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## Trapped surfaces in spherical expanding open universes

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Consider spherically symmetric initial data for a cosmology which, in the large part, approximates an open  $k = -1$ ,  $\Lambda = 0$  Friedmann-Lemaître universe. Further assume that the data are chosen so that the trace of the extrinsic curvature is a constant and that the matter field is at rest at this instant of time. One expects that no trapped surfaces appear in the data if no significant clump of excess matter is to be found. This Brief Report confirms this belief by displaying a necessary condition for the existence of trapped surfaces. This necessary condition, simply stated, says that a relatively large amount of excess matter must be concentrated in a small volume for trapped surfaces to appear.

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Over the last few years, we have been investigating spherically symmetric solutions to the Einstein equations, both asymptotically flat [1–3] and cosmological [4,5], with the intention of relating the amount of matter inside some chosen sphere to the possibility of trapped surfaces forming. The only situation which has not yet been dealt with is the case where we have a cosmological model with unbounded spatial slices which, on average, has a negative scalar curvature. These are generalizations of the “open” (hyperboloidal) Friedmann universes. In this work we will derive a necessary condition for the appearance of a trapped surface in such a cosmology. This is very similar to equivalent necessary conditions we have derived in all the other cases. Interestingly enough, we have failed, in this open case, to find the matching sufficient condition which we were able to derive in all the other cases. We comment further on this at the end of

this work.

Initial data for solutions to the Einstein equations consist of four objects (see [6])  $[g_{ab}, K_{ab}, \rho$  (energy density) and  $J_b$  (matter current density)], where  $g_{ab}$  is a three-dimensional, Riemannian (positive definite) metric which describes a three-dimensional-manifold  $\Sigma$ ;  $\Sigma$  is to be regarded as a spacelike slice through the four-manifold which is the solution of the Einstein equations and  $K_{ab}$  is a symmetric three-tensor which is the extrinsic curvature of  $\Sigma$  as an embedded surface in the four-manifold. Consider a spacelike two-surface  $S$  embedded in  $\Sigma$ . If the surface is orientable, we can identify a unit vector  $n^a$  as the “outgoing” direction orthogonal to  $S$  in  $\Sigma$ . Now  $J_b n^b$  measures the outflux of matter through  $S$ .

These data are not independent. They must satisfy the constraints

$$^{(3)}R[g] - K_{ab}K^{ab} + (K^a_a)^2 = 16\pi\rho \quad (\text{Hamiltonian constraint}), \quad (1)$$

$$D_a K^a_b - D_b K^a_a = -8\pi J_b \quad (\text{momentum constraint}), \quad (2)$$

where  $^{(3)}R[g]$  is the scalar curvature of  $\Sigma$ ,  $D$  the covariant derivative compatible with  $g_{ab}$ . The trace  $(g^{ab}K_{ab})$  of the second fundamental form  $K_{ab}$  is equal to the (positive) rate of change of the three-dimensional volume in a timelike direction  $\frac{d}{dt}(dV) = g^{ab}K_{ab}dV$ .

Here we wish to consider initial data for a spherically symmetric cosmology which in the large looks like an open Friedmann universe. The standard time slice through an open (hyperboloidal) Friedmann-Lemaître

universe is given by a hyperboloid of constant negative curvature where the line element is given by

$$ds^2 = a^2(t) (dr^2 + \sinh^2 r [d\theta^2 + \sin^2 \theta d\varphi^2]), \quad (3)$$

where  $a$  is constant on  $\Sigma$ ;  $\hat{K}_{ab}$  is pure trace  $\hat{K}_{ab} = H\hat{g}_{ab}$ ,  $H = \frac{da}{dt}/a$  is the Hubble constant; the (background) energy-density  $\hat{\rho}$  is a constant,  $\hat{J}^a = 0$ . We will use these data as a background against which we will compare our

actual spherical solution. Background quantities will be denoted by a caret.

The Hamiltonian constraint (2) now gives

$$-\frac{6}{a^2} + 6H^2 = 16\pi\hat{\rho} \quad (4)$$

while the momentum constraint (2) is identically satisfied.

We wish to consider a spherically symmetric spacetime whose spatial topology agrees with that of the open Robertson-Walker geometry.<sup>1</sup> We focus on a spacelike slice through the spacetime which is itself spherically symmetric and on which the trace of the extrinsic curvature is a constant. This is not as restrictive a condition as it appears. If one is given a spacetime with a spatial symmetry, the  $\text{tr}K=\text{const}$  slices respect this symmetry [7].

