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HOLOMORPHIC GEOMETRIC STRUCTURES ON KÄHLER–EINSTEIN MANIFOLDS

BENJAMIN MCKAY

Abstract. We prove that the compact Kähler manifolds with $c_1 \geqslant 0$ that admit holomorphic parabolic geometries are the flat bundles of rational homogeneous varieties over complex tori. We also prove that the compact Kähler manifolds with $c_1 < 0$ that admit holomorphic cominiscule geometries are the locally Hermitian symmetric varieties.

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1. Introduction

Parabolic geometries are geometric structures modelled on generalized flag varieties; for example: projective connections are modelled on projective space, while conformal connections are modelled on the sphere with its standard conformal structure. Parabolic geometries arise in Penrose's twistor programme, in hypercomplex analysis, in the study of the large scale behaviour of geodesics, in conformal geometry and in CR geometry; see [10] for a comprehensive introduction to parabolic geometries. There is a long literature on the obstruction theory of various types of holomorphic geometric structure on compact complex manifolds; see [26] for a survey of some of these results, many concerning holomorphic parabolic geometries. However, there is as yet no classification, or conjectured classification, of the smooth projective varieties admitting holomorphic parabolic geometries.

Definitions of our terminology appear in section 2. We will classify all holomorphic parabolic geometries on compact Kähler manifolds with $c_1 \geqslant 0$; these turn out to be constructed from flat bundles on complex tori. This motivates the problem of classification for $c_1 < 0$; we will only solve part of this problem. Every locally Hermitian symmetric variety has a holomorphic cominiscule geometry. We prove that a compact complex manifold M with $c_1 < 0$ admits a holomorphic cominiscule geometry just when M is a locally Hermitian symmetric variety, and we prove that the moduli space of cominiscule geometries on M is a finite dimensional complex vector space. Moreover, we find that the normal cominiscule geometries on locally Hermitian symmetric varieties are precisely the obvious flat ones.

There is a subtlety in the study of holomorphic Cartan geometries because we need to be careful about holomorphic moduli of principal bundles, an issue which does not arise in the real smooth category where these geometric structures have been thoroughly studied. Principal bundles of holomorphic Cartan geometries with a fixed model and defined on a fixed complex manifold can vary in the moduli space of holomorphic principal bundles [35]. We prove that cominiscule geometries with fixed underlying first order structure all have isomorphic principal bundles. We prove that any holomorphic parabolic geometry with reducible model on any smooth projective variety gives that variety a holomorphic local product structure.

The classification of normal holomorphic cominiscule geometries on Kähler–Einstein manifolds is known for geometries with certain models [28, 29, 30, 31]. Similarly, the rigidity of *flat* holomorphic cominiscule geometries with irreducible models on locally Hermitian symmetric varieties is known [23]. We largely follow the methods of those papers. After this paper was released on the arXiv, Antonio Di Scala pointed out that it is similar to [11], which had appeared earlier on the arXiv.

We outline the arguments of the paper. If a compact Kähler manifold M has $c_1 \leqslant 0$, it is known to be a holomorphic bundle of rational homogeneous varieties over a complex torus, up to finite covering. In a previous paper [4], we explained how to reduce the classification problem of holomorphic parabolic geometries on compact

Kähler manifolds to the same problem on those which contain no rational curves. In section 3, we apply that work to classify holomorphic parabolic geometries on M.

We develop an obstruction theory to cominiscule geometries on any complex manifold in sections 4, 5, reducing the problem to the construction of holomorphic tensors satisfying suitable linear relations in each tangent space, and the vanishing of a certain cocycle in the sheaf cohomology of a certain associated vector bundle.

Take a compact Kähler manifold M with $c_1 < 0$ and a holomorphic cominiscule geometry, i.e. modelled on a cominiscule variety (X, G). In section 6, we prove that every holomorphic cominiscule geometry on M splits TM analogously to each G-invariant splitting of TX. In section 8, we show that this splitting splits every Kähler–Einstein metric on M, by using the cominiscule geometry to generate information about Chern classes of the tangent bundle, and then using the expression of those Chern classes in the Kähler–Einstein metric.

In section 9, we quote a theorem of Klingler and Mok which ensures rigidity of the flat cominiscule geometry associated to a locally Hermitian symmetric variety. Finally, in section 10, we classify all holomorphic cominiscule geometries on Kähler–Einstein manifolds.

2. Definitions

In this section, we give all of the definitions of the less familiar terms. The definition of a Cartan geometry on a subbundle of the tangent bundle is new, as is the definition of a G-structure on such a subbundle. The other definitions are standard.

2.1. **Cartan geometries.** Sharpe [40] gives an introduction to Cartan geometries. If $E \to M$ is a principal right G-bundle, we will write the right G-action as $r_g e = eg$. Suppose that M is a manifold, $V_M \subset TM$ a vector bundle and $\pi\colon E \to M$ a principal bundle. Let

$$V_E = \left\{ (e, v) \;\middle|\; e \in E, v \in T_e E, \pi'(e)v \in V_M \right\}.$$

Let $H \subset G$ be a closed subgroup of a Lie group, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ and let $X = G/H$. A (X, G)-geometry, or G/H-geometry, or *Cartan geometry* modelled on (X, G), on a vector subbundle $V_M \subset TM$ on a manifold M is a choice of principal right H-bundle $E \to M$, and section ω of $V_E^* \otimes \mathfrak{g}$, called the *Cartan connection*, which satisifies all of the following conditions:

(1) $r_h^* \omega = \mathrm{Ad}_h^{-1} \omega$ for all $h \in H$.
(2) $\omega_e\colon V_e E \to \mathfrak{g}$ is a linear isomorphism at each point $e \in E$.
(3) For each $A \in \mathfrak{g}$, define a vector field $\vec{A}$ on E by the equation $\vec{A} \lrcorner \omega = A$; the vector fields $\vec{A}$ for $A \in \mathfrak{h}$ generate the right H-action.

A *Cartan geometry on a manifold* M means a Cartan geometry on $V_M = TM$. Similarly a Cartan geometry on a nonsingular foliation F means a Cartan geometry on the tangent bundle of F. For example, the bundle $G \to X = G/H$ is the total space of a Cartan geometry on X, with Cartan connection $\omega = g^{-1}\, dg$ the left invariant Maurer–Cartan 1-form on G; this geometry is called the *model Cartan geometry*.

If $E_0 \to M_0$ and $E_1 \to M_1$ are Cartan geometries with models (G_0, X_0) and (G_1, X_1) and Cartan connections ω_0 and ω_1, their *product geometry* is the Cartan geometry with bundle $E_0 \times E_1 \to M_0 \times M_1$ and Cartan connection $\omega_0 + \omega_1$.

Suppose that (X, G) is a homogeneous space and let H be the stabilizer of some point of X. Suppose that $X = \prod_i X_i$ splits G-equivariantly and let $K_i \subset G$ be the subgroup of G acting trivially on X_i. Let $G_i = G/K_i$ be the quotient of G acting effectively on X_i and $H_i \subset G_i$ the stabilizer of the associated point of X_i. Pick a (X, G)-geometry $\pi \colon E \to M$ on a subbundle $V = V_M$ on some manifold M. For each point $e \in E$ over any point $m \in M$, the Cartan connection ω induces a map $\omega + \mathfrak{h} \colon T_m M \to \mathfrak{g}/\mathfrak{h}$. The splitting $X = \prod_i X_i$ defines a splitting $V = \bigoplus_i V_i$ by the requirement that $\omega + \mathfrak{h}$ take V_i to $\mathfrak{g}_i/\mathfrak{h}_i$. Let $E_i = E/K_i$ and $\omega_i = \omega + \mathfrak{h}_i$: then $E_i \to M$ is a (X_i, G_i)-geometry on V_i.

On the other hand, suppose that various Lie groups G_i each act smoothly, transitively and effectively on various homogeneous spaces X_i and we let $X = \prod X_i$ and $G = \prod_i G_i$. Suppose that M bears vector bundles $V_i \subset TM$ and $V_i \cap V_j = \{0\}$ when $i \neq j$ and each V_i has a (X_i, G_i)-geometry $\pi_i \colon E_i \to M$. Let $V = \bigoplus V_i \subset TM$ and let E be the product bundle of the $E_i \to M$. Add up the Cartan connections to produce a (X, G)-geometry on V, the *product geometry*.

2.2. **Curvature.** If $E \to M$ is a Cartan geometry over a vector bundle $V_M \subset TM$, let $I = V_M^\perp \subset T^*M$ and form the sheaf I' of 2-forms which are locally divisible either by local sections of I or by exterior derivatives of local sections of I. Locally extend ω to an H-equivariant 1-form valued in $\mathfrak{g}$; the *curvature* of $E \to M$ is $d\omega + \frac{1}{2}[\omega, \omega] \in \left(\Omega^2(M)/I'\right) \otimes_H \mathfrak{g}$. In particular, the curvature of a Cartan geometry on a foliation is just the usual notion of curvature of a Cartan geometry on the leaves of the foliation.

2.3. **First order structures.** Suppose that V_0 is a vector space. The V_0-valued *frame bundle* FV of a vector subbundle $V = V_M \subset TM$ on a manifold M is the associated principal bundle, i.e. the set of all pairs (m, u) with $m \in M$ and $u \colon V_m \to V_0$ a linear isomorphism. Let $\pi \colon FV \to M$ be the map $(m, u) \to m$. The frame bundle is a principal right $\mathrm{GL}(V_0)$-bundle under the obvious action $r_g(m, u) = \left(m, g^{-1}u\right)$. Let $V_{FV_M} = (\pi')^{-1} V_M$. Define a section σ of $V^*_{FV_M} \otimes V_0$ on FV_M by $v \lrcorner \sigma = u\left(\pi'(m, u)v\right)$, where $v \in V_{(m,u)} FV_M$.

Suppose that $G \to \mathrm{GL}(V_0)$ is a morphism of Lie groups and that $V_M \subset TM$ is a subbundle. A *G-structure* or *first order structure* on V_M is a principal right G-bundle $B \to M$ together with a G-equivariant bundle map $B \to FV_M$. If G_0 is the kernel of the morphism $G \to \mathrm{GL}(V_0)$, then $B/G_0 \to FV$ is the *underlying immersed first order structure*, and is a submanifold of FV_M.

2.4. **Underlying first order structures.** Let $X = G/H$ be a homogeneous space. Let M be a manifold with a vector bundle $V_M \subset TM$, and $\pi \colon E \to M$ a (X, G)-geometry on V_M. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. Let $V_0 = \mathfrak{g}/\mathfrak{h}$. Let FV_M be the V_0-valued frame bundle. Let $\sigma = \omega + \mathfrak{h}$, a semibasic 1-form defined on V_E and valued in V_0. At each point $e \in E$ the 1-form σ determines a linear isomorphism $u \colon V_{M,m} \to V_0$ by the equation $u(\pi'(e)v) = v \lrcorner \sigma$. Map $e \in E \mapsto u = u(e) \in FV_M$. This map is an H-structure. The fibers of this map consist of the orbits in E of the subgroup H_1 of H acting trivially on $V_0 = \mathfrak{g}/\mathfrak{h}$. The map descends to a map $E/H_1 \to FV_M$ called the *underlying (H/H_1)-structure* or *underlying first order structure* of the (X, G)-geometry.

2.5. **Holomorphy.** From now on, all Cartan geometries in this paper will be assumed to be holomorphic Cartan geometries, i.e. G is a complex Lie group, X is

a complex manifold on which G acts holomorphically, effectively and transitively, $E \to M$ is a holomorphic principal bundle on a complex manifold M, $V_M \subset M$ is a holomorphic vector bundle, and the Cartan connection is holomorphic. Isomorphisms of Cartan geometries will therefore be biholomorphic.

2.6. **Rational homogeneous varieties and parabolic geometries.** A *rational homogeneous variety* is a pair (X, G) where G is a complex semisimple Lie group acting effectively, transitively and holomorphically on a compact complex manifold X. A *parabolic geometry* is a Cartan geometry modelled on a rational homogeneous variety. By a theorem of Chevalley [5] theorem 11.16 p. 154, if G is a complex semisimple Lie group, then every parabolic subgroup $P \subset G$ (i.e. containing a maximal solvable subgroup) is connected and equal to its normalizer in G and has compact quotient $X = G/P$ and every rational homogeneous variety arises this way. The center of G acts trivially on X, so without loss of generality G is in adjoint form.

If (X, G) is a rational homogeneous variety then, up to reordering factors, there is a unique decomposition $G = \prod_i G_i$ and $X = \prod_i X_i$ into products so that each G_i acts trivially on all X_j except X_i and (X_i, G_i) is a rational homogeneous variety which cannot be thus decomposed, i.e. is irreducible.

2.7. **Cominiscule varieties.** A *cominiscule variety* is a rational homogeneous variety (X, G) so that the stabilizer P of a point $x_0 \in X$ acts on $T_{x_0}X$ as a sum of irreducible representations; see [2, 32]. Each cominiscule variety (X, G) admits a Kähler metric, invariant under a maximal compact subgroup of G, in which X becomes a compact Hermitian symmetric space. The group G in our definition is a complex group of biholomorphisms of $M = G/P$; G does *not* preserve any metric on G/P. For example, if $G/P = \mathbb{P}^n$, we will have $G = \mathrm{PSL}(n+1, \mathbb{C})$, *not* $\mathbb{P}\mathrm{U}(n+1)$. For this reason, we use the term *cominiscule variety* rather than *compact Hermitian symmetric space*.

Example 1. Suppose that $X = G/P$ is a cominiscule variety. Denote by $X' = G'/P'$ its dual noncompact Hermitian symmetric space [18], i.e. $G' \subset G$ is a real form acting on X, and $P' = P \cap G'$ is compact, and G' acts on X with an open orbit $X' = G'/P'$. Pullback the standard flat (X, G)-geometry on X to a flat (X, G)-geometry on X'. If $\Gamma \subset G'$ is a cocompact lattice, then the flat (X, G)-geometry on X' is pulled back from a unique flat (X, G)-geometry on the manifold $M = \Gamma\backslash X'$, called the *standard geometry* on M.

Example 2. If $(X, G) = (\mathbb{P}^n, \mathrm{PSL}(n+1, \mathbb{C}))$ then any holomorphic (X, G)-geometry is called a *holomorphic projective connection*. The space of holomorphic projective connections on any Riemann surface is canonically identified with the space of holomorphic quadratic differentials [33].

