect the vector  $e_p$ , and leave  $e_1, \dots, e_{p-1}$  alone, and similarly  $D$  reflects only  $e_q$ . Then  $X \mapsto DXA^*$  reflects

$$e_p \otimes e_1, \dots, e_p \otimes e_{q-1}, e_1 \otimes e_q, \dots, e_{p-1} \otimes e_q.$$

So  $q-1+p-1 = p+q-2$  reflected directions, and all others invariant. This is a rotation of  $\mathbb{R}^p \otimes \mathbb{R}^q$ , i.e. has determinant 1, just when  $p+q$  is even. So finally if  $V = \mathbb{R}^{p+q}$ ,  $\text{Gr}_p V$  has an invariant volume form just when  $p+q$  is even.

**39.6.** Each “monomial”  $t_I = t_{i_1} t_{i_2} \dots t_{i_\ell}$ ,  $I = (i_1, \dots, i_\ell)$ , by commutativity, can be written with indices in increasing order; refer to  $I$  as the weight of  $t_I$ . No index occurs twice because  $t_i^2 = 0$ . Let  $a := t_1 + t_2 + \dots + t_n$  be the sum of all of the symbols. Take any permutation invariant element  $b$ . Alphabetically ordering its monomials by indices: write  $b = ct_I + \dots$ , for a unique symbol  $t_I$  lowest in alphabetical order, the *lowest weight term*. Permutations can lower weight unless  $I = (1, 2, \dots, k)$ , for some  $k$ . Since  $t_I$  is lowest weight and  $b$  is permutation invariant,  $t_I = t_{12\dots k}$ . Expand:

$$\omega^k = k! t_{12\dots k} + \dots,$$

so

$$b - \frac{c}{k!} \omega^k$$

has higher lowest weight term. Apply induction, noting that there are finitely many weights.

**39.9.** Since we know that each of  $\omega_1, \omega_2, \omega_3$  is preserved separately by  $\text{Sp}(V)$ , it suffices to prove that  $\Omega$  is preserved by  $\text{Sp}(1)$ . Consider the subgroup  $\{e^{i\theta}\} \subset \text{Sp}(1)$ . It acts on vectors  $z + wj \in \mathbb{H}^n$  by

$$Z + Wj := (z + wj)e^{i\theta} = e^{i\theta}z + e^{-i\theta}wj,$$

since  $ji = -ij$ . So it acts by

$$\begin{aligned} Z^\mu &= e^{i\theta} z^\mu, \\ Z^{\bar{\mu}} &= e^{-i\theta} z^{\bar{\mu}}, \\ W^\mu &= e^{-i\theta} w^\mu, \\ W^{\bar{\mu}} &= e^{i\theta} w^{\bar{\mu}}, \end{aligned}$$

So it changes from  $Z, W$  coordinates to  $z, w$  coordinates by

$$\begin{aligned} \omega_1 &= \frac{i}{2} dZ^{\mu\bar{\mu}} + \frac{i}{2} dW^{\mu\bar{\mu}}, \\ \omega_2 &= \frac{1}{2} dZ^\mu \wedge dW^{\bar{\mu}} + \frac{1}{2} dZ^{\bar{\mu}} \wedge dW^\mu, \\ \omega_3 &= \frac{i}{2} dZ^\mu \wedge dW^{\bar{\mu}} - \frac{i}{2} dZ^{\bar{\mu}} \wedge dW^\mu. \end{aligned}$$

to

$$\begin{aligned}\omega_1 &= \frac{i}{2}dz^{\mu\bar{\mu}} + \frac{i}{2}dw^{\mu\bar{\mu}}, \\ \omega_2 &= e^{2i\theta}\frac{1}{2}dz^\mu \wedge dw^{\bar{\mu}} + e^{-2i\theta}\frac{1}{2}dz^{\bar{\mu}} \wedge dw^\mu, \\ \omega_3 &= e^{2i\theta}\frac{i}{2}dz^\mu \wedge dw^{\bar{\mu}} - e^{-2i\theta}\frac{i}{2}dz^{\bar{\mu}} \wedge dw^\mu.\end{aligned}$$

Hence

$$\begin{aligned}\Omega &= \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3, \\ &= \omega_1 \wedge \omega_1 + dz^{\mu\bar{\mu}} \wedge dw^{\mu\bar{\mu}}\end{aligned}$$

is independent of  $\theta$ , so invariant. So now  $\Omega$  is invariant under  $\text{Sp}(V)S^1$ . The obvious automorphism  $i \rightarrow j \rightarrow k \rightarrow i$  of the quaternions permutes  $\omega_1, \omega_2, \omega_3$ , so we see that  $\Omega$  is invariant the closure of the subgroup of  $\text{Sp}(1)$  generated by  $e^{i\alpha}, e^{j\beta}, e^{k\gamma}$ , which has Lie algebra containing all of  $i, j, k$ , so that of  $\text{Sp}(1)$ . Since  $\text{Sp}(1)$  is connected, the subgroup is all of  $\text{Sp}(1)$ .

**39.12.** Clearly the linearity of  $F_h^*$  gives the first result. If  $\omega = \alpha \wedge \beta$ ,

$$\begin{aligned}\mathcal{L}_X(\alpha \wedge \beta) &= \lim_{h \rightarrow 0} \frac{F_h^*(\alpha \wedge \beta) - \alpha \wedge \beta}{h}, \\ &= \lim_{h \rightarrow 0} \frac{(F_h^*\alpha) \wedge (F_h^*\beta) - \alpha \wedge \beta}{h}, \\ &= \lim_{h \rightarrow 0} \frac{(F_h^*\alpha) \wedge (F_h^*\beta) - (F_h^*\alpha) \wedge \beta + (F_h^*\alpha) \wedge \beta - \alpha \wedge \beta}{h}, \\ &= \lim_{h \rightarrow 0} (F_h^*\alpha) \wedge \frac{(F_h^*\beta) - \beta}{h} + \frac{(F_h^*\alpha) - \alpha}{h} \wedge \beta, \\ &= (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X\beta).\end{aligned}$$

For any diffeomorphism  $f$ , going back through the definitions of  $f_*$  and  $f^*$ , we see  $f_*^{-1}Y \lrcorner f^*\alpha = f^*(Y \lrcorner \alpha)$ . Otherwise, it is just the same trick:

$$\begin{aligned}\mathcal{L}_X(Y \lrcorner \alpha) &= \lim_{h \rightarrow 0} \frac{F_h^*(Y \lrcorner \alpha) - Y \lrcorner \alpha}{h}, \\ &= \lim_{h \rightarrow 0} \frac{F_{h*}^{-1}Y \lrcorner F_h^*\alpha - Y \lrcorner \alpha}{h}, \\ &= \lim_{h \rightarrow 0} \frac{F_{h*}^{-1}Y \lrcorner F_h^*\alpha - Y \lrcorner F_h^*\alpha + Y \lrcorner F_h^*\alpha - Y \lrcorner \alpha}{h}, \\ &= \lim_{h \rightarrow 0} \frac{F_{h*}^{-1}Y - Y}{h} \lrcorner F_h^*\alpha + Y \lrcorner \frac{F_h^*\alpha - \alpha}{h}, \\ &= \lim_{h \rightarrow 0} F_{h*}^{-1} \frac{Y - F_{h*}Y}{h} \lrcorner F_h^*\alpha + Y \lrcorner \frac{F_h^*\alpha - \alpha}{h}, \\ &= (\mathcal{L}_X Y) \lrcorner \alpha + Y \lrcorner \mathcal{L}_X \alpha.\end{aligned}$$

**39.13.** Pick a point  $x_0$  and let  $x(t) = F_h(x_0)$ . At the point  $x = x_0$ ,

$$\begin{aligned}\mathcal{L}_X f &= \lim_{h \rightarrow 0} \frac{F_h^* f - f}{h}, \\ &= \lim_{h \rightarrow 0} \frac{f(F_h(x)) - f(x)}{h}, \\ &= \left. \frac{d}{dh} f(x(t)) \right|_{h=0}, \\ &= \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}, \\ &= df(x'(t)), \\ &= df(X).\end{aligned}$$

**39.14.** Start with the simple case of  $\omega = dx^1$  in some chart. The left hand side is zero, the right hand side:

$$0 \stackrel{?}{=} X(Y^1) - Y(X^1) - [X, Y]^1,$$

is clear. The same computation works for any of  $\omega = dx^1, dx^2, \dots, dx^n$ . Next, suppose the result is true for some  $\omega$ , and expand out to check it for  $f\omega$  for any  $C^1$  function  $f$ . Finally, the result is linear, so we can add such 1-forms.

**39.15.** If  $\omega$  is a 0-form, i.e. a function  $f$ , we want to prove

$$df(X_0) \stackrel{?}{=} X_0 f,$$

which is clear. If  $\omega$  is a 1-form, we want to prove

$$d\omega(X_0, X_1) \stackrel{?}{=} X_0(\omega(X_1)) - X_1(\omega(X_0)) - \omega([X_0, X_1]),$$

which is problem 39.14 on page 317. For any  $\omega$ , say a  $p$ -form, assume we have the result proven for  $(p-1)$ -forms. By induction on degree, if  $\eta = X_0 \lrcorner \omega$ , then

$$\begin{aligned}d\eta(X_1, \dots, X_p) &= - \sum_{0 < i} (-1)^i X_i(\eta(X_1, \dots, \hat{X}_i, \dots, X_p)) \\ &\quad + \sum_{0 < i < j} (-1)^{i+j} \eta([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).\end{aligned}$$

Expand this out to give

$$\begin{aligned}d(X_0 \lrcorner \omega)(X_1, \dots, X_p) &= - \sum_{0 < i} (-1)^i X_i(\omega(X_0, X_1, \dots, \hat{X}_i, \dots, X_p)) \\ &\quad - \sum_{0 < i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).\end{aligned}$$

By lemma 39.10 on page 317,

$$\mathcal{L}_{X_0} \omega = X_0 \lrcorner d\omega + d(X_0 \lrcorner \omega).$$



So

$$\begin{aligned}
 & (\mathcal{L}_{X_0}\omega)(X_1, \dots, X_p) - X_0 \lrcorner d\omega(X_1, \dots, X_p) \\
 &= - \sum_{0 < i} (-1)^i X_i(\omega(X_0, X_1, \dots, \hat{X}_i, \dots, X_p)) \\
 &\quad - \sum_{0 < i < j} (-1)^{i+j} \omega([X_i X_j], X_0, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p).
 \end{aligned}$$

which we write as

$$\begin{aligned}
 & d\omega(X_0, X_1, \dots, X_p) \\
 &= (\mathcal{L}_{X_0}\omega)(X_1, \dots, X_p) \\
 &\quad + \sum_{0 < i} (-1)^i X_i(\omega(X_0, X_1, \dots, \hat{X}_i, \dots, X_p)) \\
 &\quad + \sum_{0 < i < j} (-1)^{i+j} \omega([X_i X_j], X_0, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p), \\
 &= X_0(\omega(X_1, \dots, X_p)) \\
 &\quad + \sum \omega(X_1, \dots, [X_0 X_i], \dots, X_p) \\
 &\quad + \sum_{0 < i} (-1)^i X_i(\omega(X_0, X_1, \dots, \hat{X}_i, \dots, X_p)) \\
 &\quad + \sum_{0 < i < j} (-1)^{i+j} \omega([X_i X_j], X_0, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p),
 \end{aligned}$$

**39.16.**  $\psi^*\eta = (-1)^p \eta$ , so taking  $d$  of both sides:  $\psi^*d\eta = (-1)^p d\eta$ . But  $d\psi$  is also biinvariant, and a  $(p+1)$ -form, so  $\psi^*d\eta = (-1)^{p+1} d\eta$ .

