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# A proof, a consequence and an application of Boole's combinatorial identity

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Abstract. Boole's combinatorial identity is proved, and a consequence of it for analytic functions is derived that is used to evaluate a sequence of integrals in terms of Euler's secant sequence of integers.

## 1. Boole's identity

This features early on in [2], (cf. equation (6) on page 20) and states that if $n$ is a nonnegative integer, then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^n = n!. \tag{1}$$

In addition, if $n \geq 1$, and $m$ is any nonnegative integer less than $n$, then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m = 0. \tag{2}$$

Both of these statements have many proofs; consult [1], and the references cited therein.

Here's an outline of a combined proof of (1) and (2):

*Proof.* Write

$$\sigma_n(m) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m = n! \sum_{k=0}^{n} \frac{(-1)^k (n-k)^m}{k!\,(n-k)!},\ m, n = 0, 1, 2, \ldots.$$

Fix $m$, and observe that the sequence $\{\sigma_n(m)/n!,\ n = 0, 1, \ldots\}$ is the convolution of the sequences $\{(-1)^n/n!,\ n = 0, 1, \ldots\}$, and $\{n^m/n!,\ n = 0, 1, \ldots\}$. Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\sigma_n(m)}{n!} z^n &= \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{(n-k)^m}{(n-k)!} \\ &= \Big( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n \Big) \Big( \sum_{n=0}^{\infty} \frac{n^m}{n!} z^n \Big) \\ &= e^{-z} W_m(z), \end{aligned}$$

where

$$W_m(z) = \sum_{n=0}^{\infty} \frac{n^m z^n}{n!} = \Theta^m e^z,$$

$\Theta$ standing for the differential operator $z\frac{d}{dz}$, much used by Boole in his treatment of linear differential equations with variable coefficients.

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Clearly, $W_0(z) = e^z, W_1(z) = ze^z$, and the following recurrence relation holds:

$$W_{m+1}(z) = zW_m'(z) + W_m(z),\ m = 0, 1, \ldots,$$

where the prime denotes differentiation. So, $W_m(z)$ is a monic polynomial $p_m(z)$ times $e^z$, and $\deg p_m = m$, which is easy to see by induction. Hence,

$$\sum_{n=0}^{\infty} \frac{\sigma_n(m)}{n!} z^n = p_m(z),$$

from which it follows immediately that $\sigma_n(m) = 0, \forall n > m$ and $\sigma_n(n) = n!$. Thus (1) and (2) are true. □

## 2. A simple consequence

Suppose $f$ is analytic on a disc $D$ centred at 0 in the complex plane. Then, for any nonnegative integer $n$,

$$\lim_{x\to 0} \frac{1}{x^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(kx) = f^{(n)}(0). \tag{3}$$

*Proof.* Let

$$F(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(kx), \forall x \in \tfrac{1}{n} D.$$

Clearly, $F$ is analytic on a subdisc of $D$ centred at 0, on which

$$F^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(kx).$$

In particular, it follows from (2) that

$$F^{(m)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(0) = 0,\ m = 0, 1, \ldots, n-1, \tag{4}$$

and from (1) that

$$F^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} k^n f^{(n)}(0) = n! f^{(n)}(0). \tag{5}$$

Therefore, by integrating by parts multiple times, and applying (4) repeatedly,

$$F(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} F^{(n)}(t)\, dt = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} F^{(n)}(xs)\, ds.$$

Hence

$$F(x) - x^n \frac{F^{(n)}(0)}{n!} = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} [F^{(n)}(xs) - F^{(n)}(0)]\, ds.$$

Let $\epsilon > 0$. By hypothesis, there exists $\delta > 0$ such that $|F^{(n)}(z) - F^{(n)}(0)| < \epsilon$ whenever $|z| < \delta$, and so $|F^{(n)}(xs) - F^{(n)}(0)| < \epsilon$ whenever $|x| < \delta$ and $0 \le s \le 1$. Consequently, if $0 < |x| < \delta$,

$$\begin{aligned} \left| \frac{F(x)}{x^n} - \frac{F^{(n)}(0)}{n!} \right| &\le \frac{1}{(n-1)!} \int_0^1 (1-s)^{n-1} |F^{(n)}(xs) - F^{(n)}(0)|\, ds \\ &\le \frac{\epsilon}{(n-1)!} \int_0^1 (1-s)^{n-1}\, ds \\ &= \frac{\epsilon}{n!}. \end{aligned}$$

In other words,

$$\lim_{x\to 0}\frac{F(x)}{x^n} = f^{(n)}(0),$$

by (5), as claimed. □

In particular, if $f$ has a power series expansion about 0 so that, for some $r > 0$,

$$f(x) = \sum_{m=0}^{\infty} a_m x^m,\ \forall |x| < r,$$

then

$$\lim_{x\to 0}\frac{1}{x^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(kx) = n!a_n$$

by (3).

