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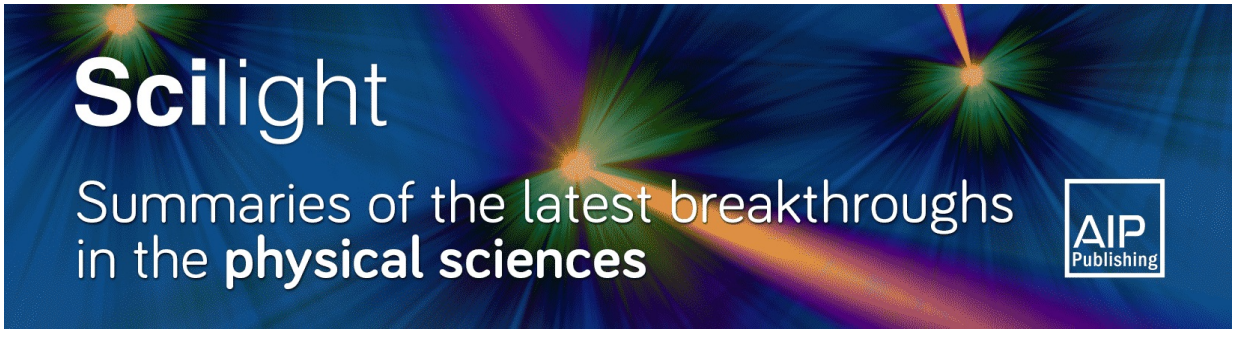
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ABSTRACT

We stabilize a prescribed cycle or an equilibrium of a difference equation using pulsed stochastic control. Our technique, inspired by Kolmogorov's law of large numbers, activates a stabilizing effect of stochastic perturbation and allows for stabilization using a much wider range for the control parameter than would be possible in the absence of noise. Our main general result applies to both prediction-based and target-oriented controls. This analysis is the first to make use of the stabilizing effects of noise for prediction-based control; the stochastic version has previously been examined in the literature, but only the destabilizing effect of noise was demonstrated. A stochastic variant of target-oriented control has never been considered, to the best of our knowledge, and we propose a specific form that uses a point equilibrium or one point on a cycle as a target. We illustrate our results numerically on the logistic, Ricker, and Maynard Smith models from population biology.

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Various linear-type methods were developed to control otherwise unstable or chaotic behavior of discrete maps. Prediction-based control (PBC) introduced by Ushio and Yamamoto in 1999 and two-parameter target-oriented control (TOC) proposed by Dattani *et al.* in 2011 are among them. Stochastic perturbations were usually considered in two different contexts: as an intrinsic part of control, which could diminish stabilization effects and thus should be kept in prescribed bounds, and as natural environmental noise, which may somehow control chaos. For instance, such noise can reduce oscillation amplitudes. First, we consider control types incorporating both deterministic and stochastic components, both of which can have a stabilizing effect. Examples illustrate that stabilization can be achieved by noise but introduction of deterministic control, which cannot stabilize in itself, can influence the bounds for stabilizing stochastic perturbations. We analyze how the effective range of stabilizing control parameters may be extended by the introduction of noise. Second, stabilization of either an unstable equilibrium or an unstable orbit of a discrete equation is investigated. In addition, both regular (applied at each step) and pulsed (applied every *k*th step) types of control are applied, and pulsed control can stabilize an equilibrium. Third, stochastic control is considered in very general settings. These results are later applied to target-oriented and prediction-based types of control involving stochastic component in the control parameter. The application of stochastic prediction-based and target-oriented controls for *k*-cycle stabilization is novel, and stochastic versions of target-oriented control have not been studied before.

I. INTRODUCTION

We investigate the use of pulsed stochastic control to stabilize a prescribed cycle of the difference equation

$$x_{n+1} = f(x_n), \quad n \in \mathbb{N}_0, \quad x_0 > 0, \tag{1}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a general class of control methods applied to (1), we reduce this problem to the stabilization of a point equilibrium at zero and present a general theorem on pulsed stabilization of the zero equilibrium to Eq. (1).

We show how this theorem may be applied for two specific control methods: prediction-based control (PBC) and a particular case

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of target-oriented control (TOC). We introduce stochastic versions of both methods and study the interplay of the underlying control with stochastic perturbation of the control parameter, establishing results that show when the introduction of noise is beneficial for stabilization. In particular, we describe the stabilization of either a point equilibrium or a cycle by noise in the context of stochastic control. We also investigate the implications of pulsed control in this setting.

Our analysis allows us to demonstrate the stabilization of cycles for three commonly used models from population biology—Ricker, logistic, and Maynard Smith—and we note that while stabilization of a point equilibrium by noise is quite a well-developed topic, stochastic stabilization of cycles is much less so.

Our technique is inspired by Kolmogorov's law of large numbers which allows us to characterize the stabilizing effect of noise in our analysis and may be stated as follows:

Lemma I.1 (Ref. 34, p. 391). *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random variables where $\mathbb{E}|v_n| < \infty$, $n \in \mathbb{N}$. Denote the common mean $\mu := \mathbb{E}v_n$, and the partial sum $S_n := \sum_{k=1}^{n} v_k$. Then, $\lim_{n\to\infty} S_n/n = \mu$, a.s.*

To the best of our knowledge, the idea of stabilization by noise goes back to 1950s for physical applications. Consider the well-known pendulum of Kapica,[22] where stochastic perturbations can stabilize its top (otherwise unstable) position. This stabilizing effect is stipulated by the type of noise and its intensity: noise that is too intense does not lead to stabilization, but noise that is insufficiently intense leads only to the preservation of the stability of the bottom equilibrium. For differential equations, a theoretical justification of stabilization by noise originated in the 1960s, see Ref. 19. For both differential and difference equations, more detailed historical notes, as well as recent results on the topic are given in Refs. 7, 9, and 20. Recently, stability and stabilization of stochastic difference equations and systems, as well as cyclic and chaotic behavior, has become a focus of many publications.[2,3,12,18,21,27,30,33] Moreover, a developed theory of random difference equations was utilized to investigate differential equations,[11] or discrete and continuous stochastic equations were considered in the framework of a single model.[14]

PBC was first introduced by Ushio and Yamamoto[37] and was studied in detail in Ref. 25. The case when the control is applied to (1) at every step, and the control parameter $\alpha \in (0, 1)$ is subject to a stochastic perturbation, can be written as

$$x_{n+1} = f(x_n) - \left(\alpha + l\xi_{n+1}\right)\left(f(x_n) - x_n\right),\ n \in \mathbb{N}_0,\ x_0 > 0, \qquad (2)$$

and was considered in Ref. 7. If the control is applied at every kth step, for $k \in \mathbb{N}$, $k > 1$, then this is called *pulsed control*, and in the case of deterministic PBC, it was investigated in Refs. 8 and 26.

Previous results on PBC view stabilization as arising from the deterministic control and in spite of the presence of stochastic perturbations of low intensity. However, we can show that it is possible to stabilize the equilibrium of (1) by stochastic PBC (2), even for values of α, which do not deliver stability in the absence of noise.

TOC, applied to (1) at every step with target T, is characterized by

$$x_{n+1} = f\left(\alpha T + (1-\alpha)x_n\right), \quad T \geq 0, \quad \alpha \in [0, 1). \qquad (3)$$

It was introduced in Ref. 13 and further investigated in Refs. 4 and 15. In Ref. 15, it was shown that TOC is topologically equivalent to the modified TOC equation,

$$x_{n+1} = \alpha T + (1-\alpha)f(x_n), \quad T \geq 0, \quad \alpha \in [0, 1). \qquad (4)$$

Note that in (2) and other stochastic control models with $f\colon [0,\infty) \to [0,\infty)$, once the control $\alpha + l\xi_{n+1} \in [0, 1]$, or, for TOC, the target is in addition non-negative, the expression in the right-hand side is non-negative. Assuming $\alpha \in [0, 1)$, $|\xi_{n+1}| \leq 1$, we get $l \leq \min\{\alpha, 1-\alpha\}$. However, most of our results are local, and we can consider parameters outside of this domain, considering the truncated version when the right-hand side is a maximum of the computed value and zero, which is quite a typical approach in population ecology.[31]

The application of modified TOC to stabilize cycles in the context of higher order or vector difference equations was considered in Refs. 5 and 6. In the present article, we find a relationship between the control parameter α and the noise intensity l, which guarantees local stability of a cycle after application of stochastic pulsed stabilization. In the case when the control parameter is such that the unperturbed model is stable, our method provides conditions on the noise intensity which preserve stability, similarly to Ref. 7. In the case when a deterministic system is unstable after application of the control, introduction of a noise with appropriate intensity guarantees stability.

Pulsed control is essential in cases where application of control at each step is either impossible or inefficient from a practical or economical point of view. Here, we consider linear types of control for nonlinear models with either one for PBC or two parameters for TOC involved. Our control results are robust as a result of the simplicity of the control structures and the continuity of the maps. While stabilization of an unstable equilibrium with a control applied at every step is always possible, once the control intensity is sufficient, pulsed stabilization with PBC in the deterministic case is problematic;[8] sometimes, it cannot be achieved for any values of the parameters. In our earlier paper,[7] a stochastic perturbation of deterministic PBC was explored but global stability was justified only in the case when all the values of the noisy control are within a range of parameters leading to stabilization in the deterministic case. In contrast with these global results, highly local results are obtained in Ref. 9, where control and stabilization are achieved solely by noise but the neighborhood of the equilibrium should be very small (sometimes less than 10^{-8}) and thus anyway another control method is required in practical applications. Here, our purpose is to combine a stabilizing effect of noise with PBC, expanding the range of allowable control parameters compared to Ref. 7 and relaxing requirements on proximity of the initial value to the equilibrium in Ref. 9.

For TOC, delayed versions and stage-structured dynamics were considered in Refs. 4–6 but its stochastic version has never been explored. Here, we obtain sufficient stabilization conditions for noisy TOC, including its pulsed version. Again, noise can expand the range of parameters for which stabilization is achieved. From this point of view, results of the present paper significantly generalize, for example, as in Ref. 4.

All stochastic sequences considered in the paper are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$, where the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is naturally generated by the sequence $(\xi_n)_{n\in\mathbb{N}}$, so that $\mathcal{F}_n = \sigma\{\xi_1, \ldots, \xi_n\}$. We use the standard abbreviations "a.s."

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for either "almost sure" or "almost surely" with respect to a fixed probability measure $\mathbb{P}$, and "i.i.d." for "independent and identically distributed," as it applies to sequences of random variables. A detailed discussion of relevant stochastic concepts and notation can be found, for example, in Ref. 34.