**40.5.** If we split  $A, B$  into the kernel of  $d$  and some complement,

$$\begin{aligned}
 A &= A_0 \oplus A_1, \\
 B &= B_0 \oplus B_1,
 \end{aligned}$$

then  $d$  takes  $A_1 \cong B_0$  and  $B_1 \cong A_0$ . So  $A \cong B$ .

**40.6.** Note that the long exact sequence maps take the left hand side to the right, and vice versa, with image of one map the kernel of the other and vice versa. Apply problem 40.5 on page 324.

**41.3.** Using a partition of unity, all compactly supported forms on  $M$  arise this way. The kernel consists of pairs  $(\omega_0, \omega_{01})$  mapped by

$$(\omega_0, \omega_{01}) \mapsto (\omega_0, -\omega_0, \omega_{01}),$$

an exact sequence

$$0 \rightarrow \Omega_{01c}^\bullet \oplus \Omega_{01c}^{\bullet-1} \rightarrow \check{\Omega}_c^\bullet \rightarrow \Omega_c^\bullet \rightarrow 0.$$

On the first term, we put the differential

$$d(\omega_0, \omega_{01}) = (d\omega_0, 2\omega_0 - d\omega_{01}).$$

Such a pair is closed just when

$$\omega_0 = \frac{1}{2} d\omega_{01}.$$

It is exact just when

$$(\omega_0, \omega_{01}) = (d\eta_0, 2\eta_0 - d\eta_{01}),$$

for some  $\eta_0, \eta_{01}$  on  $M_{01}$ . But if closed, we can take

$$\eta_0 = \frac{1}{2}\omega_{01},$$

and  $\eta_{01} = 0$ . So the first term has trivial cohomology.

**43.1.** Take  $X' := G$ ,  $G' := G \times G$  acting on  $X'$  by  $(g_1, g_2)g := g_1 g g_2^{-1}$ , with stabilizer  $H'$  the set of pairs  $(g, g)$  for  $g \in G$ . Let  $H'_0 \subset H'$  be the set of pairs  $(g, g)$  for which  $g$  acts trivially on  $\Lambda^{\text{top}} \mathfrak{g}^*$ . Take the covering space  $\hat{X}' := G'/H'_0$ . Apply our results above to the  $G'$ -homogeneous space  $X'$  to construct Haar measure. Note that we do *not* ask the resulting covering space  $\hat{X}'$  to be a Lie group, or  $\hat{X}' \rightarrow X' = G$  to be a group morphism.

**43.2.** If  $\omega$  is left invariant, then  $i^*\omega$  is right invariant and vice versa.

**43.3.** Take any  $g \in \text{U}(p)$  and write

$$\begin{pmatrix} g & 0 \\ 0 & I \end{pmatrix}$$

also as  $g$ , so  $\varphi(gM) = g\varphi(M)$ . Hence the pushforward form  $\varphi_*\Omega$  of the volume form  $\Omega$  is also left invariant, and hence biinvariant, so a constant multiple of the standard volume form. We only need to compute one integral to prove the result: the volume of the group.

**43.4.** Since all  $\varphi^g$  are smooth, so is  $\psi$ , with  $\psi(m_0) = 0$  and  $\psi'(m_0) = \varphi'(m_0)$ , so  $\psi$  is a chart on some neighborhood of  $m_0$ .

**43.5.** Compute

$$\begin{aligned} \psi^g &= L^g \circ \psi \circ g^{-1}, \\ &= L^g \circ \int_{h \in G} \varphi^h \circ g^{-1}, \\ &= L^g \circ \int_{h \in G} L^h \circ \varphi \circ h^{-1} \circ g^{-1}, \\ &= \int_{h \in G} L^{gh} \circ \varphi \circ (gh)^{-1}, \\ &= \psi. \end{aligned}$$

**47.1.** Every quadratic function on the sphere descends to a function on the real projective plane. (When is it Morse? When is it excellent Morse?) The critical points of  $xy$  on the sphere are

$$(x, y, z) = (0, 0, 1), (0, 0, -1), (a, a, 0), (a, -a, 0), (-a, a, 0), (-a, -a, 0)$$

where  $a = \frac{1}{\sqrt{2}}$ . So on the real projective plane, we have 3 critical points. Using  $x, y$  as coordinates on the sphere,  $xy$  has a saddle point at

$$(x, y, z) = (0, 0, 1), (0, 0, -1).$$

At the other points we use  $x, z$  as coordinates and find nondegenerate local minima at

$$(x, y, z) = (a, -a, 0), (-a, a, 0)$$

and nondegenerate local maxima at

$$(x, y, z) = (a, a, 0), (-a, -a, 0).$$

Hence on the real projective plane,  $xy$  is Morse with one maximum, one minimum, and one saddle point.

**47.2.** Locally we can pick out one of the two unit normal vectors  $n = n(x)$ , a map to the unit sphere. Pick a regular value  $n_0$  of  $n$ . (Careful: there is a unique normal up to sign, but *only* up to sign. We might need to change the sign of  $n$  from one open set to the next. Sard's theorem allows us to pick a value  $n_0$  so that  $\pm n_0$  are both regular, for every local choice of normal  $n(x)$ , since applying  $x \mapsto -x$  to a measure zero set gives a measure zero set, and the union of two measure zero sets is measure zero.) For any  $x = x_0$  with  $n(x_0) = \pm n_0$ , we have  $n'(x_0): T_{x_0}M \rightarrow T_{x_0}S^n$  a linear isomorphism. Rotate so that  $n_0 = (0, 1)$ , i.e.  $(0, \pm 1)$  are regular values of  $n(x)$ .

Writing points of  $\mathbb{R}^n$  as  $(x, y)$ , up to rotation, we can locally write  $M$  as the graph of a function, say  $y = y(x)$ . One of the two normal vectors is

$$n(x) = \frac{(-\nabla y, 1)}{|\nabla y|^2 + 1}.$$

If  $n(0) = (0, 1)$ , so  $\nabla y(0) = 0$ , then calculate

$$n'(0) = y''(0),$$

so being a regular value means that  $y''(0)$  is of full rank.

We can choose regular value  $n_0$  from all but a measure zero set in the unit sphere. The unit vectors linearly dependent on  $n_0$  form a measure zero set. The union of two measure zero sets is measure zero. So we can pick a regular value  $n_1$  linearly independent from  $n_0$ , and so on.

**47.3.** By the fundamental theorem of calculus in one variable:

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt} f(tx) dt, \\ &= \int_0^1 f'(tx)x dt, \\ &= \int_0^1 \sum_i \frac{\partial f}{\partial x_i}(tx)x_i dt, \\ &= \sum x_i f_i(x). \end{aligned}$$

**47.5.** The method is identical to corollary 47.2 on page 384: add bumps in Morse charts to the maxima and upside down bumps to the minima, but now we allow the bumps to be as large as needed.

**47.6.** Pick a chart near each critical point as given by the Morse lemma. Use the Euclidean metric to take the gradient of  $f$  to give such a vector field locally. Away from critical points, the result is easy locally. Glue using a partition of unity to get

a vector field  $X$ . Multiply by some positive smooth function to ensure that  $X$  is complete, as in problem 32.2 on page 257.

**47.8.** We only prove the result for mapping any one Milnor sphere  $S_\varphi^n$  to the standard sphere  $S^n$ . Recall the notation and terminology of gluings from chapter 3. Let  $Q$  be two disjoint copies of  $\mathbb{R}^n$ , which we call  $Q_0$  and  $Q_1$ , and let  $P := \mathbb{R}^n - \{0\}$ , which we glue to them by the two maps  $x \mapsto x$  and  $x \mapsto \varphi(x/|x|)/|x|$  giving us a gluing plan  $P \rightrightarrows Q$ . We want a map of the gluing  $S_\varphi^n = \bar{Q}$ . We start by imitating Ptolemaic projection on  $Q_0$ : define

$$x \in Q_0 \mapsto \varpi_n^{-1}(x) = p = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{1+|x|^2} \begin{pmatrix} 2x \\ -2 \end{pmatrix},$$

which is precisely the reverse Ptolemaic projection to the sphere. In Ptolemy's coordinate chart from the north pole, this is just  $x \mapsto x$ . The commutative diagram

$$\begin{array}{ccc} & P & \\ x \swarrow & & \searrow \varphi(x/|x|)/|x| \\ Q_0 & & Q_1 \\ \varpi_n^{-1} \searrow & & \swarrow \\ & S^n & \end{array}$$

forces us to define the map  $Q_1 \rightarrow S^n$  to get this to commute. In other words, we take any  $y \in Q_1$ , write  $y$  uniquely as

$$y = \varphi^{-1}(x/|x|)/|x|,$$

and then let  $p = \varpi_n^{-1}(x)$ , giving

$$p = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{|y|^2}{1+|y|^2} \begin{pmatrix} \frac{2\varphi^{-1}(y/|y|)}{|y|} \\ -2 \end{pmatrix}.$$

We write this as a Ptolemaic projection from the south pole:

$$p = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{1}{1+|z|^2} \begin{pmatrix} 2z \\ 2 \end{pmatrix}$$

to find easily

$$z = |y|\varphi^{-1}(y/|y|).$$

This is our expression, in Ptolemy's coordinate charts, for the map in the south pole charts, except at  $y = 0$  where this expression extends uniquely continuously to  $z = 0$ . Writing  $y = ru$  with  $|u| = 1$ , and  $z = sv$  with  $|v| = 1$ , we find our map expressed as  $s = r$  and  $v = \varphi^{-1}(u)$ . As  $r \rightarrow 0$  with fixed  $u$ , this should approach some linear map:  $z'(0)$ . But since it doesn't scale at all as we move  $r$ , the map  $z = z(y)$  is not differentiable unless  $\varphi^{-1}$  is linear, and hence  $\varphi$  is linear. But just because this particular map is not a diffeomorphism, doesn't say that there is not diffeomorphism.

**47.9.** Each component of  $M$  is compact. The restriction of  $f$  to any component has a maximum and a minimum from among the points of that component. Near a critical point, our Morse function is not constant, by the Morse lemma. So  $f$  has a local

maximum distinct from its local minimum on each component. Since  $f$  has at most two critical points in total,  $M$  is connected.

