ENERGY CONSIDERATIONS FOR NONLINEAR EQUATORIAL WATER WAVES

D. HENRY

Abstract. In this article we consider the excess kinetic and potential energies for exact nonlinear equatorial water waves. An investigation of linear waves establishes that the excess kinetic energy density is always negative, whereas the excess potential energy density is always positive, for periodic travelling irrotational water waves in the steady reference frame. For negative wavespeeds, we prove that similar inequalities must also hold for nonlinear wave solutions. Characterisations of the various excess energy densities as integrals along the wave surface profile are also derived.

1. Introduction

The dynamics of the ocean in the equatorial region presents some unique and fascinating characteristics from a modelling perspective [21, 24, 26, 48]. At the Equator there is a breakdown in mid-latitude geostrophic balance, resulting in the Equator acting as a natural waveguide leading to equatorially-trapped zonal waves [10, 24, 27]. Recently, progress has been achieved in systematically developing models which capture aspects of equatorial flows which had been hitherto captured via *ad hoc* modelling considerations (cf. [16–19, 27, 30, 31, 36], and references therein). It is noteworthy that a number of these developments do not just incorporate nonlinear effects, but they are intrinsically nonlinear [19, 28]. In this paper we address energy considerations for nonlinear two-dimensional periodic travelling waves propagating zonally in the equatorial region.

As fluid moves it must possess energy. For surface gravity waves on an inviscid fluid the total energy consists of the potential energy (resulting from the displacement of the mass of water from a position of equilibrium under the gravitational field) and the kinetic energy (due to the motion of the water particles throughout the fluid), cf. [20, 37]. Potential energy is the capacity for doing work due to the position of a body, while kinetic energy is the capacity for doing work by reason of the motion of a body. Ocean swell propagates over very long distances with relatively little loss of energy, implying that the effects of viscosity are quite negligible for these ocean waves [9, 20, 37]. Indeed this persistence of energy propagated by water waves is the primary motivation behind the desire to harness wave energy (cf. the discussions in [46]). For large-scale flows in the ocean, Coriolis effects relating to the Earth's rotation play an important role. This is the realm of geophysical fluid

2020 *Mathematics Subject Classification.* Primary: 35Q35, 35Q86, 76B15; Secondary: 35Q31.
Key words and phrases. Nonlinear water waves; wave energy; equatorial; exact solutions.

dynamics (GFD), and in this paper we consider the energy of waves whose motion is governed by the $f-$plane approximation to the geophysical governing equations of motion. A hallmark of inviscid fluids is the absence of dissipating effects, and accordingly one expects a conservation of the total energy, and a resulting balance between fluctuations in both the potential and kinetic energy [39, 40, 45]. A useful benefit of Hamiltonian formulations (which were recently constructed for equatorial water waves in [14, 15]) are the invariants of motion which arise naturally as by-products of the Hamiltonian framework; these invariants include, as expected, the total (kinetic plus potential) wave energy (cf. [14, 15]).

Due to the mathematical complexities which are intrinsic to nonlinear waves, most wave energy considerations tend to focus on the setting of linear waves, which is applicable only for waves of relatively small amplitude [22]. For large-amplitude waves in the nonlinear setting the literature is comparably sparse. The classical papers [3, 38–40, 45] present results characterising some energy properties for nonlinear waves, and more recent developments have established rigorous results concerning monotonicity properties of the kinetic energy for nonlinear waves [1, 44]. Recently, in [29], the author has proven results concerning the excess kinetic and potential energy densities for exact nonlinear periodic and travelling irrotational water waves, in the purely gravity wave setting. In this article it will be shown that these results (partially) transfer to the nonlinear wave setting for equatorial flows.

For fluids with infinite extent it is meaningless to discuss the total energy possessed by the fluid, rather we must consider suitably defined local energy densities. For water waves travelling with uniform wavespeed c we can transform to a moving reference frame in which the resulting flow is steady. In this frame, the velocity field is given by $(u-c, w)$, with $z = \eta(x)$ denoting the unknown free-surface. For periodic surface waves, define the *excess potential energy* (per unit horizontal area) over the value for the flow with an undisturbed free surface ($\eta(x) \equiv 0$) to be

$$E_p = \frac{1}{\lambda}\int_0^\lambda\int_{-d}^{\eta(x)} gz\,dz\,dx - \frac{1}{\lambda}\int_0^\lambda\int_{-d}^{0} gz\,dz\,dx, \tag{1.1}$$

whereas the *excess kinetic energy* (per unit horizontal area) over the value for the undisturbed uniform flow $(u-c, w) = (-c, 0)$ is given by

$$E_k = \frac{1}{2\lambda}\int_0^\lambda\int_{-d}^{\eta(x)} \left((u-c)^2 + w^2\right)\,dz\,dx - \frac{1}{2\lambda}\int_0^\lambda\int_{-d}^{0} c^2\,dz\,dx. \tag{1.2}$$

Here $z = -d$ is an appropriate reference level (corresponding to the flat sea-bed, for instance). In this paper we consider the excess energy densities for equatorial water waves. In the linear wave regime (corresponding to waves of relatively small amplitude a, where $a \ll d$) explicit wave solutions exist (see Section 3), and (1.1) and (1.2) can be evaluated to get

$$E_p^{lin} = \frac{ga^2}{4} + \mathcal{O}\left(a^3\right), \quad E_k^{lin} = -\frac{a^2}{4}\left(g - 2\Omega c_0\right) + \mathcal{O}(a^3), \tag{1.3}$$

where the Landau symbol $\mathcal{O}(a^3)$ denotes terms of order a^3, and higher. Hence, in the linear wave regime, the quantities E_p^{lin} and E_k^{lin} have definitive, and opposite, signs:

$$E_p^{lin} > 0, \quad E_k^{lin} < 0. \tag{1.4}$$

The relations in (1.3) have important practical implications since, while the measurement of kinetic energy is extremely difficult, estimating the wave amplitude is much more achievable (for example, cf. [4, 6–8, 11, 34] for surface-profile recovery formulae using measurements from submerged pressure transducers). The formulae in (1.3) ensures that this procedure yields an accurate estimate for the total wave energy when the wave amplitude is small. The total linear energy density (up to order $\mathcal{O}(a^2)$) is given by

$$E_{tot}^{lin} := E_p^{lin} + E_k^{lin} = \frac{a^2 \Omega c_0}{2}.$$

For purely gravity water waves (corresponding to $\Omega = 0$) there is an equipartition of energy: the excess linear energy densities E_p^{lin} and E_k^{lin} have the same magnitudes (up to order $\mathcal{O}(a^2)$) and E_{tot}^{lin} vanishes in this setting. Evidently, and unsurprisingly, wave energy considerations in the geophysical framework are more complex and involved, even in the linear regime where an equipartition of energy no longer applies.

In Section 5 we consider whether the inequalities in (1.4) are applicable also to nonlinear waves. In [29] it was proven by the author, for purely gravitational waves (in the absence of Coriolis effects), that the excess energy densities for all nonlinear wave solutions must satisfy similar inequalities to (1.4). The presence of Coriolis forces complicates matters here, and it will be shown that the answer to this question is more varied in the geophysical setting.

2. Governing equations

The governing equations for equatorial waves will be formulated in a reference frame whose origin is located at a point on the Earth's surface, and which is rotating with the Earth. The $X-$axis represents the longitudinal variable (pointing horizontally due east), the $Y-$axis the latitudinal variable (pointing horizontally due north) and the $Z-$axis points vertically upwards. The Euler equation for inviscid and incompressible fluid is given by

$$\begin{aligned}
u_t + uu_X + vu_Y + wu_Z + 2\Omega w \cos\phi - 2\Omega v \sin\phi &= -\frac{1}{\rho} P_X, \\
v_t + uv_X + vv_Y + wv_Z + 2\Omega u \sin\phi &= -\frac{1}{\rho} P_Y, \\
w_t + uw_X + vw_Y + ww_Z - 2\Omega u \cos\phi &= -\frac{1}{\rho} P_Z - g.
\end{aligned}$$

Here (u, v, w) denotes the fluid velocity field, P is the pressure and ρ is the density of the fluid, and the ϕ variable represents the latitude. The Earth is taken to be a perfect sphere of radius $R = 6378$km with constant rotational speed of $\Omega = 7.3\times10^{-5}$ rad s^{-1}, and $g = 9.81$ms^{-2} is the gravitational acceleration at the surface of the