Example 3. Call a holomorphic cominiscule geometry *standard* if, after perhaps replacing by the pull back to a finite étale covering space, it becomes a product geometry of the form $M = M' \times \prod_j M_j$, where each M_j is a compact Riemann surface with a holomorphic projective connection and $M' = \Gamma\backslash X'$ is the standard geometry, for X' the noncompact dual of a cominiscule variety.

Theorem 1. *Suppose that M is a connected compact complex manifold with $c_1(M) < 0$ bearing a holomorphic cominiscule geometry. Then M is a locally Hermitian symmetric variety. All holomorphic cominiscule geometries on M have*

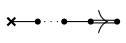
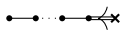
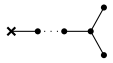
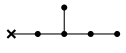
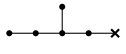
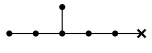
A_n		Grassmannian of k-planes in $\mathbb{C}^{n+1}$
B_n		$(2n-1)$-dimensional hyperquadric, i.e. the variety of null lines in $\mathbb{C}^{2n+1}$
C_n		space of Lagrangian n-planes in $\mathbb{C}^{2n}$
D_n		$(2n-2)$-dimensional hyperquadric, i.e. the variety of null lines in $\mathbb{C}^{2n}$
D_n		one component of the variety of maximal dimension null subspaces of $\mathbb{C}^{2n}$
D_n		the other component
E_6		complexified octave projective plane
E_6		its dual plane
E_7		the space of null octave 3-planes in octave 6-space

FIGURE 1. The classification of irreducible cominiscule varieties; every cominiscule variety is a product of these.

the same model. The moduli space of holomorphic cominiscule geometries on M is a finite dimensional complex vector space. The normal holomorphic cominiscule geometries on M form a finite dimensional linear subspace. Every holomorphic normal cominiscule geometry on M is standard. If no finite covering space of M splits into a product $M_1 \times M_2$ with M_1 a compact Riemann surface, then every holomorphic cominiscule geometry on M is normal.

Theorem 1 is a consequence of theorem 8 on page 27.

2.8. **Roots.** Helgason [18] and Knapp [24] both prove all of the results we will use on Lie algebras; we give some definitions only to fix notation; also see Landsberg [32]. Suppose that G/P is a cominiscule variety. Fix a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be the Borel subalgebra, i.e. the subalgebra consisting of the sum of the Cartan subalgebra and all of the positive root spaces. The Lie algebra $\mathfrak{p}$ can be made to contain $\mathfrak{b}$ by conjugation [14] p. 382; assume without loss of generality from now on that $\mathfrak{b} \subset \mathfrak{p}$. For each root α, let $\mathfrak{g}_\alpha$ be the root space of the root α. Write $\mathfrak{g}$ as a sum of simple Lie algebras $\mathfrak{g} = \bigoplus_i \mathfrak{g}^i$. Let $\mathfrak{p}^i = \mathfrak{g}^i \cap \mathfrak{p}$. Then $\mathfrak{p} = \bigoplus_i \mathfrak{p}^i$. A root α is *compact* if both $\mathfrak{g}_\alpha \cap \mathfrak{p} \neq \{0\}$ and $\mathfrak{g}_{-\alpha} \cap \mathfrak{p} \neq \{0\}$. Each $\mathfrak{g}^i$ has a unique noncompact simple root α^i. Write a noncompact root of $\mathfrak{g}$ as β^+ or β^- to indicate that it is a noncompact positive or noncompact negative root. If we write a root without a superscript, e.g. β, this means that it is allowed to be any of the above, or even a compact root. If β^- is any (necessarily noncompact negative) root, we will write β^+ to mean $-\beta^-$, etc.

2.9. **The grading.** Grade $\mathfrak{g}$ into a sum: $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, where $\mathfrak{g}_-$ is the sum of all noncompact negative root spaces, $\mathfrak{g}_+$ is the sum of all noncompact positive root spaces, and $\mathfrak{g}_0$ is the sum of the Cartan subalgebra of $\mathfrak{g}$ with all of the compact root spaces, so $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Each of the Lie algebras $\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_+$ is the Lie algebra of a

connected algebraic Lie subgroup G_-, G_0, G_+ of G. The subgroups G_- and G_+ are abelian and simply connected (see Knapp [24] theorem 7.129 p. 506). This grading of $\mathfrak{g}$ is only G_0-invariant, but the associated filtration is P-invariant and $P = G_0 G_+$. Identify $\mathfrak{g}_-^* = \mathfrak{g}_+$ using the Killing form. If $X_+ \in \mathfrak{g}_+$ and $Y = Y_- + Y_0 + Y_+ \in \mathfrak{g}$ then

$$\operatorname{Ad}\left(e^{X_+}\right) Y = Y_- + (Y_0 - [X_+, Y_-]) + Z_+ \tag{1}$$

where

$$Z_+ = Y_+ - [X_+, Y_0] + \frac{1}{2}[X_+, [X_+, Y_-]]\,.$$

2.10. **Chevalley bases.** A *Chevalley basis* X_α, H_α (see Serre [39]) is a basis of $\mathfrak{g}$ parameterized by roots $\alpha \in \mathfrak{h}^*$ (with $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra) for which

(1) $[H, X_\alpha] = \alpha(H) X_\alpha$ for each $H \in \mathfrak{h}$
(2) $\alpha(H_\beta) = 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}$ (measuring inner products via the Killing form)
(3) $[H_\alpha, H_\beta] = 0$,
(4)

$$[X_\alpha, X_\beta] = \begin{cases} H_\alpha, & \text{if } \alpha + \beta = 0, \\ N_{\alpha\beta} X_{\alpha+\beta}, & \text{otherwise} \end{cases}$$

with
  (a) $N_{\alpha\beta}$ an integer,
  (b) $N_{-\alpha,-\beta} = -N_{\alpha\beta}$,
  (c) If α, β, and $\alpha + \beta$ are roots, then $N_{\alpha\beta} = \pm(p+1)$, where p is the largest integer for which $\beta - p\,\alpha$ is a root,
  (d) $N_{\alpha\beta} = 0$ if $\alpha + \beta = 0$ or if any of α, β, or $\alpha + \beta$ is not a root.

3. Classification of holomorphic parabolic geometries with $c_1 \geqslant 0$

We give some examples of holomorphic parabolic geometries, and then prove (largely by citation) that these examples exhaust the holomorphic parabolic geometries with $c_1 \geqslant 0$.

Example 4. Suppose that $\bar{P} \subset \bar{Q}$ is a parabolic subgroup of a complex semisimple Lie group $\bar{G}$ and let $\bar{X} = \bar{G}/\bar{P}$. Denote the Lie algebra of $\bar{P}$ as $\bar{\mathfrak{p}}$. Take a complex linear $\bar{\mathfrak{p}}$-complementary subspace $V \subset \bar{\mathfrak{g}}$, letting $s \in V^* \otimes \mathfrak{g}$ be the inclusion map. Take a lattice $\Lambda \subset V$, let $\bar{M} = V/\Lambda$, and let $\bar{E} = (V/\Lambda) \times \bar{P}$ with elements written as $(z + \Lambda, \bar{p})$. Let $\bar{\omega} = \bar{p}^{-1}\, d\bar{p} + \operatorname{Ad}(\bar{p})^{-1} s$. Then $\bar{E} \to \bar{M}$ is a translation invariant holomorphic (X, G)-geometry on a complex torus, with holomorphic Cartan connection $\bar{\omega}$.

Example 5. Continuing with the previous example, suppose that G_0 is a complex semisimple Lie group and pick parabolic subgroups $Q_0 \subset G_0$ and $\bar{Q} \subset \bar{P}$. Let $G = G_0 \times \bar{G}$, $P = Q_0 \times \bar{Q}$ and $X = G/P$. Take any group morphism $\rho\colon \Lambda \to G_0$. Let $E' = V \times_\rho G_0 \to \bar{M}$ be the associated flat G_0-bundle. Let ω_0 be the holomorphic flat connection on $E' \to \bar{M}$. Let $E = E' \times_{\bar{M}} \bar{E}$, with obvious $G_0 \times \bar{P}$-action. Write the pullback form of ω_0 to E also as ω_0 and similarly write the pullback form of $\bar{\omega}$ to E also as $\bar{\omega}$. Let $\omega = \omega_0 + \bar{\omega}$ and let $M = E/P$. Then $E \to M$ is a holomorphic (X, G)-geometry with Cartan connection ω. Moreover, $M \to \bar{M}$ is a holomorphic fiber bundle with all fibers biholomorphic to $(G_0/Q_0) \times \left(\bar{P}/\bar{Q}\right)$. In particular, $c_1(M) \geqslant 0$.

Theorem 2. *Suppose that M is a connected compact Kähler manifold with $c_1 \geqslant 0$ and (X, G) is a rational homogeneous variety with $\dim M = \dim X$. Then M admits a holomorphic (X, G)-geometry if and only if, after perhaps replacing M by a finite étale covering space,*

1. *$M = X$ with the standard flat (X, G)-geometry or*
2. *M is a complex torus and the holomorphic (X, G)-geometry is as constructed in example 4 on page 7 or*
3. *M, with its holomorphic (X, G)-geometry, is constructed as in example 5 on page 7.*

Proof. By a theorem of Campana, Demailly and Peternell [8], every compact Kähler manifold M with $c_1 \geqslant 0$ is, up to a finite étale covering space, a holomorphic fiber bundle, with rationally connected fibers, over a compact Kähler base with $c_1 = 0$, so without loss of generality assume that M is such a bundle $M \to M'$.

If M also admits a holomorphic parabolic geometry, say modelled on a rational homogeneous variety (X, G), $X = G/P$, then M is a holomorphic fiber bundle with rational homogeneous fibers $M \to \bar{M}$ over a compact Kähler manifold $\bar{M}$ which contains no rational curves, and the holomorphic (X, G)-geometry on M drops to a holomorphic parabolic geometry on $\bar{M}$ [4] theorem 2 p. 4. This means precisely that there is a surjective morphism of complex semisimple Lie groups $G \to \bar{G}$, say with kernel G_0, with both G and $\bar{G}$ in adjoint form, carrying P to lie inside a complex parabolic subgroup $P \to \bar{P} \subset \bar{G}$, and the bundle $P \to E \to M$ of the parabolic geometry maps equivariantly $E \to \bar{E}$ to a holomorphic $\bar{P}$-bundle $\bar{P} \to \bar{E} \to \bar{M}$, and there is a holomorphic Cartan connection $\bar{\omega}$ on $\bar{E}$ which pulls back to $\omega + \mathfrak{g}_0$ and ω is a Cartan connection for $E \to \bar{M}$.

Since all rational curves in M live in the fibers of $M \to \bar{M}$, the map $M \to \bar{M}$ drops to a holomorphic fiber bundle mapping $M' \to \bar{M}$. The fibers of $M' \to \bar{M}$ have preimages in M which are the fibers of $M \to \bar{M}$, and so are rationally connected. Any immersed rational curve $C \subset F$ in a fiber $F \subset M' \to \bar{M}$ has preimage a smooth subvariety of M which lies in a fiber of $M \to \bar{M}$, so there is a copy of this variety in every fiber of $M \to \bar{M}$, so this variety has deformations moving it around in M, and so its images in M' allow that rational curve to deform freely, i.e. the ambient tangent bundle $TM'|_C$ is a sum of line bundles of nonnegative degree, one of which, TC, has positive degree. But the sum of these degrees must be $c_1(M') = 0$, a contradiction unless there are no rational curves in M', i.e. $M' = \bar{M}$.

Since $c_1\big(\bar{M}\big) = 0$ and $\bar{M}$ admits a holomorphic parabolic geometry, it follows [36] that, after perhaps replacing $\bar{M}$ by a finite étale covering space (and thereby replacing M by the pullback bundle), $\bar{M}$ is a complex torus. Moreover the parabolic geometry on $\bar{M}$ is obtained by the process given in example 5 on page 7 [36].

Because G is semisimple, the Lie algebra $\mathfrak{g}_0$ of G_0 is a sum of irreducible G-modules, i.e. of simple ideals of G, and we can inductively replace any expression of $\mathfrak{g}$ as a sum of simple ideals with an expression in which the ideals are drawn from $\mathfrak{g}_0$ or from a $\mathfrak{g}_0$-complementary G-invariant sum of simple ideals, which is then isomorphic to $\bar{\mathfrak{g}}$, so $\mathfrak{g} = \mathfrak{g}_0 \oplus \bar{\mathfrak{g}}$. Since G and $\bar{G}$ are in adjoint form, so is G_0, and therefore $G = G_0 \times \bar{G}$. We need to have P contained in the preimage of $\bar{P}$, i.e. in $G_0 \times \bar{P}$, and to be parabolic, so $P = Q_0 \times \bar{Q}$ for some parabolic subgroups $Q_0 \subset G_0$ and $\bar{Q} \subset \bar{P}$. Since ω is a Cartan connection for the principal bundle $G_0 \times \bar{P} \to E \to \bar{M}$, ω_0 is a flat connection on the bundle $G_0 \to E' = E/\bar{P} \to \bar{M}$.

Clearly we can recover the principal bundle $E \to \bar{M}$ from the two principal bundles $E' \to \bar{M}$ and $\bar{E} \to \bar{M}$ as $E = E' \times_{\bar{M}} \bar{E}$. We can recover the Cartan connection on E by pulling back the flat connection from E' and the Cartan connection from $\bar{E}$, and with obvious $G_0 \times \bar{P}$-action. We can recover M as $M = E/\left(Q_0 \times \bar{Q}\right)$. Therefore we can just pick any holomorphic principal G_0-bundle $E' \to \bar{M}$ with any flat holomorphic connection, and pick any holomorphic parabolic geometry on $\bar{M}$, and build a complex manifold M and its parabolic geometry as above.

Holomorphic G_0-bundles $E' \to \bar{M}$ with holomorphic flat connection are classified up to isomorphism by conjugacy classes of representations of the fundamental group, i.e. group morphisms $\Lambda \to G_0$, so the resulting geometry is constructed as in example 5 on page 7. □

4. Invariant tensors on cominiscule varieties

In this section, we construct a collection of invariant holomorphic tensors on every cominiscule variety (X, G). We associate to each of these tensors and to each (X, G)-geometry $E \to M$ an analogous tensor on M. We then prove that these tensors determine the underlying first order structure of the cominiscule geometry. We will later explain how to construct every cominiscule geometry as some additional information added to its first order structure.

