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# Distributed Optimization Algorithm for Discrete-Time Heterogeneous Multi-Agent Systems With Nonuniform Stepsizes

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**ABSTRACT** This paper is devoted to the distributed optimization problem of heterogeneous multi-agent systems, where the communication topology is jointly strongly connected and the dynamics of each agent is the first-order or second-order integrator. A new distributed algorithm is first designed for each agent based on the local objective function and the local neighbors' information that each agent can access. By a model transformation, the original closed-loop system is converted into a time-varying system and the system matrix of which is a stochastic matrix at any time. Then, by the properties of the stochastic matrix, it is proven that all agents' position states can converge to the optimal solution of a team objective function provided the union communication topology is strongly connected. Finally, the simulation results are provided to verify the effectiveness of the distributed algorithm proposed in this paper.

**INDEX TERMS** Distributed optimization, multi-agent systems, heterogeneous, nonuniform stepsizes.

## I. INTRODUCTION

Recently, distributed control theory has gained the fast progress due to the traditional centralized control methods, such as [1], [2], have been unable to meet the demands of control engineering. As a basic problem of distributed control, especially, consensus problems with or without constraints have received huge interests from many fields, such as systems and control, computer science and mathematics, and many excellent results have been reported. For example, consensus problems of first-order and second-order multi-agent systems with communication time-delay were solved in [3]–[6]. Then, convex and nonconvex constrained consensus problems were considered for discrete-time and continuous-time multi-agent systems in [7]–[9], some effective distributed algorithms were designed and some consensus conditions were obtained.

As the further development of consensus, the distributed optimization problem has attracted wide attention due to its

potential applications in signal processing, sensor networks, machine learning, distributed estimation, energy management problems in smart grids, event-triggered control problems, fast convergence problems and so on [10]–[20]. The main task of distributed optimization is to design proper distributed algorithm based on local information that each agent can access to force all agents cooperatively find the optimal solution of a team objective function. For first-order multi-agent systems, the subgradient method was introduced to optimize a sum of convex functions [21], [22]. Then, these results were extended to continuous-time situations [23]–[25]. Meanwhile, the continuous-time zero-gradient-sum and Newton-Raphson algorithms were proposed in [26], [27] to assure that all agents can find the optimal solution of a team objective function. Furthermore, under the balance condition, the distributed continuous-time optimization problem was solved and the corresponding convergence rate was also analyzed in [28], [29]. Taken the identical or nonidentical constraint sets into account, the distributed optimization problems were solved by a sub-gradient projection algorithm for general convex objective

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functions [30], [31]. In addition, a distributed algorithm was introduced to solve the nonuniform unbounded constrained optimization problem in [32]. For second-order multi-agent systems, by introducing intermediate variables, the optimization problems were solved in [33], [34] when the objective functions are strongly convex. To reduce the cost of communication, some distributed algorithms with nonuniform gradient stepsizes were designed to minimize the general convex team objective function [35]–[37].

Most of the existing results on distributed optimization problem assumed that all agents' dynamics are the same-order integrators. In reality, the different agents might have different dynamics due to the restriction of real environment, the corresponding system is always said to be heterogeneous systems. The consensus problems of heterogeneous multi-agent systems with or without constraints were extensively studied in [38]–[42]. However, to the best of our knowledge, there is no works on the distributed optimization algorithm of heterogeneous multi-agent systems.

In this paper, we mainly study the distributed optimization problem of heterogeneous multi-agent system with jointly strongly connected communication topologies. The main contributions of this paper are as follows:

(1) In contrast to [21]–[24], [26]–[31], [33], [34], where some algorithms were proposed to solve the distributed optimization problems, but they assumed that the objective function is strictly convex or the gradient gains are constant, this paper deals with the situation of the general convex objective function and nonuniform gradient gains, which make our analysis be more complicated.

(2) In contrast to [25], [32], [35], [37], where the distributed optimization problems were studied for first-order or second-order multi-agent systems, this paper extends these results to heterogeneous multi-agent systems including first-order and second-order integrators simultaneously, which brings us more challenges and difficulties due to the inconsistency of each agent's dynamics.

(3) The proposed algorithms and analysis methods in [21]–[35], [37] are not applicable because we not only need to deal with first-order agents but also second-order agents.

(4) In contrast to [38]–[42], where only consensus problem was considered, this paper not only considers the consensus but also optimization index, which makes our closed-loop system be nonlinear essentially due to the existence of gradient terms and brings us more challenges in convergence analysis.

To overcome these difficulties, a novel distributed algorithm with nonuniform stepsizes (gradient gains) is first designed based on the local objective function and the local neighbors' information that each agent can access. By a coordination transformation, the original closed-loop system is changed into a new one, the system matrix of which is a stochastic matrix. Then, by the properties of stochastic matrix, we prove that all agents can reach an agreement on their position states and the team objective function is minimized at the same time.

**Notations:** In this paper,  $\mathbf{R}^r$  represents Euclidean space with dimension  $r$ .  $\|x\|$  represents the Euclidean norm of a vector  $x$ . For a matrix  $A$ ,  $A^T$  represents its transpose.  $I_r$  represents identical matrix with dimension  $r$ .  $\otimes$  represents kronecker product.  $\mathbf{1}$  represent a vector with all of its entries being one.

## II. PRELIMINARIES

Let  $\mathcal{G}(k) = (\mathcal{I}_{m+n}, \mathcal{E}(k))$  be a directed graph at time  $k$ , where  $\mathcal{I}_{m+n} = \{1, \dots, m+n\}$  is the set of nodes and  $\mathcal{E}(k)$  is the set of edges at time  $k$ .  $a_{ij}(k) > 0$  if and only if  $(j, i) \in \mathcal{E}(k)$  and  $a_{ii}(k) = 0$  for all  $i$ . Let  $N_i(k) = \{j \in V : (j, i) \in \mathcal{E}(k)\}$  be the neighbor set of node  $i$ . The Laplacian  $L(k)$  of  $\mathcal{G}(k)$  is defined as  $[L(k)]_{ii} = \sum_{j=1}^n a_{ij}(k)$  and  $[L(k)]_{ij} = -a_{ij}(k)$

for all  $i \neq j$ . The union graph of some graphs is a new graph whose node set is the same as each graph and edge set is the unions of the edge sets of these graphs. A series of edges  $(v_{i_{m-1}}, v_{i_m}), (v_{i_{m-2}}, v_{i_{m-1}}), \dots, (v_{i_1}, v_{i_2})$  is called a directed path from node  $v_{i_1}$  to node  $v_{i_m}$ . A directed graph is said to be strongly connected if there exists at least one directed path between any two different nodes.

