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Constraints in spherically symmetric classical general relativity. II. Identifying the configuration space: A moment of time symmetry

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We continue our investigation of the configuration space of general relativity begun in the preceding paper. Here we examine the Hamiltonian constraint when the spatial geometry is momentarily static (MS). We begin with a heuristic description of the presence of apparent horizons and singularities. A peculiarity of MS configurations is that not only do they satisfy the positive quasilocal mass (QLM) theorem, they also satisfy its converse: the QLM is positive everywhere, if and only if the (nontrivial) spatial geometry is nonsingular. We derive an analytical expression for the spatial metric in the neighborhood of a generic singularity. The corresponding curvature singularity shows up in the traceless component of the Ricci tensor. As a consequence of the converse, if the energy density of matter is monotonically decreasing, the geometry cannot be singular. A supermetric on the configuration space which distinguishes between singular geometries and nonsingular ones is constructed explicitly. Global necessary and sufficient criteria for the formation of trapped surfaces and singularities are framed in terms of inequalities which relate some appropriate measure of the material energy content on a given support to a measure of its volume. The sufficiency criteria are cast in the following form: if the material energy exceeds some universal constant times the proper radius l_0 of the distribution, the geometry will possess an apparent horizon for one constant and a singularity for some other larger constant. A more appropriate measure of the material energy for casting the necessary criteria is the maximum value of the energy density of matter $\rho_{\max}$: if $\rho_{\max} l_0^2 < \text{some constant}$ the distribution of matter will not possess a singularity for one constant and an apparent horizon for some other smaller constant. These inequalities provide an approximate characterization of the singular (nonsingular) and trapped (nontrapped) partitions on the configuration space. Their strength is gauged by exploiting the exactly solvable piecewise constant density star as a template. Finally, we provide a more transparent derivation of the lower bound on the binding energy conjectured by Arnowitt, Deser, and Misner and proven by Bizon, Malec, and Ó Murchadha and speculate on possible improvements.

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I. INTRODUCTION

This is the second paper in a series in which we examine the solution of the constraints in general relativity under the restriction that both the spatial geometry and the extrinsic curvature be spherically symmetric. In the first paper (hereafter referred to as paper I) we examined various universal features of the constraints [1]. In particular, it was shown following [2] how the constraints can be recast in a symmetrical form with respect to the optical scalars of the theory. We introduced an appropriate large class of foliations which are in some sense the natural foliations of a spherically symmetrical spacetime. We discussed the positivity of a quasilocal mass and related local bounds on the canonical variables of the theory. Fi-

nally, a tentative realization of the configuration space was proposed.

In this paper we will specialize to solutions of the constraints when the geometry is momentarily static (MS) and asymptotically flat.¹ The extrinsic curvature and the radial material current now both vanish, $K_{ab} = 0 = J$ and the radial momentum constraint becomes vacuous. We follow the notation established in paper I. The proper (or geodesic) radius l is treated as the independent radial coordinate in our parametrization of the spatial geometry. The Hamiltonian constraint then provides an ordinary differential equation (ODE) for the circumferential radius as a function of l . We defer the examination of nonvanishing extrinsic curvature to a third paper (III) [3].

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¹A static solution is a solution possessing a surface forming timelike Killing vector. It is easy to show that the spacelike surface orthogonal to the Killing vector satisfies $K_{ab} = 0$.

Whether the solution remains static, subject to the initial conditions $K_{ab} = 0$ so that $\dot{K}_{ab} = 0$ will depend on the stresses which only show up in the dynamical equations. What is true is that if one is to construct globally static solutions, they had better satisfy the Hamiltonian constraint with $K_{ab} = 0$ to begin with.

While the constraints simplify enormously when $K_{ab} = 0$, many if not all of the subtleties involved in the solution of the initial-value problem obligingly manifest themselves. This makes such configurations important as a theoretical proving ground.

Another feature of these solutions which makes them interesting is that they are the only geometries which simultaneously satisfy the Lorentzian and Euclidean constraints. In the semiclassical approximation to tunneling they correspond to the classical turning points. They therefore represent the boundaries between the classically allowed and forbidden regimes [4].

Our goal will be to provide a characterization of the configuration space $\mathcal{C}$ of all spatial three-geometries satisfying the Hamiltonian constraint essentially by whatever means we can. From one point of view, of course, $\mathcal{C}$ could be characterized by whatever material degrees of freedom we introduce and, as such, is trivial even if the corresponding Hamiltonian is not [5]. If matter is modeled by a scalar field, for example, $\mathcal{C}$, will simply be all spherically symmetric scalar fields with an appropriate falloff. What is ignored here, however, is that to each point in the configuration space there also corresponds a spatial geometry. In particular, some configurations will correspond to geometries possessing apparent horizons, with a proper subset of these corresponding to geometries possessing singularities. The boundaries in $\mathcal{C}$ partitioning geometries with apparent horizons or singularities from those which do not are important landmarks on this space.

In Sec. II we discuss generic analytical properties of the Hamiltonian constraint. We choose, as our independent data, a spherically symmetric energy density distribution. This consists of a non-negative function ρ , given as a function of the proper radius l on the positive real axis $[0, \infty)$. The Hamiltonian constraint is then solved to give the spatial geometry. This is to be regarded as a kinematical quantity, entirely determined by the source distribution. We are not fussy about the differentiability of ρ , we would be happy with C^N where N is any integer in the interval $[0, \infty]$.

We suppose that the asymptotically flat geometry is regular at its center. Regular solutions of the Hamiltonian constraint are determined uniquely once ρ is specified. For some choices of ρ , however, we find that the circumferential radius R goes to zero at a finite value of $l = l_S$.

Let us suppose we scale some small ρ , say ρ_0 . Initially, the circumferential radius R increases monotonically with the proper radius. As the scale is raised a critical point is always reached in which the geometry develops an apparent horizon. When $K_{ab} = 0$, this corresponds to the development of an extremal embedded two-sphere. At this critical scaling, the horizon will be degenerate. As ρ is scaled further, the horizon will bifurcate

into an outer minimal two-sphere and an inner maximal two-sphere. The interpolating two-spheres which foliate the spatial region between the two will all be trapped. Typically, if ρ_0 is scaled further, the geometry will become singular at some other critical scaling corresponding to the pinching off of the minimum two-sphere to $R = 0$ at $l = l_S$. All points on this two-sphere would then be identified, the geometry would possess a bag of gold at its center. This is the only way that the spatial geometry could possess a singularity. These three-dimensional singularities are the only ones we discuss. These form only a subset of the possible four-dimensional singularities that could arise when regular initial data is evolved using the dynamical equations to find a solution of the Einstein equations. One particular class of singularities (which are simultaneously three and four dimensional) we do not discuss are those which arise when ρ at a point becomes unboundedly large.

Physically, it might be argued that the singular geometries we consider are irrelevant from the point of view of the asymptotically flat spacetime as they are disconnected from it. It could also be argued, however, that this feature is an artifact of the spherically symmetric model. If the symmetry is relaxed, the singular structures which arise will, typically, not disconnect the spatial geometry. This is not, however, our justification for considering singular structures. They are important for the reason indicated by the scaling argument. An infinitesimal change in ρ can convert a regular geometry into a singular one. The description of the configuration space is incomplete if such singular geometries are ignored.

A remarkable feature of spherically symmetrical geometries is the positivity of a well-defined quasilocal mass (QLM) whenever the spatial geometry is regular and the sources satisfy the dominant energy condition [1,6]. Its converse is not, however, generally true; there are singular solutions of the constraints with a positive everywhere quasilocal mass [3]. A peculiarity of MS configurations is that they do satisfy the converse of the positive QLM theorem. If the QLM is positive everywhere, the geometry is necessarily nonsingular. We can now exploit the definition of the QLM to derive an analytical expression for the local form of the spatial metric in the neighborhood of a generic singularity. Surprisingly, the QLM remains finite at the singularity. The constraints guarantee that the scalar curvature is also finite everywhere, even if the geometry is singular. If there is a curvature singularity, it must show up as a divergence in the traceless component of the Ricci tensor. Nongeneric radial density profiles exist in which the geometry degenerates without any curvature singularity where R pinches off. The interior region behind the singularity can be interpreted as a regular closed universe.

Our understanding of the analytic form of the metric in the neighborhood of a singularity, facilitates the identification of a supermetric on $\mathcal{C}$ in the manner of DeWitt [7]. Specifically, in Sec. III we identify a nonultralocal metric which assigns a finite norm to regular geometries and an infinite norm to singular geometries. With respect to this metric, the boundary in $\mathcal{C}$ separating singular geometries from nonsingular ones can be consigned to infinity.

A surprising consequence of the converse of the positive QLM theorem is that if ρ is strictly monotonically decreasing, the geometry cannot be singular (if it is regular at the origin). What this demonstrates is that ρ can be scaled by an arbitrarily large factor on a given support and yet the geometry remain regular everywhere. This counterexample demonstrates clearly that the scaling argument breaks down as a description of the development of singularities. For while an apparent horizon always develops at some critical scaling, a singularity does not necessarily develop. Such geometries may not be of great physical interest because we would expect that any classical singularity that might form in such a collapsing configuration would form at its center where we have assumed that the geometry is regular. However, monotonicity does provide a useful boundary in $\mathcal{C}$ separating geometries with very different properties with regard to singularity formation.

We will illustrate these points explicitly in Secs. IV and V using a piecewise constant density star.

We discuss the constant density star in greatest detail. The model does not admit curvature singularities; its only pathologies are metric degeneracies. Because of this one needs to be extra wary of making generalizations. We illustrate how a mechanical analogue can be used to study the solution space.

The local measures of energy and size which are most appropriate for measurements performed outside the star consist, respectively, of the QLM m and R at the surface of the star. What is clear already in this model, however, is that m provides a very poor measure of the material energy content just as R becomes a poor measure of the size when the geometry is highly curved. In particular, m (or R) may be infinitesimally small in geometries possessing apparent horizons.

It is true that an observer living outside the star does not have any choice but to exploit these “local” measures. Our goal, however, is to chart the configuration space, above all, in the neighborhood of configurations with apparent horizons and singularities. Because we are not asking the kind of question a remote orbiting observer in one of these geometries might ask, there is no reason why we should handicap ourselves with variables more appropriate to another scenario. The geometry behind a horizon is not out of bounds. It is in this spirit that we identify our global measures of energy content and size.

A measure which does appear to correspond quantitatively to the material energy content is the integrated energy density over the spatial volume M . The most appropriate measure of size is the proper radius of the support of matter, l_0 . The monotonicity of these two variables means that they fare better than m and R in the strong field region we are interested in. We examine in detail the dependence of M on l_0 .

A two-density model is discussed in Sec. V. We demonstrate explicitly that when the inner density exceeds the outer one, the model can never possess a singularity no matter how great the density or large its support. If, on the other hand, the outer density is the greater, singularities can appear if the star is large enough. If the outer density is tuned appropriately, the curvature singularity

becomes a metric degeneracy.

In Sec. VI we follow Bizon, Malec, and Ó Murchadha (BMÓM) by concentrating on the formulation of global necessary and sufficient conditions determining when the spatial geometry will possess apparent horizons and singularities [8–10]. The motivation behind this work was Thorne’s hoop conjecture [11]. These conditions are cast as inequalities involving appropriate global measures of the energy content of the matter distribution and of its support.

The sufficient conditions can be cast in the following form: if $M \geq Cl_0$, where C is a constant, then the star possesses a trapped surface for one constant, and a singularity for some larger constant within the sphere of proper radius l_0 . On the subset which corresponds to a monotonically decreasing density these inequalities can be strengthened. However, the latter inequality now provides a universal bound on M . To understand how this comes about suppose we are given some monotonically decreasing density, ρ_0 . While one might imagine that by scaling ρ_0 by some large constant we could make M arbitrarily large, this is misleading because the volume element appearing in M contains R^2 . Given ρ_0 , the constraint determines the value of R . When we solve the constraint with the scaled value of ρ , the net effect is to decrease R . This can occur in such a way that M , in fact, saturates.