Earth. In the Equatorial region the latitude ϕ is necessarily restricted in scope: assuming it to be fixed gives the $f-$plane approximation. The Euler equation is consequently simplified by approximating $\sin\phi \approx 0$, $\cos\phi \approx 1$, leading to

$$u_t + uu_X + vu_Y + wu_Z + 2\Omega w = -\frac{P_X}{\rho}, \tag{2.1a}$$

$$v_t + uv_X + vv_Y + wv_Z = -\frac{P_Y}{\rho}, \tag{2.1b}$$

$$w_t + uw_X + vw_Y + ww_Z - 2\Omega u = -\frac{P_Z}{\rho} - g. \tag{2.1c}$$

In modelling ocean surface waves it is reasonable to assume a constant fluid density, in which case the equation of mass conservation takes the form of the continuity equation

$$u_X + v_Y + w_Z = 0. \tag{2.1d}$$

If $Z = \eta(t, X, Y)$ denotes the (unknown) free surface of the ocean, then the pressure of the fluid is taken to match the atmospheric pressure P_{atm} at the wave surface, giving the surface dynamic boundary condition

$$P = P_{atm} \qquad \text{on} \quad Z = \eta(t, X, Y). \tag{2.1e}$$

For water waves not near breaking the free surface should consist of the same fluid particles for all time, leading to the surface kinematic boundary condition

$$w = \eta_t + u\eta_X + v\eta_Y \qquad \text{on} \quad Z = \eta(t, X, Y). \tag{2.1f}$$

In the equatorial region the Coriolis parameter $f = 2\Omega \sin\phi$ vanishes, which effectuates the equator acting as a natural wave-guide. With this in mind, we consider flows which are two-dimensional, moving solely in the zonal direction along the equator, and which are independent of the $y-$coordinate (with $v \equiv 0$ throughout the flow). We further consider waves for which the unknown free-surface $\eta(X, t)$ is a periodic and even function with respect to the spatial variable, where $\lambda > 0$ denotes the wavelength. To fix our reference frame, the condition

$$\int_0^\lambda \eta(X, t)dX = 0, \tag{2.1g}$$

locates the mean water level at $Z = 0$. In the following we assume that the effects of the surface wave motion are confined to the region $\mathcal{D}_{\eta,d} = \{(X, Z) : X \in \mathbb{R}, -d < Z < \eta(X, t)\}$, for some depth d, with the surface wave generating no appreciable vertical motion at this depth: $Z = -d$ may correspond to an impermeable flat bed, for instance. This motivates the kinematic boundary condition

$$w = 0 \qquad \text{on} \quad Z = -d. \tag{2.1h}$$

We further assume that the fluid motion is irrotational in the region $\mathcal{D}_{\eta,d}$, leading to the irrotationality condition

$$u_Z = w_X \qquad \text{in } \mathcal{D}_{\eta,d}. \tag{2.1i}$$

This supposition implies that underlying currents in this near-surface region are uniform: while non-uniform currents are prevalent in the equatorial ocean, these tend to arise at large sub-surface depths. Finally, we assume that the fluid density ρ is constant, and for convenience we set $\rho = 1$.

3. Linear water waves

3.1. **Linear solutions.** The governing equations (2.1), expressed in terms of physical variables (X, Z), are nondimensionalised using the transformation

$$X \mapsto \lambda X,\ Z \mapsto dZ,\ t \mapsto \frac{\lambda}{\sqrt{gd}}t,\ u \mapsto u\sqrt{gd},\ w \mapsto w\frac{d\sqrt{gd}}{\lambda},\ \eta \mapsto a\eta,$$

where λ is a typical wavelength and a a typical amplitude of the wave. We avoid new notation by replacing, for example, X by λX, with X now being the nondimensionalised variable. We decompose the pressure (in terms of the new nondimensional variables) as $P = P_{atm} - gdZ + gpd$, where the nondimensional pressure function p corresponds to the dynamic pressure, which measures pressure deviations from the hydrostatic distribution. These transformations lead to the boundary value problem in terms of the nondimensional variables:

$$u_t + uu_X + wu_Z + 2\Omega w = -p_X, \tag{3.1a}$$

$$\delta^2(w_t + uw_X + ww_Z) - 2\Omega u = -p_Z, \quad \text{for} \quad -1 < Z < \epsilon\eta, \tag{3.1b}$$

$$u_X + w_Z = 0 \quad \text{for} \quad -1 < Z < \epsilon\eta, \tag{3.1c}$$

$$p = \epsilon\eta \quad \text{on } Z = \epsilon\eta, \tag{3.1d}$$

$$w = \epsilon(\eta_t + u\eta_X) \quad \text{on } Z = \epsilon\eta, \tag{3.1e}$$

$$w = 0 \quad \text{on } Z = -1, \tag{3.1f}$$

$$u_z = \delta^2 w_Z \quad \text{for} \quad -1 < Z < \epsilon\eta, \tag{3.1g}$$

where $\epsilon = a/d$ is the amplitude parameter, $\delta = d/\lambda$ is the shallowness parameter, and Ω in (3.1) corresponds to a non-dimensionalised rotational speed defined by $\Omega = \sqrt{g/d} \cdot \tilde{\Omega}$. From the fourth and fifth equation in (3.1) it is obvious that both w and p, if evaluated on $Z = \epsilon\eta$, are essentially proportional to ϵ (physically, $\epsilon \to 0$ implies that wave motion on the surface is negligible, in which case we expect the associated limiting behaviour $w \to 0$ and $p \to 0$). Scaling the nondimensional variables as $p \mapsto \epsilon p$, $(u, w) \mapsto \epsilon(u, w)$, leads (again avoiding the introduction of new variables) to

$$u_t + \epsilon(uu_X + wu_Z) + 2\Omega w = -p_X, \tag{3.2a}$$

$$\delta^2\{w_t + \epsilon(uw_X + ww_Z)\} - 2\Omega u = -p_Z, \quad \text{for} \quad -1 < Z < 0, \tag{3.2b}$$

$$u_X + w_Z = 0, \quad \text{for} \quad -1 < Z < 0, \tag{3.2c}$$

$$w = \eta_t + \epsilon u\eta_X \quad \text{and} \quad p = \eta \quad \text{on } Z = \epsilon\eta, \tag{3.2d}$$

$$w = 0 \quad \text{on } Z = -1, \tag{3.2e}$$

$$u_z = \delta^2 w_Z \quad \text{for} \quad -1 < Z < 0. \tag{3.2f}$$

The linearised problem is obtained by letting $\epsilon \to 0$ in (3.2), resulting in

$$\begin{aligned} u_t + 2\Omega w = -p_x, \quad \delta^2 w_t - 2\Omega u = -p_Z, \quad &\text{for} \quad -1 < Z < 0, \\ u_x + w_Z = 0, \quad &\text{for} \quad -1 < Z < 0, \\ w = \eta_t \quad \text{and} \quad p = \eta \quad &\text{on } Z = 0, \\ w = 0 \quad &\text{on } Z = -1, \\ u_z = \delta^2 w_Z \quad &\text{for} \quad -1 < Z < 0, \end{aligned}$$

which may be solved in terms of travelling wave solutions

$$\begin{aligned} \eta(t,X) &= \cos[2\pi(X - \hat{c}_0 t)] \\ u(t,X,Z) &= 2\pi\delta\hat{c}_0 \frac{\cosh(2\pi\delta(1+Z))}{\sinh(2\pi\delta)} \cos[2\pi(X - \hat{c}_0 t)], \\ w(t,X,Z) &= 2\pi\hat{c}_0 \frac{\sinh(2\pi\delta(1+Z))}{\sinh(2\pi\delta)} \sin[2\pi(X - \hat{c}_0 t)]. \\ p(t,X,Z) &= \frac{[2\pi\hat{c}_0^2\delta\cosh(2\pi\delta(1+Z)) + 2\Omega\hat{c}_0\sinh(2\pi\delta(1+Z))]}{\sinh(2\pi\delta)} \cos[2\pi(X - \hat{c}_0 t)], \end{aligned}$$

where $\hat{c}_0$ is the nondimensionalised speed of the wave that must satisfy the linear dispersion relation

$$\hat{c}_0 = \frac{-\Omega\tanh(2\pi\delta) \pm \sqrt{\Omega^2\tanh^2(2\pi\delta) + 2\pi\delta\tanh(2\pi\delta)}}{2\pi\delta}.$$

Returning to the original physical variables via the change of variables

$$X \mapsto \frac{X}{\lambda}, \; Z \mapsto \frac{Z}{d}, \; t \mapsto t\frac{\sqrt{gd}}{\lambda}, \; u \mapsto \frac{u}{\sqrt{gd}}, \; w \mapsto w\frac{\lambda}{d\sqrt{gd}}, \; \eta \mapsto \frac{\eta}{a},$$

the linear wave solution in terms of physical variables is given by

$$\eta(t,X) = a\cos(kX - \omega t), \tag{3.3a}$$

$$u(t,X,Z) = a\omega\frac{\cosh(k(d+Z))}{\sinh kd}\cos(kX - \omega t), \tag{3.3b}$$

$$w(t,X,Z) = a\omega\frac{\sinh(k(d+Z))}{\sinh kd}\sin(kX - \omega t), \tag{3.3c}$$

$$P(t,X,Z) = P_{atm} - gZ + a\frac{kc_0^2\cosh(k(d+Z)) + 2\Omega c_0\sinh(k(d+Z))}{\sinh kd}\cos(kX - \omega t), \tag{3.3d}$$

with $k = 2\pi/\lambda$ the wavenumber, ω the wave frequency, and $c_0 = \omega/k$ the wavespeed.