**47.10.** Deform  $\varphi$  through a smooth homotopy of smooth isotopies, depending on a parameter. Our gluing constructs a family of compact manifolds, a fiber bundle over that parameter. By Ehressmann's theorem, they are all diffeomorphic. So we can replace our smooth diffeotopy by any smoothly homotopic smooth diffeotopy. Consider the diffeotopy

$$\Phi_{r,t}(u) = \varphi_{(1-t)r+tr_0}(u)$$

for some  $r_0 > 0$ . If  $\varphi_{r_0}$  is smoothly diffeotopic to the identity, we can use that diffeotopy too, to arrange that  $\varphi(u, r) = u$ . If  $\varphi_{r_0}$  reverses orientation we can let  $G(y) = -y$ .

**48.2.** Suppose that we have a Morse chart  $\varphi: U_M \rightarrow B^\lambda \times B^{n-\lambda}$ . Take any vector  $x \in \mathbb{R}^\lambda$ . Let  $t := 1 + |x|^2$  and let

$$\psi(x) = e^{-2tX} \varphi^{-1}(e^{-2t}x, 0).$$

This is smooth, and we can check that it is well defined. In our Morse chart, where  $\psi$  is defined, it is  $\psi(x) = (x, 0)$ . If  $\psi(x_1) = \psi(x_2)$ , apply the flow  $e^{2tX}$  and get into the Morse chart, to find  $x_1 = x_2$ : injective. By definition it is surjective to the ascending cell. Again applying  $e^{2tX}$ ,  $\psi$  has injective differential. So  $\psi$  is a diffeomorphism of  $\mathbb{R}^\lambda$  to the ascending cell.

**48.4.** To apply the Mayer–Vietoris theorem, it helps to note that  $M'$  is covered by two open sets: the images of  $M$  and of  $B^\lambda \times B^{n-\lambda}$ .

**48.7.** If  $A$  is symmetric and  $B$  antisymmetric,

$$\langle A, B \rangle = A_{ij}B_{ji} = -A_{ij}B_{ij} = -A_{ji}B_{ji} = -A_{ij}B_{ji},$$

so vanishes. If  $A$  is antisymmetric and  $B$  symmetric, similarly. If  $A$  is symmetric, it is orthogonally diagonalizable, so  $\langle A, A \rangle \geq 0$  and  $= 0$  just when  $A = 0$ . So this inner product is positive definite on symmetric matrices. If  $A$  is antisymmetric, there is an orthonormal basis in which  $A$  leaves invariant mutually orthogonal planes, so we need only check  $n = 2$  to see that  $\langle A, A \rangle \leq 0$  and  $= 0$  just when  $A = 0$ . So our inner product is positive definite on symmetric matrices, negative definite on antisymmetric matrices.

**48.8.** Note that

$$T_g \text{SO}(n) = \mathfrak{so}(n) = g \{ \text{antisymmetric } n \times n \text{ matrices} \}.$$

The derivative is

$$f'(g)gB = \langle A, gB \rangle = \text{tr } AgB,$$

so critical points occur just where  $Ag$  is perpendicular to all antisymmetric matrices in our inner product, i.e. just where  $Ag$  is symmetric:  $Ag = g^t A$ , or  $g^t Ag = A$ , i.e. the rotation  $g$  preserves the eigenspaces and eigenvalues of  $A$ . Since each eigenspace is 1-dimensional,  $g$  is diagonalized in a basis of  $A$ -eigenvectors. Being a rotation,  $g$  rotates each of those 1-dimensional eigenspaces, i.e. in a basis of eigenvectors

$$g = \begin{pmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$$

with an even number of minus signs, since  $g$  is a rotation, i.e. has determinant 1. We pick the first  $n - 1$  plus or minus signs, and then the last one has to equal their product.

**50.1.** on page 492

**50.3.** We have two open sets on  $\mathbb{CP}^1$ : let  $U_0 \subset \mathbb{CP}^1$  be the lines which are not the line through  $(1, 0)$ , and let  $U_1 \subset \mathbb{CP}^1$  be the lines which are not the line through  $(0, 1)$ . We have two charts on  $\mathbb{CP}^1$ : on  $U_0$  our first chart  $\varphi_0$  maps

$$\begin{bmatrix} z \\ w \end{bmatrix} \mapsto \frac{z}{w}.$$

On  $U_1$  our second chart  $\varphi_1$  maps

$$\begin{bmatrix} z \\ w \end{bmatrix} \mapsto \frac{w}{z}.$$

We have trivializations:

$$\begin{aligned} \tau_0: (z, w) &\mapsto (\zeta, e^{i\theta}) = \left( \frac{z}{w}, \frac{w}{|w|} \right) \in \mathbb{C} \times S^1, \\ \tau_1: (z, w) &\mapsto (\zeta, e^{i\theta}) = \left( \frac{w}{z}, \frac{z}{|z|} \right) \in \mathbb{C} \times S^1 \end{aligned}$$

over the open sets  $U_0, U_1$ , which we identify by our charts with  $U_0 \cong \mathbb{C}, U_1 \cong \mathbb{C}$ . Note that

$$\begin{aligned} \tau_0^{-1}: (\zeta, e^{i\theta}) &\in \mathbb{C} \times S^1 \mapsto (z, w) = \frac{e^{i\theta}}{\sqrt{1 + |\zeta|^2}} (\zeta, 1), \\ \tau_1^{-1}: (\zeta, e^{i\theta}) &\in \mathbb{C} \times S^1 \mapsto (z, w) = \frac{e^{i\theta}}{\sqrt{1 + |\zeta|^2}} (1, \zeta). \end{aligned}$$

So the Hopf fibration is a fiber bundle.

**50.6.** Differentiate to find that

$$\varphi'(g, m)(0, v) = (g'(m)v, v),$$

(if we think of  $g$  as a diffeomorphism of  $M$ ), and that

$$\varphi'(g, m)(\tilde{A}, 0) = (\tilde{A}(m), 0).$$

Hence  $\varphi'(m, g)$  has kernel precisely  $\text{Ad}_g \mathfrak{g}^m \oplus 0$ . If  $\varphi(g_0, m_0) = \varphi(g_1, m_1)$  then  $(m_0, g_0 m_0) = (m_1, g_1 m_1)$ , i.e.  $m_0 = m_1$  and  $g_1^{-1} g_0 m_0 = m_0$ , i.e.  $g_1^{-1} g_0 \in G^{m_0}$ .

**50.7.** Take a free and proper Lie group action of  $G$  on  $M$ . The map  $\varphi$  is a proper map, with image the incidence relation  $R$ . Any proper injection from a locally compact Hausdorff space is a homeomorphism to its image: [13] p. 20 corollary 2.6. Because the action is free, by problem 50.6 on page 411,  $R \subset M \times M$  is an immersed submanifold. By lemma 5.1 on page 56,  $R \subset M \times M$  is a closed embedded submanifold. Apply theorem 14.6 on page 130.

**50.8.** Suppose that action is proper. The set

$$K := \{(x_1, y_1), (x_2, y_2), \dots\} \cup \{(x, y)\} \subset M \times M$$

is compact, because it consists of the points in a convergent sequence. Hence its preimage in  $G \times M$  is also compact. Since  $y_i = g_i x_i$ , we have  $(g_i, x_i)$  in that preimage, so after perhaps replacing by a subsequence, this sequence converges to some point in that preimage, say to  $(g, x')$ . But then  $g_i \rightarrow g$  and  $x_i \rightarrow x'$  so  $x' = x$ , and so  $\varphi(g, x) = (x, gx) = (x, y)$ .

Suppose that the condition on convergent sequences is satisfied. Take a compact set  $K \subset M \times M$ . We need precisely to prove that  $\varphi^{-1}K \subset G \times M$  is also compact. Take an infinite sequence in  $\varphi^{-1}K$ , say  $(g_1, x_1), (g_2, x_2), \dots$ . We need precisely to prove that this sequence has a convergent subsequence. Apply  $\varphi$ , we get a sequence  $(x_1, y_1) = (x_1, g_1 x_1), (x_2, y_2) = (x_2, g_2 x_2), \dots$  in  $K$ . Because  $K$  is compact, after perhaps replacing by a subsequence, this sequence converges to some point in  $K$ , say  $(x, y)$ , so  $x_i \rightarrow x$  and  $g_i x_i \rightarrow y$ . By hypothesis, after perhaps replacing by a subsequence,  $g_1, g_2, \dots$  converges, say to  $g \in G$ .

**50.9.** The map  $\varphi$  in this case is  $\varphi: (h, g) \in H \times G \mapsto (g, gh^{-1}) \in G \times G$ . Take a convergent sequence  $(g_i, g_i h_i^{-1}) \rightarrow (g, k) \in G \times G$ . We need to ask when  $h_i$  converges. But then  $g_i \rightarrow g$  and  $g_i h_i^{-1} \rightarrow k$  so  $h_i^{-1} \rightarrow g^{-1}k$  converges in  $G$ . Properness of the action is precisely that it converges in  $H$ , i.e. that the limit in  $G$  lies in  $H$ , i.e. that  $H$  is closed in  $G$ .

**50.10.** With points denoted

$$\begin{pmatrix} z \\ t \end{pmatrix}$$

$M_0 = (t > -1)$  and  $M_1 = (t < 1)$ . We parameterize  $M_0$  by the map

$$x \mapsto \frac{1}{1 + |x|^2} \begin{pmatrix} 2x \\ 1 - |x|^2 \end{pmatrix}.$$

The inverse of this parameterization is the chart

$$\begin{pmatrix} z \\ t \end{pmatrix} \mapsto x = \frac{z}{1 + t}.$$

We parameterize  $M_1$  by the map

$$x \mapsto \frac{1}{1 + |x|^2} \begin{pmatrix} 2\bar{x} \\ |x|^2 - 1 \end{pmatrix}.$$

The inverse of this parameterization is the chart

$$\begin{pmatrix} z \\ t \end{pmatrix} \mapsto \frac{z}{1 - t}.$$

We let  $E = S^{2d-1}$ , mapping  $\pi: E \rightarrow M$  by

$$\pi: \begin{pmatrix} a \\ b \end{pmatrix} \in S^{2d-1} \subset k^2 \rightarrow \begin{pmatrix} 2a\bar{b} \\ |a|^2 - |b|^2 \end{pmatrix} \in S^d \subset k \oplus \mathbb{R}.$$

The  $S^{d-1}$ -action on  $S^{2d-1}$  is

$$g, \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} a\bar{g} \\ b\bar{g} \end{pmatrix}.$$

Trivialize our bundle  $S^{2d-1} \rightarrow S^d$  over  $M_0$  by parameterizing

$$(x, g) \in k \times S^{d-1} \mapsto \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{1 + |x|^2}} \begin{pmatrix} \bar{g} \\ x\bar{g} \end{pmatrix} \in E_0 \subset S^{2d-1},$$

with inverse the local trivialization

$$\tau_0: \begin{pmatrix} a \\ b \end{pmatrix} \mapsto (x, g) = (ba^{-1}, \bar{a}/|a|).$$

This trivialization is  $S^{d-1}$ -equivariant

$$\tau_0(x, gh) = g\tau_0(x, h)$$

Trivialize our bundle  $S^{2d-1} \rightarrow S^d$  over  $M_1$  by parameterizing

$$(x, g) \in k \times S^{d-1} \mapsto \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{1+|x|^2}} \begin{pmatrix} x\bar{g} \\ \bar{g} \end{pmatrix} \in E_1 \subset S^{2d-1},$$

with inverse the local trivialization

$$\tau_1: \begin{pmatrix} a \\ b \end{pmatrix} \mapsto (x, g) = (ab^{-1}, \bar{b}/|b|).$$

This trivialization is  $S^{d-1}$ -equivariant.