In the original work of BMÓM, conformally flat coordinates were exploited. The attraction of these coordinates is that they do not rely on the spherical symmetry of the problem. The super-Hamiltonian constraint reduces to a nonlinear elliptic partial differential equation (PDE) on flat R^3 . This is offset, however, by the fact that the background geometry is no longer the physical geometry and one must contend with an unphysical bifurcation in the solution space of the elliptic equation. What is more, the existence of the inequalities discovered by BMÓM is not obvious when the constraints are expressed in this way. Their derivation demonstrated a considerable level of ingenuity. By expressing the constraints directly in terms of physical quantities we are able to provide very simple alternative rederivations of the BMÓM inequalities which are both more economical and transparent than theirs.

The shortcomings of M become apparent (even in the constant density model) when we attempt to provide necessary conditions on the specification of the boundaries partitioning the configuration space. For, if $M \leq Cl_0$, C a constant, on the surface of the star, for some constant no matter how small, even if it is true that the surface is neither trapped nor singular, this is not true of the interior. What is worse, we find it impossible to provide even this weak form of necessary condition involving the surface geometry when currents flow [3]. This contrasts dramatically with the relatively straightforward generalization of the sufficiency conditions in this context.

A more appropriate measure of the energy content of the star for phrasing the necessary condition is provided by the maximum value of the energy density $\rho_{\max}$. In Sec. VII we demonstrate that if $\rho_{\max} l_0^2 < \text{constant}$, then the interior is free of trapped surfaces for one constant,

and free of singularities for some other larger constant. The proof requires the introduction of a battery of simple Sobolev and other inequalities. These bounds possess extremely nontrivial generalizations to $J \neq 0$.

As a demonstration of the effectiveness of our method, in Sec. VII we rederive the BMÓM proof of Arnowitt, Deser, and Misner's conjectured inequality placing a lower bound on the binding energy of a compact spherically symmetric system [10].

II. GENERIC ANALYTIC FEATURES OF SOLUTIONS

The Hamiltonian constraint provides a second-order nonlinear ODE for R which is regular everywhere except² at $R = 0$:

$$RR'' - \frac{1}{2}[1 - (R')^2] + 4\pi R^2 \rho = 0. \quad (2.1)$$

The closure of the geometry at its base point $R(0) = 0$, is the only boundary condition we are free to impose. We will require that the geometry be regular at this point [1], so that

$$R'(0) = \pm 1. \quad (2.2)$$

Otherwise, it is clear that R'' would be infinite and the geometry singular at the origin. R can be either positive or negative—the physical metric only depends on R^2 . However, the only way R can change sign is by passing through a singularity or degeneracy with $R = 0$ at some nonvanishing value of l . If the geometry is regular, the sign of R will not change. Our convention will be to choose the positive sign at $l = 0$.

Because Eq. (2.1) is singular at the origin we need to differentiate the equation $n - 1$ (not $n - 2$) times to determine the n th derivative at this point, $R^{(n)}(0)$. When we do this we obtain the power series expansion for $R(l)$ in the neighborhood of $l = 0$:

$$R(l) = l - \frac{4\pi\rho(0)}{9}l^3 + O(l^4). \quad (2.3)$$

The most convenient way to solve Eq. (2.1) is to exploit the fact that the equation possesses a first integral equating the quasilocal mass defined at a given value of l in terms of the interior energy content. In paper I, this was done in a completely general gauge invariant context in which neither K_{ab} nor J need be zero [1]. If we set both K_{ab} and J equal to zero in Eqs. (4.1) and (4.2) of [1] we obtain³

²The Hamiltonian constraint will also assume this form in either of the gauges, $K_R = 0$ or $K_{\mathcal{L}} + 0.5K_R = 0$ which mimic a MS configuration (MSC) (see paper I). In the latter case, however, apparent horizons will not coincide with minimal surfaces and we need to be careful when we carry over the MSC analysis.

³That this is also true in the polar gauge discussed in paper I is obvious. It is also true in the other mimicking gauge ($\alpha = 0.5$) but we must remember that what we call $m(l)$ in (2.4) is no longer the QLM.

$$(R')^2 = 1 - \frac{2m(l)}{R}, \quad (2.4)$$

where $m(l)$ is given by

$$m(l) = 4\pi \int_0^l dl R^2 R' \rho. \quad (2.5)$$

Equations (2.4) and (2.5) are a first integral of (2.1) which implement the condition of regularity at the origin, Eq. (2.2), explicitly.

Outside the support of matter $m(l)$ coincides with the Arnowitt-Deser-Misner (ADM) mass, m_∞ and

$$(R')^2 = 1 - \frac{2m_\infty}{R}, \quad (2.4')$$

If $\rho = 0$ everywhere, $m = 0$ and space will be flat, $R(l) = l$. If m_∞ is positive and R is large the second term on the right-hand side of (2.4') becomes negligible and $R \sim l$ to leading order. Asymptotic flatness is guaranteed by the constraint. The “boundary condition” at infinity does not need to be prescribed by hand. The ADM mass is encoded in the next-to-leading order:

$$R \sim l - m_\infty \ln l. \quad (2.6)$$

Because m_∞ is positive R' must tend to one from below.

The power series (2.3) would appear to suggest that, once regularity at the origin is implemented, the solution will remain regular (regardless of the singularity of the differential equation at $R = 0$) so long as $\rho(l)$ remains finite. Certainly, if ρ and its support are small, on physical grounds we would expect the solution to be regular with R increasing monotonically with l between $l = 0$ and infinity. If ρ (or its support) is sufficiently large, however, $R(l)$ need not continue to increase monotonically with l . We can imagine doing this by scaling on some initial small value. Beyond some critical scale factor, the curvature of the geometry will be sufficiently large that R' will vanish at some value of l , an apparent horizon forms, outside which $R(l)$ begins to decrease. Note that this behavior is consistent with the convexity of $R(l)$ given by (2.3) in the neighborhood of the origin. We will be more precise below.

What can go wrong beyond this point is that R might not recover but will continue decreasing until it reaches zero (where the constraint equation is singular) at some nonvanishing value of l . If we extrapolate the first two terms in Eq. (2.3) to larger l we see that $R(l)$ returns to zero when $\rho(0)l_0^2 = 9/4\pi$ —in reasonable agreement with the value for the constant density star discussed in Sec. IV. The important point is that this is the only way that the spatial geometry can be singular.⁴

In Sec. V of paper I, we proved the positivity of the QLM, m : if the geometry is regular and the weak energy

⁴Even when ρ is large, $R(l)$ cannot diverge at any finite value of l . This is because this would require R' to diverge whereas the only place it can diverge is at the singular point of the differential equation $R = 0$.

condition is satisfied everywhere then $R'^2 \leq 1$ everywhere and, as a consequence,⁵ $m \geq 0$.

Obviously, $m(l)$ goes to zero as one approaches the origin. It does so faster than R so that R' approaches unity. Starting from the origin and moving outward, we see that $m(l) = 0$ and $R' = 1$ until we run into some nonzero ρ . If the support of ρ is bounded away from the origin we get a locally flat region around the origin. Once we run into nontrivial ρ , $m(l)$ must increase to reach some positive value. If $m(l)$ is bounded from below by some positive constant value for all positive values of l outside a neighborhood of the origin then $R(l)$ cannot become small in this region if $K_{ab} = 0$. This is because this would imply that the second term on the right-hand side of Eq. (2.4) must exceed the first at some point which is impossible. Thus the geometry cannot become singular.

We can therefore claim that $m(l) > 0$ everywhere except at the origin or in a neighborhood of it if and only if the geometry is nonsingular. This neighborhood, of course, covers the whole space only in the trivial case where $\rho \equiv 0$ and we have flat space. The set of geometries with $m(l) = 0$ somewhere away from the origin is a set of zero measure and will be treated separately below after we have discussed the constant density star.

What is the analytical description of a generic singularity? Suppose that $R'(l) < -1$, or equivalently $m(l) < 0$ at some point l . Equation (2.1) tells us that $R'' < 0$. Hence R' must become more negative. R must return to zero at some finite value l_S beyond l , $R(l_S) = 0$. If ρ is bounded on the interval $[0, l_S]$ it is clear that $m(l_S)$ reaches a finite negative value. Let us assume that R goes to zero like $(l_S - l)^\gamma$, where γ is some positive constant. We then expect R' to behave like $(l_S - l)^{\gamma-1}$. From Eq. (2.4) we see that $m(l)$ goes like $(l_S - l)^{3\gamma-2}$ and that $m'(l)$ should go like $(l_S - l)^{3\gamma-3}$. However, if we differentiate Eq. (2.5) we find that $m'(l)$ equals $4\pi R^2 R' \rho$ and this means it should behave like $(l_S - l)^{3\gamma-1}$. This is inconsistent. The only way out is if $3\gamma = 2$. This means that $m(l)$ goes to a constant and when we differentiate we do not get the $3\gamma - 3$ term.

Near l_S , the second term in Eq. (2.4) will dominate so that

$$(R')^2 \sim -\frac{2m(l_S)}{R}. \quad (2.7)$$

This equation can be integrated to give

$$R \sim \left(\frac{3}{2}\right)^{2/3} [-2m(l_S)]^{1/3} (l_S - l)^{2/3}, \quad (2.8)$$

entirely consistent with the argument above. R' becomes unbounded at l_S as $-(l_S - l)^{-1/3}$. If, therefore, $R'^2 > 1$ ($R' < -1$) anywhere it must subsequently diverge. It is clear from (2.5) that not only does m remain finite at l_S ,

so also does the integrated $\rho R^2 R'$.

All points with $l = l_S$ are zero distance apart so they must be identified. The geometry now contains one of Wheeler's bags of gold [12] behind the singularity. Because the singularity cuts off the interior geometry, it is customary in the present context to discard the solution as unphysical. The exterior does, however, also contain this singularity.

Generically R' will diverge as we approach a "singularity." We will call this a strong singularity. If, however, we fine-tune the density so that $m(l_S) = 0$ then $R'(l_S) = -1$, and $R''(l_0) = 0$, the bag closes smoothly and there are no singularities in the curvature. We can always approximate such a geometry arbitrarily closely by a regular geometry. We will refer to this as a weak singularity.

A beautiful consequence of the converse of the positive QLM theorem is that it permits us to identify an important class of nonsingular geometries. We note that, by a simple integration by parts, $m(l)$ can be expressed in the alternative form

$$m(l) = \frac{4\pi}{3} R^3 \rho - 4\pi \int_0^l dl R^3 \rho'. \quad (2.5')$$

If the energy density is monotonically decreasing, $\rho' \leq 0$, then

$$m(l) \geq \frac{4\pi}{3} R^3 \rho. \quad (2.9)$$

In particular, if ρ is a monotonically decreasing function of l , and it is strictly monotonically decreasing in some neighborhood of the origin then $m(l) > 0$, and so the geometry is necessarily nonsingular.

We note that neither the converse of the positivity of the QLM nor the consequence survive when $J \neq 0$. In general, a positive value of $m(l)$ everywhere does not guarantee the nonsingularity of the geometry.

What is the geometrical nature of a generic spatial singularity? Even though the geometry might be singular we assume ρ is always finite. The constraint therefore guarantees that the scalar curvature remains finite. The metric singularity must therefore show up in the Ricci tensor. We can express

$$\mathcal{R}_{ab} = \mathcal{R}_{\mathcal{L}} n_a n_b + \mathcal{R}_R (g_{ab} - n_a n_b), \quad (2.10)$$

in the same way as we did K_{ab} . All curvature scalars can be expressed in terms of $\mathcal{R}_{\mathcal{L}}$ and $\mathcal{R}_R$. In Appendix A we show that

$$\mathcal{R}_{\mathcal{L}} = \frac{\mathcal{R}}{2} - \frac{1}{R^2} (1 - R'^2), \quad (2.11a)$$

$$\mathcal{R}_R = \frac{\mathcal{R}}{4} + \frac{1}{2R^2} (1 - R'^2). \quad (2.11b)$$

While the scalars $\mathcal{R}_{\mathcal{L}}$ and $\mathcal{R}_R$ both diverge as one approaches a strong singularity, the sum $\mathcal{R} = \mathcal{R}_{\mathcal{L}} + 2\mathcal{R}_R$ remains finite; in fact, as we approach a strong singularity, $\mathcal{R}_{\mathcal{L},R} \sim \pm R'^2/R^2 \sim \pm (l_S - l)^{-2}$. The singularity need not occur on the support of matter. At a weak singularity $\mathcal{R}_{ab}$ remains finite.