## III. MAIN RESULTS

Consider a discrete-time heterogeneous multi-agent system consisting of  $m > 0$  second-order agents and  $n > 0$  first-order agents. Let  $\mathcal{G}(k)$  be the directed communication graph among all gents at time  $k$  and  $L(k)$  be its corresponding Laplacian. Let  $L_s(k)$  and  $L_f(k)$  be the Laplacian matrices of the graphs composed of second-order agents and first-order agents respectively. Thus, the Laplacian matrix of  $\mathcal{G}(k)$  can be partitioned as

$$L(k) = \begin{bmatrix} L_s(k) + D_{sf}(k) & -A_{sf}(k) \\ -A_{fs}(k) & L_f(k) + D_{fs}(k) \end{bmatrix},$$

where  $A_{sf}(k)$  and  $A_{fs}(k)$  are the corresponding sub-blocks of  $L(k)$ ,  $N_{i,f}(k)$  and  $N_{i,s}(k)$  are the  $i$ th agent's first-order and second-order neighbor sets respectively, clearly,  $N_i(k) = N_{i,s}(k) \cup N_{i,f}(k)$ ,

$$D_{sf}(k) = \text{diag}\{\sum_{j \in N_{1,f}(k)} a_{1j}(k), \dots, \sum_{j \in N_{m,f}(k)} a_{mj}(k)\},$$

$$D_{fs}(k) = \text{diag}\{\sum_{j \in N_{m+1,s}(k)} a_{m+1,j}(k), \dots, \sum_{j \in N_{m+n,s}(k)} a_{nj}(k)\}.$$

Suppose the dynamics of each second-order agent is

$$\begin{aligned} x_i(k+1) &= x_i(k) + v_i(k)T \\ v_i(k+1) &= v_i(k) + u_i(k)T, \quad i \in \mathcal{I}_m \end{aligned} \quad (1)$$

where  $x_i(k)$ ,  $v_i(k)$ ,  $u_i(k) \in \mathbf{R}^q$  are the position state, velocity state and control input of the  $i$ th agent at time  $k$  respectively;  $\mathcal{I}_m = \{1, 2, \dots, m\}$ ;  $T > 0$  is the sample time. Suppose the dynamics of each first-order agent is

$$x_i(k+1) = x_i(k) + u_i(k)T, \quad i \in \mathcal{I}_{m+n} - \mathcal{I}_m, \quad (2)$$

where  $x_i(k), u_i(k) \in \mathbf{R}^q$  are the position state and control input of the  $i$ th agent at time  $k$  respectively.

The objective of this paper is to design a distributed algorithm based on local information each agent can access to drive all agents reach an agreement and minimize the following team objective function

$$f(s) = \sum_{i=1}^{m+n} f_i(s), \quad s \in \mathbf{R}^q, \quad (3)$$

where  $f_i(s) : \mathbf{R}^q \rightarrow \mathbf{R}$  is the local differentiable convex objective function only accessed by the  $i$ th agent,  $i \in \mathcal{I}_{m+n}$ . To solve this problem, the following assumptions are necessary.

**Assumption 1 [25]:** Suppose  $X_i := \{x \in \mathbf{R}^q | \nabla f_i(x) = 0\}$ ,  $i \in \mathcal{I}_{m+n}$  are all nonempty and bounded.

Let  $X$  be the optimal solution set of the team objective function  $f(s)$ , then all  $X_i$  and  $X$  are nonempty bounded closed convex sets (see [25] for details).

**Assumption 2:** There exists  $\sigma \in (0, 1)$  such that  $a_{ij}(k)T \geq \sigma$  and  $a_{ij}(k) \leq 1$  iff  $a_{ij}(k) > 0$ ;  $\sum_{i=1}^{m+n} a_{ij}(k) = \sum_{j=1}^{m+n} a_{ij}(k)$  for all  $i, j \in \mathcal{I}_{m+n}$ .

**Assumption 3:** Suppose  $0 = k_0 < k_1 < \dots$  is an infinite sequence of switching times and the union graph of all graphs over every interval  $[k_m, k_{m+1})$  is strongly connected and  $k_{m+1} - k_m \leq B$  for all positive integer  $m$ , where  $B > 0$  is a positive constant.

**Remark 1:** Assumption 1 guarantees the distributed optimization problem is solvable, where the assumption is made only for the local objective functions and the global information (the team objective function) doesn't be used. Assumption 2 guarantees the interaction weights cannot be vanishing as time goes to infinity. Assumption 3 guarantees that each pair of agents can communicate directly or indirectly infinite many times and at least once in each  $B$  time interval.

In this paper, we design the following distributed algorithm with nonuniform stepsizes:

For  $i \in \mathcal{I}_m$ ,

$$\begin{aligned} u_i(k) = & -v_i(k) + \sum_{j \in N_{i,s}(k)} a_{ij}(k)[(x_j(k) - x_i(k)) \\ & + (v_j(k) - v_i(k))] + \sum_{j \in N_{i,f}(k)} a_{ij}(k)[(x_j(k) \\ & - x_i(k)) - v_i(k)] - d_i(k), \end{aligned} \quad (4)$$

where

$$d_i(k) = \begin{cases} 0, & \sqrt{p_i(k)} \leq \|\nabla f_i(w_i^{sy}(k))\| \\ \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}}, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} p_i(k+1) = & p_i(k) + \arctan(e^{\|x_i(k)+v_i(k)\|})T, \quad p_i(0) > 0, \text{ and,} \\ w_i^{sy}(k) = & x_i(k) + v_i(k) + \sum_{j \in N_{i,s}(k)} a_{ij}(k)[(x_j(k) - x_i(k)) \\ & + (v_j(k) - v_i(k))]T + \sum_{j \in N_{i,f}(k)} a_{ij}(k)[(x_j(k) - x_i(k)) - v_i(k)]T. \end{aligned}$$

For  $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$ ,

$$u_i(k) = \sum_{j \in N_{i,s}(k)} a_{ij}(k)[(x_j(k) - x_i(k)) + v_j(k)]$$

$$+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)[(x_j(k) - x_i(k))] - d_i(k), \quad (5)$$

where

$$d_i(k) = \begin{cases} 0, & \sqrt{p_i(k)} \leq \|\nabla f_i(w_i^{fx}(k))\|^2 \\ \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}}, & \text{otherwise,} \end{cases}$$

$$\begin{aligned} p_i(k+1) = & p_i(k) + \arctan(e^{\|x_i(k)\|})T, \quad p_i(0) > 0, \text{ and,} \\ w_i^{fx}(k) = & x_i(k) + \sum_{j \in N_{i,s}(k)} a_{ij}(k)[(x_j(k) - x_i(k)) + v_j(k)]T + \\ & \sum_{j \in N_{i,f}(k)} a_{ij}(k)[(x_j(k) - x_i(k))]T. \end{aligned}$$

**Remark 2:** In the above algorithms, only the local information (the state information of its neighbors) is used by each agent, i.e., the corresponding data set includes all the state information of its neighbors.

Let  $y_i(k) = x_i(k) + v_i(k)$ ,  $i \in \mathcal{I}_m$ , then for  $i \in \mathcal{I}_m$  and  $j \in \mathcal{I}_{m+n} - \mathcal{I}_m$ ,

$$\begin{aligned} x_i(k+1) = & (1-T)x_i(k) + Ty_i(k) \\ y_i(k+1) = & w_i^{sy}(k) - d_i(k)T \\ x_j(k+1) = & w_j^{fx}(k) - d_j(k)T. \end{aligned} \quad (6)$$

Define  $Z(k) = [x^s(k)^T, y^s(k)^T, x^f(k)^T]^T$ ,  $x^s(k) = [x_1(k)^T, \dots, x_m(k)^T]^T$ ,  $y^s(k) = [y_1(k)^T, \dots, y_m(k)^T]^T$ ,  $x^f(k) = [x_{m+1}(k)^T, \dots, x_{m+n}(k)^T]^T$ ,  $D(k) = [0, D^s(k)^T, D^f(k)^T]^T$ ,  $D^s(k) = [d_1(k)^T, \dots, d_m(k)^T]^T$ ,  $D^f(k) = [d_{m+1}(k)^T, \dots, d_{m+n}(k)^T]^T$ . Then the closed-loop system (6) can be changed into the following form:

$$Z(k+1) = [\Phi(k) \otimes I_q]Z(k) + D(k), \quad (7)$$

where

$$\Phi(k) = \begin{bmatrix} (1-T)I_m & TI_m & 0 \\ 0 & \phi_1(k) & A_{sf}(k)T \\ 0 & A_{fs}(k)T & \phi_2(k) \end{bmatrix},$$

and  $\phi_1(k) = I_m - L_s(k)T - D_{sf}(k)T$  and  $\phi_2(k) = I_n - L_f(k)T - D_{fs}(k)T$ .