4.1. **The fundamental tensor of a cominiscule variety.** There is a well known invariant holomorphic tensor on any cominiscule variety G/P; see Kobayashi and Ochiai [30]. Take any elements $X_- \in \mathfrak{g}_-$ and $\xi^- \in \mathfrak{g}_-^*$. Pick out the element $Y_+ \in \mathfrak{g}_+$ dual to ξ^- under the Killing form of $\mathfrak{g}$. Define $\tau\left(X_-, \xi^-\right) = \operatorname{ad}\left[X_-, Y_+\right] : \mathfrak{g}_- \to \mathfrak{g}_-$; $\tau \in \mathfrak{g}_-^* \otimes \mathfrak{g}_- \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_-$. We extend τ by G-equivariance to a tensor on G/P, a holomorphic section of $T^* \otimes T \otimes T^* \otimes T$, where $T = T(G/P)$ is the holomorphic tangent bundle.

Lemma 1. *Suppose that $P \subset G$ is a parabolic subgroup of a complex semisimple Lie group G in adjoint form. Split P into the Langlands decomposition $P = G_0 G_+$ as above. The adjoint action of G_0 on $\mathfrak{g}/\mathfrak{p}$ is faithful.*

Proof. Elements of G from different simple factors commute, so we can restrict our attention to the case where G is a simple Lie group. Take an element $g_0 \in G_0$ acting trivially on $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}_-$. By invariance of the Killing form, g_0 acts trivially on $\mathfrak{g}_+$ too, and so acts trivially on the Lie subalgebra $\mathfrak{h}$ of G generated by $\mathfrak{g}_- \cup \mathfrak{g}_+$. So $\mathfrak{h}$ contains the root spaces of the noncompact roots. Since both $\mathfrak{g}_-$ and $\mathfrak{g}_+$ are invariant under reflection in the compact roots, so is $\mathfrak{h}$. By the relation $\left[X_{\alpha^-}, X_{\alpha^+}\right] = H_{\alpha^-}$, $\mathfrak{h}$ contains the coroots of the noncompact roots. By the relation $\left[H_{\alpha^-}, X_\gamma\right] = \gamma\left(H_{\alpha^-}\right) X_\gamma$, $\mathfrak{h}$ contains the root spaces of all compact roots γ so that γ is not perpendicular to some noncompact root α^-. Applying the same equation inductively, $\mathfrak{h}$ contains the root space of each simple root γ which lies in the same component of the Dynkin diagram as some noncompact root. Since G acts effectively on $X = G/P$, every component of the Dynkin diagram of G contains a noncompact root. Therefore $\mathfrak{h} = \mathfrak{g}$. So g_0 acts trivially on $\mathfrak{g}$, i.e. lies in the center of G, which is $\{1\}$ by hypothesis. □

Lemma 2. *If τ is the fundamental tensor of a cominiscule variety G/P then $\tau \in \mathfrak{g}_-^* \otimes \mathfrak{g}_- \otimes \mathfrak{g}_0$. As we vary X_- and ξ^-, the values of $\tau\left(X_-, \xi^-\right)$ span $\mathfrak{g}_0$.*

Proof. It suffices to prove the result under the assumption that G/P is irreducible, i.e. G is simple. Let $V \subset \mathfrak{g}_0$ be the span of all vectors in $\mathfrak{g}_0$ of the form $[X_-, Y_+]$. We have only to prove that $V = \mathfrak{g}_0$. For any $Z_0 \in \mathfrak{g}_0$,

$$[Z_0, [X_-, Y_+]] = [[Z_0, X_-], Y_+] + [X_-, [Z_0, Y_+]]$$

lies in V. So V is an ideal in $\mathfrak{g}_0$.

If we take the Dynkin diagram of $\mathfrak{g}$ and cut out the compact simple roots, we get the Dynkin diagram of $\mathfrak{g}_0$. The connected components in the Dynkin diagram of $\mathfrak{g}_0$ are the Dynkin diagrams of simple subalgebras of $\mathfrak{g}_0$. The Lie algebra $\mathfrak{g}_0$ also contains all of the noncompact coroots of $\mathfrak{g}$, which sum to an abelian subalgebra. Moreover, $\mathfrak{g}_0$ is the sum of this abelian subalgebra with those simple subalgebras. Take a Chevalley basis. For each root α^+, $H_{\alpha^+} = [X_{\alpha^+}, X_{\alpha^-}] \in V$. So V is an ideal in $\mathfrak{g}_0$ containing the Cartan subalgebra, and therefore containing the abelian summand of $\mathfrak{g}_0$ and intersecting all of the simple summands. Since V intersects each simple summand of $\mathfrak{g}_0$, and is an ideal, it must contain each simple summand, and contains the abelian summand, so $V = \mathfrak{g}_0$. □

Suppose that G/P is a cominiscule variety. Let Γ be the group of automorphisms of the Dynkin diagram of G/P, i.e. automorphisms of the Dynkin diagram of G which leave each component invariant and which fix every noncompact root. With a Cartan subalgebra of G chosen, we can canonically realize Γ as a subgroup of G leaving the Cartan subalgebra invariant; see Fulton and Harris [14] p. 498. To be specific, we just take a Chevalley basis and move around the labels on the simple roots according to Γ. Since Γ acts trivially on the noncompact root, it acts as isomorphisms of the compact roots, and as isomorphisms of the Lie groups G, G_-, G_0 and G_+.

Example 6. Among the Dynkin diagrams of Grassmannians, only those of the form $\operatorname{Gr}\left(p, \mathbb{C}^{2p}\right)$ have a symmetry in their Dynkin diagram, interchanging the ends, fixing the noncompact root at the middle. The Dynkin diagram of the maximal semisimple subgroup of G_0 is precisely the Dynkin diagram of G/P with the noncompact root removed. So in this case, G_0 has disconnected Dynkin diagram, and so $\mathfrak{g}_0$ is a direct sum $\mathbb{C} \oplus \mathfrak{sl}(p, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C})$. The group $\Gamma = \mathbb{Z}_2$ interchanges these two subalgebras, while the adjoint G_0-action leaves them each invariant. The Lie algebra $\mathfrak{g}_0$ imbeds as a subalgebra of $\operatorname{GL}(\mathfrak{g}/\mathfrak{p}) = \operatorname{GL}\left(p^2, \mathbb{C}\right)$. Therefore the projection operator $\alpha \in \mathfrak{gl}\left(p^2, \mathbb{C}\right)^* \otimes \mathfrak{gl}\left(p^2, \mathbb{C}\right)$ which projects onto the first of these $\mathfrak{sl}(p, \mathbb{C})$ subalgebras is invariant under G_0, but not invariant under Γ. To be explicit, write each linear map $s \colon \mathfrak{gl}(p, \mathbb{C}) \to \mathfrak{gl}(p, \mathbb{C})$ as $s(A) = s^{i\ell}_{jk} A^k_\ell$, and then $\alpha(s) = t$ where $t^{i\ell}_{jk} = \frac{1}{n}\left(s^{ip}_{pk} - s^{pq}_{pq}\delta^i_k\right)\delta^\ell_j$.

Example 7. Quadric hypersurfaces $Q^{2n-2} \subset \mathbb{P}^{2n-1}$ have an automorphism from their Dynkin diagram, interchanging the upper and lower roots on the right hand side of the diagram. The groups are

$$G = \mathbb{P}\operatorname{O}(2n, \mathbb{C}), G_0 = \mathbb{C}^\times \operatorname{SO}(2n-2, \mathbb{C}), \mathfrak{g}_- = \mathfrak{g}/\mathfrak{p} = \mathbb{C}^{2n-2}.$$

The group G_0 acts in the obvious representation on $\mathfrak{g}_-$. This automorphism swaps the two representations in the splitting

$$\Lambda^{n-1}(\mathfrak{g}/\mathfrak{p})^* = \Lambda^{n-1}\left(\mathbb{C}^{2n-2}\right)^* = \Lambda^+ \oplus \Lambda^-$$

into self-dual and anti-self-dual n-forms. So the group Γ of automorphisms of the Dynkin diagram will not preserve the Λ^+ component. Let $\alpha \in \Lambda^{n-1}\left(\mathbb{C}^{2n-2}\right) \otimes \Lambda^{n-1}\left(\mathbb{C}^{2n-2}\right)^*$ be the projection to Λ^+. So α is invariant under G_0 but not under Γ. Since $\Lambda^{n-1}(\mathfrak{g}_-) \subset \bigotimes^{n-1} \mathfrak{g}_-$ we think of α as an element of

$$\left(\bigotimes^n \mathfrak{g}_-\right) \otimes \left(\bigotimes^n \mathfrak{g}_-^*\right) = \bigotimes^n \left(\mathfrak{g}_- \otimes \mathfrak{g}_-^*\right).$$

Suppose that G/P is an irreducible cominiscule variety. If $G = A_{2p-1} = \mathrm{PSL}(2p, \mathbb{C})$ and $G/P = \mathrm{Gr}(p, 2p)$, then let α be the tensor defined in example 6 on page 10. If instead $G = D_n = \mathrm{SO}(2n, \mathbb{C})$ so that $G/P = Q^{2n-2} \subset \mathbb{P}^{2n-1}$, then let α be the tensor defined in example 7 on page 10. Otherwise let $\alpha = 0$.

Suppose that G/P is a cominiscule variety, factoring into irreducibles as

$$G/P = \prod_i G^i/P^i.$$

Define the *barnacle tensor* to be the tensor that is the formal sum of the various tensors α on each factor.

Lemma 3. *Suppose that G/P is a cominiscule variety. The kernel of the obvious morphism $\rho\colon P \to \mathrm{GL}(\mathfrak{g}/\mathfrak{p})$ is G_+ and the image is G_0. The group G_0 is precisely the group of linear transformations of $\mathfrak{g}/\mathfrak{p}$ which stabilize the fundamental tensor and the barnacle tensor and preserve the splitting*

$$\mathfrak{g}/\mathfrak{p} = \bigoplus_i \mathfrak{g}^i/\mathfrak{p}^i.$$

Note that the splitting can be defined by a collection of complex linear projection operators.

Proof. Since G and G_0 preserve the splitting, we can safely assume that G is a simple Lie group. Let G_0' be the group of linear transformations of $\mathfrak{g}/\mathfrak{p}$ which stabilize the fundamental tensor and the barnacle tensor and preserve the splitting. Since G_0' is by definition a linear algebraic group, it has finitely many components. The group G_+ lies in the kernel of $P \to G_0'$.

Suppose that $r \in G_0'$. The image of the fundamental tensor is the Lie algebra $\mathfrak{g}_0 \subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{p})$. Therefore $\mathrm{Ad}_r\, \mathfrak{g}_0 = \mathfrak{g}_0$ inside $\mathfrak{gl}(\mathfrak{g}/\mathfrak{p})$. Since G_0 is connected, $\mathrm{Ad}_r\, G_0 = G_0$ inside $\mathrm{GL}(\mathfrak{g}/\mathfrak{p})$, i.e. $G_0 \subset G_0'$ is a normal subgroup.

Define a Lie algebra $\mathfrak{g}'' = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-^*$, by using the usual brackets on $\mathfrak{g}_0$, and by making $\mathfrak{g}_-$ and $\mathfrak{g}_-^*$ abelian (so 0 brackets), and by making the brackets of $\mathfrak{g}_0$ on $\mathfrak{g}_-$ and on $\mathfrak{g}_-^*$ be the obvious action as linear transformations. Then finally define $[X_-, \xi^-] = \tau(X_-, \xi^-)$ for $X_- \in \mathfrak{g}_-$ and $\xi^- \in \mathfrak{g}_-^*$. This bracket is G_0'-invariant, and $\mathfrak{g}'' = \mathfrak{g}$ are isomorphic Lie algebras. Moreover, $\mathfrak{p} \subset \mathfrak{g}$ is an G_0'-invariant subalgebra.

So G_0' acts as automorphisms of the connected Lie group G preserving the connected subgroups G_0, G_+, G_-, P, say by a morphism $\sigma\colon G_0' \to \mathrm{Aut}\, G$. This morphism σ fixes G_0 to give the usual morphism $\mathrm{Ad}\colon G_0 \to G \subset \mathrm{Aut}\, G$. The automorphism group $\mathrm{Aut}\, G$ is a finite extension of $\mathrm{Ad}\, G$ (where $\mathrm{Ad}\, G$ acts on G by inner automorphisms), since G is semisimple (see Fulton and Harris [14] p. 498). Since G acts effectively on $X = G/P$, $G = \mathrm{Ad}\, G$, so $\mathrm{Aut}\, G$ is a finite extension of G. The morphism σ takes $\mathfrak{g}_0 \subset \mathfrak{g}_0' \to \mathfrak{g}_0$, so takes $G_0 \subset G_0' \to G_0 \subset G$, since G_0 is connected, so G_0 is embedded as a subgroup of $\mathrm{GL}(\mathfrak{g}/\mathfrak{p})$.

The inclusion map $G_0' \to \mathrm{Aut}\, G$ gives an inclusion $\mathfrak{g}_0' \to \mathfrak{g}$ on Lie algebras, extending the inclusion $\mathfrak{g}_0 \to \mathfrak{g}$. The Cartan subalgebra of $\mathfrak{g}$ is a subalgebra of

$\mathfrak{g}_0'$, so $\mathfrak{g}_0'$ is a sum of root spaces. The noncompact root spaces don't preserve the subspaces $\mathfrak{g}_-, \mathfrak{g}_+ \subset \mathfrak{g}$, so none of these root spaces lie in $\mathfrak{g}_0'$. Since $\mathfrak{g}_0$ is an ideal of $\mathfrak{g}_0'$, the compact root spaces lie inside $\mathfrak{g}_0'$, so $\mathfrak{g}_0' = \mathfrak{g}_0$. In particular G_0 is the identity component of G_0'.

Every automorphism of a complex semisimple Lie group G factors uniquely as kg where $g \in G$ and k is an automorphism of the Dynkin diagram; see Fulton and Harris [14] p. 498. So each element of G_0' must factor as kg where k lies in the automorphism group of the Dynkin diagram of G, and $g \in G$. This same element must act as an automorphism of G_0.

The group of inner automorphisms of a complex affine group G_0 acts transitively on the Cartan subgroups; Borel [5] p. 156. Therefore if we pick any element $g_0' \in G_0'$, we can arrange after multiplication with some $g_0 \in G_0$ that g_0' acts on G_0 preserving its Cartan subgroup, which is also the Cartan subgroup of G. The element g_0' therefore acts as an automorphism of the Dynkin diagram of G, leaving invariant the roots whose root spaces belong to $\mathfrak{g}_0$, i.e. as an automorphism of the Dynkin diagram of G/P. Examining the Dynkin diagrams, there are only two cases of irreducible cominiscule varieties whose Dynkin diagrams have nontrivial automorphism groups: for A_{2p-1}, the Grassmannian $\operatorname{Gr}\left(p, \mathbb{C}^{2p}\right)$, and D_n, the quadric Q^{2n-2}. In each case the automorphism has order 2, and does not preserve the barnacle tensor. Therefore $G_0' = G_0$. □

The group of linear transformations of $\mathfrak{g}/\mathfrak{p}$ preserving the fundamental tensor and splitting is already a finite extension of G_0. The barnacle tensor is defined here in order to get rid of the order two automorphisms of the Grassmannians $\operatorname{Gr}\left(p, \mathbb{C}^{2p}\right)$ and the quadric hypersurfaces.