⁵The contrapositive of this statement is that if $m(l) < 0$ at some l , the geometry must become singular at some point beyond l .

III. TO THE SINGULAR BOUNDARY OF CONFIGURATION SPACE

A. Supermetrics

In this section we construct a supermetric on the configuration space $\mathcal{C}$ of spherically symmetric asymptotically flat geometries satisfying the Hamiltonian constraint at a moment of time symmetry. Such geometries are completely characterized by the function $\rho(l)$. We can introduce a flat line element on $\mathcal{C}$,

$$\|\delta\rho\|^2 = \int_0^\infty dl \delta\rho(l)^2, \quad (3.1)$$

which is ultralocal with respect to ρ . The flatness of the metric makes life very convenient in that it is then simple to determine the “distance” between finitely separated geometries. The shortcoming of the metric (3.1) is that it does not tell us anything about the structure of the underlying geometry.

To do this we need to construct a line element which depends explicitly on the geometry. In the full theory, the commonly accepted choice is the ultralocal DeWitt supermetric

$$\|\delta g\|_{\text{DeW0}}^2 = \int d^2x \sqrt{g} G^{ab\,cd} \delta g_{ab} \delta g_{cd}, \quad (3.2)$$

where $G^{ab\,cd} = \frac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc}) + Cg^{ab}g^{cd}$ and C is some constant which is determined by the requirement that $G^{ab\,cd}$ be positive definite. On the subset of spherically symmetric geometries, $\|\delta R\|_{\text{DeW0}}$ is given, up to an irrelevant constant, by

$$\|\delta R\|_{\text{DeW0}}^2 = \int_0^\infty dl \delta R(l)^2, \quad (3.3)$$

which, remarkably, is also flat. Note that the DeWitt supermetric is not itself flat. The metric it induces on the subspace of spherically symmetric geometries is flat.

We note that with respect to any other parametrization of the spatial metric, the form assumed by the DeWitt supermetric will be more complicated. For example, with respect to the Schwarzschild parametrization (when valid) in the notation of I,

$$\|\delta\mathcal{L}\|_{\text{DeW0}}^2 = \int_0^\infty dR \frac{\delta\mathcal{L}(R)^2}{\mathcal{L}}. \quad (3.4)$$

Unfortunately, no matter how we parametrize it, the norm on any asymptotically geometry, even the flat geometry, is infinite with respect to the ultralocal metric equation (3.2) simply because $R \sim l$ at infinity. In fact the distance between any two geometries with differing values of m will also be infinite with respect to $\|\delta R\|_{\text{DeW}}$. The reason for this is that the asymptotic behavior given by Eq. (2.6) implies that $R_1 - R_2 \sim (m_{\infty 1} - m_{\infty 2}) \ln l$ which is not square integrable. It is not, however, clear if this is anything to worry about. m_∞ , after all, is conserved and, classically at least, different values of m correspond to subsets of $\mathcal{C}$ which do not intersect. Even

quantum mechanically, it would appear that the theory must be constructed with a fixed value of m_∞ .

It is, however, simple to construct a metric which gives a finite norm between geometries with different m_∞ 's. We do this by relaxing the criterion of nonultralocality by considering a metric involving a finite number of derivatives of R . If this seems like a high price to pay, one should note that in terms of ρ , even the DeWitt ultralocal metric is extremely nonlocal. Consider the metric

$$\|\delta R\|_{\text{DeW1}}^2 = \int_0^\infty dl |\delta R'(l)|^2. \quad (3.5)$$

Now $R'_1 - R'_2 \sim (m_{\infty 1} - m_{\infty 2})/l$ which is square integrable. Even still, the norm of any geometry is still infinite with respect to the nonultralocal metric we have introduced. Furthermore, none of the metrics we have considered so far, discriminates against the kind of singular geometries which show up as solutions of the constraints. In particular, it is clear from Eq. (2.8) that R' is square integrable over any compact interval, whether the geometry is singular or not. The simplest metric that solves both problems is one which is second order in derivatives⁶

$$\|\delta R\|_{\text{DeW2}}^2 = \int_0^\infty dl |\delta R''(l)|^2. \quad (3.6)$$

Now $\|\delta R\|_{\text{DeW2}}$ is finite on any nonsingular geometry that satisfies the constraints. Equation (2.8) indicates that, with respect to this metric, not only are strongly singular geometries pushed out to infinity, they are also rendered infinitely distant from any nonsingular geometry. What is also very attractive is that, with respect to this metric, the norm of the flat geometry $R(l) = l$ is zero. However, this might be disconcerting for those who advocate a zero geometry as the ground state. While the flat metric in ρ also does this, it fails to penalize singularities.

It is possible to introduce a metric which is geometrically more satisfying, which is also second order in derivatives by casting it directly in terms of differences in the curvature scalars. Unfortunately, the price we pay is that the resulting metric is no longer flat. In addition, the simplest such choice

$$\|\delta\mathcal{R}\|^2 = \int_0^\infty dl R^2 |\delta\mathcal{R}(l)|^2 \quad (3.7)$$

is a step back because it fails to discriminate against singular geometries. This is solved by considering the metric ($C = 0$ in the DeWitt supermetric)

⁶In general, on any compact support, these metrics satisfy the sequence of (Poincaré) inequalities: $\|\delta R\|_{\text{DeW0}} \leq \|\delta R\|_{\text{DeW1}} \leq \|\delta R\|_{\text{DeW2}} \leq \dots$

$$\begin{aligned} \|\delta\mathcal{R}_{\text{Rice}}\|^2 &= \int_0^\infty dl R^2 G^{ab cd} \delta\mathcal{R}_{ab} \delta\mathcal{R}_{cd} \\ &= \int_0^\infty dl R^2 [|\delta\mathcal{R}_{\mathcal{L}}(l)|^2 + 2|\delta\mathcal{R}_{\mathcal{R}}(l)|^2]. \end{aligned} \quad (3.8)$$

B. The singular boundary

Perhaps, the most important partition of $\mathcal{C}$ is into geometries which possess one or more singularities and those which do not, $\mathcal{C}_0$. $\mathcal{C}_0$ is clearly open. Within $\mathcal{C}_0$, lie all geometries corresponding to (strictly) monotonically decreasing $\rho(l)$. With respect to any of the metrics we have introduced on $\mathcal{C}$, two stars with monotonically decreasing energy density profiles can be arbitrarily far apart. $\mathcal{C}_0$ is therefore unbounded, at least in some directions, with respect to this metric. The boundary $\mathcal{S}$ appears then to be disconnected. Our task will be to begin to characterize it.

It is not clear if we should reject singular geometries as acceptable points on the configuration space. Should we treat the surface separating singular geometries from nonsingular ones as the boundary of the physical configuration space? If this is so then we are faced with a major problem. Technically, it appears unlikely that we will ever be in a position to identify $\mathcal{S}$ exactly. In the quantum theory we would be required to implement a boundary condition on the wave function on $\mathcal{S}$. What is to be hoped for is that the potential determining the dynamics in the configuration space will be sufficiently steep in almost all directions as we enter the grey area in the neighborhood of this boundary that wave packets will get reflected back onto the support of “good” geometries. An analogy which might be useful is the manner in which all wave packets which enter the Taub channels in homogeneous minisuperspace quantum gravity appear to get reflected back into the generic mixmaster minisuperspace [13].

There is another natural partition within $\mathcal{C}_0$, with boundary $\mathcal{T}$ into geometries which possess trapped surfaces and those which do not, $\mathcal{D}_0$. One would expect that $\mathcal{D}_0$ is bounded with respect to any reasonable norm. A geometry can only be singular if $R' = 0$ somewhere so that it also possesses a trapped surface. That there is no natural supermetric which discriminates against geometries possessing an apparent horizon indicates the very different natures of the boundaries $\mathcal{S}$ and $\mathcal{T}$.

IV. THE MOMENTARILY STATIC CONSTANT DENSITY STAR

A. The exact solution

In general, we cannot solve Eq. (2.1) exactly for arbitrary $\rho(l)$. What we can do is solve a few simple exactly solved problems consisting of piecewise constant values of ρ . Such models can clearly approximate a generic energy profile arbitrarily closely. We can therefore exploit them as a guide to identifying the generic features of solutions.

The simplest model there is is the constant density star.⁷ If ρ is a constant in some interval $l \leq l_0$ containing the origin we can recast Eqs. (2.4) and (2.5) as the single equation

$$R'^2 + \frac{8\pi\rho}{3} R^2 = 1 \quad (4.1)$$

in this region. Outside the support of matter,

$$R'^2 + \frac{2m_\infty}{R} = 1, \quad (4.2)$$

where $m_\infty = (4\pi/3)\rho R(l_0)^3$ is the ADM mass. We note that both R and R' are continuous across $l = l_0$.

We can analyze Eqs. (4.1) and (4.2) by drawing on the analogy with the one-dimensional motion of a particle (l is time) with unit “energy” in a time dependent potential $V(R, l)$ which changes from a harmonic oscillator to a Coulomb potential at $l = l_0$:

$$V(R, l) = \begin{cases} \frac{8\pi\rho}{3} R^2, & 0 \leq l \leq l_0, \\ \frac{2m_\infty}{R}, & l_0 \leq l. \end{cases} \quad (4.3)$$

This analogy will be particularly useful below when we examine less trivial models. In the present case, the particle starts out at the origin $R = 0$ at $l = 0$ with unit “velocity” $R' = 1$. The classical turning point marking the position of an apparent horizon (if one exists) is given by $R' = 0$ or $R_c^2 = 3/(8\pi\rho)$. This represents the Schwarzschild radius of the inner apparent horizon. Inside matter, $R = R(l)$ is given by

$$l = \int_0^R \frac{dR}{\sqrt{\left[1 - \left(\frac{R}{R_c}\right)^2\right]^{1/2}}} = R_c \arcsin\left(\frac{R}{R_c}\right) \quad (4.4)$$

or $R = R_c \sin(\pi l/2l_c)$, where l_c is given by $l_c = (\pi/2)R_c$. This is simply the metric on a round three-sphere S^3 of radius R_c . We note that the interior of the constant density star is an Einstein geometry, $\mathcal{R}_{ab} = \frac{1}{3}\mathcal{R}g_{ab}$. Thus the nonsingularity of $\mathcal{R}$ guarantees that of $\mathcal{R}_{ab}$. This is the unique such geometry. This is because all Einstein manifolds must possess constant $\mathcal{R}$ in dimensions greater than two (for otherwise it would be inconsistent with the contracted Bianchi identities) and in $d = 3$, the unique spherically symmetric solution with positive $\mathcal{R}$ and with the given topology is part of the round S^3 .

The qualitative behavior of the solution depends on the size of the star, l_0 .

If $l_0 < l_c$, R increases monotonically both inside and out and there is no apparent horizon.

If $l_0 = l_c$, an apparent horizon forms on the surface of the star; R is monotonic everywhere. We note that R'' is discontinuous here. Generically, we would expect both R' and R'' to vanish at this bifurcation.

⁷We should point out that while the concept of a constant density does not depend on the spatial system of coordinates it does depend on the foliation.

If $l_0 > l_c$, R increases monotonically inside the star until it reaches a maximum at the (inner) apparent horizon at $l = l_c$ at which $R = R_c$. Beyond l_c , R begins to decrease inside the star.

If $l_0 \geq 2l_c$, R returns to zero at $l = 2l_c$ inside the star at which point $R' = -1$. There is a bag of gold singularity contained within the support of matter. This is the only way that the geometry can exhibit a singularity in the uniform density model. The model does not therefore display generic singular behavior. This is its principle shortcoming.