## 3. An application

Consider the sequence of integrals

$$I_n = \int_0^{\infty}\frac{(\ln(x))^n}{1+x^2}\,dx,\ n = 0, 1, 2\ldots$$

It's familiar that $I_0 = \pi/2$, and clear that

$$\begin{aligned} I_n &= \int_0^1 \frac{(\ln(x))^n}{1+x^2}\,dx + \int_1^{\infty}\frac{(\ln(x))^n}{1+x^2}\,dx \\ &= \int_0^1 \frac{(\ln(x))^n}{1+x^2}\,dx + \int_0^1 \frac{(\ln(\frac{1}{x}))^n}{1+x^2}\,dx \\ &= (1+(-1)^n)\int_0^1 \frac{(\ln(x))^n}{1+x^2}\,dx. \end{aligned}$$

Hence, $I_{2n+1} = 0,\ n = 0, 1, 2, \ldots$. It's an exercise on page 134 in [3] (Titchmarsh's Theory of Functions) that $I_2 = \pi^3/8$, while the computer package MAPLE spews out values of $I_{2n}$ for $n = 2, 3, 4, 5, 6$, according to which

$$I_4 = \frac{5\pi^5}{2^5}, I_6 = \frac{61\pi^7}{2^7}, I_8 = \frac{1385\pi^9}{2^9}, I_{10} = \frac{50521\pi^{11}}{2^{11}}, I_{12} = \frac{13936098\pi^{13}}{2^{13}}.$$

The numbers $1, 5, 61, 1385, 50521, 139360981$ are the first six terms of the integer sequence named Euler's secant sequence, and numbered A000364 in [4] (Sloane's online encyclopedia of integer sequences). If $a(n)$ denotes the $n$th term of this sequence, it's tempting to conjecture that

$$I_{2n} = \frac{a(n)\pi^{2n+1}}{2^{2n+1}}, n = 0, 1, 2, \ldots.$$

One way to confirm this is as follows.

*Proof.* Recall that, for $x > 0$, $\ln x$ is the limit of the decreasing sequence, $m(\sqrt[m]{x} - 1)$, $m = 1, 2, \ldots$. Hence

$$\begin{aligned} I_n &= \lim_{m\to\infty} m^n \int_0^{\infty}\frac{(x^{1/m}-1)^n}{1+x^2}\,dx \\ &= \lim_{m\to\infty} m^n \int_0^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\frac{x^{k/m}}{1+x^2}\,dx \\ &= \lim_{m\to\infty} m^n \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}J(k/m), \end{aligned}$$

where, for $|\Re\alpha| < 1$,

$$J(\alpha) = \int_0^\infty \frac{x^\alpha}{1+x^2}\,dx = \frac{\pi}{2}\sec\left(\frac{\pi\alpha}{2}\right).$$

Since sec admits of a power series expansion about 0 of the form

$$\sec x = \sum_{n=0}^{\infty} \frac{a(n)}{(2n)!}x^{2n},$$

that is valid for all $|x| < \pi/2$, it follows that

$$\begin{aligned} I_n &= \frac{\pi}{2}\lim_{m\to\infty} m^n \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sec\left(\frac{k\pi m}{2n}\right) \\ &= \frac{\pi^{2n+1}}{2^{2n+1}}\sec^{(n)}(0), \end{aligned}$$

by (3), and so, in particular, $I_{2n+1} = 0$, $n = 0, 1, \ldots$, as we noted above, and

$$I_{2n} = \frac{a(n)\pi^{2n+1}}{2^{2n+1}},$$

as desired. □

**Remark 3.1.** The connection between the values of the sequence $I_n$ of integrals, and terms of the sequence A000364, doesn't appear to have been noticed before.

## References

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