4.2. **The curvature boundary operator.** Suppose that G/P is a cominiscule variety. Following Calderbank and Diemer [7], given any representation $\rho \colon P \to \operatorname{GL}(W)$, we define an associated space of *chains* $C_k(W) = \Lambda^k(\mathfrak{g}_-)^* \otimes W$, and a linear map which we call the *boundary operator* $\delta \colon C_k(W) \to C_{k-1}(W)$, by

$$\delta(\beta \otimes w) = \sum_{\alpha_-} \left(X_{\alpha_-} \lrcorner \beta\right) \otimes \rho\left(X_{\alpha_+}\right) w,$$

for $\beta \in \Lambda^k(\mathfrak{g}_-)^*$ and $w \in W$.

4.3. **The torsion operator.** Define the *torsion operator*

$$D \colon \mathfrak{g}_-^* \otimes \mathfrak{g}_0 \to \Lambda^2(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-,$$

as follows: for any $a \in \mathfrak{g}_-^* \otimes \mathfrak{gl}(\mathfrak{g}_-)$, and any $X, Y \in \mathfrak{g}_-$, let

$$D(a)(X \wedge Y) = [a(X), Y] - [a(Y), X].$$

Lemma 4. *Suppose that G/P is a cominiscule variety, with grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. The morphism $\operatorname{ad} \colon \mathfrak{g}_+ \to \mathfrak{g}_-^* \otimes \mathfrak{g}_0, X_+ \mapsto \operatorname{ad}(X_+)$ is injective.*

Proof. Suppose that K_+ is the kernel of this morphism. It follows from the Leibnitz identity that K_+ is a $\mathfrak{g}_0$-module. Therefore K_+ is a sum of root spaces of $\mathfrak{g}$. So it is enough to prove that for any root α^+, X_{α^+} is not in K_+, which follows from $[X_{\alpha^+}, X_{\alpha^-}] = H_{\alpha^+} \neq 0$. □

Lemma 5. *The map $\operatorname{ad} \colon \mathfrak{g}_+ \to \mathfrak{g}_-^* \otimes \mathfrak{g}_0$ has image inside $\ker D$.*

Proof. Pick a root γ. Pick X_γ and $X_{-\gamma}$ bases of the root spaces $\mathfrak{g}_\gamma$ and $\mathfrak{g}_{-\gamma}$. The kernel of ad must be a sum of root spaces, by equivariance under the Cartan subalgebra. By the Jacobi identity, for $X_+ \in \mathfrak{g}_+$ and $Y_-, Z_- \in \mathfrak{g}_-$,

$$\begin{aligned}\left[\mathrm{ad}_{X_+}(Y_-), Z_-\right] &= -\left[[Z_-, X_+], Y_-\right] - \left[[Y_-, Z_-], X_+\right] \\ &= \left[\mathrm{ad}_{X_+}(Z_-), Y_-\right].\end{aligned}$$

□

4.4. **The torsion operator in a Chevalley basis.** Let

$$H = \sum_{\beta^-} H_{\beta^-}.$$

Then for any $a \in \mathfrak{g}_-^* \otimes \mathfrak{g}_0$, define $\operatorname{tr} a \in \mathfrak{g}_+$, the *trace* of a, by

$$\operatorname{tr} a = \sum_{\beta^-, \delta^-} \frac{\left\langle \left[a\left(X_{\beta^-}\right), X_{\beta^+}\right], X_{\delta^-}\right\rangle}{\delta^-(H)\left\langle X_{\delta^-}, X_{\delta^+}\right\rangle} X_{\delta^+}.$$

It is easy to check that for any $X_+ \in \mathfrak{g}_+$, if $a = \mathrm{ad}_{X_+}$ then $\operatorname{tr} a = X_+$.

Lemma 6. *If $a \in \mathfrak{g}_-^* \otimes \mathfrak{g}_0$ and $Da = 0$ then $a = \mathrm{ad}_{X_+} + b$, for a unique $X_+ \in \mathfrak{g}_+$ (given by $X_+ = \operatorname{tr} a$) and a unique $b \in \mathfrak{g}_-^* \otimes \mathfrak{g}_0$ for which $Db = 0$ and $\operatorname{tr} b = 0$.*

Proof. Let $X_+ = \operatorname{tr} a$ and let $b = a - \mathrm{ad}_{X_+}$. □

4.5. **Tensor invariants in cominiscule geometries.** Suppose that $P \to E \to M$ is a cominiscule geometry, with Cartan connection ω, over a vector subbundle $V_M \subset TM$. The identification

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{vertical}_e & \longrightarrow & V_eE & \longrightarrow & V_mM & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \omega_e} & & \downarrow{\scriptstyle \omega_e} & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{p} & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{p} & \longrightarrow & 0 \end{array}$$

identifies the fundamental tensor τ of G/P with a tensor on V_mM, which we will also call τ. Because τ is a P-invariant tensor on $\mathfrak{g}/\mathfrak{p}$, this identification is independent of the point $e \in E$, determining a holomorphic section τ of $V^* \otimes V \otimes V^* \otimes V$, where $V = V_M$.

If $\mathfrak{g} = \bigoplus_i \mathfrak{g}^i$, let $p_i \colon \mathfrak{g}/\mathfrak{p} \to \mathfrak{g}^i/\mathfrak{p}^i$ be the obvious projection. The same identification yields a splitting $V = \bigoplus_i V_i$ corresponding to the splitting $\mathfrak{g}/\mathfrak{p} = \bigoplus_i \mathfrak{g}^i/\mathfrak{p}^i$, and a holomorphic projection tensor $\varpi^i \in V^* \otimes V_i$ identified with p_i. It also yields a barnacle tensor α, a section of $V^* \otimes V_i$ on M. A cominiscule geometry on a foliation is *normal* if its curvature is δ-closed.

Lemma 7. *Suppose that $E \to M$ is a cominiscule geometry on a vector bundle $V_M \subset TM$, with model G/P. The underlying first order structure of $E \to M$ is the G_0-structure $E/G_+ \to FV_M$. Its image in FV_M consists precisely in the linear isomorphisms $u \colon V_mM \to \mathfrak{g}/\mathfrak{p}$ preserving the canonical splitting tensors, fundamental tensor and barnacle tensor. In particular, two cominiscule geometries have the same underlying first order structure just precisely when they have the same canonical splitting, fundamental tensor and barnacle tensor.*

Proof. See lemma 3. □

5. Cominiscule geometries and underlying first order structures

Now that we can describe the underlying first order structure of any cominiscule geometry as a collection of holomorphic tensors, we want to see how a cominiscule geometry is built from its first order geometry and some additional data. We will show that two cominiscule geometries $E \to M$, $E' \to M$ with the same first order data (i.e. the same associated holomorphic tensors) have canonically isomorphic total spaces, with Cartan connections differing by a canonical section of an associated vector bundle. We thereby reduce the obstruction problem for cominiscule geometries on a complex manifold M with a given model (X, G) to two problems: (1) the obstruction problem for the existence of holomorphic tensors on M "analogous" to the G-invariant tensors found on X, and (2) the vanishing of a canonically associated element of a certain sheaf cohomology group arising from any projective space factors in X, say $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times X'$.

Suppose that $\pi \colon E \to M$ is a Cartan geometry over a vector bundle V_M, with model $X = G/H$. Write the composition $E \to E/H_1 \to FV_M$ as $\pi^{(1)}$. Suppose that $E \to M$ and $E' \to M$ are two holomorphic Cartan geometries on the same vector bundle $V_M \subset TM$, with the same model $X = G/H$, and with the same underlying first order structure. Let $E \times_{FV_M} E'$ be the set of all tuples (m, u, e, e'), so that $m \in M$ and $u \colon V_m M \to \mathfrak{g}/\mathfrak{h}$ is a linear isomorphism and $e \in E$ and $e' \in E'$ and

$$u = \pi^{(1)}(e) = \left(\pi'\right)^{(1)}\left(e'\right).$$

The bundle $E \times_{FV_M} E' \to M$ is a holomorphic principal right $H \times_{H/H_1} H$-bundle. The obvious maps $E \times_{FV_M} E' \to E$ and $E \times_{FV_M} E' \to E'$ are each holomorphic principal right H_1-bundles.

Lemma 8. *Suppose that $E \to M$ and $E' \to M$ are two Cartan geometries over the same vector bundle $V_M \subset TM$ with the same model $X = G/H$, and the same underlying first order structure. If we write the pullback of any form by the same symbol, then on $E \times_{FV_M} E'$, $\sigma = \omega' + \mathfrak{h} = \omega + \mathfrak{h}$.*

Proof. By symmetry it is enough to prove that $\omega + \mathfrak{h} = \sigma$, and it is enough to prove this on E, which we see by unwinding definitions. □

Proposition 1. *Suppose that $E \to M$ and $E' \to M$ are two cominiscule geometries with the same model G/P, on the same vector bundle $V_M \subset TM$ and with the same fundamental, splitting and barnacle tensors. On $E \times_{FV_M} E'$, there are unique holomorphic functions $s \colon E \times_{FV_M} E' \to \mathfrak{g}_-^* \otimes \mathfrak{g}_0$, and $X_+ \colon E \times_{FV_M} E' \to \mathfrak{g}_+$, so that*

(1) $\operatorname{tr} s = 0$,
(2) $\omega_0' = \omega_0 + (s + \operatorname{ad}(X_+))\, \sigma$,
(3) s and X_+ are equivariant under the $P \times_{G_0} P$-action.

Proof. From lemma 8, $\omega_- = \omega_-' = \sigma$. Taking the exterior derivative of some local extension of ω and ω' to equivariant differential forms:

$$0 = \frac{1}{2}\left[\omega_0 - \omega_0', \omega_-\right] + \frac{1}{2}\left[\omega_-, \omega_0 - \omega_0'\right] - \frac{1}{2}\left(K_- - K_-'\right) \sigma \wedge \sigma,$$

where $K\sigma \wedge \sigma$ and $K'\sigma \wedge \sigma$ are the curvatures, and K_- and K_-' are the components of K and K' valued in $\mathfrak{g}_-$. Pick any vector $v \in V_{E \times_{FV_M} E'}$ for which $v \lrcorner \sigma = 0$. Let

$A_0 = v \lrcorner (\omega_0 - \omega_0')$. Then plug in v to the above to find

$$0 = \frac{1}{2}\left[A_0, \omega_-\right] + \frac{1}{2}\left[\omega_-, A_0\right].$$

In other words, if $B_- \in \mathfrak{g}_-$, then $0 = \left[A_0, B_-\right]$. By lemma 3 on page 11, $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g}_-)$, ie. this forces $A_0 = 0$. Therefore on the kernel of $\omega_- = \sigma$, we have $\omega_0 = \omega_0'$, so $\omega_0' = \omega_0 + \left(s + \operatorname{ad}_{X_+}\right)\sigma$, where $s \in \mathfrak{g}_-^* \otimes \mathfrak{g}_0$ and $\operatorname{tr} s = 0$. □

Lemma 9. *Suppose that $E \to M$ is a cominiscule geometry with model G/P and Cartan connection ω. If $X_+ \in \mathfrak{g}_+$ and we let $g_+ = e^{X_+}$, then*

$$r_{g_+}^* \begin{pmatrix} \omega_- \\ \omega_0 \\ \omega_+ \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ -\operatorname{ad}(X_+) & I & 0 \\ \frac{1}{2}\operatorname{ad}(X_+)^2 & -\operatorname{ad}(X_+) & I \end{pmatrix} \begin{pmatrix} \omega_- \\ \omega_0 \\ \omega_+ \end{pmatrix}$$

Proof. From the definition of a Cartan connection, $r_{g_+}^* \omega = \operatorname{Ad}(g_+)^{-1} \omega$. The result then follows from eq. (1) on page 7. □

Theorem 3. *Any two cominiscule geometries on the same vector bundle, with the same model and underlying first order structure have canonically isomorphic total space. To be more specific, suppose that $E \to M$ and $E' \to M$ are two cominiscule geometries with the same model G/P and on the same holomorphic vector bundle $V_M \subset TM$. Suppose that E and E' have the same fundamental tensor, barnacle tensor and splitting. Then there exists a unique bundle isomorphism $E \cong E'$ so that $\omega' = \omega + S\,\sigma$ for a unique holomorphic P-equivariant function $S\colon E \to (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ with $\operatorname{tr} S = 0$. The two cominiscule geometries are isomorphic, via an isomorphism that is the identity on M, if and only if $S = 0$.*

Conversely, suppose we have a cominiscule geometry $E \to M$ with model G/P and Cartan connection ω. Every holomorphic section S of $T^*M \otimes \operatorname{ad}(E)$ satisfying $\operatorname{tr} S = 0$ determines a Cartan connection $\omega' = \omega + S\,\sigma$, which has the same fundamental, splitting and barnacle tensors.

Call S the *obstruction* of the pair E, E'. Let $\text{Ш}_0 \subset (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ be the P-submodule consisting of all $s \in (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ so that $\operatorname{tr} s = 0$. For any holomorphic G_0-structure, let $\text{Ш} = E \times_P \text{Ш}_0$ so that $\text{Ш} \to M$ is a holomorphic vector bundle. Similarly, if we have a local biholomorphism $f\colon M \to M'$ and G/P-geometries $E \to M$ and $E' \to M'$, and f locally identifies their underlying first order structures, define the obstruction of f to mean the obstruction of the pair (E, f^*E').

Proof. Under the $P \times_{G_0} P$-action on $E \times_{FV_M} E$, the action on ω and ω' is obvious. Pick $(g_0, g_0) \in G_0 \times_{G_0} G_0 = G_0$. Let $\rho_-\colon G_0 \to \operatorname{GL}(g_-)$ and $\rho_+\colon G_0 \to \operatorname{GL}(g_+)$ be the obvious representations. Then check that

$$r_{(g_0,g_0)}^* S_0 = \operatorname{Ad}(g_0)^{-1}\left(S_0 \circ \rho_-(g_0)\right).$$

Similarly, $r_{(g_0,g_0)}^* X_+ = \rho_+(g_0)^{-1} X_+$.