Since equations in the present paper are motivated by population models, we will assume bounded stochastic perturbations in the following sense:

Assumption I.2. *$(\xi_n)_{n\in\mathbb{N}}$ is a sequence of independent identically distributed random variables, each satisfying $|\xi_n| \le 1$.*

The paper is organized as follows. The main stabilization theorem is presented in Sec. II in its most general form. A stochastic TOC method is introduced and discussed in Sec. II A. Results obtained in Sec. II B for stochastic PBC are generalizations of Ref. 7. In Sec. III, we illustrate some of our results with computer simulations. Section IV contains a brief summary and discussion of potential directions for future research. All proofs are deferred to the Appendix.

II. LOCAL STABILIZATION OF A POINT EQUILIBRIUM AT ZERO BY PULSED STOCHASTIC CONTROL

In this section, we present a generalized control theorem that will be applied to specific classes of model and control-type in the remainder of the article. Consider the difference equation

$$z_{n+1} = g(z_n), \quad n \in \mathbb{N}_0, \quad z_0 > 0, \tag{5}$$

where the function g satisfies a Lipschitz-type condition locally around zero:

Assumption II.1. *For some $u_0 > 0$, there exists $L \ge 1$ such that*

$$|g(z)| \le L|z|, \quad |z| \le u_0. \tag{6}$$

Condition (6) in Assumption II.1 is sufficient to ensure that Eq. (5) has a point equilibrium at zero, which we aim to stabilize by the application of pulsed stochastic control at each kth step, starting with the step $k-1$. In this article, we are not concerned with the case where $L < 1$, since it would immediately follow that $\lim_{n\to\infty} z_n = 0$, for $|z_0| \le u_0$, and a control is unnecessary.

First, we characterize the control, which may depend on the function g, on a deterministic control parameter $\alpha \in [0,1)$, and on a coefficient $l > 0$ describing the amplitude of a one-dimensional stochastic perturbation, satisfying Assumption I.2. Note that if $\alpha = 0$, any achievable control is due only to this perturbation.

Suppose we apply a general stochastic control to the right-hand side of (5) at the $n = sk-1$ step, for each $s \in \mathbb{N}$, and represent the resulting stochastically controlled map by the function $G : \mathbb{R}\times[0,1]\times[0,l_0]\times[-1,1] \to \mathbb{R}$ for some $l_0 > 0$. Then, the stochastically controlled difference equation becomes (again with $z_0 > 0$)

$$z_{n+1} = \begin{cases} G(z_n, \alpha, l, \xi_{n+1}), & n = sk-1,\ s \in \mathbb{N}, \\ g(z_n), & \text{otherwise.} \end{cases} \tag{7}$$

Next, we place constraints on the form of the stochastically controlled map G under which we will prove our main result in this section.

Assumption II.2. *Define the region $\mathcal{B} := \{\alpha \in [0,1), l \in [0,l_0], |v| \le 1\}$, and suppose that (6) in the statement of Assumption II.1 holds. There exists a continuous function $\mathcal{L} : [0,1]\times[0,l_0]\times[-1,1]\times[0,\infty) \to (0,\infty)$ such that*

(i) *for $(\alpha,l,v) \in \mathcal{B}$,*

$$\begin{aligned} &|G(z,\alpha,l,v)| \le \mathcal{L}(\alpha,l,v,u)|z|, \\ &|z| \le u \le u_0, \quad (\alpha,l,v) \in \mathcal{B}; \end{aligned} \tag{8}$$

(ii) *for some $M > 0$*

$$\sup\{\mathcal{L}(\alpha,l,v,u) : (\alpha,l,v) \in \mathcal{B},\ |u| \le u_0\} = M; \tag{9}$$

(iii) *for ξ satisfying Assumption I.2, for L as given in (6), for k as given in (7), and for some $\alpha \in [0,1)$, $l \in [0,l_0]$,*

$$\lambda := -\mathbb{E}\ln\mathcal{L}(\alpha,l,\xi,0) > (k-1)\ln L. \tag{10}$$

In our applications, G is the form in which the right-hand side of the equation takes after the shift of the equilibrium to zero, or after some other transformations, and after application of control. The function $\mathcal{L}$ is a local Lipschitz constant of G at zero. Since the control is random, we have both G and $\mathcal{L}$ random.

Remark II.3. *Due to the continuity of $\mathcal{L}$ in u, condition (10) implies that for some $u_1 \in (0,u_0]$, $0 \le u \le u_1$,*

$$\lambda(u) := -\mathbb{E}\ln\mathcal{L}(\alpha,l,\xi,u) > (k-1)\ln L. \tag{11}$$

Inequality (10) is the main assumption of the paper, and it immediately implies (11). The fact that (10) and (11) guarantee stability of the zero equilibrium is a consequence of Kolmogorov's law of large numbers. This applies to models that without stochasticity will be unstable and can be connected to the illustration with Kapica's pendulum.[22] Mathematically, this can be roughly described as possible decrease of Lyapunov exponents by introducing a random component with a zero mean. This approach goes back to Kesten in 1960–1970s;[17,23] see Ref. 9 for more details. Note that conditions (10) and (11) are quite close to necessary.[1] All the results of the present paper are proved under this condition, and control parameters for all the examples are chosen to satisfy this assumption.

Now, we present the main result of this section.

Theorem II.4. *Let Assumptions I.2, II.1, and II.2 hold, and let $\gamma \in (0,1)$. Then, there exist $\delta_0 > 0$ and $\Omega_\gamma \subseteq \Omega$, $\mathbb{P}(\Omega_\gamma) > 1-\gamma$, such that for each solution z to Eq. (7) with initial value $|z_0| \le \delta_0$, we have*

$$\lim_{n\to\infty} z_n(\omega) = 0,\ \omega \in \Omega_\gamma,$$

where $z_n(\omega)$ is a sample path of the solution z_n. If we additionally suppose that $ML^{k-1} < 1$ and $|z_0| \le \frac{u_0}{L^{k-1}}$, then

$$\lim_{n\to\infty} z_n(\omega) = 0, \quad \omega \in \Omega.$$

Remark II.5. *The case $ML^{k-1} < 1$ can hold when G is nonrandom, or when $\mathcal{L}$ is nonrandom. In particular, it is the case for PBC with a control that is either deterministic or subject to low-intensity stochastic perturbation (see Ref. 7). We discuss this in more detail in Secs. II A and II B; see also Remarks II.12 and II.19.*

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In this section, we apply Theorem II.4 to stabilize cycles of

$$x_{n+1} = f(x_n), \quad n \in \mathbb{N}_0, \quad x_0 > 0.$$

First, we specify the structure of the map f. Then, we impose upon f stochastic versions of TOC and PBC. The resulting stochastically controlled maps can be converted to form (7). For each model, we then derive assumptions on the control parameter α and the noise intensity l, which ensure condition (10) in the statement of Assumption II.2, allowing us to apply Theorem II.4.

Suppose that f is a real-valued and non-negative function possessing a cycle of period $d \in \mathbb{N}$, and it satisfies a Lipschitz-type condition locally around each point in the cycle:

Assumption II.6. *For some $u_0 > 0$ and $d \in \mathbb{N}$, the continuous function $f : \mathbb{R} \to [0, \infty)$ is such that*

(a) *$f(K_i) = K_{i+1}$ for $i = 1, \dots, d$, where $K_{d+1} := K_1$;*
(b) *there exist $L_i > 0$, $i = 1, \dots, d$, such that*

$$|f(x) - K_{i+1}| \le L_i |x - K_i|, \; x \in [K_i - u_0, K_i + u_0]. \tag{12}$$

Remark II.7. *If f satisfies Assumption II.6, it is not necessarily differentiable at K_i; consider, for example, $f(x) = |x|$. However, if $f'(K_i)$ exists, then for each $u \in (0, u_0)$ and $x \in [K_i - u, K_i + u]$, (12) is satisfied with $L_i = |f'(K_i)| + \varepsilon(u)$, where $\lim_{u \to 0} \varepsilon(u) = 0$.*

Notationally, set $f^2(x) = f(f(x))$, $f^j(x) = f(f^{j-1}(x))$, $j \in \mathbb{N}$ and note that under Assumption II.6, each point of the set $\{K_1, K_2, \dots, K_d\}$ is an equilibrium for f^d. It follows that f^d satisfies a generalized Lipschitz-type condition locally around each K_i:

Lemma II.8. *Let Assumption II.6 hold and*

$$L(d) := \prod_{i=1}^{d} \max\{1, L_i\}. \tag{13}$$

Then, for $i = 1, \dots d$,

$$|f^d(x) - K_i| \le L(d)|x - K_i| \quad \text{for} \quad |x - K_i| \le \frac{u_0}{L(d)}. \tag{14}$$

Next, we assume a Lipschitz-type relationship between the position of x in the vicinity of a point in the d-cycle K_i, and the relative position of $f(x)$ to the next point in the d-cycle K_{i+1}.

Assumption II.9. *For some $u_0 > 0$ and $d \in \mathbb{N}$, the function $f : \mathbb{R} \to [0, \infty)$ satisfies Part (a) of Assumption II.6. There exist constants $\mathcal{A}_i \in \mathbb{R}$ and functions $\phi_i : \mathbb{R} \to \mathbb{R}$, $\psi_i : \mathbb{R} \to [0, \infty)$, $i = 1, \dots, d$, such that*

(i) *$\psi_i(u) \to 0$ as $u \to 0$;*
(ii) *for each $u \in (0, u_0)$ and $x \in [K_i - u, K_i + u]$, $i = 1, 2, \dots, d$,*

$$\begin{aligned} f^i(x) &= K_{i+1} + \mathcal{A}_i(x - K_i) + \phi_i(x); \\ |\phi_i(x)| &\le \psi_i(|x - K_i|)|x - K_i|. \end{aligned} \tag{15}$$

Since $K_{d+1} = K_1$, it follows that x and $f^d(x)$ have a similar relationship in the vicinity of K_1:

Lemma II.10. *Suppose that Assumption II.9 holds, and define for $u \in (0, 1)$,*

$$\mathcal{A}(d) := \prod_{i=1}^{d} \mathcal{A}_i, \; u(d) := \frac{u}{\prod_{i=1}^{d} \max\{|\mathcal{A}_i| + \psi_i(u), \, 1\}}. \tag{16}$$

Then, there exist functions $\bar{\phi} : \mathbb{R} \to \mathbb{R}$ and $\bar{\psi} : \mathbb{R} \to [0, \infty)$ such that for $x \in [K_1 - u(d), K_1 + u(d)]$,

$$\begin{aligned} f^d(x) &= K_1 + \mathcal{A}(d)(x - K_1) + \bar{\phi}(x), \\ |\bar{\phi}(x)| &\le \bar{\psi}(|x - K_1|)|x - K_1|, \end{aligned} \tag{17}$$

where $\bar{\psi}(u) \to 0$ as $u \to 0$.

