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A proof, a consequence and an application of Boole's combinatorial identity

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Abstract. Boole's combinatorial identity is proved, and a consequence of it for analytic functions is derived that is used to evaluate a sequence of integrals in terms of Euler's secant sequence of integers.

1. Boole's identity

This features early on in [2], (cf. equation (6) on page 20) and states that if n is a nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^n = n!. \tag{1}$$

In addition, if $n \geq 1$, and m is any nonnegative integer less than n, then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m = 0. \tag{2}$$

Both of these statements have many proofs; consult [1], and the references cited therein.

Here's an outline of a combined proof of (1) and (2):

Proof. Write

$$\sigma_n(m) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}k^m = n!\sum_{k=0}^{n}\frac{(-1)^k(n-k)^m}{k!\,(n-k)!},\ \ m, n = 0, 1, 2, \ldots.$$

Fix m, and observe that the sequence $\{\sigma_n(m)/n!,\ n = 0, 1, \ldots\}$ is the convolution of the sequences $\{(-1)^n/n!,\ n = 0, 1, \ldots\}$, and $\{n^m/n!,\ n = 0, 1, \ldots\}$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{\sigma_n(m)}{n!}z^n &= \sum_{n=0}^{\infty}z^n\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{(n-k)^m}{(n-k)!}\\ &= \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n\right)\left(\sum_{n=0}^{\infty}\frac{n^m}{n!}z^n\right)\\ &= e^{-z}W_m(z),\end{aligned}$$

where

$$W_m(z) = \sum_{n=0}^{\infty}\frac{n^m z^n}{n!} = \Theta^m e^z,$$

Θ standing for the differential operator $z\frac{d}{dz}$, much used by Boole in his treatment of linear differential equations with variable coefficients.

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Clearly, $W_0(z) = e^z, W_1(z) = ze^z$, and the following recurrence relation holds:

$$W_{m+1}(z) = zW'_m(z) + W_m(z),\ m = 0, 1, \dots,$$

where the prime denotes differentiation. So, $W_m(z)$ is a monic polynomial $p_m(z)$ times e^z, and $\deg p_m = m$, which is easy to see by induction. Hence,

$$\sum_{n=0}^{\infty} \frac{\sigma_n(m)}{n!} z^n = p_m(z),$$

from which it follows immediately that $\sigma_n(m) = 0, \forall n > m$ and $\sigma_n(n) = n!$. Thus (1) and (2) are true. □

2. A simple consequence

Suppose f is analytic on a disc D centred at 0 in the complex plane. Then, for any nonnegative integer n,

$$\lim_{x\to 0} \frac{1}{x^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(kx) = f^{(n)}(0). \tag{3}$$

Proof. Let

$$F(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(kx), \forall x \in \tfrac{1}{n} D.$$

Clearly, F is analytic on a subdisc of D centred at 0, on which

$$F^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(kx).$$

In particular, it follows from (2) that

$$F^{(m)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(0) = 0,\ m = 0, 1, \dots, n-1, \tag{4}$$

and from (1) that

$$F^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^n f^{(n)}(0) = n! f^{(n)}(0). \tag{5}$$

Therefore, by integrating by parts multiple times, and applying (4) repeatedly,

$$F(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} F^{(n)}(t)\, dt = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} F^{(n)}(xs)\, ds.$$

Hence

$$F(x) - x^n \frac{F^{(n)}(0)}{n!} = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} [F^{(n)}(xs) - F^{(n)}(0)]\, ds.$$

Let $\epsilon > 0$. By hypothesis, there exists $\delta > 0$ such that $|F^{(n)}(z) - F^{(n)}(0)| < \epsilon$ whenever $|z| < \delta$, and so $|F^{(n)}(xs) - F^{(n)}(0)| < \epsilon$ whenever $|x| < \delta$ and $0 \le s \le 1$. Consequently, if $0 < |x| < \delta$,

$$\begin{aligned}
\left| \frac{F(x)}{x^n} - \frac{F^{(n)}(0)}{n!} \right| &\le \frac{1}{(n-1)!} \int_0^1 (1-s)^{n-1} |F^{(n)}(xs) - F^{(n)}(0)|\, ds \\
&\le \frac{\epsilon}{(n-1)!} \int_0^1 (1-s)^{n-1}\, ds \\
&= \frac{\epsilon}{n!}.
\end{aligned}$$