Pick $Y_+, Y'_+ \in \mathfrak{g}_+$. Then let $g_+ = e^{Y_+}, g'_+ = e^{Y'_+}$. By lemma 9 on page 15, on $E \times_{FV_M} E'$,

$$\begin{aligned} r^*_{(g_+, g'_+)} \left(\omega'_0 - \omega_0\right) &= r^*_{(g_+, g'_+)} \left(\left(S + \operatorname{ad}\left(X_+\right)\right) \omega_- \right) \\ &= \left(S + \operatorname{ad}\left(X_+ - Y'_+ + Y_+\right)\right) \omega_- \\ &= \left(r^*_{(g_+, g'_+)} S + \operatorname{ad}\left(r^*_{(g_+, g'_+)} X_+ \right) \right) \omega_-. \end{aligned}$$

Let F be the set of points of $E \times_{FV_M} E'$ at which the function X_+ (defined in proposition 1 on page 14) vanishes. The set F is a principal right P-subbundle. Indeed its tangent space is a maximal integral manifold of the Pfaffian system $\operatorname{tr}\left(\omega'_0 - \omega_0\right) = 0$. Therefore both maps $F \to E$ and $F \to E'$ are bundle isomorphisms. We henceforth identify E with E' by $E \to F \to E'$.

The function $S_0 \colon E \to \mathfrak{g}_-^* \otimes \mathfrak{g}_0$ is P-equivariant, and therefore G_+-invariant. Consequently if v is any tangent vector to E for which $0 = v \lrcorner \omega_- = v \lrcorner \omega_0$ then $v \lrcorner dS_0 = 0$ and S_0 is G_0-equivariant. If v is any tangent vector to E for which $0 = v \lrcorner \omega_-$ and $v \lrcorner \omega_0 = A_0$, then $v \lrcorner dS_0 = -\operatorname{ad}\left(A_0\right) S_0 + S_0 \circ \rho\left(A_0\right)$. Therefore

$$dS_0 = -\operatorname{ad}\left(\omega_0\right) S_0 + S_0 \circ \rho\left(\omega_0\right) + \nabla S_0 \, \omega_-, \tag{2}$$

where $\nabla S_0 \colon E \to \mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}_0$ is a holomorphic P-equivariant map. The equation $\operatorname{tr} S = 0$ forces $\operatorname{tr} dS = 0$, so forces $\operatorname{tr} \nabla S = 0$.

Taking the exterior derivative of both sides of the equation $\omega'_0 = \omega_0 + S\,\omega_-$,

$$\begin{aligned} 0 = &-\frac{1}{2}\left[S\,\omega_-, \omega_0\right] - \frac{1}{2}\left[\omega_0, S\,\omega_-\right] - \frac{1}{2}\left[S\,\omega_-, S\,\omega_-\right] - \frac{1}{2}\left[\omega_-, \omega'_+ - \omega_+\right] - \frac{1}{2}\left[\omega'_+ - \omega_+, \omega_-\right] \\ &+ \frac{1}{2}\left(K'_0 - K_0\right)\omega_- \wedge \omega_- - dS \wedge \omega_- + S\left(\frac{1}{2}\left[\omega_-, \omega_0\right] + \frac{1}{2}\left[\omega_0, \omega_-\right] - \frac{1}{2}\left[\omega_-, \omega_0\right]\right). \end{aligned}$$

Plugging in what we know about dS from eq. (2),

$$\begin{aligned} 0 = &-\frac{1}{2}\left[S\,\omega_-, S\,\omega_-\right] - \frac{1}{2}\left[\omega_-, \omega'_+ - \omega_+\right] - \frac{1}{2}\left[\omega'_+ - \omega_+, \omega_-\right] \\ &+ \frac{1}{2}\left(K'_0 - K_0 - SK_-\right)\omega_- \wedge \omega_- - \left(\nabla S \omega_-\right) \wedge \omega_-. \end{aligned}$$

Pick a vector v tangent to E for which $0 = v \lrcorner \omega_-$. Suppose that $v \lrcorner \omega_+ = A_+$ and $v \lrcorner \omega'_+ = A'_+$. Plug in to find

$$0 = \frac{1}{2}\left[\omega_-, A'_+ - A_+\right] - \frac{1}{2}\left[A'_+ - A_+, \omega_-\right].$$

If we then pick any $B_- \in \mathfrak{g}_-$ and take a vector w tangent to E with $w \lrcorner \omega_- = B_-$, we find $0 = \left[B_-, A'_+ - A_+\right]$, i.e. $\operatorname{ad}\left(A'_+ - A_+\right)$ acts trivially as a linear map in $\mathfrak{g}_-^* \otimes \mathfrak{g}_0$. By lemma 4 on page 12, $A'_+ = A_+$. Therefore $\omega'_+ = \omega_+$ modulo ω_-. So $\omega'_+ = \omega_+ + b\,\omega_-$ for some unique holomorphic function $S_+ \colon E \to \mathfrak{g}_-^* \otimes \mathfrak{g}_+$. Set $S = S_0 + S_+$; the required morphism exists.

Suppose that there are two isomorphisms

$$\begin{aligned} f &\colon E \to E', f^* \omega' = \omega + S\sigma, \\ \tilde{f} &\colon E \to E', \tilde{f}^* \omega' = \omega + \tilde{S}\sigma. \end{aligned}$$

Since $f^* \omega'_- = \omega_- = \sigma$, it follows that E and E' have the same underlying first order structure. Construct the set F as above: the set of points of $E \times_{FV_M} E'$ at which the function X_+ (defined in proposition 1 on page 14) vanishes. It follows that the

graphs of f and $\tilde{f}$ both lie inside F. Being principal P-bundles over M, they must therefore be equal to F and to one another. □

A *projective factor* of a cominiscule variety

$$G/P = \left(G^1/P^1\right) \times \left(G^2/P^2\right) \times \cdots \times \left(G^s/P^s\right).$$

is a factor $G^i/P^i = \mathbb{P}^n$ with $G^i = \mathrm{PSL}(n+1, \mathbb{C})$. A *projective line factor* is a projective factor $G^i/P^i = \mathbb{P}^1$.

Lemma 10 (Čap and Slovak [10] p. 277 theorem 3.1.16, Goncharov [16])**.** *Suppose that (X, G) is a cominiscule variety. If a complex manifold admits a holomorphic (X, G)-geometry, then the underlying holomorphic first order structure is a holomorphic G_0-structure. Conversely, if a complex manifold admits a holomorphic G_0-structure, then this G_0-structure is locally the underlying first order structure of a normal holomorphic (X, G)-geometry. If (X, G) contains no projective factors, then the normal holomorphic (X, G)-geometry is unique.*

The proofs of [10] and [16] are in the real category, but they work identically in the holomorphic category. The algorithm of Čap and Slovak assumes a fundamental tensor and splitting, but doesn't assume a barnacle tensor. However, the Cartan geometry is defined on a covering space corresponding precisely to the possible choices of barnacle tensor. It is trivial to modify the proof of Čap and Slovak to include a barnacle tensor.

Lemma 11. *Suppose that $X = G/P$ is a cominiscule variety, M is a complex manifold and $V_M \subset TM$ is a holomorphic subbundle. If a V_M admits a holomorphic (X, G)-geometry, then the underlying holomorphic first order structure is a holomorphic G_0-structure. Conversely, if V_M admits a holomorphic G_0-structure, then every G_0-structure on V_M is locally the underlying first order structure of a holomorphic (X, G)-geometry. If V_M is bracket closed and admits a holomorphic G_0-structure, then every G_0-structure on V_M is locally the underlying first order structure of a normal holomorphic G/P-geometry. If G/P contains no projective factors, then then normal holomorphic (X, G)-geometry is unique. For any two local choices of holomorphic (X, G)-geometries with the given G_0-structure, wherever both are defined, there is a unique holomorphic bundle isomorphism between their principal bundles so that they differ by an obstruction.*

Proof. If $V_M = TM$ then this is just lemma 10. Next, suppose that M is a product $M = X \times Y$ of complex manifolds, and that $V_M = TF$ is the tangent bundle of the foliation F whose leaves are $\{x\} \times Y$ for $x \in X$. Then we carry out the same algorithm as Čap and Schichl [9], but carrying around a point $x \in X$ with us as we do. Moreover the algorithm specifies a choice of projective connection (for any projective factors) using a choice of some additional local data. An arbitrary foliation is locally a product, and our result is local, so the proof for V_M the tangent bundle of any holomorphic foliation is immediate.

Next suppose that we have an arbitrary holomorphic subbundle $V_M \subset TM$. Since our problem is local, we can suppose that $M = M_1 \times M_2$, splitting the tangent bundle into $T = T^1 \oplus T^2$ with $T^j_{(m_1,m_2)} = T_{m_j} M_j$, so that $V = V_M$ is complementary to T^1 at every point. Make the obvious linear identification $V_m \to T^2_m$. Use this to push the first order structure on V to a first order structure on T^2. Use the above process to locally generate a (X, G)-geometry $E \to M$ with Cartan connection ω'.

Locally trivialize $E = M_1 \times M_2 \times P$. If $v \in V_E$, write $v = (v_1, v_2, A) \in T^1 \oplus T^2 \oplus TP$ and define ω by $\omega(v) = \omega'(0, v_2, A)$. □

Corollary 1. *Suppose that $X = G/P$ is a cominiscule variety. Suppose that $V_M \subset TM$ is a holomorphic vector bundle on a complex manifold M bearing a holomorphic G_0-structure. There is a holomorphic principal P-bundle $E \to M$ and a holomorphic map $E \to FV_M$ with image equal to the G_0-structure, so that for any local biholomorphism $M' \to M$, every holomorphic (X, G)-geometry on M' with the pullback G_0-structure as its underlying first order structure has bundle canonically holomorphically isomorphic to the pullback of E.*

Proof. We construct bundles E_α locally by the previous corollary, and then we (must) glue them together as in theorem 3 on page 15. □

Cover M in open sets U_α, each equipped with a holomorphic G/P-geometry $E_\alpha \to U_\alpha$ for which the G_0-structure is the underlying first order structure. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and if $\phi \colon X \to M$ is any bundle, write X_α to mean $\phi^{-1} U_\alpha$ and $X_{\alpha\beta}$ to mean $\phi^{-1} U_{\alpha\beta}$. Over $U_{\alpha\beta}$, identify E_α with E_β as in corollary 1 to make a bundle E, arranging that

$$\omega^\beta = \omega^\alpha + S^{\alpha\beta}\sigma$$

so that $S^{\alpha\beta}$ is a holomorphic section of $\text{Ш}_{\alpha\beta}$. The obstruction $S = \left(S^{\alpha\beta}\right)$ is a 1-cocycle $S \in H^1(M, \text{Ш})$. Since each ω^α is a holomorphic partial connection on $E_\alpha \times_P G$ over V_M, the $S^{\alpha\beta}$ are differences between holomorphic partial connections on $E \times_P G$. Thus the linear embedding $\text{Ш}_0 \to (\mathfrak{g}/\mathfrak{p}) \otimes \mathfrak{g}$ induces a sheaf morphism $\text{Ш} \to V_M^* \otimes \operatorname{ad}(E \times_P G)$ which takes the obstruction cocycle S to the Atiyah class $a(M, V_M, E \times_P G)$, giving the obstruction to constructing a holomorphic partial connection on $E \times_P G$ over V_M.

Lemma 12. *Suppose that (X, G) is a cominiscule variety. A holomorphic G_0-structure B on a holomorphic vector bundle $V_M \subset TM$ is the underlying first order structure of a holomorphic (X, G)-geometry if and only if the obstruction of that structure vanishes in $H^1(M, \text{Ш})$. If the obstruction vanishes, then the space of all holomorphic (X, G)-geometries with B as underlying first order structure, modulo any isomorphism which is the identity on M, is a complex affine space modelled on the complex vector space $H^0(M, \text{Ш})$.*

For example, if M is a Stein manifold then any holomorphic G_0-structure on M is induced by a holomorphic G/P-geometry, since the cohomology groups in dimension 1 or higher of any coherent sheaf on any Stein manifold all vanish.

If G/P has no projective factors, then the obstruction of any holomorphic G_0-structure vanishes, because the uniqueness of the normal holomorphic (X, G)-geometry gives a canonical global choice of a normal (X, G)-geometry, and the obstruction 1-cocycle class in H^1 is precisely the obstruction to such a choice. Therefore for any rational homogeneous variety G/P, the obstruction of any holomorphic G_0-structure is precisely identified with the sum of the obstructions of the projective factors: if these vanish, we can then put a product normal holomorphic G/P-geometry on the manifold.

5.1. **The Schwarzian derivative and projective connections on foliations.** If $Z = f(z)$ is a local biholomorphism between open sets in the complex plane, the

Schwarzian derivative is

$$s(f) = \left(\frac{Z'''(z)}{Z'(z)} - \frac{3}{2} \left(\frac{Z''(z)}{Z'(z)} \right)^2 \right) dz^2.$$

On the other hand, if $Z = f(z)$ is a local biholomorphism between open sets of $\mathbb{C}^p$ for some $p \geqslant 2$, then let $J = \log \det Z'(z)$ and the *Schwarzian derivative* is

$$s(f) = \left(\frac{\partial_{ij} Z^\ell}{\partial_k Z^\ell} - \frac{\delta^k_j \partial_i J}{p+1} - \frac{\delta^k_i \partial_j J}{p+1} \right) dz^i \otimes dz^j \otimes \partial_k.$$

Gunning [17] pp. 48–50 proves that $Z = f(z)$ is a projective linear transformation just when $s(f)$ vanishes. More generally, if $(Z, W) = f(z, w)$ is a local biholomorphism between open sets of $\mathbb{C}^p \times \mathbb{C}^q$, and f takes the level sets of w to the level sets of W, we use the same expression to define $s(f)$ but viewing Z as if it were a function of z only. Then $s(f) = 0$ just when $(Z, W) = f(z, w)$ has $Z(z, w)$ a holomorphic projective transformation of z for each fixed w.

Suppose that F is a holomorphic nonsingular foliation of a complex manifold M. A Frobenius coordinate chart on M is a coordinate chart (z, w) defined on a Stein domain in M so that each leaf of F intersects the coordinate chart in a union of level sets of w. If (z, w) and (Z, W) are any two Frobenius coordinate charts, on the overlap where both are defined we can compute $s(f)$ where $(Z, W) = f(z, w)$. Pick Frobenius coordinate charts (z_α, w_α) on open sets U_α covering M, so that any finite intersection of these open sets is a Stein domain. Write the overlaps of charts as $(z_\beta, w_\beta) = f_{\alpha\beta}(z_\beta, w_\beta)$ and let $s_F^{\alpha\beta} = s(f_{\alpha\beta})$. By Gunning's result, there is a flat holomorphic projective connection on M just when there are Frobenius charts (z_α, w_α) whose domains cover M so that $s^{\alpha\beta} = 0$.