A. Target-oriented control

Deterministic modified TOC control is characterized in general by Eq. (4). Consider a particular case when the target T coincides with the equilibrium K of f, and where the control parameter α is stochastically perturbed by an additive noise of intensity l. Then, (4) becomes

$$x_{n+1} = (1 - \alpha - l\xi_{n+1})f(x_n) + (\alpha + l\xi_{n+1})K \tag{18}$$

for $n \in \mathbb{N}_0$. In fact, we apply (4) not for all $n \in \mathbb{N}_0$ but at each kth step, and we aim at either a point or a cycle stabilization. To the best of our knowledge, a combination of TOC with either pulsed control, stochastic control, or the use of part of a cycle as a target is novel and we tackle here all three tasks.

In Secs. II A 1 and II A 2, we present equations and conditions for the local stabilization of a point equilibrium and d-cycle, respectively, using stochastic TOC. In Sec. II A 3, we investigate global stabilization.

1. Pulsed stochastic TOC: Stabilization of a point equilibrium

Suppose $f(K) = K$, for $K > 0$. Consider the stochastic TOC model, pulsed at each kth step, with target K,

$$x_{n+1} = \begin{cases} (1 - \alpha - l\xi_{n+1})f(x_n) + (\alpha + l\xi_{n+1})K, \\ \quad n = sk - 1, \quad s \in \mathbb{N}, \\ f(x_n), \quad \text{otherwise}, \end{cases} \tag{19}$$

$$|x_0 - K| < \delta.$$

If we denote

$$z_n := x_n - K, \quad g(z) := f(z + K) - K, \tag{20}$$

then (19) takes the form of (7) with

$$G(z_n, \alpha, l, \xi_{n+1}) = (1 - \alpha - l\xi_{n+1})g(z_n). \tag{21}$$

Assuming that

$$|f(z + K) - K| \le L|z|, \quad |z| \le u_0, \tag{22}$$

we get $\mathcal{L}(\alpha, l, v, u) = |1 - \alpha - lv|L$, with $M = L \max_{|v| \le 1} |1 - \alpha - lv|$, and condition (10) takes the form

$$\lambda = -\mathbb{E} \ln |1 - \alpha - l\xi| > k \ln L. \tag{23}$$

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Here, we also assume that $\xi \neq \frac{1-\alpha}{l}$ in the case when ξ has a discrete distribution; see more details in Ref. 9. The following result now follows directly by an application of Theorem II.4:

Theorem II.11. *Let Assumption I.2 and conditions (22) and (23) hold. Then, for each $\gamma \in (0,1)$, there exist $\delta_0 > 0$ and $\Omega_\gamma \subset \Omega$, where $\mathbb{P}(\Omega_\gamma) > 1-\gamma$, such that for each solution $(x_n)_{n\in\mathbb{N}}$ to Eq. (19) with initial value satisfying $|x_0 - K| \le \delta_0$, we have*

$$\lim_{n\to\infty} x_n(\omega) = 0, \quad \omega \in \Omega_\gamma.$$

Remark II.12. *Note that for $\alpha < 1-l$, we have $M = L(1-\alpha+l)$, and condition (23) holds if $1-\alpha+l \le L^{-k}$, which gives the following ranges for the parameter α and for the noise intensity l to ensure stabilization:*

$$\alpha \in \left(1-L^{-k},\, 1\right), \quad l \le \min\left\{1-\alpha,\, \alpha - 1 + L^{-k}\right\}. \tag{24}$$

For a large value of L, the parameter α needs to be close to one, and l needs to be small. In this case, stabilization is due to the deterministic control α, and only a small stochastic disturbance is allowed.

It is also possible to demonstrate active stabilization by noise, when l is bigger than in (24). For Bernoulli-distributed ξ (taking each of the values ± 1 with the probability of 0.5), we have $-\lambda = \frac{1}{2}\ln\left|(1-\alpha)^2 - l^2\right|$, and (23) holds if

$$\sqrt{(1-\alpha)^2 - L^{-2k}} < l < \sqrt{(1-\alpha)^2 + L^{-2k}}. \tag{25}$$

Note that (25) can remain valid even for large L and $\alpha = 0$, but then $l < 1$ should be close to 1.

2. Pulsed stochastic TOC: Stabilization of a d-cycle

Let $k = md$ for some $m \in \mathbb{N}$, and let f satisfy Assumption II.6. Recall that each K_i, $i = 1,\dots,d$, is a fixed point of f^d, and therefore of f^{md}. For simplicity, we focus only on K_1, but our analysis applies equally to any other point in the cycle.

Consider the equation

$$x_{n+1} = \begin{cases} (1-\alpha-l\xi_{n+1})f(x_n) + (\alpha + l\xi_{n+1})K_1, \\ \quad n = smd-1,\ s \in \mathbb{N}, \\ f(x_n), \quad \text{otherwise}, \quad |x_0 - K_1| \le \delta. \end{cases} \tag{26}$$

Set $y_s := x_{(s-1)md}$, $\bar{\xi}_s := \xi_{(s-1)md}$, for $s \in \mathbb{N}$, and note that the sequence $(\bar{\xi}_s)_{s\in\mathbb{N}}$ satisfies Assumption I.2. For $n = smd-1$, $s \in \mathbb{N}$, we have

$$\begin{aligned} &x_n = x_{smd-1}, x_{n+1} = x_{smd} = y_{s+1}, \quad y_1 = x_0, \\ &f(x_n) = f^{md}(x_{n-md+1}) \\ &\quad = f^{md}(x_{(s-1)md}) = f^{md}(y_s). \end{aligned} \tag{27}$$

Thus, (26) can be transformed to

$$\begin{aligned} &y_{s+1} = (1-\alpha-l\bar{\xi}_{s+1})f^{md}(y_s) + (\alpha + l\bar{\xi}_{s+1})K_1, \\ &s \in \mathbb{N}, \quad |y_1 - K_1| \le \delta, \end{aligned} \tag{28}$$

which is in the form of (19) with $k = 1$, $K = K_1$, f^{md} instead of f, y_1 instead of x_0 and s starting from 1.

Equation (28), in turn, can be transformed to (7) if we set

$$z_s := y_s - K_1, \quad g(z) := f^{md}(z + K_1) - K_1.$$

Note that, by Lemma II.8, g satisfies (6) with constant $L(md) := L^m(d)$, where $L(d)$ is defined by (13). Recall that $L(d) \ge 1$. For G defined as in (21), $\mathcal{L}(\alpha, l, v, u) = |1-\alpha-lv|L^m(d)$, $M := (1-\alpha+l)L^m(d)$, condition (10) takes the form

$$\lambda := -\mathbb{E}\ln|1-\alpha-l\xi| > m\ln L(d). \tag{29}$$

Therefore, Theorem II.4 implies $\lim_{s\to\infty} y_s = \lim_{s\to\infty} x_{(s-1)md} = K_1$, with any given probability $1-\gamma$ and small enough δ_0.

To extend this result to show that $\lim_{n\to\infty} x_{nd+\bar{j}} = K_{\bar{j}}$, for each $\bar{j} = 0, 1, \dots, d-1$, we require the next lemma:

Lemma II.13. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution of Eq. (26). Let Assumptions I.2 and II.6, and condition (29) hold. Then, for each $\gamma \in (0,1)$, there exist $\delta_0 > 0$ and $\Omega_\gamma \subset \Omega$, with $\mathbb{P}(\Omega_\gamma) > 1-\gamma$, such that if $|x_0 - K_1| \le \delta_0$, and s_0 is such that*

$$|x_{smd} - K_1| < u_0 L^{-m}(d)|1-\alpha+l|^{-m+1}, \quad s \ge s_0,$$

then for $j = qd + \bar{j}$, $\bar{j} = 0, 1, \dots, d-1$, $q = 0, 1, \dots, m-1$, we have

$$|x_{smd+j} - K_{\bar{j}+1}| \le |1-\alpha+l|^{m-1}L^m(d)|x_{smd} - K_1|, \ s \ge s_0.$$

All the above brings us to the following theorem:

Theorem II.14. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution of Eq. (26). Let Assumptions I.2 and II.6, and condition (29) hold. Then, for each $\gamma \in (0,1)$ there exist $\delta_0 > 0$ and $\Omega_\gamma \subset \Omega$, with $\mathbb{P}(\Omega_\gamma) > 1-\gamma$, such that if $|x_0 - K_1| \le \delta_0$,*

$$\lim_{n\to\infty} x_{nd+\bar{j}}(\omega) = K_{\bar{j}}, \quad \omega \in \Omega_\gamma, \quad \bar{j} = 0, 1, \dots, d-1.$$

3. Global stabilization of a d-cycle by stochastic TOC

Observe that if (22) [when $d = 1$, (12) otherwise] holds globally on $\mathbb{R}$, condition (23) [when $d = 1$, (29) otherwise] also holds. It is then possible to show that stochastic TOC (19) [respectively, (26)] globally stabilizes the equilibrium K (or a d-cycle).

Theorem II.15. *Theorem II.11 (respectively, Theorem II.14) holds for any $x_0 > 0$ if, in conditions (22) and (23) [respectively, conditions (12) and (29)], local Lipschitz constants are replaced with global Lipschitz constants.*

The proof modifies that of Theorem II.4 so that solutions are not required to stay in some neighborhood of the initial value. Note, however, that the global Lipschitz constant $\bar{L}$ at the point K (or in the case of a d-cycle, $\bar{L}_i$ at each K_i, $i = 1, 2 \dots, d$) can be quite large, reaching up to $\sup_{s\in\mathbb{R}}|f'(s)|$. Nonetheless, we will see in Example III.1 that, in the case of Bernoulli ξ, condition (23) [respectively, (29)] holds for large $\bar{L}$ [or $\bar{L}^m(d)$] even with $\alpha = 0$ if l satisfies (25), where we replace L by $\bar{L}$ [or in the case of a d-cycle, L is replaced by $\bar{L}^m(d)$ and k by md].

B. Predictive-based control

The application of stochastic PBC is characterized in general by Eq. (2). Following the order of investigation in Sec. II A, we will apply pulsed stochastic PBC at each kth step to stabilize a point equilibrium in Sec. II B 1 and to stabilize a d-cycle in Sec. II B 2.

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The results of this section are illustrated for a point equilibrium in Example III.3, where even local stabilization is not possible for any $\alpha \in (0,1)$ in the absence of a stochastic perturbation, and in Example III.5, where global stabilization is considered. The application of pulsed stochastic PBC to stabilize a two-cycle is illustrated in Example III.4.