If $2l_c \geq l_0 \geq l_c$ we exit the star before a singularity is reached. However, R' is continuous at the boundary so that R continues decreasing outside the star until an exterior horizon is reached at a value of R determined by $R' = 0$ in Eq. (4b) or $R = 2m_\infty$.

We note that the constraints only admit single bubble nonsingular configurations. Any higher number of bubbles is necessarily singular.

1. Trapped surfaces and singularities: Necessary and sufficient conditions

What is most interesting about this model is that it indicates that there exist bounds on the product of the energy density by the square of the maximum support of a given uniform distribution of matter beyond which the associated geometry (i) possesses a trapped surface and (ii) is singular.

The surface of the star first becomes trapped (an apparent horizon forms) when $l_0 = \pi R_c/2$ so that $\rho l_0^2 = 3\pi/32$. If $\rho l_0^2 \geq 3\pi/32$ then the star will contain a trapped surface (either the surface of the star will be trapped or a trapped surface is contained in its interior) and conversely.⁸

The surface of the star will be singular when $l_0 = \pi R_c/2$ or $\rho l_0^2 = 3\pi/8$. If $\rho l_0^2 \geq 3\pi/8$ then the star will contain a (weak) singularity (either on its surface or in its interior) and conversely. We summarize:

$$\rho l_0^2 \geq \frac{3\pi}{32} \text{ iff (i); } \rho l_0^2 \geq \frac{3\pi}{8} \text{ iff (ii).} \quad (4.5)$$

The necessary and sufficient conditions formulated at l_0 not only coincide but they also make a nontrivial statement about the interior physics.

B. M vs l_0 , m vs l_0 , and the characterization of trapped surfaces and singularities in terms of these variables

Unfortunately, the unambiguous partitions of the configuration space represented by Eq. (4.5) are a very spe-

cial feature of the constant density star. If the density is not constant, it will not be possible in general to phrase both the necessary and the sufficient conditions with respect to the same set of variables. When it is possible, the inequalities analogous to Eq. (4.5) typically will differ. The if statement will be stronger than the only if statement. In addition, our ability to make nontrivial statements about the interior based on surface measurements will, in general, depend on the choice of variables.

We need to identify appropriate variables to characterize the material energy contained within a sphere of fixed proper radius when the energy density is not constant. Typically, this will be some (weighted) mean value of ρ over the interior volume. The two measures we introduced in paper I were the QLM m and the material energy M .

The simplest global measure of the material energy content of a spherically symmetric system is

$$M(l_0) = 4\pi \int_0^{l_0} dl R^2 \rho. \quad (4.6)$$

Let us examine M as a function of l_0 at constant ρ in the constant density model. We have

$$\begin{aligned} M &= 4\pi\rho \int_0^{l_0} dl R^2 \\ &= \frac{3l_0}{4} \left[1 - \frac{R_c}{2l_0} \sin\left(\frac{2l_0}{R_c}\right) \right]. \end{aligned} \quad (4.7)$$

What is more relevant, however, is the ratio $M(l_0)/l_0$. This is illustrated in Fig. 1.

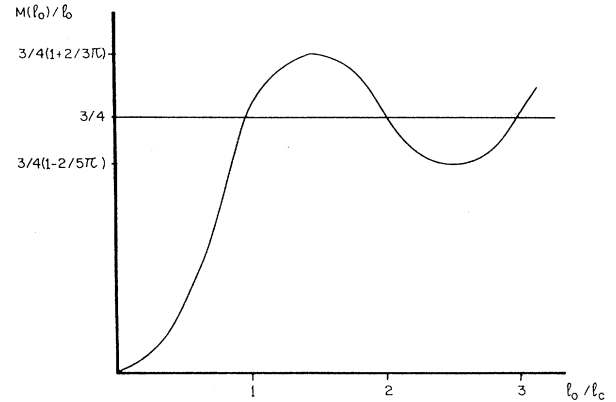


FIG. 1. $M(l_0)/l_0$ vs l_0 . Whereas M is monotonic in l_0 by construction, the ratio M/l_0 is not—even before the geometry becomes singular it begins to oscillate. The ratio rises monotonically for small values of l_0 , regardless of the formation of an apparent horizon until a maximum value is reached when the surface becomes a decreasing point of inflection of $R(l)$. Contrary to what we might have expected, the maximum does not obtain with the appearance of a singularity. As the star is made even larger, the ratio begins to decrease until it reassumes the values $\frac{3}{4}$ as the surface becomes singular. Subsequently, the ratio oscillates with decreasing amplitude with one complete oscillation between each occurrence of a singularity. As $l_0 \rightarrow \infty$, the ratio tends to the constant value $\frac{3}{4}$ independent of ρ .

⁸We need not worry about trapped surfaces in the region exterior to the star as they will always be attended by an interior one.

In this model it is already evident that crucial information is lost in replacing ρ by M to characterize the energy content of the star. The function $l \rightarrow M(l_0)/l_0$ is neither surjective nor injective—it is bounded and (because the ratio oscillates) it is not unique. The reason for this, as we will discuss further below, is the folding of R into the definition of M .

Let us attempt to formulate necessary and sufficient conditions for the presence of trapped surfaces and singularities in a constant density star using $M(l_0)$ as our measure of material energy to see how it fares. We should not be surprised that we are unable to do as well as we did with $\rho - \rho$ and l_0 after all completely characterize the star, whereas $M(l_0)$ and l_0 do not.

1. Trapped surfaces

An apparent horizon forms at the surface of the star when $M(l_0) = 3l_0/4$.

Sufficiency: If

$$M(l_0) \geq 3l_0/4 \quad (4.8)$$

the star will contain a trapped surface. This condition is sharp. The surface itself need not be trapped. However, if it is not then the interior must be singular and consequently must contain an apparent horizon. So far so good.

The tightest inequality describing necessity is considerably weaker: If

$$M(l_0) < \frac{3l_0}{4} \left(1 - \frac{2}{5\pi}\right), \quad (4.9)$$

the star does not contain a trapped surface. An important point is that the inequality (4.9) describing necessity is weaker than the one describing sufficiency. In fact, inspection of Fig. 1 indicates that this inequality is not sharp and, as such, of limited value for there are stars with $M(l_0)$ exceeding the value on the right-hand side (RHS) of Eq. (4.9) without any trapped surfaces. If it had made sense to demand that the interior be nonsingular in the hypothesis, then, when $M(l_0) < 3l_0/4$, the star would not contain a trapped surface. The possibility of singularities lurking in the interior beyond which the ratio $M(l_0)/l_0$ is decreasing is what really limits our ability to formulate the necessary condition in terms of $M(l_0)$ and l_0 .

For trapped surfaces we can conclude that M is a good variable for the formulation of a sufficiency condition but not for the formulation of a necessary one.

2. Singularities

The surface of the star will be singular when the ratio of $M(l_0)$ to l_0 is $\frac{3}{4}$, which is the same as the apparent horizon ratio. This first occurs, however, when $l_0 = \pi R_c/2$ or $\rho l_0^2 = 3\pi/8$ which is different from the apparent horizon value. The description in terms of ρ and l_0 clearly

provides a more precise characterization. When we attempt to provide necessary and sufficient conditions on $M(l_0)$ and l_0 to describe singularities we discover unforeseen complications. Not only are we unable to formulate conditions which coincide, we find that we cannot even formulate a consistent (nonvacuous) sufficiency condition of the form: if $M(l_0) \geq \alpha l_0$ for some α , then the star will contain a singularity. What we possess instead is a universal bound on the ratio

$$M(l_0)/l_0 \leq \frac{3}{4} \left(1 + \frac{2}{3\pi}\right) \quad (4.10)$$

which holds for all geometries which satisfy the constraints, whether they are (weakly) singular or not. The geometry which saturates the inequality is unique. It is the configuration packing the maximum material energy into a given proper radius. It is nonsingular but does contain a trapped surface.

The only necessary condition we can provide with these variables is the somewhat trivial one implied by the corresponding trapped surface condition. If the star contains a singularity, it will also contain a trapped surface. However, if this criterion was poor for trapped surfaces, it is even poorer as a characterization of singularities.

The reason why we are unable to provide a sufficiency condition in terms of $M(l_0)$ and l_0 which is satisfied by a nonempty set is subtle and not only a reflection of the inadequacy of these variables to describe nonsingular geometries. We have already noted that a star with a monotonically decreasing ρ cannot be strongly singular. If the decrease in ρ is strict, neither can it be even weakly singular. We will demonstrate in Sec. VI that a sufficient condition for the presence of a strong singularity is $M(l_0) > 2l_0$. We can conclude that for all geometries with⁹ $\rho' \leq 0$, $M(l_0) \leq 2l_0$. The sufficiency condition provides a universal bound on this subset of the configuration space. The existence of the bound is not simply an artifact of the constant density model.

In this regard we should stress another distinction between the choice of ρ and that of $M(l_0)$ as a measure of the material energy which is that with ρ one can distinguish between weakly singular and nonsingular geometries whereas with $M(l_0)$ one cannot. The sufficiency condition $\rho l_0^2 \geq 3\pi/8$ is, however, unstable. For if we subject the constant density to a strictly monotonically decreasing perturbation, the subset satisfying this condition also becomes empty.

3. $m(l_0)$ vs l_0

If $M(l_0)$ fares poorly for the purpose of characterizing the interior geometry, the QLM $m(l_0)$ will fare even more poorly (see Sec. VII of paper I). To demonstrate this explicitly let us compare the analytical form for M , Eq. (4.7), with that for m . We have

⁹In Sec. VIB we will show that we can do better.

$$m(l_0) = \frac{4\pi}{3} \rho R(l_0)^3 = \frac{R_c}{2} \sin^3 \frac{l_0}{R_c}. \quad (4.11)$$

$m(l_0)$ is bounded for a given ρ , $m \leq R_c/2 = \sqrt{3/(32\pi\rho)}$, assuming its maximum when a horizon forms at l_0 , decreasing thereafter as a function of l_0 as matter is deposited behind the outer horizon. It falls to zero with the appearance of a (weak) singularity. There are two nonsingular configurations corresponding to each value of $m(l_0)$ —one with, the other without an apparent horizon.

C. Binding energy in the constant density star

In Sec. VII of paper I, we saw that the “binding energy” $m - M$ is always negative. In the weak field limit we note that

$$M \sim \frac{1}{2} \frac{l_0^3}{R_c^2} - \frac{1}{10} \frac{l_0^5}{R_c^4} + \dots, \quad (4.12a)$$

while

$$m \sim \frac{1}{2} \frac{l_0^3}{R_c^2} - \frac{1}{4} \frac{l_0^5}{R_c^4} + \dots. \quad (4.12b)$$

The leading cubic terms in l_0 in both coincide. It is just the “rest” mass $m_0 = 4\pi\rho l_0^3/3$ associated with a uniform spherically symmetric distribution of matter in special relativity. Thus to leading order, the difference is given by

$$m - M \sim -\frac{3}{5} \frac{M^2}{l_0} \quad (4.13)$$

which is the Newtonian binding energy of a spherical uniform density star. We note that at the other extreme where the geometry is about to turn singular, $l_0 \sim \pi R_c$, $m = 0$, and $M \sim \frac{4}{3} M^2/l_0$. We note that in any nonsingular geometry the binding energy is bounded from below as follows [10] (see Sec. VII below):

$$M - m \geq \frac{1}{2} \frac{M^2}{l_0}.$$

For a constant density star, there is the tighter lower bound, $M - m \geq \frac{3}{5} M^2/l_0$. The coefficient on the RHS increases monotonically with l_0 . We conjecture that whenever $\rho' \leq 0$ this inequality will hold.

V. MSC'S OF THE TWO-PIECEWISE CONSTANT DENSITY STAR

The constant density model lies on the boundary of the subset of monotonically decreasing ρ , admitting weakly but not strongly singular geometries. Such weak singularities occur with zero measure in the set of all possible singular geometries. The constancy of ρ provides very strong global information on the surface concerning the interior.