We can ask for much less than a flat holomorphic projective connection. Let TF be the tangent bundle of F, i.e. the bundle of tangent vectors tangent to the leaves of F. If we take any tensor products of TF and T^*F, we write $\otimes_0$ to mean the subbundle of all tensors that are traceless in all possible ways, corresponding to an irreducible representation of the general linear group. If the leaves of F are 1-dimensional, then s_F determines a 1-cocycle $s_F \in H^1(M, -2\,TF)$. On the other hand, if the leaves of F have dimension $p \geqslant 2$, then s_F determines a 1-cocycle $s_F \in H^1\big(M, TF \otimes_0 \operatorname{Sym}^2(T^*F)\big)$. The vanishing of s_F as a 1-cocycle is of course a weaker condition than requiring that s_F vanish on each overlap of coordinate charts, as it only requires that s_F equal the differences on overlaps of a local section on each chart. We can also consider s_F as a 1-cocycle in larger sheaves than $TF \otimes_0 \operatorname{Sym}^2(TF)^*$. Molzon and Mortensen [37] p. 3024 construct a holomorphic projective connection explicitly in coordinates out of any local choice of 0-cochain in $TF \otimes \operatorname{Sym}^2(TF)^*$ with s_F as coboundary, but now viewing s_F as a 1-cocycle in $TF \otimes \operatorname{Sym}^2(TF)^*$ rather than in $TF \otimes_0 \operatorname{Sym}^2(TF)^*$. (The same computations work for foliations as for manifolds, so we can make the obvious reinterpretations of the work of Molzon and Mortensen.) They also prove that the family of holomorphic projective connections on a given complex manifold is parameterized by the 0-cycles in that sheaf. However, on p. 3019 they show explicitly in coordinates how to alter a 0-cochain in $TF \otimes \operatorname{Sym}^2(TF)^*$ with coboundary s_F to obtain a 0-cochain in $TF \otimes_0 \operatorname{Sym}^2(TF)^*$ with the coboundary s_F. Their construction of a holomorphic projective connection then yields a holomorphic normal projective

connection. Therefore under the splitting

$$TF \otimes \operatorname{Sym}^2 (TF)^* = T^*F + \left(TF \otimes_0 \operatorname{Sym}^2 (TF)^* \right),$$

the 1-cocycle s_F viewed as valued in the left hand side splits into two pieces, but the first vanishes, so s_F is valued in $TF \otimes_0 \operatorname{Sym}^2 (TF)^*$. Warning: Molzon and Mortensen use the term *normal* differently than the definition that we have used, since they don't require that their 0-cochain be traceless. We use the same definition as [7, 9] which is the standard definition today. One sees easily from the explicit expression of Molzon and Mortensen that the obstruction $s = \left(s^{\alpha\beta} \right)$ between the Cartan connections on overlaps defined earlier is exactly the Schwarzian derivative cocycle written explicitly above together with a complicated expression in the first derivative of the Schwarzian derivative.

Lemma 13. *The Schwarzian derivative 1-cocycle s_F of a holomorphic foliation F on a complex manifold M vanishes just when the obstruction 1-cocycle $S(TF)$ obstructing a holomorphic projective connection vanishes. A holomorphic foliation F on a complex manifold M admits a holomorphic projective connection just when it admits a normal holomorphic projective connection, and the space of normal holomorphic projective connections is then a complex affine space modelled on $H^0\left(M, TF \otimes_0 \operatorname{Sym}^2 (TF)^*\right)$.*

Proof. If this Schwarzian derivative 1-cocycle s_F vanishes, then there is a holomorphic projective connection constructed by Molzon and Mortensen, and so the obstruction 1-cocycle vanishes. On the other hand, by the work of Molzon and Mortensen described above, the Schwarzian derivative 1-cocycle $s_F = s(TF)$ is a quotient of the obstruction 1-cocycle, so if the obstruction 1-cocycle vanishes then the Schwarzian derivative 1-cocycle vanishes as well. Consequently, a holomorphic normal projective connection appears precisely when a holomorphic projective connection appears. Once we have one such, say ω on $E \to M$, then any other, say ω' on $E' \to M$, will have its bundle identified, $E' \cong E$, by adding the requirement that $\omega' = \omega + S\sigma$ with $\operatorname{tr} S = 0$ as in theorem 3 on page 15. But to have ω' also normal requires precisely the vanishing of the antisymmetric and trace parts of S: see Kobayashi and Nagano [27] p. 227. □

It is not known how to relate the classical Schwarzian derivative 1-cocycle to the obstruction 1-cocycle of a vector bundle if the vector bundle is not bracket closed.

5.2. **The Atiyah class and projective connections on foliations.** If $E \to M$ is any holomorphic principal bundle, and $V \subset TM$ is any holomorphic subbundle, the Atiyah class $a(M, V, E) \in H^1(M, V^* \otimes \operatorname{ad}(E))$ is the obstruction to the construction of a partial holomorphic connection on E over V, given explicitly as the cocycle consisting of differences between choices of local holomorphic partial connection. The Atiyah class $a(M, V, W)$ of a vector bundle W is defined to be the Atiyah class of the associated principal bundle. In particular, if $V = TF$ is the tangent bundle of a holomorphic foliation, then the Atiyah class of $a(M, TF, TF)$ is represented by the cochain $\frac{\partial_{jk} Z^\ell}{\partial_i Z^\ell}$ and the Atiyah class of $a\left(M, TF, \det T^*F\right)$ is represented by

$$\frac{\partial \log \det \frac{\partial Z}{\partial z}}{\partial z^i}.$$

Therefore if F has rank p then

$$s_F = a(M, TF, TF) - \frac{I}{p+1} \otimes a\left(M, TF, \det TF^*\right) - a\left(M, TF, \det TF^*\right) \otimes \frac{I}{p+1},$$

represents the Schwarzian derivative in more familiar terms for algebraic geometers. There is no known analoguous interpretation of the Schwarzian derivative of a rank 1 foliation, or of a holomorphic subbundle of TM which is not bracket closed.

5.3. **Vanishing of all other obstructions.**

Proposition 2. *Suppose that M is a complex manifold, and $X = G/P$ a cominiscule variety. Split up $X = \prod_i X_i$ into irreducibles, with $G = \prod_i G_i$ and $P = \prod_i P_i$. Then M has a holomorphic (X, G)-geometry if and only if M has*

1. *a holomorphic vector bundle $V \subset TM$ and*
2. *a splitting $V = \bigoplus_i V_i$ so that the rank of V_i is the dimension of X_i and*
3. *for each X_i which is not a projective factor, holomorphic tensors on V_i pointwise complex-linearly isomorphic to the fundamental tensor and barnacle tensor of X_i and*
4. *for each projective factor X_i, the obstruction 1-cocycle of V_i vanishes; if it should happen that V_i is bracket-closed then this occurs just when the classical Schwarzian derivative cocycle of V_i vanishes.*

In particular, if all of the V_i associated to projective factors X_i are bracket closed, then there is a holomorphic (X, G)-geometry inducing the underlying holomorphic G_0-structure just when there is a normal *holomorphic (X, G)-geometry inducing the underlying holomorphic G_0-structure.*

Proof. Once we have the bundle V and the splitting, the splitting $X = \prod_i X_i$ splits the obstruction 1-cocycle into a sum $S = \sum_i S_i$, $S_i \in H^1(M, \text{Ш}_i)$. If each of these S_i vanish, then there is a (X_i, G_i)-geometry on V_i, and if this holds for every S_i, then the product geometry is a (X, G)-geometry on V. So without loss of generality, X is irreducible, and then the existence of an inducing (X, G)-geometry holds as explained above.

If X_i is not a projective factor, then we have already ensured that the (X_i, G_i)-geometry is normal by using lemma 11 on page 17 to construct the geometry. For each projective factor X_i, if V_i is bracket-closed, then the existence of a holomorphic projective connection (which we have hypothesized) is equivalent to the vanishing of the obstruction 1-cocycle S_i by lemma 12, which is equivalent to the vanishing of the classical Schwarzian derivative by lemma 13 on page 20, which is the obstruction to the existence of a normal holomorphic projective connection (as explained by Molzon and Mortensen [37]). □

6. Splitting of the tangent bundle

In this section, we address whether, if a homogeneous model splits into a product, there is some associated splitting of every geometry with that model. If the model (X, G) of a Cartan geometry splits

$$X = X_1 \times X_2,$$
$$G = G_1 \times G_2,$$

with (X_1, G_1) and (X_2, G_2) homogeneous spaces, then every (X, G)-geometry on any manifold M splits the tangent bundle of M, but this infinitesimal splitting does not necessarily arise from a local or global splitting of M into a product.

In this section only we will allow parabolic geometries whose model is not necessarily effective. If $P \subset Q \subset G$ are parabolic subgroups of a complex semisimple Lie group G, and $E \to M$ is a holomorphic G/Q-geometry, then let $\hat{M} = E/P$ and then $E \to \hat{M}$ is a holomorphic G/P-geometry, called the *lift* of $E \to M$. Conversely, we say that a holomorphic G/P-geometry *drops* to a holomorphic G/Q-geometry if it is isomorphic to the lift of a G/Q-geometry.

Theorem 4. *Suppose that (X, G) is a complex homogeneous space, G acting effectively, holomorphically and transitively. Suppose that G preserves a splitting of the tangent bundle of X. Then clearly any holomorphic (X, G)-geometry on any complex manifold M induces a corresponding splitting of the tangent bundle of M. If M is a smooth complex projective variety, then the splitting arises from a local product structure on M or else the geometry drops to a lower dimensional holomorphic Cartan geometry with model (Y, G), where $X \to Y$ is a G-equivariant holomorphic fiber bundle.*

Proof. The Cartan geometry on M drops to a lower dimensional Cartan geometry just when M contains a rational curve [4] p. 2 theorem 2. If a holomorphic splitting on a smooth projective variety M has a summand which is not bracket closed, then M is uniruled, i.e. covered by rational curves in an algebraic family [19] p. 466 theorem 1.3, and therefore the geometry drops and the relevant summand in the splitting maps to a new summand in the new splitting. If the new summand is bracket closed, then so is the original one, and vice versa, since the two geometries have the same curvature tensor. Therefore the geometry continues to drop until we have eliminated the relevant factor, i.e. it becomes a $(G/G, G/G)$-geometry, ensuring that its curvature vanishes, which forces bracket closure in the original geometry. □

Theorem 5. *Suppose that (X, G) is a rational homogeneous variety which splits into $X = \prod_i X_i$, $G = \prod_i G_i$ into rational homogeneous varieties (X_i, G_i). Then clearly any holomorphic (X, G)-geometry on any complex manifold M induces a corresponding splitting of the tangent bundle of M. If M is a smooth complex projective variety then M is locally a product of complex manifolds with these subbundles as tangent bundles.*

Proof. By theorem 4 the geometry drops. The dropped geometry is still a holomorphic parabolic geometry, so we can repeat until all of the subbundles in the resulting splitting are bracket closed, say for a (Y, G)-geometry where $Y = \prod_i Y_i$, splitting into just as many varieties, except that perhaps some of these Y_i are not rational homogeneous varieties anymore; this happens precisely when $Y_i = G_i$. The original (X, G)-geometry and the resulting (Y, G)-geometry have the same total space and Cartan connection. For each factor $Y_i = G_i$, the curvature vanishes and the (X, G)-geometry is a product of a model geometry (Y_i, G_i) and a geometry modelled on the remaining factors. A product geometry has bracket-closed subbundles corresponding to the factors in the product. So we can assume that $Y_i \neq G_i$ and that (Y_i, G_i) is a rational homogeneous variety. But then the summands in the splitting of the tangent bundle of the original (X, G)-geometry are each bracket closed clearly just when the same is true for the (Y, G)-geometry, since they project to those. □

Theorem 6. *Suppose that M is a smooth complex projective variety and M has split tangent bundle $TM = \bigoplus V_i$. For factor V_i of rank 1, there is a holomorphic projective connection on V_i. In other words the obstruction 1-cocycle of V_i vanishes.*

Proof. Without loss of generality we can assume $TM = V_1 \oplus V_2$ and that V_1 has rank 1. If V_2 is bracket closed, then Brunella, Vitório Pereira and Touzet ([6] theorem 1.1 p. 241) tell us that the universal covering space $\pi\colon \tilde{M} \to M$ splits as $\tilde{M} = C \times N$ where C is a connected and simply connected Riemann surface, with $\pi^* V_1 = TC$ and $\pi^* V_2 = TN$. We can assume that C is the unit disk in the complex plane, the complex plane, or the projective line. In any case, the automorphisms of $\tilde{M}$ each preserve the standard holomorphic projective connection, so induce a projective connection on V_1.

On the other hand, if V_2 is *not* bracket closed, then the same theorem of Brunella, Vitório Pereira and Touzet shows that V_1 is tangent the to fibers of a holomorphic $\mathbb{P}^1$-fiber bundle, for which the existence of a holomorphic projective connection on V_1 is clear. □

7. Holonomy groups

We recall the de Rham holonomy splitting theorem in the case of Kähler manifolds.

Lemma 14. *Suppose that M is a connected complete simply connected Kähler manifold. Write $M = M_1 \times M_2 \times \cdots \times M_s$ as a product of Kähler manifolds, $s \geqslant 1$, which are themselves not products of lower dimensional Kähler manifolds. Pick a point $m = (m_1, m_2, \ldots, m_s) \in M$. Let $T = T_m M$ and $T_i = T_{m_i} M_i$. Then each M_i has some holonomy group H_i at m_i, and the holonomy group of M at m is $H = H_1 \times H_2 \times \cdots \times H_s$, with H_i acting trivially on T_j for $j \neq i$. Each group H_i preserves a complex structure and a Hermitian metric on T_i, so complexifies to a subgroup $H_i^{\mathbb{C}}$ of the complex linear group on T_i. Either*

1. *H_i is the holonomy group of a unique irreducible compact Hermitian symmetric space or,*
2. *$H_i^{\mathbb{C}}$ is of infinite type;*

and both possibilities occur precisely for $H_i = \mathrm{U}(n)$.