Since the three-metric is spherically symmetric we know that it is conformally flat and the line element can be written as

$$ds^2 = \phi^4(r) d\hat{s}^2, \quad (5)$$

where  $\phi \geq 0$ . We can write the extrinsic curvature (assuming only spherical symmetry) as

$$K_{ab} = Hg_{ab} + \left(n_a n_b - \frac{g_{ab}}{3}\right) K(r), \quad (6)$$

where  $n^a$  is the unit radial vector. We further assume  $J^a = J(r)n^a$ .

Since  $H$  is assumed constant, only  $K(r)$  appears in the momentum constraint. As in previous articles, the momentum constraint can be written as an explicit integral to give

$$K(R_S) = \frac{-12a\pi}{\phi^6(R_S) \sinh^3(R_S)} \int_0^{R_S} \phi^8(r) \sinh^3(r) J(r) dr. \quad (7)$$

One immediate consequence of (7) is that if the matter is at rest in the slice ( $J = 0$ ) we have that  $K = 0$  and so the extrinsic curvature is pure trace. For the rest of this work we will assume that this holds. Therefore the momentum constraint is automatically satisfied and we need not consider it further.

To evaluate the Hamiltonian constraint we first need to calculate the scalar curvature of the metric. To do so we use the background metric  $\hat{g}_{ab}$  and that  $\hat{R} = -\frac{6}{a^2}$ . We can use the fact that  $g_{ab} = \phi^4 \hat{g}_{ab}$  to write

$${}^{(3)}R[g] = -8\phi^{-5} \hat{\Delta} \phi - \frac{6}{a^2} \phi^{-4}. \quad (8)$$

Hence the Hamiltonian constraint reads

$$\phi^{-5} \hat{\Delta} \phi = -2\pi\delta\rho - \frac{3}{4a^2} (1 - \phi^4) \phi^{-4}, \quad (9)$$

<sup>1</sup>If we refer to the geometrical (or kinematical) properties of this spacetime we use the term Robertson-Walker, while for dynamical aspects we use the term Friedmann-Lemaître.

where  $\delta\rho$  denotes  $\rho - \hat{\rho}$ . Let us stress here that we make no assumption that  $\delta\rho$  is either small or has a fixed sign.

We wish to consider whether such a system will gravitationally collapse to form a black hole. Evidence would be the appearance of trapped surfaces in the initial data. Consider a spherically symmetric two-surface  $S$  in the initial data set. The expansion of outgoing null rays from this surface  $S$  is given by

$$\theta = D_a n^a + (g^{ab} - n^a n^b) K_{ab}, \quad (10)$$

where  $n^a$  is the unit (spacelike) normal to this surface. If  $\theta < 0$  we say that the surface is (outer) trapped. One of the singularity theorems of general relativity [8,9] tells us that the appearance of trapped surfaces is an indication that gravitational collapse is occurring. The expansion of a spherical surface in the open Friedmann universe is given by

$$\theta = \hat{D}_a \hat{n}^a + 2H = \frac{2 \coth r + 2aH}{a}, \quad \hat{n}^a = (a^{-1}, 0, 0). \quad (11)$$

Therefore if the universe is expanding ( $H > 0$ ) no trapped surfaces exist.

In the spherically symmetric but nonhomogeneous model we are considering here it is easy to show that the expansion can be written as

$$\theta = D_a n^a + 2H = \phi^{-3} (4\phi' + 2\phi \coth r) + 2H, \quad (12)$$

$$n^a = (\phi^{-1} a^{-1}, 0, 0),$$

where  $\phi' = \frac{d\phi}{dr}$ .

To relate this expression to the matter content of the universe one integrates (9) in the physical space over a sphere  $S$  of coordinate radius  $R_S$ . Using spherical symmetry, integration by parts and Gauss law the left-hand side gives

$$\begin{aligned} \int_V \phi^{-5} \hat{\Delta} \phi dV &= \int_V \phi^{-5} \hat{\Delta} \phi \phi^6 d\hat{V} \\ &= 4\pi \phi \phi' \sinh^2 r|_R - 4\pi \int_0^{R_S} \phi'^2 \sinh^2 r \, r dr. \end{aligned} \quad (13)$$