**Lemma 1:** Under Assumption 2, if  $T < \frac{1-\sigma}{n+m}$ , then  $\Phi(k)$  is a stochastic matrix and  $\Phi_1(k)$  is a double stochastic matrix for all  $k$  with  $[\Phi_1(k)]_{ij} \geq \sigma$  for all nonzero entry  $[\Phi_1(k)]_{ij}$ ,  $i, j \in \mathcal{I}_{m+n}$ , where  $\Phi_1(k) = \begin{bmatrix} \phi_1(k) & A_{sf}(k)T \\ A_{fs}(k)T & \phi_2(k) \end{bmatrix}$ , i.e.,  $\Phi(k)\mathbf{1} = \mathbf{1}$ ,  $\Phi_1(k)\mathbf{1} = \mathbf{1}$  and  $\Phi_1^T(k)\mathbf{1} = \mathbf{1}$ .

**Proof:** From the definitions of  $\Phi(k)$  and  $\Phi_1(k)$ , it is easy to verify the conclusion by simple calculation.

**Theorem 1:** Under Assumptions 1, 2 and 3. If  $T < \frac{1-\sigma}{m+n}$ , then all agents of systems (1) and (2) can reach an agreement on their position states and the team objective function is minimized under Algorithms (4) and (5).

**Proof:** First, we will prove that  $x_i(k)$  and  $y_j(k)$  are bounded for all  $k \geq 0$ ,  $i \in \mathcal{I}_{m+n}$  and  $j \in \mathcal{I}_m$ . It follows from  $\frac{\pi}{4} \leq \arctan(e^{\|y_i(k)\|}) \leq \frac{\pi}{2}$  and  $\frac{\pi}{4} \leq \arctan(e^{\|x_i(k)\|}) \leq \frac{\pi}{2}$  that there exists  $K_0 > 0$  such that  $\frac{k\pi T}{4} \leq p_i(k) \leq k\pi T$  for any  $k \geq K_0$ . Note that  $X$  and  $X_i$  are all nonempty and bounded, we can choose a constant  $\delta_1 > 0$  such that  $X \subset Y_1, X_i \subset Y_1$  for each  $i$ , where  $Y_1 = \{s \in \mathbf{R}^q | \|s - z\| \leq \delta_1\}$  for some  $z \in X$ ,  $f_i(w_i^{sy}(k)) - f_i(z) \geq 2[f_i(z) - f_i(z_i)] + T$

for all  $z \in X$ ,  $z_i \in X_i$  and  $w_i^{sy}(k) \notin Y_1$ ,  $i \in \mathcal{I}_m$ , and  $f_i(w_i^{fx}(k)) - f_i(z) \geq 2[f_i(z) - f_i(z_i)] + T$  for all  $z \in X$ ,  $z_i \in X_i$  and  $w_i^{fx}(k) \notin Y_1$ ,  $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$ . Choose Lyapunov function  $V_1(k) = \max_{i \in \mathcal{I}_{m+n}, j \in \mathcal{I}_m} \{\|x_i(k) - z\|^2, \|y_j(k) - z\|^2\}$ .

For  $i \in \mathcal{I}_m$ , by the convexity, we have

$$\begin{aligned} \|x_i(k+1) - z\|^2 &= \|(1-T)x_i(k) + Ty_i(k) - z\|^2 \\ &\leq (1-T)\|x_i(k) - z\|^2 + T\|y_i(k) - z\|^2 \leq V_1(k). \end{aligned}$$

When  $d_i(k) = 0$ , we have

$$\begin{aligned} \|y_i(k+1) - z\|^2 &= \|w_i^{sy}(k) - d_i(k)T - z\|^2 = \|y_i(k) \\ &+ \sum_{j \in N_{i,s}(k)} a_{ij}(k)[y_j(k) - y_i(k)]T \\ &+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)[x_j(k) - y_i(k)]T - z\|^2 \\ &\leq (1 - \sum_{j \in N_i(k)} a_{ij}(k)T)\|y_i(k) - z\|^2 \\ &+ \sum_{j \in N_{i,s}(k)} a_{ij}(k)T\|y_j(k) - z\|^2 \\ &+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)T\|x_j(k) - z\|^2 \leq V_1(k). \end{aligned}$$

When  $d_i(k) \neq 0$ ,  $\sqrt{p_i(k)} > \|\nabla f_i(w_i^{sy}(k))\|^2$ , we have

$$\begin{aligned} \|y_i(k+1) - z\|^2 &= \|w_i^{sy}(k) - d_i(k)T - z\|^2 = \|w_i^{sy}(k) - z\|^2 \\ &+ \|d_i(k)T\|^2 - 2(w_i^{sy}(k) - z)^T \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\leq (1 - \sum_{j \in N_i(k)} a_{ij}(k)T)\|y_i(k) - z\|^2 \\ &+ \sum_{j \in N_{i,s}(k)} a_{ij}(k)T\|y_j(k) - z\|^2 \\ &+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)T\|x_j(k) - z\|^2 + \frac{T^2}{\sqrt{p_i(k)}} \\ &- 2[f_i(w_i^{sy}(k)) - f_i(z)] \frac{T}{\sqrt{p_i(k)}} \\ &\leq (1 - \sum_{j \in N_i(k)} a_{ij}(k)T)\|y_i(k) - z\|^2 \\ &+ \sum_{j \in N_{i,s}(k)} a_{ij}(k)T\|y_j(k) - z\|^2 \\ &+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)T\|x_j(k) - z\|^2 + \frac{2T^2}{\sqrt{k\pi T}} \\ &+ \frac{4T}{\sqrt{k\pi T}}[f_i(z) - f_i(z_i)] \\ &- 2[f_i(w_i^{sy}(k)) - f_i(z_i)] \frac{T}{\sqrt{k\pi T}}, \end{aligned}$$

where  $z_i \in X_i$ .

If  $w_i^{sy}(k) \in Y_1$ , then

$$\|y_i(k+1) - z\|^2 \leq \delta_1^2 + \frac{2T^2}{\sqrt{k\pi T}} + \frac{4T}{k\pi T}[f_i(z) - f_i(z_i)],$$

which is bounded.

If  $w_i^{sy}(k) \notin Y_1$ , then  $f_i(w_i^{sy}(k)) - f_i(z) \geq 2[f_i(z) - f_i(z_i)] + T$ . Thus,  $\|y_i(k+1) - z\|^2 \leq V_1(k)$ .