Proof. By Berger's classification of the holonomy groups of Riemannian manifolds [21] p. 53 theorem 3.4.1 the holonomy groups H_i of complete simply connected Kähler manifolds which are not products and not locally symmetric must be among

$$\mathrm{U}(n)\,, \mathrm{SU}(n)\,, \mathrm{Sp}(n)\,, \mathrm{Sp}(n)\,\mathrm{U}(1)\,.$$

Their complexifications are of infinite type. Again by Berger's classification, only $\mathrm{U}(n)$ occurs as the holonomy of a compact Hermitian symmetric space, and then only for $\mathbb{P}^n$. The rest of the result is the de Rham splitting theorem [21] p. 47 theorem 3.2.7. □

8. Cominiscule geometries on Kähler–Einstein manifolds

In this section, we use well known results about Kähler–Einstein metrics to prove that, on any compact complex manifold M with $c_1 < 0$, the local splitting from any cominiscule geometry splits the universal covering space into a product of Kähler–Einstein metrics.

If M is a compact complex manifold with $c_1(M) < 0$ then M admits a Kähler–Einstein metric g, which is unique up to constant rescaling [1, 42, 43]. Suppose that

M is a compact complex manifold admitting a Kähler–Einstein metric. Let $T = TM$ be the holomorphic tangent bundle. Every holomorphic section s of $(T^* \otimes T)^{\otimes}$ is parallel [41] p. 142, [25] p. 7.

Corollary 2. *If M is a compact complex manifold with a Kähler–Einstein metric and a cominiscule geometry $E \to M$, then the fundamental tensor, splitting tensors and barnacle tensor of $E \to M$ are parallel for the Kähler–Einstein metric.*

As noticed by Catanese and Di Scala [11], if a holomorphic tensor in $(T^* \otimes T)^{\otimes}$ is isomorphic to a fundamental tensor at just one point of a compact complex manifold M with $c_1(M) < 0$, then, since it is parallel, it is isomorphic at every point. In particular, it determines an underlying holomorphic first order structure, so locally arises from a holomorphic cominiscule geometry.

Proposition 3. *Suppose that M is a connected compact Kähler manifold with $c_1 < 0$ and split tangent bundle $TM = T_1 \oplus T_2$ and suppose that T_1 has rank $n_1 \geqslant 2$ and let*

$$\eta = 2\,(n_1 + 1)\; c_2(T_1)\, c_1(T_1)^{n_1 - 2} - n_1\, c_1(T_1)^{n_1}.$$

Then $\eta = 0$ just when all of the following occur:

1. *the universal covering space $\tilde{M}$ of M splits into a product $\tilde{M} = M_1 \times M_2$ of complex manifolds and*
2. *the T_i pull back to the TM_i, and*
3. *every Kähler–Einstein metric on M pulls back to a a product metric on $M_1 \times M_2$ and a constant holomorphic sectional curvature metric on M_1.*

Proof. Because $c_1(M) < 0$ and M is Kähler, M admits a Kähler–Einstein metric, unique up to constant rescaling [1, 42, 43]. The holomorphic projections $T \to T_i \subset T$ are parallel in the Kähler–Einstein metric and so the subbundles $T_i \subset TM$ are parallel. By the de Rham splitting theorem [13], the universal covering space of M splits into a direct product $\tilde{M} = M_1 \times M_2$, with the metric a local direct product metric, and T_i pulls back to TM_i for each i., Consequently, the subbundles T_i are both bracket closed. The holonomy group of the Kähler–Einstein metric reduces to a product $H = H_1 \times H_2$, with H_i the holonomy group of M_i. The complex structure is also a holomorphic tensor and so each H_i preserves the complex structure on T_i.

Because the splitting is locally a product, the curvature of the Kähler–Einstein metric vanishes on orthonormal pairs of vectors v_1, v_2 with $v_i \in T_i$. On some open set we pick a local unitary coframing, say ω_i for T_1 and a local unitary coframing ω_I for T_2, and take the Chern connection $\omega_{\mu\bar{\nu}}$. The induced metrics on M_1 and M_2 are Kähler–Einstein, since the Ricci curvature of the restriction of the metric on $\tilde{M}$ is the restriction of the Ricci curvature.

We will see that $\eta = 0$ just when the sectional curvature of the induced metric on T_1 is constant, as in [12, 28, 26]. The Ricci tensor has components $R_{i\bar{j}} = R_{i\bar{j}k\bar{k}}, R_{I\bar{J}} = R_{I\bar{J}K\bar{K}}$ and all other components vanish. The scalar curvature of M is $s = R_{i\bar{\imath}} + R_{I\bar{I}}$, while that of M_1 is $s_1 = R_{i\bar{\imath}}$. Of course, being Kähler–Einstein, $R_{i\bar{j}} = \lambda\, \delta_{i\bar{j}}, R_{I\bar{J}} = \lambda\, \delta_{I\bar{J}}$. The scalar curvature is then $s = (n_1 + n_2)\, \lambda$, while

$s_1 = n_1\lambda$. We write out the Chern forms of the subbundle T_1:

$$\begin{aligned} c_1(T_1) &= \frac{\sqrt{-1}}{2\pi}\nabla\omega_{i\bar{\imath}}. \\ &= \frac{\sqrt{-1}}{2\pi}R_{i\bar{\jmath}}\omega_{\bar{\imath}}\wedge\omega_j \\ c_2(T_1) &= -\frac{1}{8\pi^2}\left(\nabla\omega_{i\bar{\imath}}\nabla\omega_{j\bar{\jmath}} - \nabla\omega_{i\bar{\jmath}}\nabla\omega_{j\bar{\imath}}\right) \end{aligned}$$

The Kähler form restricted to T_1 is

$$\Omega_1 = \frac{\sqrt{-1}}{2\pi}\omega_{\bar{\imath}}\wedge\omega_i.$$

Without even using the Kähler–Einstein condition,

$$\begin{aligned} \nabla\omega_{i\bar{\imath}}\wedge\nabla\omega_{j\bar{\jmath}}\wedge\Omega_1^{n_1-2} &= \left(R_{i\bar{\imath}k\bar{l}}\omega_{\bar{k}}\wedge\omega_l\right)\wedge\left(R_{j\bar{\jmath}p\bar{q}}\omega_{\bar{p}}\wedge\omega_q\right)\wedge\Omega_1^{n_1-2} \\ &= \left(s_1^2 - R_{i\bar{\jmath}}R_{j\bar{\imath}}\right)\Omega_1^{n_1}. \end{aligned}$$

Plugging in the expressions for $\nabla\omega$ in terms of R, we find

$$\begin{aligned} c_1(T_1)^2\wedge\Omega_1^{n_1-2} &= \frac{1}{4\pi^2 n_1\left(n_1-1\right)}\left(s_1^2 - R_{i\bar{\jmath}}R_{j\bar{\imath}}\right)\Omega_1^{n_1}. \\ c_2(T_1)\wedge\Omega_1^{n_1-2} &= \frac{1}{8\pi^2\, n_1\left(n_1-1\right)}\left(R_{i\bar{\jmath}k\bar{l}}R_{j\bar{\imath}l\bar{k}} + s_1^2 - 2R_{i\bar{\jmath}}R_{j\bar{\imath}}\right)\Omega_1^{n_1}. \end{aligned}$$

The holomorphic sectional curvature on M_1 is constant just when

$$R_{i\bar{\jmath}k\bar{l}} = \frac{s_1}{n_1\left(n_1+1\right)}\left(\delta_{i\bar{\jmath}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{\jmath}}\right).$$

This motivates the definition of a tensor

$$T_{i\bar{\jmath}k\bar{l}} = R_{i\bar{\jmath}k\bar{l}} - \frac{s_1}{n_1\left(n_1+1\right)}\left(\delta_{i\bar{\jmath}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{\jmath}}\right).$$

Computation gives

$$T_{i\bar{\jmath}k\bar{l}}T_{j\bar{\imath}l\bar{k}} = R_{i\bar{\jmath}k\bar{l}}R_{j\bar{\imath}l\bar{k}} - \frac{2\,s_1^2}{n_1\left(n_1+1\right)}.$$

In particular, the metric on M_1 has constant holomorphic sectional curvature just when

$$R_{i\bar{\jmath}k\bar{l}}R_{j\bar{\imath}l\bar{k}} = \frac{2\,s_1^2}{n_1\left(n_1+1\right)}.$$

If we assume next the Kähler–Einstein condition, that $R_{i\bar{\jmath}} = \lambda\,\delta_{i\bar{\jmath}}$, then

$$c_1(T_1) = \frac{s_1}{2\pi n_1}\Omega_1,$$

and

$$R_{i\bar{\jmath}}R_{j\bar{\imath}} = \frac{s_1^2}{n_1}.$$

Plugging this in to our Chern class expressions,

$$\begin{aligned} c_1(T_1)^{n_1} &= \left(\frac{s_1}{2\pi n_1}\right)^{n_1} \Omega_1^{n_1} \\ &= \left(\frac{s_1}{2\pi n_1}\right)^{n_1-2} c_1(T_1)^2 \wedge \Omega_1^{n_1-2} \\ &= \left(\frac{s_1}{2\pi n_1}\right)^{n_1-2} \frac{1}{4\,\pi^2\,n_1\,(n_1-1)} \left(s_1^2 - R_{i\bar{j}} R_{j\bar{i}}\right) \Omega_1^{n_1}. \end{aligned}$$

Similarly,

$$c_2(T_1) c_1(T_1)^{n_1-2} = \frac{1}{8\,\pi^2\,n_1\,(n_1-1)} \left(\frac{s_1}{2\pi n_1}\right)^{n_1-2} \left(R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}} + \frac{n_1-2}{n_1} s_1^2\right) \Omega_1^{n_1}.$$

Therefore

$$\begin{aligned} \eta &= 2\,(n_1+1)\, c_2(T_1) c_1(T_1)^{n_1-2} - n_1\, c_1(T_1)^{n_1} \\ &= \frac{1}{4\,\pi^2\,n_1\,(n_1-1)} \left(\frac{s_1}{2\pi n_1}\right)^{n_1-2} \left((n+1)\, R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}} - \frac{2}{n} s_1^2\right) \Omega_1^{n_1} \\ &= \frac{1}{4\,\pi^2\,n_1\,(n_1-1)} \left(\frac{s_1}{2\pi n_1}\right)^{n_1-2} \left((n+1)\, T_{i\bar{j}k\bar{l}} T_{j\bar{i}l\bar{k}}\right) \Omega_1^{n_1}. \end{aligned}$$

So finally, we see that η is a nonnegative differential form. In particular, $\int_M \eta \wedge \Omega_2^{n_2} \geqslant 0$ and equality occurs just when $\eta = 0$, which occurs just when $T = 0$, which occurs just when M_1 has constant sectional curvature. □

Lemma 15 (McKay [34])**.** *Suppose that G is a complex semisimple Lie group and $P \subset G$ is a parabolic subgroup. Suppose that $W_1, W_2, \dots, W_t$ are P-submodules and M is a compact Kähler manifold bearing a holomorphic G/P-geometry $E \to M$. Let $V_i = E \times_P W_i$, and also write V_i for the corresponding vector bundles $G \times_P W_i$ on the model. Then every polynomial equation*

$$0 = p\left(c\left(V_1\right), c\left(V_2\right), \dots, c\left(V_t\right)\right)$$

in Chern classes satisfied on the model G/P must be satisfied on M.

Corollary 3. *Suppose that M is a compact Kähler manifold with $c_1 < 0$, bearing a cominiscule geometry with model*

$$G/P = (G_1/P_1) \times (G_2/P_2) \times \cdots \times (G_s/P_s).$$

This geometry splits the tangent bundle, say as

$$TM = T^1 \oplus T^2 \oplus \cdots \oplus T^s.$$

The universal covering space $\pi \colon \tilde{M} \to M$ splits:

$$\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_s,$$

so that $\pi^* T_i = T\tilde{M}_i$. Every Kähler–Einstein metric on M lifts to a product metric on $\tilde{M}$, a product of Kähler–Einstein metrics on each of these $\tilde{M}_i$. If G_i/P_i is a projective factor of complex dimension 2 or more, then $\tilde{M}_i$ is complex hyperbolic space.

Proof. The holonomy group preserves the splitting by corollary 2 on page 24, so the de Rham splitting theorem ensures that the universal covering space splits as indicated. We could also derive the local splitting from theorem 4 on page 22, because $c_1(M) < 0$ so M is a smooth projective variety. The Chern class inequality in proposition 3 on page 24 follows from lemma 15 on page 26. Therefore $\tilde{M}_i$ has a constant negative holomorphic sectional curvature metric, which we can assume has the same sectional curvature as complex hyperbolic space, so $\tilde{M}_i$ must be isomorphic to complex hyperbolic space. □

9. The Klingler–Mok vanishing theorem

In this section, we quote a theorem of Klingler and Mok, translating the notation and terminology as needed, which ensures rigidity of the flat cominiscule geometry associated to a locally Hermitian symmetric variety.

Suppose that G/P is a cominiscule variety. Decompose $\mathfrak{g}$ into simple Lie algebras $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \cdots \oplus \mathfrak{g}^N$. For each simple factor $\mathfrak{g}^i \subset \mathfrak{g}$ let γ_i be the lowest weight whose weight space belongs to $\mathfrak{g}^i/\mathfrak{p}^i = \mathfrak{g}^i_-$. This weight γ_i is of course a root, since $\mathfrak{g}_-$ is a sum of root spaces of $\mathfrak{g}$. Note that γ_i is *not* the negative noncompact simple root in general. Let γ be the weight $\gamma = \sum_i \gamma_i$ (not a root unless $\mathfrak{g}$ is simple). Call γ the *Klingler weight* of G/P.

Theorem 7 (Klingler [23] p. 212)**.** *Suppose that G/P is a cominiscule variety with Klingler weight γ. Suppose that λ is the lowest weight of a finite dimensional irreducible G_0-module V. Suppose that $\lambda \neq 0$ and $\langle \gamma, \lambda \rangle \geqslant 0$. Suppose that M is a compact complex manifold whose universal covering space is the noncompact dual Hermitian symmetric space to G/P. Take the flat G/P-geometry $E \to M$ (as defined in example 1 on page 5. Then $0 = H^0(M, E \times_P V)$.*

Klinger only states this result when G/P is an irreducible cominiscule variety; however, his proof works verbatim for any cominiscule G/P. (His sign conventions differ from ours, so his statement looks a little different.)

10. Holonomy and cominiscule geometries

In this section, we put all of our previous results together to finish the classification of cominiscule geometries on compact complex manifolds with $c_1 < 0$.