While it is a very useful model to get us started, it is important to recognize these limitations in order to avoid making erroneous extrapolations. These limitations will be highlighted further by examining a slightly more complicated model.

A concrete model which does display generic behavior is a star consisting of an inner spherical ball of radius l_1 with energy density ρ_1 and an outer shell of thickness $l_2 - l_1$ with energy density ρ_2 :

$$\rho(l) = \begin{cases} \rho_1, & 0 \leq l \leq l_1, \\ \rho_2, & l_1 < l \leq l_2, \\ 0, & l_2 < l. \end{cases} \quad (5.1)$$

In the region $l \leq l_1$, the solution is just one of the solutions we obtained in our examination of the constant density star. In $l_1 < l \leq l_2$,

$$R'^2 + V(R) = 1, \quad (5.2)$$

where

$$V(R) = \frac{2\Delta m}{R} + \frac{8\pi\rho_2}{3} R^2, \quad (5.3)$$

$$\Delta m = \frac{4\pi}{3} R_1^3 (\rho_1 - \rho_2), \quad (5.4)$$

$$R_1 = R_{c1} \sin \left(\frac{l_1}{R_{c1}} \right), \quad (5.5)$$

and $R_{c1}^2 = 3/(8\pi\rho_1)$. Δm is a measure of the energy excess in the interior.

As we will see, the qualitative behavior of the solution in the region $l_1 < l \leq l_2$ depends very crucially on the sign of Δm . Let us therefore examine the two signs one at a time ($\Delta m = 0$ is trivial). We will suppose that the interior geometry is regular in both cases.

A. $\Delta m > 0$

Let us first suppose that $\Delta m > 0$. When $\Delta m > 0$, the potential $V(R)$ is bounded from below with a positive minimum at R_m given by

$$R_m = \left(\frac{3\Delta m}{8\pi\rho_2} \right)^{1/3}. \quad (5.6)$$

$V(R) = 1$ possesses two positive roots and $V(R_m) < 1$ whenever $\Delta m > 0$. This confirms that there is no critical positive value of Δm beyond which no solution exists. One way to show this is to note that $V(R_m) = 3\Delta m/R_m$ so that $V(R_m) \geq 1$ implies $\Delta m^2 \geq 1/72\pi\rho_2$. If we define $x = \rho_2/\rho_1$, the inequality $V(R_c) \geq 1$ can be expressed in the dimensionless form

$$\sin^3 \left(\frac{l_1}{R_{c1}} \right) (1 - x)^2 \geq \frac{4}{27} \frac{1}{x}. \quad (5.7)$$

Now, if $\Delta m > 0$ then $x < 1$ which is inconsistent with

Eq. (5.7).

The most important feature displayed by this solution is that the geometry cannot be singular on the interval $l_1 < l \leq l_2$ or outside the star. The “potential” provides a barrier keeping the solution away from $R = 0$. No matter how large the value of l_2 , the solution is periodic and nonsingular on the interval $l_1 < l \leq l_2$ oscillating between turning points given by the two positive roots of $V(R) = 1$. This provides a useful model of a (strictly somewhere) monotonically decreasing energy density. This should be contrasted with the uniform density star which turns singular beyond some critical value of l_0 . The inner higher density region has a stabilizing effect on the geometry of the outer region.

We note that the region $l \leq l_1$ can be viewed as a momentarily static slice through de Sitter space. The geometry in the interval $l_1 < l \leq l_2$ is the momentarily static slice through a Schwarzschild–de Sitter spacetime. If ρ_1 and ρ_2 are both constant in time, this initial data will generate these spacetimes in their respective intervals. There will be, in fact, an infinite number of such geometries with the same ADM mass which we can label by the number of maxima of R . The existence of such geometries might have implications in semiclassical quantum gravity. The semiclassical calculation of tunneling amplitudes involves a summation over e^{-S} , where S is the classical action evaluated on the Euclidean interpolation from one MSC to one of the infinite number of MSC’s with the same ADM mass. This sum could potentially result in a nontrivial amplification of the transition amplitude.

There is a theorem due to Schoen and Yau stating that if the volume (defined in terms of a certain embedded torus) supporting a given strictly positive ρ is made large in all directions, the geometry must be singular [14]. It would appear that our ability to make l_2 arbitrarily large without incurring any singularity is inconsistent with this result. However, a closer examination shows that this is not so. This is because even though the proper volume associated with a large value of l is large (increasingly linear with l), it is not large in all directions. It is essentially a cylinder of bounded radius. Thus, a large torus in the sense of Schoen and Yau does not fit.

B. $\Delta m < 0$

What is more dramatic is the geometry which corresponds to $\Delta m < 0$. The potential is now monotonic and unbounded from below within the interval $l_1 < l \leq l_2$.

If $l_2 < l_s$, where l_s is the proper radius at which $V(R(l_s)) = 0$ the geometry will be regular both inside and out. Let $R_s = R(l_s)$. R_s is given by

$$R_s = \left(\frac{3\Delta m}{4\pi\rho_2} \right)^{1/3} \quad (5.8)$$

and

$$l_s = l_1 + \int_{R_1}^{R_s} \frac{dR}{\sqrt{1-V(R)}}, \quad (5.9)$$

where $V(R)$ is given by (5.3) and (5.4), and R_1 is given by Eq. (5.5). The ADM mass $m_\infty = m(l_2)$. Therefore

$$\frac{2m_\infty}{R_2} = \frac{2\Delta m}{R_2} + \frac{8\pi\rho_2}{3}R_2^2. \quad (5.10)$$

When $l = l_s$, $R' = -1$ and the QLM vanishes, $m(l_s) = 0$. If $l_2 = l_s$ the geometry is flat outside. However, before it can become asymptotically flat, the geometry must close in a bag of gold. The empty top of the bag consists of a flat cap (in which $R' = -1$). $R(l) = 0$ at

$$l_S = l_s + R_s. \quad (5.11)$$

This is the same (nongeneric) kind of singularity we encountered in the uniform density model.

Recall that a regular closed bag requires the integrability condition $m(l_S) = 0$ to be satisfied. A necessary condition is that ρ assume a maximum (or a minimum) somewhere. The simplest realization of such a configuration involves three constant density slabs with $\rho_1 < \rho_2 > \rho_3$ (or $\rho_1 > \rho_2 < \rho_3$). To satisfy the integrability condition we will be required to tune the relative values of these densities and their supports appropriately. The model we are considering with $\rho_1 < \rho_2$ with a vacuum exterior represents a special case. The value l_S given by Eq. (5.11) is the unique critical value of l_0 supporting an everywhere regular closed cosmology for the given parameter values.

If $l_2 > l_s$, $R' < -1$ at $l = l_2$ and $m(l)$ will be negative for all subsequent l until a singularity is reached at some point l_S . If, in addition, $l_s \geq l_S$ where

$$l_S = l_s + \int_0^{R_s} \frac{dR}{\sqrt{1-V(R)}}$$

is finite, the geometry closes with a metric singularity $R' \rightarrow -\infty$ before we ever reach l_2 . In the neighborhood of $R = 0$, R satisfies Eq. (2.7) with singular solution (2.8). The local value of the QLM at the singularity is given by $m(l_S) = \Delta m$. Note that the quasilocal mass at the singularity is finite and determined completely by the remote behavior of the sources. If $l_2 < l_S$, the interior is fitted with a negative m exterior vacuum cap with the same singular behavior.

VI. APPROXIMATING THE BOUNDARIES IN $\mathcal{C}$

A. Inequalities of sufficiency and necessity

It is clear that a generic energy profile can, in principle, be approximated arbitrarily closely by a piecewise constant density model along the lines of our examination of the two simplest such models in Secs. IV and V. Even still, however, it is not obvious how to exploit such models to characterize the boundaries partitioning $\mathcal{C}$ which were discussed in Sec. III. The approach we will take is to establish necessary and sufficient conditions determining the circumstances under which the geometry can possess trapped surfaces or singularities [8,9]. If we could solve the problem exactly, as we did on the piecewise constant density subset, presumably the necessity and sufficiency

conditions could be formulated as if and only if statements with an appropriate choice of variables.¹⁰

Let us examine in very general terms how closely these necessary and sufficient conditions identify the (singular or nonsingular, say) partition $\mathcal{S}$ of $\mathcal{C}$. The necessary condition will provide us with a surface $\mathcal{S}_{\text{nec}}$ bounding some region in the configuration space $\mathcal{C}_{\text{nec}}$ interior to which we are assured that the geometry is nonsingular. The sufficiency condition will provide us with some other surface $\mathcal{S}_{\text{suff}}$ bounding a region in the configuration space $\mathcal{C}_{\text{suff}}$ exterior to which the geometry is always singular. $\mathcal{C}_{\text{nec}}$ will be some proper subset of $\mathcal{C}_{\text{suff}}$. The set difference of $\mathcal{C}_{\text{suff}} - \mathcal{C}_{\text{nec}}$ will not be empty, representing a grey area containing both singular and nonsingular geometries. As the constant density model illustrates, this discrepancy might be a consequence not only of our inability to solve the problem exactly but also of our inability to identify an optimal set of variables.

Ideally, we would like both the necessary and the sufficient conditions to be sharp, namely, that a configuration exists saturating the condition, though we do not necessarily expect the two to be sharp simultaneously.

B. Sufficiency and necessity in terms of M and l_0

Let us rewrite Eq. (2.1) in the form

$$4\pi\rho R^2 + (RR')' = \frac{1}{2}[1 + (R')^2] \quad (6.1)$$

and integrate out from $l = 0$ up to the surface value l_0 ,

$$M(l_0) + RR'|_{l_0} = \Gamma, \quad (6.2)$$

where we define

$$\Gamma \equiv \frac{1}{2} \int_0^{l_0} dl [1 + (R')^2]. \quad (6.3)$$

We now have two different representations for the first integral of the constraints. Earlier we exploited the expression Eq. (2.4), (2.5) which involves the definition of the QLM. Equation (6.2) involves $M(l_0)$ and will provide the basis for our derivations of all inequalities involving M .

1. Trapped surfaces: Sufficiency

In our examination of the constant density star we found that the sufficient condition for the presence of a trapped surface is as good as one could hope for; the

inequality (4.11) is sharp. This success is repeated for generic density profiles.

We begin by eliminating the term involving $R'(l_0)$ from Eq. (6.2) in favor of the expansion of the optical scalar $\Theta_- = \Theta = \Theta_+$ (see Appendix A of I): $\Theta = 2R'/R$. If the surface is not trapped, $\Theta(l_0) > 0$ and thus $M(l_0) < \Gamma$. We now require an upper bound on Γ . When $\rho \geq 0$ and the interior is regular, then $(R')^2 \leq 1$ so that

$$\Gamma \leq l_0. \quad (6.4)$$

Thus

$$M(l_0) < l_0. \quad (6.5)$$

In other words, if $M(l_0) \geq l_0$ the surface must be trapped or the interior must be singular in which case it also possesses a trapped surface.¹¹ BMÓM have demonstrated that this inequality is sharp. They do this by constructing a model in which $M(l_0) = l_0(1 - \epsilon)$ but the surface is not trapped.

It is possible to tighten the sufficiency condition whenever $\rho' \leq 0$. We exploit Eq. (2.4) to express Γ as a functional of $m(l)$:

$$\begin{aligned} \Gamma &= \frac{1}{2} \int_0^{l_0} dl [1 + (R')^2] \\ &= \int_0^{l_0} dl \left\{ 1 - \frac{1}{2} [1 - (R')^2] \right\} \\ &= \int_0^{l_0} dl \left[1 - \frac{m(l)}{R} \right]. \end{aligned} \quad (6.6)$$

We now exploit Eq. (2.9) which holds whenever $\rho' \leq 0$ to obtain

$$\Gamma \leq l_0 - \frac{4\pi}{3} \int_0^{l_0} dl \rho R^2 = l_0 - \frac{M(l_0)}{3}, \quad (6.7)$$

which is a tighter bound on Γ . Thus, in place of Eq. (6.5) we have

$$\frac{4}{3} M(l_0) < l_0. \quad (6.5')$$

Thus, if $M(l_0) \geq \frac{3}{4} l_0$ the surface of the monotonic configuration must be trapped or it contains a trapped surface.