For  $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$ , when  $d_i(k) = 0$ , we have

$$\begin{aligned} \|x_i(k+1) - z\|^2 &= \|w_i^{fx}(k) - z\|^2 \\ &\leq (1 - \sum_{j \in N_i(k)} a_{ij}(k)T)\|x_i(k) - z\|^2 \\ &+ \sum_{j \in N_{i,s}(k)} a_{ij}(k)T\|y_j(k) - z\|^2 \\ &+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)T\|x_j(k) - z\|^2 \\ &\leq V_1(k). \end{aligned}$$

When  $d_i(k) \neq 0$ ,  $\sqrt{p_i(k)} > \|\nabla f_i(w_i^{fx}(k))\|^2$ , and

$$\begin{aligned} \|x_i(k+1) - z\|^2 &= \|w_i^{fx}(k) - d_i(k)T - z\|^2 \\ &\leq \|w_i^{fx}(k) - z\|^2 + \|d_i(k)T\|^2 \\ &- 2(w_i^{fx}(k) - z)^T \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T \\ &\leq (1 - \sum_{j \in N_i(k)} a_{ij}(k)T)\|x_i(k) - z\|^2 \\ &+ \sum_{j \in N_{i,s}(k)} a_{ij}(k)T\|y_j(k) - z\|^2 \\ &+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)T\|x_j(k) - z\|^2 + \frac{T^2}{\sqrt{p_i(k)}} \\ &- 2[f_i(w_i^{fx}(k)) - f_i(z)] \frac{T}{\sqrt{p_i(k)}} \\ &\leq (1 - \sum_{j \in N_i(k)} a_{ij}(k)T)\|x_i(k) - z\|^2 \\ &+ \sum_{j \in N_{i,s}(k)} a_{ij}(k)T\|y_j(k) - z\|^2 \\ &+ \sum_{j \in N_{i,f}(k)} a_{ij}(k)T\|x_j(k) - z\|^2 + \frac{2T^2}{\sqrt{k\pi T}} \\ &+ \frac{4T}{\sqrt{k\pi T}}[f_i(z) - f_i(z_i)] \\ &- 2[f_i(w_i^{fx}(k)) - f_i(z_i)] \frac{T}{\sqrt{k\pi T}}, \end{aligned}$$

where  $z_i \in X_i$ .

If  $w_i^{fx}(k) \in Y_1$ , then

$$\|x_i(k+1) - z\|^2 \leq \delta_1^2 + \frac{2T^2}{\sqrt{k\pi T}} + \frac{4T}{k\pi T}[f_i(z) - f_i(z_i)],$$

which is bounded.

If  $w_i^{fx}(k) \notin Y_1$ , then  $f_i(w_i^{fx}(k)) - f_i(z) \geq 2[f_i(z) - f_i(z_i)] + T$ . Thus,  $\|x_i(k+1) - z\|^2 \leq V_1(k)$ .

From the preceding analysis, we can conclude that  $V_1(k)$  is bounded. Hence,  $x_i(k)$  and  $y_j(k)$  are bounded for all  $i \in \mathcal{I}_{m+n}$  and  $j \in \mathcal{I}_m$ . Note that  $f_i(s)$ ,  $i \in \mathcal{I}_{m+n}$  are differentiable, we get that  $\|\nabla f_i(w_i^{sy}(k))\|$  and  $\|\nabla f_j(w_j^{fx}(k))\|$  are bounded for all  $i \in \mathcal{I}_m$  and  $j \in \mathcal{I}_{m+n} - \mathcal{I}_m$ . Together with  $\lim_{t \rightarrow \infty} p_i(k) = \infty$ , there exists  $K_1 > K_0$  such that  $d_i(k) = \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}}$  and  $d_j(k) = \frac{\nabla f_j(w_j^{fx}(k))}{\sqrt{p_i(k)}}$  for all  $i \in \mathcal{I}_m$ ,  $j \in \mathcal{I}_{m+n} - \mathcal{I}_m$  and  $k \geq K_1$ .

Next, let us analyze the stability of the closed-loop system. Define  $z^*(k) = \frac{1}{m+n}[\sum_{i=1}^m y_i(k) + \sum_{i=m+1}^{m+n} x_i(k)]$ ,  $k \geq K_1$ , we will prove that  $\lim_{k \rightarrow \infty} \|Z(k) - \mathbf{1}_{2m+n} \otimes z^*(k)\| = 0$ .

From (7), we have

$$\begin{aligned} z^*(k+1) &= \frac{1}{m+n} [\sum_{i=1}^m y_i(k+1) + \sum_{i=m+1}^{m+n} x_i(k+1)] \\ &= \frac{1}{m+n} [\sum_{i=1}^m y_i(k) + \sum_{i=m+1}^{m+n} x_i(k) \\ &\quad - \sum_{i=1}^{m+n} d_i(k)T] \\ &= z^*(k) - \frac{1}{m+n} [\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T]. \end{aligned}$$

Take the following Lyapunov function  $V_2(k) = \sum_{i=1}^m \|y_i(k) - z^*(k)\|^2 + \sum_{i=m+1}^{m+n} \|x_i(k) - z^*(k)\|^2$ . Then

$$\begin{aligned} V_2(k+1) &= \sum_{i=1}^m \|y_i(k+1) - z^*(k+1)\|^2 \\ &\quad + \sum_{i=m+1}^{m+n} \|x_i(k+1) - z^*(k+1)\|^2 \\ &= \sum_{i=1}^m \|w_i^{sy}(k) - d_i(k)T - z^*(k)\|^2 \\ &\quad + \frac{1}{m+n} [\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T]^2 \\ &\quad + \sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - d_i(k)T - z^*(k)\|^2 \\ &\quad + \frac{1}{m+n} [\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T]^2 \\ &= \sum_{i=1}^m \|w_i^{sy}(k) - d_i(k)T - z^*(k)\|^2 \\ &\quad + \frac{m}{(m+n)^2} \|\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T\|^2 \\ &\quad + 2 \sum_{i=1}^m (w_i^{sy}(k) - d_i(k)T - z^*(k))^T \frac{1}{m+n} \\ &\quad \times [\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T] \\ &\quad + \sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - d_i(k)T - z^*(k)\|^2 \\ &\quad + \frac{n}{(m+n)^2} \|\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T\|^2 \\ &\quad + 2 \sum_{i=m+1}^{m+n} (w_i^{fx}(k) - d_i(k)T - z^*(k))^T \frac{1}{m+n} \\ &\quad \times [\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T] \end{aligned}$$

$$\begin{aligned} &= \sum_{i=1}^m \|w_i^{sy}(k) - d_i(k)T - z^*(k)\|^2 \\ &\quad + \sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - d_i(k)T - z^*(k)\|^2 \\ &\quad - \frac{1}{m+n} \|\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T\|^2. \end{aligned}$$

Besides,

$$\begin{aligned} &\sum_{i=1}^m \|w_i^{sy}(k) - d_i(k)T - z^*(k)\|^2 \\ &= \sum_{i=1}^m \|w_i^{sy}(k) - z^*(k)\|^2 + \sum_{i=1}^m \|d_i(k)T\|^2 \\ &\quad - 2 \sum_{i=1}^m (w_i^{sy}(k) - z^*(k))^T d_i(k)T \\ &\leq \sum_{i=1}^m \|w_i^{sy}(k) - z^*(k)\|^2 + \sum_{i=1}^m \|d_i(k)T\|^2 \\ &\quad - 2 \sum_{i=1}^m [f_i(w_i^{sy}(k)) - f_i(z^*(k))] \frac{T}{\sqrt{p_i(k)}}, \\ &\sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - d_i(k)T - z^*(k)\|^2 \\ &= \sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - z^*(k)\|^2 + \sum_{i=m+1}^{m+n} \|d_i(k)T\|^2 \\ &\quad - 2 \sum_{i=m+1}^{m+n} (w_i^{fx}(k) - z^*(k))^T d_i(k)T \\ &\leq \sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - z^*(k)\|^2 + \sum_{i=m+1}^{m+n} \|d_i(k)T\|^2 \\ &\quad - 2 \sum_{i=m+1}^{m+n} [f_i(w_i^{fx}(k)) - f_i(z^*(k))] \frac{T}{\sqrt{p_i(k)}}. \end{aligned}$$

Since  $\frac{k\pi T}{4} \leq p_i(k) \leq k\pi T$ ,  $\nabla f_i(w_i^{sy}(k))$  and  $f_i(w_i^{fx}(k))$  are bounded for all  $k \geq K_1$ , there must exist  $c_1, c_2 > 0$  such that  $V_2(k+1) \leq \sum_{i=1}^m \|w_i^{sy}(k) - z^*(k)\|^2 + \sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - z^*(k)\|^2 + \frac{c_1}{k} + \frac{c_2}{\sqrt{k}}$ .