Composing trivializations, the transition map is

$$(x, g) \mapsto \left( \frac{1}{x}, g \frac{\bar{x}}{|x|} \right).$$

So the clutching function as a principal bundle is

$$x \mapsto \bar{x}|x|.$$

**51.2.**

$$\begin{aligned} & \tau_{ca,m}^f \tau_{bc,m}^f \tau_{ab,m}^f \\ &= f_{a,m} \tau_{ca,m} f_{c,m}^{-1} f_{c,m} \tau_{bc,m} f_{b,m}^{-1} f_{b,m} \tau_{ab,m} f_{a,m}^{-1}, \\ &= \tau_{ca,m}^f \tau_{bc,m}^f \tau_{ab,m}^f \end{aligned}$$

**51.3.**

$$\begin{aligned} (m, \tau'_{ab,m} x) &= t'_b t_a'^{-1} (m, x), \\ &= (t'_b \varphi t_b^{-1}) t_b t_a'^{-1} [(t'_a \varphi t_a^{-1})]^{-1} (m, x), \\ &= (t'_b \varphi t_b^{-1}) t_b t_a'^{-1} (m, f_{a,m}^{-1} x), \\ &= (t'_b \varphi t_b^{-1}) (m, \tau_{ab,m} f_{a,m}^{-1} x), \\ &= (m, f_{b,m} \tau_{ab,m} f_{a,m}^{-1} x), \\ &= (m, \tau_{ab,m}^f x). \end{aligned}$$

**52.1.** Create a vector field, defined only near the boundary, which pushes the boundary inside. Then compose its flow with any smooth map  $S \rightarrow M$  of any manifold  $S$ ; after composing with that flow, that smooth map is not onto. So has degree zero. In particular, this works for  $S$  a sphere of the same dimension as  $M$ .



**52.2.** Take a smooth increasing function  $h(t)$  vanishing with all derivatives at  $t = 0$ , and equal to  $t$  not far away from  $t = 0$ . Let

$$\Phi(x, y) = \left( \frac{x}{|x|} \sqrt{1 - h(|y|^2)}, h(|y|) \varphi(y/|y|) \right).$$

At regular points  $y$  of  $\varphi$  where  $h(t) = t$ ,  $\Phi$  has degree equal to the degree of  $\varphi$ , i.e. the same number of preimages with the same orientations.

**52.3.** Take a smooth nondecreasing function  $h(t)$ ,  $-1 \leq t \leq 1$ , so that  $h(-1) = -1$  and  $h(t) = 1$  for  $t \geq 0$ . Consider the homotopy

$$\Psi_t(x, y) = (h((2-t)x + t - 1), \sqrt{1 - h((2-t)x + t - 1)^2} y).$$

Check that  $\Psi_0(x, y) = (x, y)$  and that  $\Psi_1$  takes the lower hemisphere to the entire sphere, taking the upper hemisphere to the north pole.

Our map  $\varphi$  takes  $\varphi^{-1}(\bar{H}_-)$  to  $\bar{H}_-$ ,  $\varphi^{-1}(\bar{H}_+)$  to  $\bar{H}_+$ . So  $\Psi_t \circ \varphi$  is a homotopy taking  $\varphi$  to a map which maps  $\varphi^{-1}\bar{H}_-$  to the entire sphere, and  $\varphi^{-1}\bar{H}_+$  to the north pole.

**53.4.** Pick any local coordinates, with the origin at our chosen point, and define  $\lambda, A$  as in the proof of lemma 53.3 on page 443. Use notation  $B'$  to represent a Riemannian ball. By lemma 53.2 on page 443,

$$B'_r \subseteq B_{r/\sqrt{\lambda}},$$

and

$$B_{R/\sqrt{A}} \subseteq B'_R.$$

Take the straight line Euclidean path between two points  $x_0, x_1 \in B'_r$ : its Euclidean length is  $|x_0 - x_1| \leq 2r/\sqrt{\lambda}$ , so its Riemannian length is at most

$$\sqrt{A}|x_0 - x_1| \leq 2r\sqrt{\frac{A}{\lambda}}.$$

On the other hand, if we take a path between  $x_0, x_1$  that leaves  $B'_R$ , it also leaves  $B_{R/\sqrt{A}}$ , and starts inside  $B_{r/\sqrt{\lambda}}$ , so has Euclidean length at least

$$2\left(\frac{R}{\sqrt{A}} - \frac{r}{\sqrt{\lambda}}\right).$$

We let the reader check that this length is longer than our straight line path as long as

$$r < \frac{R\sqrt{\frac{\lambda}{A}}}{1 + \sqrt{\frac{A}{\lambda}}}.$$

So under this inequality, we can ensure that the Euclidean straight path is a smooth path of length at most  $2C(r)r$ , where

$$C(r) \geq \sqrt{\frac{A}{\lambda}}.$$

on  $\bar{B}'_R$ . Any shorter path doesn't leave  $B'_R$ .

If we use a smaller  $r$  than needed in the argument above, we can repeat the argument with a smaller  $R$ , as long as

$$r < \frac{R\sqrt{\frac{\lambda}{\Lambda}}}{1 + \sqrt{\frac{\Lambda}{\lambda}}},$$

and so we can arrange that

$$C(r) \geq \sqrt{\frac{\Lambda}{\lambda}}$$

for the new choice of  $R = R(r) \rightarrow 0$  as  $r \rightarrow 0$ . In particular, with our coordinates chosen to match up the Euclidean and Riemannian metrics at the origin, we can arrange  $C(r) \rightarrow 1$ .

**54.1.** Calculate the usual derivative in time:

$$\begin{aligned} \dot{v}(t) &= \frac{d}{dt}v(t), \\ &= \frac{d}{dt} \begin{pmatrix} t^2 \sin t \\ -t^2 \cos t \\ t^2 \end{pmatrix} \\ &= \frac{d}{dt} \begin{pmatrix} 2t \sin t + t^2 \cos t \\ -2t \cos t + t^2 \sin t \\ 2t \end{pmatrix}. \end{aligned}$$

We need to find the tangent part of this. Note that

$$\langle \dot{v}(t), x(t) \rangle = t^2.$$

Since  $x(t)$  is a unit vector, and the tangent space to the sphere is just the set of vectors perpendicular to  $x(t)$ , the projection of  $\dot{v}(t)$  to the tangent space is

$$\begin{aligned} \nabla_{\dot{x}(t)} v &= \dot{v}(t) - \langle \dot{v}(t), x(t) \rangle x(t), \\ &= \begin{pmatrix} 2t \sin t + t^2 \cos t \\ -2t \cos t + t^2 \sin t \\ 2t \end{pmatrix} - t^2 \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}, \\ &= \begin{pmatrix} 2t \sin t \\ -2t \cos t \\ 2t \end{pmatrix}. \end{aligned}$$

**54.2.** Rescale to get the Earth to have unit radius, and place its centre at the origin. Write the latitude of each point as  $\phi$ , and the longitude as  $\theta$ , in radians. Each point  $(x, y, z)$  of the surface is

$$(x, y, z) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi).$$

Note that this maps  $\theta, \phi$  to the sphere, and is a local diffeomorphism (hence  $\theta, \phi$  are local coordinates) away from the north and south poles. We stay away from the poles, to avoid complications. Construct two unit vectors tangent to the sphere as

$$\begin{aligned} u_1 &:= (-\sin \theta, \cos \theta, 0), \\ u_2 &:= (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi), \end{aligned}$$

defined everywhere except at the poles, and orthonormal where defined. We let  $u_3 := (x, y, z)$ , to make an orthonormal basis. The vector  $u_1$  is tangent to the horizontal and points the way along the lines of latitude, i.e. with fixed  $\phi$  at unit speed. The vector  $u_2$  is a right angle rotation of  $u_1$  in the tangent plane at each point. Differentiate to find that as functions of  $\theta$  with fixed  $\phi$ :

$$\begin{aligned}\dot{u}_1 &= -\cos \phi u_2 - \sin \phi u_3, \\ \dot{u}_2 &= \cos \phi u_1.\end{aligned}$$

Any tangent vector  $v(\theta)$  to the sphere expands in that basis:

$$v(\theta) = c_1(\theta)u_1 + c_2(\theta)u_2.$$

Differentiate in  $\theta$ :

$$\begin{aligned}\dot{v}(\theta) &= \dot{c}_1 u_1 + c_1(-\cos \phi u_2 - \sin \phi u_3) + \dot{c}_2 u_2 + c_2 \cos \phi u_1, \\ &= (\dot{c}_1 + c_2 \cos \phi)u_1 + (-c_1 \cos \phi + \dot{c}_2)u_2 - c_1 \sin \phi u_3.\end{aligned}$$

To have  $\dot{v}$  normal, we need to get rid of the  $u_1$  and  $u_2$  terms, keeping only the  $(x, y, z)$  term, and noting that  $z = \cos \phi$ :

$$\frac{d}{d\theta} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = z \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Since  $z$  is constant on the lines of latitude, this is easy to solve:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(\theta) = \begin{pmatrix} \cos(z\theta) & \sin(z\theta) \\ -\sin(z\theta) & \cos(z\theta) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(0).$$

So the vector  $v$  turns, relative to the horizontal by a total angle  $2\pi z$  as we move it along the line of latitude of a given  $z$  value. At  $z = 0$ , on the equator, any vector  $v$  undergoing parallel transport retains a constant angle to the equator. As we get close to the north pole, the tiny line of latitude turns in a tight little circle around the north pole, and the parallel transport turns a vector nearly in a complete rotation, twice striking the tangent direction to the little circle. Similarly near the south pole, but turning the opposite direction.

