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Boolean Rings are Definitely Commutative!

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Abstract. A ring $\{R, +, .\}$ is called Boolean if $r^2 = r$ for all $r \in R$. We present four proofs that a Boolean ring is commutative.

A ring $\{R, +, .\}$ is called Boolean if $r^2 = r$ for all $r \in R$. In this bicentenary year of Boole's birth we present four proofs that a Boolean ring is commutative. Our first proof is the standard one found in many textbooks.

Proof 1. For all $r \in R$ we have $r = r^2 = (-r)^2 = -r$, so $r + r = 0$. Next, for all x and y in R, $x+y = (x+y)^2 = x^2+xy+yx+y^2$, so by cancellation in the group $\{R, +\}$, we have $xy+yx = 0 = xy+xy$, by the above. Again by cancellation we have $xy = yx$, as required. □

Proof 2. As in Proof 1, $xy + yx = 0$, for all x and y in R. Since for all $r \in R$, $0.r = 0 = r.0$ we have $(xy + yx)x = x(xy + yx)$ or $xyx + y.x^2 = x^2.y + xyx$. Cancelling xyx and remembering that $x^2 = x$, we get $xy = yx$, as required. □

Proof 3. Since for all r, $r^2 = r$ it follows that if $r^2 = 0$ then $r = 0$. Now for all x and y in R we have $(xy - xyx)^2 = xyxy + xyxxyx - xyxyx - xyxxy = xyxy + xyxyx - xyxyx - xyxy = 0$. So $xy - xyx = 0$ and $xy = xyx$. Then $(yx - xyx)^2 = yxyx + xyxxyx - yxxyx - xyxyx = yxyx + xyxyx - yxyx - xyxyx = 0$. So $yx - xyx = 0$ and $yx = xyx$. Thus $xy = yx$ as required. □

Proof 4. For $a, b \in R$ if $ab = 0$, then $ba = (ba)^2 = b(ab)a = 0$. Now, $0 = xy - xy = xy - x^2y = x(y - xy)$, so $0 = (y - xy)x = yx - xyx$. Also, $0 = yx - yx = yx - yx^2 = (y - yx)x$, so $0 = x(y - yx) = xy - xyx$. Thus $xy = yx$ for all x and y in R. □

We note it is immediate in all four proofs that $xy = yx = xyx = yxy$, for all x and y.

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