Remark 3.1. The linear surface profile (3.3a) is sinusoidal and the wave amplitude a is simply the distance between the wave-crest and the mean water level (which equals the distance between the wave trough and the mean water level). Hence the wave height, which is defined to be the overall vertical change in height between the wave crest and the wave trough, is simply twice the wave amplitude in the linear setting. Nonlinear periodic waves observed in the sea tend to have sharper elevations, and flatter depressions, than their linear counterparts. The amplitude a of a wave is defined as the maximum deviation of the wave surface from the mean water level:

$$a = \sup_{X\in[0,\lambda]} \{\eta(X,t)\},$$

for any fixed-time t. The amplitude a is usually attained at the wave-crest and, for nonlinear waves, typically exceeds the distance between the wave trough and the mean water level.

3.2. **Linear wavespeed: dispersion relations.** For the linear wave solution (3.3) the speed $c_0 = \sqrt{gd}\cdot\hat{c}_0$ satisfies the dispersion relation

$$kc_0^2 + 2\Omega\tanh kdc_0 = g\tanh kd \tag{3.4}$$

which ensures the linear pressure solution (3.3d) fulfils the dynamic surface condition (2.1e) for $Z = \eta(t,X)$ as given by (3.3a). Relation (3.4) can be solved directly to get

$$c_0^{\pm} = \frac{1}{k}\left(-\Omega\tanh kd \pm \sqrt{\Omega^2\tanh^2 kd + gk\tanh kd}\right). \tag{3.5}$$

Thus, there are two possible linear wavespeeds corresponding to the linear wave solution (3.3): $c_0^- < 0$, and $c_0^+ > 0$. If we denote the linear wavespeed for gravity waves (corresponding to setting $\Omega = 0$ in c_0^+) by

$$c_0^{grav} = \omega^{grav}/k = \sqrt{g\tanh(kd)/k}, \tag{3.6}$$

then $0 < c_0^+ < c_0^{grav} < |c_0^-|$. Furthermore, since $\Omega^2/g \approx 5.7\times10^{-10}$, for non-negligible values of k then $c_0^+ \lesssim c_0^{grav}$, $c_0^- \lesssim -c_0^{grav} < 0$. There are two corresponding wave frequencies $\omega^{\pm} = k\cdot c_0^{\pm}$ given by

$$\omega^{\pm} = -\Omega\tanh kd \pm \sqrt{\Omega^2\tanh^2 kd + gk\tanh kd}, \tag{3.7}$$

or, equivalently,

$$(\omega^{\pm})^2 = gk\tanh kd - 2\omega^{\pm}\Omega\tanh kd = \left(g - 2c_0^{\pm}\Omega\right)k\tanh kd. \tag{3.8}$$

Requiring $(\omega^{\pm})^2 > 0$ in (3.8) leads to the condition

$$g > 2\Omega c_0^{\pm} \tag{3.9}$$

which, considering the magnitudes of the respective physical parameters, is eminently reasonable. Interesting discussions of the particle trajectories prescribed by (3.3) for various choices in the dispersion relation can be found in [35].

An important quantity in linear energy considerations is the group velocity, which is defined as $c_g := \frac{d\omega}{dk}$, and we compute from (3.7)

$$
\begin{aligned}
c_g^{\pm} &= -d\Omega\,\mathrm{sech}^2 kd + \frac{1}{2(\omega^{\pm} + \Omega\tanh kd)}\left\{g\tanh kd + d\left[2\Omega^2\tanh kd + gk\right]\mathrm{sech}^2 kd\right\} \\
&= \pm\frac{1}{2\sqrt{\Omega^2\tanh^2 kd + gk\tanh kd}}\left\{g\tanh kd + kd\left[g - 2\Omega c_0\right]\mathrm{sech}^2 kd\right\} \\
&= \frac{1}{2(\omega^{\pm} + \Omega\tanh kd)}\left\{g\tanh kd + \frac{2d\,(\omega^{\pm})^2}{\sinh 2kd}\right\}.
\end{aligned}
\tag{3.10}
$$

In the absence of Coriolis forces expression (3.10) reduces to

$$
c_g^{grav} = \frac{1}{2\omega^{grav}}\left\{g\tanh kd + \frac{2d\,(\omega^{grav})^2}{\sinh 2kd}\right\} \overset{(3.6)}{=} \frac{g}{2\omega^{grav}}\left\{\tanh kd + dk\,\mathrm{sech}^2 kd\right\}.
$$

3.3. **Linear waves: energy densities and flux.**

3.3.1. *Potential energy V.* The potential energy V of a water wave at a fixed point X, measured relative to the undisturbed water level $Z = 0$, is given by

$$
V = \int_0^{\eta(X,t)} gZ\,dZ = \frac{1}{2}g\eta^2.
$$

For the linear wave solution (3.3) we have $V = \frac{1}{2}ga^2\cos^2(kX - \omega t)$, and we see that the potential energy is a quantity which varies with respect to both x and t. For fluids with infinite extent it is meaningless to discuss the total energy possessed by the fluid, instead we consider suitably defined local energy densities. The average of V over a wave-period, denoted $\bar{V}$, will be independent of both space and time and so serves as a more useful measure of the potential energy of the wave. Since the average of $\cos^2(kX - \omega t)$ (and $\sin^2(kX - \omega t)$) over a period is $1/2$, the potential energy density is

$$
\bar{V} = \frac{1}{4}ga^2.
$$

3.3.2. *Kinetic energy T.* The kinetic energy T at a fixed point X can be evaluated (to $\mathcal{O}(a^2)$) to be

$$\begin{aligned} T = \frac{1}{2}\int_{-d}^{0}\left(u^2+w^2\right)dz &= \frac{a^2\omega^2\cos^2(kX-\omega t)}{2\sinh^2(kd)}\int_{-d}^{0}\cosh^2\left[k(d+Z)\right]dZ \\ &\quad + \frac{a^2\omega^2\sin^2(kX-\omega t)}{2\sinh^2(kd)}\int_{-d}^{0}\sinh^2\left[k(d+Z)\right]dZ \\ &= \frac{a^2\omega^2\cos^2(kX-\omega t)}{4\sinh^2(kd)}\int_{-d}^{0}\left(\cosh\left[2k(d+Z)\right]+1\right)dZ \\ &\quad + \frac{a^2\omega^2\sin^2(kX-\omega t)}{4\sinh^2(kd)}\int_{-d}^{0}\left(\cosh\left[2k(d+Z)\right]-1\right)dZ \\ = \frac{a^2\omega^2\cos^2(kX-\omega t)}{4\sinh^2(kd)}\left(\frac{\sinh(2kd)}{2k}+d\right) &+ \frac{a^2\omega^2\sin^2(kX-\omega t)}{4\sinh^2(kd)}\left(\frac{\sinh(2kd)}{2k}-d\right) \\ &= \frac{a^2\omega^2\sinh(2kd)}{8k\sinh^2(kd)} + \frac{a^2\omega^2 d}{4\sinh^2(kd)}\left(\cos^2(kX-\omega t)-\sin^2(kX-\omega t)\right) \\ &= \frac{a^2\omega^2}{4\sinh^2(kd)}\left[\frac{1}{2k}\sinh(2kd)+d\cos\left[2(kX-\omega t)\right]\right]. \end{aligned}$$

Averaging over a wave-period gives the kinetic energy density, denoted $\bar{T}$:

$$\bar{T} = \frac{a^2\omega^2}{8k\sinh^2(kd)}\sinh(2kd) = \frac{a^2\omega^2}{4k\tanh(kd)} = \frac{a^2}{4}\left(g-2\Omega c_0\right), \tag{3.11}$$

where we have used (3.8) in deriving the last equality, and $\bar{T}>0$ due to (3.9).

Remark 3.2. If $\bar{T}^{grav}$ denotes the kinetic energy density for purely gravity waves (obtained by setting $\Omega=0$ in (3.11)) then

$$\bar{T} < \bar{T}^{grav} \ \text{ for } c_0 = c_0^+, \quad \text{while} \quad \bar{T} > \bar{T}^{grav} \ \text{ for } c_0 = c_0^-.$$

Hence the mean kinetic energy of a linear water wave is enhanced by Coriolis effects for left-moving waves, and diminished by Coriolis effects for right-moving waves. The magnitude of these perturbations is small bearing in mind that $\Omega = 7.3\times 10^{-5}$ rad s^{-1}, while the representative value $|c_0| = 3$m s^{-1} would be excessive for many equatorial Rossby and Kelvin surface waves, cf. [24].