**54.8.** Define  $C_{ijk}$  by

$$[e_i e_j] = -2C_{kij} e_k.$$

Use the properties of Lie derivatives:

$$\begin{aligned}0 &= \mathcal{L}_{e_i}(e_j \lrcorner \xi_k), \\ &= (\mathcal{L}_{e_i} e_j) \lrcorner \xi_k + e_j \lrcorner \mathcal{L}_{e_i} \xi_k, \\ &= [e_i e_j] \lrcorner \xi_k + e_j \lrcorner (e_i \lrcorner d\xi_k + d(e_i \lrcorner \xi_k)), \\ &= [e_i e_j] \lrcorner \xi_k + e_j \lrcorner (e_i \lrcorner d\xi_k + d(e_i \lrcorner \xi_k)), \\ &= -2C_{kij} + e_j \lrcorner (e_i \lrcorner (-\eta_{k\ell} \wedge \xi_\ell) + 0), \\ &= -2C_{kij} - (e_i \lrcorner \eta_{k\ell})(e_j \lrcorner \xi_\ell) + (e_j \lrcorner \eta_{k\ell})(e_i \lrcorner \xi_\ell) \\ &= -2C_{kij} - b_{kji} + b_{kij} \\ &= -2C_{ijk} + 2C_{kij}.\end{aligned}$$

**55.4.** In polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ , check that

$$\frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\theta^2}{r^2 \sin^2 \theta}.$$

Let  $w := -1/z$ , and write  $z = re^{i\theta}$ , so

$$w = -\frac{1}{re^{i\theta}} = -\frac{1}{r}e^{-i\theta} = \frac{1}{r}e^{i(\pi-\theta)}.$$

If  $R := 1/r$  and  $\Theta := \pi - \theta$ , then

$$\begin{aligned} \frac{dR^2 + R^2 d\Theta^2}{R^2 \sin^2 \Theta} &= \frac{(-r^{-2}dr)^2 + r^{-2}d\theta^2}{r^{-2} \sin^2 \theta}, \\ &= \frac{r^{-4}dr^2 + r^{-2}d\theta^2}{r^{-2} \sin^2 \theta}, \\ &= \frac{dr^2 + r^2 d\theta^2}{r^2 \sin^2 \theta}. \end{aligned}$$

**55.5.** Inversion takes the vertical line through the origin to itself:  $iy \mapsto i/y$ , but takes all other vertical lines to circles. Indeed under  $w = -1/z$ , the line  $x = x_0$  is taken to

$$\begin{aligned} x_0 &= \frac{z + \bar{z}}{2}, \\ &= \frac{1}{2} \left( \frac{1}{w} + \frac{1}{\bar{w}} \right), \\ &= \frac{w + \bar{w}}{2|w|^2}. \end{aligned}$$

If  $x_0 = 0$  this is still just the line on which the real part of  $w$  is zero. If  $x_0 \neq 0$ ,

$$|w|^2 = \frac{w + \bar{w}}{2x_0},$$

so if  $w = u + iv$  in real and imaginary parts, this is

$$u^2 + v^2 = \frac{u}{x_0},$$

and we complete the square to find

$$(u - u_0)^2 + v^2 = u_0^2,$$

where

$$u_0 = \frac{1}{2x_0}.$$

Hence the vertical lines become the circles orthogonal to the real axis and passing through the origin. Each vertical half line going up from the real axis becomes a half of the corresponding circle, the half lying in the upper half plane.

For a circle:

$$(x - x_0)^2 + y^2 = r_0^2,$$

expand out and write it, in terms of  $z = x + iy$ , as

$$|z|^2 - x_0(z + \bar{z}) = r_0^2 - x_0^2.$$

Invert:  $w = -1/z$ ,

$$\frac{1}{|w|^2} + x_0 \left( \frac{1}{w} + \frac{1}{\bar{w}} \right) = r_0^2 - x_0^2.$$

Clear out denominators:

$$1 + x_0(\bar{w} + w) = (r_0^2 - x_0^2)|w|^2.$$

If  $r_0 = \pm x_0$ , our circle passed through the origin, and the inversion is a vertical line. Otherwise,

$$|w|^2 - \frac{x_0}{r_0^2 - x_0^2}(w + \bar{w}) = \frac{1}{r_0^2 - x_0^2}.$$

Write in components:  $w = u + iv$ ,

$$\left( u - \frac{x_0}{r_0^2 - x_0^2} \right)^2 + v^2 = \frac{1}{r_0^2 - x_0^2} + \left( \frac{x_0}{r_0^2 - x_0^2} \right)^2.$$

**55.9.** We proved that there is a Riemannian metric on any manifold, say  $g_0$ . Let

$$g(v, w) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} g_0(\gamma_* v, \gamma_* w).$$

Then  $g$  is a  $\Gamma$ -invariant Riemannian metric. Hence the exponential map  $f(v) = \exp_{m_0} v$  is  $\Gamma$ -equivariant, and a diffeomorphism near  $0 \in T_{m_0} M$ . But  $\Gamma$  acts on  $T_{m_0} M$  by the differentials  $\gamma'(m_0)$  of the various diffeomorphisms  $\gamma \in \Gamma$ , i.e.  $f$  identifies some open set around  $0$  in the vector space  $T_{m_0} M$  with some open set around  $m_0$  in  $M$ , and turns the linear transformation  $\gamma'(m_0)$  into  $\gamma$ . The map  $\gamma'(m_0)$  preserves the metric, so in any orthonormal basis it is an orthogonal matrix.

**55.10.** Take a smooth map  $f: X \rightarrow S$ . Pick a point  $s_0 \in S$ . Let  $F_t(x) := \exp_{s_0} tv(s_0, f(x))$ , a homotopy between the constant map  $F_0(x) = s_0$ , and the map  $F_1(x) = f(x)$ .

**55.13.** Each unit speed geodesic is defined for at least a time equal to the injectivity radius, since we can rescale it to be a geodesic of slower speed but lasting for longer time. The injectivity radius has a positive lower bound by lower semicontinuity. Hence each unit speed geodesic continues to be defined for some more time, no matter how we ride along it. Hence it is defined for all time. Rescale the time variable to get the same result for any geodesic.

**56.4.** For  $w \neq z$ , compute

$$d\omega = -\frac{1}{2\pi i} f_{\bar{w}}(w) \frac{dw \wedge d\bar{w}}{w - z}.$$

Take a disk  $D \subseteq U$  around the point  $z$ . By Stokes's theorem,

$$\int_{U-D} d\omega = \int_{\partial U} \omega - \int_{\partial D} \omega,$$

which expands out to

$$-\frac{1}{2\pi i} \int_{U-D} f_{\bar{w}}(w) \frac{dw \wedge d\bar{w}}{w - z} = \frac{1}{2\pi i} \int_{\partial U} \frac{f(w)dw}{w - z} - \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)dw}{w - z}.$$

Write  $w$  as  $z + re^{i\theta}$ . Consider integrating  $\omega$  inside the disk:

$$\frac{1}{2\pi i} \int_{\partial D} f_{\bar{w}}(w) \frac{dw \wedge d\bar{w}}{w - z} = \frac{1}{\pi} \int_{\partial D} f_{\bar{w}}(w) e^{-i\theta} d\theta \wedge dr.$$

Note that  $d\omega$  is therefore locally integrable. Since  $f$  is  $C^1$ , the derivative expression is bounded, and the result is proportional to the radius of disk, so vanishes as the disk gets small.

The boundary term around the small circle: again  $w = z + re^{i\theta}$  but  $r$  is constant:

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)dw}{w - z} = \frac{1}{2\pi} \int_{\partial D} f(z + re^{i\theta}) d\theta,$$

is the average value of  $f$  around the circle, which approaches  $f(z)$  as the radius get small.

**57.1.** If  $f: C \rightarrow \mathbb{C}$  is holomorphic,  $|f|$  is continuous, so has a maximum value at some point  $p_0 \in C$ . Take a holomorphic chart near  $p_0$  in which  $p_0$  is the origin, so  $f$  becomes a holomorphic function  $f(z)$  of a complex variable in that chart. Apply the maximum modulus principle to  $f(z)$ .

**57.2.** There are finitely many poles  $p_1, \dots, p_n$ , since they can't accumulate. Cover each pole  $p_i$  with an embedded closed disk  $\bar{D}_i$ . By Stokes's theorem

$$\begin{aligned} \sum_{p \in C} \text{res}_p \omega &= \sum_{p_i} \text{res}_{p_i} \omega, \\ &= \frac{1}{2\pi i} \sum_i \int_{\partial D_i} \omega, \\ &= -\frac{1}{2\pi i} \int_{\partial(C - \bigcup_i D_i)} \omega, \\ &= -\frac{1}{2\pi i} \int_{C - \bigcup_i D_i} d\omega, \\ &= 0. \end{aligned}$$

**57.5.** By de Rham's theorem (theorem 40.2 on page 329), this class vanishes just when the integrals of the 1-form vanish over all 1-cycles, which occurs just when  $\omega = dg$  for some complex valued function  $g$ . But  $\omega$  is complex linear, so  $\bar{\partial}g = 0$ , i.e.  $g$  is holomorphic. In particular, if  $C$  is compact,  $g$  is constant and so  $\omega = 0$ . In other words, each holomorphic 1-form is determined by its cohomology class, i.e. by its integrals over 1-cycles.

**58.8.** Suppose that the vector bundle has rank  $k$  and the base manifold has dimension  $n$ . The codimension of  $A$  is the same near any point, so we can assume the vector bundle is trivial and the base manifold is an open set  $U \subset [0, \infty)^n$  in the standard corner in Euclidean space. So sections are just functions  $y = y(x)$  of variables  $x^i$ ,  $y = y^a$ , with 1-jets parameterized by  $(x, y, p)$  where  $p = (p_i^a)$  describes the first derivatives of a section. Clearly  $A$  consists of the points  $(x, y, p)$  for which  $y = 0$  and  $p$  is not surjective. If  $k > n$ , then  $A$  has codimension more than  $n$ , so transversality implies that there are no zeroes of the generic section. So we can assume that  $k \leq n$ . So  $A = U \times V$  where  $V$  is the set of all nonsurjective  $k \times n$  matrices. This  $V$  is the

union of determinantal varieties

$$V = \bigcup_{\ell=0}^{k-1} V_{k,n,\ell},$$

(whose dimensions we have calculated) so of dimension  $(k-1)(n+1)$ . So  $A$  has dimension  $n + (k-1)(n+1)$ , so codimension

$$n + k + kn - (n + (k-1)(n+1)) = n + 1.$$

**59.4.** Take a product neighborhood  $\Phi$ . We want to trivialize the normal bundle of  $\varphi$ . Since  $\Phi$  is a diffeomorphism to its image, which contains the image of  $\varphi$ , it is enough to assume that  $Q = P \times \bar{B}$  and  $\Phi$  is the identity map. The standard basis vectors  $e_1, \dots, e_q$  give sections  $(0, e_i)$  of the tangent bundle, trivializing the normal bundle.

**59.5.** For each point  $p_0 \in P$ , we can find open set  $U \subset P$  and some ball  $B$  centered at the origin so that  $\Phi$  is a diffeomorphism on  $U \times B$ . By compactness of  $P$ , we can find a single radius of ball, so a single ball  $B$  centered at the origin, for all those  $U$ , so  $\Phi$  is a local diffeomorphism on  $P \times B$ . Shrinking  $B$ ,  $\Phi$  is a local diffeomorphism on  $P \times \bar{B}$ .

**59.6.** Suppose that for any ball  $\bar{B}$ ,  $\Phi$  fails to be a diffeomorphism. Take a sequence of balls  $\bar{B}_i$  whose radii go to zero. We can find points

$$(x_i, p_i) \neq (x'_i, p'_i) \in \bar{B}_i \times P$$

so that  $\Phi(x_i, p_i) = \Phi(x'_i, p'_i)$ . Take a convergent subsequence, so  $p_i \rightarrow p$  and  $p'_i \rightarrow p'$  and  $x_i, x'_i \rightarrow 0$ :

$$\varphi(p) = \Phi(p, 0) = \Phi(p', 0) = \varphi(p'),$$

so  $p = p'$ . But  $\Phi'(p)$  is a linear isomorphism, so  $\Phi$  is a diffeomorphism on some open set  $U \times \bar{B}$ ,  $U \subseteq P$  open. Far enough down our sequences,

$$(x_i, p_i) \neq (x'_i, p'_i) \in \bar{B} \times U$$

so  $(x_i, p_i) = (x'_i, p'_i)$  a contradiction.