1. Pulsed stochastic PBC: Stabilization of a point equilibrium

Suppose $f(K) = K$ for $K > 0$, and consider the stochastic PBC model, pulsed at each kth step,

$$x_{n+1} = \begin{cases} (1-\alpha-l\xi_{n+1})f(x_n) + (\alpha + l\xi_{n+1})x_n, \\ \quad n = sk-1,\ s \in \mathbb{N}, \\ f(x_n), \quad \text{otherwise}, \quad |x_0 - K| \le \delta, \end{cases} \tag{30}$$

which, if we again use notation defined by (20), takes the form of (7), with

$$G(z,\alpha,l,v) := (1-\alpha-lv)g(z) + (\alpha+lv)z. \tag{31}$$

We may identify constraints on f that ensure condition (10) holds. Suppose first that for $|x-K| \le u \le u_0$, f admits expansion (15) with $d=1$, $K_i \equiv K$, $\mathcal{A}_i \equiv \mathcal{A}$, $\phi_i(u) \equiv \phi(u)$, and $\psi_i(u) \equiv \psi(u)$. In particular, this means that f is differentiable at K with derivative $\mathcal{A}$. Then,

$$\begin{aligned} G(z_n,\alpha,l,\xi_{n+1}) &:= \left[(1-\alpha-l\xi_{n+1})\mathcal{A} + \alpha + l\xi_{n+1}\right]z_n \\ &\quad + [1-\alpha-l\xi_n]\phi(z_n), \\ \mathcal{L}(\alpha,l,v,u) &:= \left|(1-\alpha)\mathcal{A} + \alpha + (1-\mathcal{A})lv\right| \\ &\quad + |1-\alpha+l|\psi(u), \\ L &:= \mathcal{A} + \psi(u), \end{aligned}$$

and conditions (10) and (by Remark II.3) (11) hold if

$$-\mathbb{E}\ln\left|(1-\alpha)\mathcal{A} + \alpha + (1-\mathcal{A})l\xi\right| > (k-1)\ln|\mathcal{A}|. \tag{32}$$

The following theorem then follows immediately:

Theorem II.16. *Let Assumptions I.2 and II.9 with $d=1$, and condition (32) hold, and let $(x_n)_{n\in\mathbb{N}}$ be a solution of (30). Then, for each $\gamma \in (0,1)$, there exist $\delta_0 > 0$ and $\Omega_\gamma \subset \Omega$ where $\mathbb{P}(\Omega_\gamma) > 1-\gamma$, such that, if $|x_0 - K| \le \delta_0$, we have*

$$\lim_{n\to\infty} x_n(\omega) = 0, \quad \omega \in \Omega_\gamma.$$

Remark II.17. *Relation (32) fails if $\mathcal{A} > 1$ and $l = 0$ for any $\alpha \in (0,1)$ and $k \in \mathbb{N}$. However, the presence of noise with $l > 0$ can ensure local stability even for $\alpha = 0$ and large $\mathcal{A}$. To see this, assume that ξ is Bernoulli distributed. Then,*

$$\lambda = -\frac{1}{2}\ln\left|[(1-\alpha)\mathcal{A}+\alpha]^2 - (1-\mathcal{A})^2 l^2\right|,$$

and (32) holds if

$$l_{low} := \frac{[(1-\alpha)\mathcal{A}+\alpha]^2 - \mathcal{A}^{-2(k-1)}}{(1-\mathcal{A})^2} < l^2 < \frac{[(1-\alpha)\mathcal{A}+\alpha]^2 + \mathcal{A}^{-2(k-1)}}{(1-\mathcal{A})^2}.$$

If $\alpha = 0$ and $k = 1$, the lower bound on l is given by $l_{low} = \left(\frac{\mathcal{A}^2-1}{(1-\mathcal{A})}\right)^2 = 1 + \frac{2}{\mathcal{A}-1} > 1$, for each $\mathcal{A} > 1$. For example, if $\mathcal{A} = 2$, $\alpha = 0$, and $k = 1$, we required $1.73 \approx \sqrt{3} < l < \sqrt{5}$ for (32) to hold. So, it is reasonable to combine a nonzero control parameter α with nonzero noise intensity l.

Note that for $k = 1$ and any $\mathcal{A} \neq 1$ the lower bound on l satisfies

$$l_{low} = \frac{(1-\alpha)\left[(1-\alpha)(\mathcal{A}-1)+2\right]}{\mathcal{A}-1} \to 0, \quad \text{as } \alpha \to 1,$$

while for $k > 1$

$$l_{low} = \frac{[(1-\alpha)(\mathcal{A}-1)+1]^2 - 1}{(1-\mathcal{A})^2} + \frac{1-\mathcal{A}^{-2(k-1)}}{(1-\mathcal{A})^2} \to \frac{1-\mathcal{A}^{-2(k-1)}}{(1-\mathcal{A})^2} \quad \text{as } \alpha \to 1.$$

Therefore, when $k = 1$ and $\mathcal{A} \neq 1$, for any $\varepsilon > 0$, we can choose $\alpha \in (0,1)$ such that the l_{low} satisfies $l_{low} < \varepsilon$. In other words, small noise stabilizes the equilibrium if α is close to 1. When $k > 1$ and $\mathcal{A} > 2$,

$$\frac{1-\mathcal{A}^{-2(k-1)}}{(1-\mathcal{A})^2} < \frac{1}{(1-\mathcal{A})^2} < 1,$$

so there exists $\alpha \in (0,1)$ such that $l_{low} < 1$.

We can relax the assumption that f is differentiable at K, instead requiring only that $f(x) - K$ changes sign from positive to negative as x increases through some neighborhood of K. This corresponds to the case $\mathcal{A} < -1$ if f is differentiable at K.

Theorem II.18. *Let $(x_n)_{n\in\mathbb{N}}$ be a solution to (30), where (22) holds and suppose that $f(x) > K$ for $x \in [K-u_0, K]$, $f(x) < K$ for $x \in [K, K+u_0]$, and $l \in (0, \min\{\alpha, 1-\alpha\})$.*

(i) *If either*

$$k = 1, \quad \alpha > 1 - L^{-1}, \quad l < L^{-1} - 1 + \alpha, \tag{33}$$

or

$$k > 1,\ 1 < L^k < L+1,\ \alpha \in \left(1-L^{-k}, L^{-k+1}\right), \quad l \in \left(0, \min\left\{L^{-k} - 1 + \alpha,\ \ L^{-k+1} - \alpha\right\}\right); \tag{34}$$

then, for $|x_0 - K| \le \delta \le u_0/L^k$,

$$\lim_{n\to\infty} x_n(\omega) = K, \quad \textit{for all} \quad \omega \in \Omega.$$

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(ii) *If*

$$\lambda := -\mathbb{E}\max\left\{\ln(|\alpha + l\xi|), \ln\left(|1-\alpha-l\xi|L\right)\right\} > (k-1)\ln L, \tag{35}$$

then, for any $\gamma \in (0,1)$, *there exist* $\delta > 0$ *and* $\Omega_\gamma \subset \Omega$ *with* $\mathbb{P}(\Omega_\gamma) > 1-\gamma$, *such that, for* $|x_0 - K| \le \delta$, *we have*

$$\lim_{n\to\infty} x_n(\omega) = K, \quad \omega \in \Omega_\gamma.$$

Remark II.19. *In Theorem II.18, part (i) describes a situation, where stabilization is due to the action of the underlying deterministic control, and the noise intensity* l *is kept small to preserve this effect. The case of* $k = 1$ *in part (i) of Theorem II.18, where (33) holds was covered in Ref. 7 and this analysis included establishing the global stability. However, for local stability, Theorem II.18 generalizes the results of Ref. 7 to the case where the noise plays an active role in achieving stability: it applies to the situations where, for a chosen* $\alpha \in (0,1)$ *and* $l = 0$, *the point equilibrium is unstable. The use of pulsed control here to achieve stabilization is also novel.*

In part (ii), the noise also plays an active role, and (35) gives a set of stabilizing parameters different from those in part (i). Here, we present an example where (35) is fulfilled, deferring a more detailed description and illustrative numerical simulation until Example III.5.

The inequality $\mathbb{E}\ln[1-\alpha-l\xi] < k\ln L$ *implies (35) if* $(1-\alpha-lv)L \ge \alpha + lv$ *holds for each* $|v| \le 1$. *The latter is true when* $\alpha + l < 1 - (1+L)^{-1}$. *In the case of Bernoulli-distributed* ξ, *this gives a lower bound for* l^2 *as* $l_{low} = (1-\alpha)^2 - L^{-2k}$. *It can be shown that the Ricker model with* $r = 2.41$, $L = 1.5$, *and the control with* $\alpha = 0.3$, $l = 0.24$, *and* $k = 1$, *satisfies (35) in part (ii) but not (33) in part (i) of Theorem II.18.*

Suppose more specifically that $(1-\alpha-lv)L = \alpha + lv$ *for some* $v \in (-1,1)$, *and* $\alpha + l > 1 - (L+1)^{-1}$. *Then, in order to satisfy (35), we need* $\alpha(1-\alpha) + l(1+l) < L^{-k}$. *It can be shown that for the Ricker model with* $r = 2.2$, *the values* $L = 1.2$, $\alpha = 0.28$, $l = 0.27$, $k = 2$ *satisfy (35) in part (ii), but not (34) in part (i) of Theorem II.18. More details may be found in Example III.5.*

Remark II.20. *There are cases (applicable to both Ricker and logistic models) for which local stability implies global stability. Suppose* $k = 1$, *so that control is applied at every step. Then, in the deterministic case, we have*

$$f_\alpha(x) := (1-\alpha)xe^{r(1-x)} + \alpha x,$$
$$f'_\alpha(x) = (1-\alpha)(1-rx)e^{r(1-x)} + \alpha.$$

The controlled map f_α *is unimodal with a negative Schwarzian derivative, and so equilibria of the controlled deterministic equation are globally stable once they are locally stable. The general form of this result is due to Singer[35] (see also Ref. 24). For deterministic PBC, the result is in Ref. 25 and some extensions of the idea can be found in Ref. 16. The point equilibrium* $K = 1$ *for the Ricker model is locally stable if*

$$f'_\alpha(1) = (1-\alpha)(1-r) + \alpha > -1,$$

or $\alpha \in (\alpha^*, 1)$, *where* $\alpha^* = (r-2)/r$. *According to Ref. 7, stabilization is achieved once* $(\alpha - l, \alpha + l) \subseteq (\alpha^*, 1)$.

2. Pulsed stochastic PBC: Stabilization of a d-cycle

Suppose that $k = md$, $m \in \mathbb{N}$, and Assumption II.9 holds. Consider the equation

$$x_{n+1} = \begin{cases} (1-\alpha-l\xi_{n+1})f(x_n) + (\alpha + l\xi_{n+1})x_{n-md+1}, \\ \quad n = smd - 1,\ s \in \mathbb{N}, \\ f(x_n), \quad \text{otherwise}, \end{cases} \qquad |x_0 - K_1| \le \delta. \tag{36}$$

To this model, we apply transformation (27) using notation as in Sec. II A 2, to get

$$y_{s+1} = \left(1-\alpha-l\bar{\xi}_{s+1}\right) f^{md}(y_s) + \left(\alpha + l\bar{\xi}_{s+1}\right) y_s,$$
$$s \in \mathbb{N}, \quad |y_1 - K_1| \le \delta,$$

which is covered by the case discussed in Sec. II B 1. To see this, substitute f^{dm} for f, and $\bar{\xi}_s := \xi_{sdm}$ for ξ_n and $k = 1$. Note that, by Lemma II.10, f^{md} admits the expansion (17), substituting md for d and $\mathcal{A}(md) = \mathcal{A}^m(d)$, where $\mathcal{A}(d)$ is defined as in (16). Therefore, condition (10) has the form

$$-\mathbb{E}\ln\left|(1-\alpha)\,\mathcal{A}^m(d) + \alpha + \left(1-\mathcal{A}^m(d)\right)l\xi\right| > 0. \tag{37}$$

Following the arguments of Sec. II A 2, we obtain the following theorem.