Note that

$$\begin{aligned} &\sum_{i=1}^m \|w_i^{sy}(k) - z^*(k)\|^2 + \sum_{i=m+1}^{m+n} \|w_i^{fx}(k) - z^*(k)\|^2 \\ &= \|\Phi_1(k) \otimes I_p [y^s(k)^T, x^f(k)^T]^T - \mathbf{1}z^*(k)\|^2 \\ &= \|\Phi_1(k) \otimes I_p ([y^s(k)^T, x^f(k)^T]^T - \mathbf{1}z^*(k))\|^2, \end{aligned}$$

we have

$$V_2(k+1) - V_2(k) \leq -\phi(k) + \frac{c_1}{k} + \frac{c_2}{\sqrt{k}},$$

where  $\phi(k) = ([y^s(k)^T, x^f(k)^T]^T - \mathbf{1}z^*(k))^T [I - \Phi_1(k)^T \Phi_1(k) \otimes I_p] ([y^s(k)^T, x^f(k)^T]^T - \mathbf{1}z^*(k)) \geq 0$ .

Summing the above inequality from  $k_m$  to  $k_{m+1} - 1$ , we have

$$V_2(k_{m+1} - 1) - V_2(k_m) \leq -\sum_{k=k_m}^{k_{m+1}-1} \phi(k) + B(\frac{c_1}{k} + \frac{c_2}{\sqrt{k}}).$$

Since  $\lim_{t \rightarrow \infty} B(\frac{c_1}{k} + \frac{c_2}{\sqrt{k}}) = 0$ , for any  $\epsilon_1 > 0$ , there exists  $K_2 > K_1$  such that  $B(\frac{c_1}{k} + \frac{c_2}{\sqrt{k}}) < \epsilon_1$  for all  $k > K_2$ .

If there exists  $k \in [k_m, k_{m+1})$ ,  $k_m > K_2$ , such that  $\phi(k) > 2\epsilon_1$ , then  $V_2(k_{m+1} - 1) - V_2(k_m) < -\epsilon_1$ . If  $\phi(k) \leq 2\epsilon_1$  for all  $k \in [k_m, k_{m+1})$ ,  $k_m > K_2$ , we need to estimate  $V_2(k)$ . From



Assumption 2, each nonzero entry of  $\Phi_1(k)$  is no smaller than  $\sigma$ , hence,

$$\begin{aligned} & ([y^s(k)^T, x^f(k)^T]^T - 1 z^*(k))^T [-I + \Phi_1(k)^T \Phi_1(k) \\ & \quad \otimes I_p] ([y^s(k)^T, x^f(k)^T]^T - 1 z^*(k)) \\ & \leq -[\sum_{i=1}^m \left( \sum_{j \in N_{i,s}(k)} \sigma^2 T^2 \|y_j(k) - y_i(k)\|^2 \right. \\ & \quad \left. + \sum_{j \in N_{i,f}(k)} \sigma^2 T^2 \|x_j(k) - y_i(k)\|^2 \right) \\ & \quad + \sum_{i=m+1}^{m+n} \left( \sum_{j \in N_{i,s}(k)} \sigma^2 T^2 \|y_j(k) - x_i(k)\|^2 \right. \\ & \quad \left. + \sum_{j \in N_{i,f}(k)} \sigma^2 T^2 \|x_j(k) - x_i(k)\|^2 \right)]. \end{aligned}$$

By similar calculation as [32], it is easy to see that there exists  $c_3 > 0$  such that  $\|y_i(k) - z^*(k)\| \leq c_3(\sqrt{\epsilon_1} + \epsilon_1)$  and  $\|x_j(k) - z^*(k)\| \leq c_3(\sqrt{\epsilon_1} + \epsilon_1)$  for all  $i \in \mathcal{I}_m, j \in \mathcal{I}_{m+n} - \mathcal{I}_m$ . Hence,  $V_2(k) \leq c_3(\sqrt{\epsilon_1} + \epsilon_1)$  for all  $k \in [k_m, k_{m+1})$ ,  $k_m > K_2$  when  $\phi(k) \leq 2\epsilon_1$ .

Summarizing the preceding two cases, for any  $\epsilon_2 > 0$ , there exists  $K_3 > K_2$  such that  $V_2(k_m) < \epsilon_2$  for all  $k_m > K_3$ , which implies that  $\lim_{m \rightarrow \infty} V_2(k_m) < \epsilon_2$ . According to the arbitrariness of  $\epsilon_2$ , we can conclude that  $\lim_{m \rightarrow \infty} V_2(k_m) = 0$ . Note that  $V_2(k+1) - V_2(k) \leq -\phi(k) + \frac{c_1}{k} + \frac{c_2}{\sqrt{k}} \leq \frac{c_1}{k} + \frac{c_2}{\sqrt{k}}$  for all  $k > K_2$ , we have  $\lim_{k \rightarrow \infty} V_2(k)$  exists by Cauchy convergence theorem. Therefore,

$$\lim_{k \rightarrow \infty} V_2(k) = \lim_{m \rightarrow \infty} V_2(k_m) = 0,$$

i.e.,

$$\begin{aligned} \lim_{k \rightarrow \infty} \|y_i(k) - z^*(k)\| &= 0, \\ \lim_{k \rightarrow \infty} \|x_j(k) - z^*(k)\| &= 0, \end{aligned}$$

for all  $i \in \mathcal{I}_m, j \in \mathcal{I}_{m+n} - \mathcal{I}_m$ .

Let  $l_i(k) = \sum_{j \in N_{i,s}(k)} a_{ij}(k)[y_j(k) - y_i(k)]T + \sum_{j \in N_{i,f}(k)} [x_j(k) - y_i(k)]T - d_i(k)T$  for  $i \in \mathcal{I}_m$ . Then  $\lim_{k \rightarrow \infty} l_i(k) = 0$ ,  $|l_i(k)| < M$  and  $v_i(k+1) = (1-T)v_i(k) + l_i(k)$ , where  $M > 0$  is a constant. For any  $\epsilon > 0$ , there exists  $K > 0$  such that  $(1-T)^{K+1} < \frac{T}{M}\epsilon$ . Hence,

$$\begin{aligned} & \|v_i(k+1)\| \\ &= \|(1-T)^{k+1}v_i(0) + \sum_{s=0}^k (1-T)^s l_i(k-s)\| \\ &\leq (1-T)^{k+1}\|v_i(0)\| + \sum_{s=0}^k (1-T)^s \|l_i(k-s)\| \\ &\leq (1-T)^{k+1}\|v_i(0)\| + \sum_{s=0}^K (1-T)^s \|l_i(k-s)\| \\ &\quad + M \sum_{s=K+1}^k (1-T)^s \\ &= (1-T)^{k+1}\|v_i(0)\| + \sum_{s=0}^K (1-T)^s \|l_i(k-s)\| \\ &\quad + \frac{M}{T} (1-T)^{K+1} (1 - (1-T)^{k-K}). \end{aligned}$$

Hence,  $\lim_{k \rightarrow \infty} \|v_i(k+1)\| \leq \frac{M}{T} (1-T)^{K+1} < \epsilon$ . It follows from the arbitrariness of  $\epsilon$  that  $\lim_{k \rightarrow \infty} \|v_i(k)\| = 0$ . Therefore,  $\lim_{k \rightarrow \infty} \|x_i(k) - z^*(k)\| = 0$  for all  $i \in \mathcal{I}_{m+n}$ , i.e., all agents reach an agreement on their position state.