**59.7.** We can assume by scaling that  $\bar{B}$  has radius 1. Every vector bundle on a closed ball is trivial, so the normal bundle is trivial. Pick a product neighborhood  $\Phi: \bar{B} \times \bar{B}'' \rightarrow M$ , on a product of unit balls, and restrict to the unit ball inside the product of unit balls.

**60.4.** The connection  $\dot{\nabla}$  is

$$\dot{\nabla}s = \frac{\partial s}{\partial t} dt + (1-t)\nabla_0 s + t\nabla_1 s.$$

Take some basis of local sections  $e_i$  of  $V$ . These are not varying in time  $t$ . Compute

$$\begin{aligned} \dot{\nabla}e_i &= (1-t)\nabla_0 e_i + t\nabla_1 e_i, \\ &= ((1-t)\gamma_{0j}^i + t\gamma_{1j}^i)e_j. \end{aligned}$$

So

$$\dot{\gamma}_j^i = (1-t)\gamma_{0j}^i + t\gamma_{1j}^i.$$

So the curvature of  $\dot{\nabla}$  is

$$\begin{aligned}\dot{\Omega}_j^i &= d\dot{\gamma}_j^i + \dot{\gamma}_k^i \wedge \dot{\gamma}_j^k, \\ &= dt \wedge (\gamma_{1j}^i - \gamma_{0j}^i) + (1-t)d\gamma_{0j}^i + t d\gamma_{1j}^i \\ &\quad + ((1-t)\gamma_{0k}^i + t\gamma_{1k}^i) \wedge ((1-t)\gamma_{0j}^k + t\gamma_{1j}^k),\end{aligned}$$

**60.5.**

$$P(\nabla) = \text{tr } \Omega.$$

So now when we have two connections on the same vector bundle, we have

$$\pi(t, m) = m.$$

So

$$\pi_* P(\dot{\nabla}) = \pi_* \text{tr } \dot{\Omega}.$$

Recall from problem 60.4 on page 520

$$\dot{\Omega} = dt \wedge (\gamma_{1j}^i - \gamma_{0j}^i) + \dots$$

where the  $\dots$  have no  $dt$ , so  $\pi_* = 0$  on them, hence

$$\pi_* \text{tr } \dot{\Omega} = \text{tr } \gamma_{1j}^i - \text{tr } \gamma_{0j}^i.$$

Note that this expression is only local, as it depends on choice of some basis of local sections in which we compute out differential forms  $\gamma_0, \gamma_1$ . But finally we have the invariant description

$$\pi_* \text{tr } \dot{\Omega} = \text{tr}(\nabla_1 - \nabla_0).$$

**60.6.** There are  $(2n)!$  permutations of  $1, 2, \dots, 2n$ , and each partition is associated to  $n!2^n$  different permutations, so there are  $(2n)!/(n!2^n)$  different partitions of  $1, 2, \dots, 2n$ .

**60.7.** In alphabetical order, the first pair must always start with 1. Then we can choose any number to sit beside the 1, and the smallest number not chosen starts the second pair, etc.

- (a)  $\{1, 2\}$
- (b)  $\{1, 2\}, \{3, 4\}$   
 $\{1, 3\}, \{2, 4\}$   
 $\{1, 4\}, \{2, 3\}$
- (c)  $\{1, 2\}, \{3, 4\}, \{5, 6\}$   
 $\{1, 2\}, \{3, 5\}, \{4, 6\}$   
 $\{1, 2\}, \{3, 6\}, \{4, 5\}$   
 $\{1, 3\}, \{2, 4\}, \{5, 6\}$   
 $\{1, 3\}, \{2, 5\}, \{4, 6\}$   
 $\{1, 3\}, \{2, 6\}, \{4, 5\}$   
 $\{1, 4\}, \{2, 3\}, \{5, 6\}$   
 $\{1, 4\}, \{2, 5\}, \{3, 6\}$   
 $\{1, 4\}, \{2, 6\}, \{3, 5\}$   
 $\{1, 5\}, \{2, 3\}, \{4, 6\}$



$\{1, 5\}, \{2, 4\}, \{3, 6\}$   
 $\{1, 5\}, \{2, 6\}, \{3, 4\}$   
 $\{1, 6\}, \{2, 3\}, \{4, 5\}$   
 $\{1, 6\}, \{2, 4\}, \{3, 5\}$   
 $\{1, 6\}, \{2, 5\}, \{3, 4\}$



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*If those books are in agreement with the Quran, we have no need of them; and if they are opposed to the Quran, destroy them.*

— Omar

*... , i libri si parlano tra loro, ...*

*... , books speak with one another, ...*

— Umberto Eco

IL NOME DELLA ROSA (THE NAME OF THE ROSE)

*We are tired of works of truth.*

— QUMRAN SCROLLS 4Q418 69 II 13-14

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## List of notation

|                              |                                                                    |
|------------------------------|--------------------------------------------------------------------|
| $\mathbb{R}^n$               | $n$ -dimensional Euclidean space, 2                                |
| $\mathbb{RP}^n$              | real projective space, 5                                           |
| $\mathbb{R}^0$               | Zero dimensional Euclidean space, 5                                |
| $S^n$                        | the unit sphere in $\mathbb{R}^{n+1}$ , 12                         |
| $\mathbb{P}^n$               | projective space, 20                                               |
| $\mathrm{Gr}_p V$            | Grassmannian of $p$ -dimensional linear subspaces, 21              |
| $\nu_S$                      | normal bundle, 43                                                  |
| $\nu_{S,x}$                  | normal space, 43                                                   |
| $A^t$                        | transpose of a matrix $A$ , 50                                     |
| $\mathbb{H}$                 | quaternions, 57                                                    |
| $O(n)$                       | orthogonal group, 62                                               |
| $\mathrm{Sp}(n)$             | symplectic group, 62                                               |
| $U(n)$                       | unitary group, 62                                                  |
| $Xf$                         | Lie derivative of function $f$ along flow of vector field $X$ , 71 |
| $e^{tX}$                     | time $t$ flow of vector field $X$ , 77                             |
| $1$                          | identity element of any Lie group, 83                              |
| $\mathrm{GL}(n, k)$          | general linear group, 84                                           |
| $\mathrm{SL}(n, \mathbb{C})$ | special linear group, 84                                           |
| $\mathrm{SL}(n, \mathbb{R})$ | special linear group, 84                                           |
| $\mathfrak{aff}_n$           | Lie algebra of the affine group, 115                               |
| $G \backslash M$             | quotient space, 129                                                |
| $M/G$                        | quotient space of right action, 129                                |
| $\mathbb{PSL}(n, k)$         | projective linear group, 138                                       |
| $T_{m_0}^* M$                | cotangent space of a manifold $M$ at point $m_0$ , 143             |
| $\mathbb{P}T_x^* M$          | projectivised cotangent space, 146                                 |
| $TM$                         | tangent bundle of submanifold $M$ , 146                            |
| $\mathbb{P}T^* M$            | projectivised cotangent bundle, 147                                |
| $T^* M$                      | cotangent bundle of manifold $M$ , 147                             |
| $S_{s_0}^\perp$              | conormal space of a submanifold $S$ at a point $s_0$ , 150         |
| $\varphi_p^\perp$            | conormal space of a map, 151                                       |

|                                 |                                                                             |
|---------------------------------|-----------------------------------------------------------------------------|
| $P \times_M Q$                  | fiber product, 152                                                          |
| $S_{M,s_0}^+$                   | coface of a submanifold $S$ at a point $s_0$ , 175                          |
| $S_{M,s_0}^\perp$               | conormal space of a submanifold $S \subseteq M$ at a point $s_0$ , 175      |
| $P \times_M Q$                  | fiber product, 178                                                          |
| $\partial M$                    | boundary of a manifold with corners, 181                                    |
| $\deg_q \varphi$                | degree of a proper map $\varphi$ at a regular value $q$ , 185               |
| $\deg \varphi$                  | degree of a proper map, 191                                                 |
| $\wedge$                        | wedge product of differential forms, 199                                    |
| $F^* \omega$                    | pullback of differential form, 201                                          |
| $d\theta$                       | angle 1-form, 221                                                           |
| $\lrcorner$                     | hook, 223                                                                   |
| $\varphi_*$                     | pushforward, 237                                                            |
| $\partial P$                    | vertical boundary, 239                                                      |
| $\langle \varphi \rangle$       | mean of a map to Euclidean space, 263                                       |
| $f * g$                         | convolution, 270                                                            |
| $T^*P \boxtimes TQ$             | the set of linear isomorphisms between tangent spaces of two manifolds, 281 |
| $H_{\text{dR}}^p(M)$            | the $p$ -th cohomology group of $M$ , 295                                   |
| $[\omega]$                      | cohomology class, 295                                                       |
| $H_{\text{dR}}^{\text{top}}(M)$ | top dimensional cohomology of a manifold $M$ , 299                          |
| $H_c^*(M)$                      | compactly supported cohomology, 300                                         |
| $H_c^{\text{top}}(M)$           | top dimensional compact supported cohomology of a manifold $M$ , 301        |
| $\mathcal{L}_X Y$               | Lie derivative of vector field, 314                                         |
| $\mathcal{L}_X \omega$          | Lie derivative of form, 315                                                 |
| $\chi$                          | Euler characteristic, 322                                                   |
| $\text{supp } \omega$           | support of a form $\omega$ , 351                                            |
| $f_* u$                         | pushforward of a current by a smooth map, 351                               |
| $f_p^{(k)}$                     | $k$ -jet of $f$ at $p$ , 362                                                |
| $H_o^\lambda$                   | rim, 389                                                                    |
| $H^\lambda$                     | handle, 389                                                                 |
| $f^* g$                         | pullback metric, 440                                                        |
| $\nabla$                        | covariant derivative, 445                                                   |
| $\exp_{x_0} v_0$                | exponential map of a Riemannian metric, 462                                 |
| $V \oplus W$                    | sum of vector bundles, 491                                                  |
| $V \otimes W$                   | tensor product of vector bundles, 491                                       |
| $V^*$                           | dual vector bundle, 491                                                     |

|                         |                                                         |
|-------------------------|---------------------------------------------------------|
| $\mathrm{Sym}^k(V)$     | symmetric product of vector bundles, 492                |
| $V _P$                  | pullback to submanifold, 492                            |
| $\Lambda^k V$           | exterior product of vector bundles, 492                 |
| $\varphi^* V$           | pullback of vector bundle, 492                          |
| $\dot{P}(\nabla)$       | Bott difference form, 518                               |
| $c_j(\nabla \cdot)$     | Čech–de Rham Chern form, 519                            |
| $\mathrm{ch}_j(\nabla)$ | Chern character form, 520                               |
| $\mathrm{ch}_j(V)$      | Chern character, 520                                    |
| $\mathrm{ch}(V)$        | total Chern character, 520                              |
| $c_j(V)$                | Chern class, 520                                        |
| $c_j(\nabla)$           | Chern form of connection $\nabla$ on vector bundle, 520 |
| $e(\nabla)$             | Euler class, 525                                        |
| $e(V)$                  | Euler class, 525                                        |