Theorem II.21. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a solution of Eq. (36). Let Assumptions I.2, II.9, and condition (37) hold. Then, for each* $\gamma \in (0,1)$, *there exist* $\delta_0 > 0$ *and* $\Omega_\gamma \subset \Omega$, *with* $\mathbb{P}(\Omega_\gamma) > 1-\gamma$, *such that if* $|x_0 - K_1| \le \delta_0$,

$$\lim_{n\to\infty} x_{nd+\bar{j}}(\omega) = K_{\bar{j}}, \quad \omega \in \Omega_\gamma, \quad \bar{j} = 0, 1, \ldots, d-1.$$

III. EXAMPLES AND COMPUTER SIMULATIONS

In all simulations presented in this section, we truncate the controlled map

$$x_{n+1} = \max\left\{(1-\alpha-l\xi_{n+1})f(x_n) + (\alpha + l\xi_{n+1})K, 0\right\}$$

in order to avoid negative values x_n.

We illustrate the results of Secs. II A and II B using difference equations associated with the Ricker function

$$f_1(x) = xe^{r(1-x)}, \quad x \ge 0, \tag{38}$$

the logistic map

$$f_2(x) = rx(1-x), \quad x \ge 0, \tag{39}$$

and the Maynard Smith model[36] with

$$f_3(x) = \frac{3x}{2 + (x-3)^2}, \quad x \ge 0. \tag{40}$$

Here, we simulate continuous uniformly distributed on $[0,1]$ and Bernoulli random variables ξ_n to illustrate some cases from Secs. II A and II B. Each plot incorporating stochastic perturbations ($l > 0$) was generated with three runs, with a single run used to generate deterministic plots.

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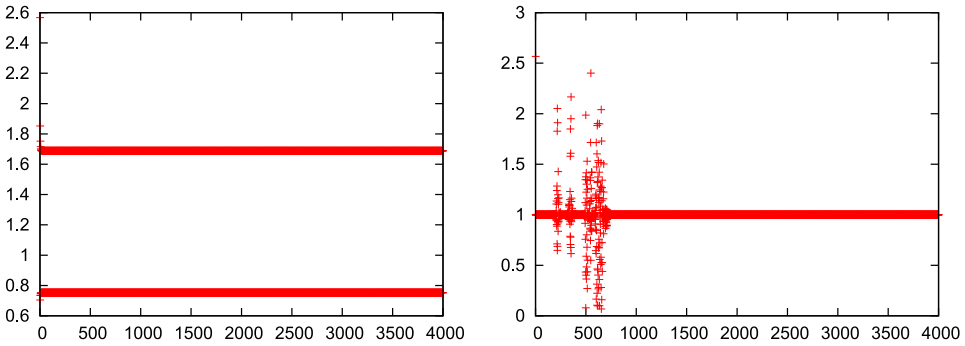


FIG. 1. Model (19) with $f = f_1$ from (38) with $r \approx 3.2716$, $\alpha = 0.7$, $m = 2$, $d = 1$, $x_0 = 0.5$, and (top, left) no noise, (top, right) $l = 0.32$, (bottom, left) $l = 0.4$, and (bottom, right) $l = 0.42$.

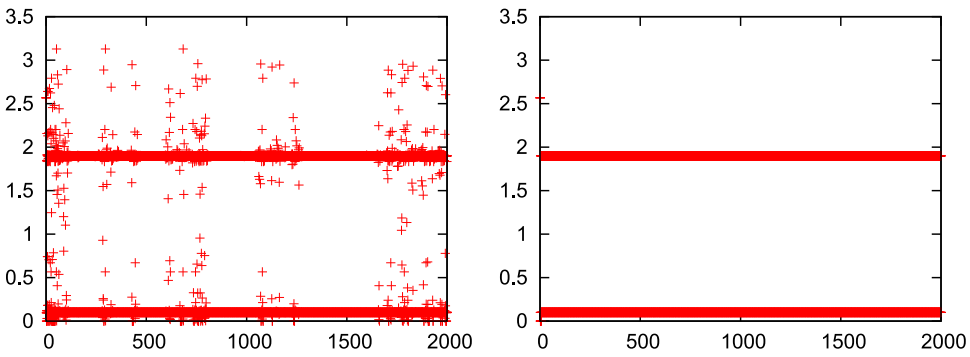


FIG. 2. Model (26) with $f = f_1$ from (38) with $r \approx 3.2716$, $\alpha = 0.3$, $m = 1$, $d = 2$, $x_0 = 0.5$, and $l = 0.65, 0.7$.

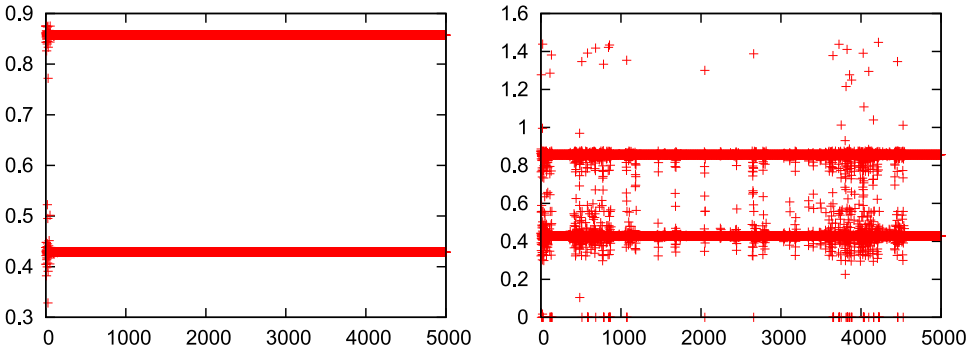


FIG. 3. Model (19) with $f = f_2$ from (39), $r = 3.5$, $\alpha = 0$, $m = 1$, $d = 2$, $x_0 = 0.5$, and (from left to right) $l = 1.2, 1.3$.

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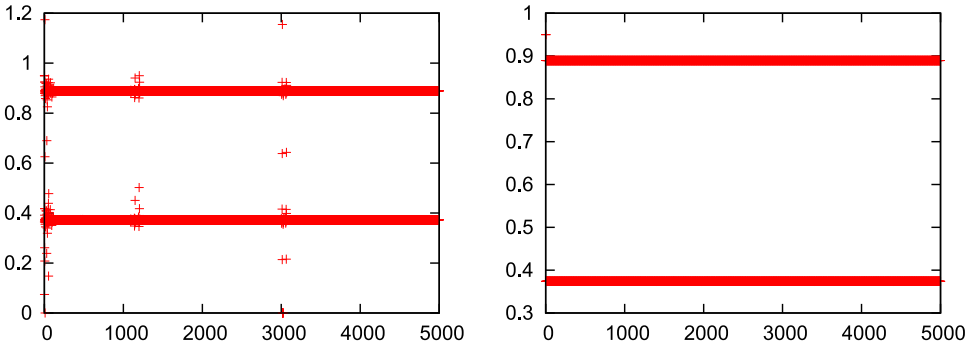


FIG. 4. Model (19) with $f = f_2$ from (39), $\alpha = 0.2$, $m = 1$, $d = 2$, $x_0 = 0.5$, and (from left to right) $l = 0.75, 0.8$.

We start with the TOC method. Examples III.1 and III.2 illustrate Theorem II.21, $d = 2$, applied to Ricker and logistic functions.

Example III.1. *First, we consider stochastic TOC (19) applied at alternate steps ($k = 2$) to a chaotic Ricker map f_1 satisfying (38) with $r \approx 3.2716$ and a continuous uniformly distributed on $[-1, 1]$ noise. Without noise, $\alpha = 0.7$ guarantees pulsed cycle stabilization, while uniformly distributed noise with $l = 0.4$ leads to stabilization of $K = 1$ (see Fig. 1).*

Next, apply a Bernoulli noise with a smaller $\alpha = 0.3$ to stabilize 2-cycle $K_1 = 0.1$, $K_2 = 1.9$ using K_1 as the target. Figure 2(right) presents stabilization for noise intensity $l = 0.7$. For $l = 0.7$, Fig. 2(left) illustrates that there is no convergence to this two-cycle, and similar results are obtained for $l > 0.75$; the range of values of l that allow stabilization is quite narrow.

Example III.2. *Consider stochastic TOC (19) applied at alternate steps ($k = 2$) to a logistic map f_2 satisfying (39) with $r = 3.5$. We can globally stabilize an unstable two-cycle. Figure 3 shows convergence with $\alpha = 0$, $r = 3.5$, and $l = 1.2$. For significantly smaller l, there is no convergence, and the effective stabilizing range for l is narrow. Increasing to $\alpha = 0.2$ in Fig. 4 leads to a higher convergence speed; see Fig. 4 for fast stabilization of a two-cycle for $l = 0.75$ and $l = 0.8$.*

Now, we proceed to the PBC method. Examples III.3 and III.4 illustrate Theorem II.21, $m = 1$, with $d = 1$ and $d = 2$, respectively.