At last, let us prove all agents' states could converge to the optimal solution of the team objective function. Define  $V_3(k) = \|z^*(k) - P_X(z^*(k))\|^2$ ,  $k > K_2$ . Then

$$\begin{aligned} V_3(k+1) &= \|z^*(k+1) - P_X(z^*(k+1))\|^2 \\ &= \|z^*(k+1) - P_X(z^*(k)) + P_X(z^*(k)) \\ &\quad - P_X(z^*(k+1))\|^2 \\ &= \|z^*(k+1) - P_X(z^*(k))\|^2 + \|P_X(z^*(k)) \\ &\quad - P_X(z^*(k+1))\|^2 + 2[z^*(k+1) - P_X(z^*(k))]^T \\ &\quad \times [P_X(z^*(k)) - P_X(z^*(k+1))]. \end{aligned}$$

Note that  $[z^*(k+1) - P_X(z^*(k))]^T [P_X(z^*(k)) - P_X(z^*(k+1))] \leq 0$  and  $\|P_X(z^*(k)) - P_X(z^*(k+1))\| \leq \|z^*(k) - z^*(k+1)\|$ , we have

$$\begin{aligned} V_3(k+1) &\leq \|z^*(k+1) - P_X(z^*(k))\|^2 + \|z^*(k) - z^*(k+1)\|^2 \\ &= \|z^*(k+1) - z^*(k) + z^*(k) - P_X(z^*(k))\|^2 \\ &\quad + \|z^*(k) - z^*(k+1)\|^2 \\ &= \|z^*(k) - P_X(z^*(k))\|^2 + 2\|z^*(k) - z^*(k+1)\|^2 \\ &\quad + 2(z^*(k) - P_X(z^*(k)))^T (z^*(k+1) - z^*(k)). \end{aligned}$$

Since  $y_i(k), x_j(k)$  are bounded for all  $i \in \mathcal{I}_m, j \in \mathcal{I}_{m+n} - \mathcal{I}_m$ , there must exist  $\gamma > 0$  such that  $\|z^*(k+1) - z^*(k)\|^2 = \|\frac{1}{m+n} [\sum_{i=1}^m \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} + \sum_{i=m+1}^{m+n} \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}}]\|^2 \leq \frac{\gamma}{k}$ . Therefore,

$$\begin{aligned} V_3(k+1) &\leq \|z^*(k) - P_X(z^*(k))\|^2 + \frac{2\gamma}{k} \\ &\quad - \frac{2}{m+n} \sum_{i=1}^m [z^*(k) - P_X(z^*(k))]^T \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad - \frac{2}{m+n} \sum_{i=m+1}^{m+n} [z^*(k) - P_X(z^*(k))]^T \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T \\ &= V_3(k) + \frac{2\gamma}{k} \\ &\quad - \frac{2}{m+n} \sum_{i=1}^m [z^*(k) - w_i^{sy}(k)]^T \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad - \frac{2}{m+n} \sum_{i=m+1}^{m+n} [z^*(k) - w_i^{fx}(k)]^T \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T \\ &\quad - \frac{2}{m+n} \sum_{i=1}^m [w_i^{sy}(k) - P_X(z^*(k))]^T \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T \\ &\quad - \frac{2}{m+n} \sum_{i=m+1}^{m+n} [w_i^{fx}(k) - P_X(z^*(k))]^T \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_i(k)}} T \end{aligned}$$

From the definitions of  $w_i^{sy}(k)$  and  $w_j^{fx}(k)$ , it is easy to see that  $\lim_{k \rightarrow \infty} \|z^*(k) - w_i^{sy}(k)\| = 0$  and  $\lim_{k \rightarrow \infty} \|z^*(k) - w_j^{fx}(k)\| = 0$  for all  $i \in \mathcal{I}_m$  and  $j \in \mathcal{I}_{m+n} - \mathcal{I}_m$ . Note that  $\nabla f_i(w_i^{sy}(k)), \nabla f_j(w_j^{fx}(k))$  are bounded and  $\frac{k\pi T}{4} \leq p_i(k) \leq k\pi T$ , we can conclude that for any  $\epsilon_3 > 0$ , there exist  $\gamma_1 > 0$  and  $K_3 > K_2$  such that  $-\frac{2}{m+n} \sum_{i=1}^m [z^*(k) - w_i^{sy}(k)]^T \frac{\nabla f_i(w_i^{sy}(k))}{\sqrt{p_i(k)}} T -$

$$\frac{2}{m+n} \sum_{i=m+1}^{m+n} [z^*(k) - w_i^{fx}(k)]^T \frac{\nabla f_i(w_i^{fx}(k))}{\sqrt{p_1(k)}} T - \frac{2}{m+n} \sum_{i=1}^m [f_i(w_i^{sy}(k)) - f_i(z^*(k))] \frac{T}{\sqrt{p_1(k)}} - \frac{2}{m+n} \sum_{i=m+1}^{m+n} [f_i(w_i^{fx}(k)) - f_i(z^*(k))] \frac{T}{\sqrt{p_1(k)}} \leq \frac{\gamma_1 \epsilon_3}{\sqrt{k}} \text{ for all } k > K_3. \text{ Hence, we have}$$

$$\begin{aligned} V_3(k+1) - V_3(k) &\leq \frac{2\gamma}{k} + \frac{\gamma_1 \epsilon_3}{\sqrt{k}} - \frac{2T}{(m+n)\sqrt{p_1(k)}} \\ &\quad \times \sum_{i=1}^{m+n} [f_i(z^*(k)) - f_i(P_X(z^*(k)))] \\ &\quad - \frac{2T}{(m+n)\sqrt{p_1(k)}} \sum_{i=1}^{m+n} \left( \frac{\sqrt{p_1(k)}}{\sqrt{p_i(k)}} - 1 \right) \\ &\quad \times [f_i(z^*(k)) - f_i(P_X(z^*(k)))]. \end{aligned}$$

For  $i \in \mathcal{I}_m$ , let  $\pi_i(k) = p_i(0) - p_1(0) + \sum_{s=0}^k [\arctan(e^{\|y_i(s)\|}) - \arctan(e^{\|y_1(s)\|})]T$ . Then  $p_i(k) = p_1(k)[1 + \frac{\pi_i(k)}{p_1(k)}]$ . Since  $\lim_{k \rightarrow \infty} \|y_i(k) - y_1(k)\| = 0$ , for any  $\epsilon' > 0$ , there exists  $K' > K_0$  such that  $|\arctan(e^{\|y_i(s)\|}) - \arctan(e^{\|y_1(s)\|})| < \epsilon'$  for all  $k > K'$ . Note that

$$\frac{\sum_{s=K_0+1}^k [\arctan(e^{\|y_i(s)\|}) - \arctan(e^{\|y_1(s)\|})]T}{p_1(k)} \leq \frac{4\epsilon'}{\pi T},$$

we can get  $\lim_{k \rightarrow \infty} \frac{\pi_i(k)}{p_1(k)} \leq \frac{4\epsilon'}{\pi T}$ . According to the arbitrariness of  $\epsilon'$ , we have  $\lim_{k \rightarrow \infty} \frac{\pi_i(k)}{p_1(k)} = 0$  and  $\lim_{k \rightarrow \infty} \frac{p_i(k)}{p_1(k)} = 1$ . Similarly, we can prove that  $\lim_{k \rightarrow \infty} \frac{p_i(k)}{p_1(k)} = 1$  for all  $i \in \mathcal{I}_{m+n} - \mathcal{I}_m$ . Hence, there exists  $K_4 > K_3$  such that  $-\frac{2T}{(m+n)\sqrt{p_1(k)}} \sum_{i=1}^{m+n} \left( \frac{\sqrt{p_1(k)}}{\sqrt{p_i(k)}} - 1 \right) [f_i(z^*(k)) - f_i(P_X(z^*(k)))] \leq \frac{\epsilon_3}{\sqrt{k}}$  and  $\frac{2\gamma}{k} < \frac{\epsilon_3}{\sqrt{k}}$  for all  $k > K_4$ . Thus, for  $k > K_4$ ,