In other words,

$$\lim_{x\to 0}\frac{F(x)}{x^n}=f^{(n)}(0),$$

by (5), as claimed. □

In particular, if f has a power series expansion about 0 so that, for some $r>0$,

$$f(x)=\sum_{m=0}^{\infty}a_m x^m,\ \ \forall|x|<r,$$

then

$$\lim_{x\to 0}\frac{1}{x^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(kx)=n!a_n$$

by (3).

3. An application

Consider the sequence of integrals

$$I_n=\int_0^\infty\frac{(\ln(x))^n}{1+x^2}\,dx,\ \ n=0,1,2\ldots$$

It's familiar that $I_0=\pi/2$, and clear that

$$\begin{aligned}I_n &= \int_0^1\frac{(\ln(x))^n}{1+x^2}\,dx+\int_1^\infty\frac{(\ln(x))^n}{1+x^2}\,dx\\ &= \int_0^1\frac{(\ln(x))^n}{1+x^2}\,dx+\int_0^1\frac{(\ln(\frac{1}{x}))^n}{1+x^2}\,dx\\ &= (1+(-1)^n)\int_0^1\frac{(\ln(x))^n}{1+x^2}\,dx.\end{aligned}$$

Hence, $I_{2n+1}=0,\ n=0,1,2,\ldots$. It's an exercise on page 134 in [3] (Titchmarsh's Theory of Functions) that $I_2=\pi^3/8$, while the computer package MAPLE spews out values of I_{2n} for $n=2,3,4,5,6$, according to which

$$I_4=\frac{5\pi^5}{2^5},I_6=\frac{61\pi^7}{2^7},I_8=\frac{1385\pi^9}{2^9},\ I_{10}=\frac{50521\pi^{11}}{2^{11}},\ I_{12}=\frac{13936098\pi^{13}}{2^{13}}.$$

The numbers $1,5,61,1385,50521,139360981$ are the first six terms of the integer sequence named Euler's secant sequence, and numbered A000364 in [4] (Sloane's online encyclopedia of integer sequences). If $a(n)$ denotes the nth term of this sequence, it's tempting to conjecture that

$$I_{2n}=\frac{a(n)\pi^{2n+1}}{2^{2n+1}},n=0,1,2,\ldots.$$

One way to confirm this is as follows.

Proof. Recall that, for $x>0$, $\ln x$ is the limit of the decreasing sequence, $m(\sqrt[m]{x}-1),\ m=1,2,\ldots$. Hence

$$\begin{aligned}I_n &= \lim_{m\to\infty}m^n\int_0^\infty\frac{(x^{1/m}-1)^n}{1+x^2}\,dx\\ &= \lim_{m\to\infty}m^n\int_0^\infty\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{x^{k/m}}{1+x^2}\,dx\\ &= \lim_{m\to\infty}m^n\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}J(k/m),\end{aligned}$$

where, for $|\Re\alpha| < 1$,

$$J(\alpha) = \int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \frac{\pi}{2}\sec\left(\frac{\pi\alpha}{2}\right).$$

Since sec admits of a power series expansion about 0 of the form

$$\sec x = \sum_{n=0}^{\infty} \frac{a(n)}{(2n)!} x^{2n},$$

that is valid for all $|x| < \pi/2$, it follows that

$$\begin{aligned} I_n &= \frac{\pi}{2} \lim_{m\to\infty} m^n \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sec\left(\frac{k\pi m}{2n}\right) \\ &= \frac{\pi^{2n+1}}{2^{2n+1}} \sec^{(n)}(0), \end{aligned}$$

by (3), and so, in particular, $I_{2n+1} = 0$, $n = 0, 1, \ldots$, as we noted above, and

$$I_{2n} = \frac{a(n)\pi^{2n+1}}{2^{2n+1}},$$

as desired. □

Remark 3.1. The connection between the values of the sequence I_n of integrals, and terms of the sequence A000364, doesn't appear to have been noticed before.

References

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