Example III.3. *Consider stochastic PBC (30) applied at every step to a Maynard–Smith model f_3 satisfying (40) and note that this model is chaotic for $\alpha = 0$, $l = 0$. There are two positive equilibria*

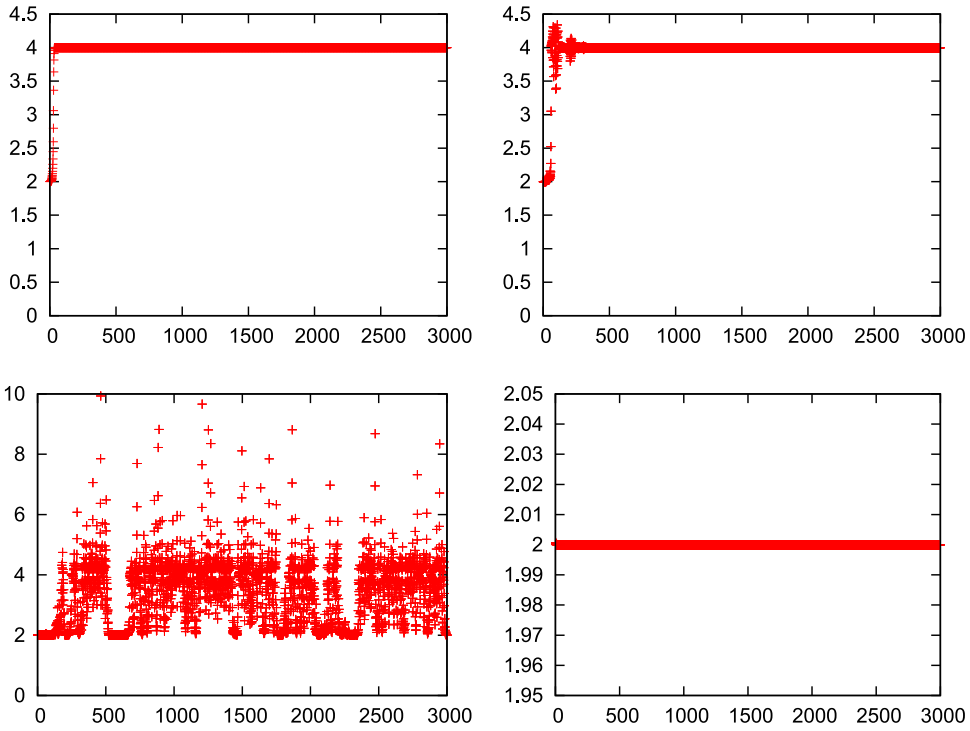


FIG. 5. Model (30) with $f = f_3$ from (40), $k = 1$, $\alpha = 0.2$, $x_0 = 2.001$, and (top) $l = 04$, $l = 0.4$ and (bottom) $l = 0.5$, $l = 0.7$.

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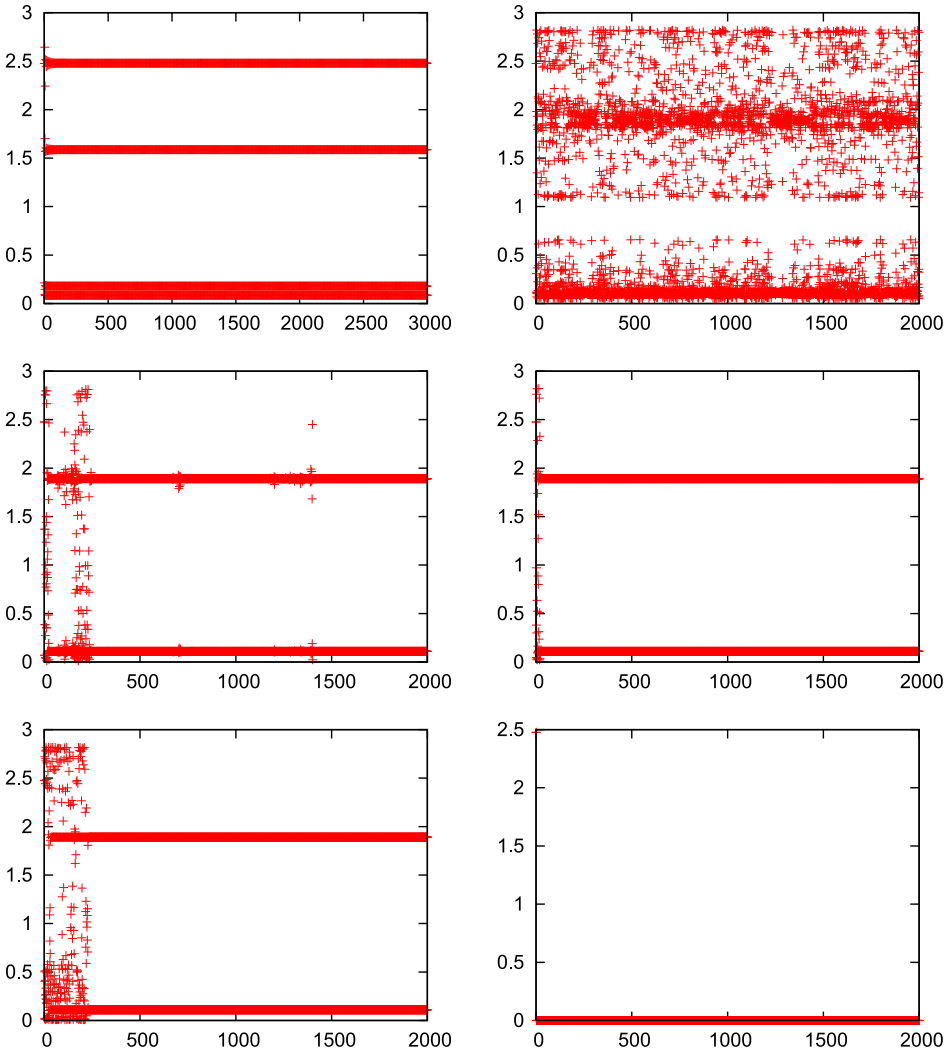


FIG. 6. Model (36) with $f = f_1$ from (38) with $r = 3.2$, $k = 2$, $\alpha = 0.4$, $x_0 = 0.5$, and (from left to right, top to bottom) $l = 0, 0.2, 0.3, 0.35, 0.4, 0.45$.

at $x \equiv 2$ and $x \equiv 4$, and $f_3'(2) = \frac{7}{3} > 1$. Note also that even local stabilization is not possible for any $\alpha \in (0,1)$ in the absence of noise ($l = 0$). Figure 5 illustrates local stabilization of the equilibrium $x \equiv 2$ with $\alpha = 0.8$ and $l = 0.4, 0.5, 0.7$. The solution $x \equiv 4$ is stable for $l = 0$, and there is no stabilization of $x \equiv 2$ for $l = 0.4$, we observe wandering between the two equilibria for $l = 0.5$ and stabilization of $x \equiv 2$ for $l = 0.7$.

Example III.4. *Let us illustrate Theorem II.21 for $d = 2$, $m = 1$. Applying pulsed stochastic PBC to stabilize a two-cycle, we consider (36), "delayed" stabilization, applied to a Ricker map f_1 satisfying (38) with $r = 3.2$, $\alpha = 0.4$. Figure 6 shows how an appropriately chosen noise intensity l leads to global stabilization of the 2-cycle $\{K_1 \approx 0.11, K_2 \approx 1.89\}$, and l changing from zero (no noise) to $l = 0.45$.*

The next example illustrates Theorem II.18.

Example III.5. *Consider stochastic PBC (30) applied at each step to a Ricker map f_1 satisfying (38) with $r = 2.41$. The global Lipschitz constant is $L = 1.5$. Note that according to Remark II.20, for $r = 2.41$, we get*

$$f_\alpha'(1) = -1.41(1-\alpha) + \alpha = 2.41\alpha - 1.41 > -1$$
$$\Leftrightarrow \alpha > \frac{0.41}{2.41} \approx 0.170\,124\,48.$$

Thus, the stabilization bound is $\alpha^* = 0.170\,1245$, and by Ref. 7, stabilization is achieved once $\alpha - l > \alpha^*$ and $\alpha + l < 1$. For $\alpha = 0.3$, $l = 0.24$, the first inequality is not satisfied $\alpha - l = 0.06 < \alpha^*$, so our previous result in Ref. 7 does not allow us to establish stability of the controlled model. It is possible to illustrate global stabilization of the equilibrium $K = 1$ with $k = 1$, $\alpha = 0.3$, and $l = 0.24$.

We can also make $K = 1$ stable with a stochastic pulsed control. Global stabilization by pulsed stochastic PBC (30) applied at alternate steps ($k = 2$) to a Ricker map f_1 satisfying (38) with $r = 2.2$ is demonstrated in Fig. 7. Only local stability conditions from Theorem II.16 hold, but global stability is observed. This indicates a possible direction for future research.

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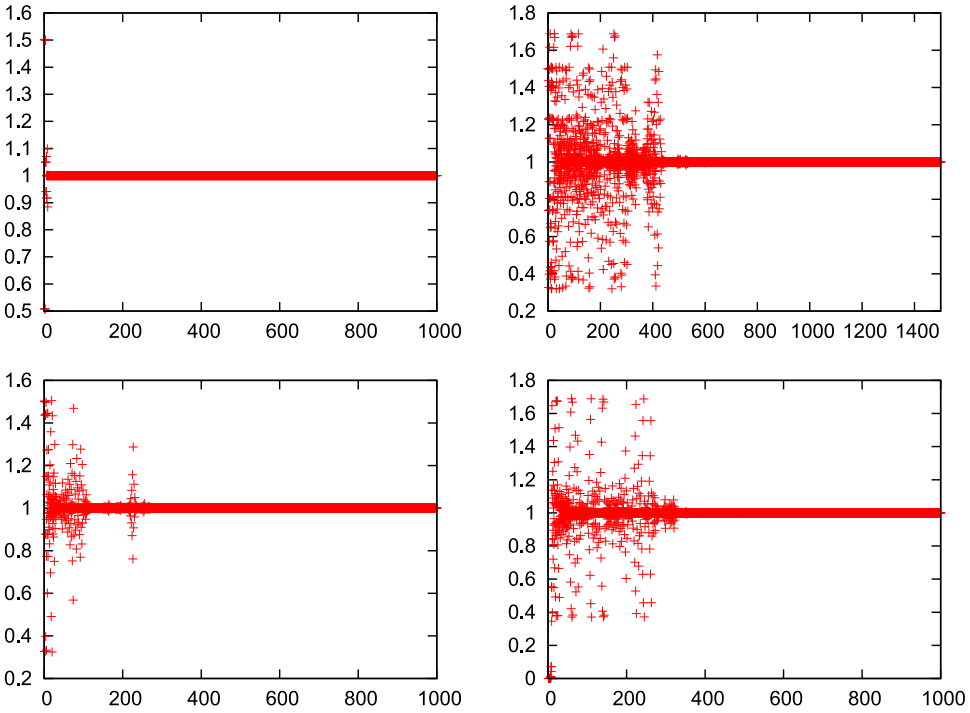


FIG. 7. Model (30) with $f = f_1$ from (38), $k = 2$, $r = 2.2$, and (top left) $\alpha = 0.28$, $l = 0.27$, $x_0 = 0.5$, (right) $l = 0.45$, $x_0 = 0.5$, (bottom) $\alpha = 0.1$, $l = 0.27$, (left) $x_0 = 0.5$, and (right) $x_0 = 10$.

IV. CONCLUSIONS

A. Summary of results

We have presented a general framework for stabilization of iterative systems by the application of control. The method demonstrates how the effective range of control parameters can be extended by stochastic perturbation.