A nice feature of Eq. (6.5') is that not only is it sharp, but we also know that it is saturated by the constant density star. In addition we note that it is easier to form a trapped surface when ρ is monotonically decreasing. This is to be contrasted with the impossibility of finding a singularity when ρ is monotonically decreasing. If we scale the value of ρ or its support, a critical point will always be reached at which a trapped surface forms. However, there is no corresponding point marking the formation of a singular geometry. This exposes the inadequacy of the

¹⁰Of course, one could argue that whenever the problem is exactly solvable, in principle at least, one could simply tick off geometries with $R' = 0$ or $R = 0$ at some finite value of l . We do not consider this as a solution to the problem because it is telling us nothing about the physics conspiring to produce these coincidences.

¹¹Note that in the derivation we did not have to assume that the interior did not contain a trapped surface.

intuitively reasonable notion that by increasing ρ on a constant support we first produce a trapped surface and then a singularity.

We note that the sufficiency condition can be cast in a weaker form in terms of the minimum energy density $\rho_{\min}$ using $\rho_{\min} V_0 \leq M(l_0)$ where V_0 is the volume of the star. In the monotonic star, this will be slightly more useful on account of the fact that $\rho_{\min}$ will be the surface energy density. More useful inequalities involving $\rho_{\min}$ can be derived by exploiting the calculus of variations in the manner of Schoen and Yau [14].

2. Trapped surfaces: Necessity

In the constant density star we found that M was a poor variable for formulating necessary conditions for the presence of trapped surfaces and of singularities.

In general, if the surface is trapped, $\Theta(l_0) \leq 0$. Then Eq. (6.2) implies that $M(l_0) \geq \Gamma$. We now require a lower bound on Γ . The weak lower bound on Γ , $\Gamma \geq l_0/2$, relies only on the fact that $(R')^2 \geq 0$ which is independent of the positivity of ρ . We get $M(l_0) \geq l_0/2$ on a trapped surface. Thus if

$$M(l_0) < \frac{l_0}{2} \quad (6.8)$$

the surface is not trapped. It is not sharp and it fails to prevent a trapped surface showing up in the interior. Compare Eq. (6.6) with the marginally better Eq. (4.9) we obtained for the constant density model which, in addition, guaranteed that no trapped surfaces showed up in the interior. The reason for the latter stronger statement is the fact that we were able to exploit the exact solution to feed global information into the surface condition. As an extreme example, consider a star with a high density core of radius l_1 and a large low density mantle. We might then have $M(l_1) \geq l_1$ but $M(l_0) \leq l_0/2$ on the surface. In other words, the surface information provides no clue as to the interior physics. For this reason the condition $\rho' < 0$ does not allow us to “tighten” the necessary condition the way it did the sufficiency condition. Technically, the inequality is operating the wrong way. What is possible is to tighten the necessary condition in a configuration with a radially increasing energy density $\rho' \geq 0$ to

$$M(l_0) \leq 3l_0/5. \quad (6.8')$$

This is the opposite extreme to the central high density core. It is harder to produce a trapped surface. However, even on this subset, we still cannot provide a strong form of necessary condition prohibiting trapped surfaces showing up in the interior. We note that the constant density star is also a special case of Eq. (6.8'). Equation (6.8') is not sharp because Eq. (4.9) is not. We will return to the issue of formulating necessary conditions below using $\rho_{\max}$ instead of M .

We will find a necessary condition analogous to the constant density result holds when we use $\rho_{\max}$ instead of M . The reason is that $\rho_{\max}$ taps into global information

without the distortions introduced by folding R into the definition of M .

3. Singular geometries: Inequality of sufficiency

Because M does not discriminate between the regular and the weakly singular geometries exhibited by the constant density model, we found that instead of a sufficiency condition for the presence of singularities, the inequality (4.10) represented a universal bound.

In general M does discriminate between geometries which are strongly singular and those which are not. Therefore it can provide a nonvacuous sufficiency condition for the presence of singularities. This bound was obtained by BMÓM [8,9]. On the monotonically decreasing density subset, it again provides a universal bound though not a sharp one.

To reproduce the sufficiency condition, let us suppose that the geometry is not strongly singular. Then $R'^2 \leq 1$, so that $\Gamma \leq l_0$ and Eq. (6.2) implies

$$M(l_0) + RR'|_{l_0} \leq l_0. \quad (6.9)$$

In addition, $R(l) \leq l$ everywhere and $R' \geq -1$. The surface term is therefore bounded from below by $-l_0$ and

$$M(l_0) \leq 2l_0. \quad (6.10)$$

If, in particular $\rho' \leq 0$, then this condition represents a universal bound on M in a MSC. It is not possible in such a model to raise the value of M indefinitely by increasing ρ while maintaining l_0 constant. What happens when we do this is that $\rho R_{\max}^2$ saturates within the star and as a consequence the increase in $M(l_0)$ is at most linear in l_0 . We will see how this occurs explicitly below in the context of a two density model. In a large outer low density region R does not grow with l_0 but oscillates between fixed minimum and maximum values. This is folded into the definition of M .

We can tighten the inequality whenever $\rho' \leq 0$ just as we did the trapped surface sufficiency condition. Let us do it in a slightly different way which exploits the Schwarzschild inequality explicitly. The Schwarzschild radius $R(l)$ provides a bound on the quasilocal mass $2m(l) \leq R(l)$ whenever the constraints are satisfied (whether the geometry is singular or not). This is a straightforward consequence of Eq. (2.4):

$$m - \frac{R}{2} = -(R')^2 \frac{R}{2}.$$

The right-hand side is bounded above by zero ($R'^2 \geq 0$). Thus

$$m - \frac{R}{2} \leq 0. \quad (6.11)$$

Equality obtains on an apparent horizon $R' = 0$ and when $R = 0$ where $m = 0$.

The condition (2.7) and the Schwarzschild inequality (6.11) together then imply that $\rho R^2 \leq 3/(8\pi)$ inside matter. Thus

$$M(l_0) = 4\pi \int_0^{l_0} dl R^2 \rho \leq \frac{3}{2} l_0.$$

It is curious that, unlike the corresponding trapped surface condition, this inequality is not sharp.

4. Singular geometries: Necessity

Suppose that the geometry possesses a singular surface at l_0 . Then $R(l_0) = 0$. What's more the product RR' appearing in Eq. (6.2) also vanishes at a singularity even though R' itself might diverge. Thus

$$M(l_0) = \Gamma \geq \frac{l_0}{2} \quad (6.12)$$

which is the same as the necessary condition for the formation of trapped surfaces. This condition is weak—it does not even demand the positivity of ρ . Neither does it say anything about the interior.

The necessary conditions we have obtained are far weaker statements than we would like, failing to eliminate the possibility of trapped surfaces or singularities lurking in the interior. What is clear by now is that $M(l_0)$ and l_0 are not the ideal variables for this purpose.

We turn now to the task of formulating alternative necessary conditions.

C. Necessity in terms of $\rho_{\max}$ and l_0

In the context of the constant density star we discovered that the inequalities which describe the trapped (untrapped), singular (nonsingular) partitions of geometries is expressed with greater precision in terms of ρ and l , the more so for the inequalities of necessity. One reason why this is so is that the single constant value encodes global information. However, another reason is that the inequalities do not involve R . What therefore are the appropriate generalizations of the constant ρ which are independent of R ?

The generalization which permits us to formulate necessary conditions is the maximum value of ρ on the support of matter, $\rho_{\max}$. The dimensionless combination of $\rho_{\max}$ and l_0 is $\rho_{\max} l_0^2$. Such conditions can be cast in the form: trapped and/or singular on surface or in the interior implies $\rho_{\max} l_0^2 \geq$ some constant. What is more the constants will be different in the two cases just as they were for a constant density star. Not only are these necessary conditions stronger than those obtained earlier in terms of M for characterizing the surface, they also detect trapped surfaces and/or singularities in the interior if they are present.

1. Singular geometries

If the interior geometry is singular then $R(l_1) = 0$ at some $l_0 \geq l_1 > 0$. We integrate Eq. (6.1) from $l = 0$ up to l_1 . The integral over the divergence results in a surface term which vanishes due to the boundary condition on R

at l_1 . We obtain

$$\int_0^{l_1} dl (1 + R'^2) = 8\pi \int_0^{l_1} dl \rho R^2. \quad (6.13)$$

It is crucial for the successful implementation of the integration that the product RR' vanishes at a singularity even though R' itself might diverge. We now apply the Hölder inequality with the L^∞ norm on ρ to obtain the bound

$$\int_0^{l_1} dl \rho R^2 \leq \rho_{\max} \int_0^{l_1} dl R^2 \quad (6.14)$$

on the left-hand side. The bound is clearly exact if ρ is constant but is a weak estimate if ρ varies a lot. Hence we get

$$\int_0^{l_1} dl (1 + R'^2) \leq 8\pi \rho_{\max} \int_0^{l_1} dl R^2. \quad (6.15)$$

The nice thing about this expression is that we can exploit a Sobolev inequality to get a second bound on the quantities occurring in it. For functions on the finite interval $(0, l_1)$ which vanish at the end points, there is a positive constant S_0 such that

$$S_0 \int_0^{l_1} dl R^2 \leq \int_0^{l_1} dl R'^2. \quad (6.16)$$

The inequality is saturated by the trigonometric function $R(l) = \sin(\pi l/l_1)$ which also determines the optimal value of S_0 : $S_0 = \pi^2/l_1^2$. S_0 is just the ground-state energy of a quantum mechanical particle in a box of width l_1 . In a constant density star $R(l)$ is exactly of this form so that the second estimate involved here is also exact. Substituting Eq. (6.16) in (6.15) we get

$$\int_0^{l_1} dl = l_1 \leq \left(8\pi \rho_{\max} - \frac{\pi^2}{l_1^2} \right) \int_0^{l_1} dl R^2. \quad (6.17)$$

While we do not have a lower bound for R' , we do have an upper bound which is that $R' \leq 1$. This then gives us that $R \leq l$ and that

$$\int_0^{l_1} dl R^2 < \frac{l_1^3}{3}. \quad (6.18)$$

When this is substituted into Eq. (6.17) we get

$$\rho_{\max} l_1^2 > \frac{\pi}{8} + \frac{3}{8\pi}. \quad (6.19)$$

Hence we conclude that if $\rho_{\max} l_1^2 < \pi/8 + 3/(8\pi)$ we will not reach a singularity within a proper distance l_1 from the origin. This estimate is only one of a family of such estimates that can be derived. Let us begin by multiplying Eq. (6.1) across by R^a where a is some, as yet undetermined, constant. We can write (6.1) as

$$R^a + (2a + 1)R^a(R')^2 - 2(R^{a+1}R')' = 8\pi\rho R^{a+2}. \quad (6.20)$$

We integrate this out to a singularity at l at l_1 . The divergence gives us a surface term which will vanish so long as $a > -\frac{1}{2}$. Thus we get

$$\begin{aligned} \int_0^{l_1} dl R^a + \frac{4(2a+1)}{(a+2)^2} \int_0^{l_1} dl ([R^{(a+2)/2}]')^2 \\ = 8\pi \int_0^{l_1} dl \rho R^{a+2} . \end{aligned} \quad (6.21)$$

We now repeat the argument as above, replace ρ with $\rho_{\max}$, and use the Sobolev inequality (with the same constant) to get

$$\int_0^{l_1} dl ([R^{(a+2)/2}]')^2 \geq \frac{\pi^2}{l_1^2} \int_0^{l_1} dl R^{a+2} . \quad (6.22)$$

When these are substituted into Eq. (6.21) we get

$$\begin{aligned} \int_0^{l_1} dl R^a \\ \leq \left[8\pi\rho_{\max} - \frac{4(2a+1)\pi^2}{(a+2)^2 l_1^2} \right] \int_0^{l_1} dl R^{a+2} . \end{aligned} \quad (6.23)$$

The last estimate needed is a bound on the ratio

$$\int_0^{l_1} dl R^{a+2} / \int_0^{l_1} dl R^a . \quad (6.24)$$

The very crude bound l_1^2 follows immediately from the fact that $R' \leq 1$ which gives us that $R < l_1$. Because R appears in the denominator we cannot naively exploit the bound $R \leq l$ as we could in Eq. (6.18). One can, however, show that the linear function $R = l$ maximizes the ratio (6.24) for all functions which satisfy the boundary condition $R(0) = 0$, and the two constraints $R' \leq 1$ and $R \geq 0$ (see Appendix B). Thus

$$\frac{\int_0^{l_1} dl R^{a+2}}{\int_0^{l_1} dl R^a} < \frac{1+a}{3+a} l_1^2 \quad (6.25)$$

holds. We obtain

$$\rho_{\max} l_1^2 \geq \frac{(1+2a)\pi}{(2+a)^2} \frac{1}{2} + \frac{3+a}{1+a} \frac{1}{8\pi} . \quad (6.26)$$

The dominant term on the RHS is the first. The best value for it occurs when $a = 1$. We get

$$\rho_{\max} l_1^2 > \frac{\pi}{6} + \frac{1}{4\pi} . \quad (6.27)$$

The RHS of Eq. (6.27) is slightly larger than that of Eq. (6.19). Note that $a = 1$ corresponds to an integrand RR'^2 which tends exactly to a constant at a singularity. More accurately, the RHS of (6.26) is maximized when $a \sim 0.8$. This is less than 1% better than Eq. (6.27). None of these numbers are particularly impressive, they are, at best, only half the value we get in the constant density star.