$$\begin{aligned} V_3(k+1) - V_3(k) &\leq \frac{2\epsilon_3}{\sqrt{k}} + \frac{\gamma_1 \epsilon_3}{\sqrt{k}} - \frac{2T}{m+n} \frac{1}{\sqrt{p_1(k)}} \\ &\quad \times [\sum_{i=1}^{m+n} f_i(z^*(k)) - \sum_{i=1}^{m+n} f_i(P_X(z^*(k)))] \\ &\leq -\frac{2\sqrt{T}}{(m+n)\sqrt{\pi k}} [\sum_{i=1}^{m+n} f_i(z^*(k)) \\ &\quad - \sum_{i=1}^{m+n} f_i(P_X(z^*(k))) - \gamma_2 \epsilon_3], \end{aligned}$$

where  $\gamma_2 = \frac{(m+n)\sqrt{\pi}}{2\sqrt{T}}(2 + \gamma_1) > 0$ .

Since  $f_i(s)$  are differentiable convex function, we can choose  $c_1 > 0$  and  $C_1 = \{s | \|s - P_X(s)\| \leq c_1\}$  such that  $\sum_{i=1}^{m+n} f_i(s_1) - \sum_{i=1}^{m+n} f_i(s_2) > \gamma_2 \epsilon_3 + 2\epsilon_4$  for all  $s_1 \notin C_1$  and  $s_2 \in C_1$ , where  $\epsilon_4 > 0$ . Set  $c_2 = c_1 + 2\epsilon_4$  and  $C_2 = \{s | \|s - P_X(s)\| \leq c_2\}$ . Note that  $\nabla f_i(w_i^{sy}(k))$  and  $\nabla f_j(w_j^{fx}(k))$  are bounded, there exists  $K_5 > K_4$  such that  $\|z^*(k+1) - z^*(k)\| < \epsilon_4$  for all  $k > K_5$ . For  $k > K_5$ , if  $z^*(k) \in C_1$ , then  $z^*(k+1) \in C_2$ . If  $z^*(k) \notin C_1$  and  $z^*(k) \in C_2$ , then  $V_3(k+1) - V_3(k) < 0$ , which implies  $z^*(k+1) \in C_2$ . If  $z^*(k) \notin C_2$  for  $k > K_5$ , then  $V_3(k+1) - V_3(k) < -\frac{2\sqrt{T}\epsilon_4}{(m+n)\sqrt{\pi k}}$ . So there exists  $K_6 > K_5$  such that  $z^*(k) \in C_2$ , i.e.,  $\sum_{i=1}^{m+n} [f_i(z^*(k)) - f_i(P_X(z^*(k)))] < c_1 + 2\epsilon_4$  for all  $k > K_6$ . Letting  $\epsilon_4 \rightarrow 0$ , we can get  $\lim_{k \rightarrow \infty} z^*(k) \in C_1$ , which suggests  $\lim_{k \rightarrow \infty} \sum_{i=1}^{m+n} [f_i(z^*(k)) - f_i(P_X(z^*(k)))] \leq \gamma_2 \epsilon_3$ . By the arbitrariness of  $\epsilon_3$ , we conclude that  $\lim_{k \rightarrow \infty} \sum_{i=1}^{m+n} [f_i(z^*(k)) - f_i(P_X(z^*(k)))] = 0$ , which means all agents' states could

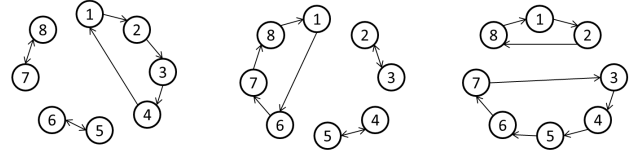


FIGURE 1. Communication graphs.

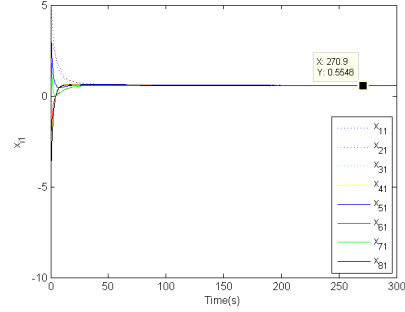


FIGURE 2. The first component of position states.

converge to the optimal solution of team objective function as  $k$  tends to infinity.

#### IV. SIMULATIONS

In this section, numerical examples are given to verify the effectiveness of the algorithm proposed in this paper. Suppose there are four second-order agents (1 to 4) and four first-order agents (5 to 8), and the communication topology switches among three graphs, each of them doesn't be strongly connected, see Fig. 1. The sample time is chosen as  $T = 0.001$ s. Take initial values  $x_1(0) = (5, 5)^T$ ,  $x_2(0) = (3, 2)^T$ ,  $x_3(0) = (1, 1.5)^T$ ,  $x_4(0) = (-2.5, -2)^T$ ,  $x_5(0) = (4, 3)^T$ ,  $x_6(0) = (-3, -2)^T$ ,  $x_7(0) = (3, 1)^T$ ,  $x_8(0) = (-5, -3)^T$ ,  $v_i(0) = (0, 0)^T$ ,  $i = 1, 2, 3, 4$ .

##### A. EXAMPLE 1

The local objective functions are chosen as  $f_1(s) = (s_1 - 2)^2 + s_2^2$ ,  $f_2(s) = s_1^4 + s_2^2$ ,  $f_3(s) = s_1^2 + s_2^4$ ,  $f_4(s) = s_1^2 + (s_2 - 2)^2$ ,  $f_5(s) = (s_1 - 2)^2 + s_2^2$ ,  $f_6(s) = s_1^4 + s_2^2$ ,  $f_7(s) = s_1^2 + (s_2 - 2)^2$ ,  $f_8(s) = s_1^2 + s_2^4$ , where  $s = (s_1, s_2)^T \in \mathbf{R}^2$ . By simple computation, we can obtain that the optimal solution of the team objective function  $f(s)$  is  $(0.5536, 0.5536)$ . The initial values of the nonuniform stepsizes (gradient gains) are chosen as  $p_i(0) = 2$ ,  $i = 1, 2, \dots, 8$ . By taking algorithms (4) and (5), the position trajectories of all agents are shown in Fig. 2 and Fig. 3, from which we could see that the position states of all agents can reach an agreement and converge to the optimal solution of the team objective function with an extremely small error. But they must can converge to the optimal solution of the team objective function as time tends to infinity. Fig. 4 and Fig. 5 show the velocity states of all second-order agents, from which we could see that velocities of all second-order agents converge to zero asymptotically.



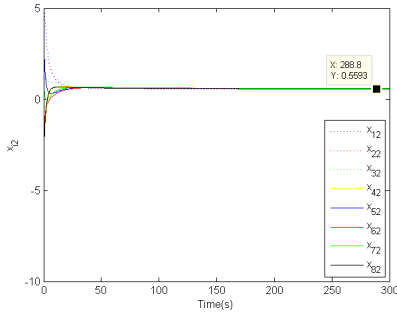


FIGURE 3. The second component of position states.