Our results may be summarized as follows:

- We present a broad general characterization of control that allows us to consider mechanisms incorporating both deterministic and stochastic components. This characterization includes stochastic forms of prediction-based control and target-oriented control; the latter is considered here for the first time.
- Both regular (applied at each step) and pulsed (applied every kth step) types of control were investigated. Pulsed TOC applied to one-dimensional models has not been studied before. Further, while pulsed stabilization of an equilibrium or a cycle by deterministic control has attracted some attention (see, for example, Refs. 8 and 26), the use of stochastic PBC and TOC for k-cycle stabilization is novel.
- The analysis extends that of Ref. 7, which also showed how the effective range of stabilizing control parameters may be extended by the introduction of noise. However, in this article, we additionally explore pulsed stochastic control and the stabilization of unstable k-cycles.

B. Future directions for research

Future research will follow one (or more) of the four directions:

1. **Study of sharp and/or global stabilization conditions.** Note that some of our results are essentially local, and we observe this in simulation. Nevertheless, we believe it is still possible to get global stabilization results for the PBC method if we introduce some additional restrictions on f and choose the noise intensity appropriately. Example III.5 in fact illustrates the global stabilization for a wide range of parameters; however, theoretical justification is yet to be obtained. Also, the conditions for stabilization presented in this article are sufficient but not necessary. It is desirable to obtain necessary and sufficient conditions, as are available in the deterministic setting (see Refs. 15, 24, and 25).
2. **Explore the dependency of parameter bounds on the noise distribution type.** Most of our examples assume Bernoulli-distributed noise perturbation. Creating a library of sufficient estimates for control parameters and noise amplitudes under various types of noise distribution would be interesting and useful. For stabilization with noise only, such results can be found in Ref. 10.
3. **Generalization of scalar results to systems or higher-order difference equations.** In Refs. 5 and 6, stabilization of high-order and vector difference equations was considered. This allowed us to analyze stage-structured and delayed population dynamics models. Even without formal control, the introduction of noise can improve population dynamics, for example, reduce oscillation amplitudes. The next stage of research is to incorporate the ideas and methods of the present paper in the controlled systems of difference equations. For example, if a system describes a network, together with stabilization of periodic orbits,[26] the problem of synchronization is of importance,

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especially if there are delays and/or stochastic component in communications.[28,29]

4. **Qualitative analysis of positive effect of noise: ecological perspective.** Finally, it would be interesting, similarly to the present paper, to qualitatively evaluate possible positive effect of noise on stability and population survival; see Ref. 32 for analysis of population interactions and the influence of stochasticity on survival, extinction, and coexistence.

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APPENDIX: PROOFS OF THE RESULTS

1. Proof of Theorem II.4

Consider two cases in turn: (a) $ML^{k-1} < 1$ and (b) $ML^{k-1} \geq 1$.

Case (a): If $ML^{k-1} < 1$, then (11) holds for $u_1 = u_0$. Since Eq. (6) in Assumption II.1 holds with $L \geq 1$, we must have $M < 1$. Fix $u \in (0, u_0]$ and set $\delta_0 \leq u/L^{k-1}$. Then, for $|z_0| \leq \delta_0$,

$$\begin{aligned}
&|z_1| \leq L|z_0| \leq \frac{u}{L^{k-2}} \leq u,\\
&|z_2| \leq L|z_1| \leq L^2|z_0| < \frac{u}{L^{k-3}} \leq u,\\
&\quad\vdots\\
&|z_{k-1}| \leq L|z_{k-2}| \leq L^{k-1}|z_0| \leq u,\\
&|z_k| \leq M|z_{k-1}| \leq ML^{k-1}|z_0| \leq Mu < u,\\
&|z_{2k}| \leq \left(ML^{k-1}\right)^2 u \leq u.
\end{aligned}$$

Reasoning recursively, we conclude that, for $i \in \mathbb{N}$,

$$\begin{aligned}
&|z_{ik}| \leq \left(ML^{k-1}\right)^i u \leq u,\\
&|z_{ik+j}| \leq (ML^{k-1})^i u \leq u,\ j = 1, \ldots, k-1,
\end{aligned}$$

where $ML^{k-1} < 1$, so, $\lim_{i\to\infty} |z_{ik+j}(\omega)| = 0$ for all $\omega \in \Omega$, which concludes the proof of Part (a).

Case (b): Let α and l be chosen as in (10), and u_1 be defined as in (11). Fix $u \leq u_1$ and $\gamma \in (0, 1)$.

Applying Lemma I.1 and condition (11), we conclude that, for $\varepsilon := 1/2[\lambda - (k-1)\ln L] > 0$, there exists a random $\mathcal{N} = \mathcal{N}(\gamma, \lambda, u)$ such that

$$\left|\frac{1}{n}\sum_{i=0}^{n} \ln \mathcal{L}(\alpha, \xi_{ik}, u) + \lambda\right| < \varepsilon, \quad n \geq \mathcal{N}.$$

Then, there exist a nonrandom $N = N(\gamma, \lambda, u)$ and $\Omega_\gamma \subset \Omega$ with $\mathbb{P}(\Omega_\gamma) > 1 - \gamma$, such that

$$\begin{aligned}
&\prod_{i=0}^{n} \mathcal{L}(\alpha, l, \xi_{ik}(\omega), u)\\
&\quad < \exp\left\{-\frac{n}{2}(\lambda + (k-1)\ln L)\right\},\ n \geq N,\ \omega \in \Omega_\gamma. \qquad \text{(A1)}
\end{aligned}$$

Suppose that, when N is chosen so that (A1) holds, δ_0 satisfies

$$\delta_0 \leq \frac{u}{(\bar{M}L^{k-1})^N}, \quad \text{where} \quad \bar{M} := \max\{1, M\}. \qquad \text{(A2)}$$

Since $\bar{M} \geq 1$, we have, for all $i = 1, \ldots, k-1$,

$$\begin{aligned}
\bar{M}^N L^{N(k-1)-i} &= \bar{M}^N (L^{k-1})^{N-\frac{i}{k-1}}\\
&\geq \bar{M}^{N-i/k-1} (L^{k-1})^{N-\frac{i}{k-1}} \geq 1.
\end{aligned}$$

By (6), (7), and (A2), we have, for $|z_0| \leq \delta_0 \leq u$,

$$|z_1| = |g(z_0)| \leq \frac{Lu}{(\bar{M}L^{k-1})^N} \leq \frac{u}{(\bar{M}L^{k-1})^{N-\frac{1}{k-1}}} \leq u,$$

and inductively, for all $i = 0, \ldots, k-1$,

$$|z_i| \leq \frac{L^i u}{(\bar{M}L^{k-1})^N} \leq \frac{u}{(\bar{M}L^{k-1})^{N-\frac{i}{k-1}}} \leq u.$$

So,

$$\begin{aligned}
|z_k| &\leq \mathcal{L}(\alpha, l, \xi_k, u)|z_{k-1}|\\
&\leq \frac{\bar{M}L^{k-1}u}{(\bar{M}L^{k-1})^N} = \frac{u}{(\bar{M}L^{k-1})^{N-1}} \leq u.
\end{aligned}$$

Similarly, for any $j < N$,

$$\begin{aligned}
|z_{jk}| &\leq \mathcal{L}(\alpha, l, \xi_{jk}, u)|z_{jk-1}|\\
&\leq \mathcal{L}(\alpha, l, \xi_{jk}, u)L^{k-1}|z_{j(k-1)}|\\
&\leq (L^{k-1})^j \prod_{i=1}^{j} \mathcal{L}(\alpha, l, \xi_{ik}, u)|z_0|\\
&< (\bar{M}L^{k-1})^j |z_0| < \frac{u}{(\bar{M}L^{k-1})^{N-j}} \leq u.
\end{aligned}$$

Denoting $\bar{\lambda} := [\lambda - (k-1)\ln L]/2$, and applying (A1), we get, on Ω_γ,

$$\begin{aligned}
|z_{kN}| &\leq L^{(k-1)N}|z_0| \prod_{i=1}^{N} \mathcal{L}(\alpha, l, \xi_{ik}, u)\\
&\leq |z_0| e^{((k-1)N)\ln L} e^{-\frac{[\lambda+(k-1)\ln L]N}{2}} = e^{-\bar{\lambda}N}|z_0| < u,
\end{aligned}$$

and then, for each $j = 1, 2, \ldots, k-1$,

$$\begin{aligned}
|z_{kN+j}| &< L^j |z_{kN}| < e^{-\bar{\lambda}N} L^j |z_0|\\
&< e^{-\bar{\lambda}N} \frac{u}{\bar{M}^N L^{(k-1)N-j}} < u.
\end{aligned}$$

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Similarly, for any $n = km + j$, where $j = 1, 2, \ldots, k-1$, $m > N$, we get, for $\omega \in \Omega_\gamma$,

$$|z_n(\omega)| < L^j |z_{km}(\omega)| < e^{-\bar{\lambda}km} L^j |z_0| \le e^{-\bar{\lambda}(n-j)} L^k |z_0| < e^{-\bar{\lambda}(n-j)} \frac{u}{\bar{M}^N L^{(k-1)N-k}} < u e^{-\bar{\lambda}(n-j)} < u,$$

which implies that $\lim_{n\to\infty} z_n(\omega) = 0$, when $\omega \in \Omega_\gamma$ and concludes the proof of Case (b).

Remark. If $ML^{k-1} > 1$ but $M < 1$, we may use much bigger initial interval than it was suggested in the proof of Theorem II.4 assuming $\delta_0 \le \frac{u}{ML^{k-1}}$ for part (a) and $\delta_0 \le \frac{u}{(ML^{k-1})^N}$ for part (b) of the proof. However, in this case, we need to change model (7) slightly, considering instead

$$z_{n+1} = \begin{cases} G(z_n, \alpha, l, \xi_{n+1}), & n = k(s-1), \quad s \in \mathbb{N}, \\ g(z_n), & \text{otherwise}, \quad z_0 \in \mathbb{R}. \end{cases} \tag{A3}$$

In (A3), the application of stochastic control starts from $n = 0$ rather than $n = s - 1$. Note that in both cases, the solution z_n remains in $[-u, u]$ for all $n \in \mathbb{N}$. The proof differs only in the estimation of the first k (respectively Nk) iterations.

2. Proof of Lemma II.8

Let $i = 1$, other cases are similar. Note that $\prod_{s=j}^{d} \max\{L_s, 1\} \ge 1$ for $j = 0, \ldots, d$, and

$$K_1 = K_{d+1} = f(K_d) = f^{d-j}(K_{d-j+1}) = f^d(K_1).$$

For $|x - K_1| \le u_0 \left(\prod_{s=1}^{d} \max\{L_s, 1\}\right)^{-1}$, we have

$$|f(x) - K_2| \le \max\{L_1, 1\}|x - K_1| \le u_0 \left(\prod_{s=2}^{d} \max\{L_s, 1\}\right)^{-1} \le u_0,$$

and, inductively, for each $j = 0, \ldots, d-1$,

$$|f^{d-j}(x) - K_{d-j+1}| \le \prod_{s=1}^{d-j} \max\{L_s, 1\}|x - K_1| \le u_0 \left(\prod_{s=d-j+1}^{d} \max\{L_s, 1\}\right)^{-1} \le u_0,$$

which, for $j = 0$, implies (14).