Despite our effort, it is clear that the estimate equation (6.25) is the weakest link in the argument.

2. Apparent horizons

Suppose now that the geometry possesses an apparent horizon, $R'(l_1) = 0$ at some $l_0 \geq l_1 > 0$. We again integrate Eq. (5.2) from $l = 0$ up to l_1 : As before, the boundary condition kills the surface term and we are left with an identical equation:

$$\int_0^{l_1} dl (1 + R'^2) = 8\pi \int_0^{l_1} dl \rho R^2 . \quad (6.28)$$

The first approximation proceeds identically

$$\int_0^{l_1} dl (1 + R'^2) \leq 8\pi\rho_{\max} \int_0^{l_1} dl R^2 . \quad (6.29)$$

For the second approximation we again exploit a Sobolev inequality involving an appropriate function space on the right-hand side. If $R(l)$ is a function on the finite interval $(0, l_1)$ such that $R(0) = 0$ and $R'(l_1) = 0$, then there is some constant S_1 such that

$$S_1 \int_0^{l_1} dl R^2 \leq \int_0^{l_1} dl R'^2 . \quad (6.30)$$

The inequality is again saturated by a sine function, $R(l) = \sin(\pi l/2l_1)$ giving $S_1 = \pi^2/4l_1^2$. In the case of a constant density star, both approximations are exact. So

$$\int_0^{l_1} dl = l_1 \leq \left(8\pi\rho_{\max} - \frac{\pi^2}{4l_1^2} \right) \int_0^{l_1} dl R^2 . \quad (6.31)$$

We again use Eq. (6.18) and we finally get

$$\rho_{\max} l_1^2 > \frac{\pi}{32} + \frac{3}{8\pi} . \quad (6.32)$$

This is a much better estimate than the corresponding bound (6.19) or its improvements for the singularity, largely due to the fact that (6.18) is much sharper in this case. In fact it is about $\frac{3}{4}$ of the number that we get in the constant density star.

An expression equivalent to Eq. (6.26) can also be written down. We have that

$$\rho_{\max} l_1^2 \geq \frac{(1+2a)\pi}{(2+a)^2} \frac{1}{8} + \frac{1}{8\pi} \frac{3+a}{1+a} . \quad (6.33)$$

The maximum of this expression occurs at about $a = 0.25$ and this improves on the bound equation (6.32) again by about 1%.

If ρ is bounded from below within the star, its minimum value, $\rho_{\min}$ can be exploited to provide alternative sufficiency conditions. However, if ρ vanishes anywhere this fails. In addition, because the inequalities of sufficiency are derived by assuming that the star does not possess a trapped surface and/or singularity we cannot

exploit Sobolev inequalities in the way did for the necessary conditions.

VII. A LOWER BOUND ON THE BINDING ENERGY OF A SPHERICALLY SYMMETRIC DISTRIBUTION OF MATTER

A. Momentarily static spherically symmetric shell

An important special case of the two-density model is the shell with¹²

$$\rho(l) = \sigma \delta(l - l_0) . \quad (7.1)$$

This model will be useful in two regards. First, it represents the configuration with least binding energy. Second, when we generalize our analysis to configurations with nonvanishing current, it provides a useful exactly solvable model in which we can examine the behavior of the geometry near singular points [3].

Inside the shell, space is flat so that $R = l$. As a consequence the material energy M is equal to its Newtonian value

$$M = 4\pi\sigma l_0^2 . \quad (7.2)$$

By integrating Eq. (6.2) across $l = l_0$ we find that R' suffers a discontinuity at the shell given by

$$\Delta R' = -4\pi\sigma l_0 . \quad (7.3)$$

Outside $R'(l_{0+}) = 1 + \Delta R'$. If σ is positive $R'(l_{0+})$ will be bounded above by one. As we have seen M does not see the discontinuity in R' . The ADM mass m_∞ , however, does:

$$m_\infty = m(l_{0+}) = \frac{l_0}{2} [1 - R'^2(l_{0+})] . \quad (7.4)$$

A horizon must form at some point outside the shell if $4\pi\sigma l_0 \geq 1$. If $4\pi\sigma l_0 < 2$, $R' > -1$, m_∞ is positive and the geometry is regular. When $4\pi\sigma l_0 = 2$, $R' = -1$, $m_\infty = 0$ and the geometry is weakly singular. At this value, $R'(l_{0+}) = -1$, the ADM mass m vanishes, and the geometry outside is flat and closed as well. The bag of gold consists of two flat caps sewn together along the shell. If $4\pi\sigma l_0 > 2$, $R'(l_{0+}) < -1$, m_∞ is negative and the geometry suffers from a strong singularity in the same way as it does in the two-density model discussed in Sec. V.

A useful expression for m_∞ is obtained by writing $\Delta R'$ in terms of M and l_0 . We use (7.2) to express (7.3) $\Delta R' = -M/l_0$ so that Eq. (7.4) can be rewritten in the form

$$m = M - \frac{M^2}{2l_0} . \quad (7.5)$$

The binding energy $M - m = M^2/(2l_0)$ is the Newtonian value. The distribution of matter with the least binding energy in Newtonian gravity is a shell.

B. Lower bound on the binding energy

It was conjectured by Arnowitt, Deser, and Misner but only proven recently by BMÓM that, in general [15,10]

$$M - m \geq \frac{M^2}{2l} . \quad (7.6)$$

This provides a lower bound on the binding energy of a spherically symmetric distribution of matter of fixed M and l_0 . The inequality is sharp. The distribution of matter which saturates this inequality is the shell [compare Eq. (3.17)].

Note that if l is decreased while M is kept fixed, the binding energy increases—the more compact the material system the larger its binding energy. BMÓM's proof relied on the use of a conformally flat coordinate system. We will reproduce the proof using the proper radius directly. The trick, as demonstrated by BMÓM is to mimic the calculation for the shell. We rewrite Eq. (6.2)

$$R' = \frac{1}{R}\Gamma - \frac{M}{R} . \quad (7.7)$$

We now substitute for R' in the expression (2.4) for m :

$$\begin{aligned} m &= \frac{R}{2}(1 - R'^2) \\ &= -\frac{M^2}{2R} + \frac{R}{2}(1 - F^2) + FM , \end{aligned} \quad (7.8)$$

where we introduce the dimensionless quantity $F := \Gamma/R$. We now express the difference as the sum of the lower bound we wish to establish and a remainder Q :

$$M - m = \frac{M^2}{2l} + Q , \quad (7.9)$$

where

$$Q \equiv \frac{M^2}{2} \left(\frac{1}{R} - \frac{1}{l} \right) + (1 - F)M + \frac{R}{2}(F^2 - 1) . \quad (7.10)$$

We will show that Q is always positive. The key to doing this is to note that if the geometry is nonsingular, then

$$1 \leq F \leq \frac{l}{R} . \quad (7.11)$$

The upper bound is simply¹³ Eq. (6.4). To obtain the appropriate lower bound, let us first examine the difference between the proper radius and the circumferential radius:

¹²In the notation of Sec. V, the shell corresponds to $\rho_1 = 0$, and the limiting value $\sigma = \rho_2(l_2 - l_1)$.

¹³We also have already got a weak lower bound on F which does not rely on the positivity of ρ , $l/(2R) \leq F$.

$$l - R = \int_0^l dl(1 - R') .$$

We can place a lower bound on this difference using the inequality $1 + R' \leq 2$:

$$l - R \geq \frac{1}{2} \int_0^l dl[1 - (R')^2] .$$

This can be inverted to provide the lower bound

$$\frac{1}{2} \int_0^l dl(R')^2 \geq R - \frac{l}{2} .$$

From this we can deduce that $F \geq 1$. We therefore have established both an upper bound and a lower bound on F .

We can now demonstrate that Q is always positive. We note that both the first and third terms are positive. The discriminant of Q is given by $(F-1)(F+1-2l/R)$ and is negative. The quadratic therefore possesses no real root. Thus Q is positive everywhere.

Note that Eq. (7.6) implies $m \geq 0$ iff $M \leq 2l$. This reproduces the sufficiency condition, Eq. (6.10).

VIII. CONCLUSIONS

In this paper we have attempted to identify generic features of asymptotically flat spherically symmetric solutions to the Hamiltonian constraint when the spatial geometry is momentarily static.

Our focus has been on the characterization of the two strong field features of such geometries, apparent horizons and singularities. The simple exactly solvable models consisting of piecewise constant energy density profiles provide a useful guideline for the choice of appropriate variables to characterize these features. Sufficiency conditions describing their existence can be cast as inequalities between M and l_0 . Matching inequalities of necessity are better cast in terms of $\rho_{\max}$ and l_0 . We found that the latter bounds could be improved using simple functional inequalities. Because these inequalities are not sharp, it is likely that the bounds can be improved. This work might provide clues toward the identification of appropriate variables with respect to which to formulate the necessary part of the hoop conjecture [11].

The techniques introduced in this paper prepare the ground for the examination of the constraints when $K_{ab} \neq 0$. In paper III, we will examine the analogous problem in this more general case.

At the end of paper I, we introduced the optical scalar plane as a representation of the phase space of the theory. In this representation nonsingular solutions of the constraints appear as bounded closed trajectories each containing the point $P \equiv (2, 2)$. The moment of time symmetry solutions we have considered in detail in this paper correspond to degenerate trajectories that run along the diagonal, $\omega_+ = \omega_-$ starting out at the point P . If the solution possesses a trapped surface it will cross the origin, $(0, 0)$, corresponding to $R' = 0$ at least once. If it

is nonsingular, it will make an even number of crossings before returning to the point P . In particular, we saw that when the energy density profile is monotonically decreasing, a configuration can cross the origin an arbitrarily large even number of times. If the solution is singular it will cross the origin an odd number of times before ultimately proceeding toward the point $Q \equiv (-2, -2)$. At this point, the QLM vanishes and $R' = -1$. Only in situations of zero measure occurring when the energy density is fine-tuned in such a way that the QLM tends monotonically to zero will the trajectory terminate at this point. In general, once Q is breached the trajectory cannot reenter the bounded interval along the diagonal between Q and P . What must occur is that it continues monotonically along the diagonal toward unboundedly large negative values of R' .

When $K_{ab} \neq 0$, there are many more possibilities [3]. Solutions may possess a future or a past apparent horizon. These horizons will no longer generally coincide with the extremal surfaces of the three-geometry. Neither does the existence of the former necessarily imply the existence of the latter.