3.3.3. *Total energy E:* For linear geophysical waves there is no equipartition of energy between the (mean) potential and kinetic energies: $\bar{V}\neq\bar{T}$. Indeed, $\bar{V}<\bar{T}$ for the wavespeed c_0^+, while $\bar{V}>\bar{T}$ for the wavespeed c_0^-. This contrasts with the case of purely gravity waves ($\Omega=0$) whereby $\bar{T}^{grav}=\bar{V}^{grav}$. The total energy $E=V+T$ has the mean-value

$$\mathcal{E} = \bar{E} = \bar{V}+\bar{T} = \frac{1}{2}a^2\left(g-\Omega c_0\right). \tag{3.12}$$

If $\mathcal{E}^{grav}$ denotes the total energy density for purely gravity waves (obtained by setting $\Omega=0$ in (3.12)) then

$$\mathcal{E} < \mathcal{E}^{grav} \ \text{ for } c_0 = c_0^+, \quad \text{while} \quad \mathcal{E} > \mathcal{E}^{grav} \ \text{ for } c_0 = c_0^-.$$

3.3.4. *Energy propagation.* The energy flux across a vertical plane, in the direction of motion of the wave crests (positive $X-$direction), is given by the expression:

$$
\begin{aligned}
E_f = &\text{ Rate of doing work on the surface of the plane } + \text{ convection of energy}\\
&= \int_{-d}^{\eta} P\,u\,dZ + \int_{-d}^{\eta} E\,u\,dZ = \int_{-d}^{0} P\,u\,dZ + \mathcal{O}(a^3)\\
= &\left[\int_{-d}^{0} P_{atm} a\omega \frac{\cosh(k(d+Z))}{\sinh kd} dZ - \int_{-d}^{0} gZa\omega \frac{\cosh(k(d+Z))}{\sinh kd} dZ\right]\cos(kX-\omega t)\\
&+a^2\omega k c_0^2 \int_{-d}^{0} \frac{\cosh(k(d+Z))}{\sinh kd}\frac{\cosh(k(d+Z))}{\sinh kd} dZ \cos^2(kX-\omega t)\\
&+2a^2\omega\Omega c_0 \int_{-d}^{0} \frac{\sinh(k(d+Z))}{\sinh kd}\frac{\cosh(k(d+Z))}{\sinh kd} dZ \cos^2(kX-\omega t)\\
= &\left[\int_{-d}^{0} P_{atm} a\omega \frac{\cosh(k(d+Z))}{\sinh kd} dZ - \int_{-d}^{0} gZa\omega \frac{\cosh(k(d+Z))}{\sinh kd} dZ\right]\cos(kX-\omega t)\\
&+\frac{a^2\omega k c_0^2}{2\sinh^2 kd}\left[d+\frac{\sinh 2kd}{2k}\right]\cos^2(kX-\omega t) + \frac{a^2\omega\Omega c_0}{\sinh^2 kd}\left[\frac{\cosh 2kd}{2k}-\frac{1}{2k}\right]\cos^2(kX-\omega t).
\end{aligned}
$$

Ignoring terms of $\mathcal{O}(a^3)$ and higher, and noting that the average of all terms which are proportional to $\cos(kX-\omega t)$ vanish, we evaluate the average energy propagation

$$
\begin{aligned}
\mathcal{E}_f = \bar{E}_f &= \frac{a^2\omega k c_0^2}{4\sinh^2 kd}\left[d+\frac{\sinh 2kd}{2k}\right] + \frac{a^2\omega\Omega c_0}{4k\sinh^2 kd}\left[\cosh 2kd - 1\right]\\
&= d\frac{a^2 k\omega}{4\sinh^2 kd}c_0^2 + \frac{a^2\omega}{4\tanh kd}c_0^2 - \frac{a^2\Omega}{2\sinh^2 kd}c_0^2 + \frac{a^2\Omega}{2\tanh^2 kd}c_0^2\\
&= \frac{a^2}{2}(g-2c_0\Omega)\left[\frac{kdc_0}{\sinh 2kd}+\frac{c_0}{2}-\frac{2\Omega}{k\sinh 2kd}+\frac{\Omega}{k\tanh kd}\right]\\
&= 2\bar{T}\cdot\left[\frac{kdc_0}{\sinh 2kd}+\frac{c_0}{2}+\frac{\Omega}{k\tanh kd}\left(1-\frac{1}{\cosh^2 kd}\right)\right],
\end{aligned}
\tag{3.13}
$$

making use of the expressions (3.4), (3.5), (3.7) and (3.8).

Remark 3.3. In the setting of purely gravitational waves ($\Omega = 0$) expression (3.13) reduces to the familiar relation (cf. [20, 22, 37])

$$
\begin{aligned}
\mathcal{E}_f^{grav} = \frac{1}{2}ga^2\cdot\left[\frac{kdc_0^{grav}}{\sinh 2kd}+\frac{c_0^{grav}}{2}\right] &\overset{(3.6)}{=} \frac{1}{2}ga^2\cdot\frac{g}{2\omega}\left[\tanh(kd)+\frac{kd}{\cosh^2(kd)}\right]\\
&\overset{(3.10)}{=} \mathcal{E}^{grav}\cdot c_g^{grav}.
\end{aligned}
\tag{3.14}
$$

Thus when the restoration force is purely gravity, the mean-energy flux for a linear water wave is given by $\mathcal{E}_f^{grav} = \mathcal{E}^{grav}\cdot c_g^{grav}$: the energy flux propagates the total energy density $\mathcal{E}^{grav}$ with group velocity c_g^{grav}. The mean kinetic and potential energies are equal for linear gravity waves, $\bar{V}^{grav} = \bar{T}^{grav}$, and hence indistinguishable in $\mathcal{E}^{grav}$ due to the equipartition of energy. However in the geophysical setting, where

$\bar{T}$ and $\bar{V}$ are unequal and distinct, expression (3.13) suggests that the energy density which is being propagated comprises solely of the kinetic energy density.

Remark 3.4. An alternative expression of the energy flux is given by

$$\begin{aligned}\mathcal{E}_f &= \frac{a^2\omega c_0^2}{4\sinh^2 kd}\left[kd + \frac{\sinh 2kd}{2}\right] + \frac{a^2\Omega c_0^2}{2\sinh^2 kd}\cdot\frac{\cosh 2kd - 1}{2} \\ &= \frac{a^2 k c_0^3}{4\sinh^2 kd}\left[kd + \frac{\sinh 2kd}{2}\right] + \Omega\frac{a^2c_0^2}{2}. \end{aligned} \tag{3.15}$$

Formally expanding quantities in (3.15) with respect to Ω, neglecting terms of $\mathcal{O}(\Omega^2)$, leads to $c_0^{\pm} = \pm c_0^{grav} - \frac{\Omega\tanh(kd)}{k} + \mathcal{O}(\Omega^2)$, with $\Omega(c_0^{\pm})^2 = \Omega(c_0^{grav})^2 + \mathcal{O}(\Omega^2)$, and $(c_0^{\pm})^3 = \pm(c_0^{grav})^3\left(1 \mp \frac{3\Omega\tanh kd}{\sqrt{gk\tanh kd}}\right) + \mathcal{O}(\Omega^2)$ which, substituting into (3.15), gives

$$\mathcal{E}_f^{\pm} = (\mathcal{E}_f^{grav})^{\pm} - \Omega\frac{a^2(c_0^{grav})^2}{4}\left(1 + \frac{6kd}{\sin 2kd}\right) + \mathcal{O}(\Omega^2), \tag{3.16}$$

where $(\mathcal{E}_f^{grav})^{\pm}$ is given by (cf.)

$$(\mathcal{E}_f^{grav})^{\pm} = \pm\frac{1}{2}ga^2\cdot\left[\frac{kdc_0^{grav}}{\sinh 2kd} + \frac{c_0^{grav}}{2}\right].$$

Similarly, implementing a perturbative expansion in the group velocity (3.10) gives

$$c_g^{\pm} = \pm\frac{c_0^{grav}}{2}\left(1 + \frac{2kd}{\sinh(2kd)}\right) - \Omega\frac{(c_0^{grav})^2}{g}\frac{2kd}{\sinh(2kd)} + \mathcal{O}(\Omega^2),$$

from which it can be easily seen that

$$2\bar{T}\cdot c_g^{\pm} = \mathcal{E}_f^{\pm} + \mathcal{O}(\Omega^2), \tag{3.17}$$

where $\mathcal{E}_f^{\pm}$ is given by (3.16). Therefore, up to first order in a perturbative expansion involving Ω, the parenthesis on the right-hand side of expression (3.13) matches the group velocity (3.10). Hence, at this level of approximation the energy flux $\mathcal{E}_f$ satisfies equation (3.17), which is identical to (3.14) (except featuring $2\bar{T}$ in place of $\mathcal{E}$) which pertains to the simpler gravity wave setting (cf. Remark 3.3). Note that the sign of the Coriolis contribution towards the energy flow in (3.16) is always negative.

3.4. **Linear waves: excess energy densities.** To estimate the excess potential energy density for small amplitude waves we substitute the linear wave solution (3.3a), $\eta(x) = a\cos(kx)$, into (5.1) to get

$$E_p^{lin} = \frac{ga^2}{4} + \mathcal{O}\left(a^3\right) > 0,$$

where the inequality clearly holds for sufficiently small amplitudes a.