Theorem 8. *Suppose that M is a connected compact complex manifold with $c_1 < 0$ bearing a cominiscule geometry. Then M admits a normal cominiscule geometry, and every normal cominiscule geometry is standard. All (either normal or abnormal) cominiscule geometries on M have the same model and the same holomorphic principal bundle $E \to M$ with the same holomorphic map $E \to FM$ to the underlying first order structure. If the geometry has Cartan connection ω, then each cominiscule geometry has a Cartan connection of the form $\omega' = \omega + s\omega_-$ for some unique $s \in H^0(M, \amalg)$. Thus the moduli space of cominiscule geometries on M is a finite dimensional complex vector space, canonically identified with $H^0(M, \amalg)$.*

Proof. Suppose that the geometry has model $X = G/P$. Split the model into irreducibles as $X = \prod_i X_i$, $G = \prod_i G_i$, $P = \prod_i P_i$. The universal covering space $\tilde{M}$ of M has split tangent bundle. The splitting is invariant parallel transport by corollary 2 on page 24. By the de Rham splitting theorem [21] p. 47 theorem 3.2.7, the universal covering $\tilde{M}$ of M then splits as a corresponding product $\tilde{M} = \prod_i \tilde{M}_i$.

We can assume that the geometry is normal, by proposition 2 on page 21. Each factor $\tilde{M}_i$ inherits a normal cominiscule geometry with irreducible model, invariant under parallel transport.

If the factor X_i is not projective, the holonomy group is the holonomy group of the G_i'-invariant metric on the noncompact Hermitian symmetric space, X_i' and $\tilde{M}_i = X_i'$. Moreover, the underlying first order structure is invariant under parallel transport. The complexified holonomy group is precisely the stabilizer of that first order structure, and also precisely the stabilizer of the standard first order structure. The proof of lemma 3 on page 11 easily adapts to prove that the data of the first order structure (the fundamental tensor, barnacle tensor and splitting) are determined completely by the action of the group G_0, so that any two first order structures with the same stabilizer are identical, ensuring the uniqueness of the first order structure on $\tilde{M}_i$. Klingler [23] p. 211 propositions 4.9 and 4.10 gives an alternative proof of the rigidity of the underlying first order structure.

If the factor (X_i, G_i) is a projective factor, say $X_i = \mathbb{P}^n$ for $n \geqslant 2$, then corollary 3 on page 26 and lemma 15 on page 26 together prove that $\tilde{M}_i$ is complex hyperbolic space and the metric on $\tilde{M}$ is a product metric. We need to establish that the projective connection on $\tilde{M}_i$ is the standard flat one. As shown in lemma 13 on page 20, given any two normal holomorphic projective connections on M, one can be written in terms of the other and a holomorphic section of $T^i \otimes_0 \operatorname{Sym}^2 \left(T^i\right)^*$, the bundle of traceless symmetric vector-valued 2-tensors, after a suitable holomorphic bundle isomorphism. (This is also noted with less detail by Klingler [23] p. 17). It follows from theorem 7 on page 27 that there are no global nonzero sections of this bundle; see Klingler [23] p. 17 for more details. Therefore the normal holomorphic projective connection on $\tilde{M}_i$ is the usual flat one on complex hyperbolic space, and this is the unique holomorphic normal projective connection on T^i.

If $X_i = \mathbb{P}^1$ then $\tilde{M}_i$ inherits a holomorphic projective connection given by a holomorphic quadratic differential, say η_i.

The product of all of these geometries on these various factors is a normal holomorphic cominiscule geometry with the same underlying first order structure as the pullback geometry, and inducing the same projective connections on the various vector subbundles. If there are two holomorphic normal (X, G)-geometries on M with the given underlying first order structure and inducing the same holomorphic projective connections on each T^i with rank 1, then they each induce the same normal geometries on each of the T^i, and so the Cartan connection restricts to the same Cartan connection above each of these T^i, on the same holomorphic bundle (corollary 1 on page 18), forcing them to agree. Therefore the product geometry is the unique normal holomorphic cominiscule geometry on M with the given model and projective connections on the $\mathbb{P}^1$-factors, and the geometry on M is flat. We still need to prove that it is standard, i.e. that we can replace M by a finite covering space to split M into a product of compact Riemann surfaces with holomorphic quadratic differentials and some locally Hermitian symmetric variety with its standard cominiscule geometry. If $\dim M = 1$, then the theorem is clear, so we can assume that G has rank at least 2.

We call Γ *reducible* if Γ has a finite index subgroup $\Gamma' \times \Gamma''$ with each of Γ' and Γ'' lying in different factors in the product $G = \prod_i G_i$. Consequently, some finite cover of M splits into a corresponding product $M = M' \times M''$, and the product geometries on the universal covering space of M then descend to products of geometries on

each factor and our proof is finished by induction. So we can assume that Γ is irreducible without loss of generality. The problem arises that there might exist some projective line factor $X_i = \mathbb{P}^1$ and some nonzero quadratic differential η_i on the associated factor $\tilde{M}_i$.

We follow Klingler [23] p. 205. Suppose that there is such a quadratic differential, invariant under Γ. The set of Γ-invariant quadratic differentials is a complex vector space. We have only to prove that it has dimension zero. The geometry on M is flat, so there is a developing map $\delta\colon \tilde{M} \to X$ to the model and a morphism $h\colon \pi_1(M) \to G$ so that $\delta\circ\gamma = h(\gamma)\delta$ for all $\gamma \in \pi_1(M)$. By the Ehresmann–Thurston deformation theorem [15], the complex analytic stack of flat (X, G)-geometries on the real manifold M is locally isomorphic to the complex analytic stack of representations of the fundamental group, by taking each geometry to its holonomy morphism h. The local rigidity of the flat holomorphic projective connection therefore follows if we can prove the local rigidity of the representation h. A theorem of Weil tells us that the vanishing of $H^1(\Gamma, \mathfrak{g})$ implies local rigidity of this representation [38] p. 91 theorem 6.7. Weil's rigidity theorem [38] p. 137 theorem 7.66 implies the vanishing of $H^1(\Gamma, \mathfrak{g})$, and therefore the normal geometry on M is locally rigid. The Γ-invariant quadratic differentials on the various 1-dimensional factors are the deformations of the geometry on M among normal geometries. Since there are no deformations, this vector space has dimension zero: the geometry is the unique normal holomorphic projective connection on M.

There are no cominiscule geometries on M with different models, because the model is the compact Hermitian symmetric space which is the compact form of the universal covering space of M, with the identity component of its biholomorphism group acting on it. The identification of the moduli space follows by lemma 11 on page 17. □

11. Conclusion

Our results above generalize those of [3, 23, 28, 29, 30, 31]. The dimension of the moduli space of holomorphic cominiscule (not necessarily normal) geometries on a general Kähler–Einstein manifold is still unknown, although we have identified it with the space of holomorphic sections of Ш. It seems unlikely that a compact complex manifold with $c_1 < 0$ can admit a *noncominiscule* holomorphic parabolic geometry. Smooth projective curves, surfaces and 3-folds admitting holomorphic projective connections on are classified by Klingler [22] and Jahnke and Radloff [20]; besides the model, the translation invariant examples on tori, and the locally complex hyperbolic spaces (as above), there are also certain torus bundles over curves. It should be feasible to classify holomorphic projective connections on torus bundles over Kähler–Einstein manifolds in any dimension.

References

1. Thierry Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95. MR MR494932 (81d:53047) 23, 24
2. Robert J. Baston and Michael G. Eastwood, *The Penrose transform*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1989, Its interaction with representation theory, Oxford Science Publications. MR MR1038279 (92j:32112) 5
3. Arnaud Beauville, *Complex manifolds with split tangent bundle*, Complex analysis and algebraic geometry, de Gruyter, Berlin, 2000, pp. 61–70. MR MR1760872 (2001f:32030) 29

4. Indranil. Biswas and Benjamin McKay, *Holomorphic Cartan geometries and rational curves*, ArXiv e-prints (2010). 2, 8, 22
5. Armand Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR MR1102012 (92d:20001) 5, 12
6. Marco Brunella, Jorge Vitório Pereira, and Fréderic Touzet, *Kähler manifolds with split tangent bundle*, Bull. Soc. Math. France **134** (2006), no. 2, 241–252. 23
7. David M. J. Calderbank and Tammo Diemer, *Differential invariants and curved Bernstein-Gelfand-Gelfand sequences*, J. Reine Angew. Math. **537** (2001), 67–103. MR MR1856258 (2002k:58048) 12, 20
8. Frédéric Campana, Jean-Pierre Demailly, and Thomas Peternell, *Rationally connected manifolds and semipositivity of the Ricci curvature*, ArXiv e-prints (2012). 8
9. Andreas Čap and Hermann Schichl, *Parabolic geometries and canonical Cartan connections*, Hokkaido Math. J. **29** (2000), no. 3, 453–505. MR MR1795487 (2002f:53036) 17, 20
10. Andreas Čap and Jan Slovák, *Parabolic geometries. I*, Mathematical Surveys and Monographs, vol. 154, American Mathematical Society, Providence, RI, 2009, Background and general theory. MR 2532439 (2010j:53037) 2, 17
11. F. Catanese and A. José Di Scala, *A characterization of varieties whose universal cover is a bounded symmetric domain without ball factors*, ArXiv e-prints (2013). 2, 24
12. Bang-Yen Chen and Koichi Ogiue, *Some characterizations of complex space forms in terms of Chern classes*, Q. J. Math **26** (1975), 459–464. 24
13. Georges de Rham, *Sur la reductibilité d'un espace de Riemann*, Comment. Math. Helv. **26** (1952), 328–344. MR MR0052177 (14,584a) 24
14. William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR MR1153249 (93a:20069) 6, 10, 11, 12
15. William M. Goldman, *Locally homogeneous geometric manifolds*, Proceedings of the International Congress of Mathematicians (Hyderabad) (Rajendra Bhatia, ed.), vol. 2, International Congress of Mathematicians, Hindawi, 2010, pp. 717–744. 29
16. Alexander B. Goncharov, *Generalized conformal structures on manifolds*, Selecta Math. Soviet. **6** (1987), no. 4, 307–340, Selected translations. MR MR925263 (89e:53050) 17
17. Robert C. Gunning, *On uniformization of complex manifolds: the role of connections*, Mathematical Notes, vol. 22, Princeton University Press, Princeton, N.J., 1978. MR 82e:32034 19
18. Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original. MR MR1834454 (2002b:53081) 5, 6
19. Andreas Höring, *Uniruled varieties with split tangent bundle*, Math. Z. **256** (2007), no. 3, 465–479. MR 2299565 (2008f:32024) 22
20. Priska Jahnke and Ivo Radloff, *Splitting submanifolds of families of fake elliptic curves*, Sci. China Math. **54** (2011), no. 5, 949–958. MR 2800919 29
21. Dominic D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics, vol. 12, Oxford University Press, Oxford, 2007. MR 2292510 (2008a:53050) 23, 27
22. Bruno Klingler, *Structures affines et projectives sur les surfaces complexes*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 2, 441–477. MR MR1625606 (99c:32038) 29
23. ———, *Un théorème de rigidité non-métrique pour les variétés localement symétriques hermitiennes*, Comment. Math. Helv. **76** (2001), no. 2, 200–217. MR MR1839345 (2002k:53076) 2, 27, 28, 29
24. Anthony W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002. MR MR1920389 (2003c:22001) 6, 7
25. Shôshichi Kobayashi, *First Chern class and holomorphic tensor fields*, Nagoya Math. J. **77** (1980), 5–11. MR MR556302 (80m:53050) 24
26. Shôshichi Kobayashi and Camilla Horst, *Topics in complex differential geometry*, Complex differential geometry, DMV Sem., vol. 3, Birkhäuser, Basel, 1983, pp. 4–66. MR MR826252 (87g:53097) 2, 24
27. Shôshichi Kobayashi and Tadashi Nagano, *On projective connections*, J. Math. Mech. **13** (1964), 215–235. MR 28 #2501 20

28. Shôshichi Kobayashi and Takushiro Ochiai, *Holomorphic projective structures on compact complex surfaces*, Math. Ann. **249** (1980), no. 1, 75–94. MR 81g:32021 2, 24, 29
29. ———, *Holomorphic projective structures on compact complex surfaces. II*, Math. Ann. **255** (1981), no. 4, 519–521. MR MR618182 (83a:32025) 2, 29
30. ———, *Holomorphic structures modeled after compact Hermitian symmetric spaces*, Manifolds and Lie groups (Notre Dame, Ind., 1980), Progr. Math., vol. 14, Birkhäuser Boston, Mass., 1981, pp. 207–222. MR MR642859 (84i:53051) 2, 9, 29
31. ———, *Holomorphic structures modeled after hyperquadrics*, Tôhoku Math. J. (2) **34** (1982), no. 4, 587–629. MR MR685426 (84b:32039) 2, 29
32. J. M. Landsberg, *Secant varieties, shadows and the universal Lie algebra: Perspectives on the geometry of rational homogeneous varieties*, From 2004 lectures at Harvard University, to appear, 2005. 5, 6
33. Frank Loray and David Marín Pérez, *Projective structures and projective bundles over compact Riemann surfaces*, Astérisque (2009), no. 323, 223–252. MR 2647972 5
34. Benjamin McKay, *Characteristic forms of complex Cartan geometries*, arXiv:math/0704.2555, April 2007. 26
35. ———, *Holomorphic Cartan geometries on uniruled surfaces*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 15-16, 893–896. MR 2835898 2
36. ———, *Holomorphic parabolic geometries and Calabi-Yau manifolds*, SIGMA Symmetry Integrability Geom. Methods Appl. **7** (2011), Paper 090, 11. MR 2861186 8
37. Robert Molzon and Karen Pinney Mortensen, *The Schwarzian derivative for maps between manifolds with complex projective connections*, Trans. Amer. Math. Soc. **348** (1996), no. 8, 3015–3036. MR 1348154 (96j:32028) 19, 21
38. M. S. Raghunathan, *Discrete subgroups of Lie groups*, Springer-Verlag, New York, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. MR 0507234 (58 #22394a) 29
39. Jean-Pierre Serre, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001, Translated from the French by G. A. Jones, Reprint of the 1987 edition. MR 2001h:17001 7
40. Richard W. Sharpe, *An introduction to Cartan geometries*, Proceedings of the 21st Winter School "Geometry and Physics" (Srní, 2001), no. 69, 2002, pp. 61–75. MR 2004f:53023 3
41. K. Yano and S. Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies, No. 32, Princeton University Press, Princeton, N. J., 1953. MR MR0062505 (15,989f) 24
42. Shing Tung Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799. MR MR0451180 (56 #9467) 23, 24
43. ———, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411. MR MR480350 (81d:53045) 23, 24

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