Whereas in the MSC's, the only approach to singularity is through the point Q , there is now a wide range of possibilities. In general, a nonvanishing extrinsic curvature can lead to more severe singularities than those encountered at a moment of time symmetry. In addition, the constraints no longer imply that the scalar curvature is finite. The converse of the positive QLM theorem is no longer true. Indeed, m can be positive everywhere and yet the geometry be singular. m will, however, always be finite if the sources are finite.

What is remarkable is that it is possible to generalize the necessary and sufficient conditions examined in this paper with the appropriate generalizations of M and $\rho_{\max}$ and an appropriate gauge. Not surprisingly, the gauges that do work are precisely the α -parametrized linear extrinsic curvature gauges introduced in paper I, with α in the range $0.5 < \alpha < \infty$ [1].

The bags of gold which occur behind singularities are physically disconnected from the exterior and, as such, of little more than curiosity value in the present context, at least, at the level of the classical theory. In cosmology, however, the bag of gold can be interpreted as a closed universe.

If $l = 0$ is the north pole, $l = l_S$ is just the south pole of the closed universe. This closed universe will generally not be regular. In fact, the argument presented in Sec. II suggests that regular closed universes constitute a subset of zero measure in the set of all closed universes. This universe will be singularity free if and only if the QLM vanishing at one pole also vanishes at the other pole. We can think of the value $M(l_S)$ assumed by M at the south pole as a measure of the binding energy of this regular closed universe.

Using Eq. (2.5') the integrability condition $m(l_S) = 0$ on a regular bag of gold implies

$$\int_0^{l_S} dl R^3 \rho' = 0 . \quad (8.1)$$

A regular spherically symmetric closed universe is impossible if ρ is strictly monotonic. ρ must possess a maximum or a minimum away from the poles. We will examine the configuration space of regular spherically symmetric cosmologies in a subsequent paper [16].

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APPENDIX A: THE RICCI TENSOR FROM THE SCALAR CURVATURE

It is always possible to construct the Ricci tensor once we know the scalar curvature. We will do this for a general spherically symmetric geometry by appealing to a minisuperspace Hilbert variational principle. Let

$$I = \int d^3x \sqrt{g} \mathcal{R}. \quad (\text{A1})$$

Then

$$\frac{1}{\sqrt{g}} \frac{\delta I}{\delta g_{ab}} = \mathcal{R}^{ab} - \frac{1}{2} \mathcal{R} g^{ab}. \quad (\text{A2})$$

To implement the variational argument we need to restore the metric coefficient $\mathcal{L}^2$ (defined in I) so that the spatial coordinate system is only defined up to spherical symmetry. This is because we require as many independent metric components as there are independent components of $\mathcal{R}_{ab}$ which is two in this case. Up to a divergence which is irrelevant for our purposes (here a prime refers to differentiation with respect to r),

$$I = 2 \int d^3x \left[\frac{R'^2}{\mathcal{L}} + \mathcal{L} \right]. \quad (\text{A3})$$

Thus

$$\frac{\delta I}{\delta R} = -4 \left(\frac{R'}{\mathcal{L}} \right)', \quad \frac{\delta I}{\delta \mathcal{L}} = - \left[\left(\frac{R'}{\mathcal{L}} \right)^2 - \mathcal{L} \right]. \quad (\text{A4})$$

We now note that

$$\frac{\delta I}{\delta g_{rr}} = \frac{1}{2\mathcal{L}} \left(\frac{\delta I}{\delta \mathcal{L}} \right), \quad \frac{\delta I}{\delta g_{\theta\theta}} = \frac{1}{4R} \left(\frac{\delta I}{\delta \mathcal{L}} \right). \quad (\text{A5})$$

Having taken all variations we can safely set $\mathcal{L} = 1$. We use Eqs. (A2), (A4), and (A5) to express the nonvanishing components of the Ricci tensor in the form

$$\mathcal{R}^{rr} = \frac{1}{2} \mathcal{R} + \frac{1}{R^2} (R'^2 - 1), \quad \mathcal{R}^{\theta\theta} = \frac{1}{2R^2} \mathcal{R} + \frac{R''}{R^3}. \quad (\text{A6})$$

We now eliminate the second derivative in the second term in favor of $\mathcal{R}$ and lower derivatives of R . Equations (2.11a) and (2.11b) for the scalars $\mathcal{R}_{\mathcal{L}}$ and $\mathcal{R}_R$ are then given by the appropriate projections.

In any regular solution of the constraints $\mathcal{R}_R \geq \mathcal{R}/4$, and $\mathcal{R}_{\mathcal{L}} \leq \mathcal{R}/2$ everywhere.

We note that there is no nontrivial regular everywhere solution of the Hamiltonian constraint in which either $\mathcal{R}_R$ or $\mathcal{R}_{\mathcal{L}}$ vanishes. In the case of $\mathcal{R}_R = 0$ this is obvious on account of the above bound. In the latter case, consistency with the constraint (at a moment of time symmetry) then implies that

$$\rho R^2 = \frac{1}{R} \int_0^l \rho R^2 R' dl,$$

so that $\rho \sim R^{-3}$ which is singular.

APPENDIX B: PROOF OF THE BOUND (6.25)

In this appendix we prove the validity of the bound, Eq. (6.25) in the text for all $a > -1$. For simplicity we will first provide a proof for $a = 1$. The generalization will then be clear. We define

$$F(R, l_1) = \int_0^{l_1} R^3 dl / \int_0^{l_1} R dl. \quad (\text{B1})$$

We will prove that

$$F(R, l_1) \leq \frac{l_1^2}{2} \quad (\text{B2})$$

is true, for functions that satisfy

$$R(0) = 0, \quad R'(l) \leq 1, \quad R \geq 0. \quad (\text{B3})$$

Let us calculate the first variation of F :

$$\delta F = \int_0^{l_1} (3R^2 - F) \delta R dl / \left(\int_0^{l_1} R dl \right)^2. \quad (\text{B4})$$

We claim that the maximum is achieved by $R = l$.

In this variational problem we can only consider variations which satisfy Eq. (B3), in other words we start with a function R_0 which satisfies Eq. (B3) and look for a family of functions R_t , such that $R_{t=0} = R_0$ and that for some range of t , $0 \rightarrow \Delta t$, $R(t)$ also satisfies Eq. (B3). The δR in Eq. (B4) must be the first derivative of such a sequence R_t with respect to t , evaluated at $t = 0$.

In particular, the allowed variations around $R = l$ must satisfy

$$\delta R' \leq 0, \quad \delta R(0) = 0, \quad (\text{B5})$$

which forces δR to be negative and monotonically decreasing on the interval $(0, l)$. Now let us consider the integral in Eq. (B4), with $R = l$ and $F = l^2/2$ if we replace δR with a negative constant $-C$. It is clearly negative, equaling $-Cl_1^2/2$. Notice that $3R^2 - F$ is negative in the range $(0, l_1/\sqrt{6})$ and positive on the rest. This means that, since δR must be negative, the integrand in Eq. (B4) is positive on the interval $(0, l_1/\sqrt{6})$ and negative elsewhere.

Since δR is monotonically decreasing if we replace δR

in the integral by its value at $l = l_1/\sqrt{6}$ we make the integral more positive. However, it is still negative. Therefore the actual integral is negative and all the allowed variations reduce F . Hence $R = l$ is a local maximum.

This is not enough, however. We need to show that no other maximum occurs. To do this we note that $F < R_{\max}^2$, where $R_{\max}$ is the maximum achieved by R on the interval $(0, l_1)$. This can be shown by extracting $R_{\max}^2$ from the R^3 integral. We immediately get that $(3R^2 - F')$ must be positive near the maximum of R . If we further have that $R' < 1$ at $R_{\max}$ we can find a positive variation δR , localized near the maximum of R , which satisfies the $R' < 1$ condition. Such a variation will increase F . Thus,

in particular, R cannot possess an interior maximum. The maximum must be assumed at $l = l_1$ and at this maximum, $R' = 1$.

Thus the only case we have to consider is the case where R achieves its maximum at l_1 and $R' = 1$ there. We need only to show that if $R_{\max} < l_1$, there exists a variation increasing F .

Suppose that $R_{\max} < l_1$. We cannot have $R' = 1$ on the whole interval, for this would imply $R = l$. Therefore we have a point l_2 where $R(l_2)' < 1$. An allowed variation is then $\delta R = 0$ on the interval $(0, l_2)$ and $\delta R = 1$ on (l_2, l_1) . For l_2 close to l_1 , the integral in Eq. (B4) can be approximated:

$$\begin{aligned} \int_{l_2}^{l_1} [3(l - l_1 + R_{\max})^2 - F] dl &= R_{\max}^3 - (l_2 - l_1 + R_{\max})^3 - F(l_1 - l_2) \\ &= (l_1 - l_2)[(l_1 - l_2)^2 - 3R_{\max}(l_1 - l_2) + 3R_{\max}^2 - F]. \end{aligned} \quad (B6)$$

We know that we can choose l_2 such that $l_1 - l_2 < R_{\max}$ because of the positivity of R . But this, in conjunction with $F < R_{\max}^2$, is sufficient to show that the expression in Eq. (B6) is positive. This completes the proof.

The generalization of this proof is straightforward. We can prove that

$$F_a(R, l_1) = \frac{\int_0^{l_1} R^{2+a} dl}{\int_0^{l_1} R^a dl} \leq \frac{(1+a)}{3+a} l_1^2, \quad (B7)$$

for any $a > -1$. The technique is as before, first of all we prove that $R = l$ is a local maximum, then we prove that if $R_{\max} < l_1$ there exists an allowed variation which increases F .

We have

$$\delta F_a = \frac{\int_0^{l_1} ([2+a]R^{1+a} - aFR^{-1+a})\delta R dl}{\left(\int_0^{l_1} R^a dl\right)^2}. \quad (B8)$$

If $a < 0$ we immediately see that increasing R increases F . Therefore $R = l$ is the maximum. We can now restrict our attention to the case where $a \geq 0$.

If we replace δR in Eq. (B8) with a negative constant $-C$ we can do the integration and show that it is clearly

negative, equaling $-2Cl_1^2/(3+a)$. It is clear that the allowed δR 's must be negative and monotonically decreasing. The integrand in Eq. (B8) is positive in the range $(0, l_2)$ and negative in (l_2, l_1) , where l_2 is given by

$$l_2^2 = \frac{a(1+a)}{(2+a)(3+a)} l_1^2 < l_1^2.$$

Therefore we can replace δR by its value at l_2 and make the integral more positive. However, since the integral with a negative constant is negative, the correct integral is even more negative. Therefore $R = l$ is a local maximum.

The rest of the proof is just as before. We know that $F < R_{\max}^2$ so therefore the quantity $([2+a]R^{1+a} - aFR^{-1+a})$ is positive near the maximum of R . Therefore a positive variation, localized near the maximum, increases F . The only situation we need to deal with is where the maximum of R occurs at $l = l_1$ and simultaneously $R' = 1$ there. Let us find the point $l = l_2$ where $R'(l_2) < 1$ and $R = l - l_1 + R_{\max}$ in the interval (l_2, l_1) . Now an allowed variation is one which is zero on the interval $(0, l_2)$ and a positive constant on the interval (l_2, l_1) . For such a variation we can do the integration in Eq. (B8) to give

$$\begin{aligned} \delta F_a \left(\int_0^{l_1} R^a dl \right)^2 &= R_{\max}^{2+a} - (l_2 + R_{\max} - l_1)^{2+a} - FR_{\max}^a + F(l_2 + R_{\max} - l_1)^a \\ &= R_{\max}^a [R_{\max}^2 - F] - (l_2 + R_{\max} - l_1)^a [(l_2 + R_{\max} - l_1)^2 - F]. \end{aligned} \quad (B9)$$

The first term in on the second line Eq. (B9) is positive and also the first term is greater than the second term because $R_{\max} > l_2 + R_{\max} - l_1$. Therefore the variation of F is positive.

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