## Index

- abelian
  - Lie algebra, **101**
- action, **87**
  - circle, **242, 301**
  - closed, **129**
  - group, **29**
  - Lie algebra, **114**
  - proper, **411**
  - right, **129**
  - torus, **301**
- addition laws, **59**
- adjoint representation, **106**
- Ado's theorem, **114**
- affine
  - span, **266**
  - subspace, **266**
- affine connection, **448**
  - metric compatible, **448**
- affine group, **84, 99**
- affine transformation, **84**
- algebra
  - fundamental theorem of, **94, 102, 190, 222**
- almost complex structure, **475**
- alphabetical order
  - see partition, alphabetical order, **526**
- analytic, **162**
- angle form, **220**
- angle function, **24**
- angular velocity, **96**
- antishear, **565**
- argument principle, **222**
- ascend, **389**
- ascending cell, **389**
- atlas, **3, 166**
  - equivalent, **8**
  - maximal, **8, 166**
- ball
  - coordinate, **248**
- banjax, **103**
- base
  - of fiber bundle, **407**
- basis, **59**
  - of open sets, **10**
  - of short exact sequence, **153**
- Beltrami equation, **478**
- Bianchi classification of 3-dimensional Lie algebras, **115**
- Bianchi identity, **517, 519**
- biholomorphism, **91, 476**
- biinvariant, **318, 444**
- birth, **389**
- Borel theorem, **173**
- Borel–Moore homology, **306**
- Bott difference form, **520**
- boundary, **183, 304, 306**
  - function, **165**
- boundary attaching map, **183**
- box
  - flow, **70**
- bracket closed, **123**
- Brothers Karamazov, **17**
- Brouwer fixed point theorem, **171, 226, 277**
- bump function, **23**
- bundle
  - principal, **411, 411**
  - projectivised cotangent, **147**
- bundle of tangent hyperplanes, **166**
- calculus
  - fundamental theorem of, **210, 243**
- calibration, **457**
- Cartan 3-form, **319**
- Cartan embedding, **63**
- Cauchy
  - integral theorem, **221**
- Cauchy integral theorem
  - with remainder, **478**
- Cauchy–Schwarz inequality, **593**

- Cayley transform, *18*
- Čech–de Rham differential, *341*
- cell
  - ascending, *389*
  - descending, *389*
- center, *105*
- centralizer, *105*
- centre, *42, 137, 263*
- chain
  - long, *351*
- character, *112*
- characteristic
  - class, *520*
- characteristic form, *519*
- chart, *2*
  - compatible, *3*
  - corner, *166*
  - isothermic, *474*
- Chern
  - character
    - total, *522*
  - character form, *522*
  - class, *522*
    - total, *522*
  - form, *522*
- Chern character, *522*
- circle action, *242, 301*
- circle bundle, *423*
- closed
  - form, *206*
- closed action, *129*
- clutching function, *412, 495*
- clutching isomorphism, *495*
- coboundary
  - Čech–de Rham, *341*
- cocharacter, *112*
- cocycle
  - Čech–de Rham, *341*
- codimension, *149, 151*
- coface, *177*
- coffee, *227*
- coframing, *475*
  - orthonormal, *449*
- cohomology, *323*
  - Čech–de Rham, *341*
  - class, *297*
  - compactly supported, *302*
  - de Rham, *297*
  - group, *297*
  - relative, *345*
- collar, *259*
- commutative diagram, *33*
- compact
  - relatively, *248*
- compactly supported
  - cohomology, *302*
- compactness radius, *470*
- compatible, *3, 166*
- complete
  - vector field, *69*
- completely reducible representation, *307*
- complex, *323*
  - analytic, *476*
  - curve, *476*
- complex manifold, *476*
- complex structure
  - almost, *475*
- cone, *52*
- conformal, *470, 475*
- conformal map, *474*
- conformal structure, *474*
- conjugate linear, *475*
- conjugation, *58*
- connected
  - simply, *291*
- connected sum, *36*
  - data, *36*
- connection, *505*
  - affine, *448*
  - affine, metric compatible, *448*
  - Levi–Civita, *449*
  - pullback, *506*
  - standard flat, *505*
- connection forms, *505*
- conormal, *150, 151, 177*
- conormal space, *151*
- continuously differentiable
  - piecewise, *291*
- contractible, *249*
- contraction, *249*
- convex, *42, 263, 266, 268, 466*
  - combination, *266*
  - function, *249*
  - hull, *266*
  - set, *249*
  - strongly, *466*
- convexity
  - geodesic, *466*
- convexity radius, *466*
  - strong, *466*

- convolution, [245](#)
- coordinate ball, [248](#)
- coordinates, [24](#), [166](#)
  - cylindrical, [216](#)
  - spherical, [216](#)
- corner chart, [166](#)
- cotangent
  - bundle, [147](#)
  - line, [145](#)
  - space, [143](#), [166](#)
- covariant derivative, [447](#), [448](#)
- covector, [143](#)
- cover
  - fine, [251](#)
  - goo, [346](#)
  - good, [343](#)
- covering map, [25](#), [100](#), [188](#)
  - Riemannian, [444](#)
- critical
  - point, [44](#), [169](#)
  - value, [44](#)
- critical point
  - index, [383](#)
- cube, [261](#)
- curl, [208](#)
- current, [349](#)
  - Dirac, [350](#)
- curvature
  - Gauss, [231](#)
  - of connection, [517](#)
- cycle, [236](#), [305](#)
- cylindrical coordinates, [216](#)
  
- death, [389](#)
- deformation retraction, [44](#)
- degree, [187](#)
- descend, [389](#)
- descending cell, [389](#)
- descent, [49](#), [132](#)
- determinant, [48](#), [85](#), [200](#), [201](#)
- determinantal manifold, [13](#)
- determinantal variety, [52](#), [162](#), [625](#)
- diffeomorphism, [8](#)
  - local, [25](#), [72](#)
  - support, [192](#)
- diffeotopic, [191](#)
- diffeotopy, [191](#)
  - support, [192](#)
- differential, [323](#)
- differential form, [200](#)
  - convergence, [349](#)
  - locally integrable, [478](#)
- differential relation, [369](#), [370](#)
- dilation, [461](#), [577](#)
  - infinitesimal, [444](#)
- dimension, [60](#)
  - of subset, [161](#), [403](#)
- disjoint union, [12](#)
- distance, [442](#)
- distribution, [126](#)
- divergence, [208](#)
- Dolbeault differential, [487](#)
- domain, [165](#)
- Dostoyevsky, Fyodor, [17](#)
- double pendulum, [1](#)
- double point, [9](#), [35](#)
- duality
  - Poincaré–Lefschetz, [333](#)
  
- eikonal equation, [457](#)
- Einstein
  - notation, [65](#), [505](#)
  - summation notation, [66](#)
- embedded submanifold, [12](#)
- embedding, [12](#), [56](#)
- equivariant, [87](#), [128](#)
- Euclidean
  - metric, [440](#)
- Euler characteristic, [324](#)
- Euler class, [527](#)
- Euler form, [527](#)
- evenly covered, [25](#)
- exact, [297](#)
  - form, [206](#)
  - sequence, [151](#), [179](#), [323](#)
    - Mayer–Vietoris, [326](#)
- exact sequence, [153](#)
  - orientation, [153](#)
- excellent
  - Morse function, [384](#)
- exhaustion, [158](#), [248](#)
- exotic sphere, [388](#)
- exponential map, [464](#)
  - Lie group, [99](#)
- extension theorem
  - Seely, [174](#)
- exterior power
  - vector bundle, [495](#)
- exterior product, [494](#)

- feuille, 117
- fiber, **240**
- fiber bundle, **407**, 413
  - base, **407**
  - homotopy, **423**
  - map, **407**
  - pullback, **422**
  - trivial, **407**
- fiber bundle morphism, **410**
- fiber product, **152**, **180**, **410**
- fine cover, **251**
- fine topology, **365**
- flow
  - box, **70**
  - line, **69**
  - of vector field, **69**, **77**
- foliation
  - Reeb, **119**
  - tangent bundle, **123**
  - tangent space, **123**
- form, **200**
  - closed, **206**
  - exact, **206**
- fraktur, 94
- free
  - group action, **29**
- Frobenius, **126**
- Fubini's theorem, **239**
- Fubini–Study form, **316**
- Fubini–Tonelli theorem, 156
- function
  - meromorphic, **485**
- fundamental
  - cocycle, **237**
  - cycle, **236**
  - lemma of Riemannian geometry, **451**
  - theorem of algebra, **94**, **102**, **190**, **222**
- fundamental theorem of calculus, 210, 243
- Gauss curvature, **231**
- Gauss map, **230**
- Gauss–Bonnet theorem, **231**
- Gaussian, **215**
- genus, **47**, 167, 231
- geodesic, **455**
  - minimal, **457**
- geodesic convexity, 466
- good cover, 343, 346
- gradient, **207**
- Grass, Günther, 93
- Grassmannian, **21**, **87**, 502
- Gronwall's inequality, **70**
- group
  - action, **29**
  - orthogonal, **62**
  - symplectic, **62**
  - unitary, **62**
- Hadamard's lemma, **383**
- hairy ball theorem, **193**, 370
- handle, **391**
- Heisenberg group, **97**, 99, 115
- hexagonal torus, **31**
- Hilbert–Poincaré series, **325**, 399
- holomorphic, **221**, **476**
- holonomy, **120**, **508**
- homogeneous space, **87**
- homology, **325**
  - Borel–Moore, **306**
- homology class, **305**
- homotopy, **120**, **190**, 278
  - of fiber bundles, **423**
  - vector bundle, **503**
- homotopy group, **401**
- Hopf fibration, **24**, 136, 424, **427**
- Hopf invariant, **428**
- Hopf manifold, **131**
- hyperbolic
  - metric, **441**
  - plane, 451
  - space, **441**
- hyperbolic plane, **474**
- ideal, **106**
- idempotent, **61**
- identity, Jacobi, **113**
- image, **60**
- immersed submanifold, **55**
- immersion, **55**
- implicit function theorem, **39**
  - with corners, **171**
- impulse response, **271**
- incidence relation, **129**
- index
  - critical point, **383**
  - of zero, **499**
- induced