The excess kinetic energy density (1.2), evaluated for the linear wave solution prescribed by (3.3), becomes

$$E_k^{lin} = \frac{1}{2\lambda}\int_0^\lambda\int_{-d}^0 \left(u^2+w^2\right)\, dz\, dx - \frac{c_0}{\lambda}\int_0^\lambda\int_{-d}^{a\cos kx} u\, dz\, dx + \frac{1}{2\lambda}\int_0^\lambda\int_0^{a\cos kx} c_0^2\, dz\, dx + \mathcal{O}(a^3). \tag{3.18}$$

The first term on the right-hand side of (3.18) is given by (3.11), the third term is zero due to (2.1g), and the second term in (3.18) is found (using (3.3)) to be

$$\begin{aligned}\frac{c_0}{\lambda}a\omega\int_0^\lambda\int_{-d}^{a\cos kx}\frac{\cosh(k(d+z))}{\sinh(kd)}\cos kx\, dz\, dx &= \frac{c_0\omega}{\lambda k}a\int_0^\lambda \frac{\sinh(k(d+a\cos kx))}{\sinh kd}\cos kx\, dx\\ &= \frac{c_0\omega}{\lambda}\frac{\cosh kd}{\sinh kd}a^2\int_0^\lambda \cos^2 kx\, dx + \mathcal{O}(a^3) = \frac{\omega^2}{2k\tanh kd}a^2 \overset{(3.8)}{=} \frac{a^2}{2}\left(g-2\Omega c_0\right),\end{aligned}$$

where we have used the identity

$$\sinh(k(d+a\cos x)) = \sinh kd + \cosh kd\cdot ka\cos x + \frac{\sinh kd}{2!}(ka\cos x)^2 + \ldots \tag{3.19}$$

Hence (3.18) can be explicitly computed in the linear setting to get

$$E_k^{lin} = -\frac{a^2}{4}\left(g-2\Omega c_0\right) + \mathcal{O}(a^3) < 0, \tag{3.20}$$

where the inequality in (3.20) holds due to (3.9) regardless of the sign of c_0.

Finally, the total excess energy density is

$$E_{tot}^{lin} = E_p^{lin} + E_k^{lin} = \frac{1}{2}\Omega c_0 a^2 + \mathcal{O}(a^3). \tag{3.21}$$

The sign of E_{tot}^{lin} in (3.21) depends on the choice of $c_0^{\pm}$: $E_{tot}^{lin}<0$ for c_0^-, while $E_{tot}^{lin}>0$ for c_0^+, for sufficiently small amplitude waves.

4. Nonlinear travelling water waves

In the analysis of nonlinear waves, our focus will be restricted to travelling waves whereby the X and t variables have a functional dependence of the form $X-ct$. The dispersion relation (3.5) suggests that c may be taken to be either positive or negative, corresponding to right–moving (respectively, left–moving) waves. Making this choice affects not just the magnitude of the wavespeed (as can be seen explicitly in the linear setting in (3.5)) but also the magnitude of the wave energy densities (as seen explicitly in (3.11) and (3.12) for the linear regime). This situation contrasts with the setting of purely gravity waves ($\Omega=0$) where the choice of sign is quite immaterial: the different choices simply correspond to choosing different directions of wave motion. As seen in Proposition 2 below, the choice $c<0$ is most propitious for the analysis of the excess energy for nonlinear waves and, unless otherwise stated, in subsequence considerations we take $c<0$ (although Remarks 4.1 and 5.3 outline how various considerations can be applied to the case $c>0$).

Define new variables x and z in the reference frame moving with speed c via

$$x = X - ct, \qquad z = Z. \tag{4.1}$$

The governing equations (2.1) for fluid motion in the moving reference frame are then transformed to

$$u_x + w_z = 0, \tag{4.2a}$$
$$(u-c)u_x + wu_z + 2\Omega w = -P_x, \tag{4.2b}$$
$$(u-c)w_x + ww_z - 2\Omega u = -P_z - g \qquad \text{in } \mathcal{D}_{\eta(x),d}, \tag{4.2c}$$

with the kinematic and dynamic boundary conditions

$$w = (u-c)\eta_x, \tag{4.2d}$$
$$P = P_{atm} \qquad\qquad \text{on } z = \eta(x), \tag{4.2e}$$
$$w = 0 \qquad\qquad \text{on } z = -d. \tag{4.2f}$$

In the following we analyse smooth exact solutions to the governing equations (4.2) for which η, u, w, P have period λ in the $x-$variable. Moreover, there is a single crest and trough per period, and the profile η is decreasing from crest to trough with $\eta'(x) \neq 0$ except at the maximum (crest) or minimum (trough). The functions η, u, P are symmetric while v is antisymmetric about the crest. The existence of such waves was rigorously established using bifurcation theory methods in [12, 32] (permitting general underlying vorticity distributions, but without stagnation points) with further generalisations achieved in [41, 42]. The assumption we make regarding the symmetry of the unknown wave-surface is not restrictive in the sense that it can be proven that if the surface profile is monotonic between troughs and crests, then it must in fact be symmetric [33], even in the presence of stagnation points [2]. We choose the crest to lie on $x = 0$, with the trough located at $x = \pm\lambda/2$. Additionally, we assume that there is no underlying constant current, that is

$$\kappa = \int_{-\lambda/2}^{\lambda/2} u(x, -d)\, dx = 0. \tag{4.3}$$

(A detailed mathematical analysis of the underlying fluid motion for nonlinear equatorial waves, which also accommodates a non-zero current term $\kappa \neq 0$, can be found in [43].) The stream function ψ is defined (up to a constant) by

$$\psi_z = u - c, \quad \psi_x = -w, \tag{4.4}$$

and we fix the constant by setting $\psi = 0$ on $z = \eta(x)$. Relations (4.2d) and (2.1h) imply that ψ is constant on both boundaries of $\mathcal{D}_{\eta(x),d}$, and so it follows from integrating (4.4) that $\psi = m$ on $z = -d$, where

$$m = \int_{-d}^{\eta(x)} (u(x,z) - c) dz. \tag{4.5}$$

The above expression gives the relative mass flux, and it is seen by direct calculation that m is an invariant of the flow. Since

$$\psi(x,z) = -m + \int_{-d}^{z} (u(x,s) - c) ds,$$

we deduce that ψ is a periodic function, with period λ. Furthermore, the level sets of the stream function $\psi(x,z)$ describe the streamlines of the flow. An important consequence of the irrotationality condition (2.1i) is that the stream function ψ, and hence also ψ_z and u, are harmonic functions throughout the fluid domain $\mathcal{D}_{\eta(x),d}$. The strong maximum principle for harmonic functions [25] implies that $m \neq 0$, unless the flow is trivial. Since $\psi = 0$ on $z = \eta$, applying the strong maximum principle for harmonic functions to ψ, and in turn ψ_z, we can infer that $m > 0$. Furthermore, it follows that there are no stagnation points in $\mathcal{D}_{\eta(x),d}$, that is,

$$\psi_z = u(x,z) - c > 0 \quad \text{in } \mathcal{D}_{\eta(x),d}. \tag{4.6}$$

The absence of stagnation points is a physically realistic scenario for water waves without underlying currents containing strong non-uniformities, and which are not near breaking: in this situation the maximal horizontal velocity u typically has a magnitude of about 10% of the wavespeed. Along the lines of purely gravitational waves ($\Omega = 0$, cf. [9]), it can be shown that the maximum of the harmonic function u is attained precisely at the wave-crest: the limiting case whereby a stagnation point occurs at the wavecrest ($u = c$) is known as an extreme wave. Define the relative hydraulic head Q by the expression

$$Q := \frac{(u-c)^2 + w^2}{2} + (g - 2\Omega c)z + \frac{P}{\rho} - 2\Omega\psi \qquad \text{in } \mathcal{D}_{\eta(x),d}.$$

It follows from taking the curl of equations (4.2b) and (4.2c) that Q is constant in $\mathcal{D}_{\eta(x),d}$: this is the $f-$plane version of Bernoulli's law. Accordingly, we can reformulate (4.2) in terms of the functions η, ψ in the moving reference frame as

$$\Delta\psi = 0 \qquad \text{in } \ -d < z < \eta(x), \tag{4.7a}$$

$$|\nabla\psi|^2 + 2(g - 2\Omega c)z = Q \qquad \text{on } z = \eta(x), \tag{4.7b}$$

$$\psi = 0 \qquad \text{on } z = \eta(x), \tag{4.7c}$$

$$\psi = -m \qquad \text{on } z = -d. \tag{4.7d}$$

It is clear from (4.7b) that $Q > 0$ for non-trivial waves ($\eta \not\equiv 0$) in the setting $c < 0$.

The governing equations (4.2) and (4.7) have been greatly simplified by being formulated in the moving reference frame wherein they describe steady fluid motion. Transforming to the moving frame is an apparently trivial exercise mathematically, simply requiring the change of coordinates (4.1). Yet this innocuous-looking change of variables belies a hidden complication that must be addressed before we can consider the physical flow characteristics and parameters of the underlying wave motion, namely: what is the wavespeed c, and how do we determine it? The issue of determining the wavespeed from the kinematics of a travelling wave is a surprisingly complex matter both from the mathematical [9, 13], and physical [47], perspectives.