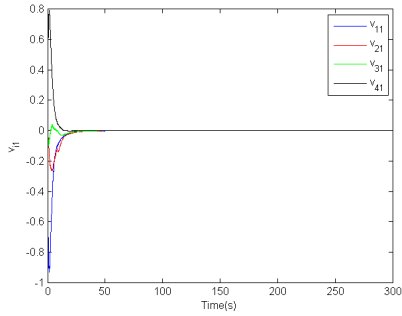


FIGURE 4. The first component of velocity states.

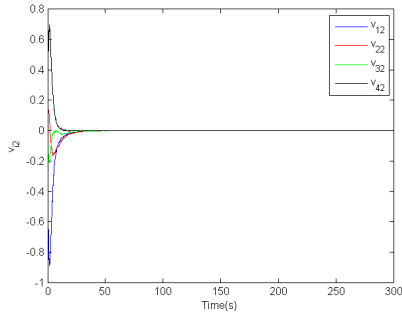


FIGURE 5. The second component of velocity states.

## B. EXAMPLE 2

In this subsection, to show the effectiveness of the nonuniform gradient gains, we make a comparison with the algorithm that has uniform gradient gains. The gradient gains in algorithms (4) and (5) are chosen very large such as  $p_i(k) = 5 * 10^5, i = 1, 2, \dots, 8$  for all  $k$  (If the gradient gains are chosen to be small, then the agreement cannot be reached). Then by taking algorithms (4) and (5), the position trajectories of all agents are shown in Fig. 6 and Fig. 7, from which we could see that the position states of all agents can reach an agreement and converge to the optimal solution of the team objective function. Moreover, the velocity states of all second-order agents are shown in Fig. 8 and Fig. 9, from which we could see that velocities of all second-order agents converge to zero. However, it can see from Fig. 6, 7, 8, 9 that the distributed optimization problem of heterogeneous multi-agent systems is solved after about 50 seconds, while from Fig. 3, 4, 5, 6 we could see that the problem is solved after nearly 50 seconds with greater convergence accuracy.

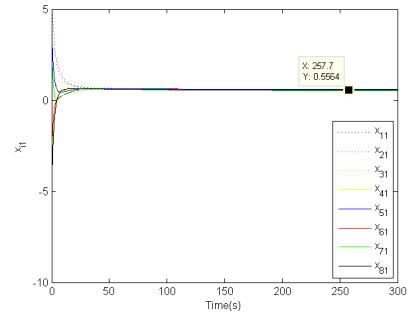


FIGURE 6. The first component of position states.

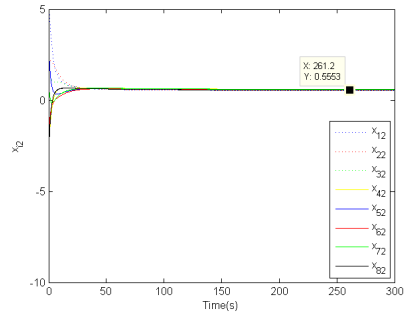


FIGURE 7. The second component of position states.

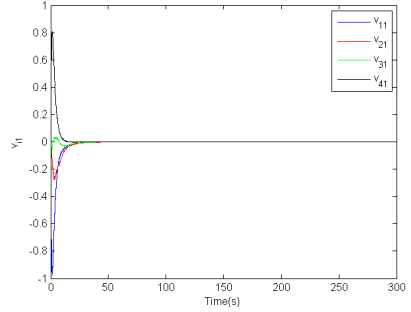


FIGURE 8. The first component of velocity states.

Therefore, by taking the algorithms with nonuniform gradient gains, the distributed optimization problem of the heterogeneous multi-agent systems can be solved and consume less time, while the convergence accuracy could be greatly improved.

## C. EXAMPLE 3

In this subsection, the convex function is chosen as  $f_1(s) : \mathbf{R}^2 \rightarrow \mathbf{R}$ :

$$f_1(s) = \begin{cases} 0, & \text{if } \|s\| \leq 1, \\ 0.5(\|s\| - 1)^2, & \text{otherwise.} \end{cases}$$

By simple calculations, we have

$$\Delta f_1(s) = \begin{cases} [0, 0]^T, & \text{if } \|s\| \leq 1, \\ (\|s\| - 1) \frac{s}{\|s\|} \neq [0, 0]^T, & \text{otherwise.} \end{cases}$$

$$f_2(s) = s_1^4 + s_2^2, f_3(s) = s_1^2 + s_2^4, f_4(s) = s_1^2 + (s_2 - 2)^2, f_5(s) = (s_1 - 2)^2 + s_2^2, f_6(s) = s_1^4 + s_2^2, f_7(s) = s_1^2 + (s_2 - 2)^2,$$

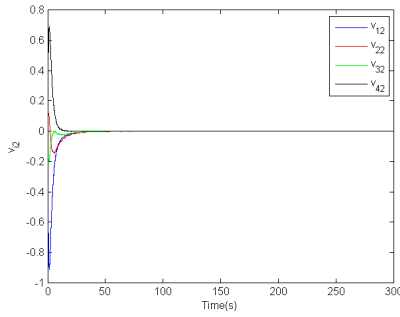


FIGURE 9. The second component of velocity states.

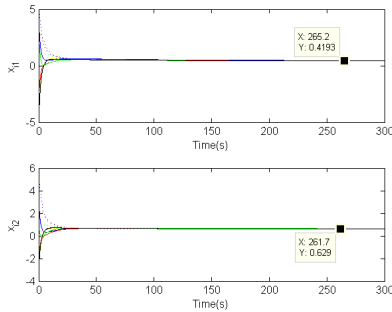


FIGURE 10. The position states.

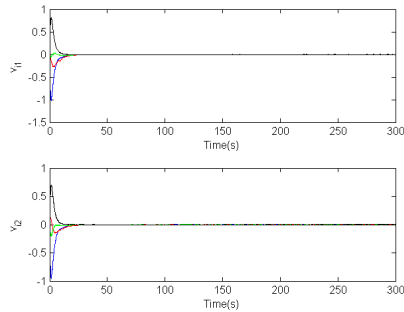


FIGURE 11. The velocity states.

$f_8(s) = s_1^2 + s_2^4$ . The initial values of the nonuniform gradient gains are chosen as  $p_i(0) = 2$ ,  $i = 1, 2, \dots, 8$ . By simple computation, we can obtain that the optimal solution of the function  $\sum_{i=2}^8 f_i(s)$  is  $s = (0.362, 0.6144)$ . By calculation  $\|s\| = 0.7131 < 1$ , therefore we have  $f_1(s) = 0$ . Since local differentiable convex objective function  $f_1(s) \geq 0$ , then we could conclude that the optimal solution of the team objective function  $f(s)$  is  $(0.362, 0.6144)$ . By taking algorithms (4) and (5) with the nonuniform gradient gains, the position trajectories of all agents are shown in Fig. 10, from which we could see that the position states of all agents can reach an agreement and converge to the optimal solution of the team objective function with an extremely small error. Fig. 11 show the velocity states of all second-order agents, from which we could see that velocities of all second-order agents converge to zero asymptotically. Therefore, the proposed algorithms are available for non-smooth functions.

## V. CONCLUSION

In this paper, we studied the distributed optimization problem of heterogeneous multi-agent systems with switching jointly strongly connected topologies, where each agent has first-order or second-order dynamics and the team objective function is the sum of finite local convex function. To solve this problem, a distributed algorithm is proposed for each agent based on the local information accessed by each agent. Then, by a coordination transformation, the closed-loop system is converted into an equivalent system, the system matrix of which is stochastic matrix. Based on the properties of stochastic matrix, it is shown that the consensus can be achieved and the team objective function can be minimized simultaneously. Finally, simulation results were given to demonstrate the correctness of the theoretical results. In the future, the distributed optimization problems with time-delay will be considered.

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