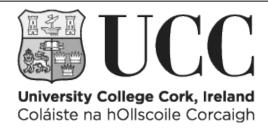


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Boolean Rings are Definitely Commutative!

DESMOND MACHALE

ABSTRACT. A ring $\{R, +, .\}$ is called Boolean if $r^2 = r$ for all $r \in R$. We present four proofs that a Boolean ring is commutative.

A ring $\{R, +, .\}$ is called Boolean if $r^2 = r$ for all $r \in R$. In this bicentenary year of Boole's birth we present four proofs that a Boolean ring is commutative. Our first proof is the standard one found in many textbooks.

Proof 1. For all $r \in R$ we have $r = r^2 = (-r)^2 = -r$, so r + r = 0. Next, for all x and y in R, $x + y = (x + y)^2 = x^2 + xy + yx + y^2$, so by cancellation in the group $\{R, +\}$, we have xy + yx = 0 = xy + xy, by the above. Again by cancellation we have xy = yx, as required. \square

Proof 2. As in Proof 1, xy + yx = 0, for all x and y in R. Since for all $r \in R$, 0.r = 0 = r.0 we have (xy + yx)x = x(xy + yx) or $xyx + y.x^2 = x^2.y + xyx$. Cancelling xyx and remembering that $x^2 = x$, we get xy = yx, as required.

Proof 3. Since for all r, $r^2 = r$ it follows that if $r^2 = 0$ then r = 0. Now for all x and y in R we have $(xy - xyx)^2 = xyxy + xyxxyx - xyxyx = 0$. So xy - xyx = 0 and xy = xyx. Then $(yx - xyx)^2 = yxyx + xyxxyx - yxxyx - xyxyx = 0$. So yx - xyx = 0 and yx = xyx. Thus xy = yx as required.

Proof 4. For $a, b \in R$ if ab = 0, then $ba = (ba)^2 = b(ab)a = 0$. Now, $0 = xy - xy = xy - x^2y = x(y - xy)$, so 0 = (y - xy)x = yx - xyx. Also, $0 = yx - yx = yx - yx^2 = (y - yx)x$, so 0 = x(y - yx) = xy - xyx. Thus xy = yx for all x and y in x. □

We note it is immediate in all four proofs that xy = yx = xyx = yxy, for all x and y.

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