This question is particularly apposite when posed in the moving reference frame (wherein mathematical analysis is performed) in which the steady flow is time-independent. We note that, unlike for purely gravity waves (whereby $\Omega = 0$), the wavespeed c appears explicitly in the governing equation formulations (4.7). There is no canonical definition of the wavespeed, however (bearing in mind relations (4.3) and (4.4)) *Stokes' first definition* of the wavespeed defines the wavespeed by the expression

$$c = -\frac{1}{\lambda}\int_0^\lambda \psi_z(x, z_0)\,dx < 0, \tag{4.8}$$

where the last inequality follows from (4.6). The wavespeed c is thus defined by (4.8) to be the mean horizontal velocity of the fluid in the moving frame of reference for which the wave is stationary. It can be seen (cf. [43]) that expression (4.8) is independent of the (fixed) depth z_0 beneath the wave trough level.

Remark 4.1. For the choice of wavespeed $c > 0$, the condition ensuring an absence of stagnation points corresponding to (4.6) is given by

$$\psi_z = u(x, z) - c < 0 \quad \text{ in } \mathcal{D}_{\eta(x),d}, \tag{4.9}$$

and it follows that the mass-flux $m < 0$. If we make the physically reasonable assumption that condition (3.9) holds for the choice of nonlinear wavespeed $c > 0$, it follows that $Q > 0$ also in this case. Stokes first definition of the wavespeed also applies to this setting, except with a reversal of the inequality in (4.8) above.

The irrotationality condition (2.1i) enables the definition (up to a constant) of a velocity potential $\phi(x, y)$ by way of the relations

$$\phi_x := u - c = \psi_z, \quad \phi_z := w = -\psi_x. \tag{4.10}$$

Hence the physical system is conservative, and it follows from (4.2a) and relation (4.10) that ϕ is a harmonic function. Fixing $\phi = 0$ on the crest line we can express

$$\phi(x, y) = \int_0^x [u(l, -d) - c]\,dl + \int_{-d}^y w(x, s)ds,$$

from which we deduce that $\phi(x, z) + cx$ has period λ in x, ϕ is odd in the $x-$variable and vanishes at $x = 0$, and $\phi(\lambda n, z) = -c\lambda n$ for any integer n. The existence of a velocity potential enables us to define the hodograph coordinate transformation

$$(x, z) \mapsto (q, p) = (-\phi(x, z), -\psi(x, z))\,. \tag{4.11}$$

The governing equations (4.7) can be further reformulated in terms of the height function

$$h(q, p) = z + d,$$

as follows. The mapping (4.11) is conformal since ϕ and ψ are harmonic conjugates (4.10), hence $\Delta_{(x,z)}h = 0$ implies that $\Delta_{(q,p)}h = 0$. Furthermore (4.11) transforms the fluid domain $\mathcal{D}_{\eta(x),d}$ with an unknown free boundary into the fixed rectangular strip $\mathbb{R} \times [-m, 0]$. We have

$$\begin{cases} \partial_q = h_p \partial_x + h_q \partial_z, \\ \partial_p = -h_q \partial_x + h_p \partial_z, \end{cases}$$

with

$$\begin{cases} \partial_x = (c-u)\partial_q + w\partial_p, \\ \partial_z = -w\partial_q + (c-u)\partial_p, \end{cases}$$

while

$$h_q = -\frac{w}{(c-u)^2 + w^2} = -\frac{\partial x}{\partial p} = \frac{\partial z}{\partial q}, \quad h_p = \frac{c-u}{(c-u)^2 + w^2} = \frac{\partial x}{\partial q} = \frac{\partial z}{\partial p}, \tag{4.12}$$

noting that condition (4.6) implies that $h_p > 0$. The governing equations (4.7) have the following reformulation in terms of h:

$$\Delta h = 0 \qquad \text{for } -m < p < 0, \tag{4.13a}$$

$$[Q + 2(d-h)(g - 2\Omega c)]\left(h_q^2 + h_p^2\right) = 1 \qquad \text{on } p = 0, \tag{4.13b}$$

$$h = 0 \qquad \text{on } p = -m. \tag{4.13c}$$

5. Wave energy densities

5.1. **Excess potential energy density.** Our first result concerning the excess potential energy E_p follows immediately from the definition (1.1).

Proposition 1. The excess potential energy per unit horizontal area, E_p, can be expressed as

$$E_p = \frac{g}{\lambda} \int_0^{\lambda/2} \eta^2(x) dx. \tag{5.1}$$

Hence, $E_p > 0$ is positive for all (non-trivial) water wave solutions of the nonlinear governing equations (4.2).

This result is independent of the sign of the wavespeed c, and it reflects the fact that both a raised, and depressed, free-surface serve to increase the potential energy by an amount proportional to the square of the displacement from the mean water level. The raised surface increases the potential energy through adding new fluid above the position of the mean water level, whereas a depressed surface increases the potential energy through the removal of fluid beneath the mean water level.

Proof. It is extremely straightforward to show that the presence of free-surface waves increases the potential energy of a flow. Expression (1.1) reduces to

$$E_p = \frac{1}{\lambda} \int_0^{\lambda} \int_0^{\eta(x)} gz \, dz \, dx = \frac{g}{2\lambda} \int_0^{\lambda} \eta^2(x) dx > 0,$$

with equality holding only in the absence of free-surface waves ($\eta \equiv 0$), in which case the potential energy for the flow is minimised. Expression (5.1) follows from symmetry considerations. □

5.2. **Excess kinetic energy density.** Unlike the situation for potential energy, considerations relating to the excess kinetic energy for nonlinear waves are affected, and complicated, by the presence of Coriolis terms involving Ω. Nevertheless, for negative wavespeeds $c<0$ we can prove that the presence of waves serves to decrease the kinetic energy in the moving frame: the excess kinetic energy is negative in this case. This result generalises inequality (3.20), which is applicable for linear water waves, and which explicitly shows that E_k^{lin} is negative. Since (3.20) pertains only to waves of relatively small amplitude ($a \ll d$), it can not be used to infer that a similar inequality holds for larger amplitude nonlinear waves. Instead, this result follows from Proposition 2.

Proposition 2. The excess kinetic energy per unit horizontal area, E_k, can be expressed as

$$E_k = -\frac{c}{\lambda}\int_0^{\lambda/2} \eta(x)u(x,\eta(x))dx. \tag{5.2}$$

Additionally, for the choice of wavespeed $c<0$, the excess kinetic energy $E_k < 0$ is negative for all (non-trivial) water wave solutions of the nonlinear governing equations (4.2).

To prove the second part of this statement we use the following:

Lemma 5.1. (i) *If $c<0$, the function $u(x,\eta(x))+2\Omega\eta(x)$ is strictly increasing along the free-surface from crest to trough, that is,*

$$\partial_x\left[u(x,\eta(x))+2\Omega\eta(x)\right]>0 \text{ for } x\in(0,\lambda/2).$$

Since $\eta'(x)<0$ for $x\in(0,\lambda/2)$, we conclude that $\partial_x u(x,\eta(x))>0$.

(ii) *If $c>0$, the function $u(x,\eta(x))+2\Omega\eta(x)$ is strictly decreasing along the free-surface from crest to trough, that is,*

$$\partial_x\left[u(x,\eta(x))+2\Omega\eta(x)\right]<0 \text{ for } x\in(0,\lambda/2).$$

Proof. Direct calculation from (4.2b) and (4.2c) shows that $\Delta P = -2u_x - 2u_y \le 0$, where the Coriolis terms vanish due to the irrotationality condition (2.1i). Hence the pressure function P is superharmonic and its minimum must be attained on the boundary of $\mathcal{D}_{\eta(x),d}$, cf. [25]. On the boundary $z=-d$, (4.2c) and (2.1h) show that $P_z = -g+2\Omega u<0$, where the inequality comes from (4.6) and the natural assumption that c satisfies (3.9), therefore Hopf's maximum principle implies that the minimum of P cannot occur here [25]. The pressure $P=P_{atm}$ is constant along the free-surface $z=\eta(x)$, which itself is strictly decreasing for $x\in(0,\lambda/2)$. Accordingly, Hopf's minimum principle implies that $P_x(x,\eta(x))<0$ for $x\in(0,\lambda/2)$, which in conjunction with (4.2b) and (4.2d) gives

$$\begin{aligned}(u(x,\eta(x))-c)\,u_x(x,\eta(x)) &+ w(x,\eta(x))u_z(x,\eta(x))+2\Omega w(x,\eta(x))\\ &= (u(x,\eta(x))-c)\left[u_x(x,\eta(x))+u_z(x,\eta(x))\eta'(x)+2\Omega\eta'(x)\right]\\ &= (u(x,\eta(x))-c)\,\partial_x\left[u(x,\eta(x))+2\Omega\eta(x)\right]>0.\end{aligned}$$