To do this integration one needs to use that  $dV = \phi^6 d\hat{V} = \phi^6 a^3 \sinh^2 r \sin \varphi$ . The right-hand side of (9) gives

$$\begin{aligned} \int_V \phi^{-5} \hat{\Delta} \phi dV &= 2\pi \delta M_S + \frac{3}{4a^2} V \\ &\quad - \frac{3}{4a^2} 4\pi \int_0^{R_S} \phi^2 \sinh^2 r \, r dr. \end{aligned} \quad (14)$$

We equate (14) with (13), divide by  $2\pi$ , recognize that the area  $A$  of a sphere  $S$  at a given radius  $R$  is given by  $A = \int_S \sqrt{|g|} d^2x = 4\pi a^2 \phi^4 \sinh^2 R_S$  and use (12) to finally get

$$\begin{aligned}
\frac{A}{8\pi}\theta &= -\delta M_S + \int_0^{R_S} [2\phi' \sinh^2 r + (\phi^2 \cosh r \sinh r)' - \frac{3}{2}\phi^2 \sinh^2 r]adr + \frac{3}{8\pi a^2}V + \frac{HA}{4\pi} \\
&= -\delta M_S + \int_0^{R_S} [\frac{1}{2}\phi^2 + \frac{1}{2}(\phi \cosh r + 2\phi' \sinh r)^2]adr + \frac{3}{8\pi a^2}V + \frac{HA}{4\pi}.
\end{aligned} \tag{15}$$

Equation (15) is the appropriate equation to show a necessary condition; we assume  $S$  to be trapped, i.e.,  $\theta < 0$ , which leads to

$$\begin{aligned}
0 &> -\delta M_S + \int_0^{R_S} [\frac{1}{2}\phi^2 + \frac{1}{2}(\phi \cosh r + 2\phi' \sinh r)^2]adr + \frac{3}{8\pi a^2}V + \frac{HA}{4\pi} \\
&\geq -\delta M_S + \frac{1}{2}L_S + \frac{3}{4a^2}V + \frac{HA}{4\pi}.
\end{aligned} \tag{16}$$

The proper radius of the sphere of coordinate radius  $R_S$  is given by

$$L_S = \int_0^{R_S} \sqrt{g_{rr}}dr = \int_0^{R_S} \phi^2 adr. \tag{17}$$

Hence, if a given surface is trapped, we get

$$\delta M_S \geq \frac{1}{2}L_S + \frac{3}{8\pi a^2}V + \frac{HA}{4\pi}. \tag{18}$$

Finally we have the following theorem.

*Theorem 1. (A necessary condition.) Assume one is given initial data for a spherically symmetric open cosmology, such that*

$$(1) K^a_a = \text{const} = 3H,$$

$$(2) J_b = 0,$$

*and (background) Friedmann-Lemaître data  $(a, H, \hat{\rho})$  satisfying Eq. (4).*

*If for a spherical surface  $S$ , its proper radius  $L_S$ , its area  $A$ , the amount of the excess mass  $\delta M_S$ , and the volume  $V$  inside  $S$  satisfy*

$$\delta M_S < \frac{1}{2}L_S + \frac{AH}{4\pi} + \frac{3V}{8\pi a^2} \tag{19}$$

*then  $S$  is not trapped.*

An obvious reorganization of Eq. (19) can be achieved by moving the term  $\hat{\rho}V$  from the left- to the right-hand

side. We then can use Eq. (4) to give

$$\hat{\rho}V + \frac{3}{8\pi a^2}V = \frac{3H^2V}{8\pi}. \tag{20}$$

This gives us a condition for the nonappearance of trapped surfaces which is independent of any choice of background.

*Theorem 2. Assume one is given initial data as in theorem 1. If for a spherical surface  $S$ , its proper radius  $L_S$ , its area  $A$ , the total amount of matter  $M_S$  enclosed by  $S$ , and the volume  $V$  inside  $S$  satisfy*

$$M_S < \frac{1}{2}L_S + \frac{AH}{4\pi} + \frac{3H^2V}{8\pi} \tag{21}$$

*then  $S$  is not trapped.*

This result forms the basis of a simple test which may allow a determination of the global topology of our Universe [10].

In the other cosmological models and in the case of asymptotically flat data we have found essentially identical necessary conditions and we have also found sufficient conditions expressed in terms of the same parameters. Surprisingly, we have failed to date to find, in the open case, a sufficient condition. This should not lead one to jump to the hasty conclusion that trapped surfaces are absent from spherical expanding open universes. It is easy to find examples with trapped surfaces. Because of the spherical symmetry, it is a simple matter to construct a model which consists of a constant density sphere embedded in an open cosmology. If the density is large enough, trapped surfaces appear.

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