3. Proof of Lemma II.10

Let $u(d)$ be defined as in (16). Under Assumption II.9, we have, for $x \in [K_i - u(d), K_i + u(d)]$,

$$|f(x) - K_{i+1}| \le |\mathcal{A}_i(x - K_i)| + |\phi_i(x)| \le [|\mathcal{A}_i| + \psi_i(u)]|x - K_i|,$$

so (12) holds for $L_i := |\mathcal{A}_i| + \psi_i(u)$. Acting as in the proof of Lemma II.8, we obtain that, for $j = 1, \ldots, d$,

$$|f^j(x) - K_{j+1}| \le \prod_{i=1}^{j} [|\mathcal{A}_i| + \psi_i(u)]\,|x - K_1| \le \frac{u}{\prod_{i=j+1}^{d} \max\{|\mathcal{A}_i| + \psi_i(u),\, 1\}} \le u,$$

where notationally, $\prod_i^j \cdot = 1$ for any $i > j$. Now apply (15) recursively, for $|x - K_1| < u(d)$,

$$\begin{aligned} f^d(x) - K_1 &= \mathcal{A}_d\Big(\mathcal{A}_{d-1}(f^{d-2}(x) - K_{d-1}) + \phi_{d-1}(f^{d-2}(x))\Big) + \phi_d(f^{d-1}(x)) \\ &\vdots \\ &= \left(\prod_{i=1}^{d} \mathcal{A}_i\right)(x - K_1) + \sum_{j=1}^{d-1} \left(\prod_{i=j+1}^{d} \mathcal{A}_i\right) \phi_j(f^{j-1}(x)) \\ &\quad + \phi_d(f^{d-1}(x)), \end{aligned}$$

where $f^0(x) := x$. Define

$$\bar{\phi}(x) := \sum_{j=1}^{d} \left(\prod_{s=j+1}^{d} \mathcal{A}_s\right) \phi_j(f^{j-1}(x)).$$

Acting as above we get, for $j = 1, \ldots, d$,

$$\begin{aligned} &|\phi_j(f^{j-1}(x))| \\ &\le \psi_j(|f^{j-1}(x) - K_j|)|f^{j-1}(x) - K_j| \\ &\le \psi_j(|f^{j-1}(x) - K_j|)\left[|\mathcal{A}_{j-1}| + \psi_{j-1}(u)\right] \\ &\quad \times \left|f^{j-2}(x) - K_{j-1}\right| \\ &\le \psi_j(|f^{j-1}(x) - K_j|) \prod_{i=1}^{j-1} [|\mathcal{A}_i| + \psi_i(u)]\,|x - K_1|. \end{aligned}$$

So, we can set

$$\bar{\psi}(x) := b \sum_{j=1}^{d} \left(\prod_{s=j+1}^{d} \mathcal{A}_s\right) \psi_j(|f^{j-1}(x) - K_j|) \times \prod_{i=1}^{j-1} (|\mathcal{A}_i| + \psi_i(u)),$$

which completes the proof of (17).

4. Proof of Lemma II.13

An application of Theorem II.4 implies the existence of s_0 and $\Omega_\gamma \subset \Omega$ with $\mathbb{P}(\Omega_\gamma) \ge 1 - \gamma$ such that, for sufficiently small $\delta > 0$,

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$s \geq s_0, \omega \in \Omega_\gamma$

$$|x_{smd}(\omega) - K_1| < u_0 L^{-m}(d)|1-\alpha+l|^{-m+1}.$$

We need to show that, for each $s \geq s_0$, $j = qd + \bar{j}$, $\bar{j} = 1, \ldots, d-2$, and $q \leq m-1$, we have $|x_{smd+j}(\omega) - K_{\bar{j}+1}| \leq u_0$ when $\omega \in \Omega_\gamma$, which allows us to apply (12) on each step.

Indeed, $|x_{smd+1} - K_2| = |f(x_{smd}) - f(K_1)| \leq L_1|x_{smd} - K_1| \leq u_0$. Reasoning inductively, we have for each $j = 1, \ldots, d-1$ and $\omega \in \Omega_\gamma$,

$$\begin{aligned}
|x_{smd+j}(\omega) - K_{j+1}| &= |f(x_{smd+j-1}(\omega)) - f(K_j)| \\
&\leq \prod_{i=1}^{j} L_i |x_{smd}(\omega) - K_1| \leq u_0, \\
|x_{smd+d}(\omega) - K_{d+1}| &\\
&\leq |1-\alpha+l|L(d)|x_{smd}(\omega) - K_1| \leq u_0,
\end{aligned}$$

and for $q = d+1, \ldots, 2d-1$ and again $\omega \in \Omega_\gamma$,

$$\begin{aligned}
&|x_{smd+q}(\omega) - K_{q-d+1}| \\
&\quad \leq L_{q-d}|x_{smd+q-1}(\omega) - K_{q-d}| \\
&\quad \leq \cdots \leq \prod_{\theta=1}^{q-d} L_\theta |x_{smd+q-d}(\omega) - K_1| \\
&\quad \leq L(d)|1-\alpha+l| \prod_{\theta=1}^{q-d} L_\theta |x_{smd}(\omega) - K_1| \leq u_0.
\end{aligned}$$

Similarly, for $j = qd + \bar{j}$, $\bar{j} = 1, \ldots, d-2$, $q \leq m-1$, and for $\omega \in \Omega_\gamma$,

$$\begin{aligned}
&|x_{smd+j}(\omega) - K_{\bar{j}+1}| \\
&\quad \leq L_{\bar{j}}|x_{smd+j-1}(\omega) - K_{\bar{j}}| \\
&\quad \leq \cdots \leq |1-\alpha+l|^q \prod_{\theta=1}^{\bar{j}} L_\theta L^q(d)|x_{smd}(\omega) - K_1| \\
&\quad \leq |1-\alpha+l|^q L^{q+1}(d)|x_{smd}(\omega) - K_1| \\
&\quad \leq |1-\alpha+l|^{m-1} L^m(d)|x_{smd} - K_1| \leq u_0.
\end{aligned}$$

5. Proof of Theorem II.15

Consider (19) with arbitrary x_0 and assume that (23) holds. Then, for any $\gamma \in (0,1)$, there exists $\Omega_\gamma \subset \Omega$ with $\mathbb{P}(\Omega_\gamma) > 1-\gamma$ and $N \in \mathbb{N}$ such that for $\bar{\lambda} = \frac{1}{2}(\lambda - k \ln \bar{L})$, we have on Ω_γ, for $n = tk + j$, $j = 1, 2, \ldots, k-1$, and $t = \lfloor n/k \rfloor > N$, where $\lfloor q \rfloor$ is an integer part of $q \in [0, \infty)$,

$$\begin{aligned}
&\left(L^k|1-\alpha-l\xi_n|\right)^t \leq e^{-\bar{\lambda}t}, \\
&|x_n - K| \leq L^{tk+j}|1-\alpha-l\xi_n|^t|x_0 - K| \\
&\leq e^{-\bar{\lambda}t}L^k|x_0 - K|,
\end{aligned}$$

which tends to zero as $n \to \infty$ (so that $k \to \infty$). Here, note that condition (23) holds with L substituted by the global constant $\bar{L}$, which implies that $\lambda - k \ln \bar{L} > 0$, and then $\bar{\lambda} > 0$.

Analogously, consider (26) with arbitrary x_0, and assume that (29) holds. Then, condition (29) is satisfied with $L(d)$ substituted by the global constant $\bar{L}(d)$, which implies that $\lambda - m \ln \bar{L}(d) > 0$. Applying Lemma II.13, we get, for $\bar{\lambda} = (\lambda - m \ln L(d))/2$, $n = tmd + j$, $t = \lfloor u/(md) \rfloor > N$, $j = qd + \bar{j}$, $\bar{j} = 0, 1, \ldots, d-1$, $q = 0, 1, \ldots, m-1$,

$$\begin{aligned}
&\left(L^m(d)|1-\alpha-l\xi_n|\right)^t \leq e^{-\bar{\lambda}t}, \\
&|x_n - K_{\bar{j}+1}| = |x_{tmd+j} - K_{\bar{j}+1}| \\
&\quad \leq |1-\alpha+l|^{m-1} L^m(d)|x_{tmd} - K_1| \\
&\quad \leq e^{-\bar{\lambda}t}|1-\alpha+l|^{m-1} L^m(d)|x_0 - K| \to 0, \quad \text{as } m \to \infty,
\end{aligned}$$

and, therefore, $n \to \infty$.

If $ML^k < 1$ (respectively, $ML(d) < 1$), each of limits above holds for all $\omega \in \Omega$.

6. Proof of Theorem II.18

Let $G(z, \alpha, l, v)$ be defined as in (31) and denote, for simplicity, $G := G(z, \alpha, l, v)$. Let $z \in [-u_0, 0]$ and $G > 0$, then

$$\begin{aligned}
|G| &= (1-\alpha-lv)g(z) + (\alpha+lv)z < (1-\alpha-lv)g(z) \\
&= (1-\alpha-lv)g(z) \leq (1-\alpha-lv)L|z|.
\end{aligned}$$

If $G \leq 0$, we have

$$|G| = -(1-\alpha-lv)g(z) - (\alpha+lv)z < (\alpha+lv)|z|.$$

Now, let $z \in [0, u_0]$ and $G > 0$, then

$$|G| = (1-\alpha-lv)g(z) + (\alpha+lv)z < (\alpha+lv)|z|.$$

If $G \leq 0$,

$$\begin{aligned}
|G| &= -(1-\alpha-lv)g(z) - (\alpha+lv)z \\
&< (1-\alpha-lv)|g(z)| \leq (1-\alpha-lv)L|z|.
\end{aligned}$$

So, (8) holds for $\mathcal{L}(\alpha, l, v, u) = \max_{|v| \leq 1}\{(1-\alpha-lv)L, \alpha+lv\}$, and (9) is satisfied for $M = \max\{(1-\alpha+l)L, \alpha+l\}$. Condition (10) then takes the form

$$\min\left\{-\mathbb{E}\ln|1+\alpha|, -\mathbb{E}\ln|1-\alpha-l\xi|L\right\} \geq (k-1)\ln L.$$

A direct application of Theorem II.4 proves Part (ii).

The condition $ML^{k-1} < 1$ takes the form $\max\{(1-\alpha+l)L, \alpha+l\}L^{k-1} < 1$. In the case where $k = 1$ and all $L > 1$, this implies the conditions listed in (33) in Part (i). In the case where $k > 1$ and for all L satisfying $1 < L^k < L+1$, this implies the conditions in (34) in Part (i). Another application of Theorem II.4 concludes the proof.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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