Statement (i) follows from (4.6), while statement (ii) follows from (4.9). □

Proof of Proposition 2. In the nonlinear wave regime, we work as follows. The hodograph change of variables (4.11) transforms the region $\mathcal{D}^{\lambda}_{\eta(x),d} = \mathcal{D}_{\eta(x),d} \cap \{x \in [0,\lambda]\}$ to the finite rectangle $R = [0, c\lambda] \times [-m, 0]$. From relations (4.12) we can re-express the excess kinetic energy (1.2) as

$$\begin{aligned} E_k &= \frac{1}{2\lambda}\int_0^{\lambda}\int_{-d}^{\eta(x)} \left((u-c)^2 + v^2\right)\, dy\, dx - \frac{c^2 d}{2} \\ &= \frac{1}{2\lambda}\int_0^{c\lambda}\int_m^0 \frac{1}{h_p^2 + h_q^2}\left|\frac{\partial(x,y)}{\partial(q,p)}\right| dp\, dq - \frac{c^2 d}{2} \\ &= \frac{1}{2\lambda}\int_0^{c\lambda}\int_m^0 \frac{1}{h_p^2 + h_q^2}\left(h_p^2 + h_q^2\right) dp\, dq - \frac{c^2 d}{2} = -\frac{c}{2}(m + cd) = -\frac{cd}{2}(c - \tilde{c}). \end{aligned} \tag{5.3}$$

Here $\tilde{c}$ corresponds to *Stokes' second definition* of the wavespeed, which is defined to be equal to the depth-averaged horizontal fluid velocity in the moving frame:

$$\tilde{c} = -\frac{1}{\lambda d}\int_0^{\lambda}\int_{-d}^{\eta(x)} \psi_z(x, z_0)\, dz dx = -\frac{m}{d}. \tag{5.4}$$

The second equality in (5.4) follows from (4.4) and (4.5). Along the lines of [13], it can be shown that

$$c - \tilde{c} = \frac{1}{\lambda d}\int_0^{\lambda} \eta(x)\left(u(x,\eta(x)) - c\right) dx = \frac{2}{\lambda d}\int_0^{\lambda/2} \eta(x) u(x,\eta(x)) dx, \tag{5.5}$$

where the second equality follows from an implementation of (2.1g), and symmetry considerations. Substituting relation (5.5) into (5.3) leads to the expression (5.2).

The proof that $E_k < 0$ for $c < 0$ follows from (5.3) by showing that $c < \tilde{c} < 0$ for all wave solutions of (4.2). This result was first established in the gravity wave setting in [13], and the analysis replicated for equatorial waves in [23]. If x_0 is the unique point where $\eta(x_0) = 0$, then re-expressing (5.5) gives

$$\begin{aligned} c - \tilde{c} &= \frac{2}{\lambda d}\int_0^{\lambda/2} \eta(x)\left[u(x,\eta(x)) + 2\Omega\eta(x)\right] dx - \frac{4\Omega}{\lambda d}\int_0^{\lambda/2} \eta^2(x) dx \\ &= \frac{2}{\lambda d}\int_0^{x_0} \eta(x)\left[u(x,\eta(x)) + 2\Omega\eta(x)\right] dx + \frac{2}{\lambda d}\int_{x_0}^{\lambda/2} \eta(x)\left[u(x,\eta(x)) + 2\Omega\eta(x)\right] dx \\ &\quad - \frac{4\Omega}{\lambda d}\int_0^{\lambda/2} \eta^2(x) dx \overset{(\dagger)}{<} \frac{2}{\lambda d}\int_0^{x_0} \eta(x) u(x_0,\eta(x_0)) dx + \frac{2}{\lambda d}\int_{x_0}^{\lambda/2} \eta(x) u(x_0,\eta(x_0)) dx \\ &\quad - \frac{4\Omega}{\lambda d}\int_0^{\lambda/2} \eta^2(x) dx = \frac{2u(x_0,\eta(x_0))}{\lambda d}\int_0^{\lambda/2} \eta(x) dx - \frac{4\Omega}{\lambda d}\int_0^{\lambda/2} \eta^2(x) dx \\ &= -\frac{4\Omega}{\lambda d}\int_0^{\lambda/2} \eta^2(x) dx < 0. \end{aligned} \tag{5.6}$$

The first inequality (†) in (5.6) follows from an application of statement (i) in Lemma 5.1, using the fact that $\eta'(x) < 0$ in this region. Now using relation (5.6) in (5.3) we conclude that

$$E_k = -\frac{c}{2\lambda}\int_0^{\lambda} \eta(x)u(x,\eta(x))dx < 0.$$

Expression (5.2) follows from symmetry considerations. $\square$

Remark 5.2. Stokes' first definition of the wavespeed (4.8) can be evaluated for the linear wave solution (3.3) to get

$$c \approx -\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} a\omega\frac{\cosh(k(d+Z))}{\sinh kd}\cos(x)dx + \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\frac{\omega}{k}\,dx = c_0,$$

where $c_0 = \omega/k$ is the wavespeed given by the linear dispersion relation (3.5). Evaluating Stokes' second definition of the wavespeed (5.4) for the linear wave solution (3.3) gives

$$\begin{aligned}
\tilde{c} \approx -\frac{1}{\lambda d}\int_{-\lambda/2}^{\lambda/2}\int_{-d}^{\eta(x)} a\omega\frac{\cosh(k(d+Z))}{\sinh kd}\cos(x)dzdx + \frac{1}{\lambda d}\int_{-\lambda/2}^{\lambda/2}\int_{-d}^{\eta(x)}\frac{\omega}{k}\,dzdx \\
= -\frac{ac_0}{\lambda d}\int_{-\lambda/2}^{\lambda/2}\frac{\sinh(k(d+a\cos x))}{\sinh kd}\cos(x)dx + c_0 \\
= -\frac{ac_0}{\lambda d}\int_{-\lambda/2}^{\lambda/2}\left(\cos x + \frac{\cosh kd}{\sinh kd}ka\cos^2 x + \frac{1}{2!}(ka)^2\cos^3 x\right)dx + c_0 \\
= c_0 - \frac{c_0 ka^2}{2d}\coth kd + \mathcal{O}(a^3). \quad (5.7)
\end{aligned}$$

where we have used the identity (3.19). Hence $c = \tilde{c} + \mathcal{O}(a^2)$, and definitions (4.8) and (5.4) 'agree' up to $\mathcal{O}(a^2)$: they coincide in the regime of linear water waves.

Remark 5.3. Regarding the case where the wavespeed is positive, $c > 0$, relation (3.20) implies that $E_k < 0$ for sufficiently small amplitude waves, whatever the sign of c, once the (physically plausible) condition (3.9) is satisfied. For larger amplitude nonlinear waves, considerations similar to those in (5.6) carry over to the case $c > 0$, except with a reversal of inequality (†) due to statement (ii) in Lemma 5.1. Hence, for $c, \tilde{c} > 0$, we can show that

$$c - \tilde{c} > -\frac{4\Omega}{\lambda d}\int_0^{\lambda/2}\eta^2(x)dx. \quad (5.8)$$

In the case of purely gravity waves ($\Omega = 0$) we recover the result (cf. [13]) that $c - \tilde{c} > 0$ for all nonlinear waves. Although Ω is numerically very small, and the identity (5.7) implies that $c - \tilde{c} > 0$ for small amplitude wave solutions of (2.1), unfortunately we are unable to infer from (5.8) that a similar result holds for nonlinear wave solutions of (2.1). Proving this would enable us to conclude, by way of (5.3), that $E_k < 0$ holds also when $c > 0$.

5.3. **Total excess energy density.** The total excess energy density, defined by $E_{tot} = E_p + E_k$, can be characterised for nonlinear waves as follows.

Proposition 3. The total excess energy for nonlinear waves is given by

$$E_{tot} = -\frac{1}{2\lambda}\int_0^{\lambda/2} \eta(x)\left(u^2(x,\eta(x)) + v^2(x,\eta(x))\right)dx + \frac{\Omega c}{\lambda}\int_0^{\lambda/2} \eta^2(x)dx. \tag{5.9}$$

Proof. For nonlinear water waves, (5.1) and (5.2) give

$$\begin{aligned} E_{tot} = E_p + E_k &= \frac{1}{2\lambda}\int_0^{\lambda} \eta(x)\left(g\eta(x) - cu(x,\eta(x))\right)dx. \\ &= -\frac{1}{4\lambda}\int_0^{\lambda} \eta(x)\left(u^2(x,\eta(x)) + v^2(x,\eta(x)) - 2\Omega c\eta(x)\right)dx. \end{aligned}$$

The last equality follows from the Bernoulli relation (4.7b) combined with an application of (2.1g). Expression (5.9) now follows from symmetry considerations. □

The total excess energy for water waves is expressed in (5.9) as the mean of the kinetic energy along the wave surface profile, weighted by the wave surface profile itself, plus an additional term which depends on both the wavespeed, and the Coriolis term Ω. Expression (5.9) matches (3.21) for linear waves solutions (3.3), where we note that the first weighted integral term is zero (at order $\mathcal{O}(a^2)$).