- metric
  - Riemannian, **442**
- inequality
  - Cauchy–Schwarz, 593
- infinitesimal dilation, **444**
- initial
  - submanifold, **57**
- injectivity radius, **467**
- inner product, **61**
- inner product space, **62**
- integrable
  - plane field, **126**
- integral
  - manifold, **123**
  - of form, **209**
- interior
  - of a convex set, **267**
- intersection number, **154, 375**
- inverse function theorem, **39**
  - with corners, **171**
- inversion, **461**
- irreducible representation, **307**
- isometry, **84**
  - group, **440**
  - Riemannian, **440**
- isomorphism
  - Lie group, **86**
- isothermic
  - chart, **474**
- Jacobi identity, **78, 113**
- jet bundle, **370**
- Kähler form, **311**
- kernel, **60**
- Klein bottle, **29, 31, 67, 408**
- Künneth formula, **346**
- lacunary principle, **400**
- leaf space, **120**
- left invariant, **96**
- left multiplication, **96**
- left translation, **96**
- Leibnitz rule, **71, 185, 204, 210**
- lemma
  - fundamental, of Riemannian geometry, **451**
  - Poincaré, **243, 479**
- length, 439
  - Riemannian, **440**
- Levi–Civita connection, **449**
- Lie algebra, **96, 113**
  - abelian, **101**
  - action, **114**
  - morphism, **114**
  - semisimple, **127**
- Lie bracket, **78**
- Lie derivative, **317**
- Lie group, **83**
  - isomorphism, **86**
  - morphism, **86**
- Lie subgroup, **83**
- line
  - projective, **18**
- line bundle, **500**
- linear
  - combination, **59**
  - fractional transformation, **461**
  - isometry, **62**
  - map, **60**
  - relation, **59**
- linear fractional transformation, **111**
- linearly independent, **59**
- Lipschitz, **577**
  - locally, **577**
- local
  - parameterization, **2**
- local complement, **120**
- local diffeomorphism, **25**
- locally integrable, **478**
- locally Lipschitz, **577**
- long cycle, **306**
- loop, **291**
- Möbius strip, **32, 492**
- manifold, **11, 13**
  - complex, **476**
  - integral, **123**
  - Riemannian, **440**
  - topological, **10, 13**
  - with corners, **166**
- map
  - covering, 100, **188**
  - Riemannian covering, **444**
  - tidy, **169**
- maximal atlas, **8**
- Mayer–Vietoris sequence
  - homology, **376**
- mean, **265**
- measure zero, **155, 156**

- meromorphic function, **485**
- metric, **442**
  - Euclidean, **440**
  - vector bundle, **500**
- metrizable, **13**, **443**
- mille-feuille, **117**
- Milnor sphere, **387**
- minimal geodesic, **457**
- model
  - trivial vector bundle, **492**
- Möbius strip, **26**, **408**, **513**
- morphism
  - fiber bundle, **410**
  - Lie algebra, **114**
  - Lie group, **86**
  - of complexes, **323**
  - of homogeneous spaces, **128**
  - principal, **411**
- Morse chart, **383**
- Morse function, **372**
  - excellent, **384**
- Morse inequalities, **399**
- multiplication laws, **59**
- multiplicity, **222**, **513**
- neighborhood
  - strong, **365**
- nilpotent, **114**
- normal bundle, **43**, **75**, **491**
- normal map, **43**
- normal space, **75**
- normalizer, **105**
- null homotopic, **466**
- one parameter subgroup, **100**
- open
  - cover, **247**
    - locally finite, **247**
  - set, **166**
- orbit, **29**, **87**, **124**
- order
  - of a current, **351**
- ordered, **384**
- orientation, **26**
  - exact sequence, **153**
  - of a curve, **26**
  - of a surface, **26**
  - vector space, **153**
- oriented
  - vector bundle, **498**
- orthogonal group, **62**
- orthonormal
  - coframing, **449**
- paper, **227**
- parallel
  - parking, **81**
  - section, **506**
- parallel transport, **448**, **507**
- parameterization
  - local, **2**
- parking, parallel, **81**
- partition, **525**
  - alphabetical order, **526**
  - associated to a permutation, **525**
- partition of unity, **211**, **257**
- path
  - vector field along, **455**
- pendulum, **1**
  - double, **1**
- permutation
  - natural, of partition, **526**
- Picard's theorem, **69**
- piecewise
  - continuously differentiable, **291**
- plane field, **123**
  - bracket closed, **123**
  - integrable, **126**
- Poincaré lemma, **243**, **479**
- Poincaré–Lefschetz duality, **333**
- point
  - critical, **44**, **169**
  - regular, **44**, **169**
- pole, **485**
- positively oriented basis, **53**
- potential, **287**
- precompact, **240**, **248**
- pregradient vector field, **386**
- principal bundle, **411**, **411**
- principal bundle morphism, **411**
- projection, **6**, **146**, **147**
- projective line, **18**
- projective linear group, **138**
- projective plane, **5**
- projective space, **20**
- projectivised cotangent bundle, **147**
- projectivised cotangent space, **146**
- proper, **56**
  - action, **411**
  - group action, **29**

- map, [187](#)
- Ptolemaic projection, [7](#)
- pullback, [203](#)
  - connection, [506](#)
  - fiber bundle, [422](#)
  - form, [203](#)
  - metric, [442](#)
  - vector bundle, [494](#)
- pushforward, [239](#)
  - of a current, [353](#)
- quaternionic Kaehler form
  - quaternionic Kähler form, [314](#)
- quaternions, [57](#)
- quotient map, [129](#)
- quotient space, [129](#)
- radius
  - compactness, [470](#)
- rank theorem, [40](#), [50](#)
- real
  - projective space, [145](#)
- reductive
  - Lie group, [319](#)
- regular
  - point, [44](#), [169](#)
  - value, [44](#), [169](#)
- relative integral, [338](#)
- relatively
  - compact, [248](#)
- representation, [88](#)
  - completely reducible, [307](#)
  - irreducible, [307](#)
- residue, [486](#)
- retract
  - smooth, [44](#)
- Riemann sphere, [5](#)
- Riemann surface, [476](#)
- Riemannian
  - covering map, [444](#)
  - geometry, [442](#)
  - isometry, [440](#)
  - manifold, [440](#)
  - metric, [440](#)
    - induced, [442](#)
- right action, [129](#)
- right invariant, [96](#)
- right multiplication, [96](#)
- right translation, [96](#)
- rigid motion, [440](#)
- rim, [391](#)
- ring
  - thin, [248](#)
- ripple, [249](#), [280](#)
- rose, [214](#)
- rotation, [84](#)
  - matrix, [84](#)
- Sard's theorem, [158](#), [160](#), [171](#)
- Schwartz function, [482](#)
- section, [123](#), [369](#), [413](#), [491](#)
- Seeley
  - extension theorem, [174](#)
- semisimple
  - Lie algebra, [127](#)
- shear, [565](#)
- sheet, [25](#), [188](#), [235](#), [444](#)
- sign
  - of map between manifolds of equal dimension, [187](#)
- simplex, [167](#)
- simply connected, [291](#)
- simultaneous flow box
  - theorem, [82](#)
- skating, [80](#)
- smile, [213](#)
- smooth, [2](#), [23](#), [24](#), [166](#)
  - deformation retraction, [44](#)
  - spectral theorem, [33](#)
  - structure, [8](#), [166](#)
- smooth retract, [44](#)
- smoothing homotopy, [277](#)
- solid angle form, [224](#), [226](#), [287](#)
- solvable, [114](#)
- span, [59](#)
- spectral theorem
  - smooth, [33](#)
- spherical coordinates, [216](#)
- stabiliser subgroup, [129](#)
- stable, [282](#), [283](#)
- stable manifold, [400](#)
- standard corner, [165](#)
- standard flat connection, [505](#)
- standard homotopy, [520](#)
- star-shaped, [42](#), [263](#)
- Stokes's theorem, [219](#)
- strictly convex, [466](#)
- strong
  - neighborhood, [365](#)
  - topology, [365](#)

- strong convexity radius, **466**
- strongly convex, **466**
- subbundle
  - vector, **493**
- subgroup
  - Lie, **83**
  - one parameter, **100**
- sublevel set, **250, 268**
- submanifold
  - embedded, **12**
  - immersed, **55**
  - initial, **57**
  - weakly embedded, **57**
- submersion, **44, 169**
- submersion connection
  - trivial, **417**
- subordinate
  - partition of unity, **211**
- subset dimension, **161, 403**
- subspace, **59**
- support, **23**
  - diffeomorphism, **192**
  - diffeotopy, **192**
  - of a current, **350**
- Sütterlin, **94**
- symmetric power
  - vector bundle, **495**
- symmetric product, **494**
- symplectic group, **62**
- tangent
  - bundle, **146, 578**
  - foliation, **123**
  - space, **41**
  - foliation, **123**
  - vector, **41**
- tangent space, **166**
- tangent vector, **166**
- test form, **349**
- test function, **482**
- tetrahedron, **266**
- The Tin Drum, **93**
- theorem
  - Ado, **114**
  - Borel, **173**
  - Brouwer fixed point, **171, 226, 277**
  - Cauchy integral, **221**
  - Cauchy integral with remainder, **478**
  - flow box, **70**
  - Frobenius, **123**
  - Fubini, **239**
  - Fubini–Tonelli, **156**
  - fundamental, of algebra, **94, 102, 190, 222**
  - fundamental, of calculus, **210, 243**
  - Gauss–Bonnet, **231**
  - hairy ball, **193, 370**
  - implicit function, **39, 171**
  - inverse function, **39, 171**
  - Picard, **69**
  - rank, **40, 50**
  - Sard, **158, 160, 171**
  - Seeley extension, **174**
  - simultaneous flow box, **82**
  - Stokes’s, **219**
  - Thom transversality, **370**
  - Whitney decomposition, **261**
  - Whitney embedding, **258, 259**
  - Whitney immersion, **258, 372**
- thin ring, **248**
- Thom transversality, **370**
- tidy, **275**
  - map, **169**
- topological manifold, **10**
  - broad sense, **13**
- topologist’s sine curve, **56**
- topology
  - fine, **365**
  - strong, **365**
  - Whitney, **365**
- torsion, **329, 449**
- torus
  - action, **301**
  - hexagonal, **31**
- total Chern
  - character, **522**
  - class, **522**
- transition map, **166, 419**
- translation, **461**
  - left, **96**
  - right, **96**
- transverse, **150, 152, 162, 181**
  - submanifolds, **178**
- trivial
  - vector bundle, **493**
  - model, **492**
- trivialization, **407**
- Trotter product formula, **104**
- twisted handle, **397**



- umbrella, Whitney, *48, 53, 162*
- unitary group, *62*
- universal bundle, *502*
- unstable manifold, *400*
  
- value
  - critical, *44*
  - regular, *44, 169*
- vector bundle, *491*
  - metric, *500*
  - oriented, *498*
  - pullback, *494*
  - trivial, *493*
- vector field, *66, 166*
  - along a path, *455*
  - commuting, *79*
  - complete, *69*
- vector subbundle, *493*
- velocity, *41*
- volume form, *265*
  
- weakly embedded submanifold, *57*
- wedge product, *201*
- Whitney
  - decomposition, *261, 263*
  - decomposition theorem, *261*
  - embedding theorem, *258, 259*
  - immersion theorem, *258, 372*
  - topology, *365*
  - umbrella, *48, 53, 162*
  
- zero section, *491*