Determining the sign of (5.9) provides insight into whether the kinetic or potential energies predominate for a given wave solution and, for nonlinear waves whereby relation (3.21) is not applicable, establishing this analytically is not a trivial exercise. We know that $\eta(0) > 0$, and $\eta(\lambda/2) < 0$, with $\eta'(x) < 0$ for $x \in (0, \lambda/2)$, and furthermore nonlinear waves tend to have sharper crest elevations and flatter depressions compared to linear waves (see Remark 3.1). Rigorous results establishing monotonicity properties of the kinetic energy for nonlinear waves do exist [1,38,44], however these establish an exponential decrease of the kinetic energy with respect to vertical depth (for fluid motion beneath the wave trough) and do not pertain to the behaviour along the free surface. Some monotonicity properties can be established for the horizontal velocity component u along the free-surface (see Lemma 5.1) but, even in the simpler setting of gravity waves, little is known analytically about the behaviour of the vertical velocity v, cf. [5] for numerical investigations which provide some insight into this question. Similarly, insight into expression (5.9) could be achieved in the first instance by way of numerical investigations.

References

[1] A. Aleman and A. Constantin, On the decrease of kinetic energy with depth in wave-current interactions, *Math. Ann.* **378** (2020), 853–872.

[2] A. Aivaliotis, On the symmetry of equatorial travelling water waves with constant vorticity and stagnation points, *Nonlinear Anal. Real World Appl.* **34** (2017), 159–171.

[3] T. B. Benjamin and P. J. Olver, Hamiltonian structure, symmetries and conservation laws of water waves, *J. Fluid Mech.* **125** (1982), 137–185.

[4] P. Bonneton and D. Lannes, Recovering water wave elevation from pressure measurements, *J. Fluid Mech.* **833** (2017), 399–429.
[5] D. Clamond, Note on the velocity and related fields of steady irrotational two-dimensional surface gravity waves, *Philos. Trans. R. Soc. Lond. Ser. A* **370** (2012), 1572–1586.
[6] D. Clamond, New exact relations for easy recovery of steady wave profiles from bottom pressure measurements, *J. Fluid Mech.* **726** (2013), 547–558.
[7] D. Clamond and A. Constantin, Recovery of steady periodic wave profiles from pressure measurements at the bed, *J. Fluid Mech.* **714** (2013), 463–475.
[8] D. Clamond and D. Henry, Extreme water-wave profile recovery from pressure measurements at the seabed, *J. Fluid Mech.* **903** (2020), R3, 12 pp.
[9] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Conference Series in Applied Mathematics, Vol. 81, SIAM, Philadelphia, 2011.
[10] A. Constantin, An exact solution for equatorially trapped waves, *J. Geophys. Res.: Oceans* **117** (2012), C05029.
[11] A. Constantin, On the recovery of solitary wave profiles from pressure measurements, *J. Fluid Mech.* **699** (2012), 376–384.
[12] A. Constantin, On equatorial wind waves, *Differential and Integral Equations* **26** (2013), 237–252
[13] A. Constantin, Mean velocities in a Stokes wave, *Arch. Ration. Mech. Anal.* **207** (2013), 907–917.
[14] A. Constantin and R. I. Ivanov, Equatorial wave-current interactions, *Comm. Math. Phys.* **370** (2019), 1–48.
[15] A. Constantin, R. I. Ivanov, and C. I. Martin, Hamiltonian formulation for wave–current interactions in stratified rotational flows, *Arch. Ration. Mech. Anal.* **221** (2016), 1417–1447.
[16] A. Constantin and R. S. Johnson, The dynamics of waves interacting with the Equatorial Undercurrent, *Geophys. Astrophys. Fluid Dyn.* **109** (2015), 311–358.
[17] A. Constantin and R. S. Johnson, An exact, steady, purely azimuthal equatorial flow with a free surface, *J. Phys. Oceanogr.* **46**(2016), 1935–1945.
[18] A. Constantin and R. S. Johnson, Current and future prospects for the application of systematic theoretical methods to the study of problems in physical oceanography, *Phys. Lett. A* **380** (2016), 3007–3012.
[19] A. Constantin and R. S. Johnson, A nonlinear, three-dimensional model for ocean flows, motivated by some observations of the Pacific Equatorial Undercurrent and thermocline, *Phys. Fluids* **29** (2017), 056604.
[20] G. D. Crapper, *Introduction to water waves*, Ellis Horwood, Chichester, 1984.
[21] B. Cushman-Roisin and J.-M. Beckers, *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects*, Academic, Waltham, Mass., 2011.
[22] R. G. Dean and R. A. Dalrymple, *Water wave mechanics for engineers and scientists*, World Scientific Publishing, 1991.
[23] L. Fan, Mean velocities in an irrotational equatorial wind wave, *Appl. Numer. Math.* **141** (2019), 158–166.
[24] A. V. Fedorov and J. N. Brown, Equatorial waves, in *Encyclopedia of Ocean Sciences*, edited by J. Steele, pp. 3679–3695, Academic, San Diego, Calif., 2009.

[25] L. E. Fraenkel, *An introduction to maximum principles and symmetry in elliptic problems*, Cambridge University Press, Cambridge, 2000.
[26] A. Gill, *Atmosphere-ocean dynamics*, Academic Press, New York, 1982.
[27] D. Henry, On three-dimensional Gerstner-like equatorial water waves, *Philos. Trans. Roy. Soc. A* **376** (2018), 20170088, 16 pp.
[28] D. Henry, On nonlinearity in three-dimensional equatorial flows, *J. Nonlinear Math. Phys.* **25** (2018), 351–357.
[29] D. Henry, On the energy of nonlinear water waves, *Proc. R. Soc. A.* **477** (2021), 20210544.
[30] D. Henry and C. I. Martin, Exact, free-surface equatorial flows with general stratification in spherical coordinates, *Arch. Ration. Mech. Anal.* **233** (2019), 497–512.
[31] D. Henry and C. I. Martin, Stratified equatorial flows in cylindrical coordinates, *Nonlinearity* **33** (2020), 3889–3904.
[32] D. Henry and A.-V. Matioc, On the existence of equatorial wind waves. *Nonlinear Anal.* **101** (2014), 113–123.
[33] D. Henry and A.-V. Matioc, On the symmetry of steady equatorial wind waves, *Nonlinear Anal. Real World Appl.* **18** (2014), 50–56.
[34] D. Henry and G. P. Thomas, Prediction of the free-surface elevation for rotational water waves using the recovery of pressure at the bed, *Philos. Trans. Roy. Soc. A* **376** (2018), 20170102, 21 pp.
[35] D. Ionescu-Kruse and A.-V. Matioc, Small-amplitude equatorial water waves with constant vorticity: dispersion relations and particle trajectories, *Discrete Contin. Dyn. Syst.* **34** (2014), 3045–3060.
[36] R. S. Johnson, Application of the ideas and techniques of classical fluid mechanics to some problems in physical oceanography, *Phil. Trans. R. Soc. A* **376** (2018) 20170092.
[37] J. Lighthill, *Waves in fluids*, Cambridge University Press, Cambridge-New York, 1978.
[38] M. S. Longuet-Higgins, On the decrease of velocity with depth in an irrotational water wave, *Math. Proc. Camb. Philos. Soc.* **49** (1953), 552–560.
[39] M. S. Longuet-Higgins, Integral properties of periodic gravity waves of finite amplitude, *Proc. R. Soc. Lond. A* **342** (1975), 157–174.
[40] M. S. Longuet-Higgins, New integral relations for gravity waves of finite amplitude, *J. Fluid Mech.* **149** (1984), 205–215.
[41] C. I. Martin, Equatorial wind waves with capillary effects and stagnation points, *Nonlinear Anal. TMA* **96** (2014), 1–17.
[42] C. I. Martin, On periodic geophysical water flows with discontinuous vorticity in the equatorial $f-$plane approximation, *Philos. Trans. Roy. Soc. A* **376** (2018), 20170096, 23 pp.
[43] R. Quirchmayr, On irrotational flows beneath periodic traveling equatorial waves, *J. Math. Fluid Mech.* **19** (2017), 283–304
[44] L. Roberti, On the decrease of velocity with depth in irrotational periodic water waves, *Monatsh. Math.* **193** (2020), 671–682.
[45] V. T. Starr, Momentum and energy integrals for gravity waves of finite height, *J. Mar. Res.* **6** (1947), 175–193.
[46] G. P. Thomas, The Theory Behind the Conversion of Ocean Wave Energy: a Review, In: J. Cruz (ed.) *Ocean Wave Energy: Current status and future perspectives*, 41–91, Springer, Berlin-Heidelberg, 2008.

[47] G. Thomas and G. Klopman, Wave-current interactions in the nearshore region, *Gravity waves in water of finite depth*, 215–319, Advances in Fluid Mechanics, 10. WIT, Southhampton, United Kingdom, 1997.

[48] G. K. Vallis, *Atmospheric and Oceanic Fluid Dynamics*, Cambridge University Press, 2006.

School of Mathematical Sciences, University College Cork, Cork, Ireland.
Email address: d.henry@ucc.ie